Edge-weightings and the chromatic number

By

Cory T. Palmer

Submitted to

Central European University

Department of Mathematics and its Applications

In partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics and its Applications

Supervisor: Ervin Győri

Budapest, Hungary

May, 2008

Table of Contents

Abstract

Motivated by the difficulty of determining bounds on the chromatic number of a graph, we examine several new graph parameters that are related to the standard chromatic number. These parameters are based on edge-weightings and are interesting in their own right but they also have potential consequences for the chromatic number. The core of the thesis will be dedicated to the particular problem to determine the minimum number of weights needed to assign to the edges of a graph G with no component K_2 so that any two adjacent vertices have distinct sets of weights on their incident edges. The main result is that this minimum is at most $\lceil \log_2 \chi(G) \rceil + 1$. This upper-bound is best possible for $\chi(G) \geq 3$. We also characterize the case when $\chi(G) = 2$.

Acknowledgments

First of all I indebted to my supervisor Ervin Győri for his helpful guidance and advice for the entirety of my graduate studies. I would also like to thank him for introducing me to the topic of graph theory and for many useful discussions (mathematical and otherwise). The mathematics Ph.D. program is a joint endeavor between the Rényi Institute of Mathematics and Central European University. I am grateful for both institutions and the many opportunities they provided during my studies. I would also like to express my gratitude to the Department of Mathematical Sciences at the University of Memphis for having me as a guest during the Fall 2007 semester. I also would like to thank my family who have encouraged me at every step of my education. Finally, I would like to thank my wife Zsuzsa for her constant love, support and understanding.

Chapter 1

Introduction

1.1 Background

The origins of graph theory can probably be narrowed down to two classical problems; the seven bridges of Königsberg problem and the four-color problem. The bridges problem motivated the study of Eulerian circuits and by the 1880s we had a characterization of graphs with an Eulerian circuit (by Hierholzer [43]) as well as Fleury's [32] polynomial-time algorithm to find such a circuit in a given graph. This nearly closed this chapter of graph theory. On the other hand, the four-color problem motivated the study of general graph colorings; a topic very actively researched today.

Graph coloring problems come in many varieties but at their most general they ask us to partition the objects of a graph (vertices, edges, faces, etc.) into different classes so that some given constraints are satisfied. The most classic graph coloring problem is to assign colors to the vertices of a graph so that any two vertices connected by an edge have different colors. Formally, if a graph is a pair $G = (V, E)$ where V is a set of vertices and E is a set of edges between pairs of vertices in V, then a *vertex coloring of* G is a map $f : V \to C$ where C is a set of colors. We say that such a coloring is *proper* if f has the following property: if $xy \in E$ then $f(x) \neq f(y)$ i.e. adjacent vertices are assigned different colors by f. Given a graph G we are typically concerned with determining the minimum possible size of C such that an f with the desired property exists; we denote this minimum by $\chi(G)$ and refer to it as the *chromatic number of* G. Graphs that can be colored with k colors are *k-colorable* and graphs that cannot be colored with fewer than k colors are *k-chromatic*.

The most famous graph coloring problem is the *four-color problem* which asks if every planar graph (a graph that can be drawn in the plane with no crossing edges) has a proper vertex coloring with 4 colors. The four-color problem has an extensive history (see, for example, the book by Saaty and Kainen [66]). In 1976, it was answered in the positive by Appel, Haken and Koch [7, 10] by expanding on the ideas of Kempe [54] and Heesch [42]. The proof is a major achievement, but relies on a computer to help check a large number of cases and has been the subject of some controversy. Some of the controversy has been addressed by Appel and Haken in updated versions of the proof [8, 9] and by a simpler version by Robertson, Sanders, Seymour, Thomas [65]. However, these newer proofs still rely on examining (by computer) a large number of cases.

Before the four-color problem was solved, the study of the chromatic number of a graph became a major area of graph theory in its own right. Naturally, attempts were made to characterize $\chi(G)$. In 1916, König [55] showed that a characterization is possible for 2-colorable graphs.

Theorem 1. *A graph G is 2-colorable if and only if G contains no odd cycle as a subgraph.*

Despite this promising beginning, the chromatic number has proved very difficult to characterize in general. If $\omega(G)$ denotes the *clique number* i.e. the size of the largest complete subgraph of G, then obviously $\chi(G) \geq \omega(G)$. However, Zykov [79] showed that in general any significant connection between $\chi(G)$ and $\omega(G)$ is hopeless.

Theorem 2. *Given a natural number $k \geq 2$ there exists a graph G such that $\chi(G) = k$ and $\omega(G) = 2$.*

In particular, this means that we can find triangle-free graphs of any chromatic number. Later, Mycielski [60] constructed triangle-free graphs of arbitrary chromatic number with fewer edges and vertices than required by Zykov's construction. These constructions both have an exponential number of edges. Erdős [27] was able to construct a triangle-free graph of arbitrary chromatic number with a polynomial number of edges. The length of the shortest cycle in a graph is called the *girth*. The above constructions all have girth 4. Tutte (under the pseudonym Blanche Descartes) [24, 25] and independently Kelly and Kelly [53] go a step further and construct graphs of arbitrary chromatic number with girth 6. (In fact, Tutte's result predates Theorem 2).

In a naive sense we expect that a graph of large chromatic number should have smaller subgraphs of large chromatic number. Zykov's theorem suggests this is not necessary and the famous application of random graphs by Erdős [28] totally defeats this expectation.

Theorem 3. *Given natural numbers $g \geq 3$ and $k \geq 2$, then there exists a graph G such that $\chi(G) = k$ and G has no cycle of length at most g.*

The proof is one of th earliest uses of the probabilistic technique and is non-constructive. However, Lovász [57] and later Nešetřil and Rödl [61] give explicit inductive constructions of graphs (also uniform hypergraphs) with arbitrary chromatic number and girth. If we think of the chromatic number as a kind of global density function, then the remarkable consequence of these theorems is that graphs can be as "locally sparse" as we like while being arbitrarily "globally dense."

What about the relationship between $\chi(G)$ and other standard graph parameters? The *independence number*, $\alpha(G)$, of a graph G is the size of the largest independent set of vertices in G. Obviously no color may be repeated more than $\alpha(G)$ times in a proper vertex coloring of G, so we have the easy lower bound $\chi(G) \geq n/\alpha(G)$ where n is the number of vertices of G. However, for a graph of girth g it is clear that $\alpha(G) \geq \lfloor g/2 \rfloor$. Thus

Theorem 3 implies the existence of graphs with arbitrarily large chromatic number and independence number.

We call a graph G *connected* if for any two vertices $x, y \in V(G)$ there is a path in G between them. The *connectivity* $\kappa(G)$ of a graph G is the maximum integer such that after the removal any set of $\kappa(G)$ vertices from G the resulting graph is still connected. High connectivity seems to be a necessary requirement for high chromatic number, but it is easy to see that it is far from sufficient. For any n consider the complete bipartite graph $K_{n,n}$ which, by definition, is 2-colorable, but has connectivity n. This graph also shows that large minimum degree, average degree or maximum degree are insufficient conditions for high chromatic number.

The complete bipartite graphs have many more edges than are necessary for chromatic number 2 (a single edge is enough). What if we restrict our attention to graphs where the removal of any edge (or vertex) yield a graph with strictly smaller chromatic number? A graph G is *k-critical* (in general *critical*) if $\chi(G) = k$ and for any proper subgraph $G' \subset G$ we have $\chi(G') < k$. Observe that any k-chromatic graph must have a k-critical subgraph. Thus a robust understanding of critical graphs can help us understand the chromatic number. We might expect (or at least hope) that critical graphs to have comparatively simple structure, but this is not the case. Dirac [23] first showed the existence of critical graphs with $O(n^2)$ edges. Even more, Brown and Moon [18] proved that critical graphs can have large independence number and Simonovits [67] and Toft [69] independently proved that critical graphs can have large minimum degree.

Despite huge amounts of effort and progress, the cause for high chromatic number remains somewhat inexplicable. Because it is so difficult to characterize, alternative descriptions of or parameters depending on the chromatic number can be valuable. This will be the main motivation of the present thesis. In Chapter 2 we will define a new graph parameter called the general neighbor-distinguishing index that initially seems to be quite distinct from the chromatic number. However, we will show that this new parameter depends strongly on the chromatic number. As a result, the it provides bounds on $\chi(G)$ that many other graph parameters cannot. Furthermore, the analysis given in this thesis of the general neighbor-distinguishing index is essentially complete. We are able to characterize this new parameter in terms of $\chi(G)$ in such a way that no improvements to our upper bounds are possible.

Before introducing this new graph parameter, let us continue our general discussion of the importance of the chromatic number and some of its variants.

1.2 Further details

It is well known that the vertex coloring problem is **NP**-complete. Given the difficulty of characterization this is a reasonable situation In fact, the complexity of determining the chromatic number of a graph is one of Karp's original 21 reductions [52]. In Karp's notation, 3-SAT can be reduced to CHROMATIC NUMBER.

Because the chromatic number appears to be an independent graph parameter it has remained an important topic of study in graph theory. Indeed, classification of certain graphs by their chromatic number is a standard research problem. However, graph colorings have proved to have important motivation elsewhere; for instance in the study of matchings, connectivity, and Hamiltonian cycles. Sometimes very unexpected applications arise. The Erdős-Stone-Simonovits Theorem [30, 29] tells us that the maximum number of edges in a graph without some specified subgraph depends strongly on the chromatic number of the subgraph in question.

Theorem 4. *The maximum number of edges of a graph F on n vertices without a subgraph G of chromatic number $\chi(G)$ is*

$$\mathrm{ex}(n; F) = \frac{\chi(G) - 2}{\chi(G) - 1}\binom{n}{2} + o(n^2).$$

Furthermore, graph coloring problems are of major importance in applied problems. Many types of scheduling and pattern matching problems can be reformulated in the language of graph colorings. Many recreational problems are also graph coloring problems in disguise (e.g. Sudoku).

Despite its difficulty, much is known about the chromatic number. We mention a few favorite examples. An important upper bound on $\chi(G)$ is given by Brooks [17].

Theorem 5. *If G is a connected graph with maximum degree $\Delta(G)$, then*

$\chi(G) \le \Delta(G) + 1$ *with equality holding if and only if* G *is an odd cycle or a complete graph.*

The maximum degree $\Delta(G)$ of a graph is easy to determine (examine each vertex once), so there is little hope to obtain a much better characterization of $\chi(G)$ in terms of $\Delta(G)$. However, good improvements have been made for graphs with certain excluded subgraphs.

A graph on n vertices can have chromatic number of any integer value from 0 to n. However, on average this is not the case. Bollobás [16] shows that the chromatic number of random graphs is generally restricted to a small range.

Theorem 6. *Let* $G_{n,p}$ *be a random graph on* n *vertices with edge probability* $0 < p < 1$*, then asymptotically almost always*

$$\chi(G_{n,p}) = \left(\frac{1}{2} + o(1)\right) \frac{\log n}{\log\left(1/(1-p)\right)}.$$

The study of graph colorings is not necessarily restricted to finite graphs. However, de Brujin and Erdős [19] show that it is enough to consider only finite graphs when dealing with a finite number of colors.

Theorem 7. *For fixed finite* k*, if all finite subgraphs of an infinite graph* G *are* k*-colorable, then* G *is* k*-colorable.*

There is a special class of graphs where the chromatic number can be determined easily. This fundamental class of graphs are the so-called perfect graphs. We say that a graph G is *perfect* if for every induced subgraph H of

G we have that the clique number $\omega(H)$ is equal to the chromatic number $\chi(H)$. As an example, it is trivial that the bipartite graphs are perfect. Berge [15] conjectured what is now called the Perfect Graph Theorem of Chudnovsky, Robertson, Seymour and Thomas [22].

Theorem 8. *A graph G is perfect if, and only if, neither G nor its complement $\overline{G}$ contains an induced odd cycle of length at least 5 as a subgraph.*

Earlier, Lovász [58] proved an important weaker version of the conjecture (see also Fulkerson [34]).

Theorem 9. *The complement of a perfect graph is perfect.*

Grötschel, Lovász and Schrijver [37] also showed that the chromatic number of a perfect graph can be determined in polynomial time. This is important as perfect graphs have many connections to other combinatorial problems.

In addition to vertex coloring problems, many other types of graph coloring problems have been formulated. However, the significance of vertex coloring problems is often emphasized as many of these new problems can be reformulated in terms of vertex colorings or depend strongly on the chromatic number.

After vertex colorings it is natural to study *edge colorings*. In this problem we ask for the minimum number of colors necessary to assign to the edges of a graph such that any two incident edges are colored with different colors. This parameter is called the *chromatic index* and is denoted $\chi'(G)$. Immediately we have that the maximum degree, $\Delta(G)$, is a lower bound on the chromatic index. In fact, Vizing [72] shows that this trivial lower bound is half of the story.

Theorem 10. *A graph G with maximal degree $\Delta(G)$ has chromatic index equal to $\Delta(G)$ or $\Delta(G) + 1$.*

Despite the apparent difference, edge colorings turn out to be a special case of vertex colorings. The line graph $L(G)$ of G is the graph formed when replacing all edges of G with vertices and connecting two vertices of $L(G)$ by an edge if their corresponding edges in G were adjacent. Because properly coloring the edges of a graph G is exactly equivalent to properly coloring the vertices of the line graph $L(G)$, we know that edge colorings are merely the restriction of vertex colorings to the class of line graphs. Edge colorings are interesting in their own right, but it is notable that a seemingly different coloring problem is just a restriction of the classical question. Amazingly enough, despite just two possible values, Holyer [44] proved that to determine the chromatic index of a graph is **NP**-complete. The study of edge colorings will be of special importance in the later chapters.

If edge colorings are just vertex colorings, then what coloring problems are truly new questions? *List-colorings* are one well-known example. Here, we assign a list of possible colors to each vertex and ask if we can properly color the graph from the given lists. The *list-chromatic number* (or *choosability*) of G is defined as the smallest k such that no matter how we assign sets of k colors to vertices of G we can find a proper vertex coloring of G from the given lists. This parameter is denoted $\mathrm{ch}(G)$. If we let all the lists be the same k colors then we are just asking if G is k-chromatic, so we have that $\chi(G) \leq \mathrm{ch}(G)$. That this is not equality can be seen by examining the complete bipartite graph $K_{3,3}$. A clever assignment of lists of size 2 to the vertices (using a total of 3 different colors) shows that $\mathrm{ch}(K_{3,3}) > 2$. A similar trick can be employed for the infinite class of graphs $K_{n,n}$.

Unlike edge colorings, list-colorings are not easily translatable to vertex colorings on some special class of graphs. Alon [6] emphasizes the difference between the list-coloring number and the chromatic number by showing that $\mathrm{ch}(G)$ can be related to the average degree of a graph.

Theorem 11. *If s is a natural number and G is a graph of average degree $d(G) > 4\binom{s^4}{s}\log(2\binom{s^4}{s})$, then $\mathrm{ch}(G) > s$.*

No relationship between the average degree and the chromatic number of the above form is possible. However, that the list-chromatic number is an upper bound on the chromatic number can be very useful as Thomassen demonstrates in [68].

Theorem 12. *Every planar graph has list-chromatic number at most 5.*

This bounds immediately implies that planar graphs are 5-colorable. Thomassen's proof is a beautiful example of the technique of strengthening a hypothesis to allow an inductive argument. More importantly, however, is that his proof completely avoids Euler's formula and thus offers a completely different technique for attacking vertex coloring problems. Unfortunately an improvement of this technique cannot be used to prove the four-color theorem as there exist planar graphs that are have list-chromatic number larger than 4 (see Voigt [74]).

List colorings have also been applied to the edges of a graph giving the *list-edge-chromatic number* (or *edge-choosability*) of a graph; in notation $\mathrm{ch}'(G)$. As before, the problem can be translated to the original list coloring problems on the vertices of the corresponding line graph. However, unlike vertex list-colorings, it is conjectured that the edge version is truly not a new question.

Conjecture 1. *For any graph G we have* $\mathrm{ch}'(G) = \chi'(G)$.

This so-called *List-Coloring Conjecture* has been confirmed for bipartite graphs by Galvin [36] and it has been shown by Kahn [50] that for any $\epsilon > 0$ and $\Delta(G)$ large enough that $\mathrm{ch}'(G) \leq (1+\epsilon)\Delta(G)$.

Instead of separating the vertices and edges into their own problems we may attempt to color them at the same time. A *total coloring* of a graph G is an assignment of colors to the vertices and edges of G such that any two adjacent edges have different colors, any two incident edges have different colors and any incident edge and vertex have different colors. The minimum number of colors needed for such a coloring is often denoted $\chi''(G)$. Much like edge colorings, total colorings can be translated to vertex colorings. It is enough to consider the so-called *total graph* $T(G)$ of G i.e. replace all edges and vertices of G with vertices and connect two vertices of $T(G)$ if they should receive different colors in the total coloring of G. Clearly, $\chi''(G) \geq \Delta(G) + 1$ and Behzad [14] and Vizing [73] independently conjecture that only one more color is needed beyond the requirement of Vizing's Theorem.

Conjecture 2. *If G is a graph, then $\chi''(G) \leq \Delta(G) + 2$.*

Molloy and Reed [59] give the upper bound $\chi''(G) \leq \Delta(G) + 10^{26}$ thus confirming the conjecture is of the proper order (the authors also remark that this constant can be reduced considerably). Other partial results are known, but the conjecture remains standing.

It is clear that the field of graph coloring problems is a rich one. Many surveys of the topic and of open problems are available. Toft's survey [70] in the Handbook of Combinatorics and the extensive problem book of Jensen and Toft [49] provided particular inspiration for this introduction and are excellent resources for further background.

Now let us return to the topic of edge colorings. As we have seen, an edge coloring of a graph G is equivalent to a vertex coloring of the *line graph* $L(G)$. In the next chapter we will discuss how an edge coloring of G can be used to produce a type of vertex coloring of G. This is a general question and has many different approaches, we will survey several of these approaches and outline what is known. At their most general we will refer to these types of problems as "edge weighting problems."

Chapter 2

Edge-weightings

2.1 Introduction

We begin the general formulation of edge-weighting problems with some fundamental definitions. Given a graph G with vertex set $V(G)$ and edge set $E(G)$ an *edge-weighting* is a map $\varphi : E(G) \to W$. In general our task will be to minimize the size of W such that there exists a map φ satisfying several given constraints. Most often, W will be the set $\{1, 2, 3, \ldots, k\}$. In this case we will call φ a *k-edge-weighting*.

Given a graph G and an edge-weighting φ of G, for $v \in V(G)$ let $S_\varphi(v)$ be the set of edge-weights appearing on edges incident to v under the edge-weighting φ. Formally, $S_\varphi(v) = \{\varphi(e) : e \ni v\}$. We will make frequent use of the notion of weight-sets when defining different types of edge-weighting problems.

The class of edge-weighting problems can roughly be split into two parts. First are the *proper* edge-weightings i.e. edge-weightings of G where any two incident edges must get mapped to different elements of W by φ. Without additional constraints this is just the classical edge-coloring problem. Second are the *non-proper* (sometimes *general*) edge-weightings i.e. edge-weightings where we do not require that incident edges get mapped to different elements of W by φ.

Let us take an aside to justify the term "edge-weighting." Our reasons are twofold. First, we will often speak simultaneously of edge-weightings and vertex colorings. To avoid confusion, the term "weighting" will always refer to edges (even if the weighting can be thought of as a coloring) while the term "coloring" will always refer to vertices. Second, the reader may be aware that our edge-weightings are sometimes called edge labellings. We prefer the term "weighting" for historical reasons as it refers to the origins of many problems of the type to be discussed. In particular, in the study of irregularity strength the weight of an edge in a graph can be thought of as the multiplicity of that edge in the corresponding multigraph.

2.2 Proper weightings

If only proper edge-weightings are considered we begin with the classical edge-coloring problem. Trivially, we know that for a graph G we need at least as many edge weights as the maximum degree, $\Delta(G)$ for a proper edge-weighting. Furthermore, Vizing's Theorem 10 tells us that any graph G needs either $\Delta(G)$ or $\Delta(G)+1$ different edge weights for a proper edge-weighting. We call graphs that have proper edge-weightings with $\Delta(G)$ edge weights *Class* 1; otherwise, they are *Class* 2. Because the problems considered in this section will be more restrictive than the classical edge-coloring problem, the value given by a graph's Class will be a lower bound on any possible upper bounds.

2.2.1 Neighbor-distinguishing index

We call a proper edge-weighting φ of G *neighbor-distinguishing* (also called *adjacent vertex-distinguishing*) if for any two adjacent vertices x, y (i.e. xy is an edge) the set of edge-weights on edges incident to x is different from the set of edge-weights on edges incident to y i.e. $S_\varphi(x) \neq S_\varphi(y)$.

Immediately we must remark that if G contains an isolated edge, this parameter is not well defined. Clearly, any weight assigned to the isolated edge will force its endpoints to have the same weight set. This cannot be avoided and as a result we will restrict our analysis to graphs without isolated edges. In fact, isolated edges will cause a similar problem in all other edge-weightings to be discussed and henceforth we will assume all graphs have no isolated edges.

The *neighbor-distinguishing index* of G, in notation $\chi'_a(G)$, is the smallest k such that there exists a proper k-edge-weighting which is neighbor-distinguishing. This graph parameter was introduced by Zhang, Liu and

Wang [78]. It is easy to see that $\chi'_a(C_5) = 5$ and Zhang et al. [78] conjecture that $\chi'_a(G) \leq \Delta(G) + 2$ for any connected graph $G \notin \{K_2, C_5\}$.

The conjecture has been confirmed by Balister, Győri, Lehel and Schelp [12] for bipartite graphs and for graphs of maximum degree 3. They also prove general upper bound on the parameter $\chi'_a(G)$.

Theorem 13. *If G is a graph without an edge component, then*

$$\chi'_a(G) \leq \Delta(G) + \mathrm{O}(\log \chi(G)).$$

Hatami [41] gives an asymptotically-stronger upper bound on $\chi'_a(G)$ by using a random edge-weighting technique.

Theorem 14. *If G is a graph without an edge component and $\Delta(G) > 10^{20}$, then*

$$\chi'_a(G) \leq \Delta(G) + 300.$$

Despite the large maximum degree requirement, the result is of particular importance as it shows that a simple additive constant is enough for an upper bound on $\chi'_a(G)$.

Edwards, Horňák and Woźniak [26] have shown that $\chi'_a(G) \leq \Delta(G) + 1$ if G is bipartite, planar, and of maximum degree $\Delta(G) \geq 12$. In particular, this gives infinitely many graphs with $\chi'_a(G)$ less than the conjectured upper bound.

2.2.2 Strong edge colorings

Instead of requiring that only adjacent vertices have different weight sets, we can require that any two (not necessarily adjacent) vertices have different weight sets. We call a proper edge-weighting satisfying the above condition *vertex-distinguishing* (this notion is also called a *strong edge coloring*). For a graph G we denote the minimum k such that there is a proper k-edge-weighting which is vertex-distinguishing by $\chi'_s(G)$.

This graph parameter is introduced by Burris and Schelp [20]. The authors prove the following general upper bound.

Theorem 15. *If G is a graph on n vertices with n_i vertices of degree i, then there exists a constant C depending on $\Delta(G)$ such that*

$$\chi'_s(G) \leq C \max\{n_i^{1/i} \,|\, 1 \leq i \leq \Delta(G)\}.$$

The authors point out that a simple counting argument shows that this upper bound is of the correct order. They also conjecture two different upper bounds of different strength. The weaker conjecture has been confirmed by Bazgan, Harkat-Benhamdine, Li and Woźniak [13].

Theorem 16. *If G is a graph on n vertices without an edge component and at most one isolated vertex, then*

$$\chi'_s(G) \leq n + 1.$$

The sharpness of Theorem 16 can be confirmed by considering the complete graph on n vertices. The stronger conjecture of Burris and Schelp [20] corresponds to Theorem 16 when G is a complete graph and remains open.

Conjecture 3. *If G is a graph without an edge component and at most one isolated vertex and k is the minimum integer such that for all d the number of vertices of degree d is not greater than $\binom{k}{d}$, then $\chi'_s(G)$ is equal to k or $k+1$.*

This conjecture has been confirmed by Balister, Bollobás and Schelp [11] for graphs that consist of just paths or of just cycles.

2.3 Non-proper weightings

We now turn our attention to non-proper edge-weightings. Unlike in the case of proper edge-weightings, there is no underlying lower bound corresponding to Vizing's Theorem. Indeed, without additional restraints a non-proper edge-weighting has little meaning.

2.3.1 Irregularity strength

The first question about non-proper edge-weightings is the so-called *irregularity strength* of a graph. This question, introduced by Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba [21], asks for the smallest k such that there is a k-edge-weighting φ of a given graph G such that for any two vertices x, y we have

$$\sum_{e \ni u} \varphi(e) \neq \sum_{e \ni v} \varphi(e).$$

In other words, for any two vertices $u, v \in V(G)$ the sum of weights on the edges incident to u should be different from the sum of weights on the edges incident to v. This graph parameter is denoted $s(G)$. Chartrand et al. [21] show that for any graph G on $n \geq 3$ vertices that $s(G) \leq 2n - 3$. This bound is later improved by Nierhoff [62] by refining the ideas of Aigner and Triesch [4] to $s(G) \leq n - 1$ for any graph G on $n \geq 4$ vertices.

In the case of regular graphs, stronger upper bounds on $s(G)$ are known. In particular, Faudree and Lehel [31] have shown that if G is regular, then $s(G) \leq \lceil \frac{n}{2} \rceil + 9$. The authors further speculate that if G is r-regular, then $s(G) \leq \frac{n}{r} + c$ for some absolute constant c.

In this direction, Frieze, Gould, Karoński and Pfender [33] use probabilistic techniques to show that for a graph G with minimum degree δ and maximum degree Δ that $s(G) \leq c_1 n/\delta$ if $\Delta \leq n^{1/2}$ and that $s(G) \leq c_2 (\log n) n/\delta$ if $\Delta > n^{1/2}$.

Many other results can be found in the excellent survey of Lehel [56]. This survey also discusses the extension of irregularity strength to hypergraphs.

If we modify the irregularity strength problem so that we map each edge to an element of a set W of algebraically independent real numbers, then we get a new edge-weighting problem. It is easy to see that in this case, two sums of the form $\sum_{e \ni u} \varphi(e)$ will be the same only when for every $\alpha \in W$ the term α occurs in both sums exactly the same number of times. It is easy to see that this is the same question if we replace W with $\{1, 2, \ldots, k\}$ and for an edge-weighting $\varphi : E(G) \to \{1, 2, \ldots, k\}$ of a graph G we require that for any two vertices $u, v \in V(G)$ the multiset of all edge weights appearing on edges incident to u is different from the multiset of all edge weights appearing on edges incident to v (where the multiplicity of an element is exactly how many times that edge weight appears on edges incident to the corresponding vertex). For a given graph G we will denote the smallest k such that a map satisfying the conditions exists by $c(G)$. It is easy to see that $c(G) \leq s(G)$. Aigner, Triesch and Tuza introduce this problem in [5]. The authors prove the following result for regular graphs.

Theorem 17. *If G is a r-regular graph, then there exist constants C_1, C_2 such that*

$$C_1 n^{1/r} \leq c(G) \leq C_2 n^{1/r}.$$

2.3.2 1,2,3-Conjecture

Motivated by the results on irregularity strength, Karoński, Łuczak and Thomason [51] propose the study of non-proper edge-weightings where we only require that for adjacent vertices u, v that we have

$$\sum_{e \ni u} \varphi(e) \neq \sum_{e \ni v} \varphi(e).$$

In other words, for any edge uv of G the sum of weights on the edges incident to u should be different from the sum of weights on the edges incident to v. We denote the smallest k such that there is a φ satisfying the above condition by $\chi_w^e(G)$ (this notation is introduced in [39]). Karoński et al. [51] prove that $\chi_w^e(G) \leq 3$ for graphs G without an edge component and chromatic number

$\chi(G) \leq 3$ and conjecture that $\chi_w^e(G) \leq 3$ for all graphs G without an edge component. This is sometimes referred to as the $1, 2, 3$-Conjecture.

Conjecture 4. *If G is a graph without an edge component, then*

$$\chi_w^e(G) \leq 3.$$

Conjecture 4 has been attacked primarily in two ways. The first is to show it is true for smaller classes of graphs the second is to find general upper bound on $\chi_w^e(G)$. Karoński et al. [51] also prove that for all graphs G without an edge component that $\chi_w^e(G) \leq 213$. Addario-Berry, Dalal, McDiarmid, Reed and Thomason [2] improved this upper bound to $\chi_w^e(G) \leq 30$. Later, Addario-Berry, Dalal and Reed [3] proved what is the current best-known upper bound.

Theorem 18. *If G is a graph without an edge component, then*

$$\chi_w^e(G) \leq 16.$$

The proof of their result is based on finding a spanning subgraph of G called H where $d_H(u) \neq d_H(v)$ for any edge $uv \in E(G)$. If one can find such a subgraph (they do not always exist), then weighting all edges of H with weight 1 and all other edges in G with weight 0 would give a 2-edge-weighting of G with the desired property. In the same paper, the authors show an asymptotic version where 2 edge-weights is sufficient. We present a slightly weaker version of the theorem.

Theorem 19. *Let $G_{n,p}$ be a random graph on n vertices with edge probability $0 < p < 1$, then asymptotically almost always*

$$\chi_w^e(G) = 2.$$

Similar to the problem posed in [5], we can ask for a multiset version of the $1, 2, 3$-conjecture. In this case, we require that for any edge $uv \in E(G)$ that the multiset of weights on edges incident to u is different from the multiset of weights on edges incident to v. We denote the minimum k such that there is a k-edge-weighting satisfying the above condition by $\chi_m^e(G)$. It is easy to see that $\chi_m^e(G) \leq \chi_w^e(G)$. Addario-Berry, Aldred, Dalal and Reed [1] prove the following theorems concerning this version of edge-weighting.

Theorem 20. *If G is a graph without an edge component, then*

$$\chi_m^e(G) \leq 4.$$

Theorem 21. *If G is a graph without an edge component and G has minimum degree at least 1000, then*

$$\chi_m^e(G) \leq 3.$$

2.3.3 Vertex-distinguishing chromatic index

Similar to the problem posed by Burris and Schelp [20], we may ask for a proper edge-weighting such that all vertices have different weight sets. Formally, the question is to determine the minimum k such that there exists a k-edge-weighting such that for any two vertices $u, v \in V(G)$ the set of edge-weights in edges incident to u is different from the set of edge-weights on edges incident to v. This problem is introduced by Harary and Plantholt [40] and for a graph G the parameter is denoted $\chi_0(G)$. Harary and Plantholt [40] proved, among other things, that $\chi_0(K_n) = \lceil \log_2 n \rceil + 1$ for any $n \geq 3$. In spite of the fact that the structure of complete bipartite graphs is simple, it seems that the problem of determining $\chi_0(K_{m,n})$ is not easy, especially in the case $m = n$, as documented by papers of Zagaglia Salvi [75], [76], Horňák and Soták [45], [46] and Horňák and Zagaglia Salvi [48].

2.3.4 General neighbor-distinguishing index

After considering the two previous edge-weighting problems, it is natural to study the corresponding problem for **sets** of edge-weights. Formally, we are looking for a k-edge-weighting such that for all edges $uv \in E(G)$ the set of edge-weights on edges incident to u is different from the set of edge-weights on edges incident to v. This graph parameter can be seen as a descendant of the irregularity strength of a graph as well as both the vertex-distinguishing index and the neighbor-distinguishing index of a graph. For a general graph G we will refer to this parameter as the *general neighbor-distinguishing index* of G and denote it by $\chi_s^e(G)$ (it is also denoted $\mathrm{gndi}(G)$). This question is introduced by Győri, Horňák, Palmer and Woźniak [38] who prove the important result that for a bipartite graph G we have $\chi_s^e(G) \leq 3$ and show that in some cases determining the value of the parameter can be tied strongly to the property-$\mathcal{B}$ for hypergraphs. Later, Győri and Palmer [39] give the best possible upper bound for this parameter for all graphs depending on the classic chromatic number $\chi(G)$.

Theorem 22. *If G is a graph without an edge component and $\chi(G) \geq 3$, then*

$$\chi_s^e(G) = \lceil \log_2 \chi(G) \rceil + 1.$$

Chapter 3 will be a study of the general neighbor-distinguishing index and is based on the results of Győri et al. [38] and Győri and Palmer [39].

2.4 Related problems

At this point it is clear that many more edge-weighing problems of a similar nature can be defined. Furthermore, many of the previous problems, particularly the non-proper edge-weightings, can be fit into the broad theory of graph labellings. Rather than go into this direction we point the interested reader to the exhaustive survey of Gallian [35] (available online) of graph labeling problems. The 11th edition of this survey is 190 pages and covers over 800 papers on the topic of labeling problems! (We remark that the problems discussed in this chapter have yet to appear together in a published survey paper.)

To conclude this chapter we would like to introduce two more weighting problems of personal interest. Both of these problems extend the edge-weightings described in the previous sections to their total coloring analogue.

Now we weight both edges and vertices. Formally, a *k-total-weighting* of a graph G is a map $\psi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$. As before, we can ask that this map be proper or non-proper. A k-total-weighting is *proper* if for any pair of incident edges e, f we have $\psi(e) \neq \psi(f)$, for any vertex v and incident edge e we have $\psi(v) \neq \psi(e)$ and for any two adjacent vertices u, v we have $\psi(u) \neq \psi(v)$. Without any additional constraints this is just the total coloring problem. Similar to before, let $S_\psi(v)$ be the set of weights on edges incident to v as well as the weight on v under ψ. Formally,

$$S_\psi(v) = \left(\bigcup_{e \ni v} \psi(e) \right) \cup \psi(v).$$

In [77] Zhang, Chen, Li, Yao, Lu and Wang ask for the the smallest k such that there exists a proper k-total-coloring ψ of a given graph G such that for any edge $uv \in E(G)$ we have $S_\psi(u) \neq S_\psi(v)$. The authors denote this parameter $\chi_{at}(G)$ and determine its value for some simple classes of graphs including cycles and complete graphs. The authors also conjecture that just one more weight is necessary beyond the $\Delta(G) + 2$ required by the Total Coloring Conjecture.

Conjecture 5. *If G is a connected graph without an edge component, then*

$$\chi_{at}(G) \leq \Delta(G) + 3.$$

Finally, the authors ask about the monotonicity of their parameter.

Problem 1. *If H is a subgraph of G, when do we have $\chi_{at}(H) \leq \chi_{at}(G)$?*

Przybyło and Woźniak [63] recently posed a new non-proper k-total-weighting question in the spirit of the problem posed by Karoński et al. [51]. The question is to find, for a given graph G, the minimum k such that there exists a non-proper k-total-weighting ψ such that for any edge $uv \in E(G)$ we have

$$\psi(u) + \sum_{e \ni u} \psi(e) \neq \psi(v) + \sum_{e \ni v} \psi(e).$$

The authors denote this parameter by $\tau(G)$ and pose the so-called $1,2$-Conjecture.

Conjecture 6. *If G is a graph without an edge component, then*

$$\tau(G) \leq 2.$$

In other words, the weights 1 and 2 are sufficient to find a total-weighting satisfying the given condition. The authors prove a weaker form of their conjecture, that 11 or $\lfloor \frac{\chi(G)}{2} \rfloor + 1$ weights are enough for such a weighting. In a second paper, Przybyło and Woźniak [64] have improved the 11 weights to 7 for regular graphs. This $1,2$-Conjecture is strongly related to the $1,2,3$-Conjecture of Karoński et al. [51] as the authors suggest. This is further reinforced in that both papers rely on the techniques of Addario-Berry et al. [3] on the $1,2,3$-Conjecture.

Now we return our attention to the general neighbor-distinguishing index introduced by Győri et al. [38].

Chapter 3

General neighbor-distinguishing index

3.1 Introduction

All graphs we discuss are simple (note that most results hold for multigraphs too) and finite. Let G be a graph and k a non-negative integer. A *k-edge-weighting* of G is a map $\varphi : E(G) \to \{1, 2, \dots, k\}$. (In this section the term "weighting" will always refer to edges while "coloring" will always refer to vertices.) The *weight set* (with respect to the map φ) of a vertex $x \in V(G)$ is the set $S_\varphi(x)$ of weights of edges incident to x (the subscript φ can be omitted when it does not cause confusion). Formally, $S_\varphi(x) = \{\varphi(e) : e \ni x\}$. A k-edge-weighting φ is *vertex-coloring by sets* if $S_\varphi(x) \neq S_\varphi(y)$ whenever vertices x, y are adjacent (typically we will omit the phrase "by sets"). We will often refer to φ as a *vertex-coloring edge-weighting*. For a graph G we are interested in the minimum k such that there exists a k-edge-weighting of G that is vertex-coloring. We will denote this parameter by $\chi^e_s(G)$. If G has a component K_2, then G cannot have a vertex-coloring edge-weighting, so we (have to) assume that G has no such component. If G is a graph with components $G_1, \ldots, G_n$, then we can take the maximum of these minima componentwise, so the analysis of the vertex-coloring edge-weightings can be restricted to connected graphs. Therefore all graphs will be assumed to be connected unless otherwise stated.

The main results of this thesis are as follows.

Theorem 23. *If G is a bipartite graph without an edge component, then*

$$\chi^e_s(G) \leq 3.$$

Theorem 24. *If G is a graph without an edge component and $\chi(G) \geq 3$, then*

$$\chi^e_s(G) = \lceil \log_2 \chi(G) \rceil + 1.$$

We will begin by proving Theorem 23 which will be integral to the proof of Theorem 24. The proof of Theorem 24 will be separated into three parts. First we prove Theorem 24 for $\chi(G) \leq 4$, then for $5 \leq \chi(G) \leq 8$, and finally for $\chi(G) \geq 8$. The next sections will be concerned with establishing the upper bounds for Theorem 23 and Theorem 24. The lower bound is a simple observation and will be used implicitly in the proofs in this chapter.

Remark 25. *If G is a graph without an edge component, then $\chi_s^e(G) \geq \lceil \log_2 \chi(G) \rceil + 1$.*

Proof. Assume that we have a vertex-coloring edge-weighting of G with $k = \chi_e^s(G)$ weights, and so we have at most 2^k different weight sets appearing in G. This naturally gives us a proper vertex-coloring of G with 2^k colors. However, it is clear that a vertex with weight set S and a vertex with weight set $\{1, 2, \ldots, k\} - S$ cannot be neighbors as the weight sets of neighbors must have a nonempty intersection (the weight of the edge connecting neighbors is necessarily in the intersection of their weight sets). Therefore we can color such vertices with the same color and thus at most 2^{k-1} different colors are needed to color G. So, $\chi(G) \leq 2^{k-1}$ yields $\lceil \log_2 \chi(G) \rceil \leq k - 1$. $\square$

3.2 Paths, cycles and bipartite graphs

We begin our analysis of vertex-coloring edge-weightings with a few trivial remarks. First of all, it is clear that if we have a graph G with $\chi_s^e(G) = 0$ then G must have no edges i.e. G is a collection of isolated vertices. Second, it is not possible for a graph G to have $\chi_s^e(G) = 1$ as this yields a single non-empty weight set. So, the study of $\chi_s^e(G)$ is only interesting from $\chi_s^e(G) = 2$. Our first proposition characterizes this case and has some interesting consequences.

Proposition 26. *For any graph G the following statements are equivalent:*

(i) $\chi_s^e(G) = 2$.

(ii) G is bipartite and there is a bipartition $\{X_1 \cup X_2, Y\}$ of $V(G)$ such that $X_1 \cap X_2 = \emptyset$ and any vertex of Y has a neighbor in each of X_1 and X_2.

Proof. $(i) \Rightarrow (ii)$: Consider a vertex-coloring 2-edge-weighting φ of G. Under φ the only possible weight sets are $\{1\}$, $\{2\}$ and $\{1,2\}$ (indeed, isolated vertices get weight set $\{\}$, but we may ignore them). Since $\{1\} \cap \{2\} = \emptyset$, for any $xy \in E(G)$ exactly one of $S_\varphi(x)$ and $S_\varphi(y)$ is equal to $\{1,2\}$. Let $Y := \{y \in V(G) : S_\varphi(y) = \{1,2\}\}$, let $X_1 := \{x \in V(G) : S_\varphi(x) = \{1\}\}$ and let $X_2 := \{x \in V(G) : S_\varphi(x) = \{2\}\}$. Clearly, the sets X_1, X_2, Y are pairwise disjoint. So, any edge of G joins a vertex of $X_1 \cup X_2$ to a vertex of Y, and any vertex of Y has a neighbor in each of X_1 and X_2. Thus $\{X_1 \cup X_2, Y\}$ is the desired bipartition of $V(G)$.

$(ii) \Rightarrow (i)$: Let the 2-edge-weighting φ of G be defined as follows for $xy \in E(G)$:

$$\varphi(xy) = \begin{cases} 1 & \text{if } x \in X_1 \text{ and } y \in Y, \\ 2 & \text{if } x \in X_2 \text{ and } y \in Y. \end{cases}$$

Then we have $S_\varphi(x) = \{1\}$ for $x \in X_1$, $S_\varphi(x) = \{2\}$ for $x \in X_2$ and $S_\varphi(y) = \{1,2\}$ for $y \in Y$, and thus φ is vertex-coloring. $\square$

Proposition 26 suggests the difficulty of determining $\chi_s^e(G)$. Merely to decide if $\chi_s^e(G) = 2$ for a bipartite graph G is **NP**-complete. In fact, it is equivalent to determine if a hypergraph has property-$\mathcal{B}$ i.e. is 2-colorable. A hypergraph is a generalization of a graph where edges may contain more than 2 vertices. Formally, a hypergraph is a pair $(\mathcal{V}, \mathcal{E})$ of vertices $\mathcal{V}$ and hyperedges $\mathcal{E}$ such that a hyperedge $E \in \mathcal{E}$ is a subset of the vertices $E \subset \mathcal{V}$. A classic topic concerning hypergraphs is the so-called property-$\mathcal{B}$. A hypergraph is said to have *property-$\mathcal{B}$* if the vertex set $\mathcal{V}$ can be colored with 2 colors such that no hyperedge in $\mathcal{E}$ is monochromatic. Determining if a hypergraph has property-$\mathcal{B}$ is well-known to be **NP**-complete (in complexity theory this problem is often referred to as SET-SPLITTING).

Any hypergraph can easily be represented by a bipartite graph in the following way. Let $(\mathcal{V}, \mathcal{E})$ be a hypergraph and we will construct a bipartite graph with vertex classes $V_{\mathcal{V}} = \mathcal{V}$ and $V_{\mathcal{E}} = \mathcal{E}$. We connect a vertex $v \in V_{\mathcal{V}}$ to a vertex $E \in V_{\mathcal{E}}$ exactly if $v \in E$ in the original hypergraph. From here it is easy to establish the equivalence of property-$\mathcal{B}$ and if a bipartite graph has $\chi_s^e(G) = 2$.

In particular, if a hypergraph has property-$\mathcal{B}$ then the vertex set can be split into two classes such that each edge meets each of the two classes of vertices. In bipartite graph notation this is exactly condition (ii) of Proposition 26. In other words, if we can determine in general if a bipartite graph G has $\chi_s^e(G) = 2$ then we can determine if a given hypergraph has property-$\mathcal{B}$.

As we will discuss later, determining $\chi_s^e(G)$ for non-bipartite graphs is also **NP**-complete.

Let us introduce some additional notation. Let $X \subset V(G)$ be an independent set in G (e.g. a color class in a coloring of G) and let $S_\varphi(X)$ be the family of weight sets appearing on vertices in X under the vertex-coloring edge-weighting φ. Formally, $S_\varphi(X) = \{S_\varphi(x) : x \in X\}$. We will call an edge-weighting of G *canonical* if there is a proper coloring of the vertices with $\chi(G)$ colors such that the family of weight sets appearing on vertices in any color class is strictly disjoint from the family of weight sets appearing on vertices in another color class. In other words, an edge-weighting φ is canonical if for any two color classes X and Y in a coloring of G, we have $S_\varphi(X) \cap S_\varphi(Y) = \emptyset$. Note that a canonical edge-weighting is necessarily vertex-coloring, but a vertex-coloring edge-weighting need not be canonical.

In this section we will concern ourselves with a very specific canonical edge-weighting. In particular, if G is a bipartite graph our aim will be to find a bipartition of $V(G) = (X, Y)$ and an edge-weighting φ such that $S_\varphi(X) \subseteq \mathcal{S}_1 := \{\{3\}, \{1, 2\}\}$ and $S_\varphi(Y) \subseteq \mathcal{S}_2 := \{\{1\}, \{2\}, \{1, 3\}, \{2, 3\}\}$ The set $\mathcal{S}_2$ has the following important property: whenever $S \in \mathcal{S}_2$, then $S \cup \{3\} \in \mathcal{S}_2$.

Proposition 27. *Let P_n be a path on $n \geq 3$ vertices, then*

$$\chi_s^e(P_n) = \begin{cases} 2 & \textit{if } n \textit{ is odd}, \\ 3 & \textit{if } n \textit{ is even}. \end{cases}$$

Furthermore, there is a canonical edge-weighting φ and a bipartition X, Y of $V(P_n)$ such that,

$$S_\varphi(X) \subseteq \{\{3\}, \{1, 2\}\},$$
$$S_\varphi(Y) \subseteq \{\{1\}, \{2\}, \{1, 3\}, \{2, 3\}\},$$

and at least one endpoint of P_n has weight set different from $\{3\}$.

Proof. Suppose that $n = 2k+1$ is odd. Let us arrange the vertices of $P_{2k+1} = x_1x_2\ldots x_{2k+1}$ into three classes X_1, X_2, Y as follows. For i even put x_i in Y, for $i = 1, 3, 5, \ldots, 2k+1$ alternate x_i between X_1 and X_2. Now we have a bipartition P_n as described by Proposition 26, thus $\chi_s^e(P_{2k+1}) = 2$.

Alternatively, for n odd we can distinguish two cases i.e. $n = 4k+1$ and $n = 4k+3$. In the first case, we have $4k$ consecutive edges i.e. k consecutive 4-edge-paths. Let us weight each 4-edge-path $e_1e_2e_3e_4$ with weights $1, 2, 2, 1$ respectively. In the second case we have $4k+2$ consecutive edges i.e. k consecutive 4-edge-paths followed by a single 2-edge-path. Let us weight each 4-edge-path $e_1e_2e_3e_4$ with weights $1, 2, 2, 1$ respectively and the final 2-edge-path f_1f_2 with weights $1, 2$ respectively.

Now suppose that $n = 2k$ is even. We can distinguish two cases $n = 4k$ and $n = 4k+2$. In the first case, we have $4k-1$ consecutive edges i.e. $k-1$ consecutive 4-edge-paths followed by a single 3-edge-path. Let us weight each 4-edge-path $e_1e_2e_3e_4$ with weights $1, 2, 2, 1$ respectively and the final 3-edge-path $f_1f_2f_3$ with weights $1, 2, 3$ respectively. In the second case, we have $4k+1$ consecutive edges i.e. k consecutive 4-edge-paths followed by a single edge. Let us weight each 4-edge-path $e_1e_2e_3e_4$ with weights $1, 2, 2, 1$ respectively and the final edge f_1 with weight 3.

It is a simple matter to check that indeed the given edge-weighings satisfy the conditions of the proposition. The bipartition X, Y comes immediately from the weight sets formed under φ.

$\square$

Proposition 28. *Let C_n be a cycle on $n \geq 3$ vertices, then*

$$\chi_s^e(C_n) = \begin{cases} 2 & \textit{if } n \equiv 0 \pmod 4, \\ 3 & \textit{if } n \not\equiv 0 \pmod 4. \end{cases}$$

Proof. The proposition will follow from four simple cases. Let us consider the number of vertices n modulo 4. Note that we always have the same number of edges as vertices. Let us label the edges by $e_1e_2\ldots e_n$. In each case, when we weight several edges in a sequence, we always mean to weight the edges in the order determined by their index (i.e. lowest index first, highest index last).

1. If $n = 4k-1$, then we have $k-1$ consecutive 4-edge-paths and a single 3-edge-path. Let us weight each 4-edge-path with weights $1, 2, 2, 1$ and the 3-edge-path with weights $1, 2, 3$.

2. If $n = 4k$, then we have k consecutive 4-edge-paths. Let us weight each 4-edge path with weights $1, 2, 2, 1$ respectively. In this case, the final weight sets will alternate between $\{1, 2\}$ and $\{1\}$ or $\{2\}$. Also note that this case fits appropriately with Proposition 26.

3. If $n = 4k+1$, then let us break the cycle into $k-1$ consecutive 4-edge-paths and a single 5-edge-path. Let us weight each 4-edge-path with weights $1, 2, 2, 1$ and the 5-edge-path with weights $1, 2, 2, 3, 1$.

4. If $n = 4k+2$, then let us break the cycle into $k-1$ consecutive 4-edge-paths and a single 6-edge-path. Let us weight each 4-edge-path with weights $1, 2, 2, 1$ and the 6-edge-path with weights $1, 2, 3, 1, 2, 3$.

It is simple to check that the given edge weightings are vertex-coloring. When $k = 1$ all 4 cases can be checked easily. Observe that for $k > 1$ it is enough to consider each case for $k = 2$ as any additional 4-edge-paths have no chance to prevent the edge-weighting from being vertex-coloring. The case $k = 2$ can also be checked easily.

In the 3 cases where $n \not\equiv 0 \pmod 4$ it follows from Proposition 26 that 2 edge weights are not sufficient. It is easy to see that condition (ii) of Proposition 26 requires that C_n have length a multiple of 4.

$\square$

At this point we can see that, in general, $\chi_s^e(G)$ is not a monotone graph parameter under the addition of edges. In particular, the path on $4k$ vertices, P_{4k}, has $\chi_s^e(P_{4k}) = 3$ but the cycle on $4k$ vertices, C_{4k}, has $\chi_s^e(C_{4k}) = 2$. Beyond these examples we do not have a better understanding of when $\chi_s^e(G)$ changes from 3 to 2 under the addition of an edge. A full characterization may be difficult as it could shed considerable light on the case when $\chi_s^e(G) = 2$ which is **NP**-complete.

However, our main result for graphs G with $\chi(G) \geq 3$ that $\chi_s^e(G) = \lceil \log_2 \chi(G) \rceil + 1$ implies monotonicity beyond the bipartite case as we know that $\chi(G)$ is a monotone graph parameter under the addition of edges. Oddly enough, it is this theorem that implies monotonicity. So far, proving directly that $\chi_s^e(G)$ is monotone has been elusive. A direct proof of monotonicity would be interesting, but at this point the only consequence would be to simplify the proof of the main theorem. Let us refer to the fact that the proof of the upper bound for 3-chromatic graphs is considerably more difficult than the proof for 4-chromatic graphs. If we know directly that $\chi_s^e(G)$ is monotone then the separate proof for 3-chromatic graphs would be unnecessary.

We introduce some additional notation to aid in the analysis of bipartite graphs. Let G be a graph and let $x \in V(G)$, then by $d_G(x)$ we denote the degree of x in G i.e. the number of edges incident to x. A *branch* of a tree T is a minimal length subpath P of T from a vertex of degree 1 to a vertex of degree 1 or at least 3 in T i.e. a maximal length subpath P of T such that the internal vertices of P all have degree 2 in T and one endpoint has degree 1 in T. Let $b(T)$ denote the number of branches of T. If T is an n-vertex path P_n, then $b(T) = 1$ and T itself is the only branch of T. On the other hand, if $\Delta(T) \geq 3$, any branch P of T has one endvertex of degree one, the other of degree at least three and $b(T)$ is equal to the number of vertices of T of degree 1.

Theorem 29. *If T is a tree without an edge component, then $\chi_s^e(T) \leq 3$. Furthermore, there is a canonical edge-weighting φ and a bipartition X, Y of $V(T)$ such that,*

$$S_\varphi(X) \subseteq \{\{3\}, \{1,2\}\},$$

$$S_\varphi(Y) \subseteq \{\{1\}, \{2\}, \{1,3\}, \{2,3\}\},$$

and no vertex $v \in V(T)$ with $d_T(v) > 1$ has weight set $S_\varphi(v) = \{3\}$.

Proof. We proceed by induction on the number of branches $b(T)$. The base case holds as if $b(T) = 1$, then $n \geq 3$ and $T \simeq P_n$ and we are done by Proposition 27.

So, let $b(T) > 1$ and assume the statement of the theorem holds for any tree T' without an edge component and with $b(T') < b(T)$. Fix a vertex $x \in V(T)$ with $d_T(x) = 1$ and choose a vertex $y \in V(T)$ with $d_T(y) \geq 3$ such that the length of the path from x to y is minimal. The subpath P of T with endvertices x and y is a branch of T. Put $T' := T - (V(P) - \{y\})$. Clearly, T' is a subtree of T with $b(T') = b(T) - 1$ and $|E(T')| \geq 2$. By induction there is a 3-edge-weighting φ' and a bipartition X', Y' of $V(T')$ such that

$$S_{\varphi'}(X') \subseteq \mathcal{S}_1\{\{3\}, \{1,2\}\},$$
$$S_{\varphi'}(Y') \subseteq \mathcal{S}_2\{\{1\}, \{2\}, \{1,3\}, \{2,3\}\}.$$

We will construct a 3-edge-weighting φ of T that satisfies the statement of the theorem by extending φ'. So, for $e \in E(T')$ put $\varphi(e) = \varphi'(e)$. Now it remains to define φ for the edges of P and find bipartition of $V(T)$ satisfying the theorem. We distinguish 2 main cases and a number of subcases.

1. The branch P is a single edge xy.

1.1. If $S_{\varphi'}(y) \neq \{1,2\}$, then $S_{\varphi'}(y) \in \mathcal{S}_2$. Defining $\varphi(xy) := 3$ yields $S_{\varphi}(y) = S_{\varphi'}(y) \cup \{3\} \in \mathcal{S}_2$, $S_{\varphi}(x) = \{3\} \in \mathcal{S}_1$ and φ and the bipartition $X', Y' \cup \{x\}$ of $V(T)$ satisfy the conditions of the theorem.

1.2. If $S_{\varphi'}(y) = \{1,2\}$, set $\varphi(xy) := 1$. Then $S_{\varphi}(x) = \{1\} \in \mathcal{S}_2$, $S_{\varphi}(y) =$

$\{1,2\} \in \mathcal{S}_1$ and φ and the bipartition $X' \cup \{x\}, Y'$ of $V(T)$ satisfy the conditions of the theorem.

2. The branch P has length at least 3. Let z be the unique neighbor of y in P. Since $d_{T'}(y) = d_T(y) - 1 \geq 2$ the edge-weighting φ' gives that there is $i \in S_{\varphi'}(y) \cap \{1,2\}$. Let us consider the path P and let φ'' be the 3-edge-weighting and X'', Y'' be the partition of $V(P)$ given by applying Proposition 27. Without loss of generality we may assume $\varphi''(yz) = i$ (by permuting the weights 1 and 2 if necessary).

2.1. If $S_{\varphi'}(y) \neq \{1,2\}$, let

$$\varphi(e) = \begin{cases} \varphi'(e) & \text{if } e \in T', \\ \varphi''(e) & \text{if } e \in P. \end{cases}$$

In such a case $S_\varphi(v) = S_{\varphi'}(v)$ for any $v \in V(T')$, $S_\varphi(v) = S_\varphi(v)$ for any $v \in V(A) - \{y\}$ and thus φ and the bipartition $X' \cup X'', Y' \cup Y''$ of $V(T)$ satisfy the conditions of the theorem.

2.2. If $S_{\varphi'}(y) = \{1,2\}$, then $y \in Y'$.

2.2.1. If $V(P) = \{x, z, y\}$, set $\varphi(yz) := 2$ and $\varphi(zx) := 3$ to obtain $S_\varphi(y) = \{1,2\} \in \mathcal{S}_1$, $S_\varphi(z) = \{2,3\} \in \mathcal{S}_1$ and $S_\varphi(x) = \{3\} \in \mathcal{S}_2$; thus φ and the bipartition $X' \cup \{z\}, Y' \cup \{x\}$ of $V(T)$ satisfy the conditions of the theorem.

2.2.2. If $|V(P)| \geq 4$, then $P' := P - y$ is a path on $|V(P)| - 1 \geq 3$ vertices. By Proposition 27 there is a 3-edge-weighting φ'' of P' such that $S_{\varphi''}(z) =$

$\{1\}$; if X'', Y'' is the bipartition of $V(P')$ given by Proposition 27, then $z \in X''$. Let

$$\varphi(e) = \begin{cases} \varphi'(e) \text{ if } e \in T', \\ \varphi''(e) \text{ if } e \in P', \\ 1 \text{ if } e = yz. \end{cases}$$

In such a case $S_\varphi(v) = S_{\varphi'}(v)$ for any $v \in V(T')$, $S_\varphi(v) = S_{\varphi''}(v)$ for any $v \in V(A')$ and thus φ and the bipartition $X' \cup X'', Y' \cup Y''$ of $V(P)$ satisfy the conditions of the theorem.

□

Theorem 30. *If G is a connected bipartite graph on $n \geq 3$ vertices, then $\chi_s^e(G) \leq 3$. Furthermore, there is a canonical edge-weighting φ and a bipartition X, Y of $V(G)$ such that,*

$$S_\varphi(X) \subseteq \{\{3\}, \{1,2\}\},$$
$$S_\varphi(Y) \subseteq \{\{1\}, \{2\}, \{1,3\}, \{2,3\}\}.$$

Proof. We proceed by induction on the cyclomatic number $\mu(G) := |E(G)| - |V(G)| + 1$. The base case $\mu(G) = 0$ holds as G is a tree and we can use Theorem 29. So, let $\mu(G) > 0$ and assume the statement of the theorem holds for any connected bipartite graph H on at least 3 vertices with $\mu(H) < \mu(G)$. From $\mu(G) > 0$ it follows that there is a cycle C in G (of even length). Let $xy \in E(C)$ be an edge of C, then the subgraph $H := G - xy$ is connected, has at least 3 vertices and $\mu(H) = \mu(G) - 1$. By induction there exists a canonical 3-edge-weighting ψ of H and a bipartition X, Y of $V(H)$ with

$$S_\psi(X) \subseteq \{\{3\}, \{1,2\}\},$$
$$S_\psi(Y) \subseteq \{\{1\}, \{2\}, \{1,3\}, \{2,3\}\}.$$

Note that x and y cannot be the same class of the bipartition as they are endpoints of an even-length path. Therefore, without loss of generality, we may suppose that $x \in X$ and $y \in Y$.

Let us construct a 3-edge-weighting φ of G extending ψ i.e. $\varphi(e) = \psi(e)$ if $e \in E(H)$. Now it remains to choose a weight for the remaining edge $xy \in G$. If $S_\psi(x) \cap S_\psi(y) \neq \emptyset$, put $\varphi(xy) \in S_\psi(x) \cap S_\psi(y)$ which gives $S_\varphi(x) = S_\psi(x)$ and $S_\varphi(y) = S_\psi(y)$ and thus φ satisfies the conditions of the theorem. If $S_\psi(x) \cap S_\psi(y) = \emptyset$, then there is $i \in \{1,2\}$ such that $S_\psi(x) = \{i\}$ and $S_\varphi(y) = \{3\}$; in this case setting $\varphi(xy) := 3$ yields $S_\varphi(x) = \{i, 3\}$ and $S_\varphi(y) = \{3\}$ and thus φ satisfies the conditions of the theorem. $\square$

Although we will soon improve it, we will prove the main result of Győri et al. [38] as the technique seems to be useful for achieving partial results in related problems. We use the following lemma proved by Balister et al. [12].

Lemma 31. *If G is a graph having neither K_2 nor K_3 as a component, then G can be written as an edge-disjoint union of $\lceil \log_2 \chi(G) \rceil$ bipartite graphs, each of which has no component K_2.*

From here it is an easy step to get a general bound on $\chi_s^e(G)$.

Theorem 32. *If G is a graph without an edge component, then*

$$\chi_s^e(G) \leq 2\lceil \log_2 \chi(G) \rceil + 1.$$

Proof. Without loss of generality we may suppose that G is connected. If $G = K_1$, then $\chi_s^e(G) = 0$. For $G = K_3 = C_3$ Proposition 28 yields $\chi_s^e(G) = 3$. If $G \notin \{K_1, K_3\}$, put $r := \lceil \log_2 \chi(G) \rceil$. By Lemma 31 we know that G can be written as an edge-disjoint union of r bipartite graphs, each of which has no component K_2. Let $B_1, \dots, B_r$ be such an edge-disjoint decomposition of G. By Theorem 30, for any $i \in \{1, 2, 3, \dots, r\}$ there is a canonical edge-weighting $\varphi_i : E(B_i) \to \{1, 2i, 2i+1\}$ and a bipartition X_i, Y_i of $V(B_i)$ such that

$$S_{\varphi_i}(X_i) \subseteq \{\{1\}, \{2i, 2i+1\}\},$$
$$S_{\varphi_i}(Y_i) \subseteq \{\{2i\}, \{2i+1\}, \{1, 2i\}, \{1, 2i+1\}\}.$$

Now let $\varphi := \bigcup_{i=1}^r \varphi_i$, be the common continuation of all the φ_i's. Let us confirm that φ is vertex-coloring. For any edge $e \in E(G)$ there is a unique $i \in \{1, 2, 3, \dots, r\}$ such that $e \in E(B_i)$, and so $e = xy$ with $x \in X_i$ and $y \in Y_i$. Trivially, $S_{\varphi_i}(x) \subseteq S_\varphi(x)$ and $S_{\varphi_i}(y) \subseteq S_\varphi(y)$. Therefore, $S_\varphi(x)$ contains exactly one of the weights $2i, 2i+1$ and $S_\varphi(y)$ contains either both weights $2i, 2i+1$ or none of them. Hence we have $S_\varphi(x) \neq S_\varphi(y)$. Thus, the edge-weighting $\varphi : E(G) \to \{1, 2, 3, \dots, 2r+1\}$ shows that $\chi_s^e(G) \leq 2r+1 = 2\lceil \log_2 \chi(G) \rceil + 1$. □

Before we turn our attention to the best possible upper bound, let us add a few remarks about the previous upper bounds. Theorem 32 was the best known upper bound when [38] was written. Later Győri and Palmer (unpublished) were able to improve the upper bound to $\frac{3}{2}\lceil \log_2 \chi(G) \rceil + 1$ by proving a more general version of Lemma 31 and their result (to be proved in the next section) that for a 4-colorable graph we can always find a vertex-coloring 3-edge weighting. Several improvements of the constant multiple of $\lceil \log_2 \chi(G) \rceil$ were possible by further progress in this way. A breakthrough was achieved by the same authors (also unpublished) that reduces the multiplicative factor to 1 and gives a general upper bound of $\lceil \log_2 \chi(G) \rceil + 5$. Eventually that proof was improved to give the very satisfactory answer that the upper bound is the best possible $\lceil \log_2 \chi(G) \rceil + 1$. The next section will concern the proof of this upper bound and is based on the work of Győri and Palmer [39].

3.3 Non-bipartite graphs

Additional notation used in the proof is mostly standard. In particular, the neighbors of a vertex v are denoted by $N(v)$. Similarly, the set of all neighbors of a set of vertices X is denoted by $N(X) = \bigcup_{v \in X} N(v) - X$. The set of all edges between two disjoint sets of vertices X and Y is denoted by $E(X, Y)$.

Theorem 24 will follow from three lemmas.

Lemma 33. *If G is a graph without an edge component and $3 \leq \chi(G) \leq 4$,* *then*

$$\chi_s^e(G) = 3.$$

Lemma 34. *If G is a graph without an edge component and $5 \leq \chi(G) \leq 8$, then*

$$\chi_s^e(G) = 4.$$

Lemma 35. *If G is a graph without an edge component and $\chi(G) \geq 8$, then*

$$\chi_s^e(G) = \lceil \log_2 \chi(G) \rceil + 1.$$

Lemma 33 will follow from separate statements for 3-chromatic and 4-chromatic graphs. Lemma 34 is somewhat more straightforward and it is essentially enough to focus on the case when $\chi(G) = 8$. Lemma 35 will follow from a slightly stronger statement that does not hold for graphs of smaller chromatic number and is the reason why Lemma 33 and Lemma 34 are proved independently.

3.3.1 4-colorable graphs

This section will be concerned with the proof of Lemma 33. Let G be a 3-chromatic graph. We call a 3-coloring X, Y, Z of $V(G)$ *stable* if the following conditions hold:

(i) if $x \in X$, then $N(x) \cap Y \neq \emptyset$

(ii) if $y \in Y$, then $N(y) \cap Z \neq \emptyset$

(iii) if $z \in Z$, then $N(z) \cap X \neq \emptyset$

In other words, each vertex in X must have a neighbor in Y, each vertex in

Y must have a neighbor in Z and each vertex in Z must have a neighbor in X. If a 3-coloring is not stable, then call it *unstable* and observe that there must be a vertex that fails to satisfy the above requirement. With an abuse of notation, we will call such a vertex *unstable*, otherwise a vertex is *stable*.

Proposition 36. *If G is 3-chromatic and has a stable 3-coloring X, Y, Z, then $\chi_s^e(G) = 3$. Furthermore, there is a canonical edge-weighting φ such that,*

$$S_\varphi(X) \subseteq \{\{3\}, \{1,3\}\},$$
$$S_\varphi(Y) \subseteq \{\{2\}, \{2,3\}\},$$
$$S_\varphi(Z) \subseteq \{\{1\}, \{1,2\}\}.$$

Proof. If X, Y, Z is a stable 3-coloring of G then weight all $E(X,Y)$ edges with 3, all $E(Y,Z)$ edges with 2 and all $E(X,Z)$ edges with 1. Notice that among vertices of X the only possible weight sets are $\{3\}$ and $\{1,3\}$, among vertices of Y the only possible weight sets are $\{2\}$ and $\{2,3\}$ and among vertices of Z the only possible weight sets are $\{1\}$ and $\{1,2\}$. Clearly this edge-weighting yields φ with the desired properties. $\square$

Therefore to show a 3-chromatic graph, G, has $\chi_s^e(G) = 3$ it is sufficient to find a stable 3-coloring.

Proposition 37. *If G is a connected 3-chromatic graph and contains a triangle, then G has a stable 3-coloring.*

Proof. Let X, Y, Z be the color classes of a 3-coloring of G. Orient the edges of G from X to Y, from Y to Z and from Z to X. Define $D \subset V(G)$ to be the set of vertices v for which either v is on a directed cycle in G or v is on a directed path to a directed cycle in G. Note that every vertex in D has outdegree at least 1 and therefore is stable.

Now choose a 3-coloring X, Y, Z that maximizes the size of D. If $D = V(G)$, then all vertices of G are stable and we are done. So, let us assume $|D| < |V(G)|$. Now let us recolor the vertices of G as follows: if $v \in D \cap X$ then put v in X', if $v \in D \cap Y$ then put v in Y', if $v \in D \cap Z$ then put v in Z'; if $v \in X - D$ put v in Y', if $v \in Y - D$ then put v in Z', if $v \in Z - D$ then put v in X'. In other words, vertices in D keep their color and vertices of $V(G) - D$ are moved to the "next" color class. Now, orient the edges in this recoloring from X' to Y', from Y' to Z' and from Z' to X'.

Let us confirm that this new coloring is proper. For the sake of contradiction assume it is not proper. For the coloring to be not proper there must be some edge contained in one of the color classes. Edges with both endpoints in D remain between different color classes as both endpoints stay in different color classes. Edges with both endpoints in $V(G) - D$ remain between different color classes as both endpoints move to their respective "next" classes. Therefore, we must examine edges of the form uw where $u \in D$ and $w \in V(G) - D$. Without loss of generality, let us assume that after recoloring the vertices of $V(G) - D$ that u and w are both in color class X'. This means that w moved from Z to X' and that before recoloring, the edge uw was oriented from w to u. But this means that w should have been in D not $V(G) - D$. This is a contradiction, therefore the recoloring is a proper coloring.

Now let us examine the maximality of D. After recoloring, let us define D' as the set of vertices on a directed cycle or on a directed path to a directed cycle under the new coloring X', Y', Z'. Note that all vertices in D are in D' as no vertices of D are recolored and therefore maintain their original orientation. Because G is connected and D is nonempty (it contains at least a triangle), there was a directed edge $\overrightarrow{vu}$ such that $v \in D$ and $u \in V(G) - D$ under the original orientation. Without loss of generality, assume that $v \in X$. Thus, $u \in Y$ to force the appropriate orientation of vu. After recoloring, v is in X' and u is in Z'. Therefore, the edge vu is now oriented from u to v which implies that $u \in D'$. But $u \notin D$, so $|D'| > |D|$ contradicting the maximality of D. Therefore, we must have that $D = V(G)$ and we have a stable 3-coloring X, Y, Z. $\square$

We note that this proof is essentially the same as the proof of Lemma 2.1 in [1]. We also note that in Proposition 37, a triangle as an induced subgraph was important only in that it is a stable graph. With appropriate adjustments to the proof, the proposition can be restated with any stable induced subgraph instead of a triangle. However, as stated the proposition is strong enough for the remaining proofs.

A number of stable and unstable graphs are known. In particular, the

Petersen Graph and odd cycles of length divisible by 3 are stable, while odd cycles of length not divisible by 3 are unstable.

We introduce the notion of an almost-canonical edge-weighting of a 3-chromatic graph to assist us in the proof of Theorem 24. We call an edge-weighting, φ, of a 3-chromatic graph *almost-canonical* if there exists a 3-coloring X, Y, Z and a vertex $x \in X$ such that the families of weight sets $S_\varphi(X - x), S_\varphi(Y), S_\varphi(Z)$ are pairwise disjoint and $S_\varphi(x) \notin S_\varphi(X)$. Note that an almost-canonical edge-weighting need not be vertex-coloring.

Proposition 38. *If G is an unstable connected 3-chromatic graph and x is an arbitrary vertex of G, then G has an almost-canonical 3-edge-weighting φ and a 3-coloring X, Y, Z with $x \in X$ such that,*

$$S_\varphi(X - x) \subseteq \{\{3\}, \{1, 3\}\},$$
$$S_\varphi(Y) \subseteq \{\{2\}, \{2, 3\}\},$$
$$S_\varphi(Z) \subseteq \{\{1\}, \{1, 2\}\},$$
$$S_\varphi(x) = \{1\}.$$

Proof. Let x be an arbitrary vertex of G. Let us add two new vertices v and w and three new edges xv, xw, and vw to G thus creating a triangle in this new graph. By Proposition 37, this new graph has a stable 3-coloring. Let X, Y, Z be the color classes of this coloring after removing v and w (and their incident edges) and assume without loss of generality that $x \in X$. Because we only removed neighbors of x from the stable graph, x is the only unstable vertex in the coloring of G i.e. x has no neighbors in Y.

Now, weight all edges in $E(X, Y)$ with weight 3, all edges in $E(Y, Z)$ with weight 2 and all edges in $E(X, Z)$ with weight 1. The only possible weight sets in $X - x$ are $\{3\}$ and $\{1, 3\}$, the only possible weight sets in Y are $\{2\}$ and $\{2, 3\}$ and the only possible weight sets in Z are $\{1\}$ and $\{1, 2\}$ while x must have weight set $\{1\}$. Clearly this edge-weighting is almost-canonical and yields φ with the desired properties. $\square$

Proposition 39. *If G is an unstable connected 3-chromatic graph with a vertex of degree 1, then $\chi_s^e(G) = 3$.*

Proof. Let x be a vertex of degree 1 in G. Call z the single neighbor of x. The vertex z has degree at least 2, otherwise xz is an isolated edge. Let us add two new vertices v and w and three new edges xv, xw, and vw to G thus creating a triangle in this new graph. By Proposition 37, this new graph has a stable 3-coloring. Let X, Y, Z be the color classes of this coloring after removing v and w (and their incident edges) and assume without loss of generality that $x \in X$. Because we only removed neighbors of x from the stable graph, x is the only unstable vertex in the coloring of G i.e. x has no neighbors in Y. Therefore $z \in Z$. If the neighbors of z are all in X then we can move x to Y thus making x stable and keeping z stable. This gives a stable 3-coloring of G, a contradiction. So, we may assume z has a neighbor in Y.

Now, weight all edges in $E(X, Y)$ with weight 3, all edges in $E(Y, Z)$ with weight 2 and all edges in $E(X, Z)$ with weight 1. The only possible weight sets in $X - x$ are $\{3\}$ and $\{1,3\}$, the only possible weight sets in Y are $\{2\}$ and $\{2,3\}$, and the only possible weight sets in Z are $\{1\}$ and $\{1,2\}$ while x must have weight set $\{1\}$. However, the only neighbor of x is z which has weight set $\{1,2\}$. Therefore, this 3-edge-weighting is vertex-coloring. $\square$

Proposition 40. *If G is a connected 3-chromatic graph, then $\chi_s^e(G) = 3$.*

Proof. By the previous propositions, we may assume that G is triangle-free, the minimum degree of G is at least 2 and G does not have a stable 3-coloring. Let x be an arbitrary vertex of G. Let us add two new vertices v and w and three new edges xv, xw, and vw to G thus creating a triangle in this new graph. By Proposition 37, this new graph has a stable 3-coloring. Let X, Y, Z be the color classes of this coloring after removing v and w (and their incident edges) and assume without loss of generality that $x \in X$. Because we only removed neighbors of x from the stable graph, x is the only unstable vertex in the coloring of G i.e. x has no neighbors in Y. Now we will construct a vertex-coloring edge-weighting of G. Let $L \subset N(x) \subset Z$ be the set neighbors of x that themselves have no neighbors in Y (note that each vertex in L must have at least one neighbor other than x by the minimum degree assumption). Let $M = N(L) - x \subset X$ be the neighbors of L (they are necessarily in X) excluding x.

Claim 41. *There exists a vertex-coloring* 3*-edge-weighting* φ *of* G *such that,*

$$S_\varphi(X - M - x) \subseteq \{\{3\}, \{1,3\}\},$$

$$S_\varphi(M) \subseteq \{\{3\}, \{1,3\}, \{1,2,3\}\},$$

$$S_\varphi(x) = \{1\},$$

$$S_\varphi(Z - L) \subseteq \{\{1\}, \{1,2\}\},$$

$$S_\varphi(L) \subseteq \{\{1,2\}, \{1,3\}\},$$

$$S_\varphi(Y) \subseteq \{\{2\}, \{2,3\}\}.$$

Clearly the above claim implies the proposition. We will construct a 3-edge-weighting with the following families of weight sets and then confirm that it is indeed vertex-coloring by checking that no vertices in X with weight set $\{1\}$ or $\{1,3\}$ have neighbors in Z with the same weight set. We will weight the edges of G by the following steps taking care to note possible weight sets after each step (if they could have been changed).

1. Weight all edges incident to x with weight 1. This immediately gives $S_\varphi(x) = \{1\}$. Also at this point, vertices in L have weight set $\{1\}$; vertices in $Z - L$ have weight set $\emptyset$ or $\{1\}$; all other weight sets are $\emptyset$.

2. Consider the induced bipartite subgraph $G[M \cup Z]$. For every $v_0 \in M$ let $v_0v_1v_2v_3 \ldots v_r$ be a shortest path from v_0 to a vertex $v_r \in Z - L$ if such a path exists. Note that because $G[M \cup Z]$ is bipartite, r is odd. Weight the edges $v_0v_1, v_2v_3, v_4v_5, \ldots, v_{r-1}v_r$ with weight 1 and weight

the edges $v_1v_2, v_3v_4, \ldots, v_{r-2}v_{r-1}$ with weight 2 for all such minimum-length paths. Note that weighting edges in this way will never force an edge to get weight 1 and 2 at the same time as this would contradict minimality of the path lengths. Furthermore, any vertex in L on such a minimal path will necessarily have all of its incident edges weighted in this step. (The last two statements can be checked easily by the reader using contradiction arguments.) At this point, vertices in M have weight set $\emptyset$ or $\{1\}$ or $\{1,2\}$; vertices in L have weight set $\{1\}$ or $\{1,2\}$; vertices in $Z-L$ have weight set $\emptyset$ or $\{1\}$.

3. Weight all unweighted edges in $G[M \cup L]$ with weight 3. At this point, vertices in M have weight set $\{3\}$ or $\{1\}$ or $\{1,3\}$ or $\{1,2\}$ or $\{1,2,3\}$; vertices in L have weight set $\{1,3\}$ or $\{1,2\}$. After this step all edges incident to vertices of L have been weighted.

4. Weight all edges between X and Y with weight 3. Except for x every vertex of X has a neighbor in Y, so the vertices in M have weight set $\{3\}$ or $\{1,3\}$ or $\{1,2,3\}$; vertices in $X-M-x$ have weight set $\{3\}$; vertices in Y have weight set $\emptyset$ or $\{3\}$. After this step all edges incident to vertices of M have been weighted.

5. Weight all edges between $X-M-x$ and Z with weight 1 (all edges between M and Z are already weighted). At this point, vertices in $X-M-x$ have weight set $\{3\}$ or $\{1,3\}$; all vertices in Z have a neighbor in X, so the vertices of $Z-L$ have weight set $\{1\}$. After this

step all edges incident to vertices of X have been weighted.

6. Weight all edges between Z and Y with weight 2. All vertices in Y have a neighbor in Z, so the vertices of Y have weight set $\{2\}$ or $\{2,3\}$; the vertices in $Z - L$ have weight set $\{1\}$ or $\{1,2\}$. After this step all edges have been weighted.

At this point we have achieved the weight sets necessary for Claim 41. Now it remains to confirm that the edge-weighting given is vertex-coloring. In most cases this is immediate from the construction of the weighting. However, we must check that no vertex in X with weight set $\{1\}$ or $\{1,3\}$ has a neighbor in Z with the same weight set. We distinguish two cases.

1. Weight set $\{1\}$. This weight set appears in X only on the vertex x. The neighbors of x are either in L or $Z - L$. The weight set $\{1\}$ does not appear in L. In $Z - L$ the weight set $\{1\}$ does appear, but only on vertices that have no neighbors in Y. If such a vertex were a neighbor of x then it would have been in L initially. So, there are no edges in G with weight set $\{1\}$ on both endpoints.

2. Weight set $\{1,3\}$. This weight set appears in M, $X - M - x$ and $L \subset Z$. However, M consists of the neighbors of L (excluding x) in X, so there are no edges between $X - M - x$ and L and we may restrict our analysis to edges between M and L. Let $l \in L$ and let $M_l \subset M$ be the neighbors of l in M. The only way for an edge with an endpoint

l to get weight set $\{1, 3\}$ is to not be on a minimal path from M to $Z - L$. Otherwise, the weight set of l would necessarily contain a 2 as we alternate weights 1 and 2 along such minimal paths (see Step 2). This means that every vertex of M_l does not have a path to $Z - L$ and therefore every edge with an endpoint in M_l gets weight 3. So, the vertices of M_l all have weight set $\{3\}$. Therefore, there are no edges in G with weight set on $\{1, 3\}$ on both endpoints.

Therefore the given 3-edge-weighting is vertex-coloring, thus proving our claim and proposition. $\square$

That there exist almost-canonical edge-weightings of 3-chromatic graphs will help us show that for 4-chromatic graphs G we have $\chi_s^e(G) = 3$.

Proposition 42. *If G is a connected 4-chromatic graph, then $\chi_s^e(G) = 3$. Furthermore, there is a canonical edge-weighting φ and a 4-coloring X, Y, Z, W such that,*

$$S_\varphi(X) \subseteq \{\{3\}, \{1, 3\}\},$$

$$S_\varphi(Y) \subseteq \{\{2\}, \{2, 3\}\},$$

$$S_\varphi(Z) \subseteq \{\{1\}, \{1, 2\}\},$$

$$S_\varphi(W) = \{\{1, 2, 3\}\}.$$

Proof. Let us 4-color G such that a color class W is minimal over all 4-colorings. This implies that any vertex of W has a neighbor in each of the other three color classes (even if they are properly recolored). Consider the induced subgraph $G' = G[V(G) - W]$ on the other three color classes. Let us assume G' is connected (if it is not the next steps should be performed for each component). If G' is stable, then by Proposition 36 we have a stable 3-coloring X, Y, Z of G' and a canonical edge-weighting φ such that,

$$S_\varphi(X) \subseteq \{\{3\}, \{1,3\}\},$$
$$S_\varphi(Y) \subseteq \{\{2\}, \{2,3\}\},$$
$$S_\varphi(Z) \subseteq \{\{1\}, \{1,2\}\}.$$

If G' is unstable, let x be a vertex of G with a neighbor in W. By Proposition 38, we have a 3-coloring X, Y, Z of G' and an almost-canonical edge-weighting φ such that,

$$S_\varphi(X - x) \subseteq \{\{3\}, \{1,3\}\},$$
$$S_\varphi(Y) \subseteq \{\{2\}, \{2,3\}\},$$
$$S_\varphi(Z) \subseteq \{\{1\}, \{1,2\}\},$$
$$S_\varphi(x) = \{1\}.$$

In both cases, we extend the edge-weighting φ by weighting all $E(X, W)$ edges with 3, all $E(Y, W)$ edges with 2 and all $E(Z, W)$ edges with 1. This does not change the families on weight sets of vertices in $X - x$, Y or Z. Furthermore, the weight set of x will necessarily become $\{1, 3\}$ as x has a neighbor in W. Because each vertex of W has a neighbor in X, Y, Z every vertex in W will necessarily have weight set $\{1, 2, 3\}$. This gives φ that satisfies the conditions of the proposition. □

Clearly, Proposition 40 and Proposition 42 imply Lemma 33.

3.3.2 8-colorable graphs

Before proving Lemma 34, we introduce some definitions. Let $X_1, X_2, \ldots, X_t$ be pairwise disjoint independent sets of $V(G)$ (e.g. color classes in a coloring of G), let $w_1, w_2, \ldots, w_t$ be distinct edge weights and let $S_\varphi(X_i)$ be the family of weight sets appearing on vertices in X_i under a vertex-coloring edge-weighting φ of G. For $i = 1, 2, \ldots, t$ we say that each X_i is w_i*-safe* if for any pair $a, b \in [1, t]$ where $a \neq b$ and any pair of weight sets $S_1 \in S_\varphi(X_a)$ and $S_2 \in S_\varphi(X_b)$ we have that $S_2 \neq S_1 \cup \{w_a\} \neq S_2 \cup \{w_b\} \neq S_1$. In particular, this implies that for all $i \in [1, t]$ we may add weight w_i to any weight sets in $S_\varphi(X_i)$ and not disrupt the vertex-coloring property. This tool will allow us to weight the edges of a graph in an inductive way without ruining previously "good" weightings.

Additionally, an independent set X is called i*-free* (with respect to φ) if $i \notin S$ for all $S \in S_\varphi(X)$ i.e. i never appears in the weight set of any vertex $x \in X$. The following proposition implies Lemma 34 and will form the base case of the inductive proof of Lemma 35.

Proposition 43. *If G is a graph such that $5 \leq \chi(G) \leq 8$, then $\chi_s^e(G) = 4$. Furthermore, if $\chi(G) = 8$ then there is a vertex-coloring edge-weighting of G and an 8-coloring of G with distinct color classes X_1, X_2, X_3, X_4, Y such that X_i is i-safe (for $i = 1, 2, 3, 4$) and Y is 4-free.*

Proof. We explicitly construct an edge-weighting of G where $\chi(G) = 8$. It will be clear from the proof that our weighting will also work for graphs of chromatic number between 5 and 8. Color G with 8 colors in such a way as to maximize the size of the subgraph H induced by the first four colors. Let $F = G[V(G) - V(H)]$ be the graph induced by the remaining color classes. Therefore, $|V(F)|$ is minimal over all colorings and no vertex of F can be colored with a color from H i.e. each vertex in F has a neighbor in each color class of H.

The subgraph H is not necessarily connected. We will distinguish 5 types of components of H. Let H_4 be an arbitrary 4-chromatic component of H, let H_3 be an arbitrary 3-chromatic component of H, let $H_2 \neq K_2$ be an arbitrary bipartite component of H, let xy be an arbitrary isolated edge of H and let v be an arbitrary isolated vertex of H. We will describe how to weight edges among these subgraphs. This technique should be followed for all such components.

By Proposition 42 we have a vertex-coloring edge-weighting, φ_4, of H_4 with $\chi_s^e(H_4) = 3$ weights and a 4-coloring X_4, Y_4, Z_4, W_4 such that,

$$S_{\varphi_4}(X_4) \subseteq \{\{3\}, \{1, 3\}\},$$

$$S_{\varphi_4}(Y_4) \subseteq \{\{2\}, \{2, 3\}\},$$

$$S_{\varphi_4}(Z_4) \subseteq \{\{1\}, \{1, 2\}\},$$

$$S_{\varphi_4}(W_4) = \{\{1, 2, 3\}\}.$$

If H_3 has a stable 3-coloring then by Proposition 36 we have a vertex-coloring edge-weighting, φ_3, of H_3 with $\chi_s^e(H_3) = 3$ weights and a 3-coloring X_3, Y_3, Z_3 such that,

$$S_{\varphi_3}(X_3) \subseteq \{\{3\}, \{1,3\}\},$$
$$S_{\varphi_3}(Y_3) \subseteq \{\{2\}, \{2,3\}\},$$
$$S_{\varphi_3}(Z_3) \subseteq \{\{1\}, \{1,2\}\}.$$

If H_3 does not have a stable 3-coloring then choose a vertex u in H_3 that has a neighbor in F (such a vertex exists as G is connected). Now, by Proposition 38 we have an almost-canonical edge-weighting, φ_3, of H_3 with $\chi_s^e(H_3) = 3$ weights and a 3-coloring X_3, Y_3, Z_3 with $u \in X_3$ such that,

$$S_{\varphi_3}(X_3 - u) \subseteq \{\{3\}, \{1,3\}\},$$
$$S_{\varphi_3}(Y_3) \subseteq \{\{2\}, \{2,3\}\},$$
$$S_{\varphi_3}(Z_3) \subseteq \{\{1\}, \{1,2\}\},$$
$$S_{\varphi_3}(u) = \{1\}.$$

Note that u may have the same weight set as some of its neighbors in Z_3. (We will resolve this conflict later when weighting the edges from u to F.)

By Theorem 23 we have a vertex-coloring edge-weighting, φ_2, of H_2 with $\chi_s^e(H_2) = 3$ weights and a bipartition X_2, Y_2 such that,

$$S_{\varphi_2}(X_2) \subseteq \{\{3\}, \{1,2\}\},$$
$$S_{\varphi_2}(Y_2) \subseteq \{\{1\}, \{2\}, \{1,3\}, \{2,3\}\}.$$

Finally, we will weight all isolated edges xy with weight 2. We refer to this edge-weighting of all of the components of H (and hence H itself) as φ.

Now let:

$$X = X_4 \cup X_3 \cup X_2 \cup \{x\} \cup \{v\},$$

$$Y = Y_4 \cup Y_3 \cup Y_2 \cup \{y\},$$

$$Z = Z_4 \cup Z_3,$$

$$W = W_4.$$

Note that X, Y, Z, W is a 4-coloring of H. By our choice of H, each vertex of F has a neighbor in each of X, Y, Z, W. Let A_1, A_2, A_3, A_4 be the color classes of F (if $\chi(G) < 8$ we just follow the given edge-weighting and ignore the steps involving the appropriate color classes A_i). Let us assume the color classes of F are colored such that each vertex in A_i has a neighbor in each A_j for all $j < i$. Let us weight all edges $E(A_4, A_3)$ with weight 2. Let us weight all other edges in F with weight 4.

Now it remains to weight all edges in $E(H, F)$. The table below describes how to weight edges between different color classes and shows what the possible weight sets are in each color class after weighting all of the edges of G. In particular, the first column represents each color class (some are split into distinct components). Recall that each vertex in A_i has a neighbor in each of X, Y, Z, W. Columns two through five represent the weight to give edges between A_i and the corresponding color class in a given row (when such an edge exists). We refer to the weighting of all of the edges of G by ψ. The final column represents the possible weight sets appearing in the corresponding color class in a given row.

					S_ψ – possible weight sets
	A_1	A_2	A_3	A_4	
X_4, X_3	3	3	3	3	$\{3\}, \{1,3\}$
X_2	3	3	3	3	$\{3\}, \{1,2\}, \{1,2,3\}$
$\{x\}$	3	3	3	3	$\{2\}, \{2,3\}$
$\{v\}$	3	3	3	3	$\{3\}$
Y_4, Y_3	3	3	3	3	$\{2\}, \{2,3\}$
Y_2	3	3	3	3	$\{1\}, \{2\}, \{1,3\}, \{2,3\}$
$\{y\}$	4	4	4	4	$\{2\}, \{2,4\}$
Z_4, Z_3	4	4	4	4	$\{1\}, \{1,2\}, \{1,4\}, \{1,2,4\}$
W_4	3	1	2	1	$\{1,2,3\}$
A_1		4	4	4	$\{3,4\}$
A_2	4		4	4	$\{1,3,4\}$
A_3	4	4		2	$\{2,3,4\}$
A_4	4	4	2		$\{1,2,3,4\}$

Now, by examining the table, we will verify that ψ is vertex-coloring. Clearly the single weight set appearing in A_i (with respect to ψ) only appears in A_i for each i. It remains to check that under the weighting ψ the weight sets on vertices in H are distinct from the weight sets on their neighbors in H. To show this we must confirm that there is no edge where both endpoints have the same weight set under ψ. Recall that each edge is between some pair of color classes X, Y, Z, W while being contained in one of the component types described above i.e. H_4, H_3, H_2, xy. With this and the contents of the last column of the table, the only case where an edge might get the same weight set on both of its endpoints is when it is of the form $K_2 = xy$ in H. For such a component, we have $S_\psi(x) \in \{\{2\}, \{2,4\}\}$ and $S_\psi(y) \in \{\{2\}, \{2,3\}\}$ but both x and y cannot get weight set $\{2\}$ as G has no K_2 components i.e. at least one of x or y has a neighbor in F (and all edges from F to x get weight 3 and all edges from F to y get weight 4). In no other case is there danger of an edge having the same weight set on both endpoints. Therefore, our edge-weighting is vertex-coloring.

Furthermore, it is clear that when $\chi(G) = 8$ we have that A_2 is 1-safe, A_3 is 2-safe, A_4 is 3-safe, A_1 is 4-safe and W is 4-free (all with respect to the final weighting ψ) thus giving the proposition. $\square$

3.3.3 Graphs of higher chromatic number

For technical reasons we prove the following stronger version of Lemma 35.

Proposition 44. *Suppose G is a graph such that $\chi(G) \geq 8$ and let $k = \lceil \log_2 \chi(G) \rceil$, then $\chi_s^e(G) \leq k+1$. Furthermore, there exists a vertex-coloring $(k+1)$-edge-weighting of G and a $\chi(G)$-coloring of G with distinct color classes $X_1, X_2, \ldots, X_{k+1}, Y$ such that X_i is i-safe (for $i = 1, 2, \ldots, k+1$) and Y is $(k+1)$-free.*

Proof. Let G be a graph with chromatic number $\chi(G)$. There exists an integer k such that $2^{k-1} < \chi(G) \leq 2^k$. We proceed by induction on $\chi(G)$.

The base case $\chi(G) = 8$ holds by Proposition 43. So, let $\chi(G) > 8$ and assume the statement of the proposition for all graphs H with $\chi(H) < \chi(G)$.

Color G with $\chi(G)$ colors in such a way as to maximize the size of the subgraph H induced by the first 2^{k-1} colors. Let $F = G[V(G) - V(H)]$ be the graph induced by the remaining color classes. Therefore, $|V(F)|$ is minimal over all colorings and no vertex of F can be colored with a color from H i.e. every vertex in F has a neighbor in each color class of H. By induction we have $\chi_s^e(H) = k$ and we have a k-edge-weighting of H and a (2^{k-1})-coloring of H with distinct color classes $X_1, X_2, \ldots, X_k, Y$ such that X_i is i-safe (for $i = 1, 2, \ldots, k$) and Y is k-free. Let us keep this edge-weighting of $H \subset G$ and weight the remaining edges of G.

First, weight all edges in F with (new) weight $k+1$. Now it remains to weight the edges between H and F. Label the color classes of F with $(k-1)$-length binary strings from 0 to $\chi(G) - 2^{k-1}$. Let $v \in F$ be an arbitrary vertex in F. By construction of H and F, v has a neighbor in each color class of H (notably in each X_i and Y). If the binary string corresponding to the color class of v has a 1 in the i-th binary digit (i ranges from 1 to $k-1$) then weight all edges between v and X_i with weight i. Next, weight all edges between v and X_k with k and all edges between v and Y with $k+1$ (this guarantees that each weight set in F has both weights k and $k+1$). Finally, for all remaining unweighted edges $vw \in E(F, H)$ we weight vw as follows: if $w \in H$ is incident to an edge with weight k then weight vw with weight k. Otherwise, weight vw with weight $k+1$. In this way, we guarantee that every weight set in H has at most one of the weights k and $k+1$. This immediately distinguishes the weight sets in H from those in F. Clearly, each color class in F will have a single unique weight set corresponding to its (unique) binary string (and the weights k and $k+1$). The color classes of H were already distinguished by the first k weights. The edges between F and H only added weight i to i-safe color classes of H (for $1 \leq i \leq k$) or a new weight $k+1$, so weight sets of any pair adjacent vertices in H remain distinct. This gives a vertex-coloring $(k+1)$-edge-weighting of G where $k+1 = \lceil \log_2 \chi(G) \rceil + 1$. Furthermore, for $i = 1, 2, \ldots, k-1$ the color class X_i remains i-safe, the first class of F (its corresponding binary string is $00\ldots0$) is k-safe, class Y is now $(k+1)$-safe class and X_k is $(k+1)$-free (as all edges between X_k and F got weight k). $\square$

Now that we have achieved the best possible bound on $\chi_s^e(G)$ let us discuss the remaining open details and some possible generalizations.

Chapter 4

Generalizations and concluding remarks

As we can see from the main result (Theorem 24) there is not much more to say about $\chi_s^e(G)$. The parameter is essentially characterized in terms of $\chi(G)$. Efforts to characterize $\chi_s^e(G)$ in terms of other graph parameters would in turn give a good characterization of $\chi(G)$ and as a result are probably too optimistic (as outlined in the introduction). Furthermore, a deeper understanding of when $\chi_s^e(G) = 2$ would also be interesting, but this is equivalent to the property-$\mathcal{B}$ for hypergraphs which is **NP**-complete. This connection, however, may be relevant for progress on either problem and also suggests a new way to generalize the property-$\mathcal{B}$. One problem of particular interest is a direct proof that $\chi_s^e(G)$ is monotone for graphs with chromatic number 3 or greater. This could potentially simplify the proof of the main result.

Because of the strict relationship between the two parameters, it is conceivable that $\chi_s^e(G)$ can be used as a tool to find $\chi(G)$ for certain classes of graphs. The relationship is logarithmic, so determining $\chi_s^e(G)$ for some G would give an upper bound on $\chi(G)$. For example, describing $\chi_s^e(G)$ for Kneser graphs would give upper bounds on the chromatic number of Kneser graphs by a completely new method. More tempting, perhaps, would be to attempt an alternate proof of the 4-color theorem with $\chi_s^e(G)$. Indeed, if we can show directly that $\chi_s^e(G) = 3$ for any planar graph G then we would achieve that $\chi(G) = 4$ for any planar graph. This is not likely to be an easy endeavor. The proofs in Chapter 3 that determine $\chi_s^e(G)$ depend strongly on knowing the chromatic number of G. Alternate techniques are necessary if we want to use $\chi_s^e(G)$ to say anything directly about $\chi(G)$.

Throughout the main chapter we avoided discussion of multigraphs. This was merely for the sake of simplicity. In fact, Theorem 24 holds for multigraphs and no improvement is possible. Clearly a multigraph has the same chromatic number as its underlying simple graph, so the lower bound (Remark 25) on $\chi_s^e(G)$ holds as before. Furthermore, all proofs of upper bounds can be adapted to multigraphs. If G is a multigraph, we can follow the steps of any proof by ignoring edge multiplicity, this will give a vertex coloring edge-weighting of the underlying simple graph of G. To weight a multiedge in G we just repeat the weight that appears on its corresponding edge in the underlying simple graph of G. Obviously, the weight set of a vertex v in G and in the underlying simple graph of G will be the same as the multiplicity (in excess of 1) of a specific weight has no additional impact on the weight set of v.

When considering a map to a certain class of graph objects (in our case, the edges), it is natural to consider dual-type problems with a domain of a different class of objects (e.g. edge colorings are a kind of dual of vertex colorings). One problem of this type is to weight the vertices rather than the edges. In this case, we have a map $\varphi : V(G) \to \{1, 2, \ldots, k\}$. Now how do we form the notion of weight set $S_\varphi(v)$ of a vertex $v \in V(G)$ in this problem? Naturally, for any neighbor $u \in V(G)$ of v we should have $\varphi(u) \in S_\varphi(v)$. But should the weight of a vertex v be included in its own weight set? If we do include the weight we get an interesting situation. It is easy to see that the complete graph K_n will be impossible to weight so that adjacent vertices have different weight sets. Indeed, all vertices of K_n will have the same weight set no matter how we choose the weighting.

On the other hand, if we do not include $\varphi(v)$ in $S_\varphi(v)$ we get a different (but equally interesting) situation. For any graph G, we can consider a standard proper coloring of G as a weighting φ of G. It is easy to see that the weight sets of any two adjacent vertices are different under this weighting φ; in particular, for $xy \in E(G)$ we have that $S_\varphi(x) \not\ni \varphi(x) \in S_\varphi(y)$. This gives an upper bound of $\chi(G)$ on this new graph parameter. We are not required to have a proper weighting of the vertices; so can we do better than $\chi(G)$ different weights? In general we cannot. If we again consider complete graphs, we have no alternative but to use n different weights on the vertices as if the same weight appears on two vertices then these vertices will have the same weight sets. In fact, no graphs (under a cursory examination) are known where this parameter is not equal to $\chi(G)$. It would be interesting to resolve this.

From here we can go into many different directions. Let us conclude with a discussion of two (unstudied) problems of personal interest. The more natural problem is to generalize the vertex-coloring edge-weighting problem to hypergraphs. This has been done for irregularity strength; the 1,2-Conjecture has also been reformulated in terms of hypergraphs. The second question is to consider an analogue to list-edge-colorings.

Typically, generalizations of graph problems to hypergraphs are remarkably difficult. Turan's Theorem [71] is a good example; it is not even clear what the corresponding question for hypergraphs should be. Thus, the first issue in generalizing our graph parameter to hypergraph is the question of how to do so. First let us restrict our attention to uniform hypergraphs i.e. hypergraphs where all hyperedges have the same number of vertices (although we may ask the same question for non-uniform hypergraphs).

If we let a *t-uniform hypergraph* $\mathcal{H}$ be a set of vertices $\mathcal{V}$ and a set of hyperedges $\mathcal{E} \subset \binom{\mathcal{V}}{t}$, then we can define a k-edge-weighting of $\mathcal{H}$ to be a map $\varphi : \mathcal{E} \to \{1, 2, \ldots, k\}$ as in the case of graphs. However, the corresponding definition of vertex coloring is tricky. Should this mean that any two vertices contained in a hyperedge get different sets of weights on their incident hyperedges (thus coloring each vertex in an edge with different colors i.e. a rainbow coloring) or should it mean that in any hyperedge there should be two vertices with different sets of weights on their incident hyperedges (thus avoiding only monochromatic edges)? Both notions of hypergraph coloring are valid. The easiest (?) way to avoid this difficulty is to consider both problems. Essentially nothing has been done in this direction. Unfortunately, the techniques developed in Chapter 3 will probably not work here as the general construction of a vertex-coloring edge-weighting relied implicitly on the fact that an edge was between exactly two color classes in a proper coloring.

List coloring versions of some of the edge-weighting problems discussed in Chapter 2 have generally remained unasked. However, Horňák and Woźniak [47] examine the list-coloring analogue of the neighbor-distinguishing index. In the case of the general neighbor-distinguishing index such an analogue has not been considered. If for every edge $e \in E(G)$ we define a *list* $L(e)$ of weights then we say a *list-edge-weighting* of G is a map φ where $\varphi(e) \in L(e)$ i.e. for each edge $e \in E(G)$ we weight e with one of the weights from its list $L(e)$. Such an edge-weighting is *vertex-coloring* if for every pair of adjacent vertices $u, v \in V(G)$ we have that $S_\varphi(v) \neq S_\varphi(u)$ where $S_\varphi(v)$ is the set of edge weights appearing on the edges incident to v under φ. The question here is to determine the minimum k such that for any collection of lists all of size k we can find a vertex coloring edge-weighting of G. Much like the original list coloring concept, if we set all the lists to the same k weights we just get the general neighbor-distinguishing index problem. Therefore, this parameter is bounded below by $\chi_s^e(G)$. A first step toward understanding this question would be to attempt to construct a graph G where this parameter is not equal to $\chi_s^e(G)$. If we have lists assigned to all edges then at each vertex we can construct a list of possible weight sets based on the possible choices of edge weights. There are additional difficulties, but this immediately resembles the original list-coloring problem on the vertices. A more concrete connection between this parameter and the list-chromatic number would be fascinating.

Bibliography

[1] L. Addario-Berry, R.E.L. Aldred, K. Dalal and B. Reed, Vertex colouring edge partitions. *J. Combin. Theory Ser. B*, **94** (2005) no. 2, 237–244

[2] L. Addario-Berry, K. Dalal, C. McDiarmid, B. A. Reed and A. Thomason, Vertex-colouring edge-weightings. *Combinatorica*, **27** (2007) no. 1, 1–12

[3] L. Addario-Berry, K. Dalal and B. A. Reed, Degree constrained subgraphs. *Proceedings of GRACO2005*, 257–263 (electronic), *Electron. Notes Discrete Math.*, **19** Elsevier, Amsterdam (2005)

[4] M. Aigner and E. Triesch, Irregular assignments of trees and forests. *SIAM J. Discrete Math.*, **3** (1990) no. 4, 439–449

[5] M. Aigner, E. Triesch and Z. Tuza, Irregular assignments and vertex-distinguishing edge-colorings of graphs. *Combinatorics '90* (Gaeta, 1990), 1–9, *Ann. Discrete Math.*, **52**, North-Holland, Amsterdam (1992)

[6] N. Alon, Restricted colorings of graphs. *Surveys in combinatorics*, *London Math. Soc. Lecture Note Ser.*, **187**, 1–33, Cambridge Univ. Press, Cambridge, ISBN:0-521-44857-3 (1993)

[7] K. Appel and W. Haken, Every planar map is four colorable. I. Discharging. *Illinois J. Math.*, **21** (1977) no. 3, 429–490

[8] K. Appel and W. Haken, The four color proof suffices. *Math. Intelligencer*, **8** (1986) no. 1, 10–20

[9] K. Appel and W. Haken, Every planar map is four colorable. *(Contemporary Mathematics, 98)*, With the collaboration of J. Koch. American Mathematical Society, Providence, RI, ISBN: 0-8218-5103-9 (1989) xvi+741 pp.

[10] K. Appel, W. Haken and J. Koch, Every planar map is four colorable. II. Reducibility. *Illinois J. Math.*, **21** (1977) no. 3, 491–567

[11] P. N. Balister, B. Bollobás and R. H. Schelp, Vertex distinguishing colorings of graphs with $\Delta(G) = 2$. *Discrete Math.*, **252** (2002) no. 1–3, 17–29

[12] P. N. Balister, E. Győri, J. Lehel and R. H. Schelp, Adjacent vertex distinguishing edge-colorings. *SIAM J. Discrete Math.*, **21** (2007) no. 1, 237–250

[13] C. Bazgan, A. Harkat-Benhamdine, H. Li and M. Woźniak, On the vertex-distinguishing proper edge-colorings of graphs. *J. Combin. Theory Ser. B*, **75** (1999) no. 2, 288–301

[14] D. Behzad, Graphs and their chromatic numbers. *Doctoral Thesis*, Michigan University (1965)

[15] C. Berge, Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Nat. Reihe*, **10** (1961) 114–115

[16] B. Bollobás, The chromatic number of random graphs. *Combinatorica*, **8** (1988) no. 1, 49–55

[17] R. L. Brooks, On colouring the nodes of a network. *Proc. Cambridge Phil. Soc.*, **37** (1941) 194–197

[18] W. G. Brown and J. W. Moon, Sur les ensembles de sommets indpendants dans les graphes chromatiques minimaux. (in French), *Canad. J. Math.*, **21** (1969) 274–278

[19] N. G. de Bruijn and P. Erdős, A colour problem for infinite graphs and a problem in the theory of relations. *Nederl. Akad. Wetensch. Proc. Ser. A.*, **54** = *Indagationes Math.*, **13** (1951) 369–373

[20] A. C. Burris and R. H. Schelp, Vertex-distinguishing proper edge-colorings. *J. Graph Theory*, **26** (1997) no. 2, 73–82

[21] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz and F. Saba, Irregular networks. *Congr. Numer.*, **64** (1988) 197–210

[22] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, The strong perfect graph theorem. *Ann. of Math. (2)*, **164** (2006) no. 1, 51–229

[23] G. A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.*, **27** (1952) 85–92

[24] B. Descartes, A three-colour problem. *Eureka* **9** (1947)

[25] B. Descartes, Solutions to problems in Eureka No. 9. *Eureka* **10** (1948)

[26] K. Edwards, M. Horňák and M. Woźniak, On the neighbour-distinguishing index of a graph. *Graphs Combin.*, **22** (2006) no. 3, 341–350

[27] P. Erdős, Remarks on a theorem of Ramsay. *Bull. Res. Council Israel. Sect. F*, **7F** (1957/1958) 21–24.

[28] P. Erdős, Graph theory and probability. *Canad. J. Math.*, **11** (1959) 34–38

[29] P. Erdős and M. Simonovits, A limit theorem in graph theory. *Studia Sci. Math. Hungary*, **1** (1966) 51–57

[30] P. Erdős and A. H. Stone, On the structure of linear graphs. *Bull. Amer. Math. Soc.*, **52** (1946) 1087–1091

[31] R. J. Faudree and J. Lehel, Bound on the irregularity strength of regular graphs. *Combinatorics* (Eger, 1987), *Colloq. Math. Soc. János Bolyai*, **52**, North-Holland, Amsterdam (1988) 247–256

[32] M. Fleury, Deux problemes de geometrie de situation. *Journal de mathematiques elementaires*, (1883) 257–261

[33] A. Frieze, R. J. Gould, M. Karoński and F. Pfender, On graph irregularity strength. *J. Graph Theory*, **41** (2002) no. 2, 120–137

[34] D. R. Fulkerson, Blocking and anti-blocking pairs of polyhedra. *Math. Programming*, **1** (1971) 168–194

[35] J. A. Gallian, A Dynamic Survey of Graph Labeling. *The Electronic Journal of Combinatorics*, **15** (2008) 190 pp.

[36] F. Galvin, The list chromatic index of a bipartite multigraph. *J. Combin. Theory Ser. B*, **63** (1995) no. 1, 153–158

[37] M. Grötschel, L. Lovász and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, **1** (1981) no. 2, 169–197

[38] E. Győri, M. Horňák, C. Palmer and M. Woźniak, General neighbour-distinguishing index of a graph. *Discrete Math.*, **308** (2008) no. 5–6, 827–831

[39] E. Győri and C. Palmer, A new type of edge-derived vertex coloring. (submitted to *Discrete Math.*)

[40] F. Harary and M. Plantholt, The point-distinguishing chromatic index. *Graphs and Applications*, (F. Harary and J. S. Maybee, Eds.), Wiley-Interscience, New York (1985) 147–162

[41] H. Hatami, $\Delta + 300$ is a bound on the adjacent vertex distinguishing edge chromatic number. *J. Combin. Theory Ser. B*, **95** (2005) no. 2, 246–256

[42] H. Heesch, *Untersuchungen zum Vierfarbenproblem.* (in German) B. I. Hochschulskripten, 810/810a/810b Bibliographisches Institut, Mannheim (1969) 290 pp.

[43] C. Hierholzer, Über die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechnug zu umfahren. (in German), *Mathematische Annalen*, **6** (1873) 30–32

[44] I. Holyer, The NP-completeness of edge-coloring. *SIAM J. Comput.*, **10** (1981) no. 4, 718–720

[45] M. Horňák and R. Soták, The fifth jump of the point-distinguishing chromatic index of $K_{n,n}$. *Ars Combin.*, **42** (1996) 233–242.

[46] M. Horňák and R. Soták, Localization of jumps of the point-distinguishing chromatic index of $K_{n,n}$. *Discuss. Math. Graph Theory*, **17** (1997) 243–251.

[47] M. Horňák and M. Woźniak, On neighbour-distinguishing colourings from lists. *Fifth Cracow Conference on Graph Theory, USTRON '06*, 295–297 (electronic), *Electron. Notes Discrete Math.*, **24**, Elsevier, Amsterdam (2006)

[48] M. Horňák and N. Zagaglia Salvi, On the point-distinguishing chromatic index of complete bipartite graphs. *Ars Combin.*, **80** (2006) 75–85

[49] T. R. Jensen and B. Toft, *Graph coloring problems.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York (1995) xxii+295 pp.

[50] J. Kahn, Asymptotically good list-colorings. *J. Combin. Theory Ser. A*, **73** (1996) no. 1, 1–59

[51] M. Karoński, T. Łuczak and A. Thomason, Edge weights and vertex colours. *J. Combinatorial Theory B*, **91** (2004) 151–157

[52] R. M. Karp, Reducibility among combinatorial problems. *Complexity of Computer Computations* (R.E. Miller and J.W. Thatcher, Eds.), Plenum, New York (1972) 85–103

[53] J. B. Kelly and L. M. Kelly, Paths and circuits in critical graphs. *Amer. J. Math.* **76** (1954) 786–792.

[54] A. B. Kempe, On the Geographical Problem of the Four Colours. *Amer. J. Math.*, **2** (1879) no. 3, 193–200.

[55] D. König, Über Graphen und ihre Anwendung auf Determinatentheorie und Mengenlehre. (in German), *Math. Ann.*, **77** (1916) 453–465

[56] J. Lehel, Facts and quests on degree irregular assignments. *Graph theory, combinatorics, and applications*, Vol. **2**, Kalamazoo, MI (1988) 765–781, Wiley, New York (1991)

[57] L. Lovász, On chromatic number of finite set systems *Acta. Math. Sci. Hungar.*, **19** (1968) 9–67

[58] L. Lovász, Normal hypergraphs and the perfect graph conjecture. *Discrete Math.*, **2** (1972) no. 3, 253–267

[59] M. Molloy and B. Reed, A bound on the total chromatic number. *Combinatorica* **18** (1998), 241-280

[60] J. Mycielski, Sur le coloriage des graphs. (in French), *Colloq. Math.*, **3** (1955) 161–162

[61] J. Nešetřil and V. Rödl, A short proof of the existence of highly chromatic hypergraphs without short cycles. *J. Combin. Theory Ser. B*, **27** (1979) no. 2, 225–227

[62] T. Nierhoff, A tight bound on the irregularity strength of graphs. *SIAM J. Discrete Math.*, **13** (2000) no. 3, 313–323

[63] J. Przybyło and M. Woźniak, 1, 2 Conjecture. *Matematyka Dyskretna*, Preprint MD 024 (2007), otrzymano dnia 14.02.2007.

[64] J. Przybyło and M. Woźniak, 1, 2 Conjecture, II. *Matematyka Dyskretna*, Preprint MD 026 (2007), otrzymano dnia 4.04.2007.

[65] N. Robertson, D. Sanders, P. Seymour and R. Thomas, The four-colour theorem. *J. Combin. Theory Ser. B*, **70** (1997) no. 1, 2–44

[66] T. L. Saaty and P. C. Kainen, *The four-color problem. Assaults and conquest.* McGraw-Hill International Book Co., New York (1977) ix+217 pp.

[67] M. Simonovits, On colour-critical graphs. *Studia Sci. Math. Hungar.*, **7** (1972) 67–81

[68] C. Thomassen, Every planar graph is 5-choosable. *J. Combin. Theory Ser. B*, **62** (1994) no. 1, 180–181

[69] B. Toft, Two theorems on critical 4-chromatic graphs. *Studia Sci. Math. Hungar.*, **7** (1972) 83–89

[70] B. Toft, Colouring, stable sets and perfect graphs. *Handbook of combinatorics*, Vol. **1, 2**, Elsevier, Amsterdam (1995) 233–288

[71] P. Turán, Eine Extremalaufgabe aus der Graphentheorie. (in Hungarian), *Mat. Fiz. Lapok*, **48** (1941) 436–452

[72] V. G. Vizing, On an estimate of the chromatic class of a p-graph. (in Russian), *Diskret. Analiz No.*, **3** (1964) 25–30

[73] V. G. Vizing, Some unsolved problems in graph theory. (in Russian), *Uspekhi Math. Nauk.*, **23** (1968) 117–134, English translation in *Russian Math. Surveys*, **23** (1968) 125-141

[74] M. Voigt, List colourings of planar graphs. *Discrete Math.*, **120** (1993) 215–219

[75] N. Zagaglia Salvi, On the point-distinguishing chromatic index of $K_{n,n}$. *Ars Combin.*, **25B** (1988) 93–104

[76] N. Zagaglia Salvi, On the value of the point-distinguishing chromatic index of $K_{n,n}$. *Ars Combin.*, **29B** (1990) 235–244

[77] Z. Zhang, X. Chen, J. Li, B. Yao, X. Lu and J. Wang, On adjacent-vertex-distinguishing total coloring of graphs. *Sci. China Ser. A*, **48** (2005) no. 3, 289–299

[78] Z. Zhang, L. Liu and J. Wang, Adjacent strong edge coloring of graph. *Appl. Math. Lett.*, **15** (2002) 623–626

[79] A. A. Zykov, On some properties of linear complexes. (in Russian), *Mat. Sbornik N.S.*, **24** (1952) 163–188