CEU MATHEMATICS ENTRANCE EXAM, 2016

DIRECTIONS: *There are 10 problems, 5 problems from Analysis, and 5 from Algebra (10 points each). You have 3 hours for the 10 problems. No books or notes. To receive full credit your solutions must be clear, complete and correct.*

ANALYSIS EXAM

1. Determine the value of $a \in \mathbb{R}$ for which the following series converges:

$$\sum_{n=1}^{\infty} \left(\frac{a}{n+2} - \frac{1}{n+3} \right)$$

2. Show that if $\omega = k \cdot c$ with $\omega, k, c > 0$, then the function $f(x,t) = \sin(kx - \omega t)$ satisfies the differenial equation

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 f(x,t)}{\partial t^2}.$$

3. (i) Suppose that the function $f \colon \mathbb{R} \to \mathbb{R}$ is continuous at zero, and satisfies the equation

$$f(x+y) = f(x)f(y)$$

for any $x, y \in \mathbb{R}$. Show that $f \colon \mathbb{R} \to \mathbb{R}$ is continuous everywhere.

(ii) Determine all continuous functions $f \colon \mathbb{R} \to \mathbb{R}$ satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

4. Show that if $f \colon \mathbb{R} \to \mathbb{R}$ is continuous, then for any $x \in \mathbb{R}$, we have

$$\int_0^x f(u) \cdot (x-u) du = \int_0^x \left(\int_0^u f(t) dt \right) du.$$

5. Define the function $f \colon \mathbb{R}^2 \to \mathbb{R}$ by the formula

$$f(x,y) = x \sin \frac{1}{y} + y \sin \frac{1}{x}$$

if $xy \neq 0$ and to be 0 if $xy = 0$. Show that f is continuous at the origin $(0,0) \in \mathbb{R}^2$.

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ALGEBRA EXAM

6. Let $\{v_1, v_2, v_3, v_4\}$ be an orthonormal basis of $\mathbb{C}^4$ (endowed with the standard scalar product). For a complex number c define the linear transformation $\varphi_c : \mathbb{C}^4 \to \mathbb{C}^4$ by: $\varphi_c(v_i) = cv_{i+1}$ for $1 \le i \le 3$ and $\varphi_c(v_4) = cv_1$. Determine those $c \in \mathbb{C}$ for which φ_c is unitary.

7. For a 3×3 matrix A over the field of real numbers, let $\varrho(A)$ denote its rank. Do there exist two 3×3 matrices B, C such that:

   - $\varrho(B) = 2$, $\varrho(C) = 1$;
   - $\varrho(B + C) = 3$, $\varrho(B - C) = 2$?

   Construct such B and C, or prove the impossibility.

8. True or false? Justify your answer.

   If $f(x)$ is a non-constant real polynomial for which $g(x) = f(x^2) + 1$ has a real root then $f(x)$ also has a real root.

9. Let Ω denote the set of integer tuples (a, b) where $a + b$ is odd. We define two permutations of Ω:

   $$\tau : (a, b) \mapsto (a + 1, b - 1), \qquad \sigma : (a, b) \mapsto (b, a).$$

   Let G denote the group generated by τ and σ. Construct a strictly descending infinite sequence of non-commutative subgroups $G > G_1 > G_2 > \cdots$.

10. Let p be a prime number. Let M denote the ring of 2×2 matrices over the field F of p elements. For $A \in M$, let $C(A)$ denote the set of those matrices $B \in M$ such that $AB = BA$. What are the possible values of the dimension of $C(A)$ over F for $A \in M$?