

CENTRAL EUROPEAN UNIVERSITY, Mathematics Department
ENTRANCE EXAMINATION for both MS and PhD: March 12, 2011

DIRECTIONS: There are 10 problems (10 points each). You have 3 hours. No books or notes. To receive full credit your solutions must be clear, complete and correct.

1. Determine the characteristic polynomial of the complex $n \times n$ matrix

$$M = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & 1 \\ 1 & \cdots & 1 & 1 & 0 \end{pmatrix}.$$

2. Let $f(x), g(x)$ be two integer polynomials such that the sum of the coefficients of $f(x)$ is negative, the sum of the coefficients of $g(x)$ is positive. Prove that there exists a unique rational number q such that $q \cdot f(x) + g(x) \in \mathbb{Q}[x]$ is divisible by $x - 1$.

3. Let \mathcal{M} denote the 2×2 matrices over the field of 3 elements and let $I \in \mathcal{M}$ denote the identity matrix. Determine whether the following claim is true or false. Justify your answer.

There exists $A \in \mathcal{M}$ such that $A^4 = I$ but $A^2 \neq I$.

4. Let $R = \{a/b \in \mathbb{Q} \mid a, b \in \mathbb{Z}, b \text{ odd}\}$ be the ring of rational numbers with odd denominator. Show that the non-zero ideals of R are of the form $I_m = (2^m)$ (for $m = 0, 1, \dots$).

5. Determine the finite groups with exactly three conjugacy classes.

6. Show that the sequence $(x_n)_{n \geq 1}$ defined by $x_n = \sin(2\pi(n^3 - n^2 + 1)^{1/3})$ is convergent and compute its limit.

7. Show that $x^3 > 6(x - \sin x)$ for all $x > 0$.

8. Show that $\int_{-\infty}^{+\infty} e^{-x^2+2x} dx$ converges and compute its value.

9. Show that the series $\sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!}$ converges on \mathbb{R} , its sum $s = s(x)$ is infinitely differentiable on \mathbb{R} , and $s(x) + s'(x) + s''(x) + s'''(x) = e^x$ for all $x \in \mathbb{R}$.

10. Show that there is a unique solution of the differential equation $tx'(t) = (2t^2 + 1)x(t) + t^2$, $t > 0$ which has a finite limit as $t \rightarrow +\infty$, and compute this limit.