CEU MATHEMATICS ENTRANCE EXAM, 2017

DIRECTIONS: *There are 10 problems, 5 problems from Analysis, and 5 from Algebra (10 points each). You have 3 hours for the 10 problems. No books, notes or use of internet. To receive full credit your solutions must be clear, complete and correct.*

ANALYSIS EXAM

1. Suppose that $K \subset X$ is a compact, and $L \subset X$ is a closed subset in the topological space X. Show that $K \cap L$ is compact.

2. Suppose that for the sequence $\{a_n\}$ with $a_n \in \mathbb{R} \setminus \{0\}$ we have that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = b < 1$. Show that $\lim_{n\to\infty} a_n = 0$.

3. Suppose that $f(x)$ is differentiable on $[a, b]$ and $f'(x)$ is continuous on $[a, b]$. Suppose furthermore that $f(a) = f(b) = 0$ and $\int_a^b f^2(x)dx = 1$. Show that
$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}.$$

4. Find $\int \frac{e^{\frac{1}{x}}}{x^3}\, dx$.

5. Solve the differential equation $y'(x) = y^2(x) + 1$.

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ALGEBRA EXAM

6. Determine the characteristic polynomial of the 4×4 real matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

7. Is the polynomial $f(x) = x^4 + x^2 + 1$ irreducible over the rational numbers?

8. Let A, B be two $n \times n$ real matrices. Which of the following two statements imply the other?

$$(I)\ A^T B = 0. \qquad (II)\ \mathrm{Ker}(A+B) = \mathrm{Ker}(A) \cap \mathrm{Ker}(B).$$

(The set $\mathrm{Ker}(A)$ stands for the collection of vectors that are sent to 0 by the matrix A.)

9. Recall that the set of real functions $S = \{f : \mathbb{R} \to \mathbb{R}\}$ is a commutative ring with the usual (pointwise) addition and multiplication of functions.

   Is there a ring homomorphism $\varphi : S \to \mathbb{R}$ to the ring of real numbers, such that $\varphi(\sin) = 2$? (Here 'sin' denotes the usual sinus function.)

10. Which of the following groups are isomorphic? Justify your answers. In every pair the first component is a set, the second is the group operation on it. (You need not verify that these are groups.)

$$(\mathbb{Q} \setminus \{0\}, \cdot) \quad (\mathbb{R} \setminus \{0\}, \cdot) \quad (\{a \in \mathbb{R} \mid a > 0\}, \cdot) \quad (\mathbb{C} \setminus \{0\}, \cdot)$$

$$(\mathbb{Z}, +) \qquad (\mathbb{Q}, +) \qquad (\mathbb{R}, +) \qquad (\mathbb{C}, +)$$