

CEU Mathematics Entrance Exam, 2012

Test for candidates applying for M.S. and/or Ph.D.

DIRECTIONS: There are 10 problems (10 points each). You have 3 hours. To receive full credit your solutions must be clear, complete and correct. No books, notes or electronic devices.

1. Let $M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. What is the degree of its minimal polynomial?

2. Determine the real polynomial $f(x)$ of least degree with the following three properties: $f(i) = 0$ (where i is a complex square root of -1); the sum of coefficients of f is 0; the constant term of f is 1.

3. Let I denote the 2×2 identity matrix. Suppose that A, B are two invertible 2×2 matrices for which $A^{-1}BA = B^{-1}$ and $B^{-1}AB = A^{-1}$. Show that $A^2 = B^2 = B^{-2}$. Give an example of such a pair of matrices for which $A, B \neq \pm I$.

4. Let R be the smallest subring of \mathbb{Q} (the field of rational numbers) that contains $3/10$. Does $1 \in R$?

5. Does there exist a group of 12 elements such that the orders of its elements are:

a. 12, 12, 12, 12, 6, 6, 4, 4, 3, 3, 2, 1;

b. 6, 6, 3, 3, 3, 3, 3, 3, 3, 2, 1?

6. Prove that $\ln(\ln(k+1)) - \ln(\ln k) < \frac{1}{k \ln k}$ for all $k \in \mathbb{N}$, $k \geq 2$, and use this inequality to show that $\sum_{n=2}^{\infty} \frac{1}{k \ln k} = \infty$.

7. If $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the condition $|f(x) - f(y)| \leq C|x - y|^\alpha \forall x, y \in \mathbb{R}$, where $\alpha \in (1, \infty)$ and $C \in (0, \infty)$ are given constants, then f is constant.

8. Show that $\int_{-\infty}^{\infty} e^{-x^2+x} dx$ is convergent and compute its value.

9. Show that if $a, b, c \in \mathbb{R}$, $a > 0$, $b > 0$, then all solutions of the equation $u''(t) + au'(t) + bu(t) = c$, $t \geq 0$ converge to c/b as $t \rightarrow \infty$.

10. Let $f, g : [0, 1] \rightarrow \mathbb{R}$ be continuous functions. Define $x_n(t) = f(t) + \int_0^t x_{n-1}(s) ds$, $0 \leq t \leq 1$, $n = 1, 2, \dots$, where $x_0(t) = g(t)$, $0 \leq t \leq 1$. Show that the sequence (x_n) is uniformly convergent on $[0, 1]$ and its limit is independent of g .