CEU MATHEMATICS ENTRANCE EXAM, 2014

DIRECTIONS: *There are 10 problems (10 points each). You have 3 hours. No books or notes. To receive full credit your solutions must be clear, complete and correct.*

(1) Suppose that $K \subset \mathbb{R}^n$ is a compact set. Show that $L = \{x + y \mid x, y \in K\}$ is also compact.

(2) Suppose that $\{x_k\}$ is a sequence in $\mathbb{R}^n$ with the property that the series $\sum x_k$ is absolutely convergent. Show that $\sum \cos k \cdot x_k$ is convergent.

(3) Assume that for $f\colon \mathbb{R} \to \mathbb{R}$ we have that $f(x + y) = f(x) + f(y)$, and f is continuous at the origin. Prove that f is continuous everywhere.

(4) Suppose that f is a continuous function on the interval $[a, b] \subset \mathbb{R}$, and $\int_a^b f(x)h(x)dx = 0$ for all continuous functions h on $[a, b]$. Show that $f(x) = 0$ for all $x \in [a, b]$.

(5) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function with continuous derivative. Suppose furthermore that there is $K \in \mathbb{R}$ with the property that $|f'(x)| \leq K$ for all $x \in \mathbb{R}$. Show that there is a constant $c > 0$ such that the function $x + cf(x)$ is a bijection.

(6) Suppose that $\mathcal{H} \subseteq M_n(\mathbb{R})$ is a set of real $n \times n$ matrices and $A \in M_n(\mathbb{R})$ is invertible such that $AH \in \mathcal{H}$ for every $H \in \mathcal{H}$. Does it follow that $A^{-1}H \in \mathcal{H}$ for every $H \in \mathcal{H}$? Prove or give a counterexample.

(7) For a group G let $i(G) = \{n \geq 1 | \exists g \in G,\ o(g) = n\}$ denote the set of integers that occur as order of an element in G. Which of the following is/are true? Justify your answers.
   (a) If $n, m \in i(G)$ then $gcd(n, m) \in i(G)$.
   (b) If $n, m \in i(G)$ then $lcm(n, m) \in i(G)$.
   (c) For any prime p and natural number n there exists G such that $i(G) = \{1, p, p^2, \dots, p^n\}$.

(8) Let $f(x)$ be a polynomial of degree at most 2 with rational coefficients. Suppose $f(n)$ is an integer for every integer n. Show that each coefficient of $f(x)$ is half of an integer.

(9) Let
$$R = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & d & e \end{pmatrix} \mid a, b, c, d, e \in \mathbb{R} \right\}.$$
Show that every ideal of the ring R is a real vector space and determine the ideals of dimension 1. (The subring $I \leq R$ is an ideal if $i \in I$, $r \in R$ implies ir, $ri \in I$.)

(10) Let $v, u \in \mathbb{R}^n$ be row vectors. Prove that the matrix $v^T u$ has eigenvalue 1 if and only if $uv^T = 1$.