CEU Mathematics Entrance Exam, 2009

Test for candidates applying for M.S. and/or Ph.D.

DIRECTIONS: *There are 10 problems (10 points each). You have 3 hours. To receive full credit your solutions must be clear, complete and correct. No books or notes.*

1. Let R be a commutative ring, $1 \in R$. Put $\Delta(R) = \{(a,a) \mid a \in R\} \leq R \times R$ the diagonal subring of the direct product $R \times R$.

a. Show that $\Delta(R)$ is maximal in $R \times R$ if and only if R is a field. (We say $A < B$ is maximal, if A is not contained in any intermediate subring $A < C < B$.)

b. When is $\Delta(R)$ an ideal of $R \times R$?

2. Are there real polynomials $a(x)$ and $b(x)$ such that

$$\frac{1}{t^{20}-1} = \frac{a(t)}{t^2-1} + \frac{b(t)}{t^5-1}$$

holds for every real $t \neq \pm 1$?

3. Let G be a group. Assume there are n elements $x_1, \ldots x_n$ of G such that the subgroups generated by all subsets of $\{x_1, \ldots x_n\}$ are distinct. Show that the order of G is at least 2^n. Is equality possible?

4. Denote by $z_1, z_2, \ldots z_n$ those complex numbers whose n-th power is 2. Determine the value $S_k = z_1^k + z_2^k + \cdots + z_n^k$ for every $k \in \mathbb{N}$.

5. Let V be a 4-dimensional vectorspace over $\mathbb{C}$. A set of 4 linear transformations $\varphi_1, \ldots \varphi_4$ of V is *sparse* if each has an eigenbasis but there is no common eigenvector for all 4 of them. What is the smallest possible cardinality of the set $\{\lambda \in \mathbb{C} \mid \exists i : \lambda \text{ is an eigenvalue of } \varphi_i\}$ if we consider every *sparse* set of 4 linear transformations?

6. Determine the volume of the bounded domain in $\mathbb{R}^3$ between the surfaces $z = 4x^2 + y^2$, $z = c(4x^2 + y^2)^{1/2}$, where c is a positive constant.

7. Show that every solution $u = u(t)$ of the differential equation $t^2 u'' + 3tu' + u = 0,\ t > 0$ exists on $(0, \infty)$ and converges to zero, as $t \longrightarrow \infty$.

8. Show that $x^3 > 6(x - \sin x)$ for all $x > 0$.

9. Let $f_n : [0,1] \longrightarrow \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} \frac{n}{2}x^2\,, & \text{if } \ 0 \leq x \leq \frac{1}{n}, \\ x - \frac{1}{2n}\,, & \text{if } \ \frac{1}{n} < x \leq 1. \end{cases}$$

Show that:

(a) f_n is continuously differentiable on $[0,1]$ for all $n = 1, 2, \ldots$;

(b) $(f_n)_{n\geq 1}$ is uniformly convergent on $[0,1]$;

(c) $(f_n')_{n\geq 1}$ is pointwise convergent, but not uniformly convergent on $[0,1]$.

10. For each pair $(a, \alpha) \in \mathbb{R}^2$ there exists a unique sequence of real numbers satisfying

$$u_0 = a,\ u_n + n^\alpha u_n^9 = u_{n-1},\ n = 1, 2, \ldots$$

Show that $(u_n)_{n\geq 1}$ is convergent and calculate its limit for $\alpha \geq -1$.