CENTRAL EUROPEAN UNIVERSITY, Mathematics Department

ENTRANCE EXAMINATION for both MS and PhD: March 1, 2008

DIRECTIONS: *There are 10 problems (10 points each). You have 3 hours. No books or notes. To receive full credit your solutions must be clear, complete and correct. The order of your solutions is not important, so you are free to start with any problem.*

1. Let V be a finite dimensional vector space over F, the finite field of 2 elements. Determine the sum

$$\sum_{v \in V} v.$$

2. If R is a ring then let $Z(R)$ denote its *centre*, that is $Z(R) = \{a \in R | ab = ba \, \forall b \in R\}$. Suppose that $I \triangleleft R$ is a two-sided ideal and I is commutative and has no 0-divisors. Show that $I \subseteq Z(R)$. (A nonzero element a of a commutative ring S is called *zero-divisor*, if for some other nonzero element b we have $ab = 0$.)

3. Is there a rational polynomial $f(x)$ of degree 100 such that its remainder after division by $g(x) = x^{60} - 2$ and by $h(x) = x^{49} + 3$ are the same?

4. Let $\mathbb{Z}$ denote the group of integers with respect to addition. Let G be a group, and fix an element $x \in G$. Consider the map $\varphi_x : \mathbb{Z} \to G$ defined by $\varphi_x(n) = x^n$ for every integer n. Verify that φ_x is a homomorphism.

Show that for every homomorphism $\psi : \mathbb{Z} \to G$ there exists an $x \in G$ such that $\psi = \varphi_x$.

5. Let V be a finite dimensional vector space over the real field $\mathbb{R}$. We say that a linear transformation P of V is a *projection* if $P \cdot P = P$. Let I denote the identity transformation of V. For which $c \in \mathbb{R}$ is it true that for every projection P the transformation $P + cI$ is invertible? (Hint: You may want to consider first the eigenvalues of P.)

6. Show that $\int_0^\infty x^k e^{-x^2} \, dx$ is convergent for all $k \geq 0$ and calculate its values for $k = 0$ and $k = 1$.

7. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function, with period $T > 0$, then $\forall a \in \mathbb{R}$

$$\frac{1}{t} \int_a^{a+t} f(s) \, ds \longrightarrow \frac{1}{T} \int_0^T f(s) \, ds \quad \text{as} \quad t \to \infty.$$

8. Show that $\sum_{n=0}^\infty (-1)^n \frac{2n+1}{2^n}$ is absolutely convergent and calculate its sum.

9. Let $c_n \in \mathbb{R}$, $c_n \longrightarrow c \in \mathbb{R}$, and let $a_n, , b_n : [0, T] \to \mathbb{R}$ be continuous functions such that $a_n \longrightarrow a$, $b_n \longrightarrow b$, uniformly on $[0, T]$, for a given $T > 0$. If x_n denote the solution of

$$x_n'(t) = a_n(t) x_n(t) + b_n(t), \; t \in [0, T]; \; x_n(0) = c_n,$$

then both x_n and x_n' converge uniformly on $[0, T]$.

10. If $u_n \in \mathbb{R} \setminus \{0\}$, $n = 1, 2, ...$, such that $|u_{n+1}/u_n|$ converges to some $l < 1$, then $u_n \longrightarrow 0$.