Solvability of Some Nonlinear Fourth Order Boundary Value Problems

by

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Abstract

This thesis is concerned with the study of certain classes of nonlinear fourth order boundary value problems. They are motivated by some physical problems. Sufficient conditions for the existence of solutions under various assumptions are presented.

After an introductory chapter, we discuss (in Chapter 2) a fourth order equation involving a quite general nonlinearity. A variant of the method of lower and upper solutions is presented. The existence of a solution located between suitable lower and upper solutions is proved. Practical examples show that the hypotheses of the main results hold for a wide range of differential equations. A direct application to the extended Fisher-Kolmogorov equation provides some sharp estimates for its stationary periodic solutions.

Next, in Chapter 3, a class of fourth order differential inclusions involving the p-biharmonic operator is investigated. It is related to the beam-column theory. The existence of solutions for a wide class of boundary conditions is proved. The problem is treated variationally, via the critical point theory for non-smooth functionals. Some of the results concern mountain pass type solutions.

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Contents

Introduction

This thesis is devoted to the study of certain classes of boundary value problems involving fourth order differential equations and inclusions. The main results presented herein provide sufficient conditions for the existence of solutions to such problems.

Fourth order differential equations occur in various physical problems. Some of them describe certain phenomena related to the theory of elastic stability. A classical fourth order equation arising in the beam-column theory is the following (see Timoshenko and Gere [50, 1961])

$$EI\frac{d^4u}{dx^4} + P\frac{d^2u}{dx^2} = q, \tag{1}$$

where u is the lateral deflection, q is the intensity of a distributed lateral load, P is the axial compressive force applied to the beam and EI represents the flexural rigidity in the plane of bending. Various generalizations of the equation describing the deformation of an elastic beam with different types of two point boundary conditions have been extensively studied in the last two decades via a broad range of methods. Gupta [20, 1988] and [21, 1991] studies the equation of the form

$$\frac{d^4u}{dx^4} + g\left(x, u, u', u''\right) = e\left(x\right), \quad x \in (0,1),$$

and, more generally, the equation

$$\frac{d^4u}{dx^4} + f\left(u\right)u' + g\left(x, u, u', u''\right) = e\left(x\right), \quad x \in (0,1),$$

where f is a continuous function, g is a Carathéodory function satisfying the inequalities

$$\begin{aligned} g(x,u,v,w) &\geq a(x)\,u^2 + b(x)\,|uv| + c(x)\,|uw| + d(x)\,|u|\,, \\ |g(x,u,v,w)| &\leq |\alpha(x,u,v)|\,|w|^2 + \beta(x)\,, \end{aligned}$$

with real valued functions $a(x)$, $b(x)$, $c(x)$, $d(x)$, $\alpha(x,u,v)$ and $\beta(x)$. The main tool used there is the Leray-Schauder continuation theorem. Grossinho and Tersian [19, 2000] consider the boundary value problem

$$u^{(iv)}(x) + g(u(x)) = 0, \quad x \in (0,1)$$

$$u''(0) = -f(-u'(0))\,, \quad u'''(0) = -h(u(0))\,, \quad u''(1) = u'''(1) = 0,$$

where g is a strictly monotone function that may have some discontinuities, f and h are unbounded, continuous and strictly increasing functions defined on finite open intervals. The existence of solutions is treated via the dual variational method (see Ambrosetti and Badiale [2, 1989]).

The problem

$$\begin{aligned} &u^{(iv)}(x) = f(x,u(x))\,, \quad x \in (0,1)\,, \\ &u(0) = u'(0) = u''(1) = 0, \quad u'''(1) = g(u(1)) \end{aligned}$$

is studied in the works of To Fu Ma [35, 2004] and Ma & Silva [36, 2004]. Existence of solutions is proved in [35] via the mountain pass theorem. Uniqueness of the solutions is investigated in [36] under more restrictive assumptions on function f, with the help of the fixed point theorem for contractive mappings.

Some higher order parabolic partial differential equations have been of an increasing interest during the last two decades. We would like to mention two of them, namely, the extended Fisher-Kolmogorov equation (EFK)

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \quad \gamma > 0, \tag{2}$$

and the Swift-Hohenberg equation (SH) [48, 1977]

$$\frac{\partial u}{\partial t} = \alpha u - \left(1 + \frac{\partial^2 u}{\partial x^2}\right)^2 u - u^3, \quad \alpha > 0. \tag{3}$$

The first one is proposed in Coullet, Elphick and Repaux [8, 1987] as well as in Dee and van Saarloos [10, 1988] as a higher order model and a generalization of the classical Fisher-Kolmogorov nonlinear diffusion equation (see Kolmogorov, Petrovski and Piscounov [29, 1937])

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \tag{4}$$

related to the study of a front propagation into an unstable state. The Swift-Hohenberg equation is derived from the equations for thermal convection.

The results presented in this thesis are obtained by using two main approaches: the technique of lower and upper solutions and the critical point theory due to Motreanu and Panagiotopoulos [41, 1999].

The material of the thesis is organized as follows.

Chapter 1 is preliminary. It contains some theoretical results (without proofs) we provide to make the presentation self-contained. A brief introduction to the Leray-Schauder degree theory is included in Section 1.1. A variant of the mountain pass theorem is presented in Section 1.2 in the framework of the non-smooth critical point theory.

Chapter 2 is devoted to some applications of the method of lower and upper solutions to the boundary value problem

$$u^{(iv)} = f(t, u, u', u'', u'''), \quad 0 < t < 1, \tag{5}$$

$$u(0) = u'(1) = u''(0) = u'''(1) = 0, \tag{6}$$

where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function satisfying a Nagumo-type growth assumption. The boundary conditions correspond to one endpoint simply supported and the other one sliding clamped when beam deformation is considered. The results presented in Chapter 2 improve some previous results by Grossinho and Tersian [19, 2000], Gupta [20, 1988], [21, 1991], Gyulov and Tersian [25, 2004], Ma and da Silva [36, 2004], since the nonlinearity herein is allowed to depend on all derivatives up to order three. Moreover, the present conditions on the lower and upper solutions and function f (see Definition 2.1.2) are less restrictive than the assumptions in the paper of Franco et al. [13, 2005], where the functions $\alpha,\ \beta \in C^4(I)$ are lower and upper solutions, respectively, if $\alpha \le \beta$, $\alpha'' \ge \beta''$ and

$$\begin{aligned} \alpha^{(iv)}(t) &\le f\left(t, \alpha(t), -C, \alpha''(t), \alpha'''(t)\right) \quad \text{for} \quad t \in I, \\ \beta^{(iv)}(t) &\ge f\left(t, \beta(t), C, \beta''(t), \beta'''(t)\right) \quad \text{for} \quad t \in I, \end{aligned} \tag{7}$$

with a constant $C > 0$ defined by

$$C := 2\max\left\{|\alpha'|_\infty, |\beta'|_\infty\right\} + \max\left\{|\beta(0) - \alpha(1)|, |\beta(1) - \alpha(0)|\right\}.$$

The results (see Theorem 2.1.1) presented in Chapter 2 concern the existence of solutions of the boundary value problem (5)-(6). They are located between suitable lower and upper solutions. It is assumed that function f satisfies a Nagumo-type growth condition with respect to the third derivative, namely, that there is a continuous function $h_E : \mathbb{R}_0^+ \to [a, +\infty)$, for some $a > 0$, such that

$$\int_0^{+\infty} \frac{s}{h_E(s)} ds = +\infty, \tag{8}$$

and

$$|f(t, x_0, x_1, x_2, x_3)| \le h_E(|x_3|), \quad \forall (t, x_0, x_1, x_2, x_3) \in E, \tag{9}$$

where E is a suitable subset of $[0,1] \times \mathbb{R}^4$. The proof of the main result is performed as follows. First, an appropriate parameterized (homotopic) class of boundary value problems (see (2.29)), with parameter $\lambda \in [0, 1]$, is constructed provided that there exist lower and upper solution functions. Then, it is shown that the solutions of those boundary value problems are uniformly bounded with respect to λ in the space $C^3([0, 1])$ of the real valued three times continuously differentiable functions on the interval $[0, 1]$. The estimations on the third derivatives of the solutions come up as a consequence of the Nagumo-type growth assumption (see Lemma 2.1.1 and Step 1 in the proof of Theorem 2.1.1). Next, the equivalent operator formulation reduces any of the boundary value problems from the homotopic class to the existence of a fixed point of a parameterized operator $\mathcal{T}_\lambda : C^3([0, 1]) \to C^3([0, 1])$. Since the solutions of the homotopic boundary value problems are uniformly bounded with respect to λ there exists a domain $\Omega \subset C^3([0, 1])$ such that $\mathcal{T}_\lambda x \neq x$ for any $x \in \partial\Omega$. Then, the degree $deg\,(I - \mathcal{T}_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0, 1]$. Moreover, the invariance under homotopy implies that $deg\,(I - \mathcal{T}_0, \Omega, 0) = deg\,(I - \mathcal{T}_1, \Omega, 0)$. The boundary value problem corresponding to $\lambda = 0$ is a very simple one, in fact, it is linear and the corresponding degree, $deg\,(I - \mathcal{T}_0, \Omega, 0)$, is an odd number, implying that there is solution for the problem corresponding to $\lambda = 1$. Finally, this solution is shown to be a solution of the original problem (5)-(6).

Next, it is shown that the hypotheses of the main result in Chapter 2 and the definition of lower and upper solutions can be modified in order to derive similar results for other boundary conditions. Then, an example presented at the end of Section 2.1 shows that the hypotheses of Theorem 2.1.1 are satisfied for a large class of boundary value problems.

Moreover, another application of Theorem 2.1.1 is analyzed in Section 2.2. It is shown that there is a periodic stationary solution u, with period $T = 4L$, of the extended Fisher-Kolmogorov equation (2) for any L in an interval of the form $(L_0, L_1]$, where L_0 and L_1 are suitable constants. According to Theorem 2.1.1, it is located between the following lower and upper solutions: $\alpha(x) = C_\alpha \varphi(x)$ and $\beta(x) = C_\beta \varphi(x)$, where

$$\varphi(x) := \sin\frac{\pi x}{2L} - \frac{P\left(\frac{\pi}{2L}\right)}{3P\left(\frac{3\pi}{2L}\right)} \sin\frac{3\pi x}{2L},$$

and the constants C_α and C_β are defined by

$$C_\alpha := \sqrt{\frac{-4P\left(\frac{\pi}{2L}\right)}{3\left(1-\frac{P\left(\frac{\pi}{2L}\right)}{P\left(\frac{3\pi}{2L}\right)}\right)^3}}, \quad C_\beta := \sqrt{\frac{-4P\left(\frac{\pi}{2L}\right)}{3\left(1+\frac{P\left(\frac{\pi}{2L}\right)}{3P\left(\frac{3\pi}{2L}\right)}\right)^3}}. \tag{10}$$

Here $P(\cdot)$ denotes the polynomial $P(\xi) := \gamma\xi^4 + \xi^2 - 1$.

The results of Chapter 2 are mainly based on the paper by F. Minhós, T. Gyulov and A.I. Santos [40, 2009].

Chapter 3 contains a variational treatment of the problem

$$\left(|u''|^{p-2} u''\right)'' - \left(a |u'|^{p-2} u'\right)' + b |u|^{p-2} u \in \overline{\partial} F(t,u), \tag{11}$$

$$\begin{pmatrix} -\left(|u''|^{p-2} u''\right)'(0) + a(0) |u'(0)|^{p-2} u'(0) \\ \left(|u''|^{p-2} u''\right)'(1) - a(1) |u'(1)|^{p-2} u'(1) \\ |u''(0)|^{p-2} u''(0) \\ -|u''(1)|^{p-2} u''(1) \end{pmatrix} \in \partial j \begin{pmatrix} u(0) \\ u(1) \\ u'(0) \\ u'(1) \end{pmatrix}, \tag{12}$$

where $a(t), b(t) \in C([0,1])$ are given real functions, $p > 1$, $F(t,\cdot)$ is locally Lipschitz for a.a. $t \in [0,1]$ and $j(\cdot)$ is a convex lower semicontinuous function, possibly unbounded. The set $\overline{\partial} F(t,u)$ denotes the generalized gradient of $F(t,\cdot)$ (see Subsection 1.2.1) while ∂j is the subdifferential of function j. The differential inclusion (11) can be considered as a generalization of the equation (1) if it is assumed that the bending moment depends nonlinearly on the curvature. More details on the justification of such an assumption are provided in the introductory section 3.1.

It is worth noting that condition (12) covers many special cases which are frequently considered. For example, the periodic boundary conditions $u^{(i)}(0) = u^{(i)}(1)$,

$i = 0, 1, 2, 3,$ are easily obtained if $j\left((x_1, x_2, x_3, x_4)^T\right) = 0$ when $x_1 = x_2$ and $x_3 = x_4$, and $+\infty$ otherwise.

Problem (11)-(12) is treated variationally, namely, its solutions are derived as critical points of the functional

$$I(u) := \frac{1}{p}\int_0^1 \left(|u''|^p + a\,|u'|^p + b\,|u|^p\right) dt - \int_0^1 F(t,u)\,dt + J(u),$$

where $J : W^{2,p}(0,1) \to \mathbb{R} \cup \{+\infty\}$ is a convex function defined by

$$J(u) := j\left((u(0), u(1), u'(0), u'(1))^T\right), \quad \forall u \in W^{2,p}(0,1).$$

Note that functional I may not be continuously differentiable, in fact, it may be unbounded on the Sobolev space $W^{2,p}(0,1)$. The treatment is based on a generalized critical point theory developed in Motreanu and Panagiotopoulos [41, 1999] (see Section 1.2). It can be proved that a critical point of functional I in the sense of this theory is a solution of problem (11)-(12).

Chapter 3 includes two main results. The existence of at least one solution to problem (11)-(12) is obtained under appropriate hypotheses (see Theorem 3.1.1). To be more precise, it is assumed that the growth of function $F(t,x)$ is specified by the condition

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^p} < \frac{\lambda_1}{p},$$

with the constant λ_1 given by

$$\lambda_1 := \inf\left\{\frac{\int_0^1 \left(|u''|^p + a\,|u'|^p + b\,|u|^p\right)\, dt}{\|u\|_{L^p}^p} : u \in \mathcal{D}\backslash\{0\}\right\}, \tag{13}$$

where $\mathcal{D} = D(J)$ is the effective domain of the convex functional J:

$$\mathcal{D} = \left\{u : u \in W^{2,p}(0,1),\ (u(0), u(1), u'(0), u'(1))^T \in D(j)\right\}.$$

In this case it is proved that functional I is coercive, that is $I(u) \to +\infty$ as $\|u\| \to +\infty$, and its (sequential) weak lower semicontinuity is a consequence of the compactness of the imbedding $W^{2,p}(0,1) \subset C^1([0,1])$. Then, functional I is bounded from below and there exists a critical point of I, namely its minimizer.

If function $F(t,x)$ satisfies the condition

$$\limsup_{x \to 0} \frac{F(t,x)}{|x|^p} < \frac{\lambda_1}{p},$$

and $(0,0,0,0)^T \in \partial j\left((0,0,0,0)^T\right)$, then $u(t) \equiv 0$ is a critical point of functional I. Thus, an interesting problem will be to find sufficient conditions that guarantee the existence of nonzero critical points. The second main result in Chapter 3 concerns this question (see Theorem 3.1.2). If the growth of function $F(t,x)$ at infinity is of the order of $|x|^\theta$ or higher with $\theta > p$, then it is shown (see the proof of Case (G_θ)) that functional I has a mountain pass nonzero critical point (see Theorem 1.2.2). Moreover, a slightly weaker condition can be assumed (see case (G_p) in Theorem 3.1.2) that guarantees the mountain pass geometry of functional I. It allows some special cases when the growth of function $F(t,x)$ at infinity is of the order of $|x|^p$. An example for the case when $p = 2$ is presented in Section 3.4.

The results of Chapter 3 in the particular case when $p = 2$ and a and b are constants were reported in the paper by Gyulov and Moroşanu [22, 2007]. The general case is studied in Gyulov and Moroşanu [24]. The reference Gyulov and Moroşanu [23, 2009] contains a short variant concerning the particular case when a and b are constants and $p > 1$.

Chapter 1

An Overview of Some Methods and Results from Nonlinear Analysis

This chapter contains a preliminary part that is necessary for the subsequent main part of the thesis. Some theoretical results are recalled without proofs in order to present the text in a self-contained form up to a certain extent. Their relationship to the results presented in the next chapters is explained in what follows.

Section 1.1 contains a concise introduction to the Leray-Schauder degree theory. The proof of the main results in Chapter 2 refer to the basic conclusions in that theory. It is presented here for the readers' convenience.

Section 1.2 includes a variant of the mountain pass theorem in the framework of the non-smooth critical point theory, i.e., when a variational problem is considered where the corresponding functional is the sum of a locally Lipschitz function and a convex lower semicontinuous one. The main results in Chapter 3 are derived as an application of this theory.

1.1 Topological degree theory

1.1.1 Brouwer degree

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded open set, $p \in \mathbb{R}^n$, and let the map $f \in C\left(\overline{\Omega}, \mathbb{R}^n\right)$ be such that $p \notin f(\partial\Omega)$. The Brouwer degree (see [3]), denoted by $deg(f, \Omega, p)$, assigns the "topological" number of solutions of the equation $f(x) = p$ in the domain Ω to the map f. If the map f is continuously differentiable and such that the Jacobian $J_f(x) \neq 0$ for any x in the preimage $f^{-1}(p)$, then the degree has a definite meaning - each of the solutions of $f(x) = p$ is counted up with the corresponding sign of the Jacobian $J_f(x)$. The degree has a rather blurred sense when a general continuous map f is considered. However, it is possible to construct a well-defined concept of degree via some extensions for certain classes of maps.

Definition 1.1.1 *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded open set, $p \in \mathbb{R}^n$, and let the map $f \in C^1\left(\overline{\Omega}, \mathbb{R}^n\right)$ be such that $p \notin f(\partial\Omega)$ and $J_f(x) \neq 0$ for all $x \in f^{-1}(p)$. Define the Brouwer degree $deg(f, \Omega, p)$ as the sum*

$$deg(f, \Omega, p) = \sum_{x \in f^{-1}(p)} sgn(J_f(x)),$$

where $deg(f, \Omega, p) = 0$, if $f^{-1}(p) = \emptyset$.

It is worth noting that the set $f^{-1}(p)$ contains isolated points as far as $J_f(x) \neq 0$ for any $x \in f^{-1}(p)$. Then $f^{-1}(p)$ is finite due to the boundedness of the domain Ω. Thus the summation in Definition 1.1.1 is well-defined.

Let $f \in C^2\left(\overline{\Omega}, \mathbb{R}^n\right)$, p and p' be such that $p \notin f(\partial\Omega)$, $p' \notin f(\partial\Omega)$, $|p - p'| < dist(p, f(\partial\Omega))$ and $J_f(x) \neq 0$ for all $x \in f^{-1}(p')$. It can be shown that $deg(f, \Omega, p')$ does not depend on the choice of p'. We omit the proof of that fact which can be found e.g. in O'Regan et al. [43, 2006], Fonseca and Gangbo [12, 1995], and Lloyd

[34, 1978] among the many references on the topic. Furthermore, according to Sard's lemma (see [43, Lemma 1.1.4, p. 2]), the set of all p', satisfying the above hypotheses is dense in a vicinity of p.

Definition 1.1.2 *Let* $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ *be a bounded open set* $p \in \mathbb{R}^n$ *and the map* $f \in C^2\left(\overline{\Omega}, \mathbb{R}^n\right)$ *be such that* $p \notin f(\partial\Omega)$. *Define Brouwer degree* $deg(f, \Omega, p)$ *as*

$$deg(f, \Omega, p) = deg(f, \Omega, p'),$$

where $p' \notin f(\partial\Omega)$ *is such that* $|p - p'| < dist(p, f(\partial\Omega))$ *and* $J_f(x) \neq 0$ *for all* $x \in f^{-1}(p')$.

Finally, let $f \in C\left(\overline{\Omega}, \mathbb{R}^n\right)$, $p \notin f(\partial\Omega)$ and $g \in C^2\left(\overline{\Omega}, \mathbb{R}^n\right)$ be such that $|f - g| < dist(p, f(\partial\Omega))$. Obviously, $p \notin g(\partial\Omega)$. One can prove that the degree $deg(g, \Omega, p')$ does not depend on the choice of g.

Definition 1.1.3 *Let* $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, *be a bounded open set,* $p \in \mathbb{R}^n$, *and let the map* $f \in C\left(\overline{\Omega}, \mathbb{R}^n\right)$ *be such that* $p \notin f(\partial\Omega)$. *Define the Brouwer degree* $deg(f, \Omega, p)$

$$deg(f, \Omega, p) = deg(g, \Omega, p),$$

where the function $g \in C^2\left(\overline{\Omega}, \mathbb{R}^n\right)$ *is such that* $|f - g| < dist(p, f(\partial\Omega))$.

The main properties of the Brouwer degree are presented in the following.

Theorem 1.1.1 *Let* $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, *be a bounded open set,* $p \in \mathbb{R}^n$, *and let the map* $f \in C\left(\overline{\Omega}, \mathbb{R}^n\right)$ *be such that* $p \notin f(\partial\Omega)$. *The Brouwer degree* $deg(f, \Omega, p)$ *has the following properties:*

(i) ***(Normality)*** $deg(I, \Omega, p) = 1$ *if* $p \in \Omega$ *and* $deg(I, \Omega, p) = 0$ *if* $p \in \mathbb{R}^n \setminus \overline{\Omega}$, *where* $I : \mathbb{R}^n \to \mathbb{R}^n$ *is the identity operator.*

(ii) ***(Solvability)*** *If* $deg(f,\Omega,p) \neq 0$, *then the equation* $f(x) = p$ *has at least one solution in the domain* Ω.

(iii) ***(Invariance under homotopy)*** *If the map* $F : [0,1] \times \overline{\Omega} \to \mathbb{R}^n$ *is continuous and* $p \notin \bigcup_{t\in[0,1]} F(t,\partial\Omega)$, *then* $deg(F(t,\cdot),\Omega,p)$ *does not depend on* $t \in [0,1]$.

(iv) ***(Additivity)*** *Let* Ω_1 *and* Ω_2 *be disjoint open subsets of* Ω *and* $p \in \mathbb{R}^n$ *be such that* $p \notin f\left(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)\right)$. *Then*

$$deg(f,\Omega,p) = deg(f,\Omega_1,p) + deg(f,\Omega_2,p).$$

(v) $deg(f,\Omega,p) = deg(f-p,\Omega,0)$.

(vi) $deg(f,\Omega,p)$ *is a constant on any connected component of* $\mathbb{R}^n \setminus f(\partial\Omega)$.

(vii) Let the map $g : \overline{\Omega} \to F_m$ *be continuous, where* F_m *is a subspace of* $\mathbb{R}_n$, $\dim F_m = m$ *and* $1 \leq m \leq n$. *Suppose that* y *is such that* $y \notin (I-g)(\partial\Omega)$. *Then*

$$deg(I-g,\Omega,y) = deg\left((I-g)|_{\overline{\Omega}\cap F_m},\Omega\cap F_m,y\right).$$

The properties (i)-(v) determine uniquely $deg(f,\Omega,p)$ (see LLoyd [34, 1978], Deimling [11, 1985]), i.e., there exists a unique function $d : A \to \mathbb{Z}$, satisfying these conditions, where

$$A = \left\{(f,\Omega,p) : \Omega \subset \mathbb{R}^n \text{ open and bounded, } f \in C\left(\overline{\Omega},\mathbb{R}^n\right), p \notin f(\partial\Omega)\right\}.$$

Another important property of the Brouwer degree is contained in the following odd mapping theorem (Borsuk's Theorem).

Theorem 1.1.2 *Let* $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, *be bounded, open and symmetric with respect to zero such that* $0 \in \Omega$. *If the map* $f \in C\left(\overline{\Omega},\mathbb{R}^n\right)$ *is odd and* $0 \notin f(\partial\Omega)$, *then* $deg(f,\Omega,0)$ *is an odd number.*

1.1.2 Leray-Schauder degree

In 1934 Leray and Schauder [33] suggested an extension of the Brouwer degree theory to maps of the form $I - T$, where I is the identity operator defined on a real Banach space E, $T : \overline{\Omega} \subset E \to E$ is a compact continuous operator and Ω is a bounded open set.

We are going to present the definition of the Leray-Schauder degree and its main properties. First we formulate the following auxiliary results.

Lemma 1.1.1 *Let E be a real Banach space, $B \subset E$ be a bounded and closed set and $T : B \to E$ be a compact continuous operator. Then for any $\varepsilon > 0$ there exists a finite dimensional space F_ε and a continuous map $T_\varepsilon : B \to F_\varepsilon$, such that*

$$\|T(x) - T_\varepsilon(x)\| < \varepsilon, \quad \forall x \in B.$$

Lemma 1.1.2 *Let E be a Banach space, $B \subset E$ be a bounded closed set and $T : B \to E$ be a continuous compact operator. Suppose that $Tx \neq x$ for each $x \in B$. Then there exists $\varepsilon_0 > 0$, such that $tT_{\varepsilon_1}x + (1-t)T_{\varepsilon_2}x \neq x$ for all $t \in [0,1]$ and $x \in B$, where $\varepsilon_i \in (0, \varepsilon_0)$ and the maps $T_{\varepsilon_i} : B \to F_{\varepsilon_i}$, for $i = 1, 2$, are as in Lemma 1.1.1.*

Let $\Omega \subset E$ be a bounded open subset of the Banach space E and $T : \overline{\Omega} \to E$ be a compact continuous operator such that $0 \notin (I - T)(\partial\Omega)$. We apply Lemma 1.1.2 where $B := \overline{\Omega}$. The invariance of the Brouwer degree under the homotopy $(t, x) \mapsto tT_{\varepsilon_1}x + (1-t)T_{\varepsilon_2}x$ implies that

$$deg\left(I - T_{\varepsilon_1}, \Omega \cap span\left\{F_{\varepsilon_1}, F_{\varepsilon_2}\right\}, 0\right) = deg\left(I - T_{\varepsilon_2}, \Omega \cap span\left\{F_{\varepsilon_1}, F_{\varepsilon_2}\right\}, 0\right).$$

Next, the property *(vii)* of Theorem 1.1.1 yields that

$$deg\left(I - T_{\varepsilon_i}, \Omega \cap span\left\{F_{\varepsilon_1}, F_{\varepsilon_2}\right\}, 0\right) = deg\left(I - T_{\varepsilon_i}, \Omega \cap F_{\varepsilon_i}, 0\right),$$

for $i = 1, 2$. Hence the Brouwer degree $deg\,(I - T_\varepsilon, \Omega \cap F_\varepsilon, 0)$ is well-defined and does not depend on ε, where $\varepsilon \in (0, \varepsilon_0)$. We are ready to present the definition of the Leray-Schauder degree.

Definition 1.1.4 *Let E be a real Banach space, $\Omega \subset E$ be a bounded open set and $T : \overline{\Omega} \to E$ be a compact continuous operator such that $0 \notin (I - T)(\partial\Omega)$. According to Lemma 1.1.2, there exists $\varepsilon_0 > 0$, such that $tT_{\varepsilon_1}x + (1-t)\,T_{\varepsilon_1}x \neq x$ for all $t \in [0,1]$ and $x \in \partial\Omega$, where $\varepsilon_i \in (0, \varepsilon_0)$ and the maps $T_{\varepsilon_i} : \overline{\Omega} \to F_{\varepsilon_i}$, $i = 1, 2$, are as in Lemma 1.1.1.*

Define the Leray-Schauder degree $deg\,(I - T, \Omega, 0)$ when $p = 0$ as follows

$$deg\,(I - T, \Omega, 0) = deg\,(I - T_\varepsilon, \Omega \cap F_\varepsilon, 0)\,,$$

where $\varepsilon \in (0, \varepsilon_0)$.

If $S : \overline{\Omega} \to E$ is a compact continuous operator and p is such that $p \notin (I - S)(\partial\Omega)$, then

$$deg\,(I - S, \Omega, p) := deg\,(I - S - p, \Omega, 0)\,.$$

The main properties of the Leray-Scauder degree are collected in the following:

Theorem 1.1.3 *Let $\Omega \subset E$ be a bounded open subset of a real Banach space E, and $T : \overline{\Omega} \to E$ be a compact continuous operator, such that $p \notin (I - T)(\partial\Omega)$. Then the following properties hold:*

*(i) (**Normality**) $deg\,(I, \Omega, p) = 1$, if $p \in \Omega$ and $deg\,(I, \Omega, p) = 0$, if $p \in E \setminus \overline{\Omega}$.*

*(ii) (**Solvability**) If $deg\,(I - T, \Omega, p) \neq 0$, then the equation $x = Tx + p$ has at least a solution in Ω.*

*(iii) (**Invariance under homotopy**) Let $\mathcal{T}_t : \overline{\Omega} \to E$, $t \in [0,1]$, be continuous both in t and $x \in \bar{\Omega}$ and compact, and $x \neq \mathcal{T}_t x + p$ holds for all pairs $(t, x) \in [0,1] \times \partial\Omega$. Then $\deg\,(I - \mathcal{T}_t, \Omega, p)$ does not depend on $t \in [0,1]$.*

*(iv) (**Additivity**) Let Ω_1 and Ω_2 be two disjoint open subsets of Ω and $p \in E$ be such that $p \notin (I-T)\left(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)\right)$. Then*

$$deg\,(I-T,\Omega,p) = deg\,(I-T,\Omega_1,p) + deg\,(I-T,\Omega_2,p)\,.$$

Theorem 1.1.4 *Let $\Omega \subset E$ be bounded, open and symmetric with respect to zero such that $0 \in \Omega$. If the map T is continuous, compact and odd and such that $x \neq Tx$ for any $x \in \partial\Omega$, then the Leray-Schauder degree $deg\,(I-T,\Omega,0)$ is an odd integer.*

1.2 Non-smooth critical point theory

We present a brief overview of the non-smooth critical point theory developed by Motreanu and Panagiotopoulos (see [41]). It extends the concept of a critical point of a continuously differentiable functional to the case when the sum of a locally Lipschitz functional and a convex, lower semicontinuous one is considered.

1.2.1 Clarke's gradient of a locally Lipschitz function

First, we recall the definitions of a locally Lipschitz function and its generalized directional derivative (see [14]).

Definition 1.2.1 *Let X be a Banach space and $U \subset X$ be an open set. A function $\Phi : U \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in U$ there exists a neighborhood $V \subset U$, $x \in V$, and a constant $k_V > 0$, such that*

$$|\Phi(y) - \Phi(z)| \leq k_V \|y-z\|\,, \quad \forall y, z \in V.$$

Definition 1.2.2 *Let X be a Banach space and let Φ be a locally Lipschitz function defined on an open set $U \subset X$. The generalized directional derivative $\Phi^0(u;v)$ of the* *function* Φ *at the point* $u \in U$ *in the direction* $v \in X$*, is defined by*

$$\Phi^0(u;v) := \limsup_{w \to u, s \downarrow 0} \frac{\Phi(w+sv) - \Phi(w)}{s}.$$

One can easily check the following properties of the generalized directional derivative:

Proposition 1.2.1 *Let* $\Phi : U \to \mathbb{R}$ *be a locally Lipschitz function defined on an open subset* U *of a Banach space* X*. Then following facts are true:*

(i) For every $u \in U$ *the function* $\Phi^0(u;\cdot) : X \to \mathbb{R}$ *is positively homogeneous and subadditive and satisfies*

$$\left|\Phi^0(u;v)\right| \leq k_V \|v\|, \quad \forall v \in X,$$

where V *is a vicinity of* u *and* $k_V > 0$ *is a constant corresponding to* V*;*

(ii) $\Phi^0(\cdot;\cdot) : U \times X \to \mathbb{R}$ *is upper semicontinuous;*

(iii) $\Phi^0(u;-v) = (-\Phi)^0(u;v)$ *for any* $u \in U$ *and every* $v \in X$*.*

We are now ready to introduce the following definition (see Clarke [5, 1975] and [6, 1975]).

Definition 1.2.3 *Let* $\Phi : U \to \mathbb{R}$ *be a locally Lipschitz function defined on an open subset* U *of a Banach space* X*. The generalized gradient of Clarke* $\overline{\partial}\Phi(u)$ *at a point* $u \in U$ *is defined by*

$$\overline{\partial}\Phi(u) = \left\{\eta \in X^* : \Phi^0(u;v) \geq \langle \eta, v \rangle, \ \forall v \in X\right\},$$

where X^* *is the dual of the space* X*.*

Proposition 1.2.1 (i) and Hahn-Banach theorem imply that the generalized gradient of Clarke $\overline{\partial}\Phi(u) \subset X^*$ at any point $u \in U$ is not empty. Obviously, $\overline{\partial}\Phi(u)$ is the singleton $\{\Phi'(u)\}$ when $\Phi(u)$ is continuously differentiable.

Certain properties of the generalized gradient are collected in the following theorems. They are used in the proof of the main results from Chapter 3.

Proposition 1.2.2 *Let Φ, Φ_1 and Φ_2 be locally Lipschitz functions defined on an open subset U of the Banach space X. The following facts are true:*

(i) $\overline{\partial}(\lambda\Phi)(u) = \lambda\overline{\partial}\Phi(u)$ *for any* $\lambda \in \mathbb{R}$;

(ii) $\overline{\partial}(\Phi_1 + \Phi_2)(u) \subset \overline{\partial}\Phi_1(u) + \overline{\partial}\Phi_2(u)$;

(iii) if $u \in U$ *is a local extremum point of* Φ, *then* $0 \in \overline{\partial}\Phi(u)$;

(iv) let $f : V \to U$ *be a continuously differentiable function on an open subset* V *of a Banach space* Y. *Then the map* $\Phi \circ f : V \to \mathbb{R}$ *is locally Lipschitz and*

$$\overline{\partial}(\Phi \circ f)(v) \subseteq \overline{\partial}\Phi(f(v)) \circ f'(v) := \left\{\eta \circ f'(v) : \eta \in \overline{\partial}\Phi(f(v))\right\}, \quad \forall v \in V;$$

(v) the function $\Phi_1\Phi_2$ *is locally Lipschitz and*

$$\overline{\partial}(\Phi_1\Phi_2) \subseteq \Phi_1\overline{\partial}\Phi_2 + \Phi_2\overline{\partial}\Phi_1.$$

Finally, the following mean value result is due to Lebourg [32, 1975].

Theorem 1.2.1 (Lebourg) *Let U be an open subset of a Banach space X and let x, y be two points of U such that the line segment*

$$[x, y] = \{(1-t)x + ty : 0 \leq t \leq 1\}$$

is contained in U. Assume that the function $\Phi : U \to \mathbb{R}$ is locally Lipschitz. Then there exist $u \in (x, y) := \{(1-t)x + ty : 0 < t < 1\}$ and $\xi \in \overline{\partial}\Phi(u)$ satisfying

$$\Phi(y) - \Phi(x) = \langle \xi, y - x \rangle.$$

1.2.2 Critical points and the mountain pass theorem

Let us recall the definition of a critical point as well as the Palais-Smale condition for a functional I of the form

$$I = \Phi + \psi,$$

where Φ is a locally Lipschitz functional, and $\psi : X \to (-\infty, +\infty]$ is a proper, convex and lower semicontinuous (**l. s. c.**) function.

Definition 1.2.4 *An element $u \in X$ is called a critical point of the functional I if the following inequality holds*

$$\Phi^0 (u; v - u) + \psi (v) - \psi (u) \geq 0, \quad \forall v \in X.$$

A number $c \in \mathbb{R}$ such that $I^{-1} (c)$ contains a critical point is called a critical value of I.

If $\psi \equiv 0$ and $I = \Phi \in C^1 (X, \mathbb{R})$, then the element $u \in X$ is a critical point if and only if $I' (u) = 0$, i.e., the above definition extends the usual notion of a critical point.

Another generalization is concerned with the so-called Palais-Smale condition which guarantees the compactness of the set of critical points of the functional I. In the continuously differentiable case, a functional $J \in C^1 (X; \mathbb{R})$ is said to satisfy the Palais-Smale condition [44] if every sequence $\{u_n\} \subset X$ such that $J (u_n)$ is bounded and $J' (u_n) \to 0$ in X^* has a convergent subsequence. When the more general case is considered the definition is as follows (see Motreanu and Panagiotopoulos [41, 1999]).

Definition 1.2.5 *Functional I is said to satisfy the Palais-Smale (PS) condition if every sequence $\{u_n\} \subset X$ for which $I (u_n)$ is bounded and*

$$\Phi^0 (u_n; v - u_n) + \psi (v) - \psi (u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in X \tag{1.1}$$

for a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ *with* $\varepsilon_n \to 0$, *possesses a convergent subsequence.*

Finally, the following generalized mountain pass result will be used in the thesis when certain classes of boundary value problems are treated variationally. It can be found in Motreanu and Panagiotopoulos [41, 1999] (see also Kourogenis et al. [30, 2002]).

Theorem 1.2.2 (Mountain Pass) *Suppose that* I *satisfies the* (PS) *condition,* $I(0) = 0$ *and*

(i) *there exist* $\alpha,\ \rho > 0$ *such that* $I(u) \geq \alpha$ *if* $\|u\| = \rho$,

(ii) $I(e) \leq 0$ *for some* $e \in X$, *with* $\|e\| > \rho$.

Then the number

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = e\},$$

is a critical value of I *with* $c \geq \alpha$.

Chapter 2

The Method of Lower and Upper Solutions

In this chapter an existence and location result is proved for a fourth order nonlinear differential equation of the following quite general form

$$u^{(iv)} = f\left(t, u, u', u'', u'''\right), \quad 0 < t < 1, \tag{2.1}$$

where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function satisfying a Nagumo-type growth assumption (see Definition 2.1.1), with boundary conditions

$$u\left(0\right) = u'\left(1\right) = u''\left(0\right) = u'''\left(1\right) = 0, \tag{2.2}$$

describing, in this case, a beam deformation with one endpoint simply supported and the other one sliding clamped.

This result improves Grossinho and Tersian [19, 2000], Gupta [20, 1988], [21, 1991], Gyulov and Tersian [25, 2004], Ma and da Silva [36, 2004], since it deals with a fully nonlinear differential equation, that is, the nonlinearity is allowed to depend on all derivatives up to order three. The presence of odd order derivatives raises some difficulties when the existence of solutions is treated with certain classical methods. First, the variational approach fails when a differential equation of the form (2.1) is considered. It is generally impossible to construct a functional that has sufficiently good properties in order to investigate the solutions of (2.1) as critical points. Secondly, the monotone method for lower and upper solutions does not work. There are serious obstacles in verifying the monotonicity of the corresponding sequences of successive approximations .

Although (2.2) is a particular case of the nonlinear boundary conditions contained in Franco et al. [13, 2005], the assumptions required here on the lower and upper solutions as well as on function f are more general.

The existence and location of a solution to problem (2.1)-(2.2), (see Theorem 2.1.1) is derived by applying the technique of lower and upper solutions. The dependence on the third derivative is restricted to a Nagumo-type growth condition ([42]), which provides an *a priori* bound for every solution of (2.1) (see Lemma 2.1.1) and plays an important role in our treatment. The *a priori* estimations on the solution and its derivatives allow to define an open set where the topological degree is well defined.

This kind of arguments were suggested by Coster and Habets [7, 1996] for second order boundary value problems (see also Mawhin [38, 1979]) and by Grossinho and Minhós [16, 17, 18, 2001], Minhós et al. [39, 2005] for higher order separated boundary value problems.

For other type of controls at the ends of the beam with all derivatives up to order three, similar results can be obtained. More precisely, considering equation

(2.1) with one of the following boundary conditions

$$u(0) = u'(1) = u''(1) = u'''(0) = 0, \tag{2.3}$$

$$u(1) = u'(0) = u''(0) = u'''(1) = 0, \tag{2.4}$$

$$u(1) = u'(0) = u''(1) = u'''(0) = 0 \tag{2.5}$$

analogous theorems hold, with adequate modifications, assuming the second derivatives of lower and upper solutions in the reversed order, such as in (2.2), (see Theorem 2.1.2). For boundary value problems including equation (2.1) and one of the following conditions

$$u(0) = u'(0) = u''(0) = u'''(1) = 0, \tag{2.6}$$

$$u(0) = u'(0) = u''(1) = u'''(0) = 0, \tag{2.7}$$

$$u(1) = u'(1) = u''(0) = u'''(1) = 0, \tag{2.8}$$

$$u(1) = u'(1) = u''(1) = u'''(0) = 0, \tag{2.9}$$

existence and location results are obtained for another type of lower and upper solutions with the second derivatives well ordered (see Definition 2.1.3 and Theorem 2.1.3).

In Section 2.2 we apply one of the main results (Theorem 2.1.1) to the extended Fisher-Kolmogorov equation (EFK)

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \quad \gamma > 0, \tag{2.10}$$

proposed in Coullet, Elphick and Repaux [8, 1987] as well as in Dee and van Saarloos [10, 1988] as a higher order model and a generalization of the classical Fisher-Kolmogorov nonlinear diffusion equation (see Kolmogorov , Petrovski and Piscounov [29, 1937])

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \tag{2.11}$$

related to the study of a front propagation into an unstable state. Stationary periodic solutions of (2.10) are studied in Peletier and Troy [45, 2001] by the topological shooting method and in Chaparova et al. [4, 2003], Gyulov and Tersian [25, 2004], Tersian and Grossinho [47, 2003], Tersian and Chaparova [49, 2001] by a variational technique, considering the following boundary value problem

$$-\gamma\, u^{(iv)} + u'' + u - u^3 = 0, \tag{2.12}$$

$$u(0) = u''(0) = u(2L) = u''(2L) = 0. \tag{2.13}$$

The odd extension $\tilde{u}$,

$$\tilde{u}(x) := \begin{cases} u(x), & \text{if} \quad x \in [0, 2L] \\ -u(-x), & \text{if} \quad x \in [-2L, 0], \end{cases}$$

of the solution u of problem (2.12)-(2.13) to the interval $[-2L, 2L]$ yields a $4L$-periodic stationary solution of (2.10). It is known (cf. [4, 25, 49]) that if $L < L_0$, for some real L_0, then there are only trivial solutions for (2.12)-(2.13). An existence and location result will be obtaiend for symmetric solutions of (2.12)-(2.13), for $L > L_0$, which are close to L_0 (see Lemma 2.2.1 and Proposition 2.2.1).

2.1 A class of fourth order boundary value problems

The equation (2.1) is studied in this section. An assumption of the form of a Nagumo-type condition is required. Definitions of lower and upper solutions are formulated for different types of boundary conditions. The main results are stated and proved.

2.1.1 Definitions and *a priori* bound

A Nagumo-type growth condition and the definition of lower and upper solutions will be important tools for the method used herein.

Definition 2.1.1 *A continuous function $g : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is said to satisfy the Nagumo-type condition in*

$$E = \left\{ \begin{array}{c} (t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : \gamma_i(t) \le x_i \le \Gamma_i(t), \\ i = 0, 1, 2 \end{array} \right\}, \tag{2.14}$$

with $\gamma_i(t)$ and $\Gamma_i(t)$ continuous functions such that, for $i = 0, 1, 2$ and every $t \in [0,1]$,

$$\gamma_i(t) \le \Gamma_i(t), \tag{2.15}$$

if there exists a real continuous function $h_E : \mathbb{R}_0^+ \to [a, +\infty)$, for some $a > 0$, such that

$$|g(t, x_0, x_1, x_2, x_3)| \le h_E(|x_3|), \quad \forall (t, x_0, x_1, x_2, x_3) \in E, \tag{2.16}$$

with

$$\int_0^{+\infty} \frac{s}{h_E(s)} ds = +\infty. \tag{2.17}$$

This condition provides an *a priori* estimate for the third order derivative of the solutions of problem (2.1)-(2.2).

Lemma 2.1.1 *Assume $\gamma_i, \Gamma_i \in C([0,1], \mathbb{R})$, for $i = 0, 1, 2$, such that (2.15) holds and let E be given by (2.14). Assume there exists a function $h_E \in C\left(\mathbb{R}_0^+, [a, +\infty)\right)$, with $a > 0$, such that*

$$\int_\eta^{+\infty} \frac{s}{h_E(s)} ds > \max_{t \in [0,1]} \Gamma_2(t) - \min_{t \in [0,1]} \gamma_2(t), \tag{2.18}$$

where $\eta \ge 0$ is given by

$$\eta := \max\{\Gamma_2(1) - \gamma_2(0), \Gamma_2(0) - \gamma_2(1)\}$$

Then, there exists $r > 0$ *(depending on* γ_2, Γ_2 *and* h_E*) such that for every continuous function* $f : E \to \mathbb{R}$ *verifying (2.16) and for every solution* $u(t)$ *of (2.1) such that*

$$\gamma_i(t) \leq u^{(i)}(t) \leq \Gamma_i(t), \tag{2.19}$$

for $i = 0, 1, 2$ *and every* $t \in [0,1]$, *we have*

$$\|u'''\|_\infty < r.$$

Proof. Let $u(t)$ be a solution of (2.1) verifying (2.19).

Suppose, by contradiction, that $|u'''(t)| > \eta$ for every $t \in [0,1]$. If $u'''(t) > \eta$ we have the following contradiction

$$\begin{aligned} \Gamma_2(1) - \gamma_2(0) &\geq u''(1) - u''(0) \\ &= \int_0^1 u'''(t)\,dt > \int_0^1 \eta\,dt \geq \Gamma_2(1) - \gamma_2(0). \end{aligned}$$

An analogous contradiction is obtained if it is assumed that $u'''(t) < -\eta$, for every $t \in [0,1]$. So, there exists $t \in [0,1]$ such that $|u'''(t)| \leq \eta$.

Take $r > \eta$ such that

$$\int_\eta^r \frac{s}{h_E(s)} ds > \max_{t\in[0,1]} \Gamma_2(t) - \min_{t\in[0,1]} \gamma_2(t). \tag{2.20}$$

If $|u'''(t)| \leq \eta$, for every $t \in [0,1]$, the proof is finished. If not, suppose that there is $t \in [0,1]$ such that $u'''(t) > \eta$ and consider an interval $I = [t_0, t_1]$ (or $I = [t_1, t_0]$) such that $u'''(t_0) = \eta$ and $u'''(t) > \eta$ for $t \in I \setminus \{t_0\}$. Assume $I = [t_0, t_1]$ (the other case is analogous). Applying a convenient change of variable we have, by (2.16) and

(2.20), for arbitrary $t_2 \in I \setminus \{t_0\}$,

$$\begin{aligned}\int_{u'''(t_0)}^{u'''(t_2)} \frac{s}{h_E(s)} ds &= \int_{t_0}^{t_2} \frac{u'''(t)}{h_E(u'''(t))} u^{(iv)}(t)\,dt \\ &= \int_{t_0}^{t_2} \frac{u'''(t)}{h_E(u'''(t))} f(t,u,u',u'',u''')\,dt \\ &\leq \int_{t_0}^{t_2} u'''(t)\,dt = u''(t_2) - u''(t_0) \\ &\leq \max_{t\in[0,1]} \Gamma_2(t) - \min_{t\in[0,1]} \gamma_2(t) < \int_{\eta}^{r} \frac{s}{h_E(s)} ds.\end{aligned}$$

Then $u'''(t_2) < r$ and we have $u'''(t) < r$, for every $t \in I$. Arguing as before in the intervals J, where $u'''(t) > \eta$ for $t \in J$, we obtain that $u'''(t) < r$, for every $t \in [0,1]$.

The proof of $u'''(t) > -r$, for every $t \in [0,1]$ such that $u'''(t) < -\eta$, follows by similar steps. ■

Remark. Notice that condition (2.17) implies (2.18).

The typical functions used to define the set E are lower and upper solutions of problem (2.1)-(2.2).

Definition 2.1.2 *(i) A function $\alpha(t) \in C^4((0,1]) \cap C^3([0,1])$ is a lower solution of problem (2.1)-(2.2) if*

$$\alpha^{(iv)}(t) \leq f(t, \alpha(t), \alpha'(t), \alpha''(t), \alpha'''(t)), \tag{2.21}$$

and

$$\alpha(0) \leq 0, \quad \alpha'(1) \leq 0, \quad \alpha''(0) \geq 0, \quad \alpha'''(1) \geq 0. \tag{2.22}$$

(ii) A function $\beta(t) \in C^4((0,1]) \cap C^3([0,1])$ is an upper solution of problem (2.1)-(2.2) if

$$\beta^{(iv)}(t) \geq f(t, \beta(t), \beta'(t), \beta''(t), \beta'''(t)), \tag{2.23}$$

and

$$\beta(0) \geq 0, \quad \beta'(1) \geq 0, \quad \beta''(0) \leq 0, \quad \beta'''(1) \leq 0. \tag{2.24}$$

2.1.2 Existence and location results

In the presence of lower and upper solutions of problem (2.1)-(2.2) and assuming that the nonlinearity satisfies a Nagumo-type condition we obtain not only an existence result but also some information about the solution and about its first and second derivatives as well.

Theorem 2.1.1 *Let $f:[0,1]\times\mathbb{R}^4\to\mathbb{R}$ be a continuous function. Suppose that there are lower and upper solutions of (2.1)-(2.2) α and β, respectively, such that*

$$\beta''(t)\leq\alpha''(t),\forall t\in[0,1]. \tag{2.25}$$

Assume that f satisfies Nagumo-type conditions (2.16) and (2.17) in

$$E_*=\left\{\begin{array}{c}(t,x_0,x_1,x_2,x_3)\in[0,1]\times\mathbb{R}^4:\alpha(t)\leq x_0\leq\beta(t),\\ \alpha'(t)\leq x_1\leq\beta'(t),\ \beta''(t)\leq x_2\leq\alpha''(t)\end{array}\right\},$$

and

$$\begin{aligned} f(t,\alpha(t),\alpha'(t),x_2,x_3) &\leq f(t,x_0,x_1,x_2,x_3) \\ &\leq f(t,\beta(t),\beta'(t),x_2,x_3), \end{aligned} \tag{2.26}$$

for $(t,x_2,x_3)\in[0,1]\times\mathbb{R}^2$, $\alpha(t)\leq x_0\leq\beta(t)$ and $\alpha'(t)\leq x_1\leq\beta'(t)$.
Then problem (2.1)-(2.2) has at least a solution $u(t)\in C^4([0,1])$ satisfying, for $t\in[0,1]$,

$$\alpha(t)\leq u(t)\leq\beta(t),\quad \alpha'(t)\leq u'(t)\leq\beta'(t),\quad \beta''(t)\leq u''(t)\leq\alpha''(t).$$

Remark. The relations

$$\alpha(t)\leq\beta(t),\ \alpha'(t)\leq\beta'(t),\ \forall t\in[0,1],$$

are easily obtained from (2.25) by integration and using (2.22) and (2.24).

Proof. Define the auxiliary continuous functions

$$\delta_i(t, x_i) = \begin{cases} \beta^{(i)}(t), & \text{if} \quad x_i > \beta^{(i)}(t) \\ x_i, & \text{if} \quad \alpha^{(i)}(t) \leq x_i \leq \beta^{(i)}(t) \\ \alpha^{(i)}(t), & \text{if} \quad x_i < \alpha^{(i)}(t) \end{cases}, \tag{2.27}$$

for $i = 0, 1$ and

$$\delta_2(t, x_2) = \begin{cases} \alpha''(t), & \text{if} \quad x_2 > \alpha''(t) \\ x_2, & \text{if} \quad \beta''(t) \leq x_2 \leq \alpha''(t) \\ \beta''(t), & \text{if} \quad x_2 < \beta''(t) \end{cases}. \tag{2.28}$$

Consider the homotopic equation, for $\lambda \in [0, 1]$,

$$\begin{aligned} u^{(iv)}(t) &= \lambda f(t, \delta_0(t, u(t)), \delta_1(t, u'(t)), \delta_2(t, u''(t)), u'''(t)) \\ &\quad + u''(t) - \lambda\, \delta_2(t, u''(t)), \end{aligned} \tag{2.29}$$

with boundary conditions (2.2).

Take $r_1 > 0$ such that for every $t \in [0, 1]$,

$$\begin{gathered} -r_1 < \beta''(t) \leq \alpha''(t) < r_1, \\ f(t, \alpha(t), \alpha'(t), \alpha''(t), 0) + r_1 - \alpha''(t) > 0, \\ f(t, \beta(t), \beta'(t), \beta''(t), 0) - r_1 - \beta''(t) < 0. \end{gathered} \tag{2.30}$$

Step 1. *Every solution $u(t)$ of problem (2.29)-(2.2) satisfies*

$$\left|u^{(i)}(t)\right| < r_1, \quad \forall t \in [0, 1],$$

for $i = 0, 1, 2$, independently of $\lambda \in [0, 1]$.

Assume, by contradiction, that the above estimate is not true for $i = 2$. So there exist $\lambda \in [0, 1]$, $t \in [0, 1]$ and a solution u of (2.29)-(2.2) such that

$$|u''(t)| \geq r_1.$$

In the case $u''(t) \geq r_1$ define

$$\max_{t\in[0,1]} u''(t) := u''(t_0) \geq r_1.$$

Then $t_0 \in (0,1]$. Assume first that $t_0 \in (0,1)$. Hence $u'''(t_0) = 0$ and $u^{(iv)}(t_0) \leq 0$. For $\lambda \in [0,1]$, by (2.26) and (2.30), we obtain the following contradiction

$$\begin{aligned} 0 &\geq u^{(iv)}(t_0) \\ &= \lambda f(t_0, \delta_0(t_0, u(t_0)), \delta_1(t_0, u'(t_0)), \delta_2(t_0, u''(t_0)), u'''(t_0)) \\ &\quad + u''(t_0) - \lambda \delta_2(t_0, u''(t_0)) \\ &= \lambda f(t_0, \delta_0(t_0, u(t_0)), \delta_1(t_0, u'(t_0)), \alpha''(t_0), 0) \\ &\quad + u''(t_0) - \lambda \alpha''(t_0) \\ &\geq \lambda f(t_0, \alpha(t_0), \alpha'(t_0), \alpha''(t_0), 0) + u''(t_0) - \lambda \alpha''(t_0) \\ &= \lambda [f(t_0, \alpha(t_0), \alpha'(t_0), \alpha''(t_0), 0) + r_1 - \alpha''(t_0)] \\ &\quad + u''(t_0) - \lambda r_1 > 0. \end{aligned}$$

If $t_0 = 1$, then

$$\max_{t\in[0,1]} u''(t) = u''(1) \geq r_1.$$

Since $u'''(1) = 0$, the inequality $u^{(iv)}(1) \leq 0$ holds and by the above computations a similar contradiction is achieved. The inequality $u''(t) > -r_1$ for all $t \in [0,1]$ can be proved using analogous arguments and therefore

$$|u''(t)| < r_1, \quad \forall t \in [0,1].$$

By integration and the boundary conditions (2.2) we obtain

$$|u'(t)| < r_1 \ \text{ and } \ |u(t)| < r_1, \quad \forall t \in [0,1].$$

Step 2. *There is $r_2 > 0$ such that, for every solution $u(t)$ of problem (2.29)-(2.2),*

$$|u'''(t)| < r_2, \quad \forall t \in [0,1],$$

independently of $\lambda \in [0,1]$.

Consider the set

$$E_{r_1} = \left\{ (t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : -r_1 \leq x_i \leq r_1,\ i = 0,1,2 \right\},$$

and the function $F_\lambda : E_{r_1} \to \mathbb{R}$, for $\lambda \in [0,1]$, given by

$$\begin{aligned} F_\lambda (t, x_0, x_1, x_2, x_3) &= \lambda f (t, \delta_0 (t, x_0), \delta_1 (t, x_1), \delta_2 (t, x_2), x_3) \\ &\quad + x_2 - \lambda\, \delta_2 (t, x_2). \end{aligned}$$

It will be proved that function F_λ satisfies Nagumo-type conditions (2.16) and (2.17) in E_{r_1} independently of $\lambda \in [0,1]$. Indeed, since f verifies (2.16) in E_*, then

$$\begin{aligned} |F_\lambda (t, x_0, x_1, x_2, x_3)| &\leq |f (t, \delta_0 (t, x_0), \delta_1 (t, x_1), \delta_2 (t, x_2), x_3)| \\ &\quad + |x_2| + |\delta_2 (t, x_2)| \\ &\leq h_{E_*} (|x_3|) + 2r_1 \end{aligned}$$

Define $h_{E_{r_1}} (t) := h_{E_*} (t) + 2r_1$ in $\mathbb{R}_0^+$. Then

$$\begin{aligned} \int_0^{+\infty} \frac{s}{h_{E_{r_1}} (s)} ds &= \int_0^{+\infty} \frac{s}{h_{E_*} (s) + 2r_1} ds \\ &\geq \frac{1}{1 + \frac{2r_1}{a}} \int_0^{+\infty} \frac{s}{h_{E_*} (s)} ds = +\infty \end{aligned}$$

and so F_λ satisfies assumptions (2.16), (2.17). Thus it verifies (2.18) with E and h_E replaced, respectively, by E_{r_1} and $h_{E_{r_1}}$. Let

$$\gamma_i (t) \equiv -r_1 \text{ and } \Gamma_i (t) \equiv r_1, \text{ for } i = 0,1,2.$$

Then, Step 1 and Lemma 2.1.1 imply that there is $r_2 > 0$ such that

$$|u''' (t)| < r_2, \quad \forall t \in [0,1].$$

Notice that r_2 is independent of λ, because r_1 and $h_{E_{r_1}}$ do not depend on λ.

Step 3. *Problem (2.29)-(2.2) has at least one solution $u_1(t)$ when $\lambda = 1$.*

Define the operators

$$\mathcal{L} : C^4([0,1]) \subset C^3([0,1]) \to C([0,1]) \times \mathbb{R}^4$$

and, for any $\lambda \in [0,1]$,

$$\mathcal{N}_\lambda : C^3([0,1]) \to C([0,1]) \times \mathbb{R}^4$$

as follows

$$\mathcal{L}u := \left(u^{(iv)}, u(0), u''(0), u'(1), u'''(1)\right)$$

$$\begin{aligned} \mathcal{N}_\lambda u \quad := \quad & (\lambda f(t, \delta_0(t, u(t)), \delta_1(t, u'(t)), \delta_2(t, u''(t)), u'''(t)) \\ & + u''(t) - \lambda\, \delta_2(t, u''(t)), 0, 0, 0, 0). \end{aligned}$$

Since $\mathcal{L}$ has a compact inverse we can define the completely continuous operator

$$\mathcal{T}_\lambda : C^3([0,1]) \to C^3([0,1]),$$

where

$$\mathcal{T}_\lambda(u) := \mathcal{L}^{-1}\mathcal{N}_\lambda(u).$$

Let r_2 be the constant defined in Step 2 and let Ω be the set

$$\Omega := \left\{x \in C^3([0,1]) : \left\|x^{(i)}\right\|_\infty < r_1,\ i = 0,1,2,\ \|x'''\|_\infty < r_2\right\}.$$

By Steps 1 and 2, the degree $deg(I - \mathcal{T}_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0,1]$ and, by invariance under homotopy,

$$deg(I - \mathcal{T}_0, \Omega, 0) = deg(I - \mathcal{T}_1, \Omega, 0).$$

The equation $x = \mathcal{T}_0(x)$ is equivalent to the problem

$$\begin{cases} u^{(iv)}(t) = u''(t), \\ u(0) = u'(1) = u''(0) = u'''(1) = 0, \end{cases}$$

which has only the zero solution. Moreover, the map $\mathcal{T}_0$ is odd. Therefore, by the odd mapping theorem, the degree

$$deg\left(I-\mathcal{T}_0,\Omega,0\right)$$

is an odd number. Hence, the equation $x=\mathcal{T}_1\left(x\right)$ has at least one solution, that is problem (2.29)-(2.2) with $\lambda=1$ has a solution $u_1\left(t\right)$ in Ω.

Step 4. *The function $u_1\left(t\right)$ is a solution of problem (2.1)-(2.2).*

By (2.27), (2.28) and equation (2.29) it will be enough to prove that

$$\alpha\left(t\right)\leq u_1\left(t\right)\leq\beta\left(t\right),\ \alpha'\left(t\right)\leq u_1'\left(t\right)\leq\beta'\left(t\right),\ \beta''\left(t\right)\leq u_1''\left(t\right)\leq\alpha''\left(t\right)$$

for every $t\in[0,1]$. Assume that there is $t\in[0,1]$ such that $u_1''\left(t\right)>\alpha''\left(t\right)$ and let t_2 be such that

$$u_1''\left(t_2\right)-\alpha''\left(t_2\right)=\max_{t\in[0,1]}\left[u_1''\left(t\right)-\alpha''\left(t\right)\right]>0.$$

Therefore $t_2\in(0,1]$. If $t_2\in(0,1)$ then $u_1'''\left(t_2\right)=\alpha'''\left(t_2\right)$ and

$$u_1^{(iv)}\left(t_2\right)\leq\alpha^{(iv)}\left(t_2\right). \tag{2.31}$$

By (2.21) and (2.26) it is obtained the following contradiction with (2.31):

$$\begin{aligned}
u_1^{(iv)}\left(t_2\right) &= f\left(t_2,\delta_0\left(t_2,u_1\left(t_2\right)\right),\delta_1\left(t_2,u_1'\left(t_2\right)\right),\delta_2\left(t_2,u_1''\left(t_2\right)\right),u_1'''\left(t_2\right)\right)\\
&\quad +u_1''\left(t_2\right)-\delta_2\left(t_2,u_1''\left(t_2\right)\right)\\
&= f\left(t_2,\delta_0\left(t_2,u_1\left(t_2\right)\right),\delta_1\left(t_2,u_1'\left(t_2\right)\right),\alpha''\left(t_2\right),\alpha'''\left(t_2\right)\right)\\
&\quad +u_1''\left(t_2\right)-\alpha''\left(t_2\right) \qquad (2.32)\\
&\geq f\left(t_2,\alpha\left(t_2\right),\alpha'\left(t_2\right),\alpha''\left(t_2\right),\alpha'''\left(t_2\right)\right)+u_1''\left(t_2\right)-\alpha''\left(t_2\right)\\
&> f\left(t_2,\alpha\left(t_2\right),\alpha'\left(t_2\right),\alpha''\left(t_2\right),\alpha'''\left(t_2\right)\right)\geq\alpha^{(iv)}\left(t_2\right).
\end{aligned}$$

If $t_2=1$ then

$$u_1''\left(1\right)-\alpha''\left(1\right)=\max_{t\in[0,1]}\left[u_1''\left(t\right)-\alpha''\left(t\right)\right]>0$$

and $u_1'''(1) - \alpha'''(1) \geq 0.$

By (2.2) and (2.22), $\alpha'''(1) = u_1'''(1) = 0$ and therefore $u_1^{(iv)}(1) \leq \alpha^{(iv)}(1)$. Computations as in (2.32) lead to a similar contradiction. So

$$\alpha''(t) - u_1''(t) \geq 0, \ \forall t \in [0,1],$$

$\alpha'(t) - u_1'(t)$ is a nondecreasing function and, by the boundary conditions,

$$\alpha'(t) - u_1'(t) \leq \alpha'(1) - u_1'(1) \leq 0,$$

i.e. $\alpha'(t) \leq u_1'(t)$, for every $t \in [0,1]$. Therefore, $\alpha(t) - u_1(t)$ is a nonincreasing function and so

$$\alpha(t) - u_1(t) \leq \alpha(0) - u_1(0) \leq 0,$$

i.e. $\alpha(t) \leq u_1(t)$, $\forall t \in [0,1]$.

Analogously, it can be proved that the inequalities

$$u_1''(t) \geq \beta''(t), u_1'(t) \leq \beta'(t), u_1(t) \leq \beta(t), \ \forall t \in [0,1]$$

hold and so the proof is finished. ■

We remark that Theorem 2.1.1 holds for problem (2.1)-(2.3), if the inequalities (2.22) and (2.24) are replaced with

$$\alpha(0) \leq 0, \quad \alpha'(1) \leq 0, \quad \alpha''(1) \geq 0, \quad \alpha'''(0) \leq 0, \tag{2.33}$$

and

$$\beta(0) \geq 0, \quad \beta'(1) \geq 0, \quad \beta''(1) \leq 0, \quad \beta'''(0) \geq 0, \tag{2.34}$$

respectively.

If (2.2) is replaced by (2.4), or (2.5), then it is necessary to define different boundary conditions of lower and upper solutions. More precisely, the inequalities (2.22) and (2.24) are replaced by

$$\alpha(1) \leq 0, \quad \alpha'(0) \geq 0, \quad \alpha''(0) \geq 0, \quad \alpha'''(1) \geq 0, \tag{2.35}$$

and

$$\beta(1) \geq 0, \quad \beta'(0) \leq 0, \quad \beta''(0) \leq 0, \quad \beta'''(1) \leq 0, \tag{2.36}$$

respectively, for conditions (2.4), and with

$$\alpha(1) \leq 0, \quad \alpha'(0) \geq 0, \quad \alpha''(1) \geq 0, \quad \alpha'''(0) \leq 0, \tag{2.37}$$

and

$$\beta(1) \geq 0, \quad \beta'(0) \leq 0, \quad \beta''(1) \leq 0, \quad \beta'''(0) \geq 0, \tag{2.38}$$

respectively, for conditions (2.5). Moreover a different condition on the behaviour of f must be assumed, as it is shown in the next result, whose proof follows by similar arguments.

Theorem 2.1.2 *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function. Suppose that there are lower and upper solutions of (2.1)-(2.4), α and β respectively, such that (2.25) holds. Assume that f satisfies Nagumo-type conditions (2.16) and (2.17) in*

$$E_1 = \left\{ \begin{array}{c} (t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : \alpha(t) \leq x_0 \leq \beta(t), \\ \beta'(t) \leq x_1 \leq \alpha'(t), \ \beta''(t) \leq x_2 \leq \alpha''(t) \end{array} \right\},$$

and

$$f(t, \alpha(t), \alpha'(t), x_2, x_3) \leq f(t, x_0, x_1, x_2, x_3) \leq f(t, \beta(t), \beta'(t), x_2, x_3),$$

for $(t, x_2, x_3) \in [0,1] \times \mathbb{R}^2$, $\alpha(t) \leq x_0 \leq \beta(t)$ and $\beta'(t) \leq x_1 \leq \alpha'(t)$.
Then problem (2.1)-(2.4) has at least a solution $u(t) \in C^4([0,1])$ satisfying, for $t \in [0,1]$,

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \beta'(t) \leq u'(t) \leq \alpha'(t), \quad \beta''(t) \leq u''(t) \leq \alpha''(t).$$

Concerning the cases where three assumptions on the same end-point of the beam are considered then new differential inequalities for lower and upper solutions must be assumed, as it can be seen in next definition:

Definition 2.1.3 *(i) A function $\alpha(t) \in C^4((0,1]) \cap C^3([0,1])$ is a lower solution of problem (2.1)-(2.6) if*

$$\alpha^{(iv)}(t) \geq f(t, \alpha(t), \alpha'(t), \alpha''(t), \alpha'''(t)),$$

and

$$\alpha(0) \leq 0, \quad \alpha'(0) \leq 0, \quad \alpha''(0) \leq 0, \quad \alpha'''(1) \leq 0. \tag{2.39}$$

(ii) A function $\beta(t) \in C^4((0,1]) \cap C^3([0,1])$ is an upper solution of problem (2.1)-(2.6) if

$$\beta^{(iv)}(t) \leq f(t, \beta(t), \beta'(t), \beta''(t), \beta'''(t)),$$

and

$$\beta(0) \geq 0, \quad \beta'(0) \geq 0, \quad \beta''(0) \geq 0, \quad \beta'''(1) \geq 0. \tag{2.40}$$

As a consequence of this new definition α'' and β'' are well ordered and the corresponding existence and location result is the following:

Theorem 2.1.3 *Suppose that there are α and β, respectively, lower and upper solutions of (2.1)-(2.6) such that*

$$\alpha''(t) \leq \beta''(t), \forall t \in [0,1].$$

Assume that the continuous function $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ satisfies Nagumo-type conditions (2.16) and (2.17) in

$$E_2 = \left\{ \begin{array}{c} (t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : \alpha(t) \leq x_0 \leq \beta(t), \\ \alpha'(t) \leq x_1 \leq \beta'(t), \ \alpha''(t) \leq x_2 \leq \beta''(t) \end{array} \right\},$$

and

$$f(t,\alpha(t),\alpha'(t),x_2,x_3) \geq f(t,x_0,x_1,x_2,x_3) \\ \geq f(t,\beta(t),\beta'(t),x_2,x_3), \tag{2.41}$$

for $(t,x_2,x_3) \in [0,1] \times \mathbb{R}^2$, $\alpha(t) \leq x_0 \leq \beta(t)$ *and* $\alpha'(t) \leq x_1 \leq \beta'(t)$.

Then problem (2.1)-(2.6) has at least a solution $u(t) \in C^4([0,1])$ *satisfying, for* $t \in [0,1]$,

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \alpha'(t) \leq u'(t) \leq \beta'(t), \quad \alpha''(t) \leq u''(t) \leq \beta''(t).$$

Note that Theorem 2.1.3 can be applied to problem (2.1)-(2.7), if the boundary conditions (2.39) and (2.40) are replaced with

$$\alpha(0) \leq 0, \quad \alpha'(0) \leq 0, \quad \alpha''(1) \leq 0, \quad \alpha'''(0) \geq 0, \tag{2.42}$$

and

$$\beta(0) \geq 0, \quad \beta'(0) \geq 0, \quad \beta''(1) \geq 0, \quad \beta'''(0) \leq 0, \tag{2.43}$$

respectively.

Let (2.39) and (2.40) be replaced with

$$\alpha(1) \leq 0, \quad \alpha'(1) \geq 0, \quad \alpha''(0) \leq 0, \quad \alpha'''(1) \leq 0, \tag{2.44}$$

$$\beta(1) \geq 0, \quad \beta'(1) \leq 0, \quad \beta''(0) \geq 0, \quad \beta'''(1) \geq 0, \tag{2.45}$$

for boundary conditions (2.8), or with

$$\alpha(1) \leq 0, \quad \alpha'(1) \geq 0, \quad \alpha''(1) \leq 0, \quad \alpha'''(0) \geq 0, \tag{2.46}$$

$$\beta(1) \geq 0, \quad \beta'(1) \leq 0, \quad \beta''(1) \geq 0, \quad \beta'''(0) \leq 0, \tag{2.47}$$

for boundary conditions (2.9). Then Theorem 2.1.3 holds for problems (2.1)-(2.8) and (2.1)-(2.9) with the set E_2 and condition (2.41) replaced by

$$E_3 = \left\{ \begin{array}{c} (t,x_0,x_1,x_2,x_3) \in [0,1] \times \mathbb{R}^4 : \alpha(t) \leq x_0 \leq \beta(t), \\ \beta'(t) \leq x_1 \leq \alpha'(t), \ \alpha''(t) \leq x_2 \leq \beta''(t) \end{array} \right\}$$

and

$$f(t,\alpha(t),\alpha'(t),x_2,x_3) \geq f(t,x_0,x_1,x_2,x_3) \geq f(t,\beta(t),\beta'(t),x_2,x_3),$$

for $(t,x_2,x_3) \in [0,1]\times\mathbb{R}^2$, $\alpha(t) \leq x_0 \leq \beta(t)$ and $\beta'(t) \leq x_1 \leq \alpha'(t)$.

2.1.3 An example

Consider the fourth order boundary value problem

$$\begin{cases} u^{(iv)} = u^{2m+1} + (u')^{2n+1} + 3(u'')^{2p+1}(|u'''|+1)^{\theta} + e(t), \\ \\ u(0) = u'(1) = u''(0) = u'''(1) = 0, \end{cases} \tag{2.48}$$

where $m,n,p \in \mathbb{N}\cup\{0\}$, $\theta \in [0,2]$ and $e(t) \in C([0,1])$ such that $\|e\|_{\infty} \leq \frac{3}{2}$.

The functions α, $\beta : [0,1] \to \mathbb{R}$ defined by

$$\alpha(t) := -\frac{1}{2}t(2-t) \quad \text{and} \quad \beta(t) := \frac{1}{2}t(2-t)$$

are lower and upper solutions of (2.48), respectively. In fact

$$f(t,\alpha,\alpha',\alpha'',\alpha''') \geq -\left(\frac{1}{2}\right)^{2m+1} - (1-t)^{2n+1} + 3 + e(t) \geq 0 \equiv \alpha^{(iv)}(t),$$

$$f(t,\beta,\beta',\beta'',\beta''') \leq \left(\frac{1}{2}\right)^{2m+1} + (1-t)^{2n+1} - 3 + e(t) \leq 0 \equiv \beta^{(iv)}(t),$$

and

$$\begin{aligned} \alpha(0) &= 0, \quad \alpha'(1) = 0, \quad \alpha''(0) = 1, \quad \alpha'''(1) = 0, \\ \beta(0) &= 0, \quad \beta'(1) = 0, \quad \beta''(0) = -1, \quad \beta'''(1) = 0. \end{aligned}$$

The continuous function

$$f(t,x_0,x_1,x_2,x_3) = x_0^{2m+1} + x_1^{2n+1} + 3x_2^{2p+1}(|x_3|+1)^{\theta} + e(t)$$

verifies assumption (2.26) and the Nagumo-type conditions (2.16) and (2.17) in

$$E = \left\{ \begin{array}{c} (t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : \frac{t^2 - 2t}{2} \leq x_0 \leq \frac{2t - t^2}{2}, \\ t - 1 \leq x_1 \leq 1 - t, \; -1 \leq x_2 \leq 1 \end{array} \right\}$$

and as $\alpha''(t) \equiv 1 > \beta''(t) \equiv -1$, for every $t \in [0,1]$ then, by Theorem 2.1.1, there is a solution $u(t)$ of (2.48), such that

$$-\frac{1}{2}t(2-t) \leq u(t) \leq \frac{1}{2}t(2-t), \quad t - 1 \leq u'(t) \leq 1 - t,$$
$$-1 \leq u''(t) \leq 1, \; \forall t \in [0,1].$$

2.2 An application to the EFK equation

In this section our existence and location result will be applied to the extended Fisher-Kolmogorov (EFK) equation (2.10) that generalizes to higher order the non-linear diffusion equation (2.11). This type of problems describing the propagation of a front into an unstable state arises in biology, population dynamics, pulse propagation in nerves, crystal growth and fluid flow among others (see Cross and Hohenberg [9, 1993] and the references therein).

Consider the boundary value problem (2.12)-(2.13), that describes a stationary $4L$-periodic solution of (2.10), and the linear differential operator

$$\mathcal{L}u := \gamma \, u^{(iv)} - u'' - u. \tag{2.49}$$

Let ξ_0 be the unique positive root of the equation $P(\xi) = 0$ where

$$P(\xi) := \gamma \, \xi^4 + \xi^2 - 1. \tag{2.50}$$

It is known (cf. [4] ,[25] and [49]) that if $L < L_0$, with $L_0 = \frac{\pi}{2\xi_0}$, there are no nonzero solutions of (2.12)-(2.13). To obtain existence and location results about symmetric solutions of (2.12)-(2.13), for values of $L > L_0$ which are close to L_0, we need the following lemma:

Lemma 2.2.1 *The equation*

$$\frac{-4P\left(\frac{\pi}{2L}\right)}{3+\frac{P\left(\frac{\pi}{2L}\right)}{P\left(\frac{3\pi}{2L}\right)}}=\frac{1}{3} \tag{2.51}$$

has a unique solution L_1 *in the interval* $(L_0, 3L_0)$. *Moreover, the inequality*

$$P\left(\frac{3\pi}{2L}\right) > -3P\left(\frac{\pi}{2L}\right) \tag{2.52}$$

holds for $L \in (L_0, L_1]$.

Proof. Denote $\xi(L) = \frac{\pi}{2L}$ and consider the function

$$h(\xi) := -P(\xi)\left(4+\frac{1}{3P(3\xi)}\right)-1.$$

Notice that, for $\xi \in \left(\frac{\xi_0}{3}, \xi_0\right)$, functions $P(\xi)$ and $P(3\xi)$ are increasing, $P(\xi) < 0$, $P(3\xi) > 0$ and $h(\xi)$ is decreasing. Since the right limit $h\left(\left(\frac{\xi_0}{3}\right)^+\right) = +\infty$ and $h(\xi_0) = -1$ then $h(\xi)$ has one unique zero ξ_1 in $\left(\frac{\xi_0}{3}, \xi_0\right)$. Due to the equivalence of (2.51) and the equation $h(\xi) = 0$ in $\left(\frac{\xi_0}{3}, \xi_0\right)$, $L_1 := \frac{\pi}{2\xi_1}$ is the unique root of (2.51) in $(L_0, 3L_0)$.

Suppose, by contradiction, that (2.52) does not hold, that is

$$P(3\xi) \leq -3P(\xi),$$

for some $\xi \in [\xi_1, \xi_0)$. As $P(\xi) < 0$, we have

$$0 \geq h(\xi) \geq -P(\xi)\left(4-\frac{1}{9P(\xi)}\right)-1 = -4P(\xi)-\frac{8}{9}.$$

Therefore $P(\xi) \geq -\frac{2}{9}$, $P(3\xi) \leq \frac{2}{3}$ and the following contradiction is obtained:

$$-\frac{16}{3} \geq -9P(\xi)+P(3\xi)-8 = 72\gamma\xi^4.$$

Hence, inequality (2.52) holds for any $L \in (L_0, L_1]$. ■

Proposition 2.2.1 *Consider the function*

$$\varphi(x) := \sin\frac{\pi x}{2L} - \frac{P\left(\frac{\pi}{2L}\right)}{3P\left(\frac{3\pi}{2L}\right)} \sin\frac{3\pi x}{2L}.$$

If $L \in (L_0, L_1]$, *then there exists a nontrivial solution* $u(x)$ *of problem (2.12)-(2.13) such that, for any* $x \in [0, L]$, *the inequalities*

$$C_\alpha \varphi(x) \le u(x) \le C_\beta \varphi(x), \quad C_\alpha \varphi'(x) \le u'(x) \le C_\beta \varphi'(x),$$
$$C_\beta \varphi''(x) \le u''(x) \le C_\alpha \varphi''(x),$$

hold with $C_\alpha, C_\beta > 0$ *given by*

$$C_\alpha := \sqrt{\frac{-4P\left(\frac{\pi}{2L}\right)}{3\left(1 - \frac{P\left(\frac{\pi}{2L}\right)}{P\left(\frac{3\pi}{2L}\right)}\right)^3}}, \quad C_\beta := \sqrt{\frac{-4P\left(\frac{\pi}{2L}\right)}{3\left(1 + \frac{P\left(\frac{\pi}{2L}\right)}{3P\left(\frac{3\pi}{2L}\right)}\right)^3}}. \tag{2.53}$$

Moreover, this solution $u(x)$ *is symmetric with respect to the middle point* $x = L$ *of the interval* $[0, 2L]$, *i.e.,*

$$u(x) = u(2L - x)$$

for $x \in [0, L]$.

Proof. Every solution $u(x)$ of (2.12)-(2.13) such that it is symmetric about the middle point $x = L$ of $[0, 2L]$ verifies

$$u(0) = u''(0) = u'(L) = u'''(L) = 0. \tag{2.54}$$

Conversely, if $u(x)$ satisfies (2.12) and (2.54) then its even extension $\bar{u}(x)$

$$\bar{u}(x) := \begin{cases} u(x), & \text{if} \quad x \in [0, L], \\ u(2L - x), & \text{if} \quad x \in [L, 2L], \end{cases}$$

about the point $x = L$ is a solution of (2.12)-(2.13). So we can consider equation (2.12) in $[0, L]$ with boundary conditions (2.54). We re-scale the interval $[0, L]$ by the change of variables $t = \frac{x}{L}$, $v(t) = u(Lt)$, $t \in [0,1]$. A new boundary value problem in the interval $[0,1]$ is obtained

$$v^{(iv)}(t) = \gamma^{-1}\left[L^2 v''(t) + L^4\left(v(t) - v^3(t)\right)\right], \tag{2.55}$$

$$v(0) = v'(1) = v''(0) = v'''(1) = 0. \tag{2.56}$$

Step 1. *Function* $\overline{\varphi}(t) := \varphi(tL)$ *satisfies, for* $t \in [0,1]$,

$$\overline{\varphi}^{(iv)} = \gamma^{-1}\left(L^2\overline{\varphi}'' + L^4\left(\overline{\varphi} + \frac{4P\left(\frac{\pi}{2L}\right)}{3}\sin^3\frac{\pi t}{2}\right)\right), \tag{2.57}$$

the boundary conditions (2.56) and the inequalities

$$0 \le \overline{\varphi}(t) \le 1 + \frac{P\left(\frac{\pi}{2L}\right)}{3P\left(\frac{3\pi}{2L}\right)}, \quad \overline{\varphi}'(t) \ge 0, \quad \overline{\varphi}''(t) \le 0. \tag{2.58}$$

For $\mathcal{L}$ given by (2.49) we have, by (2.50),

$$\begin{aligned}
\gamma\frac{\overline{\varphi}^{(iv)}(t)}{L^4} - \frac{\overline{\varphi}''(t)}{L^2} - \overline{\varphi}(t) &= \mathcal{L}\varphi(x) \\
&= \mathcal{L}\left(\sin\frac{\pi x}{2L}\right) - \frac{P\left(\frac{\pi}{2L}\right)}{3P\left(\frac{3\pi}{2L}\right)}\mathcal{L}\left(\sin\frac{3\pi x}{2L}\right) \\
&= \frac{P\left(\frac{\pi}{2L}\right)}{3}\left(3\sin\frac{\pi x}{2L} - \sin\frac{3\pi x}{2L}\right) \\
&= \frac{4P\left(\frac{\pi}{2L}\right)}{3}\sin^3\frac{\pi t}{2}
\end{aligned}$$

and then $\overline{\varphi}$ verifies (2.57).

Since $P\left(\frac{\pi}{2L}\right) < 0$ for $L \in (L_0, L_1]$, the inequality (2.52) implies that

$$\begin{aligned}
\overline{\varphi}''(t) &= \frac{\pi^2}{4}\left(-\sin\frac{\pi t}{2} + \frac{3P\left(\frac{\pi}{2L}\right)}{P\left(\frac{3\pi}{2L}\right)}\sin\frac{3\pi t}{2}\right) \\
&= \frac{\pi^2}{4}\frac{3P\left(\frac{\pi}{2L}\right)}{P\left(\frac{3\pi}{2L}\right)}\left(-\frac{P\left(\frac{3\pi}{2L}\right)}{3P\left(\frac{\pi}{2L}\right)}\sin\frac{\pi t}{2} + \sin\frac{3\pi t}{2}\right) \\
&\le \frac{\pi^2}{4}\frac{3P\left(\frac{\pi}{2L}\right)}{P\left(\frac{3\pi}{2L}\right)}\left(\sin\frac{\pi t}{2} + \sin\frac{3\pi t}{2}\right) \\
&= \frac{\pi^2}{2}\frac{3P\left(\frac{\pi}{2L}\right)}{P\left(\frac{3\pi}{2L}\right)}\cos\frac{\pi t}{2}\ \sin\pi t \le 0,
\end{aligned}$$

for any $t \in [0,1]$.

By easy computations, $\overline{\varphi}$ verifies boundary conditions (2.56) and, for $t \in [0,1]$,

$$0 \geq \int_t^1 \overline{\varphi}''(s)\ ds = \overline{\varphi}'(1) - \overline{\varphi}'(t) = -\overline{\varphi}'(t).$$

Therefore $\overline{\varphi}(t)$ is a nondecreasing function and so

$$0 = \overline{\varphi}(0) \leq \overline{\varphi}(t) \leq \overline{\varphi}(1) = 1 + \frac{P\left(\frac{\pi}{2L}\right)}{3P\left(\frac{3\pi}{2L}\right)}.$$

Step 2. *For C_α and C_β given by* (2.53),

$$\alpha(t) := C_\alpha \overline{\varphi}(t), \quad \beta(t) := C_\beta \overline{\varphi}(t),$$

are lower and upper solutions of problem (2.55)-(2.56), respectively.

By standard arguments, for any $t \in (0,1]$, we have

$$-1 \leq \frac{\sin\frac{3\pi t}{2}}{\sin\frac{\pi t}{2}} \leq 3.$$

Then, the inequality (2.52) implies that

$$0 < 1 + \frac{P\left(\frac{\pi}{2L}\right)}{3P\left(\frac{3\pi}{2L}\right)} \leq 1 - \frac{P\left(\frac{\pi}{2L}\right)\sin\frac{3\pi t}{2}}{3P\left(\frac{3\pi}{2L}\right)\sin\frac{\pi t}{2}} = \frac{\overline{\varphi}(t)}{\sin\frac{\pi t}{2}} \leq 1 - \frac{P\left(\frac{\pi}{2L}\right)}{P\left(\frac{3\pi}{2L}\right)}$$

and

$$\frac{-4P\left(\frac{\pi}{2L}\right)}{3\left(1 + \frac{P\left(\frac{\pi}{2L}\right)}{3P\left(\frac{3\pi}{2L}\right)}\right)^3} \geq \frac{-4P\left(\frac{\pi}{2L}\right)\sin^3\frac{\pi t}{2}}{3\ \left(\overline{\varphi}(t)\right)^3} \geq \frac{-4P\left(\frac{\pi}{2L}\right)}{3\left(1 - \frac{P\left(\frac{\pi}{2L}\right)}{P\left(\frac{3\pi}{2L}\right)}\right)^3}. \tag{2.59}$$

Therefore, the definition (2.53) of C_β yields

$$\begin{aligned}
\beta^{(iv)} &= C_\beta\ \overline{\varphi}^{(iv)}(t) \\
&= \gamma^{-1}\left(L^2\beta'' + L^4\beta + L^4 C_\beta \frac{4P\left(\frac{\pi}{2L}\right)}{3}\sin^3\frac{\pi t}{2}\right) \\
&\geq \gamma^{-1}\left[L^2\beta'' + L^4\beta - L^4 C_\beta \frac{-4P\left(\frac{\pi}{2L}\right)}{3\left(1 + \frac{P\left(\frac{\pi}{2L}\right)}{3P\left(\frac{3\pi}{2L}\right)}\right)^3}\left[\overline{\varphi}(t)\right]^3\right] \\
&= \gamma^{-1}\left[L^2\beta'' + L^4\beta - L^4\left(C_\beta\right)^3\ \left(\overline{\varphi}(t)\right)^3\right] \\
&= \gamma^{-1}\left[L^2\beta'' + L^4\left(\beta - \beta^3\right)\right],
\end{aligned}$$

proving condition (2.23). Boundary conditions (2.24) are trivially satisfied as $\overline{\varphi}(t)$ verifies (2.56).

To show that $\alpha(t) = C_\alpha\, \overline{\varphi}(t)$ is a lower solution of (2.55)-(2.56) the arguments are similar, using the second inequality in (2.59).

Step 3. *If* $L \in (L_0, L_1]$, *then*

$$0 \le \alpha(t) \le \beta(t) \le \frac{1}{\sqrt{3}} \text{ and } \beta''(t) \le \alpha''(t) \le 0,\ \forall t \in [0,1].$$

It can be easily derived from (2.53) and the inequalities (2.58), that for any $t \in [0,1]$, the inequality (2.25) from Theorem 2.1.1 is satisfied

$$\beta''(t) = C_\beta\, \overline{\varphi}''(t) \le C_\alpha\, \overline{\varphi}''(t) = \alpha''(t) \le 0.$$

Next, (2.59) and Lemma 2.2.1 imply that

$$0 \le C_\alpha\, \overline{\varphi}(t) = \alpha(t) \le \beta(t) = C_\beta\, \overline{\varphi}(t) \le C_\beta \left(1 + \frac{P\left(\frac{\pi}{2L}\right)}{3P\left(\frac{3\pi}{2L}\right)}\right) = \frac{1}{\sqrt{3}}.$$

Step 4. *There exists a solution* $v(t)$ *of problem (2.55)-(2.56) such that*

$$\alpha(t) \le v(t) \le \beta(t), \alpha'(t) \le v'(t) \le \beta'(t), \beta''(t) \le v''(t) \le \alpha''(t). \tag{2.60}$$

Define the auxiliary function

$$g(t, x_0, x_1, x_2, x_3) = \gamma^{-1}\left(L^2 x_2 + L^4\left(x_0 - x_0^3\right)\right).$$

Since $g(t, x_0, x_1, x_2, x_3)$ is constant in x_1 and it is increasing in x_0 when $|x_0| \le \frac{1}{\sqrt{3}}$, the function $g : E \to \mathbb{R}$ satisfies the inequalities (2.26) if the set E is defined as follows

$$E := \left\{ \begin{array}{c} (t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : \\ C_\alpha\, \overline{\varphi}(t) \le x_0 \le C_\beta\, \overline{\varphi}(t),\ C_\beta\, \overline{\varphi}''(t) \le x_2 \le C_\alpha\, \overline{\varphi}''(t) \end{array} \right\}.$$

Moreover,

$$|g(t, x_0, x_1, x_2, x_3)| \leq \gamma^{-1} \left(L^2 \; C_\beta \max_{t\in[0,1]} |\overline{\varphi}''(t)| + \frac{2\sqrt{3}}{9} L^4 \right)$$

and so g satisfies the Nagumo condition in E. Then, by Theorem 2.1.1, there is a solution $v(t)$ of (2.55)-(2.56) satisfying (2.60). ■

Chapter 3

The Variational Method

3.1 A class of non-smooth fourth order boundary value problems: formulation, motivation and main results

It is the purpose of this chapter to investigate the following class of nonlinear, nonsmooth, fourth order boundary value problems

$$\left(|u''(t)|^{p-2} u''(t)\right)'' - \left(a(t)\,|u'(t)|^{p-2} u'(t)\right)' \qquad (3.1)$$

$$+ b(t)\,|u(t)|^{p-2} u(t) \in \overline{\partial} F(t,u),$$

$$\begin{pmatrix} -\left(|u''|^{p-2} u''\right)'(0) + a(0)\,|u'(0)|^{p-2} u'(0) \\ \left(|u''|^{p-2} u''\right)'(1) - a(1)\,|u'(1)|^{p-2} u'(1) \\ |u''(0)|^{p-2} u''(0) \\ -|u''(1)|^{p-2} u''(1) \end{pmatrix} \in \partial j \begin{pmatrix} u(0) \\ u(1) \\ u'(0) \\ u'(1) \end{pmatrix}, \qquad (3.2)$$

where $a, b \in C([0,1])$ are given real functions, $p > 1$ and F, j are nonlinear functions satisfying some conditions which are specified below. Both equation (3.1) and the boundary condition (3.2) are sufficiently general to cover a broad range of specific problems. Our treatment here is mainly based on a variational approach.

To be more specific, let us formulate our assumptions on F and j:

(H_1) $F = F(t,\xi) : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping, satisfying in addition $F(t,0) = 0$ for a.a. $t \in (0,1)$, as well as the Lipschitz condition:

$\forall \rho > 0$ there is an $\alpha_\rho \in L^1(0,1)$ such that

$$|F(t,x) - F(t,y)| \leq \alpha_\rho(t)\,|x-y|\,,$$

for a.a. $t \in (0,1)$ and all x, y with $|x|, |y| \leq \rho$;

(H_2) Function $j : \mathbb{R}^4 \to (-\infty, +\infty]$ is proper, convex and lower semicontinuous (l.s.c), such that the null (column) vector $(0,0,0,0)^T \in D(j)$.

Now, to complete the presentation of our boundary value problem, let us explain the notation we have used above in (3.1) and (3.2). $\overline{\partial}F(t,\xi)$ denotes the generalized Clarke gradient of $F(t,\cdot)$ at $\xi \in \mathbb{R}$, while ∂j stands for the subdifferential of j.

Note that our conditions $F(t,0) = 0$ and $(0,0,0,0)^T \in D(j)$ do not restrict much the generality of the problem. In fact, the later can always be reached by a translation of u. Of course, this operation changes equation (3.1), but the treatment is similar. In particular, if either $p = 2$ or b is the null function, equation (3.1) remains unchanged and it suffices to assume that $F(t,0)$ is an L^1 function.

A classical fourth order equation arising in the beam-column theory is the following (see Timoshenko and Gere [50])

$$EI\frac{d^4u}{dx^4} + P\frac{d^2u}{dx^2} = q, \tag{3.3}$$

where u is the lateral deflection, q is the intensity of a distributed lateral load, P is the axial compressive force applied to the beam and EI represents the flexural rigidity in the plane of bending. Equation (3.3) is derived from the static equilibrium equations for any slice at distance x along the beam, namely the equilibrium of forces reads

$$q = -\frac{dV}{dx}, \tag{3.4}$$

where V is the shearing force and the equilibrium of moments is expressed by the equation

$$V = \frac{dM}{dx} - P\frac{du}{dx}, \tag{3.5}$$

where M denotes the bending moment. It is assumed that the bending moment depends linearly on the curvature. It can be expressed (if some higher order terms are neglected) as follows

$$EI\frac{d^2u}{dx^2} = -M. \tag{3.6}$$

Let us consider a more general situation, that the bending moment is a power function of the curvature with exponent $p-1$, i.e.,

$$M = -c\left|\frac{d^2u}{dx^2}\right|^{p-2}\frac{d^2u}{dx^2}, \tag{3.7}$$

where c is a constant. Then the presence of the term $\left(|u''|^{p-2}\,u''\right)''$ in (3.1) is justified if we assume (3.7) instead of (3.6) when equation (3.3) is derived. If $p = 2$, then (3.7) coincides with (3.6) where $c = EI$.

Another equation that motivates our investigation here is the following one

$$Dw^{iv} + N_x w'' + Eh\frac{w}{a^2} = q,\ \ t \in (0,1)\,. \tag{3.8}$$

It models the radial deflection w for symmetrical buckling of a cylindrical shell under uniform axial compression N_x (see [50], p. 457, [41]).

The applied lateral load q in (3.3) or (3.8) may be presented as the reaction of a support, which generally depends nonlinearly on the deflection (see [15], [19], [20],

[21], [22], [35], [36], [37], [41], [51]),

$$q(t) = f(t, u(t)),$$

or, more generally,

$$q(t) \in \overline{\partial} F(t, u(t)),$$

where F is a nonsmooth function (in particular, F may have some jumps, e.g., the case of adhesive support, see [41]).

Condition (3.2) covers many different types of boundary conditions (see [26]). For example, it is easy to check that for

$$j\left((x_1, x_2, x_3, x_4)^T\right) := \begin{cases} 0, & x_1 = x_2,\ x_3 = x_4, \\ +\infty, & \text{otherwise}, \end{cases}$$

we obtain the periodic conditions $u^{(i)}(0) = u^{(i)}(1)$, $i = 0, 1, 2, 3$, while the case of simply supported endpoints, i.e., $u(0) = u(1) = u''(0) = u''(1) = 0$, corresponds to the following choice

$$j\left((x_1, x_2, x_3, x_4)^T\right) := \begin{cases} 0, & x_1 = x_2 = 0, \\ +\infty, & \text{otherwise}. \end{cases}$$

Our aim in this chapter is to investigate the existence of solutions to problem (3.1)-(3.2). Some results related to the particular case $p = 2$ and a, b constants are reported in Gyulov and Moroşanu [22, 2007]. The treatment of the more general problem (3.1), (3.2) requires more advanced analysis. It is studied in Gyulov and Moroşanu [24]. The reference Gyulov and Moroşanu [23, 2009] contains a short variant concerning the particular case when a and b are constants and $p > 1$.

By a *solution* of this problem we mean a function $u \in W^{2,p}(0,1)$, with $\left(|u''|^{p-2} u''\right)' \in \mathcal{AC}([0,1], \mathbb{R})$, which satisfies (3.2) and for a.a. $t \in (0,1)$

$$\begin{aligned} \left(|u''(t)|^{p-2} u''(t)\right)'' &- \left(a(t) |u'(t)|^{p-2} u'(t)\right)' \\ &+ b(t) |u(t)|^{p-2} u(t) \in \overline{\partial} F(t, u). \end{aligned} \tag{3.9}$$

In fact, since $|u''|^{p-2}\, u'' =: v \in W^{2,1}(0,1)$, and $u'' = |v|^{q-2}\, v$, where q is the conjugate of p (i.e., $p^{-1} + q^{-1} = 1$), it follows that $u \in C^2([0,1])$. In particular, the values of $u,\ u',\ u''$ at $t = 0$ and $t = 1$ in (3.2) make sense. Note that if $1 < p \leq 2$ then $u \in C^3([0,1])$.

Now, we define the set

$$\mathcal{D} = \left\{ u : u \in W^{2,p}(0,1),\ \left(u(0), u(1), u'(0), u'(1)\right)^T \in D(j) \right\},$$

and the functional $J : W^{2,p}(0,1) \to \mathbb{R} \cup \{+\infty\}$,

$$J(u) := j\left(\left(u(0), u(1), u'(0), u'(1)\right)^T \right), \quad \forall u \in W^{2,p}(0,1),$$

whose effective domain is $D(J) = \mathcal{D}$.

Obviously, $\mathcal{D} \neq \emptyset$ since $(0,0,0,0)^T \in D(j)$, so J is proper, convex and l.s.c.

In order to obtain existence of solutions to problem (3.1), (3.2), we consider the following functional

$$I(u) := \frac{1}{p} \int_0^1 \left(|u''|^p + a\,|u'|^p + b\,|u|^p \right) dt - \int_0^1 F(t,u)\, dt + J(u),$$

and use the critical point theory developed in Motreanu and Panagiotopoulos [41] (see Section 1.2 from Chapter 1).

Let us define the following two constants,

$$\lambda_1 := \inf \left\{ \frac{\int_0^1 \left(|u''|^p + a\,|u'|^p + b\,|u|^p \right)\, dt}{\|u\|_{L^p}^p} : u \in \mathcal{D} \backslash \{0\} \right\}, \tag{3.10}$$

and

$$\overline{\lambda}_1 := \lim_{s \to \infty} \inf_{\substack{ru \in \mathcal{D} \\ r \geq s}} \left\{ \int_0^1 \left(|u''|^p + a\,|u'|^p + b\,|u|^p \right)\, dt + \frac{pJ(ru)}{r^p} : \|u\|_{L^p}^p = 1 \right\}. \tag{3.11}$$

Both λ_1 and $\overline{\lambda}_1$ will be important in the sequel. It is easily seen that $\lambda_1 \leq \overline{\lambda}_1$, and in most cases $\lambda_1 < \overline{\lambda}_1$.

We are now able to state the main results of the present chapter, as follows.

Theorem 3.1.1 *Assume* (H_1) *and* (H_2). *Suppose, in addition, that the following condition is satisfied*

(L_∞)

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^p} < \frac{\lambda_1}{p}, \tag{3.12}$$

uniformly for a.a. $t \in (0,1)$. *Then problem (3.1), (3.2) has at least a solution.*

In order to state our next result, we introduce a new condition on F:

(L_0)

$$\limsup_{x\to 0} \frac{F(t,x)}{|x|^p} < \frac{\lambda_1}{p},$$

uniformly for a.e. $t \in (0,1)$. Obviously this implies $0 \in \overline{\partial} F(t,0)$ for a.a. $t \in (0,1)$, so in this case $u(t) \equiv 0$ is a solution of problem (3.1), (3.2). We are interested in the existence of nontrivial solutions of problem (3.1), (3.2). We have

Theorem 3.1.2 *Assume that* $\lambda_1 > 0$ *and that* (L_0), (H_1), *and* (H_2) *are fulfilled. Suppose, in addition, that* $D(j)$ *is closed,* $(0,0,0,0)^T \in \partial j\left((0,0,0,0)^T\right)$, *and either* (G_θ) *or* $(G_p) - \left(\overline{L}_\infty\right)$ *holds, where*

(G_θ) *there exist constants* $\theta > p$, *and* $k, M > 0$, *such that*

$$\begin{aligned} j'(z;z) &\leq \theta j(z) + k, \qquad \forall z \in D(j), & (3.13)\\ 0 &< \theta F(t,x) \leq \xi x, \quad \forall \xi \in \overline{\partial} F(t,x), & (3.14) \end{aligned}$$

for all $|x| > M$, *and* *a.a.* $t \in (0,1)$,

(G_p) *there exist positive constants* c, k, M *such that*

$$\begin{aligned} j'(z;z) &\leq pj(z) + k, \qquad \forall z \in D(j), & (3.15)\\ 0 &< \left(p + \frac{c}{|x|^{p-1}}\right) F(t,x) \leq \xi x, \quad \forall \xi \in \overline{\partial} F(t,x), & (3.16) \end{aligned}$$

for all $|x| > M$, *and a.a.* $t \in (0,1)$, *and*

$(\overline{L}_\infty)$

$$\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^p} > \frac{\overline{\lambda}_1}{p}, \tag{3.17}$$

uniformly for a.a. $t \in (0,1)$.

Then problem (3.1), (3.2) has at least a nonzero solution.

Remark. It is worth pointing out that any of the conditions (3.13) and (3.15) implies that the domain $D(j)$ of functional j is a convex cone. Moreover, assumption (3.15) guarantees that $\overline{\lambda}_1 < \infty$ (see Lemma 3.2.2 below).

Remark. Assumption (3.16) allows asymptotically quadratic growth of function $F(t,x)$ and implies that functional $I(\cdot)$ satisfies the Palais-Smale condition (Lemma 3.2.3). We may think about a more general condition of Cerami type, but it seems it is difficult to apply such a condition to the case when the functional is the sum of a locally Lipschitz functional and a proper, convex, l.s.c. one.

3.2 Variational settings and auxiliary results

Our framework in the next part of the thesis will be $X = W^{2,p}(0,1)$.

Define $\psi : X \to (-\infty, +\infty]$ by

$$\begin{aligned} \psi(u) &= \tfrac{1}{p} \int_0^1 \left(|u''|^p + |u'|^p + |u|^p\right) dt + J(u) \\ &= \tfrac{1}{p} \|u\|^p_{W^{2,p}(0,1)} + J(u), \end{aligned}$$

whose effective domain is $D(\psi) = \mathcal{D}$, and

$$\Phi(u) := -\int_0^1 F(t,u)\,dt + \varphi(u), \quad u \in X = W^{2,p}(0,1), \tag{3.18}$$

where

$$\varphi(u) := \frac{1}{p}\int_0^1 \left((a(t)-1)|u'(t)|^p + (b(t)-1)|u(t)|^p\right)dt.$$

Obviously, ψ is proper, convex and l.s.c., while $\varphi \in C^1(W^{2,p}(0,1),\mathbb{R})$, and

$$\begin{aligned}\langle \varphi'(u), v\rangle = \quad & \int_0^1 \Big((a(t)-1)|u'(t)|^{p-2}u'(t)v'(t) \\ & + (b(t)-1)|u(t)|^{p-2}u(t)v(t)\Big)dt.\end{aligned}$$

We have

Proposition 3.2.1 *Assume that $F : (0,1)\times\mathbb{R}\to\mathbb{R}$ satisfies (H_1). Then the functional Φ defined by (3.18) is locally Lipschitz. Moreover, if $u \in W^{2,p}(0,1)$ and $l \in \overline{\partial}\Phi(u)$ then there is some $u_l \in L^1(0,1)$ such that $u_l(t) \in \overline{\partial}F(t,u(t))$ for a.a. $t \in (0,1)$, and*

$$\begin{aligned}\langle l, v\rangle = \quad & \int_0^1 \Big(-u_l(t)v(t) + (a(t)-1)|u'(t)|^{p-2}u'(t)v'(t) \\ & + (b(t)-1)|u(t)|^{p-2}u(t)v(t)\Big)dt, \quad \forall v \in W^{2,p}(0,1), \qquad (3.19)\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing in $W^{2,p}(0,1)$.

Proof. By the continuity of the embedding $W^{2,p}(0,1) \subset C^1([0,1])$ (see, for example, Adams and Fournier [1, 2003]) and assumption (H_1) it follows that Φ is indeed locally Lipschitz.

Define

$$R(u) := -\int_0^1 F(t,u(t))\,dt, \quad \forall u \in W^{2,p}(0,1).$$

One can prove (see Theorem 2.7.3 in [6]) that given $\xi \in \overline{\partial}R(u)$ then there exists some $v_\xi \in L^1(0,1)$ such that $v_\xi(t) \in -\overline{\partial}F(t,u(t))$ for a.a. $t \in (0,1)$, and

$$\langle \xi, v\rangle = \int_0^1 v_\xi v\,dt, \quad \forall v \in W^{2,p}(0,1). \qquad (3.20)$$

For reader's convenience, here is a complete direct proof. First, Fatou's lemma and (H_1) imply that

$$R^0(u;v) \leq \int_0^1 (-F(t,\cdot))^0 (u(t);v(t))\,dt \leq \int_0^1 \alpha_\rho(t)\,|v(t)|\,dt, \quad \forall v \in W^{2,p}(0,1), \tag{3.21}$$

where ρ is such that $\|u\|_C < \rho$, and $\alpha_\rho(t) \in L^1(0,1)$ is defined in (H_1).

Let $\xi \in \overline{\partial} R(u)$. Then

$$\langle \xi, v\rangle \leq R^0(u;v) \leq \int_0^1 \alpha_\rho(t)\,|v(t)|\,dt, \quad \forall v \in W^{2,p}(0,1),$$

and the Hahn-Banach theorem implies that ξ can be extended to a continuous linear functional $\overline{\xi}$ on $L^1((0,1);\mu) \supset W^{2,p}(0,1)$, where μ is the measure on $[0,1]$, defined by

$$\mu(A) := \int_A \alpha_\rho(t)\,dt,$$

for any set A, which is measurable with respect to the usual Lebesgues measure $m(\cdot)$. Obviously, $\alpha_\rho(t)$ can be chosen such that $\alpha_\rho(t) > 1$. So, $\mu(A) = 0$ iff $m(A) = 0$. Then, there exists $\beta_{\overline{\xi}} \in L^\infty((0,1);\mu) \equiv L^\infty(0,1)$, such that

$$\left\langle \overline{\xi}, w\right\rangle = \int_0^1 \alpha_\rho(t)\,\beta_{\overline{\xi}}(t)\,w(t)dt, \quad \forall w \in L^1((0,1);\mu),$$

so

$$\langle \xi, v\rangle = \left\langle \overline{\xi}, v\right\rangle = \int_0^1 \alpha_\rho(t)\,\beta_{\overline{\xi}}(t)\,v(t)dt, \quad \forall v \in W^{2,p}(0,1),$$

and (3.20) holds with $v_\xi := \alpha_\rho \beta_{\overline{\xi}} \in L^1(0,1)$. Now, (3.20) and (3.21) imply

$$\int_0^1 v_\xi v dt \leq \int_0^1 (-F(t,\cdot))^0 (u(t);v(t))\,dt, \quad \forall v \in W^{2,p}(0,1),$$

yielding that $v_\xi(t) \in \overline{\partial}(-F)(t,u(t)) = -\overline{\partial} F(t,u(t))$ for a.a. $t \in (0,1)$.

Now, let $l \in \overline{\partial}\Phi(u)$. By $\overline{\partial}\Phi(u) \subset \overline{\partial}R(u) + \overline{\partial}\varphi(u) = \overline{\partial}R(u) + \varphi'(u)$, there exists $\xi \in \overline{\partial}R(u)$ such that $l = \xi + \varphi'(u)$ and (3.19) is obtained with $u_l := -v_\xi$, where v_ξ is determined by ξ as above. ■

Define $I : X = W^{2,p}(0,1) \to (-\infty, +\infty]$ by

$$\begin{aligned} I(u) = &\quad \Phi(u) + \psi(u) \\ = &\quad \frac{1}{p}\int_0^1 \left(|u''|^p + a\,|u'|^p + b\,|u|^p\right) dt - \int_0^1 F(t,u)\,dt + J(u). \end{aligned}$$

Theorem 3.2.1 *If $F : (0,1) \times \mathbb{R} \to \mathbb{R}$ satisfies (H_1) and $u \in W^{2,p}(0,1)$ is a critical point of functional I, then u is a solution of problem (3.1), (3.2).*

Proof. We adapt a previous device from [28], Proposition 3.2, to the present functional (see also [22], p. 2805). If we take $v = u + sw$, $s > 0$, in the inequality

$$\Phi^0(u; v-u) + \psi(v) - \psi(u) \geq 0, \quad \forall v \in W^{2,p}(0,1),$$

we easily get

$$\Phi^0(u; w) + \int_0^1 \left(|u''|^{p-2} u''w'' + |u'|^{p-2} u'w' + |u|^{p-2} uw\right) dt + J'(u; w) \geq 0, \quad (3.22)$$

for all $w \in W^{2,p}(0,1)$, where

$$J'(u; w) = j'\left((u(0), u(1), u'(0), u'(1))^T ; (w(0), w(1), w'(0), w'(1))^T\right) \quad (3.23)$$

is the directional derivative of the convex function J at u in the direction w. For $w \in C_0^\infty(0,1) \subset W^{2,p}(0,1)$, inequality (3.22) reads

$$\Phi^0(u; w) \geq -\int_0^1 \left(|u''|^{p-2} u''w'' + |u'|^{p-2} u'w' + |u|^{p-2} uw\right) dt. \quad (3.24)$$

Function $\Phi^0(u;\cdot)$ is subadditive and positively homogeneous, so by the Hahn-Banach theorem there exists a linear functional $l : W^{2,p}(0,1) \to \mathbb{R}$, such that

$$l(w) = -\int_0^1 \left(|u''|^{p-2} u''w'' + |u'|^{p-2} u'w' + |u|^{p-2} uw \right) dt,$$

and

$$\Phi^0(u;w) \geq l(w), \ \forall w \in W^{2,p}(0,1). \tag{3.25}$$

On the other hand, there exits a constant $k > 0$, such that

$$\Phi^0(u;w) \leq k \|w\|_{W^{2,p}(0,1)}, \quad \forall w \in W^{2,p}(0,1), \tag{3.26}$$

which, together with (3.25), yields

$$|l(w)| \leq k \|w\|_{W^{2,p}(0,1)}, \quad \forall w \in W^{2,p}(0,1),$$

showing that l is continuous and $l(w) = \langle l, w \rangle$. Now, from inequality (3.25), it follows that $l \in \overline{\partial}\Phi(u)$. Using Proposition 3.2.1, we deduce that there is some $u_l \in L^1(0,1)$ such that

$$u_l(t) \in \overline{\partial}F(t, u(t)), \quad \text{for a.a. } t \in (0,1), \tag{3.27}$$

and

$$\int_0^1 \left(|u''|^{p-2} u''w'' + a|u'|^{p-2} u'w' + b|u|^{p-2} uw - u_l w \right) dt = 0, \tag{3.28}$$

for all $w \in C_0^\infty(0,1)$. As $u \in W^{2,p}(0,1)$, we have $\left(|u''|^{p-2} u''\right)' \in W^{1,1}(0,1)$, i.e. $\left(|u''|^{p-2} u''\right)'$ is absolutely continuous and

$$\left(|u''(t)|^{p-2} u''(t)\right)'' - \left(a(t)|u'(t)|^{p-2} u'(t)\right)' + b(t)|u(t)|^{p-2} u(t) = u_l(t) \tag{3.29}$$

for a.a. $t \in (0,1)$. Now, from (3.27) it follows that (3.9) holds.

Next, we prove that u satisfies (3.2). We already know that $u'' \in C^2([0,1])$. The above inclusion relation (3.27) implies that

$$u_l(t)\,w(t) \le F^0(t, u(t); w(t)) \quad \text{for a.a.} \quad t \in (0,1), \quad \forall w \in W^{2,p}(0,1).$$

Then, by (3.29), we have

$$\begin{aligned}
&\int_0^1 \left(|u''|^{p-2} u''w'' + a\,|u'|^{p-2} u'w' + b\,|u|^{p-2} uw\right) dt \\
&\quad + \left(\left(|u''|^{p-2} u''\right)'(1) - a(1)\,|u'(1)|^{p-2} u'(1)\right) w(1) \\
&\quad - \left(\left(|u''|^{p-2} u''\right)'(0) - a(0)\,|u'(0)|^{p-2} u'(0)\right) w(0) \\
&\quad - |u''(1)|^{p-2} u''(1)\,w'(1) + |u''(0)|^{p-2} u''(0)\,w'(0) \\
&\le \int_0^1 F^0(t, u(t); w(t))\,dt,
\end{aligned}$$

for all $w \in W^{2,p}(0,1)$. Thus,

$$\Phi^0(u; w) \le \int_0^1 (-F)^0(t, u(t); w(t))\,dt + \langle \varphi'(u), w\rangle,$$

and from (3.22), we get

$$\begin{aligned}
&\int_0^1 (-F)^0(t, u(t); w(t))\,dt - \int_0^1 F^0(t, u(t); w(t))\,dt + J'(u; w) \qquad (3.30)\\
&\ge \left(\left(|u''|^{p-2} u''\right)'(1) - a(1)\,|u'(1)|^{p-2} u'(1)\right) w(1) \\
&\quad - \left(\left(|u''|^{p-2} u''\right)'(0) - a(0)\,|u'(0)|^{p-2} u'(0)\right) w(0) \\
&\quad - |u''(1)|^{p-2} u''(1)\,w'(1) + |u''(0)|^{p-2} u''(0)\,w'(0),
\end{aligned}$$

for all $w \in W^{2,p}(0,1)$. Now, let $x, y, z, q \in \mathbb{R}$ be arbitrarily chosen and, for each

$n \in \mathbb{N}$, let $w_n \in W^{2,p}(0,1)$ be defined by:

$$w_n := \begin{cases} x\omega_0(nt) + \frac{y}{n}\omega_1(nt), & \text{if} \quad t \in \left[0, \frac{1}{n}\right), \\ 0, & \text{if} \quad t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right], \\ z\omega_0(n(1-t)) - \frac{q}{n}\omega_1(n(1-t)), & \text{if} \quad t \in \left(1 - \frac{1}{n}, 1\right], \end{cases}$$

where $\omega_0(s)$ and $\omega_1(s)$ are such that $\omega_0(1) = \omega_1(1) = \omega_0'(1) = \omega_1'(1) = 0$, $\omega_0(0) = \omega_1'(0) = 1$ and $\omega_0'(0) = \omega_1(1) = 0$, e.g., $\omega_0(s) := (s-1)^2(2s+1)$ and $\omega_1(s) := s(s-1)^2$.

Then, $w_n(0) = x$, $w_n'(0) = y$, $w_n(1) = z$, and $w_n'(1) = q$. From hypothesis (H_1) there is some $\alpha_\rho \in L^1(0,1)$ such that

$$\left|F^0(t, u(t); \eta)\right| \le \alpha_\rho(t)|\eta|, \quad \forall \eta \in \mathbb{R}, \quad \text{for a.a. } t \in (0,1),$$

where $\rho > 0$ depends on the supremum norm $\|u\|_C$ of u. Taking $\eta = w_n(t)$, one obtains

$$\left|F^0(t, u(t); w_n(t))\right| \le \alpha_\rho(t) \max\left\{|x| + \frac{4|y|}{27n}, |z| + \frac{4|q|}{27n}\right\}, \tag{3.31}$$

for a.a. $t \in (0,1)$.

On the other hand,

$$F^0(t, u(t); w_n(t)) \to F^0(t, u(t); 0) = 0, \text{ for a.a. } t \in (0,1).$$

This together with (3.31) implies, by Lebesgue's dominated convergence theorem, that

$$\int_0^1 F^0(t, u(t); w_n(t))\, dt \to 0, \quad \text{as } n \to \infty. \tag{3.32}$$

Similarly, one has

$$\int_0^1 (-F)^0(t, u(t); w_n(t))\, dt \to 0, \quad \text{as } n \to \infty. \tag{3.33}$$

In (3.30) we take $w = w_n$ and let $n \to \infty$. Thus, by (3.23), (3.32) and (3.33), we derive

$$\begin{aligned} & j'\left((u(0), u(1), u'(0), u'(1))^T ; (x, z, y, q)^T \right) \\ \geq & \left(\left(|u''|^{p-2} u'' \right)' (1) - a(1) |u'(1)|^{p-2} u'(1) \right) z \\ & - \left(\left(|u''|^{p-2} u'' \right)' (0) - a(0) |u'(0)|^{p-2} u'(0) \right) x \\ & - |u''(1)|^{p-2} u''(1) q + |u''(0)|^{p-2} u''(0) y, \end{aligned}$$

which, as x, y, z, and q were arbitrarily chosen, implies that u satisfies (3.2). ■

Lemma 3.2.1 *We have $\lambda_1 > -\infty$, where λ_1 is the constant defined by (3.10). Moreover, if $\lambda_1 > 0$, then there exists a constant $m > 0$ such that*

$$\int_0^1 \left(|u''|^p + a |u'|^p + b |u|^p \right) dt \geq m \|u\|^p_{W^{2,p}(0,1)}, \quad \forall u \in \mathcal{D}.$$

Proof. First, there exists a constant K such that

$$\|u'\|^p_{L^p} \leq K \left(\varepsilon \|u''\|^p_{L^p} + \varepsilon^{-1} \|u\|^p_{L^p} \right), \quad \forall \varepsilon \in (0, 1). \tag{3.34}$$

Choose ε such that $K\varepsilon \leq |a|^{-1}_\infty$ where $|a|_\infty := \max_{t\in[0,1]} |a(t)|$. Let λ be such that $b(t) + \lambda \geq |a|_\infty K\varepsilon^{-1}$. Then,

$$\|u'\|^p_{L^p} \leq K \left(\varepsilon \|u''\|^p_{L^p} + \varepsilon^{-1} \|u\|^p_{L^p} \right) \leq |a|^{-1}_\infty \int_0^1 \left(|u''|^p + (b + \lambda) |u|^p \right) dt,$$

i.e.

$$\int_0^1 \left(|u''|^p + a |u'|^p + b |u|^p \right) dt \geq \int_0^1 \left(|u''|^p - |a|_\infty |u'|^p + b |u|^p \right) dt \geq -\lambda \|u\|^p_{L^p},$$

so $\lambda_1 \geq -\lambda > -\infty$.

Next, suppose that $\lambda_1 > 0$ holds. We will prove that there is some $\mu > 0$ such that

$$\int_0^1 \left(|u''|^p + a\,|u'|^p + b\,|u|^p\right)\,dt \geq \frac{\mu}{\mu+1}\left(\|u''\|_{L^p}^p + \|u'\|_{L^p}^p + \|u\|_{L^p}^p\right), \tag{3.35}$$

for all $u \in \mathcal{D}$. The inequality (3.35) is equivalent to

$$\int_0^1 \left(|u''|^p + ((a-1)\,\mu + a)\,|u'|^p + ((b-1)\,\mu + b)\,|u|^p\right)\,dt \geq 0.$$

We will prove that there is $\mu > 0$ such that the following stronger inequality

$$\int_0^1 \left(|u''|^p + (-(|a|_\infty + 1)\,\mu + a)\,|u'|^p + (-(|b|_\infty + 1)\,\mu + b)\,|u|^p\right)\,dt \geq 0$$

holds for every $u \in \mathcal{D}$. Let K be as above and $\varepsilon,\ 0 < \varepsilon < 1$ be such that $K\varepsilon\,|a|_\infty < 1$. We denote by $\delta,\ 0 < \delta < 1$, a number that will be chosen later. We have

$$\begin{aligned}
&\textstyle\int_0^1 (|u''|^p + (-(|a|_\infty + 1)\,\mu + a)\,|u'|^p + (-(|b|_\infty + 1)\,\mu + b)\,|u|^p)\,\,dt \\
&\geq (1-\delta)\textstyle\int_0^1 (|u''|^p + a\,|u'|^p + b\,|u|^p)\,\,dt - (|b|_\infty\,\delta + (|b|_\infty + 1)\,\mu)\textstyle\int_0^1 |u|^p\,\,dt \\
&\quad +\delta\textstyle\int_0^1 |u''|^p\,\,dt - (|a|_\infty\,\delta + (|a|_\infty + 1)\,\mu)\textstyle\int_0^1 |u'|^p\,\,dt \\
&\geq ((1-\delta)\,\lambda_1 - (|b|_\infty\,\delta + (|b|_\infty + 1)\,\mu))\textstyle\int_0^1 |u|^p\,\,dt + \delta\textstyle\int_0^1 |u''|^p\,\,dt \\
&\quad - (|a|_\infty\,\delta + (|a|_\infty + 1)\,\mu)\textstyle\int_0^1 |u'|^p\,\,dt
\end{aligned}$$

for every $u \in \mathcal{D}$. We look for μ and δ such that the inequalities

$$\delta \geq (|a|_\infty\,\delta + (|a|_\infty + 1)\,\mu)\,K\varepsilon, \tag{3.36}$$

$$\begin{aligned}
A := (1-\delta)\,\lambda_1 - (|b|_\infty\,\delta + (|b|_\infty + 1)\,\mu) \\
\geq (|a|_\infty\,\delta + (|a|_\infty + 1)\,\mu)\,K\varepsilon^{-1}
\end{aligned} \tag{3.37}$$

are satisfied. The former inequality is equivalent to

$$\frac{1 - K\varepsilon\,|a|_\infty}{(|a|_\infty + 1)\,K\varepsilon}\delta \geq \mu \tag{3.38}$$

and the later one is equivalent to

$$\lambda_1 \geq \left(\lambda_1 + |b|_\infty + |a|_\infty K\varepsilon^{-1}\right)\delta + \left(1 + |b|_\infty + (|a|_\infty + 1) K\varepsilon^{-1}\right)\mu. \tag{3.39}$$

Obviously, there are μ and δ such that (3.38) and (3.39) hold as well as the inequalities (3.36) and (3.37). Therefore,

$$\begin{aligned}
&\int_0^1 \left(|u''|^p + \left(-\left(|a|_\infty + 1\right)\mu + a\right)|u'|^p + \left(-\left(|b|_\infty + 1\right)\mu + b\right)|u|^p\right)\ dt \\
&\qquad \geq A\int_0^1 |u|^p\ dt + \delta\int_0^1 |u''|^p\ dt \\
&\qquad\quad - \left(|a|_\infty \delta + \left(|a|_\infty + 1\right)\mu\right)\int_0^1 |u'|^p\ dt \\
&\qquad \geq A\int_0^1 |u|^p\ dt + \delta\int_0^1 |u''|^p\ dt \\
&\qquad\quad - \left(|a|_\infty \delta + \left(|a|_\infty + 1\right)\mu\right)\left(K\varepsilon\int_0^1 |u''|^p\ dt + K\varepsilon^{-1}\int_0^1 |u|^p\ dt\right) \geq 0.
\end{aligned}$$

■

Lemma 3.2.2 *Assume that (3.15) holds. Then* $\overline{\lambda}_1 < +\infty$.

Proof. A standard computation shows that

$$s^{-p} j(sz) \leq j(z) + \frac{k}{p}\left(1 - s^{-p}\right),\ \forall z \in D(j),\ \forall s \geq 1,$$

which yields

$$\begin{aligned}
\overline{\lambda}_1 \ \leq\ & \inf\Big\{\|u''\|_{L^p}^p + |a|_\infty \|u'\|_{L^p}^p + |b|_\infty \|u\|_{L^p}^p + pJ(u) : u \in W^{2,p}(0,1) \\
& \|u\|_{L^p}^p = 1,\quad \left(u(0), u(1), u'(0), u'(1)\right)^T \in D(j)\Big\} + k.
\end{aligned}$$

■

Lemma 3.2.3 *Assume* (H_1) *holds,* $\lambda_1 > 0$*, and either* (G_θ) *or* (G_p) *holds. If, in addition,* $D(j)$ *is closed, then functional* I *satisfies the Palais-Smale condition.*

Proof. First, we prove that each Palais-Smale sequence is bounded. Let $\{u_n\}$ be such a sequence. Then, there exists a constant C such that

$$C \geq \frac{1}{p} \int_0^1 \left(|u_n''|^p + a\,|u_n'|^p + b\,|u_n|^p\right) dt + J(u_n) + \Phi(u_n) - \varphi(u_n). \tag{3.40}$$

On the other hand, setting $v = (1+s)\,u_n$, $s > 0$, in (1.1) and taking the limit as $s \to 0^+$, we obtain

$$\Phi^0(u_n; u_n) + \psi'(u_n; u_n) \geq -\varepsilon_n \|u_n\|.$$

This inequality reads

$$\Phi^0(u_n; u_n) - \langle \varphi'(u_n), u_n\rangle + \int_0^1 \left(|u_n''|^p + a\,|u_n'|^p + b\,|u_n|^p\right) dt$$
$$+ J'(u_n; u_n) \geq -\varepsilon_n \|u_n\|. \tag{3.41}$$

We will examine separately the cases when (G_p) and (G_θ) hold.

Case 1. Let (G_θ) hold for some $\theta > p$. We will prove that

$$\theta(\Phi(u) - \varphi(u)) \geq \Phi^0(u; u) - \langle \varphi'(u), u\rangle - m_1, \quad \forall u \in W^{2,p}(0,1), \tag{3.42}$$

and

$$\theta J(u) \geq J'(u; u) - m_2, \quad \forall u \in W^{2,p}(0,1), \tag{3.43}$$

for some positive constants m_1 and m_2. Indeed, (3.43) follows from (3.13) and the definition of functional J. Now, let $l \in \overline{\partial}\Phi(u)$. By Proposition 3.2.1, there exists some $u_l \in L^1$, such that $u_l(t) \in \overline{\partial}F(t, u(t))$ for a.a. $t \in (0,1)$, and

$$\langle l, v\rangle = \int_0^1 \Big(-u_l(t)\,v(t) + (a(t)-1)\,|u'(t)|^{p-2}\,u'(t)\,v'(t)$$
$$+ (b(t)-1)\,|u(t)|^{p-2}\,u(t)\,v(t)\Big)\,dt, \quad \forall v \in W^{2,p}(0,1).$$

Hypothesis (H_1) implies that given $M>0$ there exists an $\alpha_M(t)\in L^1$ such that for each $x\in\mathbb{R}$, with $|x|\leq M$, the inequalities

$$|\xi|\leq\alpha_M(t)\,,\quad \forall\xi\in\overline{\partial}F(t,x)\,,$$

and

$$|F(t,x)|\leq M\alpha_M(t)\,,$$

are satisfied. Hence,

$$\begin{aligned}\int_0^1 u(t)\,u_l(t)\,dt &= \int_{|u(t)|>M} u(t)\,u_l(t)\,dt+\int_{|u(t)|\leq M} u(t)\,u_l(t)\,dt\\ &\geq \int_{|u(t)|>M}\theta F(t,u(t))\,dt-M\int_0^1\alpha_M(t)\,dt\\ &= \theta\left(\int_0^1 F(t,u(t))\,dt-\int_{|u(t)|\leq M}F(t,u(t))\,dt\right)\\ -M\int_0^1\alpha_M(t)\,dt &\geq \theta\int_0^1 F(t,u(t))\,dt-M(1+\theta)\int_0^1\alpha_M(t)\,dt,\end{aligned}$$

which yields

$$-\langle l,u\rangle+\langle\varphi'(u)\,,u\rangle\geq\theta(-\Phi(u)+\varphi(u))-M(1+\theta)\int_0^1\alpha_M(t)\,dt,$$

i.e.,

$$\theta(\Phi(u)-\varphi(u))\geq\langle l,u\rangle-\langle\varphi'(u)\,,u\rangle-m_1,\quad\forall l\in\overline{\partial}\Phi(u)\,,$$

where $m_1=M(1+\theta)\int_0^1\alpha_M(t)\,dt>0$. It follows that

$$\begin{aligned}\theta(\Phi(u)-\varphi(u)) &\geq \max\left\{\langle l,v\rangle : l\in\overline{\partial}\Phi(u)\right\}-\langle\varphi'(u)\,,u\rangle-m_1\\ &= \Phi^0(u;u)-\langle\varphi'(u)\,,u\rangle-m_1,\end{aligned}$$

which yields (3.42).

Now, multiplying (3.40) by $-\theta$ and adding (3.41) to (3.42) and (3.43), we find

$$\theta C + m_1 + m_2 \geq \left(\frac{\theta}{p} - 1\right) \int_0^1 \left(|u_n''|^p + a\,|u_n'|^p + b\,|u_n|^p\right) dt - \varepsilon_n \|u_n\| .$$

On the other hand, by the hypothesis $\lambda_1 > 0$ and by Lemma 3.2.1, there exists a constant $m_3 > 0$ such that

$$\theta C + m_1 + m_2 \geq m_3 \|u_n\|^p - \varepsilon_n \|u_n\| ,$$

which implies that $\{u_n\}$ is bounded.

Case 2. Let (G_p) hold. We will prove that there exists a constant $k_1 > 0$, and, given $\rho > 0$ there exists a constant $m_1 = m_1(\rho) > 0$, such that for each $u \in W^{2,p}(0,1)$, with $\|u\| \geq \rho$, the inequality

$$\left(p + \frac{k_1}{\|u\|^{p-1}}\right)(\Phi(u) - \varphi(u)) \geq \Phi^0(u;u) - \langle \varphi'(u), u\rangle - m_1 \tag{3.44}$$

holds. Let $l \in \overline{\partial}\Phi(u)$ and $u_l \in L^1$ be defined as in Case 1. Then, we have

$$\begin{aligned}
\int_0^1 u(t)\,u_l(t)\,dt &= \int_{\{|u(t)|>M\}} u(t)\,u_l(t)\,dt + \int_{\{|u(t)|\leq M\}} u(t)\,u_l(t)\,dt \\
&\geq \int_{\{|u(t)|>M\}} \left(p + \frac{c}{|u(t)|^{p-1}}\right) F(t,u(t)) - M \int_0^1 \alpha_M(t)\,dt \\
&\geq \left(p + \frac{c}{d\,\|u\|^{p-1}}\right) \int_{\{|u(t)|>M\}} F(t,u(t)) - M \int_0^1 \alpha_M(t)\,dt \\
&= \left(p + \frac{c}{d\,\|u\|^{p-1}}\right) \left(\int_0^1 F(t,u(t)) - \int_{\{|u(t)|\leq M\}} F(t,u(t))\right) \\
&\quad -M \int_0^1 \alpha_M(t)\,dt,
\end{aligned}$$

where we have used the obvious inequality

$$|u(t)| \leq d \|u\|_{W^{2,p}(0,1)}$$

for some positive constant d. Now, the above inequalities yield that

$$\int_0^1 u(t)\, u_l(t)\, dt \geq \left(p + \frac{c}{d\|u\|^{p-1}}\right) \int_0^1 F(t, u(t)) - M\left(p + 1 + \frac{c}{d\rho^{p-1}}\right) \int_0^1 \alpha_M(t)\, dt.$$

Similarly as in Case 1, we obtain that (3.44) holds with $k_1 = c/d$, and

$$m_1 = M\left(p + 1 + \frac{c}{d\rho^{p-1}}\right) \int_0^1 \alpha_M(t)\, dt.$$

Next, the inequalities (3.15) and (3.47) imply

$$\left(p + \frac{k_1}{\|u\|^{p-1}}\right) J(u) \geq J'(u;u) - k + \frac{k_1}{\|u\|^{p-1}} J(u) \geq J'(u;u) - k - k_1 \frac{c_2\|u\| + c_1}{\|u\|^{p-1}}, \quad (3.45)$$

for all $u \in W^{2,p}(0,1)$, such that $\|u\| \geq 1$.

Now, suppose that $\{u_n\}$ is unbounded. We may assume that $\|u_n\| \geq 1$ for all n. Then, applying (3.44) with $\rho = 1$ and (3.45) to (3.40) and (3.41), we get

$$pC + \frac{\tilde{C}}{\|u_n\|^{p-1}} \geq \frac{k_1}{p\|u_n\|^{p-1}} \int_0^1 \left(|u_n''|^p + a|u_n'|^p + b|u_n|^p\right) dt - \overline{m}_1 - k_1 c_2 \|u_n\|^{2-p} - \varepsilon_n \|u_n\|, \quad (3.46)$$

for some constants $\tilde{C} > 0$ and $\overline{m}_1 > 0$. Now, by $\lambda_1 > 0$ and Lemma 3.2.1, inequality (3.46) implies

$$pC + \tilde{C} + \overline{m}_1 \geq \left(\frac{k_1 m}{p} - \frac{k_1 c_2}{\|u_n\|^{p-1}} - \varepsilon_n\right) \|u_n\|,$$

a contradiction since $\varepsilon_n \to 0$ and $\|u_n\| \to \infty$. Therefore $\{u_n\}$ is bounded.

Next, let $\{u_n\}$ again be a Palais-Smale sequence. By the boundedness of $\{u_n\}$, under each of the hypotheses (G_θ) and (G_p), there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, and $u \in W^{2,p}(0,1)$, such that $u_n \rightharpoonup u$ in $W^{2,p}(0,1)$. Thus, $u_n \to u$ (for a subsequence) strongly in $C^1([0,1])$ and, as $D(j)$ is convex and closed, $u \in D(J)$. From (1.1) we derive that

$$\begin{aligned}
&\Phi^0(u_n; u-u_n) + J'(u_n; u-u_n) + \varepsilon_n \|u-u_n\| \\
&\qquad \geq -\int_0^1 \left(|u_n''|^{p-2} u_n'' u'' + |u_n'|^{p-2} u_n' u' + |u_n|^{p-2} u_n u\right) dt + \|u_n\|^p \\
&\qquad \geq \|u_n\|^{p-1} (\|u_n\| - \|u\|),
\end{aligned}$$

yielding that

$$\begin{aligned}
&\Phi^0(u_n; u-u_n) + J'(u_n; u-u_n) + \varepsilon_n \|u-u_n\| - \|u\|^{p-1}(\|u_n\| - \|u\|) \\
&\qquad \geq \left(\|u_n\|^{p-1} - \|u\|^{p-1}\right)(\|u_n\| - \|u\|).
\end{aligned}$$

The functional Φ has the trivial extension $\widetilde{\Phi}$ on the space $C^1([0,1])$ defined by (3.18). Moreover, $\widetilde{\Phi}$ is locally Lipschitz functional and obviously

$$\Phi^0(v; w) = \widetilde{\Phi}^0(v; w), \quad \forall v, w \in W^{2,p}(0,1).$$

The upper semicontinuity of $\widetilde{\Phi}^0(\cdot;\cdot)$ yields

$$\limsup_{n\to\infty} \Phi^0(u_n; u-u_n) \leq \widetilde{\Phi}^0(u; 0) = 0.$$

On the other hand,

$$\begin{aligned}
\limsup_{n\to\infty} J'(u_n; u-u_n) &\leq \limsup_{n\to\infty} (J(u) - J(u_n)) \\
&= J(u) - \liminf_{n\to\infty} J(u_n) \leq 0.
\end{aligned}$$

Hence

$$0 \geq \limsup_{n\to\infty} \left(\|u_n\|^{p-1} - \|u\|^{p-1} \right) \left(\|u_n\| - \|u\| \right)$$

i.e., $\|u_n\| \to \|u\|$. Since $(W^{2,p}(0,1), \|\cdot\|)$ is uniformly convex, $u_n \to u$ strongly in $W^{2,p}(0,1)$. Thus, I satisfies the Palais-Smale condition, as claimed. ■

3.3 Proof of the main results

Proof of Theorem 3.1.1. First, we will prove that functional I is coercive. Suppose that, on the contrary, there exist a constant $C > 0$ and a sequence $\{u_n\} \subset W^{2,p}(0,1)$, such that $\|u_n\|_{W^{2,p}(0,1)} \to \infty$ and $I(u_n) \leq C$. Denote $v_n := \frac{u_n}{\|u_n\|}$, where $\|\cdot\|$ is the norm in $W^{2,p}(0,1)$. Then, $u_n, v_n \in D(J)$. Moreover, $\|v_n\| = 1$ implies that there exists a subsequence of $\{v_n\}$ (denoted again by $\{v_n\}$) and $v \in D(J)$, such that $v_n \rightharpoonup v$ in $W^{2,p}(0,1)$. Therefore, $v_n \to v$ strongly in $C^1([0,1])$.

Since J is proper, convex and l.s.c., it is bounded from below by an affine functional. Therefore, there are some constants $c_1 > 0$ and $c_2 > 0$, such that

$$J(u) \geq -c_1 - c_2 \|u\|. \tag{3.47}$$

So, we have

$$\begin{aligned} C &\geq \frac{1}{p} \int_0^1 \left(|u_n''|^p + a|u_n'|^p + b|u_n|^p \right) dt + J(u_n) + \Phi(u_n) - \varphi(u_n) \\ &\geq \frac{1}{p} \int_0^1 \left(|u_n''|^p + a|u_n'|^p + b|u_n|^p \right) dt + \Phi(u_n) - \varphi(u_n) - c_1 - c_2 \|u_n\|, \end{aligned}$$

which implies

$$\frac{C + c_1}{\|u_n\|^p} + \frac{c_2}{\|u_n\|^{p-1}} \geq \frac{1}{p} \int_0^1 \left(|v_n''|^p + a|v_n'|^p + b|v_n|^p \right) dt + \frac{\Phi(u_n) - \varphi(u_n)}{\|u_n\|^p}. \tag{3.48}$$

If (3.12) holds, then there are constants $\sigma > 0$ and $\rho > 0$ such that

$$F(t,x) \le \frac{\lambda_1 - \sigma}{p} |x|^p \text{ for a.a. } t \in (0,1),$$

and for all x with $|x| > \rho$. Hence,

$$F(t,x) \le \rho \alpha_\rho(t) + \left| \frac{\lambda_1 - \sigma}{p} \right| \rho^p + \frac{\lambda_1 - \sigma}{p} |x|^p \quad \text{for a.a. } t \in (0,1), \ x \in \mathbb{R},$$

which gives

$$\Phi(u) - \varphi(u) = -\int_0^1 F(t,u)\,dt \ge -k - \frac{\lambda_1 - \sigma}{p} \int_0^1 |u|^p\,dt, \ \forall u \in W^{2,p}(0,1),$$

so by (3.48) it follows

$$\begin{aligned} \frac{C + c_1 + k}{\|u_n\|^p} + \frac{c_2}{\|u_n\|^{p-1}} &\ge \frac{1}{p} \int_0^1 \left(|v_n''|^p + a |v_n'|^p + (b - \lambda_1 + \sigma) |v_n|^p \right) dt \\ &\ge \frac{\sigma}{p} \int_0^1 |v_n|^p\,dt. \quad (3.49) \end{aligned}$$

Therefore, $v_n \to v = 0$ in $C^1([0,1])$, and then

$$\|v_n''\|_{L^p}^p = \|v_n\|_{W^{2,p}}^p - \|v_n'\|_{L^p}^p - \|v_n\|_{L^p}^p \to 1, \quad \text{as } n \to \infty.$$

This implies that

$$\frac{C + c_1 + k}{\|u_n\|^p} + \frac{c_2}{\|u_n\|^{p-1}} \ \ge \ \frac{1}{p} \int_0^1 \left(|v_n''|^p + a |v_n'|^p + (b - \lambda_1 + \sigma) |v_n|^p \right) dt \ \to \ \frac{1}{p}$$

a contradiction.

Finally, by the compactness of the imbedding $W^{2,p}(0,1) \subset C^1([0,1])$, functional Φ is sequentially weakly continuous, and ψ is weakly lower semicontinuous. Hence, I is sequentially weakly lower semicontinuous and its coercivity implies that it is bounded from below and attains its infimum. So I has a critical point, which by

Theorem 3.2.1 is a solution of problem (3.1)-(3.2). ■

Proof of Theorem 3.1.2. We will apply Theorem 1.1. First of all, according to Lemma 3.2.3, I satisfies the Palais-Smale condition.

Next, we can assume without any loss of generality that $j\left((0,0,0,0)^T\right) = 0$. Since $(0,0,0,0)^T \in \partial j\left((0,0,0,0)^T\right)$, we have $J(u) \geq J(0) = 0$ and so in particular $I(0) = 0$. Now, we will prove that there exist $\rho > 0$ and $\alpha(\rho) > 0$ such that $I(u) \geq \alpha$ for $\forall u \in W^{2,p}(0,1)$ with $\|u\| = \rho$, where $\|\cdot\|$ denotes the norm of $W^{2,p}(0,1)$. Indeed, if $\rho = \|u\|$ is small enough, then by $|u(t)| \leq d\|u\|$ for some constant d, and by

$$F(t,x) \leq \frac{\lambda_1 - \sigma}{p}|x|^p, \qquad \forall |x| \leq \delta,$$

with some constants $\sigma > 0$ and $\delta > 0$, we have

$$\begin{aligned} I(u) &= \frac{1}{p}\int_0^1 \left(|u''|^p + a|u'|^p + b|u|^p\right) dt - \int_0^1 F(t,u)\,dt + J(u) \\ &\geq \frac{1}{p}\int_0^1 \left(|u''|^p + a|u'|^p + b|u|^p\right) dt - \frac{\lambda_1 - \sigma}{p}\int_0^1 |u|^p\,dt \geq m\|u\|^p, \end{aligned}$$

with some constant $m > 0$. Here, we have used the inequality

$$\int_0^1 \left(|u''|^p + a|u'|^p + (b - \lambda_1 + \sigma)|u|^p\right) dt \geq \sigma \int_0^1 |u|^p\,dt,$$

as well as Lemma 3.2.1 with b replaced with $b - \lambda_1 + \sigma$.

Finally, we have to find an $e \in W^{2,p}(0,1)$ such that

$$I(e) \leq 0 \qquad (3.50)$$

and

$$\|e\| > \rho. \qquad (3.51)$$

In what follows, we will examine separately the two alternative cases of the theorem, which we denote by (a) and (b).

$(a):\ (G_\theta)$ holds.

The mapping $s \longmapsto s^{-\theta}F(t,sx)$ is locally Lipschitz for a.a. $t \in (0,1)$, so we have for each $s>0$

$$\begin{aligned}\overline{\partial}_s\left(s^{-\theta}F(t,sx)\right) &\subset \overline{\partial}_s\left(s^{-\theta}\right)F(t,sx)+s^{-\theta}\overline{\partial}_s\left(F(t,sx)\right)\\ &= s^{-\theta-1}\left(-\theta F(t,sx)+sx\overline{\partial}F(t,sx)\right).\end{aligned}$$

Let $|x|>M$, where M is the constant which appear in the statement of the theorem. Given $1\le r<s$, by Lebourg's mean value theorem and assumption (3.14), there exist $\tau\in(r,s)$ and $\xi\in\overline{\partial}_s\left(s^{-\theta}F(t,sx)\right)\big|_{s=\tau}$, $\xi\ge 0$, such that

$$s^{-\theta}F(t,sx)-r^{-\theta}F(t,rx)=\xi(s-r)\ge 0,$$

i.e.,

$$F(t,sx)\ge s^{\theta}F(t,x),\quad \text{for a.a. } t\in[0,1],\quad \forall\,|x|>M,\quad s\ge 1.$$

Now, let $h\in C_0^\infty(0,1)$ be such that $|h|>M$ on a set with positive measure. Then,

$$\begin{aligned}\int_0^1 F(t,sh)\,dt &= \int_{\{|sh|>M\}} F(t,sh)\,dt+\int_{\{|sh|\le M\}} F(t,sh)\,dt\\ &\ge \int_{\{|h|>M\}} F(t,sh)\,dt - M\int_0^1 \alpha_M(t)\,dt\\ &\ge s^{\theta}\int_{\{|h|>M\}} F(t,h)\,dt - M\int_0^1 \alpha_M(t)\,dt,\end{aligned}$$

for all $s \geq 1$. We have $J(sh) = 0$ for each s, thus

$$I(sh) = \frac{s^p}{p} \int_0^1 \left(|h''|^p + a|h'|^p + b|h|^p \right) dt - \int_0^1 F(t, sh)\, dt$$

$$\leq \frac{s^p}{p} \int_0^1 \left(|h''|^p + a|h'|^p + b|h|^p \right) dt - s^\theta \int_{\{|h|>M\}} F(t, h)\, dt + M \int_0^1 \alpha_M(t)\, dt$$

for all $s \geq 1$, i.e.,

$$I(sh) \leq s^p k_1 - s^\theta k_2 + k_3 \to -\infty, \quad \text{as } s \to \infty,$$

with $k_1, k_2, k_3 > 0$. Therefore, we can choose s_0 sufficiently large such that $I(s_0 h) \leq 0$ and $\|s_0 h\| > \rho$. Then $e := s_0 h$ satisfies conditions (3.50) and (3.51).

(b) : (G_p) and $\left(\overline{L}_\infty\right)$ hold.

Let $u_n \in \mathcal{D}$ and $s_n > 0$ be such that $\|u_n\|_{L^p} = 1$, $s_n \to \infty$,

$$\int_0^1 \left(|u_n''|^p + a|u_n'|^p + b|u_n|^p \right) dt + \frac{pJ(s_n u_n)}{s_n^p} \to \overline{\lambda}_1.$$

The inequality (3.17) implies that there exist constants $C > 0$ and $\sigma > 0$ such that

$$F(t, x) \geq \frac{\overline{\lambda}_1 + \sigma}{p} |x|^p, \quad \forall |x| > C, \qquad \text{a.a. } t \in (0, 1).$$

We have

$$\int_0^1 F(t, s_n u_n(t))\,dt = \int_{\{|s_n u_n|>C\}} F(t, s_n u_n)\,dt + \int_{\{|s_n u_n|\le C\}} F(t, s_n u_n)\,dt$$
$$\ge s_n^p \frac{\overline{\lambda}_1+\sigma}{p} \int_{\{|s_n u_n|>C\}} |u_n|^p\,dt - C\int_0^1 \alpha_C(t)\,dt$$
$$= s_n^p \frac{\overline{\lambda}_1+\sigma}{p}\left(\int_0^1 |u_n|^p\,dt - \int_{\{|s_n u_n|\le C\}} |u_n|^p\,dt\right)$$
$$-C\int_0^1 \alpha_C(t)\,dt \ge s_n^p \frac{\overline{\lambda}_1+\sigma}{p} - \left|\frac{\overline{\lambda}_1+\sigma}{p}\right| C^p - C\int_0^1 \alpha_C(t)\,dt,$$

hence

$$I(s_n u_n) \le \frac{s_n^p}{p}\int_0^1 \left(|u_n''|^p + a|u_n'|^p + b|u_n|^p\right)dt + J(s_n u_n)$$
$$-s_n^p \frac{\overline{\lambda}_1+\sigma}{p} - \left|\frac{\overline{\lambda}_1+\sigma}{p}\right| C^p - C\int_0^1 \alpha_C(t)\,dt,$$

i.e.,

$$\frac{I(s_n u_n)}{s_n^p} \le \frac{1}{p}\left(\int_0^1 \left(|u_n''|^p + a|u_n'|^p + b|u_n|^p\right)dt + p\frac{J(s_n u_n)}{s_n^p}\right)$$
$$-\frac{\left(\overline{\lambda}_1+\sigma\right)}{p} - \frac{C_1}{s_n^p},$$

which converges to $-\sigma/p$ as $n \to \infty$. Therefore, one can choose some n such that $I(s_n u_n) < 0$ and $\|s_n u_n\| > \rho$. Obviously, $e := s_n u_n$ satisfies (3.50) and (3.51). ■

3.4 An example

Consider the boundary value problem

$$u^{iv} = F'(u), \tag{3.52}$$

$$u(0) = u'(0) = u(1) = u'(1) = 0, \tag{3.53}$$

where $F(x) = \frac{c}{2} x^2 e^{-\frac{1}{|x|}}$, $c > 0$. It is easily seen that

$$F'(x) = c\left(x + \frac{\text{sign}x}{2}\right) e^{-\frac{1}{|x|}}$$

and we can choose $j\left((x_1, x_2, x_3, x_4)^T\right) = 0$, if $x_1 = x_2 = x_3 = x_4 = 0$, and $= +\infty$, otherwise.

Obviously, functions j and F satisfy assumptions (3.15), (3.16), and

$$\lim_{|x|\to\infty} \frac{F(x)}{x^2} = \frac{c}{2}.$$

Note that $\lambda_1 = \overline{\lambda}_1 > 0$. In fact, λ_1 is the first eigenvalue of the clamped beam operator.

If $c \le \lambda_1$, then $\forall u \in D(J)$,

$$I(u) = \frac{1}{2}\left(\int_0^1 u''^2 dt - c\int_0^1 u^2 e^{-\frac{1}{|u|}} dt\right),$$

and

$$\begin{aligned} 0 &= I'(u;u), \\ &= \int_0^1 \left(u''^2 - \lambda_1 u^2\right) dt \\ &\quad + \lambda_1 \int_0^1 u^2 dt - c\int_0^1 \left(u^2 + \frac{|u|}{2}\right) e^{-\frac{1}{|u|}} dt \\ &\ge c\int_0^1 u^2 \left(1 - \left(1 + \frac{1}{2|u|}\right) e^{-\frac{1}{|u|}}\right) dt \ge 0, \end{aligned}$$

where $I'(u;u)$ is the directional derivative of I at u in the direction u. It follows that problem (3.52)-(3.53) has only the null solution.

If $c > \lambda_1$, then Theorem 3.1.2 guarantees the existence of at least one nonzero solution.

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