CENTRAL EUROPEAN UNIVERSITY
Department of Mathematics and Its Applications

MATHEMATICS ENTRANCE EXAMINATION: March 5, 2005

DIRECTIONS: There are 10 problems (*10 points each*). To receive full credit your solutions must be clear, complete, and correct. You have 3 hours. No books or notes.

1. Show that the series $\sum_{n=1}^{\infty} n^{\alpha} 3^{-n}$ converges for every $\alpha \in \mathbb{R}$ and for $\alpha = -1$ determine its sum.

2. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable on $\mathbb{R}$ and f' is continuous on $\mathbb{R}$, but not differentiable everywhere.

3. Let $f : [0,1] \to \mathbb{R}$ be a given continuous function. Show that the sequence $x_n : [0,1] \to \mathbb{R}$, defined by

$$x_1(t) = f(t), \quad x_{n+1}(t) = 1 + t + \int_0^t x_n(s)\, ds, \quad n = 1, 2, \cdots$$

is uniformly convergent on $[0,1]$.

4. Show that $f : \mathbb{R} \to \mathbb{R}$,

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{for } (x,y) \neq (0,0), \\ \\ 0 & \text{for } x = y = 0, \end{cases}$$

is continuous on $\mathbb{R}^2$ and calculate $\iint_D f(x,y)\, dxdy$, where

$$D = \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq 1, \ y \geq 0\}.$$

5. Let a be a positive real number. Show that every solution $x = x(t)$ of the differential equation $x''(t) + 2x'(t) + x(t) = (a-1)^2 e^{-at}$ tends to zero as $t \longrightarrow \infty$.

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6. Let A be an $n \times n$ matrix with rational entries such that the minimal polynomial of A is $x^3 + 2x + 2$.

a. Prove that 3 divides n.

b. Show a 6×6 rational matrix A with the above minimal polynomial.

7. Consider the $\mathbb{R}$–algebra of linear transformations of the $\mathbb{R}$–vector space of polynomials in $\mathbb{R}[x]$ of degree ≤ 2. Denote by A the $\mathbb{R}$–subalgebra generated by the transformation $\frac{\partial}{\partial x}$, and denote by B the $\mathbb{R}$–subalgebra generated by $2\frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2}$. Are the algebras A and B isomorphic?

8. a. Prove that if N is a normal subgroup of G of finite index k, then x^k is contained in N for all elements x in G.

b. Prove that the only proper subgroup of finite index in the multiplicative group of non-zero real numbers is the subgroup of positive real numbers.

9. Define the binary operation $\circ$ on the ring $\mathbb{Z}_n$ of modulo n residue classes as $a \circ b = ab + a + b$ for $a, b \in \mathbb{Z}_n$.

a. Is the operation $\circ$ associative?

b. Is $(\mathbb{Z}_n, \circ)$ a group?

10. Determine the degree of the field extension $\mathbb{Q}(\sqrt{2}, i)$ over the field $\mathbb{Q}$ of rational numbers (where i is the imaginary complex unit).