

CEU MATHEMATICS ENTRANCE EXAM, 2015

DIRECTIONS: *There are 10 problems, 5 problems from Analysis, and 5 from Algebra (10 points each). You have 3 hours for the 10 problems. No books or notes. To receive full credit your solutions must be clear, complete and correct.*

ANALYSIS EXAM

- (1) Assume that f is differentiable and $F(x, y) = yf(x^2 - y^2)$. (Here $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$.) Prove that $y^2 \cdot \frac{\partial F}{\partial x} + xy \cdot \frac{\partial F}{\partial y} = xF$.
- (2) Compute $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$. Determine whether $\sum_{n=1}^{\infty} 2^{-\frac{1}{n}}$ is convergent or not.
- (3) Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous periodic function with period $T > 0$, that is, $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. Show that for any $a \in \mathbb{R}$

$$\frac{1}{t} \int_a^{a+t} f(s) ds \longrightarrow \frac{1}{T} \int_0^T f(s) ds$$

as $t \rightarrow \infty$.

- (4) Define the Fourier coefficients of a continuous function f on $[-\pi, \pi]$ as

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos ntdt, \quad (n = 0, 1, 2, \dots) \quad \text{and}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin ntdt, \quad (n = 1, 2, \dots).$$

Suppose that the Fourier coefficients of the continuous functions f and F are given as a_n, b_n and A_n, B_n respectively. Show that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(t)F(t)dt = \frac{1}{2}a_0A_0 + \sum_{n=1}^{\infty} (a_nA_n + b_nB_n).$$

- (5) Suppose that f is continuous and $\int_B f = 0$ for any subinterval $B \subset [a, b]$. Show that in this case $f = 0$ on $[a, b]$.

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ALGEBRA EXAM

- (6) Let φ be the linear transformation of \mathbb{R}^3 whose matrix with respect to the standard basis of \mathbb{R}^3 is

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}.$$

Determine a basis in which the matrix of φ is diagonal or show that no such basis exists.

- (7) For a ring R a derivation is a map $D : R \rightarrow R$ such that $D(a + b) = D(a) + D(b)$ and $D(ab) = D(a)b + aD(b)$. Let now $R = \mathbb{Z}[x]$ be the ring of integer polynomials. For which polynomials $f(x) \in R$ does there exist a derivation D of R such that $D(x^2 + 1) = f(x)$?
- (8) Let $G = \{r \in \mathbb{Q} \mid 0 \leq r < 1\}$ be the set of non-negative rationals smaller than 1. Define

$$a \circ b = \begin{cases} a + b, & \text{if } a + b < 1; \\ a + b - 1, & \text{if } a + b \geq 1. \end{cases}$$

Verify that G is a group with this operation.

Show that for any finite set $\{g_1, \dots, g_n\} \subseteq G$ there is a proper subgroup $H \subset G$, $H \neq G$ such that $\{g_1, \dots, g_n\} \subseteq H$.

- (9) Let $M_5(\mathbb{Z})$ denote the ring of 5×5 matrices with integer entries.
- (a):** Does $M_5(\mathbb{Z})$ have a subring isomorphic to $\mathbb{Z}[x]$, the ring of integer polynomials?
- (b):** Let $I = (x^2(x - 1)^3) \triangleleft \mathbb{Z}[x]$ be the ideal generated by $x^2(x - 1)^3$ in $\mathbb{Z}[x]$. Does $M_5(\mathbb{Z})$ have a subring isomorphic to the factor ring $\mathbb{Z}[x]/I$?
- (10) Let $v_1, v_2 \in \mathbb{R}^n$ be two vectors in a Euclidean space. Suppose that for every integer $k \in \mathbb{Z}$ we have $|v_1| \leq |v_2 + kv_1|$. Show that for every $k_1, k_2 \in \mathbb{Z}$, (not both 0) we have $|v_1| \leq |k_1v_1 + k_2v_2|$.