Central European University

Department of Mathematics and Its Applications

Entrance Examination: March 6, 2004

Directions: This exam has 10 problems (*10 points each*). To receive full credit your solution must be clear and correct. You have 3 hours. No books or notes.

1. a) Let $\{c_n\}$ be a sequence of real numbers that converges to c. Show that their "average" (arithmetic mean), $S_n = \frac{1}{n}(c_1 + c_2 + \cdots + c_n)$, also converges to c.
   b) Give an example of a sequence that does *not* converge but whose arithmetic mean does converge.

2. a) Let $A = (a_{ij})$ be a self-adjoint $n \times n$ complex matrix (so $a_{ij} = \bar{a}_{ji}$). If it satisfies $A^m = 1$ for some *odd* positive integer m, show that A is the identity matrix.
   b) Does the same conclusion hold for m even? (Justify your assertions.)

3. Let $f(x)$ be a real-valued continuous function defined for all $0 \leq x \leq 1$. Show that
$$\lim_{n\to\infty} \int_0^1 f(x^n)\, dx$$
exists and compute this limit. Justify your assertions.

4. Show that in a group any subgroup of finite index contains a normal subgroup of finite index.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with continuous derivative. Suppose $|f'(x)| \leq 2$ for all x. Show that there exists $c > 0$ such that the function $g : \mathbb{R} \to \mathbb{R}$ defined by
$$g(x) = x + cf(x)$$
is a bijection (that is, one-to-one and onto) with differentiable inverse.

6. For each of the following, either give an explicit example or else explain briefly why no such example exists.
   a) An ideal $I \subset \mathbb{R}[x]$ which is not principal.
   b) A unique factorization domain which is not a Euclidean domain.
   c) $p(x),\ q(x) \in (\mathbb{Z}/6\mathbb{Z})\,[x]$, both non-constant, such that $p(x)q(x) = x$.

7. Consider a smooth real-valued function $u(x, y) = u(r)$ that depends only on $r = \sqrt{x^2 + y^2}$.
   a) Compute the Laplacian, $\Delta u := \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.
   b) Find all such functions $u = u(r)$ that satisfy $\Delta u = 0$ (except possibly at the origin).

8. Let F be a finite field of odd order and $a,\ b,\ c$ nonzero elements of F. Show that the equation $ax^2 + by^2 = c$ has a solution over F in $x,\ y$.

9. Prove that every finitely generated subgroup of the group of rational numbers with respect to the addition is cyclic.

10. Consider the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n^x}$ where x is a real number. Assume the real sequence $\{a_n\}$ is bounded. Prove that for any $c > 1$ this series converges uniformly for x in the interval $x \geq c$.