CEU Mathematics Entrance Exam, 2013

Test for candidates applying for M.S. and/or Ph.D.

DIRECTIONS: *There are 10 problems (10 points each). You have 3 hours. No books or notes. To receive full credit your solutions must be clear, complete and correct.*

1. Let R be a ring with $1 \in R$. Multiplication by a positive integer k means multiple addition $ka = a + a + \cdots + a$. Suppose the positive integer n is such that $n1 = 0$ in R. Show that $na = 0$ for every $a \in R$.

2. Let $A = \begin{pmatrix} \lambda & 2 & 3 \\ 1 & 3 & 2 \\ 0 & 1 & \lambda \end{pmatrix}$, where λ is a complex parameter. Is there a choice for λ such that A is not invertible?

3. Let $f(x) = x^3 + x^2 + 2x + 4$ and $g(x) = x^2 + 3x + 1$. Verify that $f(x)$ has no rational root. Can we change the constant term of $g(x)$ to a suitable complex number so that $g(x)$ now divides $f(x)$?

4. Let $f_{a,b} : \mathbb{R} \to \mathbb{R}$ denote the linear function $x \mapsto ax + b$, where $a \neq 0$. Let $G = \{f_{a,b} \mid 0 \neq a \in \mathbb{R}, b \in \mathbb{R}\}$ and let $\circ$ denote composition of functions (that is, performing one map after the other). Show that G is a group with this operation $\circ$. Construct an infinite chain of non-Abelian subgroups $H_1 < H_2 < H_3 < \cdots < G$.

5. Let G be a finite group, p the smallest prime divisor of $|G|$ and $x \in G$ an element of order p. Suppose $h \in G$ is such that $h^{-1}xh = x^{10}$. Show that $p = 3$.

6. Suppose that $L \subset \mathbb{R}^n$ is a closed subset. Show that if $K \subset \mathbb{R}^n$ is compact, then $K \cap L \subset X$ is also compact.

7. Prove that the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_n = \cos^2\left(\pi\sqrt{(n+1)(n+3)}\right)$$

is convergent and determine its limit.

8. Assume that for $f\colon \mathbb{R} \to \mathbb{R}$ we have that

$$f(x+y) = f(x)f(y), \tag{1}$$

and f is continuous at the origin. Prove that f is continuous everywhere. Determine all the continuous functions satisfying Equation (1).

9. Suppose that $f(x)$ is differentiable on $[a, b]$ and $f'(x)$ is continuous on $[a, b]$. Suppose furthermore that $f(a) = f(b) = 0$ and $\int_a^b f^2(x)dx = 1$. Show that

$$\int_a^b xf(x)f'(x)dx = -\frac{1}{2}.$$

10. Solve the equation

$$x'(t) = x(t)^2 - t^2 + 1.$$