CEU MATHEMATICS ENTRANCE EXAM, 2018

DIRECTIONS: *There are 10 problems, 5 problems from Analysis, and 5 from Algebra (10 points each). You have 3 hours for the 10 problems. No books, notes or use of internet. To receive full credit your solutions must be clear, complete and correct.*

ANALYSIS EXAM

1. Suppose that $\{a_n\}$ is a sequence of real numbers converging to $a \in \mathbb{R}$. Let $b_k = \frac{1}{k}\sum_{n=1}^{k} a_n$. Show that $b_k \to a$. Give an example of a sequence $\{a_n\}$ which is not convergent, although the corresponding $\{b_k\}$ converges.

2. Show that the series $\sum_{n=1}^{\infty} n^a \cdot 2^{-n}$ is convergent for all $a \in \mathbb{R}$. Determine its value for $a = -1$.

3. Suppose that X is a topological space, $K \subset X$ is closed and $L \subset X$ is compact. Show that $K \cap L$ is compact.

4. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = \frac{x^2 y}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that f is a continuous function.

5. Solve the differential equation $x \cdot y'(x) + y(x) = -x^3$ on $\{x \in \mathbb{R} :\ x > 0\}$.

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ALGEBRA EXAM

6. Determine the rank of the matrix
$$A = \begin{pmatrix} t & 1 & 1 & 0 \\ 1 & t & 1 & 0 \\ 1 & 1 & t & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$
depending on the value of the real parameter t.

7. Let $X = \{(a_1, a_2, \ldots) \mid a_i \in \mathbb{R}\}$ denote the set of infinite real sequences and let $n = (0, 0, \ldots) \in X$ be the null-sequence. We perform two operations, σ, τ on the elements of X:
$$\sigma((a_1, a_2, \ldots)) = (b_1, b_2, \ldots), \text{ where } b_1 = 0,\ b_i = a_{i-1} \text{ for every } i \geq 2;$$
$$\tau((a_1, a_2, \ldots)) = (c_1, c_2, \ldots), \text{ where } c_{2i} = a_i,\ c_{2i-1} = 0 \text{ for every } i \geq 1.$$
Show that there exists a sequence $x \in X$ such that $\sigma\tau\sigma\tau\sigma\tau\sigma\tau\sigma\tau(x) \neq n$.

8. Let $R = M_2(\mathbb{R})$ denote the ring of 2×2 real matrices, and let I denote its identity matrix. Put
$$S = \{2nI \mid n \text{ integer}\} \subseteq R.$$
Determine explicitly the set of matrices $A \in R$ such that $A^2 = A$ and $AB \in S$ for all $B \in S$.

9. Let G be a finite group. For every positive integer n let
$$s_n = |\{x \in G \mid o(x) = n\}|$$
denote the number of elements of G of order n. Determine G if we know that $s_n \leq 1$ for every n?

10. Is there a polynomial $f(x)$ with real coefficients such that $f(3n) > f(3n+1)$ and $f(3n) > f(3n-1)$ for every positive integer n?