# ON SMALL GAPS BETWEEN PRIMES

by

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Submitted to

Central European University

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In partial fulfillment of the requirements for the degree of Doctor of Philosophy

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Budapest, Hungary

2015

# Contents

# Abstract

This work gives an account and applications of recently developed methods in the investigation of small gaps between prime numbers that extended the state of the art, namely the Goldston-Pintz-Yıldırım method and the Maynard-Tao method. We give brief expositions of the ideas involved in each in Chapters 2 and 4 respectively, and apply the Goldston-Pintz-Yıldırım method to prime elements in totally real number fields in Chapter 3 and the Maynard-Tao method to the problem of finding uniform gaps between primes in arithmetic progressions over a range of moduli in chapters 5 and 6.

## Acknowledgements

I'd first and foremost like to express my deepest gratitude for my supervisor János Pintz for accepting me as a student, his guidance, mentoring and support.

I am indebted to my second supervisor Gergely Harcos for looking out for me when I first arrived in Budapest, his instruction and encouragement.

I am grateful for the ongoing guidance and influence of my teacher Cem Yalçın Yıldırım.

I thank Selin Çağatay for her invaluable friendship over the years.

## Remark on notation

We use $c$ and $C$ to denote constants which need not be the same in each instance, so we will freely write things like $(\log x)^C \sum_{n \leq x} n^{-1} \ll (\log x)^C$. If we need to track them, we will employ subscripts or superscripts.

CHAPTER 1

# Introduction

In the last decade, the study of small gaps between prime numbers has seen breakthrough results owing to the developments of new methods. The present work aims to give some applications of these methods.

An outstanding problem in the study of the distribution of primes is the twin prime conjecture, to the effect that there are infinitely many prime pairs $p_n$, $p_{n+1}$ (with $p_n$ denoting the $n$-th prime) such that $p_{n+1} - p_n = 2$. Though probably dating back to antiquity, the assertion first appears in print in a work of de Polignac from 1849 [25], in a generalized form:

POLIGNAC'S CONJECTURE. *Every even number can be expressed in an infinitude of ways as the difference of two consecutive primes.*

Such a number, of course, has to be even, since one of any two numbers with odd difference has to be even, and can't be a prime unless it's exactly 2, which occurs only once. Generalising this trivial obstruction to more than two numbers, one is led to make the following definition.

DEFINITION. *A set $\mathcal{H} = \{h_1, \ldots, h_k\}$ of $k$ integers is called an admissible $k$-tuple if $\mathcal{H}$ does not cover all residue classes* $(\mathrm{mod}\ p)$ *for any prime $p$.*

Clearly, this condition needs to be checked only for primes up to $k$. With this definition Dickson conjectured in 1904 [5] the following generalization of Polignac's conjecture.

DICKSON'S CONJECTURE. *If a $k$-tuple $\mathcal{H} = \{h_1, \ldots, h_k\}$ is admissible, then there are infinitely many integers $n$ such that the numbers $n + h_i$, $i = 1, \ldots, k$, are simultaneously prime.*

Hardy and Littlewood refined this to conjecture an asymptotic formula for the number of such prime tuples up to $x$. For an admissible $k$-tuple $\mathcal{H}$ and a given prime $p$, write $\nu_{\mathcal{H}}(p)$ to denote the number of residue classes (mod $p$) occupied by elements of $\mathcal{H}$. Define the singular series

$$\mathfrak{S}(\mathcal{H}) = \prod_p \left(1 - \frac{\nu_{\mathcal{H}}(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-k}. \tag{1.1}$$

With these Hardy and Littlewood [13] made the following conjecture.

Hardy-Littlewood prime tuple Conjecture. *Let $\mathcal{H}$ be an admissible $k$-tuple. Then*

$$\#\{n \le x : n + h_i \text{ is prime for all } h_i \in \mathcal{H}\} \sim \mathfrak{S}(\mathcal{H}) \int_2^x \frac{du}{(\log u)^k}. \tag{1.2}$$

Even the weakest of these assertions remained virtually unassailable for a long time, so efforts were directed towards obtaining suitably weakened forms of such statements. Since we are looking for primes that are close, failing to obtain a result of the form

$$\liminf_{n\to\infty} (p_{n+1} - p_n) \le c \tag{1.3}$$

for some absolute constant $c$ (this assertion was called the Bounded Gap Conjecture, and is now a theorem), the natural thing to do is to consider, with a slowly increasing function $g(n)$, the quantity

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{g(p_n)} \le c. \tag{1.4}$$

The first natural $g(n)$ to consider is the average distance between primes around $n$, which by the Prime Number Theorem is $\log n$, so that we trivially have

$$\Delta := \liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} \le 1. \tag{1.5}$$

Thus the investigation of small gaps between primes was launched in search of results of the form $\Delta \le c < 1$. The first such result is due to Hardy and Littlewood [14] who in 1926 showed $\Delta \le 2/3$ conditionally on the Generalized Riemann Hypothesis. Again on GRH, Rankin [28] established $\Delta \le 3/5$

in 1940. That same year, Erdős [4] provided the first unconditional result $\Delta \leq 1 - c$, with an unspecified but effective constant $c$. Bombieri and Davenport [1] made a breakthrough in 1966 by substituting the Bombieri-Vinogradov Theorem for GRH and obtained $\Delta \leq (2 + \sqrt{3})/8 = 0.4665\ldots$ . The next two decades saw only small incremental improvements over this result. Finally, Maier [19] in 1988 used his matrix method to improve Huxley's [17] estimate $\Delta \leq 0.4425\ldots$ by a factor $e^{-\gamma}$, where $\gamma$ is Euler's constant, to obtain $\Delta \leq 0.2484\ldots$ .

In 2005, D. A. Goldston, J. Pintz and C. Y. Yıldırım made the breakthrough of settling the so-called Small Gap Conjecture when they proved that $\Delta = 0$ [11]. They were subsequently able to refine this [12] to the even stronger assertion

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{(\log p_n)^{1/2}(\log\log p_n)^2} < \infty. \tag{1.6}$$

Conditionally, assuming the Elliott-Halberstam conjecture, the method actually yields the Boounded Gap Conjecture in the form

$$\liminf_{n\to\infty} (p_{n+1} - p_n) \leq 16. \tag{1.7}$$

We shall give an overview of their method (abbreviated "the GPY method" in the sequel) in Chapter 2, and proceed to apply it to prove the existence of "close" primes in totally real number fields in Chapter 3.

The GPY method was refined by Zhang [29] in 2013 to obtain an unconditional proof of the Bounded Gap Conjecture, with the bound 70 000 000 (through a collaborative online Polymath project [26], the method was optimized to reduce this bound to 4680). Zhang obtained his result by proving a strengthening of the Bombieri-Vinogradov Theorem which manages to go beyond the level 1/2 for a class of certain smooth moduli. That such a theorem would imply the Bounded Gaps Conjecture was previously proved by Motohashi and Pintz [24].

Soon after Zhang, another refinement of the GPY method in a different direction was devised independently by Maynard [20] and Tao (unpublished). This approach involves a more general choice of sieve weights, and produces not only smaller gaps (namely 600, later optimized to 246 in a Polymath project [27]) between two primes, but can also show the existence of any number of primes within a bounded length interval infinitely often, while the best result concerning $p_{n+2} - p_n$ obtainable by previous methods, including Zhang's, was

$$\liminf_{n\to\infty} \frac{p_{n+2} - p_n}{\log p_n} = 0, \tag{1.8}$$

and even this was conditional on the Elliott-Halberstam Conjecture.

We will briefly discuss the Maynard-Tao method in Chapter 4 and then apply it to primes in arithmetic progressions in chapters 5 and 6.

CHAPTER 2

# The GPY method

The aim of this chapter is to give an overview of the ideas involved in the GPY method. The method can be regarded as a sieving process in that it relies on estimating a sum of certain appropriately chosen weight functions over a given set of numbers, with the aim of detecting two primes among $n+h_1, \ldots, n+h_i$ for some $n$ in the set. The basic idea is to find a non-negative function $w(n)$, such that for a given admissible $k$-tuple $\mathcal{H} = h_1, \ldots, h_k$, there holds

$$\sum_{X<n\leq 2X} \#\{1 \leq i \leq k : n+h_i \text{ prime }\} \, w(n) > \sum_{X<n\leq 2X} w(n), \tag{2.1}$$

which implies that there's at least one $n \in (x, 2x]$ with at least two of the numbers $n+h_i$ prime. Such a function should clearly be concentrated on those $n$ for which the numbers $n+h_i$ are prime. Taking a cue from the classical theory of $\pi(x) = \#\{p \leq x\}$ leading to the Prime Number Theorem, which goes through a consideration of $\sum_{n\leq x} \Lambda(n)$ with the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some prime } p, \\ 0, & \text{otherwise,} \end{cases} \tag{2.2}$$

the most natural weight to consider would be

$$\Lambda(n;\mathcal{H}) := \Lambda(n+h_1)\ldots\Lambda(n+h_k). \tag{2.3}$$

In fact in its original formulation the Hardy-Littlewood Conjecture predicts $\sum_{n\leq X} \Lambda(n,\mathcal{H}) \sim \frac{X}{(\log X)^k}(\mathfrak{S}(\mathcal{H}) + o(1))$, from which one can derive (1.2) with little effort.

No effective way to handle $\sum_{n\leq x}\Lambda(n,\mathcal{H})$ directly is known. However, motivated by the elementary identity

$$\Lambda(n)=\sum_{d|n}\mu(d)\log\frac{n}{d}, \tag{2.4}$$

one can expect the truncation

$$\Lambda_R(n):=\sum_{\substack{d|n\\ d\leq R}}\mu(d)\log\frac{R}{d} \tag{2.5}$$

to be a useful approximation to $\Lambda(n)$ and in turn to approximate $\Lambda(n;\mathcal{H})$ by

$$\Lambda_R(n+h_1)\ldots\Lambda_R(n+h_k). \tag{2.6}$$

Indeed, Goldston and Yıldırım, in a series of papers [8–10] devised methods to handle sums of the form $\sum_{n\leq X}\Lambda_R(n+h_1)\ldots\Lambda_R(n+h_k)$ and applied their results to the problem of prime tuples to obtain

$$\Delta=\liminf_{n\to\infty}\frac{p_{n+1}-p_n}{\log p_n}\leq\frac{1}{4}. \tag{2.7}$$

A different kind of approximation was needed to obtain $\Delta=0$. The generalized von Mangoldt function

$$\Lambda_k(n)=\sum_{d|n}\mu(d)\left(\log\frac{n}{d}\right)^k \tag{2.8}$$

vanishes when $n$ has more than $k$ distinct prime factors (the notational conflict between $\Lambda_k$ and $\Lambda_R$ is tolerable within the expository nature of the present chapter). Defining the polynomial

$$P(n;\mathcal{H})=(n+h_1)(n+h_2)\ldots(n+h_k), \tag{2.9}$$

we have $\Lambda_k(P(n;\mathcal{H}))\neq 0$ if $\{n+h_i\}_i$ is a prime tuple; there will be overcounts due to prime powers but the contribution of those can be expected to be negligible. Implementing also the smoothed truncation from (2.5) as well, we can hope to detect prime tuples with the weights

$$\Lambda_R(n;\mathcal{H}) = \frac{1}{k!}\sum_{\substack{d|P(n,\mathcal{H})\\ d\le R}} \mu(d)\left(\log\frac{R}{d}\right)^k. \tag{2.10}$$

This however is also not enough to obtain $\Delta = 0$. The final crucial idea is to compromise and look for at most $k+\ell$ primes dividing $P(n;\mathcal{H})$ with $\ell < k$. Thus we arrive at

$$\Lambda_R(n;\mathcal{H},\ell) = \frac{1}{(k+\ell)!}\sum_{\substack{d|P(n,\mathcal{H})\\ d\le R}} \mu(d)\left(\log\frac{R}{d}\right)^{k+\ell}. \tag{2.11}$$

Returning to (2.1) we take $w(n)$ to be $\Lambda_R(n;\mathcal{H},k+\ell)^2$, and also counting the primes among $n+h_i$ weighted by

$$\theta(n+h_i) = \begin{cases} \log(n+h_i), & \text{if } n+h_i \text{ is prime},\\ 0, & \text{otherwise}, \end{cases} \tag{2.12}$$

we arrive at the expression

$$S = \sum_{X<n\le 2X}\left(\sum_{i=1}^{k}\theta(n+h_i) - \log 3X\right)\Lambda_R(n;\mathcal{H},\ell)^2. \tag{2.13}$$

Showing that this is positive amounts to showing the existence of two primes among the numbers $n+h_i$ for some $n\in(X,2X]$. Thus one needs to be able to estimate sums of the form

$$\sum_{X<n\le 2X}\Lambda_R(n;\mathcal{H},\ell)^2 \tag{2.14}$$

and

$$\sum_{X<n\le 2X}\Lambda_R(n;\mathcal{H},\ell)^2\theta(n+h_i). \tag{2.15}$$

Such sums can be handled in an analytic fashion, by expressing them in terms of contour integrals of zeta-functions, or through purely sieve-theoretic arguments; we shall see an application of the first approach in Chapter 3 and the second in Chapter 6. Eschewing the technicalities for now, the results turn out to be

$$\sum_{X<n\leq 2X} \Lambda_R(n;\mathcal{H},\ell)^2 \sim \frac{1}{(k+2\ell)!}\binom{2\ell}{\ell}\mathfrak{S}(\mathcal{H})X(\log R)^{k+2\ell} \tag{2.16}$$

and

$$\begin{aligned}\sum_{X<n\leq 2X} &\Lambda_R(n;\mathcal{H},\ell)^2\theta(n+h_i) \\ &\sim \frac{1}{(k+2\ell+1)!}\binom{2\ell+2}{\ell+1}\mathfrak{S}(\mathcal{H})X(\log R)^{k+2\ell+1},\end{aligned} \tag{2.17}$$

so that we have

$$\begin{aligned} S \sim &\left(\frac{2k}{k+2\ell+1}\,\frac{2\ell+1}{\ell+1}\log R - \log 3X\right) \\ &\times \frac{1}{(k+2\ell)!}\binom{2\ell}{\ell}\mathfrak{S}(\mathcal{H})X(\log R)^{k+2\ell}.\end{aligned} \tag{2.18}$$

Now we need to scratch the surface of how (2.15) is handled in order to expose the relationship of the method with the distribution of primes in arithmetic progressions. Expanding the square in (2.15) and rearranging the sum, we get

$$\begin{aligned} &\sum_{X<n\leq 2X} \Lambda_R(n;\mathcal{H},\ell)^2\theta(n+h_i) \\ &= \sum_{X<n\leq 2X}\theta(n+h_i)\sum_{\substack{[d_1,d_2]\mid P(n;\mathcal{H})\\ d_1,d_2\leq R}}\mu(d_1)\mu(d_2)\left(\log\frac{R}{d_1}\right)^{k+\ell}\left(\log\frac{R}{d_2}\right)^{k+\ell} \\ &= \sum_{d_1,d_2\leq R}\mu(d_1)\mu(d_2)\left(\log\frac{R}{d_1}\right)^{k+\ell}\left(\log\frac{R}{d_2}\right)^{k+\ell}\sum_{\substack{X<n\leq 2X\\ [d_1,d_2]\mid P(n;\mathcal{H})}}\theta(n+h_i).\end{aligned} \tag{2.19}$$

Since $[d_1,d_2]$ is squarefree, the condition $[d_1,d_2] \mid P(n;\mathcal{H})$ in the innermost sum is equivalent to the condition that $p \mid P(n;\mathcal{H})$ for each prime $p \mid [d_1,d_2]$, or what is the same, that $n \pmod p$ lies in the set $\Omega(p) := \{-h_i \pmod p : i=1,\ldots,k\}$ for each $p \mid [d_1,d_2]$. By the Chinese Remainder theorem, we can extend the definition of $\Omega$ to squarefree integers multiplicatively, so that

$n \pmod{d} \in \Omega(d)$ if and only if $n \pmod{p} \in \Omega(p)$ for all $p \mid d$. With this notation, we can write the innermost sum as (ignoring an inconsequential shift by $h_i$),

$$\sum_{\substack{b\in\Omega([d_1,d_2]) \\ (b+h_i,[d_1,d_2])=1}} \sum_{\substack{x<n\leq 2X \\ n\equiv b+h_i \pmod{[d_1,d_2]}}} \theta(n). \tag{2.20}$$

Now the inner sum here is a weighted count of primes between $X$ and $2X$ lying in an arithmetic progression. It is natural to expect that the primes in $(X, 2X]$ will be more or less evenly distributed among the $\varphi(q)$ reduced residue classes modulo $q$, specifically, that for $(a, q) = 1$,

$$\sum_{\substack{x<n\leq 2X \\ n\equiv a \pmod{q}}} \theta(n) \sim \frac{X}{\varphi(q)}. \tag{2.21}$$

The Prime Number Theorem for Arithmetic Progressions grants us this uniformly for $q$ up to $(\log X)^C$, beyond which the error term becomes larger than the main term. However, in our case, the modulus $[d_1, d_2]$ runs up to $R^2$, and we'd like to be able to take $R$ much larger than a power of $(\log X)$. However, even though we can't approximate the desired counts of primes uniformly for each modulus over a larger range, we can control their deviation from the expected value *on average*, by using an estimate of the form

$$\sum_{q\leq Q} \max_{(a,q)=1} \Big| \sum_{\substack{x<n\leq 2X \\ n\equiv a \pmod{q}}} \theta(n) - \frac{X}{\varphi(q)} \Big| \ll \frac{X}{(\log X)^A}. \tag{2.22}$$

We say that the primes have *level of distribution* $\vartheta$ if an estimate of the form (2.22) holds with $Q = X^{\vartheta-\varepsilon}$ for all $\varepsilon$ and $A$. The Bombieri-Vinogradov Theorem states that primes have level of distribution $1/2$, and the Elliott-Halberstam Conjecture predicts level of distribution 1. Thus, if primes have level of distribution $\vartheta$, then in order to replace the inner sum in (2.20) by $X/\varphi([d_1, d_2])$ within acceptable error, we must have $[d_1, d_2] \leq X^{\vartheta-\varepsilon}$, which in turn forces us to take $R$ at most $X^{\frac{\vartheta}{2}-\varepsilon}$. Thus, if we return to (2.18), we see that the left hand side is positive if

$$\frac{k}{k+2\ell+1}\frac{2\ell+1}{\ell+1}\vartheta > 1. \tag{2.23}$$

As $k, \ell \to \infty$ with $\ell = o(k)$, the left hand side has limit $2\vartheta$, thus the method fails by a hair's breadth to detect bounded gaps between primes. However we can deduce conditionally that if primes have level of distribution $\vartheta$ for any $\vartheta$ strictly greater than $1/2$, then there are bounded gaps between primes.

In order to salvage $\Delta = 0$ unconditionally, one last modification is needed. We average over all $k$-tuples with diameter $\leq H$ for some parameter $H$, and consider

$$\sum_{X<n\leq 2X}\left(\sum_{1\leq h_0\leq H}\theta(n+h_0) - \log 3X\right)\sum_{\substack{1\leq h_1,\ldots,h_k\leq H\\ \text{distinct}}}\Lambda_R(n;\mathcal{H},\ell)^2. \tag{2.24}$$

Note that this time in the first sum we consider $n + h_0$ for all $h_0 \in [1, H]$ and not only the components of a given tuple. So we also need the estimate, for $h_0 \notin \mathcal{H}$,

$$\begin{aligned}\sum_{X<n\leq 2X}\Lambda_R(n;\mathcal{H},\ell)^2\theta(n+h_0)& \\ &\sim \frac{1}{(k+2\ell)!}\binom{2\ell}{\ell}\mathfrak{S}(\mathcal{H}\cup\{h_0\})X(\log R)^{k+2\ell}.\end{aligned} \tag{2.25}$$

This differs from (2.16) only in that we have $\mathfrak{S}(\mathcal{H}\cup\{h_0\})$ in place of $\mathfrak{S}(\mathcal{H})$, thus by appealing to Gallagher's average result [6],

$$\sum_{\substack{1\leq h_1,\ldots,h_k\leq H\\ \text{distinct}}}\mathfrak{S}(\mathcal{H}) = (1+o(1))H^k, \quad (H\to\infty), \tag{2.26}$$

the sums of type (2.25) produce a term with an extra factor of $H$, so that when we put everything together, the positivity of the whole expression relies on whether or not

$$\frac{H}{\log X} + \frac{k}{k+2\ell+1}\frac{2\ell+1}{\ell+1}\vartheta > 1, \tag{2.27}$$

instead of (2.23). For any $\varepsilon > 0$, if $H = \varepsilon \log X$ and $\vartheta = 1/2$, the inequality (2.27) is satisfied by taking $k$ and $\ell = o(k)$ large enough, thus establishing

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0. \tag{2.28}$$

CHAPTER 3

# An application of the GPY method to number fields

## 1. Introduction

In this chapter, we generalize the work of Goldston, Pintz and Yıldırım to totally real number fields. We show

$$\liminf_{\substack{\omega_0,\omega_1\in\mathcal{O}_K \text{ prime} \\ \omega_0\neq\omega_1}} \left|\frac{\mathbf{N}(\omega_1-\omega_0)}{\log \mathbf{N}\omega_0}\right| = 0. \tag{3.1}$$

We follow closely the exposition in their paper with Motohashi [7] which contains a simplified and more condensed proof of the result. A key ingredient is Hinz's generalization of the Bombieri-Vinogradov theorem to number fields [16], which gives the level of distribution 1/2 only in the totally real case, whence our restriction. His result in the totally real case is reproduced here as Lemma 3.10. The method requires the analogue of Gallagher's computation of the singular series [6], which is proved in Lemma 3.12. We shall also prove some estimates for the Dedekind zeta-function $\zeta_K$ that will be needed to evaluate contour integrals.

Throughout $K$ denotes a totally real number field of degree $\kappa$ over the rationals with ring of integers $\mathcal{O}_K$. Ideals of $\mathcal{O}_K$ are denoted by Gothic letters, and $\mathfrak{p}$ denotes a prime ideal. For $\operatorname{Re} s > 1$, the Dedekind zeta-function of the field $K$ is given by

$$\zeta_K(s) = \sum_{\mathfrak{a}\subseteq\mathcal{O}_K} \frac{1}{(\mathbf{N}\mathfrak{a})^s}, \tag{3.2}$$

and there admits the Euler product expansion

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{(\mathbf{N}\mathfrak{p})^s}\right)^{-1}. \tag{3.3}$$

$\mathfrak{R}_{\mathbf{y}}$ will denote the set of integers $\alpha \in \mathcal{O}_K$ such that

$$0 < \alpha^{(i)} \leq y_i, \quad i = 1, \ldots, \kappa, \tag{3.4}$$

where $\mathbf{y} = (y_1, \ldots, y_\kappa)$. We shall write $y$ for $y_1 y_2 \cdots y_\kappa$, and, multiplying by a totally positive unit if necessary (see, for instance, [15, p. 62]) we may assume without loss of generality that $y_i \asymp y^{1/\kappa}$. We shall sometimes write $A\mathfrak{R}_{\mathbf{y}}$ instead of $\mathfrak{R}_{A\mathbf{y}}$ for ease of legibility. The lower-case letter $c$ denotes a positive constant which need not be the same at every instant; when we need to keep track of a particular constant, we employ subscripts. In this chapter, the arithmetic functions $\mu$, $\Lambda$, $\phi$, etc. are the number field generalizations of their classical counterparts.

All implicit constants depend on $K$.

## 2. Lemmata

LEMMA 3.1. *Let $K$ be a totally real number field of degree $\kappa$ and discriminant $D$, and let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$. Denote by $\mathfrak{R}$ the subset of integers $\alpha$ of $\mathcal{O}_K$ satisfying*

$$0 < \alpha^{(i)} \leq y_i, \quad i = 1, \ldots, \kappa, \tag{3.5}$$

*and put $y = y_1 y_2 \ldots y_\kappa$. Then for any $\gamma \in \mathcal{O}_K$, there holds*

$$\sum_{\substack{\alpha \in \mathfrak{R} \\ \alpha \equiv \gamma \ (\mathrm{mod}\ \mathfrak{a})}} 1 = \frac{y}{\sqrt{|D|}\mathbf{N}\mathfrak{a}} + O\left(\left(\frac{y}{\mathbf{N}\mathfrak{a}}\right)^{1-\frac{1}{\kappa}}\right). \tag{3.6}$$

PROOF. Consider the canonical image of $\mathcal{O}_K$ in the Minkowski space which in our case is $\mathbb{R}^\kappa$. We may fix a fundamental parallelepiped of the lattice formed by the image of $\mathfrak{a}$, and view the space as tiled with $\mathfrak{a}$-translates of it. Since every translate of the fundamental parallelepiped contains the image of exactly one number from every equivalence class (mod $\mathfrak{a}$), our problem amounts to counting the parallelepipeds entirely contained in $R = \{(x_1, \ldots, x_\kappa) : 0 \leq x_i \leq y_i\}$, up to an error the number of parallelepipeds intersecting the boundary, for the congruence class representative in those intersecting parallelepipeds may or may not lie in $R$.

Now fix ideal class representatives $\mathfrak{A}_1, \ldots, \mathfrak{A}_{\mathbf{h}}$ of $K$, not necessarily integral ideals, where $\mathbf{h}$ is the class number of $K$. Fix also a fundamental parallelepiped associated to each $\mathfrak{A}_j$. For any one of those representatives, say $\mathfrak{A}_j$, a certain number, say $N$, of the $\mathfrak{A}_j$-translates of the fundamental parallelepiped will intersect the boundary of $R$. Call them $P_1, \ldots, P_N$ and denote by $v_i$ for $i = 1, \ldots, N$ the volume of $R \cap P_i$. Then the volume

$$y - \sum_{i=1}^{N} v_i \tag{3.7}$$

is comprised of all the parallelepipeds entirely contained in $R$, and only them. Since each has volume $\sqrt{|D|}\mathbf{N}\mathfrak{A}_j$, their number is

$$\frac{1}{\sqrt{|D|}\mathbf{N}\mathfrak{A}_j}\left(y - \sum_{i=1}^{N} v_i\right), \tag{3.8}$$

and thus the number of integers of $K$ in $\mathfrak{R}$ which are congruent to $\gamma \pmod{\mathfrak{A}_j}$ is

$$\frac{y}{\sqrt{|D|}\mathbf{N}\mathfrak{A}_j} - \frac{1}{\sqrt{|D|}\mathbf{N}\mathfrak{A}_j}\sum_{i=1}^{N} v_i + O(N). \tag{3.9}$$

The second term is also clearly

$$\frac{1}{\sqrt{|D|}\mathbf{N}\mathfrak{A}_j}\sum_{i=1}^{N} v_i \leq \frac{1}{\mathbf{N}\mathfrak{A}_j}\sum_{i=1}^{N} \mathbf{N}\mathfrak{A}_j \leq N, \tag{3.10}$$

so it suffices to estimate $N$.

Let $D_j$ be the diameter of our fundamental parallelepiped. If we expand and shrink $R$ by an amount $2D_j$ in all directions along basis vectors and form $R^+$ and $R^-$ respectively, every parallelepiped that intersects the original boundary will lie entirely outside $R^-$ and entirely inside $R^+$. Since the volume of $R^+$ is $\prod_i(y_i + 2D_j) = y + O_j\left(y^{1-\frac{1}{\kappa}}\right)$, and likewise for $R^-$, the intermediate region has volume $O_j\left(y^{1-\frac{1}{\kappa}}\right)$, and hence

$$N = O_j\left(\frac{y^{1-\frac{1}{\kappa}}}{\sqrt{|D|}\mathbf{N}\mathfrak{A}_j}\right) = O_j\left(\left(\frac{y}{\mathbf{N}\mathfrak{A}_j}\right)^{1-\frac{1}{\kappa}}\right). \tag{3.11}$$

Now let $\mathfrak{a}$ be any ideal of $\mathcal{O}_K$. We need the same parallelepiped count as before. We have $\mathfrak{a} = \beta\mathfrak{A}_j$ for some $\beta \in K$ and some $j$. As before, by multiplying with a unit if necessary, we may suppose $\beta^{(i)} \asymp \mathbf{N}(\beta)^{1/\kappa}$. Now the parallelepiped grid of $\mathfrak{a}$ is simply that of $\mathfrak{A}_j$ dilated in the $x_i$ axis by a factor of $\beta^{(i)}$. But counting $\mathfrak{A}_i$ parallelepipeds scaled by $\beta^{(i)}$'s which are contained in or intersects the boundary of $R$ is the same thing as counting original parallelepipeds which are contained in or intersects $R$ scaled by $\left(\beta^{(i)}\right)^{-1}$'s. So our count is identical to the special case but with $(\beta^{(i)})^{-1}y_i$ in place of $y_i$, which by the above is

$$\frac{y/\mathbf{N}(\beta)}{\sqrt{|D|}\mathbf{N}\mathfrak{A}_j} + O_j\left(\left(\frac{y/\mathbf{N}(\beta)}{\mathbf{N}\mathfrak{A}_j}\right)^{1-\frac{1}{\kappa}}\right) = \frac{y}{\sqrt{|D|}\mathbf{N}\mathfrak{a}} + O_j\left(\left(\frac{y}{\mathbf{N}\mathfrak{a}}\right)^{1-\frac{1}{\kappa}}\right). \tag{3.12}$$

Since there are finitely many possibilities for the ideal class $j$, we may choose the weakest implicit constant and the proof is complete. □

LEMMA 3.2. *We have, with the classical notation $s = \sigma + it$,*

$$\zeta_K(s) \ll \tau^{2-1/n-\sigma}. \tag{3.13}$$

*uniformly for $|t| \geq 1$, $\sigma \geq 1 - 1/\kappa + \delta$. Here $\tau$ denotes $|t| + 4$ and $\delta$ is an arbitrary positive constant.*

PROOF. Let $r(n)$ be the number of ideals with norm $n$, and let $M(x) = \sum_{n\leq x} r(n)$. It is well known that $M(x) = \rho_K x + O(x^{1-1/\kappa})$, where $\rho_K$ is the residue of $\zeta_K(s)$ at $s = 1$. Put $E(x) = M(x) - \rho_K x$. Then for $\sigma > 1$,

(3.14)
$$\begin{aligned}\zeta_K(s) &= \sum_{n\le x} r(n)n^{-s} + \int_x^\infty u^{-s}dM(u)\\ &= \sum_{n\le x} r(n)n^{-s} + \rho_K\int_x^\infty u^{-s}du + \int_x^\infty u^{-s}dE(u)\\ &= \sum_{n\le x} r(n)n^{-s} + \rho_K\frac{x^{1-s}}{s-1} + u^{-s}E(u)\Big|_x^\infty + s\int_x^\infty E(u)u^{-s-1}du,\end{aligned}$$

so

$$\zeta_K(s) = \sum_{n\le x} r(n)n^{-s} + \rho_K\frac{x^{1-s}}{s-1} + x^{-s}E(x) + s\int_x^\infty E(u)u^{-s-1}du. \tag{3.15}$$

In particular, since $E(1) = 1 - \rho_K$, with $x = 1$ the above gives,

$$\zeta_K(s) = \frac{\rho_K s}{s-1} + s\int_1^\infty E(u)u^{-s-1}du. \tag{3.16}$$

This furnishes an analytic continuation of $\zeta_K$ to the half plane $\sigma > 1 - 1/\kappa$. We can estimate the integral in (3.15) trivially,

$$\int_x^\infty E(u)u^{-s-1}du \ll \int_x^\infty u^{-\sigma-1/\kappa}du = \frac{x^{1-1/\kappa-\sigma}}{\sigma - 1 + 1/\kappa}. \tag{3.17}$$

Now

(3.18)
$$\sum_{n\le x} r(n)n^{-s} \ll \left(\max_{n\le x} r(n)\right)\sum_{n\le x} n^{-\sigma} \ll \left(\max_{n\le x} r(n)\right)\left(1 + \int_1^x u^{-\sigma}du\right)$$

uniformly for $\sigma \ge 0$. Clearly $r(n)$ is multiplicative. Suppose $p$ factors in $\mathcal{O}_K$ as $p = \mathfrak{p}_1^{e_1}\dots\mathfrak{p}_g^{e_g}$ with $\mathbf{N}\mathfrak{p}_i = p^{f_i}$. Then

$$\begin{aligned}\#\{\mathfrak{a}\subset\mathcal{O}_K : \mathbf{N}\mathfrak{a} = p^m\} &= \#\{(k_1,\dots,k_g) : k_1f_1 + \dots + k_gf_g = m\}\\ &\le \#\{(a_1,\dots,a_g) : a_1 + \dots + a_g = m\}\\ &= \#\{(d_1,\dots,d_g) : d_1\dots d_g = p^m\}\\ &= \tau_g(p^m)\\ &\le \tau_\kappa(p^m),\end{aligned} \tag{3.19}$$

so that $r(n) \ll n^{\varepsilon}$ for any positive $\varepsilon$. Thus

$$\sum_{n \le x} r(n) n^{-s} \ll x^{\varepsilon} \left( 1 + \int_1^x u^{-\sigma} du \right). \tag{3.20}$$

Now if $0 \le \sigma \le 1 - 1/\log x$, the integral is $\le x^{1-\sigma}/(1-\sigma)$. If $|\sigma - 1| \le 1/\log x$, then $u^{-\sigma} \asymp u^{-1}$ uniformly for $1 \le u \le x$, so that the integral is $\ll \log x$. If $\sigma \ge 1 + 1/\log x$, then the integral is $< \int_1^{\infty} u^{-\sigma} du = 1/(\sigma - 1)$. Thus

$$\sum_{n \le x} r(n) n^{-s} \ll x^{\varepsilon} (1 + x^{1-\sigma}) \min\left( |\sigma - 1|^{-1}, \log x \right) \tag{3.21}$$

uniformly for $0 \le \sigma \le 2$. Using this and (3.17) in (3.15) with $x = \tau$ yields the desired result. □

We reproduce here Lemma 6.3 of [23].

Lemma 3.3. *Suppose that $f(z)$ is analytic in a domain containing the disc $|z| \le 1$, that $|f(z)| \le M$ in this disc, and that $f(0) \ne 0$. Let $r$ and $R$ be fixed, $0 < r < R < 1$. Then for $|z| \le r$ we have*

$$\frac{f'}{f}(z) = \sum_{k=1}^{K} \frac{1}{z - z_k} + O\left( \log \frac{M}{|f(0)|} \right) \tag{3.22}$$

*where the sum is extended over all zeros $z_k$ of $f$ for which $|z_k| \le R$. (The implicit constant depends on $r$ and $R$ but is otherwise absolute.)*

Now we apply this to $\zeta_K(s)$.

Lemma 3.4. *If $|t| \ge 2$ and $1 - 1/\kappa + \delta < \sigma < 2$, we have*

$$\frac{\zeta_K'}{\zeta_K}(s) = {\sum}' \frac{1}{s - \rho} + O\left( \log \tau \right), \tag{3.23}$$

*where $\sum'$ denotes summation over those zeros $\rho$ of $\zeta_K(s)$ with $|\rho - (3/2 + it)| \le 1/2 + 1/\kappa - \delta$.*

Proof. We apply Lemma 3.3 to the function $f(z) = \zeta_K(z + (3/2 + it))$, with $R = 1/2 + 1/\kappa - \delta$ and $r = 1/2 + 1/\kappa - 2\delta$, in which case $f(0) \gg 1$

by the absolute convergence of the Euler product and $f(z) \ll \tau^2$ by Lemma 3.2. $\square$

We note here that de la Valée-Poussin's classical argument for the non-vanishing of the Riemann zeta-function on the line $\sigma = 1$ applies verbatim to $\zeta_K(s)$.

LEMMA 3.5. *For $\sigma > 1$,*

$$\operatorname{Re}\left(-3\frac{\zeta_K'}{\zeta_K}(\sigma) - 4\frac{\zeta_K'}{\zeta_K}(\sigma+it) - \frac{\zeta_K'}{\zeta_K}(\sigma+2it)\right) \geq 0. \tag{3.24}$$

LEMMA 3.6. *The function $\zeta_K(s)$ does not vanish on the line $\sigma = 1$.*

Now we are in a position to establish a zero-free region for $\zeta_K(s)$.

PROPOSITION 3.7. *There is a constant $c$ depending on $K$ such that the function $\zeta_K(s)$ does not vanish in the region*

$$\sigma \geq 1 - \frac{c}{\log \tau}. \tag{3.25}$$

PROOF. Suppose $\rho_0 = \beta_0 + i\gamma_0$ is a zero of $\zeta_K(s)$ with $1 - 1/\kappa + \delta < \beta_0 < 1$, $|\gamma_0| > 2$. Since $\operatorname{Re}\rho < 1$ for any zero $\rho$ of $\zeta_K(s)$, we have $\operatorname{Re}\frac{1}{s-\rho} > 0$ whenever $\sigma > 1$. Thus, using Lemma 3.4 with $s = 1+\varepsilon+i\gamma_0$ and $s = 1+\varepsilon+2i\gamma_0$ respectively, we obtain

$$\operatorname{Re} -\frac{\zeta_K'}{\zeta_K}(1+\varepsilon+i\gamma_0) \leq -\frac{1}{1+\varepsilon-\beta_0} + c_1\log(|\gamma_0|+4) \tag{3.26}$$

and

$$\operatorname{Re} -\frac{\zeta_K'}{\zeta_K}(1+\varepsilon+2i\gamma_0) \leq c_1\log(|2\gamma_0|+4). \tag{3.27}$$

Also, by virtue of the simple pole of $\zeta_K(s)$ at $s = 1$, we have

$$-\frac{\zeta_K'}{\zeta_K}(1+\varepsilon) = \frac{1}{\varepsilon} + O(1). \tag{3.28}$$

Using these in Lemma 3.5, we obtain

$$\frac{3}{\varepsilon} - \frac{4}{1+\varepsilon-\beta_0} + c_2\log(|\gamma_0|+4) \geq 0. \tag{3.29}$$

We take $\varepsilon = 1/(2c_2 \log(|\gamma_0| + 4))$, whence

$$7c_2 \log(|\gamma_0| + 4) \geq \frac{4}{1 + \varepsilon - \beta_0}, \tag{3.30}$$

or,

$$1 + \frac{1}{2c_2 \log(|\gamma_0| + 4)} - \beta_0 \geq \frac{4}{7c_2 \log(|\gamma_0| + 4)}, \tag{3.31}$$

so we have

$$\beta_0 \leq 1 - \frac{1}{14c_2 \log(|\gamma_0| + 4)}. \tag{3.32}$$

Choosing $c$ small enough so as to exclude the finitely many zeros with $\gamma \leq 2$, we obtain the desired result. □

PROPOSITION 3.8. *Let $c$ be the constant in Proposition 3.7. Then for $\sigma > 1 - c/(2 \log \tau)$ and $|t| \geq 2$, we have*

$$|\frac{\zeta_K'}{\zeta_K}(s)| \ll \log \tau, \tag{3.33}$$

$$|\log \zeta_K(s)| \leq \log \log \tau + O(1), \tag{3.34}$$

*and*

$$\frac{1}{\zeta_K(s)} \ll \log \tau. \tag{3.35}$$

PROOF. It is plain that for $\sigma > 1$,

$$|\frac{\zeta_K'}{\zeta_K}(s)| \leq -\frac{\zeta_K'}{\zeta_K}(\sigma) \ll \frac{1}{\sigma - 1}. \tag{3.36}$$

Let $s_1 = 1 + 1/\log \tau + it$. So

$$\frac{\zeta_K'}{\zeta_K}(s_1) \ll \log \tau. \tag{3.37}$$

Then from Lemma 3.4 we deduce that

$$\sum_{\rho}{}' \operatorname{Re} \frac{1}{s_1 - \rho} \ll \log \tau, \tag{3.38}$$

where the sum is taken over those zeros $\rho$ for which $|\rho - (3/2 + it)| \leq 1/2 + 1/\kappa - \delta$. If $1 - c/(2\log\tau) \leq \sigma \leq 1 + 1/\log\tau$, then again by Lemma 3.4 we have

$$\frac{\zeta_K'}{\zeta_K}(s) - \frac{\zeta_K'}{\zeta_K}(s_1) = \sum_{\rho}{}' \left(\frac{1}{s-\rho} - \frac{1}{s_1-\rho}\right) + O(\log\tau). \tag{3.39}$$

We have $|s-\rho| \asymp |s_1 - \rho|$ so it follows that

$$\frac{1}{s-\rho} - \frac{1}{s_1-\rho} \ll \frac{1}{|s_1-\rho|^2\log\tau} \ll \operatorname{Re}\frac{1}{s_1-\rho}, \tag{3.40}$$

whence we obtain (3.33). Now we know that for $\sigma > 1$, $\zeta_K(\sigma) \leq \rho_K/(\sigma - 1) + O(1)$. So if $s_1 = 1 + 1/\log\tau + it$,

$$\begin{aligned} |\log\zeta_K(s_1)| &\leq \sum_{\mathfrak{a}\neq\mathcal{O}_K} \frac{\Lambda(\mathfrak{a})}{\log\mathbf{N}(\mathfrak{a})}\mathbf{N}(\mathfrak{a})^{-1-1/\log\tau} \\ &= \log\zeta_K\left(1+\tfrac{1}{\log\tau}\right) \leq \log\log\tau + O(1). \end{aligned} \tag{3.41}$$

Then if $s$ is in the indicated region,

$$\log\zeta_K(s) - \log\zeta_K(s_1) = \int_{s_1}^{s}\frac{\zeta_K'}{\zeta_K}(w)dw = O(1) \tag{3.42}$$

by (3.33), and (3.34) follows. Then (3.35) follows from (3.34) by exponentiation. $\square$

## 3. Main results

Let $y$ be a parameter tending to infinity. We have four other parameters $H, R, k$ and $\ell$ obeying the following restrictions.

$$H \ll \log y \ll \log R \leq \log y, \tag{3.43}$$

and

$$\text{integers } k, \ell > 0 \text{ are arbitrary but bounded.} \tag{3.44}$$

Implicit constants may depend on these $k$ and $\ell$.

Let

$$\mathcal{H} = \{h_1, h_2, \ldots, h_k\} \subseteq \mathfrak{H}, \tag{3.45}$$

where $\mathfrak{H} = H^{1/\kappa}\mathfrak{R}_{(1,\ldots,1)}$ and the $h_i$ are distinct. For a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, we put

$$\Omega(\mathfrak{p}) = \{\text{distinct residue classes among } -h \,(\text{mod } \mathfrak{p}),\ h \in \mathcal{H}\}, \tag{3.46}$$

and write $\alpha \in \Omega(\mathfrak{p})$ as a shorthand for $\alpha \,(\text{mod } \mathfrak{p}) \in \Omega(p)$ for $\alpha \in \mathcal{O}_K$. We say that $\mathcal{H}$ is *admissible* if $|\Omega(\mathfrak{p})| < \mathbf{N}\mathfrak{p}$ for every prime ideal $\mathfrak{p}$, and assume this unless otherwise stated.

We extend $\Omega$ multiplicatively, so that $\alpha \in \Omega(\mathfrak{a})$ for a square-free ideal $\mathfrak{a}$ if and only if $\alpha \in \Omega(\mathfrak{p})$ for all $\mathfrak{p} \mid \mathfrak{a}$. This is equivalent to

$$P(\alpha;\mathcal{H}) \in \mathfrak{a}, \qquad P(\alpha;\mathcal{H}) = (\alpha+h_1)(\alpha+h_2)\cdots(\alpha+h_k). \tag{3.47}$$

We put, with $\mu$ denoting the usual generalization of the Möbius function to ideals,

$$\lambda_R(\mathfrak{a};n) = \begin{cases} 0, & \text{if } \mathbf{N}\mathfrak{a} > R, \\ \dfrac{1}{n!}\mu(\mathfrak{a})\left(\log R/\mathbf{N}\mathfrak{a}\right)^n, & \text{if } \mathbf{N}\mathfrak{a} \le R, \end{cases} \tag{3.48}$$

and

$$\Lambda_R(\alpha;\mathcal{H},n) = \sum_{\alpha\in\Omega(\mathfrak{a})} \lambda_R(\mathfrak{a};n) = \frac{1}{n!}\sum_{\substack{P(\alpha;\mathcal{H})\in\mathfrak{a}\\ \mathbf{N}\mathfrak{a}\le R}} \mu(\mathfrak{a})\left(\log R/\mathbf{N}\mathfrak{a}\right)^n. \tag{3.49}$$

With these we shall evaluate

$$\sum_{\alpha\in 2^{1/\kappa}\mathfrak{R}_{\mathbf{y}}\setminus\mathfrak{R}_{\mathbf{y}}} \Lambda_R(\alpha;\mathcal{H},k+\ell)^2, \tag{3.50}$$

which, on expanding out the square, equals

$$\sum_{\mathfrak{a}_1,\mathfrak{a}_2} \lambda_R(\mathfrak{a}_1;k+\ell)\lambda_R(\mathfrak{a}_2;k+\ell) \sum_{\substack{\alpha\in\Omega(\mathfrak{a}_1),\alpha\in\Omega(\mathfrak{a}_2)\\ \alpha\in 2^{1/\kappa}\mathfrak{R}_{\mathbf{y}}\setminus\mathfrak{R}_{\mathbf{y}}}} 1. \tag{3.51}$$

Now the condition $\alpha \in \Omega(\mathfrak{a}_1)$, $\alpha \in \Omega(\mathfrak{a}_2)$ on the inner sum is equivalent to $\alpha \in \Omega(\mathfrak{a}_1 \cap \mathfrak{a}_2)$, thus that sum can be written as

$$\sum_{h\in\Omega(\mathfrak{a}_1\cap\mathfrak{a}_2)} \sum_{\substack{\alpha\equiv h \ (\text{mod } \mathfrak{a}_1\cap\mathfrak{a}_2)\\ \alpha\in 2^{1/\kappa}\mathfrak{R}_{\mathbf{y}}\setminus\mathfrak{R}_{\mathbf{y}}}} 1. \tag{3.52}$$

The inner sum is amenable to Lemma 3.1, so we have

$$\sum_{\substack{\alpha \equiv h \ (\mathrm{mod}\ \mathfrak{a}_1 \cap \mathfrak{a}_2) \\ \alpha \in 2^{1/\kappa}\mathfrak{R}_{\mathbf{y}} \setminus \mathfrak{R}_{\mathbf{y}}}} 1 = \frac{y}{\sqrt{|D|}\mathbf{N}(\mathfrak{a}_1 \cap \mathfrak{a}_2)} + O\left(\left(\frac{y}{\mathbf{N}(\mathfrak{a}_1 \cap \mathfrak{a}_2)}\right)^{1-\frac{1}{\kappa}}\right). \tag{3.53}$$

Hence, we have

$$\begin{aligned}\sum_{\alpha \in 2^{1/\kappa}\mathfrak{R}_{\mathbf{y}} \setminus \mathfrak{R}_{\mathbf{y}}} \Lambda_R(\alpha; \mathcal{H}, k+\ell)^2 = \frac{y}{\sqrt{|D|}}\mathcal{T} \\ + O\left(y^{1-\frac{1}{\kappa}}(\log R)^{2k+2\ell} \sum_{\mathbf{N}\mathfrak{a}_1, \mathbf{N}\mathfrak{a}_2 \le R} \frac{|\Omega(\mathfrak{a}_1 \cap \mathfrak{a}_2)|}{\mathbf{N}(\mathfrak{a}_1 \cap \mathfrak{a}_2)^{1-1/\kappa}}\right),\end{aligned} \tag{3.54}$$

where

$$\mathcal{T} = \sum_{\mathfrak{a}_1, \mathfrak{a}_2} \frac{|\Omega\,(\mathfrak{a}_1 \cap \mathfrak{a}_2)|}{\mathbf{N}(\mathfrak{a}_1 \cap \mathfrak{a}_2)} \lambda_R(\mathfrak{a}_1; k+\ell)\lambda_R(\mathfrak{a}_2; k+\ell). \tag{3.55}$$

Now

$$\begin{aligned}\sum_{\mathbf{N}\mathfrak{a}_1, \mathbf{N}\mathfrak{a}_2 \le R} \frac{|\Omega(\mathfrak{a}_1 \cap \mathfrak{a}_2)|}{\mathbf{N}(\mathfrak{a}_1 \cap \mathfrak{a}_2)^{1-1/\kappa}} &\ll \sum_{\substack{\mathbf{N}\mathfrak{a} \le R^2 \\ \mathfrak{a} \text{ squarefree}}} \frac{\tau_k(\mathfrak{a})\tau_3(\mathfrak{a})}{(\mathbf{N}\mathfrak{a})^{1-1/\kappa}} \\ &\ll \sum_{\mathbf{N}\mathfrak{a} \le R^2} \frac{\tau_{3k}(\mathfrak{a})}{(\mathbf{N}\mathfrak{a})^{1-1/\kappa}} \\ &\ll R^{2/\kappa}(\log R)^{3k},\end{aligned} \tag{3.56}$$

since $\sum_{\mathfrak{a}} \tau_{3k}(\mathfrak{a})(\mathbf{N}\mathfrak{a})^{-1+1/\kappa-s} = \zeta_K^{3k}(s+1-1/\kappa)$ which has a pole of order $3k$ at $s = 1/\kappa$. Thus,

$$\sum_{\alpha \in 2^{1/\kappa}\mathfrak{R}_{\mathbf{y}} \setminus \mathfrak{R}_{\mathbf{y}}} \Lambda_R(\alpha; \mathcal{H}, k+\ell)^2 = \frac{y}{\sqrt{|D|}}\mathcal{T} + O\left(y^{1-1/\kappa}R^{2/\kappa}(\log R)^c\right). \tag{3.57}$$

We can express $\lambda_R(\mathfrak{a}, n)$ as

$$\lambda_R(\mathfrak{a}, n) = \frac{\mu(\mathfrak{a})}{2\pi i}\int_{(1)} \left(\frac{R}{\mathbf{N}\mathfrak{a}}\right)^s \frac{ds}{s^{n+1}}, \tag{3.58}$$

where $(\alpha)$ denotes the line $\operatorname{Re} s = \alpha$. Thus

$$\mathcal{T} = \frac{1}{(2\pi i)^2}\int_{(1)}\int_{(1)} F(s_1, s_2; \Omega)\frac{R^{s_1+s_2}}{(s_1 s_2)^{k+\ell+1}} ds_1 ds_2, \tag{3.59}$$

where

$$(3.60)\qquad \begin{aligned} F(s_1,s_2;\Omega) &= \sum_{\mathfrak{a}_1,\mathfrak{a}_2} \mu(\mathfrak{a}_1)\mu(\mathfrak{a}_2)\frac{|\Omega\left(\mathfrak{a}_1\cap\mathfrak{a}_2\right)|}{\mathbf{N}(\mathfrak{a}_1\cap\mathfrak{a}_2)(\mathbf{N}\mathfrak{a}_1)^{s_1}(\mathbf{N}\mathfrak{a}_2)^{s_2}} \\ &= \prod_{\mathfrak{p}}\left(1-\frac{|\Omega(\mathfrak{p})|}{\mathbf{N}\mathfrak{p}}\left(\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1}}+\frac{1}{(\mathbf{N}\mathfrak{p})^{s_2}}-\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+s_2}}\right)\right)\end{aligned}$$

in the region of absolute convergence. We put

$$(3.61)$$

$$\begin{aligned} G(s_1,s_2;\Omega) &= F(s_1,s_2;\Omega)\left(\frac{\zeta_K(s_1+1)\zeta_K(s_2+1)}{\zeta_K(s_1+s_2+1)}\right)^k \\ &= \prod_{\mathfrak{p}}\left(1-\frac{|\Omega(\mathfrak{p})|}{\mathbf{N}\mathfrak{p}}\left(\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1}}+\frac{1}{(\mathbf{N}\mathfrak{p})^{s_2}}-\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+s_2}}\right)\right) \\ &\quad\times\left(1-\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+s_2+1}}\right)^k\left(1-\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+1}}\right)^{-k}\left(1-\frac{1}{(\mathbf{N}\mathfrak{p})^{s_2+1}}\right)^{-k}.\end{aligned}$$

In particular, we have the singular series

$$(3.62)\qquad \mathfrak{S}(\mathcal{H}) = G(0,0;\Omega) = \prod_{\mathfrak{p}}\left(1-\frac{|\Omega(\mathfrak{p})|}{\mathbf{N}\mathfrak{p}}\right)\left(1-\frac{1}{\mathbf{N}\mathfrak{p}}\right)^{-k}.$$

Suppose $\min(\operatorname{Re}s_1,\operatorname{Re}s_2,0)=\sigma_0>-c$. For $\mathbf{N}\mathfrak{p}\geq 2^\kappa H$, we have $|\Omega(\mathfrak{p})|=k$, for if $\mathfrak{p}\mid h_i-h_j$ then $\mathbf{N}\mathfrak{p}\mid |\prod_\tau(h_i^\tau-h_j^\tau)|\leq 2^\kappa H$. Taking logarithms of only those factors in the above product yields

$$(3.63)$$

$$\begin{aligned} &\sum_{\substack{\mathfrak{p}\\ \mathbf{N}\mathfrak{p}\geq 2^\kappa H}}\left(\log\left(1-\frac{k}{\mathbf{N}\mathfrak{p}}\left(\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1}}+\frac{1}{(\mathbf{N}\mathfrak{p})^{s_2}}-\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+s_2}}\right)\right)\right. \\ &\quad+k\log\left(1-\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+s_2+1}}\right)-k\log\left(1-\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+1}}\right) \\ &\quad\left.-k\log\left(1-\frac{1}{(\mathbf{N}\mathfrak{p})^{s_2+1}}\right)\right) \\ &=-\sum_{\substack{\mathfrak{p}\\ \mathbf{N}\mathfrak{p}\geq 2^\kappa H}}\sum_{m=1}^\infty\frac{1}{m}\left(k^m\left(\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+1}}+\frac{1}{(\mathbf{N}\mathfrak{p})^{s_2+1}}-\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+s_2+1}}\right)^m\right. \\ &\quad\left.+k\left(\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+s_2+1}}\right)^m-k\left(\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+1}}\right)^m-k\left(\frac{1}{(\mathbf{N}\mathfrak{p})^{s_2+1}}\right)^m\right).\end{aligned}$$

Now the $m = 1$ term in the inner sum cancels to zero, so the last double series in absolute value is

$$
\begin{aligned}
&\ll \sum_{m=2}^{\infty} \frac{k^m}{m} \sum_{\substack{\mathfrak{p} \\ \mathbf{N}(\mathfrak{p}) \geq 2^{\kappa} H}} (\mathbf{N}\mathfrak{p})^{-m(2\sigma_0+1)} \\
&= \sum_{m=2}^{\infty} \frac{k^m}{m} \sum_{n \geq 2^{\kappa} H} r^*(n) n^{-m(2\sigma_0+1)} \qquad (3.64) \\
&= \sum_{m=2}^{\infty} \frac{k^m}{m} \sum_{n \geq 2^{\kappa} H} n^{-m(2\sigma_0+1)+\varepsilon},
\end{aligned}
$$

where $r^*(n)$ is the number of prime ideals with norm $n$, so that $r^*(n) \leq r(n) \ll n^{\varepsilon}$. Then provided that $2\sigma_0 + 1 > 1/2$ and $H$ is large enough, the above is

$$
\begin{aligned}
&\ll \sum_{m=2}^{\infty} \frac{k^m}{m} \frac{(2^{\kappa} H)^{-m(2\sigma_0+1)+\varepsilon+1}}{m(2\sigma_0+1) - \varepsilon - 1} \qquad (3.65) \\
&\ll \sum_{m=2}^{\infty} \frac{1}{m^2} \ll 1.
\end{aligned}
$$

Thus, the part of the product in question is uniformly bounded. Logarithm of the factors $k^2 < N\mathfrak{p} < 2^{\kappa} H$ is

(3.66)

$$
\begin{aligned}
= - \sum_{\substack{\mathfrak{p} \\ k^2 < \mathbf{N}\mathfrak{p} < 2^{\kappa} H}} \sum_{m=1}^{\infty} \frac{1}{m} \Bigg( &|\Omega(\mathfrak{p})|^m \left( \frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+1}} + \frac{1}{(\mathbf{N}\mathfrak{p})^{s_2+1}} - \frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+s_2+1}} \right)^m \\
&+ k \left( \frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+s_2+1}} \right)^m - k \left( \frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+1}} \right)^m - k \left( \frac{1}{(\mathbf{N}\mathfrak{p})^{s_2+1}} \right)^m \Bigg).
\end{aligned}
$$

The $m \geq 2$ terms above are $\ll 1$ as before, and the $m = 1$ term is

(3.67)

$$
\ll \sum_{\substack{\mathfrak{p} \\ k^2 < \mathbf{N}\mathfrak{p} < 2^{\kappa} H}} (\mathbf{N}\mathfrak{p})^{-2\sigma_0-1} \ll H^{-2\sigma_0} \sum_{\substack{\mathfrak{p} \\ \mathbf{N}\mathfrak{p} < 2^{\kappa} H}} (\mathbf{N}\mathfrak{p})^{-1} \ll H^{-2\sigma_0} \log\log H.
$$

Bounding the remaining $k^2$ terms is trivial, so we obtain

$$
G(s_1, s_2; \Omega) \ll \exp\left( c(\log y)^{-2\sigma_0} \log\log\log y \right). \qquad (3.68)
$$

Now we write (3.59) as

$$\mathcal{T} = \frac{1}{(2\pi i)^2} \int_{(1)} \int_{(1)} G(s_1, s_2; \Omega) \times \left( \frac{\zeta_K(s_1+s_2+1)}{\zeta_K(s_1+1)\zeta_K(s_2+1)} \right)^k \frac{R^{s_1+s_2}}{(s_1 s_2)^{k+\ell+1}} ds_1 ds_2. \tag{3.69}$$

We put $U = \exp\left(\sqrt{\log y}\right)$, and shift the $s_1$ and $s_2$ contours to the vertical lines $c_0(\log U)^{-1} + it$ and $c_0(2\log U)^{-1} + it$, $c_0$ being half the constant of Proposition 3.7, and truncate them to $|t| \le U$ and $|t| \le U/2$, and denote the results by $L_1$ and $L_2$ respectively. We have

$$\int_{(c_0(2\log U)^{-1})} \int_{(c_0(\log U)^{-1})} = \int_{L_2}\int_{L_1} + \int_{\substack{|t_2|>U/2 \\ \sigma_2 = c_0(2\log U)^{-1}}} \int_{L_1} + \int_{(c_0(2\log U)^{-1})} \int_{\substack{|t_1|>U \\ \sigma_1 = c_0(\log U)^{-1}}} . \tag{3.70}$$

Now using (3.68), and the fact that $\zeta_K(1+\delta+it), \zeta_K(1+\delta+it)^{-1} \ll 1/\delta$, we have

$$\begin{aligned}
&\int_{\substack{|t_2|>U/2 \\ \sigma_2 = c_0(2\log U)^{-1}}} \int_{L_1} G(s_1, s_2; \Omega) \left( \frac{\zeta_K(s_1+s_2+1)}{\zeta_K(s_1+1)\zeta_K(s_2+1)} \right)^k \frac{R^{s_1+s_2}}{(s_1 s_2)^{k+\ell+1}} ds_1 ds_2 \\
&\ll \exp\left(c \log\log\log y\right) \left(\sqrt{\log y}\right)^{3k} \int_{\substack{|t_2|>U/2 \\ \sigma_2 = c_0(2\log U)^{-1}}} \int_{L_1} \frac{|ds_1||ds_2|}{|s_1 s_2|^{k+\ell+1}} \\
&\ll \exp\left(c \log\log\log y\right) \left(\sqrt{\log y}\right)^{4k+\ell+1} \exp\left(\tfrac{3}{2}\sqrt{\log y}\right) \int_{\substack{|t_2|>U/2 \\ \sigma_2 = c_0(2\log U)^{-1}}} \frac{|ds_2|}{|s_2|^{k+\ell+1}} \\
&\ll \frac{\exp\left(c \log\log\log y\right) \left(\sqrt{\log y}\right)^{4k+\ell+1} \exp\left(\frac{3}{2}\sqrt{\log y}\right)}{\exp\left((k+\ell)\sqrt{\log y}\right)} \\
&\ll \exp\left(-c\sqrt{\log y}\right).
\end{aligned} \tag{3.71}$$

Similarly,

(3.72)
$$\int\limits_{(c_0(2\log U)^{-1})} \int\limits_{\substack{|t_1|>U \\ \sigma_1=c_0(\log U)^{-1}}} G(s_1,s_2;\Omega)\left(\frac{\zeta_K(s_1+s_2+1)}{\zeta_K(s_1+1)\zeta_K(s_2+1)}\right)^k \times \frac{R^{s_1+s_2}}{(s_1s_2)^{k+\ell+1}}ds_1ds_2$$
$$\ll \exp\left(c\log\log\log y\right)(\sqrt{\log y})^{3k}\exp\left(\tfrac{3}{2}\sqrt{\log y}\right) \times \int\limits_{(c_0(2\log U)^{-1})} \int\limits_{\substack{|t_1|>U \\ \sigma_1=c_0(\log U)^{-1}}} \frac{|ds_1||ds_2|}{(|s_1s_2|^{k+\ell+1})}.$$

Also,

$$\int\limits_{\substack{|t_1|>U \\ \sigma_1=c_0(\log U)^{-1}}} \frac{|ds_1|}{|s_1|^{k+\ell+1}} \ll U^{-k-\ell} \tag{3.73}$$

and

$$\int\limits_{(c_0(2\log U)^{-1})} \frac{|ds_1|}{|s_1|^{k+\ell+1}} \ll (\log U)^{k+\ell+1}, \tag{3.74}$$

and we obtain the same bound as before. Thus

(3.75)
$$\mathcal{T} = \frac{1}{(2\pi i)^2}\int\limits_{L_2}\int\limits_{L_1} G(s_1,s_2;\Omega)\left(\frac{\zeta_K(s_1+s_2+1)}{\zeta_K(s_1+1)\zeta_K(s_2+1)}\right)^k \frac{R^{s_1+s_2}}{(s_1s_2)^{k+\ell+1}}ds_1ds_2 + O\left(\exp\left(-c\sqrt{\log y}\right)\right).$$

Now we shift the $L_1$ contour to $L_3$: $-c_0(\log U)^{-1}+it$, $|t|<U$. We encounter singularities at $s_1=0$ and $s_1=-s_2$. Now

$$\int\limits_{L_2}\int\limits_{L_3} G(s_1,s_2;\Omega)\left(\frac{\zeta_K(s_1+s_2+1)}{\zeta_K(s_1+1)\zeta_K(s_2+1)}\right)^k \frac{R^{s_1+s_2}}{(s_1s_2)^{k+\ell+1}}ds_1ds_2 \ll \exp\left(-c\sqrt{\log y}\right) \tag{3.76}$$

as well, thus we have

$$\mathcal{T} = \frac{1}{(2\pi i)^2}\int_{L_2}\left(\operatorname*{Res}_{s_1=-s_2} + \operatorname*{Res}_{s_1=0}\right)ds_2 + O\left(\exp\left(-c\sqrt{\log y}\right)\right). \tag{3.77}$$

Now,

(3.78)

$$\operatorname*{Res}_{s_1=-s_2} = \frac{1}{2\pi i}\int\limits_{C(s_2)} G(s_1,s_2;\Omega)\left(\frac{\zeta_K(s_1+s_2+1)}{\zeta_K(s_1+1)\zeta_K(s_2+1)}\right)^k \frac{R^{s_1+s_2}}{(s_1s_2)^{k+\ell+1}}ds_1,$$

with the circle $C(s_2)$: $|s_1+s_2| = (\log y)^{-1}$. So $G(s_1,s_2;\Omega) \ll (\log\log y)^c$; $\zeta_K(s_1+s_2+1) \ll \log y$; $R^{s_1+s_2} \ll 1$. Also, since $|s_2| \ll |s_1| \ll |s_2|$, we have $(s_1\zeta_K(s_1+1))^{-1} \ll \log(|s_2|+2)(|s_2|+1)^{-1}$. Thus,

$$\operatorname*{Res}_{s_1=-s_2} \ll (\log y)^{k-1}(\log\log y)^c\left(\frac{\log(|s_2|+2)}{|s_2|+1}\right)^{2k}|s_2|^{-2\ell-2}. \tag{3.79}$$

Using this in (3.77), we obtain

$$\mathcal{T} = \frac{1}{(2\pi i)^2}\int_{L_2}\left(\operatorname*{Res}_{s_1=0}\right)ds_2 + O\left((\log y)^{k+\ell-1/2}(\log\log y)^c\right). \tag{3.80}$$

Now we put

$$Z(s_1,s_2) = G(s_1,s_2;\Omega)\left(\frac{(s_1+s_2)\zeta_K(s_1+s_2+1)}{s_1\zeta_K(s_1+1)s_2\zeta_K(s_2+1)}\right)^k. \tag{3.81}$$

This is regular around $(0,0)$. Then

$$\operatorname*{Res}_{s_1=0} = \frac{R^{s_2}}{\ell!s_2^{\ell+1}}\left(\frac{\partial}{\partial s_1}\right)^\ell_{s_1=0}\left\{\frac{Z(s_1,s_2)}{(s_1+s_2)^k}R^{s_1}\right\}. \tag{3.82}$$

We use this in (3.80) and shift the $s_2$-contour to $L_4$: $-c_0(\log U)^{-1}+it$, $|t| \le U/2$. Now using Cauchy's theorem,

$$\left(\frac{\partial}{\partial s_1}\right)^\ell_{s_1=0}\left\{\frac{Z(s_1,s_2)}{(s_1+s_2)^k}R^{s_1}\right\} = \frac{1}{2\pi i}\int_{C_1}\frac{Z(w,s_2)}{w^{\ell+1}(w+s_2)^k}R^w dw, \tag{3.83}$$

where $C_1$ is the circle around 0 with radius $c_0(2\log U)^{-1}$. Thus on $C_1$, $(w+s_2)^{-1} \ll \sqrt{\log y}$; $R^w \ll \exp\left(-\frac{c_0}{2}\sqrt{\log y}\right)$; $Z(w,s_2) \ll (\log\log y)^c$. From this we see that the integral over $L_4$ is $\ll \exp(-c\sqrt{\log y})$. Thus,

$$\begin{aligned}\mathcal{T} &= \operatorname*{Res}_{s_2=0}\operatorname*{Res}_{s_1=0} + O((\log y)^{k+\ell}) \\ &= \frac{1}{(2\pi i)^2}\int_{C_3}\int_{C_2}\frac{Z(s_1,s_2)R^{s_1+s_2}}{(s_1+s_2)^k(s_1s_2)^{\ell+1}}ds_1ds_2 + O((\log y)^{k+\ell}),\end{aligned} \tag{3.84}$$

where $C_2$ and $C_3$ are the circles $|s_1| = \eta$, $|s_2| = 2\eta$, with a small $\eta > 0$. We write $s_1 = s$, $s_2 = s\xi$. Then the double integral is equal to

$$\frac{1}{(2\pi i)^2}\int_{C_4}\int_{C_2}\frac{Z(s,s\xi)R^{s(\xi+1)}}{(\xi+1)^k\xi^{\ell+1}s^{k+2\ell+1}}dsd\xi, \tag{3.85}$$

where $C_4$ is the circle $|\xi| = 2$. This is equal to

$$\frac{Z(0,0)}{2\pi i(k+2\ell)!}(\log R)^{k+2\ell}\int_{C_4}\frac{(\xi+1)^{2\ell}}{\xi^{\ell+1}}d\xi + O\left((\log y)^{k+2\ell-1}(\log\log y)^c\right). \tag{3.86}$$

We note that $Z(0,0) = \mathfrak{S}(\mathcal{H})/\rho_K^k$ and obtain

Lemma 3.9. *If* (3.43) *and* (3.44) *hold, and* $R \leq y^{1/2}/(\log y)^C$, *where* $C$ *is a sufficiently large constant depending only on* $k$ *and* $\ell$,

$$\begin{aligned}\sum_{\alpha\in 2^{1/n}\mathfrak{R}_{\mathbf{y}}\setminus\mathfrak{R}_{\mathbf{y}}}\Lambda_R(\alpha;\mathcal{H},k+\ell)^2 &= \frac{\mathfrak{S}(\mathcal{H})}{\rho_K^k(k+2\ell)!}\binom{2\ell}{\ell}\frac{y}{\sqrt{|D|}}(\log R)^{k+2\ell} \\ &\quad + O\left(y(\log y)^{k+2\ell-1}(\log\log y)^c\right).\end{aligned} \tag{3.87}$$

Now we quote here a result generalizing the Bombieri-Vinogradov Theorem to number fields [16]. Henceforth we denote by $\omega$ prime elements of $K$.

Lemma 3.10. *Let* $y_1,\ldots,y_\kappa$ *be positive real numbers, and write* $y$ *for the product* $y_1\ldots y_\kappa$. *Then for any constant* $A > 0$,

$$\sum_{\mathbf{N}\mathfrak{q}\leq y^{1/2}/(\log y)^B}\max_{x_i\leq y_i}\max_{(\gamma,\mathfrak{q})=1}\left|\pi(\mathfrak{R}_{\mathbf{x}};\mathfrak{q},\gamma) - \frac{I_{\mathbf{x}}}{\phi(\mathfrak{q})}\right| \ll \frac{y}{(\log y)^A} \tag{3.88}$$

*where* $B$ *is some constant depending on* $A$,

$$\pi(\mathfrak{R}_{\mathbf{x}};\mathfrak{q},\gamma) = \sum_{\substack{\omega\in\mathfrak{R}_{\mathbf{x}} \\ \omega\equiv\gamma\ (\mathrm{mod}\ \mathfrak{q})}} 1. \tag{3.89}$$

*and*

$$I_{\mathbf{x}} = \frac{1}{2^{\kappa-1}\mathbf{hR}} \int_2^{x_i} \cdots \int \frac{du_1 \dots du_\kappa}{\log(u_1 \dots u_\kappa)}. \tag{3.90}$$

*Here* $\mathbf{h}$ *is the class number and* $\mathbf{R}$ *the regulator of* $K$.

If we put

$$\pi^*(\mathfrak{R}_{\mathbf{x}}; \mathfrak{q}, \gamma) = \sum_{\substack{\omega \in 2^{1/\kappa}\mathfrak{R}_{\mathbf{x}} \setminus \mathfrak{R}_{\mathbf{x}} \\ \omega \equiv \gamma \ (\mathrm{mod}\ \mathfrak{q})}} 1, \tag{3.91}$$

then the lemma clearly entails

$$\sum_{\mathbf{N}\mathfrak{q} \le y^{1/2}/(\log y)^B} \max_{x_i \le y_i} \max_{(\gamma,\mathfrak{q})=1} \left| \pi^*(\mathfrak{R}_{\mathbf{x}}; \mathfrak{q}, \gamma) - \frac{I_{2^{1/\kappa}\mathbf{x}} - I_{\mathbf{x}}}{\phi(\mathfrak{q})} \right| \ll \frac{y}{(\log y)^A}. \tag{3.92}$$

Now suppose that $\vartheta$ is an absolute constant $0 < \vartheta < 1$ such that for any $A > 0$ we have

$$\sum_{\mathbf{N}\mathfrak{q} \le y^{\vartheta}} \max_{x_i \le y_i} \max_{(\gamma,\mathfrak{q})=1} \left| \pi^*(\mathfrak{R}_{\mathbf{x}}; \mathfrak{q}, \gamma) - \frac{I_{2^{1/\kappa}\mathbf{x}} - I_{\mathbf{x}}}{\phi(\mathfrak{q})} \right| \ll \frac{y}{(\log y)^A}. \tag{3.93}$$

In particular, we may take $\vartheta$ to be any number less than $1/2$ by the preceding discussion.

Now let $\varpi(\alpha)$ be be the characteristic function of the prime elements in $\mathcal{O}_K$. We shall evaluate

$$\sum_{\alpha \in 2^{1/\kappa}\mathfrak{R}_{\mathbf{y}} \setminus \mathfrak{R}_{\mathbf{y}}} \varpi(\alpha + h)\Lambda_R(\alpha; \mathcal{H}, k + \ell)^2 \tag{3.94}$$

with an arbitrary algebraic integer $h \in \mathfrak{H}$. We note that this is equal to

$$\sum_{\alpha \in 2^{1/\kappa}\mathfrak{R}_{\mathbf{y}} \setminus \mathfrak{R}_{\mathbf{y}}} \varpi(\alpha + h)\Lambda_R(\alpha; \mathcal{H} \setminus \{h\}, k + \ell)^2 \tag{3.95}$$

if $h \in \mathcal{H}$.

Assume that $R \le y^{\vartheta/2}$ and $h \notin \mathcal{H}$. Expanding out the square in (3.94), we obtain

$$\sum_{\mathfrak{a}_1, \mathfrak{a}_2} \lambda_R(\mathfrak{a}_1; k + \ell)\lambda_R(\mathfrak{a}_2; k + \ell) \sum_{\substack{\alpha \in 2^{1/\kappa}\mathfrak{R}_{\mathbf{y}} \setminus \mathfrak{R}_{\mathbf{y}} \\ \alpha \in \Omega(\mathfrak{a}_1 \cap \mathfrak{a}_2)}} \varpi(\alpha + h), \tag{3.96}$$

which in turn equals

(3.97)
$$\sum_{\mathfrak{a}_1,\mathfrak{a}_2} \lambda_R(\mathfrak{a}_1;k+\ell)\lambda_R(\mathfrak{a}_2;k+\ell) \times \sum_{b\in\Omega(\mathfrak{a}_1\cap\mathfrak{a}_2)} \delta((b+h,\mathfrak{a}_1\cap\mathfrak{a}_2))\pi^*(\mathfrak{R}_{\mathbf{y}};\mathfrak{a}_1\cap\mathfrak{a}_2,b+h) + O(y^{1-1/\kappa}(\log y)^c),$$

where $\delta(x)$ is the unit measure placed at $x=1$. To see how the error term arises, note that if $S$ is the set $2^{1/\kappa}\mathfrak{R}_{\mathbf{y}}\setminus\mathfrak{R}_{\mathbf{y}}$ and $\Delta$ denotes symmetric difference, the change in the inner sum introduces an error

$$\sum_{\mathfrak{a}_1,\mathfrak{a}_2} \lambda_R(\mathfrak{a}_1;k+\ell)\lambda_R(\mathfrak{a}_2;k+\ell) \sum_{\substack{\alpha\in(S+h)\Delta S\\ \alpha\in\Omega(\mathfrak{a}_1\cap\mathfrak{a}_2)}} \varpi(\alpha+h). \tag{3.98}$$

Since the volume of the set we sum over is $\ll y^{1-1/\kappa}\log y$, this is majorized by

$$y^{1-1/\kappa}(\log R)^{2k+2\ell+1} \sum_{\mathbf{N}\mathfrak{a}_1,\mathbf{N}\mathfrak{a}_2\le R} \frac{|\Omega(\mathfrak{a}_1\cap\mathfrak{a}_2)|}{\mathbf{N}(\mathfrak{a}_1\cap\mathfrak{a}_2)}, \tag{3.99}$$

and we deal with this sum as we did for the error term in (3.54), this time we obtain only a log-power since the exponent of $\mathbf{N}(\mathfrak{a}_1\cap\mathfrak{a}_2)$ is $-1$ in this case. Now the main term of (3.97) is equal to

(3.100)
$$\sum_{\mathfrak{a}_1,\mathfrak{a}_2} \lambda_R(\mathfrak{a}_1;k+\ell)\lambda_R(\mathfrak{a}_2;k+\ell) \sum_{b\in\Omega(\mathfrak{a}_1\cap\mathfrak{a}_2)} \delta((b+h,\mathfrak{a}_1\cap\mathfrak{a}_2))\frac{I_{2^{1/\kappa}\mathbf{y}}-I_{\mathbf{y}}}{\phi(\mathfrak{a}_1\cap\mathfrak{a}_2)}$$
$$+O\Bigg(\sum_{\mathfrak{a}_1,\mathfrak{a}_2} \lambda_R(\mathfrak{a}_1;k+\ell)\lambda_R(\mathfrak{a}_2;k+\ell) \times \sum_{b\in\Omega(\mathfrak{a}_1\cap\mathfrak{a}_2)} \delta((b+h,\mathfrak{a}_1\cap\mathfrak{a}_2))\left(\pi^*(\mathfrak{R}_{\mathbf{y}};\mathfrak{a}_1\cap\mathfrak{a}_2,b+h)-\frac{I_{2^{1/\kappa}\mathbf{y}}-I_{\mathbf{y}}}{\phi(\mathfrak{a}_1\cap\mathfrak{a}_2)}\right)\Bigg).$$

In the error term consider those $\mathfrak{a}_1,\mathfrak{a}_2$ satisfying

$$|\Omega(\mathfrak{a}_1\cap\mathfrak{a}_2)|\le\tau_k(\mathfrak{a}_1\cap\mathfrak{a}_2)<(\log y)^{A/2}, \tag{3.101}$$

in which case we have $|\{\mathfrak{a}_1, \mathfrak{a}_2 : \mathfrak{a}_1 \cap \mathfrak{a}_2 = \mathfrak{a}\}| = \tau_3(\mathfrak{a}) < (\log y)^{\frac{A \log 3}{2 \log k}}$. Then using (3.93), their contribution is

$$\ll \frac{y}{(\log y)^{A/3}}. \tag{3.102}$$

Now for square-free $\mathfrak{a}$,

$$\begin{aligned} \frac{\phi(\mathfrak{a})}{\mathbf{N}\mathfrak{a}} = \prod_{\mathfrak{p}|\mathfrak{a}} \left(1 - \frac{1}{\mathbf{N}\mathfrak{p}}\right) &\geq \prod_{p|\mathbf{N}\mathfrak{a}} \left(1 - \frac{1}{p}\right)^{\kappa} \\ &= \left(\frac{\phi(\mathbf{N}\mathfrak{a})}{\mathbf{N}\mathfrak{a}}\right)^{\kappa} \\ &\gg \left(\frac{1}{(\log\log \mathbf{N}\mathfrak{a})^{\kappa}}\right). \end{aligned} \tag{3.103}$$

Also,

$$\pi^*(\mathfrak{R}_{\mathbf{y}}; \mathfrak{a}, \gamma) = \pi(\mathfrak{R}_{2^{1/\kappa}\mathbf{y}}; \mathfrak{a}, \gamma) - \pi(\mathfrak{R}_{\mathbf{y}}; \mathfrak{a}, \gamma) \ll \frac{I_{2^{1/\kappa}\mathbf{y}} - I_{\mathbf{y}}}{\phi(\mathfrak{a})} \tag{3.104}$$

by the main theorem of [21], and by a trivial induction $I_{\mathbf{y}} \ll y/\log y$, so,

$$\pi^*(\mathfrak{R}_{\mathbf{y}}; \mathfrak{a}, \gamma) \ll \frac{y}{\phi(\mathfrak{a}) \log y} \ll \frac{y}{\mathbf{N}\mathfrak{a}}. \tag{3.105}$$

Thus the contribution of the terms in the error not satisfying (3.101) is

$$\ll y(\log R)^{2(k+\ell)} \sum_{\mathbf{N}\mathfrak{a}_1, \mathbf{N}\mathfrak{a}_2 \leq R} \frac{\tau_k(\mathfrak{a}_1 \cap \mathfrak{a}_2)}{(\log y)^{A/2}} \frac{|\Omega(\mathfrak{a}_1 \cap \mathfrak{a}_2)|}{\mathbf{N}(\mathfrak{a}_1 \cap \mathfrak{a}_2)}. \tag{3.106}$$

Now

$$\sum_{\mathbf{N}\mathfrak{a}_1, \mathbf{N}\mathfrak{a}_2 \leq R} \frac{\tau_k(\mathfrak{a}_1 \cap \mathfrak{a}_2)|\Omega(\mathfrak{a}_1 \cap \mathfrak{a}_2)|}{\mathbf{N}(\mathfrak{a}_1 \cap \mathfrak{a}_2)} \ll \sum_{\mathbf{N}\mathfrak{a} \leq R^2} \frac{\tau_k^2(\mathfrak{a})\tau_3(\mathfrak{a})}{\mathbf{N}\mathfrak{a}}. \tag{3.107}$$

Accordingly, we consider

$$\begin{aligned} \sum_{\mathfrak{a}} \frac{\tau_k^2(\mathfrak{a})\tau_3(\mathfrak{a})}{(\mathbf{N}\mathfrak{a})^{1+s}} &= \prod_{\mathfrak{p}} \sum_{n=0}^{\infty} \frac{\tau_k^2(\mathfrak{p}^n)\tau_3(\mathfrak{p}^n)}{(\mathbf{N}\mathfrak{p})^{n(1+s)}} \\ &= \prod_{\mathfrak{p}} \left(1 + \frac{3k^2}{(\mathbf{N}\mathfrak{p})^{1+s}} + O\left((\mathbf{N}\mathfrak{p})^{-2\sigma-2}\right)\right) \\ &= \zeta_K(1+s)^{3k^2} \prod_{\mathfrak{p}} \left(1 + O\left((\mathbf{N}\mathfrak{p})^{-2\sigma-2}\right)\right), \end{aligned} \tag{3.108}$$

and using Perron's formula, we see that

$$\sum_{\mathbf{N}\mathfrak{a}\le R^2} \frac{\tau_k^2(\mathfrak{a})\tau_3(\mathfrak{a})}{\mathbf{N}\mathfrak{a}} \ll (\log R)^c. \tag{3.109}$$

Putting everything together, we obtain

$$\begin{aligned}\sum_{\mathfrak{a}_1,\mathfrak{a}_2} &\lambda_R(\mathfrak{a}_1; k+\ell)\lambda_R(\mathfrak{a}_2; k+\ell) \\ &\times \sum_{b\in\Omega(\mathfrak{a}_1\cap\mathfrak{a}_2)} \delta((b+h,\mathfrak{a}_1\cap\mathfrak{a}_2))\pi^*(\mathfrak{R}_{\mathbf{y}};\mathfrak{a}_1\cap\mathfrak{a}_2, b+h) \\ &= (I_{2^{1/\kappa}\mathbf{y}} - I_{\mathbf{y}})\mathcal{T}^* + O\left(\frac{y}{(\log y)^{A/3}}\right),\end{aligned} \tag{3.110}$$

where

$$\mathcal{T}^* = \sum_{\mathfrak{a}_1,\mathfrak{a}_2} \frac{\lambda_R(\mathfrak{a}_1; k+\ell)\lambda_R(\mathfrak{a}_1; k+\ell)}{\phi(\mathfrak{a}_1\cap\mathfrak{a}_2)} \sum_{b\in\Omega(\mathfrak{a}_1\cap\mathfrak{a}_2)} \delta((b+h,\mathfrak{a}_1\cap\mathfrak{a}_2)). \tag{3.111}$$

It remains to evaluate $\mathcal{T}^*$. The inner sum in (3.111) is equal to

$$\prod_{\mathfrak{p}|\mathfrak{a}_1\cap\mathfrak{a}_2}\left(\sum_{b\in\Omega(\mathfrak{p})} \delta((b+h,\mathfrak{p}))\right) = \prod_{\mathfrak{p}|\mathfrak{a}_1\cap\mathfrak{a}_2} (|\Omega^+(\mathfrak{p})|-1), \tag{3.112}$$

where $\Omega^+$ corresponds to the set $\mathcal{H}^+ = \mathcal{H}\cup\{h\}$. As before, we have

$$\begin{aligned}\mathcal{T}^* = \frac{1}{(2\pi i)^2}\int_{(1)}\int_{(1)}\prod_{\mathfrak{p}}\Bigg(1 - \frac{|\Omega^+(\mathfrak{p})|-1}{\mathbf{N}\mathfrak{p}-1} \\ \times\left(\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1}} + \frac{1}{(\mathbf{N}\mathfrak{p})^{s_2}} - \frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+s_2}}\right)\Bigg)\frac{R^{s_1+s_2}}{(s_1s_2)^{k+\ell+1}}ds_1ds_2.\end{aligned} \tag{3.113}$$

We consider the function

$$\begin{aligned}\prod_{\mathfrak{p}}\left(1 - \frac{|\Omega^+(\mathfrak{p})|-1}{\mathbf{N}\mathfrak{p}-1}\left(\frac{1}{(\mathbf{N}\mathfrak{p})^{s_1}} + \frac{1}{(\mathbf{N}\mathfrak{p})^{s_2}} - \frac{1}{(\mathbf{N}\mathfrak{p})^{s_1+s_2}}\right)\right) \\ \times\left(\frac{\zeta_K(s_1+1)\zeta_K(s_2+1)}{\zeta_K(s_1+s_2+1)}\right)^k.\end{aligned} \tag{3.114}$$

If $\mathcal{H}^+$ is admissible, then the singular series is $\mathfrak{S}(\mathcal{H}^+)$ and the estimation of the integrals is analogous to the previous case. Thus we obtain

$$\mathcal{T}^* = \frac{\mathfrak{S}(\mathcal{H}^+)}{\rho_K^k (k+\ell)!}\binom{2\ell}{\ell}(\log R)^{k+2\ell} + O\left((\log y)^{k+2\ell-1}(\log\log y)^c\right). \tag{3.115}$$

On the other hand, if $\mathcal{H}^+$ is not admissable, then the Euler product vanishes at $s_1 = 0$ and $s_2 = 0$ to the order equal to the number of prime ideals with $|\Omega^+(\mathfrak{p})| = \mathbf{N}\mathfrak{p}$, in which case we necessarily have $\mathbf{N}\mathfrak{p} \leq k+1$. The number of such ideals is bounded. Thus the effect of the vanishing factors in the product is only to annhiliate the main term, while the estimation of the error terms remain intact.

Finally, if $h \in \mathcal{H}$, the preceding discussion applies with the translation $k \mapsto k-1$, $\ell \mapsto \ell + 1$.

Thus, we obtain

Lemma 3.11. *Suppose* (3.43), (3.44) *and* (3.93) *hold, then for* $R \leq y^{\vartheta/2}$, *we have*

$$\sum_{\alpha \in 2^{1/\kappa}\mathfrak{R}_{\mathbf{y}} \setminus \mathfrak{R}_{\mathbf{y}}} \varpi(\alpha + h)\Lambda_R(\alpha; \mathcal{H}, k+\ell)^2 = \begin{cases} \dfrac{\mathfrak{S}(\mathcal{H}\cup\{h\})}{\rho_K^k (k+2\ell)!}\dbinom{2\ell}{\ell}(I_{2^{1/\kappa}\mathbf{y}} - I_{\mathbf{y}})(\log R)^{k+2\ell} \\ \qquad + O\left(y(\log y)^{k+2\ell-2}(\log\log y)^c\right), & \text{if } h \notin \mathcal{H}, \\ \\ \dfrac{\mathfrak{S}(\mathcal{H})}{\rho_K^{k-1}(k+2\ell+1)!}\dbinom{2(\ell+1)}{\ell+1}(I_{2^{1/\kappa}\mathbf{y}} - I_{\mathbf{y}})(\log R)^{k+2\ell+1} \\ \qquad + O\left(y(\log y)^{k+2\ell-1}(\log\log y)^c\right), & \text{if } h \in \mathcal{H}. \end{cases} \tag{3.116}$$

Now we use our lemmata to obtain our main theorem. First we need a lemma concerning the average size of $\mathfrak{S}(\mathcal{H})$. The proof follows Gallagher's [6, §2] computation for the rational case.

LEMMA 3.12. *For fixed $k$, there holds*

$$\sum_{\substack{\mathcal{H}\subseteq\mathfrak{H}\\|\mathcal{H}|=k}} \mathfrak{S}(\mathcal{H}) \sim \left(\frac{H}{\sqrt{|D|}}\right)^k \tag{3.117}$$

*as $H \to \infty$, where in the sum the permutations of elements of $\mathcal{H}$ are counted.*

PROOF. Put

$$D_{\mathcal{H}} = \prod_{i<j}(h_i - h_j). \tag{3.118}$$

Then $1 \le |\Omega(\mathfrak{p})| \le k$, with equality at the right unless $\mathfrak{p} \mid D_{\mathcal{H}}$. Now

$$\begin{aligned}\mathfrak{S}(\mathcal{H}) &= \prod_{\mathfrak{p}}\left(1-\frac{|\Omega(\mathfrak{p})|}{\mathbf{N}\mathfrak{p}}\right)\left(1-\frac{1}{\mathbf{N}\mathfrak{p}}\right)^{-k}\\ &= \prod_{\mathfrak{p}}\left(1+\frac{(\mathbf{N}\mathfrak{p})^k-|\Omega(\mathfrak{p})|(\mathbf{N}\mathfrak{p})^{k-1}-(\mathbf{N}\mathfrak{p}-1)^k}{(\mathbf{N}\mathfrak{p}-1)^k}\right) \qquad (3.119)\\ &= \prod_{\mathfrak{p}}\left(1+a(\mathfrak{p},|\Omega(\mathfrak{p})|)\right).\end{aligned}$$

The product converges by the discussion preceding the bound (3.68) for $G(s_1, s_2, \Omega)$. Also, we have the bounds

$$a(\mathfrak{p},|\Omega(\mathfrak{p})|) \ll \begin{cases}(\mathbf{N}\mathfrak{p}-1)^{-2}, & \text{if } |\Omega(\mathfrak{p})| = k,\\ (\mathbf{N}\mathfrak{p}-1)^{-1}, & \text{if } |\Omega(\mathfrak{p})| < k.\end{cases} \tag{3.120}$$

Since when $\mathcal{H}$ is fixed, the $|\Omega(\mathfrak{p})|$ are determined, we can write $a_{\mathcal{H}}(\mathfrak{p})$ for the sake of brevity. We extend $a_{\mathcal{H}}$ multiplicatively to square-free ideals, so that

$$\mathfrak{S}(\mathcal{H}) = \sum_{\mathfrak{a}} a_{\mathcal{H}}(\mathfrak{a}). \tag{3.121}$$

By the bounds in (3.120),

$$\sum_{\mathbf{N}\mathfrak{a}>x} |a_{\mathcal{H}}(\mathfrak{a})| \le \sum_{\mathbf{N}\mathfrak{a}>x} \frac{\mu^2(\mathfrak{a})c^{\omega(\mathfrak{a})}}{\phi^2(\mathfrak{a})}\phi((\mathfrak{a}, D_{\mathcal{H}})), \tag{3.122}$$

where $c$ is an implicit constant coming from (3.120), which we may assume is an integer, and $\omega(\mathfrak{a})$ is the number of distinct prime divisors of $\mathfrak{a}$ (there's no risk of ambiguity with our use of $\omega$ to denote primes due to the presence of an argument). Put $\mathfrak{a} = \mathfrak{d}\mathfrak{e}$ with $\mathfrak{d} \mid D_{\mathcal{H}}$ and $(\mathfrak{e}, D_{\mathcal{H}}) = 1$. Then the right hand side of the above equals

$$\sum_{\mathfrak{d}|D_{\mathcal{H}}} \frac{\mu^2(\mathfrak{d})c^{\omega(\mathfrak{d})}}{\phi(\mathfrak{d})} \sum_{\substack{\mathbf{N}\mathfrak{e}>x/\mathbf{N}\mathfrak{d} \\ (\mathfrak{e},D_{\mathcal{H}})=1}} \frac{\mu^2(\mathfrak{e})c^{\omega(\mathfrak{e})}}{\phi^2(\mathfrak{e})}. \tag{3.123}$$

We first deal with the inner sum. We have

$$\sum_{\substack{\mathbf{N}\mathfrak{e}>x/\mathbf{N}\mathfrak{d} \\ (\mathfrak{e},D_{\mathcal{H}})=1}} \frac{\mu^2(\mathfrak{e})c^{\omega(\mathfrak{e})}}{\phi^2(\mathfrak{e})} \ll \sum_{n=n_0}^{\infty} \sum_{2^n<\mathbf{N}\mathfrak{e}<2^{n+1}} \frac{\mu^2(\mathfrak{e})c^{\omega(\mathfrak{e})}}{\phi^2(\mathfrak{e})}, \tag{3.124}$$

where $n_0 = \left[\frac{\log(x/\mathbf{N}\mathfrak{d})}{\log 2}\right]$. Then provided

$$\sum_{n<\mathbf{N}\mathfrak{e}<2n} \frac{\mu^2(\mathfrak{e})c^{\omega(\mathfrak{e})}}{\phi^2(\mathfrak{e})} \ll \frac{(\log n)^C}{n}, \tag{3.125}$$

we get

$$\sum_{n=n_0}^{\infty} \sum_{2^n<\mathbf{N}\mathfrak{e}<2^{n+1}} \frac{\mu^2(\mathfrak{e})c^{\omega(\mathfrak{e})}}{\phi^2(\mathfrak{e})} \ll \sum_{n=n_0}^{\infty} \frac{n^C}{2^n} \ll \frac{n_0^C}{2^{n_0}} \ll \frac{\mathbf{N}\mathfrak{d}}{x}(\log x)^C. \tag{3.126}$$

Now we show (3.125). We have

$$\begin{aligned} \sum_{n<\mathbf{N}\mathfrak{e}<2n} \frac{\mu^2(\mathfrak{e})c^{\omega(\mathfrak{e})}}{\phi^2(\mathfrak{e})} &< \sum_{\mathbf{N}\mathfrak{e}<2n} \frac{\mathbf{N}\mathfrak{e}}{n} \frac{\mu^2(\mathfrak{e})c^{\omega(\mathfrak{e})}}{\phi^2(\mathfrak{e})} \\ &= \frac{1}{n} \sum_{\mathbf{N}\mathfrak{e}<2n} \frac{\mathbf{N}\mathfrak{e}\mu^2(\mathfrak{e})c^{\omega(\mathfrak{e})}}{\phi^2(\mathfrak{e})}, \end{aligned} \tag{3.127}$$

thus showing

$$\sum_{\mathbf{N}\mathfrak{e}<x} \frac{\mathbf{N}\mathfrak{e}\mu^2(\mathfrak{e})c^{\omega(\mathfrak{e})}}{\phi^2(\mathfrak{e})} \ll (\log x)^C \tag{3.128}$$

suffices. We form the corresponding Dirichlet series.

(3.129)
$$\begin{aligned}\sum_{\mathfrak{e}} \frac{\mathbf{N}\mathfrak{e}\mu^2(\mathfrak{e})c^{\omega(\mathfrak{e})}}{\phi^2(\mathfrak{e})(\mathbf{N}\mathfrak{e})^s} &= \prod_{\mathfrak{p}} \sum_{n=0}^{\infty} \frac{(\mathbf{N}\mathfrak{p})^n \mu^2(\mathfrak{p}^n)c^{\omega(\mathfrak{p}^n)}}{\phi^2(\mathfrak{p}^n)(\mathbf{N}\mathfrak{p})^{ns}} \\ &= \prod_{\mathfrak{p}} \left(1 + \frac{c\mathbf{N}\mathfrak{p}}{(\mathbf{N}\mathfrak{p}-1)^2(\mathbf{N}\mathfrak{p})^s}\right) \\ &= \prod_{\mathfrak{p}} \left(1 + \frac{c}{(\mathbf{N}\mathfrak{p})^{s+1}} \left(\frac{\mathbf{N}\mathfrak{p}}{\mathbf{N}\mathfrak{p}-1}\right)^2\right) \\ &= \prod_{\mathfrak{p}} \left(1 + \frac{c}{(\mathbf{N}\mathfrak{p})^{s+1}} + O\left(\frac{1}{(\mathbf{N}\mathfrak{p})^{2+\sigma}}\right)\right) \\ &= \zeta_K^c(s+1) \prod_{\mathfrak{p}} \left(1 + O((\mathbf{N}\mathfrak{p})^{-2-\sigma} + (\mathbf{N}\mathfrak{p})^{-2-2\sigma+\varepsilon})\right).\end{aligned}$$

The product is bounded for $\sigma > -1/4$, say, hence Perron's formula gives us (3.128). Thus (3.123) is

$$\ll \frac{(\log x)^C}{x} \sum_{\mathfrak{d}|D_{\mathcal{H}}} \frac{\mu^2(\mathfrak{d})c^{\omega(\mathfrak{d})}\mathbf{N}\mathfrak{d}}{\phi(\mathfrak{d})}, \tag{3.130}$$

and we have

$$\begin{aligned}\sum_{\mathfrak{d}|D_{\mathcal{H}}} \frac{\mu^2(\mathfrak{d})c^{\omega(\mathfrak{d})}\mathbf{N}\mathfrak{d}}{\phi(\mathfrak{d})} &= \prod_{\mathfrak{p}^j \| D_{\mathcal{H}}} \sum_{\mathfrak{d}|\mathfrak{p}^j} \frac{\mu^2(\mathfrak{d})c^{\omega(\mathfrak{d})}\mathbf{N}\mathfrak{d}}{\phi(\mathfrak{d})} \\ &= \prod_{\mathfrak{p}|D_{\mathcal{H}}} \left(1 + C + \frac{C}{\mathbf{N}\mathfrak{p}-1}\right) \\ &\le c'^{\omega(D_{\mathcal{H}})} \ll (\mathbf{N}D_{\mathcal{H}})^{\varepsilon} \ll H^{\varepsilon},\end{aligned} \tag{3.131}$$

so we obtain

$$\mathfrak{S}(\mathcal{H}) = \sum_{\mathbf{N}\mathfrak{a}\le x} a_{\mathcal{H}}(\mathfrak{a}) + O\left(\frac{(Hx)^{\varepsilon}}{x}\right). \tag{3.132}$$

We sum this over all $\mathcal{H} \subseteq \mathfrak{H}$ with $|\mathcal{H}| = k$ and get

$$\sum_{\substack{\mathcal{H}\subseteq\mathfrak{H}\\|\mathcal{H}|=k}} \mathfrak{S}(\mathcal{H}) = \sum_{\mathbf{N}\mathfrak{a}\le x} \sum_{\substack{\mathcal{H}\subseteq\mathfrak{H}\\|\mathcal{H}|=k}} a_{\mathcal{H}}(\mathfrak{a}) + O\left(H^k \frac{(Hx)^{\varepsilon}}{x}\right). \tag{3.133}$$

The inner sum here can be rewritten

$$\sum_{\nu}\prod_{\mathfrak{p}|\mathfrak{a}} a(\mathfrak{p},|\Omega(\mathfrak{p})|)\left(\sum{}' 1 + O(H^{k-1})\right). \tag{3.134}$$

Here $\nu$ runs over "vectors" $(\ldots,|\Omega(\mathfrak{p})|,\ldots)_{\mathfrak{p}|\mathfrak{a}}$ with $1 \le |\Omega(\mathfrak{p})| \le \mathbf{N}\mathfrak{p}$, and $\sum{}' 1$ is the number of $k$-tuples $h_1,\ldots,h_k \in \mathfrak{H}$ of not necessarily distinct integers which, for each prime ideal $\mathfrak{p} \mid \mathfrak{a}$ occupy exactly $|\Omega(\mathfrak{p})|$ residue classes modulo $\mathfrak{p}$. The error term arises because we dropped the restriction that the $h_i$ be distinct.

For each $\mathfrak{p} \mid \mathfrak{a}$, there are $\binom{\mathbf{N}\mathfrak{p}}{|\Omega(\mathfrak{p})|}$ ways of choosing the $|\Omega(\mathfrak{p})|$ to be occupied, and once these are chosen, $\sigma(k,|\Omega(\mathfrak{p})|)$ ways of assigning one of them to each $h_i$, where $\sigma(r,|\Omega(\mathfrak{p})|)$ is the number of surjective maps from a set of $k$ elements into a set of $|\Omega(\mathfrak{p})|$ elements. Then using the Chinese Remainder Theorem and appealing to Lemma 3.1, we see that

$$\sum{}' 1 = \left(\left(\frac{H}{\sqrt{|D|}\mathbf{N}\mathfrak{a}}\right)^k + O\left(\left(\frac{H}{\mathbf{N}\mathfrak{a}}\right)^{k-1/\kappa}\right)\right)\prod_{\mathfrak{p}|\mathfrak{a}}\binom{\mathbf{N}\mathfrak{p}}{|\Omega(\mathfrak{p})|}\sigma(k,|\Omega(\mathfrak{p})|). \tag{3.135}$$

Thus the inner sum in (3.133) is

$$\left(\frac{H}{\sqrt{|D|}\mathbf{N}\mathfrak{a}}\right)^k A(\mathfrak{a}) + O\left(\left(\frac{H}{\mathbf{N}\mathfrak{a}}\right)^{k-1/\kappa} B(\mathfrak{a})\right) + O(H^{k-1}C(\mathfrak{a})), \tag{3.136}$$

where

$$\begin{aligned}
A(\mathfrak{a}) &= \sum_{\nu}\prod_{\mathfrak{p}|\mathfrak{a}} a(\mathfrak{p},|\Omega(\mathfrak{p})|)\binom{\mathbf{N}\mathfrak{p}}{|\Omega(\mathfrak{p})|}\sigma(k,|\Omega(\mathfrak{p})|),\\
B(\mathfrak{a}) &= \sum_{\nu}\prod_{\mathfrak{p}|\mathfrak{a}} |a(\mathfrak{p},|\Omega(\mathfrak{p})|)|\binom{\mathbf{N}\mathfrak{p}}{|\Omega(\mathfrak{p})|}\sigma(k,|\Omega(\mathfrak{p})|),\\
C(\mathfrak{a}) &= \sum_{\nu}\prod_{\mathfrak{p}|\mathfrak{a}} |a(\mathfrak{p},|\Omega(\mathfrak{p})|)|.
\end{aligned} \tag{3.137}$$

Rearranging, we get

$$
\begin{aligned}
A(\mathfrak{a}) &= \prod_{\mathfrak{p}|\mathfrak{a}} \left( \sum_{\nu=1}^{\mathbf{N}\mathfrak{p}} a(\mathfrak{p},\nu) \binom{\mathbf{N}\mathfrak{p}}{\nu} \sigma(k,\nu) \right), \\
B(\mathfrak{a}) &= \prod_{\mathfrak{p}|\mathfrak{a}} \left( \sum_{\nu=1}^{\mathbf{N}\mathfrak{p}} |a(\mathfrak{p},\nu)| \binom{\mathbf{N}\mathfrak{p}}{\nu} \sigma(k,\nu) \right), \\
C(\mathfrak{a}) &= \prod_{\mathfrak{p}|\mathfrak{a}} \left( \sum_{\nu=1}^{\mathbf{N}\mathfrak{p}} |a(\mathfrak{p},\nu)| \right).
\end{aligned} \tag{3.138}
$$

Now for $\mathbf{N}\mathfrak{a} > 1$, we show that $A(\mathfrak{a}) = 0$. Plugging in the explicit expression for $a(\mathfrak{p},\nu)$, the $\mathfrak{p}$-factor becomes

$$
\begin{aligned}
(\mathbf{N}\mathfrak{p}-1)^{-k} \Bigg( \Big( (\mathbf{N}\mathfrak{p})^k - (\mathbf{N}\mathfrak{p}-1)^k \Big) \sum_{\nu=1}^{\mathbf{N}\mathfrak{p}} \binom{\mathbf{N}\mathfrak{p}}{\nu} \sigma(k,\nu) \\
- (\mathbf{N}\mathfrak{p})^{k-1} \sum_{\nu=1}^{\mathbf{N}\mathfrak{p}} \nu \binom{\mathbf{N}\mathfrak{p}}{\nu} \sigma(k,\nu) \Bigg),
\end{aligned} \tag{3.139}
$$

and using the combinatorial identities [6, (i) and (ii) of §3]

$$
\sum_{\nu=1}^{n} \binom{n}{\nu} \sigma(r,\nu) = n^r \tag{3.140}
$$

and

$$
\sum_{\nu=1}^{n} \nu \binom{n}{\nu} \sigma(r,\nu) = n^{r+1} - (n-1)^r n, \tag{3.141}
$$

the sums are $(\mathbf{N}\mathfrak{p})^k$ and $(\mathbf{N}\mathfrak{p})^{k+1} - (\mathbf{N}\mathfrak{p}-1)^k \mathbf{N}\mathfrak{p}$ respectively, so the factor vanishes. In the same manner, using the weaker bound in (3.120), namely that $a(\mathfrak{p},\nu) \ll (\mathbf{N}\mathfrak{p}-1)^{-1}$, we find that the $\mathfrak{p}$-th factor in $B(\mathfrak{a})$ is $\ll (\mathbf{N}\mathfrak{p})^k/(\mathbf{N}\mathfrak{p}-1)$, whence

$$
B(\mathfrak{a}) \leq c^{\omega(\mathfrak{a})} \frac{(\mathbf{N}\mathfrak{a})^k}{\phi(\mathfrak{a})}. \tag{3.142}
$$

Likewise, the $\mathfrak{p}$-th factor in $C(\mathfrak{a})$ is $\ll \mathbf{N}\mathfrak{p}/(\mathbf{N}\mathfrak{p}-1)$, so

$$
C(\mathfrak{a}) \leq c^{\omega(\mathfrak{a})} \frac{\mathbf{N}\mathfrak{a}}{\phi(\mathfrak{a})}. \tag{3.143}
$$

Thus we see that (3.133) is $(H/\sqrt{|D|})^k$ plus an error term, which is

$$\ll H^{k-1/\kappa} \sum_{\mathbf{N}\mathfrak{a}\le x} c^{\omega(\mathfrak{a})} \frac{\mathbf{N}\mathfrak{a}}{\phi(\mathfrak{a})} + H^k \frac{(Hx)^\varepsilon}{x}. \tag{3.144}$$

To deal with the sum, we again form the corresponding Dirichlet series with a view to using Perron's formula. We have

(3.145)

$$\begin{aligned}
\sum_{\mathfrak{a}} c^{\omega(\mathfrak{a})} \frac{\mathbf{N}\mathfrak{a}}{\phi(\mathfrak{a})(\mathbf{N}\mathfrak{a})^s} &= \prod_{\mathfrak{p}} \left(1 + \frac{c\mathbf{N}\mathfrak{p}}{\mathbf{N}\mathfrak{p}-1}\left((\mathbf{N}\mathfrak{p})^{-s} + (\mathbf{N}\mathfrak{p})^{-2s} + \cdots\right)\right)\\
&= \prod_{\mathfrak{p}} \left(1 + \left(1 + \frac{1}{\mathbf{N}\mathfrak{p}-1}\right)\left(c(\mathbf{N}\mathfrak{p})^{-s} + c(\mathbf{N}\mathfrak{p})^{-2s} + \cdots\right)\right)\\
&= \prod_{\mathfrak{p}} \left(1 + c(\mathbf{N}\mathfrak{p})^{-s} + O\left(\frac{1}{(\mathbf{N}\mathfrak{p})^{\sigma+1}} + \frac{1}{(\mathbf{N}\mathfrak{p})^{2\sigma-\varepsilon}}\right)\right)\\
&= \zeta_K^c(s) \prod_{\mathfrak{p}} \left(1 + O\left(\frac{1}{(\mathbf{N}\mathfrak{p})^{\sigma+1}} + \frac{1}{(\mathbf{N}\mathfrak{p})^{2\sigma-\varepsilon}}\right)\right),
\end{aligned}$$

and Perron's formula yields that the sum in question is $\ll x(\log x)^c \ll x^{1+\varepsilon}$. Thus (3.144) is

$$\begin{aligned}
&\ll H^{k-1/\kappa} x^{1+\varepsilon} + H^k (xH)^\varepsilon / x\\
&\ll H^{k-\frac{1}{2\kappa}+\varepsilon}
\end{aligned} \tag{3.146}$$

upon choosing $x = H^{1/2\kappa}$. This completes the proof. □

Now we are in a position to prove our main theorem. We shall evaluate

$$\sum_{\substack{\mathcal{H}\subseteq\mathfrak{H}\\ |\mathcal{H}|=k}} \sum_{\alpha\in 2^{1/\kappa}\mathfrak{R}_{\mathbf{y}}\setminus\mathfrak{R}_{\mathbf{y}}} \left(\sum_{h\in\mathfrak{H}} \varpi(\alpha+h) - 1\right) \Lambda_R(\alpha;\mathcal{H},k+\ell)^2 \tag{3.147}$$

to see that it is positive, for if this is the case, then there is an algebraic integer $\alpha \in 2^{1/\kappa}\mathfrak{R}_{\mathbf{y}} \setminus \mathfrak{R}_{\mathbf{y}}$ such that

$$\sum_{h\in\mathfrak{H}} \varpi(\alpha+h) - 1 > 0. \tag{3.148}$$

Hence

$$\min_{\substack{\omega_0,\omega_1\in\alpha+\mathfrak{H}\\ \omega_0\neq\omega_1}} |\mathbf{N}(\omega_1-\omega_0)| \le H. \tag{3.149}$$

Assume $R = y^{\vartheta/2}$, so that lemmata 3.9 and 3.12 hold. We also note that a simple induction (where the assumption that $y_i \asymp y^{1/\kappa}$ is relevant) shows that

$$I_{\mathbf{y}} = \frac{y}{2^{\kappa-1}\mathbf{hR}\log y}\left(1+o(1)\right). \tag{3.150}$$

Then using these we see that (3.147) is asymptotically equal to

$$\sum_{\substack{\mathcal{H}\subseteq\mathfrak{H}\\|\mathcal{H}|=k}}\sum_{\alpha\in 2^{1/\kappa}\mathfrak{R}_{\mathbf{y}}\setminus\mathfrak{R}_{\mathbf{y}}}\left\{\sum_{h\notin\mathfrak{H}}+\sum_{h\in\mathfrak{H}}\right\}\varpi(\alpha+h)\Lambda_R(\alpha;\mathcal{H},k+\ell)^2$$
$$-\frac{1}{(\sqrt{|D|})^{k+1}\rho_K^k(k+2\ell)!}\binom{2\ell}{\ell}yH^k(\log R)^{k+2\ell},$$

with an error of size $o(yH^k(\log y)^{k+2\ell+1})$, and using lemmata 3.11 and 3.12 together with (3.150) this is asymptotically equal in the same sense to

(3.151)
$$\begin{aligned}&\frac{1}{2^{\kappa-1}(\sqrt{|D|})^{k+1}\mathbf{hR}\rho_K^k(k+2\ell)!}\binom{2\ell}{\ell}\frac{y}{\log y}H^{k+1}(\log R)^{k+2\ell}\\&\quad+\frac{k}{2^{\kappa-1}(\sqrt{|D|})^{k}\mathbf{hR}\rho_K^{k-1}(k+2\ell+1)!}\binom{2(\ell+1)}{\ell+1}\frac{y}{\log y}H^{k}(\log R)^{k+2\ell+1}\\&\quad-\frac{1}{(\sqrt{|D|})^{k+1}\rho_K^k(k+2\ell)!}\binom{2\ell}{\ell}yH^k(\log R)^{k+2\ell}\\&=\left(\frac{H}{2^{\kappa-1}\sqrt{|D|}\mathbf{hR}\rho_K}+\frac{k}{2^{\kappa-1}\mathbf{hR}(k+2\ell+1)}\cdot\frac{2(2\ell+1)}{\ell+1}\log R\right.\\&\quad\left.-\frac{1}{\sqrt{|D|}\rho_K}\log y\right)\times\frac{1}{(\sqrt{|D|})^{k}\rho_K^{k-1}(k+2\ell)!}\binom{2\ell}{\ell}\frac{y}{\log y}H^k(\log R)^{k+2\ell},\end{aligned}$$

so it suffices to show that the expression in the parentheses is positive. Recall that the analytic class number formula in the totally real case is given by

$$\rho_K = \frac{2^{\kappa-1}\mathbf{hR}}{\sqrt{|D|}}. \tag{3.152}$$

Thus with further simplication (3.147) is positive provided

$$\frac{H}{\log y} \geq \left(1+\varepsilon-\frac{k}{k+2\ell+1}\cdot\frac{2(2\ell+1)}{\ell+1}\cdot\frac{\vartheta}{2}\right)2^{\kappa-1}\mathbf{hR}\sqrt{D}. \tag{3.153}$$

Choosing $\ell = [\sqrt{k}]$, we get

$$\liminf_{\substack{\omega_0,\omega_1\in\mathcal{O}_K\\ \omega_0\neq\omega_1}} \left|\frac{\mathbf{N}(\omega_1-\omega_0)}{\log \mathbf{N}\omega_0}\right| = 2^{\kappa-1}\mathbf{hR}\sqrt{D}\inf_{\vartheta<1/2}\max\{0,1-2\vartheta\} = 0, \tag{3.154}$$

hence we obtain our

Theorem 3.13. *Let $K$ be a totally real number field. Then*

$$\liminf_{\substack{\omega_0,\omega_1\in\mathcal{O}_K \text{ prime}\\ \omega_0\neq\omega_1}} \left|\frac{\mathbf{N}(\omega_1-\omega_0)}{\log \mathbf{N}\omega_0}\right| = 0. \tag{3.155}$$

CHAPTER 4

# The Maynard-Tao method

The aim of this chapter is to give an informal outline of the workings of the Maynard-Tao modification of the GPY method. As in Chapter 2, we will by no means aim for rigour, since the the technical details of the method will be demonstrated through application in the next chapter.

To understand the effect of the modification, we take a step back and reformulate the latter in more general terms. Recall that in the GPY method we were interested in establishing the positivity of an expression of the form

$$S = \sum_{X<n\leq 2X} \left( \sum_{i=1}^{k} \theta(n+h_i) - \log 3X \right) \left( \sum_{d \mid P(n;\mathcal{H})} \lambda(d) \right)^2 , \tag{4.1}$$

where $\lambda(d)$ was given by

$$\lambda(d) = \begin{cases} \dfrac{\mu(d)}{(k+\ell)!} \left( \log \dfrac{R}{d} \right)^{k+\ell}, & \text{if } d \leq R, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Here, for purposes of generality, we leave $\lambda(d)$ unspecified, but assume that it is supported on square-free integers $d \leq R$. Expanding the square and rearranging, we find that (4.1) equals

$$S = \sum_{d_1,d_2} \lambda(d_1)\lambda(d_2) \left( \sum_{i=1}^{k} \sum_{\substack{X<n\leq 2X \\ [d_1,d_2] \mid P(n;\mathcal{H})}} \theta(n+h_i) - (\log 3X) \sum_{\substack{X<n\leq 2X \\ [d_1,d_2] \mid P(n;\mathcal{H})}} 1 \right) \tag{4.3}$$

The second sum counts the number of $n \in (X, 2X]$ that fall into one of the $\nu_{\mathcal{H}}([d_1, d_2])$ residue classes (mod $[d_1, d_2]$), so the contribution (within acceptable error) of the inner sum is

$$X(\log 3X) \sum_{\substack{d_1,d_2 \\ D=[d_1,d_2]}} \lambda(d_1)\lambda(d_2)\frac{\nu_{\mathcal{H}}(D)}{D}. \tag{4.4}$$

The first sum, on the other hand, can be approximated for each residue class $n \pmod{[d_1,d_2]} \in \Omega([d_1,d_2])$ by $X/\phi([d_1,d_2])$ only when $n+h_i$ runs over residue classes relatively prime to $[d_1,d_2]$, or what is the same, if $n \in \Omega([d_1,d_2])$ but $n \not\equiv -h_i \pmod{p}$ for any $p \mid [d_1,d_2]$. So we must have $n \in \Omega_i(p) = \Omega(p)\setminus\{-h_i \pmod{p}\}$ for each $p \mid [d_1,d_2]$ by the Chinese Remainder Theorem. Thus $|\Omega_i(D)| = \nu_{\mathcal{H}}^*(D)$ where $\nu_{\mathcal{H}}^*(D)$ is the multiplicative function defined on primes by $\nu_{\mathcal{H}}^*(p) = \nu_{\mathcal{H}}(p)-1$. With these, the first (double) sum contributes

$$Xk \sum_{\substack{d_1,d_2 \\ D=[d_1,d_2]}} \lambda(d_1)\lambda(d_2)\frac{\nu_{\mathcal{H}}^*(D)}{\phi(D)}, \tag{4.5}$$

and our expression becomes

$$X\left(k \sum_{\substack{d_1,d_2 \\ D=[d_1,d_2]}} \lambda(d_1)\lambda(d_2)\frac{\nu_{\mathcal{H}}^*(D)}{\phi(D)} - (\log 3X) \sum_{\substack{d_1,d_2 \\ D=[d_1,d_2]}} \lambda(d_1)\lambda(d_2)\frac{\nu_{\mathcal{H}}(D)}{D}\right). \tag{4.6}$$

We have dealt with sums of type (4.4) and (4.5) in Chapter 3 using analytic methods by approximating them with integrals of zeta-functions. It is also possible to deal with them using elementary sieve-theoretic arguments. We have the following reciprocity law: Let $L(d)$ and $Y(r)$ be sequences of numbers supported on square-free integers. If

$$L(d) = \mu(d)\sum_{d|n} Y(n) \text{ for all } d \geq 1, \tag{4.7}$$

then

$$Y(r) = \mu(r)\sum_{r|m} L(m) \text{ for all } r \geq 1. \tag{4.8}$$

This is easily seen by plugging in the definition and rearranging the sums. Also, it is clear that if $L(d)$ is supported on $d \leq R$, then $Y(r)$ is supported on $r \le R$. Now put

$$L(d) = \frac{\lambda(d)\nu_{\mathcal{H}}(d)}{d}. \tag{4.9}$$

Then since $\nu_{\mathcal{H}}((d_1, d_2))\nu_{\mathcal{H}}(D) = \nu_{\mathcal{H}}(d_1)\nu_{\mathcal{H}}(d_2)$, we can write the sum in (4.4) as

$$\begin{aligned} S_1 &= \sum_{\substack{d_1, d_2 \\ D=[d_1,d_2]}} \lambda(d_1)\lambda(d_2)\frac{\nu_{\mathcal{H}}(D)}{D} \\ &= \sum_{r,s} Y(r)Y(s) \sum_{\substack{d_1,d_2 \le R \\ d_1|r,\, d_2|s}} \mu(d_1)\mu(d_2)\frac{(d_1, d_2)}{\nu_{\mathcal{H}}((d_1, d_2))}. \end{aligned} \tag{4.10}$$

The inner sum is multiplicative, and for a prime dividing only one of $r$ and $s$, it is easily worked out to be zero. Since $r$ and $s$ are square-free, it follows that the inner sum is non-vanishing only when $r = s$, in which case for a prime divisor $p$ of $r$ its value is seen to be $1 - 1 - 1 + p/\nu_{\mathcal{H}}(p)$. Letting $\omega$ be the multiplicative function defined on primes by $\omega(p) = p - \nu_{\mathcal{H}}(p)$, we obtain

$$S_1 = \sum_r Y(r)^2 \frac{\omega(r)}{\nu_{\mathcal{H}}(r)} = \sum_r \frac{y(r)^2\nu_{\mathcal{H}}(r)}{\omega(r)}, \tag{4.11}$$

where we have put $Y(r) = y(r)\nu_{\mathcal{H}}(r)/\omega(r)$. Now for a multiplicative function $g(n)$, if $g(p)$ is "sufficiently close" to a constant $k$ on primes, we have

$$\sum_{n \le x} \frac{g(n)}{n} \sim G_k(g(n), 0)\frac{(\log x)^k}{k!}, \tag{4.12}$$

where

$$G_k(g(n), s) = \prod_p \left(1 + \frac{g(p)}{p^{1+s}} + \frac{g(p^2)}{p^{2(1+s)}} + \dots\right)\left(1 - \frac{1}{p^{1+s}}\right)^k, \tag{4.13}$$

and the condition that $g(n)$ is "sufficiently close" to $k$ is essentially that $G_k(g(n), s)$ converges at $s = 0$. To see why this is so, consider the associated

Dirichlet series,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{g(n)}{n^{1+s}} &= \prod_p\left(1+\frac{g(p)}{p^{1+s}}+\frac{g(p^2)}{p^{2(1+s)}}+\dots\right)\\ &= \prod_p\left(1+\frac{g(p)}{p^{1+s}}+\frac{g(p^2)}{p^{2(1+s)}}+\dots\right)\left(1-\frac{1}{p^{1+s}}\right)^k\zeta(1+s)^k\\ &= G_k(g(n),s)\zeta(1+s)^k,\end{aligned}\tag{4.14}$$

and if $G_k(g(n),s)$ is bounded in a region $\sigma>-\delta$, Perron's formula will pick up the main term asserted in (4.12). Now we assume that in (4.11), $y(r)$ is given by

$$y(r)=F\left(\frac{\log r}{\log R}\right),\tag{4.15}$$

where $F(t)$ is some measurable function supported on $[0,1]$. Then we can use (4.12) and summation by parts to deduce that

$$\mathfrak{S}(\mathcal{H})S_1=\mathfrak{S}(\mathcal{H})\sum_r\frac{\nu_{\mathcal{H}}(r)}{\omega(r)}F\left(\frac{\log r}{\log R}\right)^2\sim(\log R)^k\int_0^1F(t)^2\frac{t^{k-1}}{(k-1)!}dt,\tag{4.16}$$

where

$$\begin{aligned}\mathfrak{S}(\mathcal{H}) &= G\left(\frac{\nu_{\mathcal{H}}(n)n}{\omega(n)},0\right)^{-1}\\ &= \prod_p\left(1+\frac{\nu_{\mathcal{H}}(p)}{p-\nu_{\mathcal{H}}(p)}\right)^{-1}\left(1-\frac{1}{p}\right)^{-k}\\ &= \prod_p\left(1-\frac{\nu_{\mathcal{H}}(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-k}.\end{aligned}\tag{4.17}$$

Using the same reciprocity law with

$$L^*(d)=\frac{\lambda(d)\nu_{\mathcal{H}}^*(d)}{\phi(d)}\tag{4.18}$$

and

$$Y^*(r)=\frac{y^*(r)\nu_{\mathcal{H}}^*(r)}{\omega(r)},\tag{4.19}$$

one obtains

$$S_2 = \sum_{\substack{d_1,d_2 \\ D=[d_1,d_2]}} \lambda(d_1)\lambda(d_2)\frac{\nu^*_{\mathcal{H}}(D)}{\phi(D)} \tag{4.20}$$
$$= \sum_r \frac{y^*(r)^2\nu^*_{\mathcal{H}}(r)}{\omega(r)}.$$

We can work out $y^*(r)$ in terms of $F$ using the reciprocity law as follows. We have

(4.21)

$$y^*(r) = \frac{\omega(r)}{\nu^*_{\mathcal{H}}(r)}Y(r) = \frac{\omega(r)}{\nu^*_{\mathcal{H}}(r)}\mu(r)\sum_{r|d} L^*(d)$$
$$= \frac{\omega(r)}{\nu^*_{\mathcal{H}}(r)}\mu(r)\sum_{r|d} \frac{\nu^*_{\mathcal{H}}(d)}{\nu_{\mathcal{H}}(d)}\frac{d}{\phi(d)}L(d)$$
$$= \frac{\omega(r)}{\nu^*_{\mathcal{H}}(r)}\mu(r)\sum_{r|d} \frac{\nu^*_{\mathcal{H}}(d)}{\nu_{\mathcal{H}}(d)}\frac{d}{\phi(d)}\mu(d)\sum_{d|n}\frac{y(n)\nu_{\mathcal{H}}(n)}{\omega(n)}$$
$$= \frac{r}{\phi(r)}\sum_{r|n}\frac{y(n)}{\omega(n/r)}\sum_{\frac{d}{r}|\frac{n}{r}}\mu(d/r)\frac{\nu^*_{\mathcal{H}}(d/r)d/r}{\phi(d/r)}\nu_{\mathcal{H}}(n/d).$$

Now the summand of the inner sum is multiplicative, so working it out prime by prime we see that it equals $\omega(n/r)/\phi(n/r)$. Hence

$$y^*(r) = r\sum_{r|n}\frac{y(n)}{\phi(n)} = \frac{r}{\phi(r)}\sum_{\substack{m\le R/r \\ (m,r)=1}}\frac{y(mr)}{\phi(r)} \tag{4.22}$$
$$\sim (\log R)\int_{\frac{\log r}{\log R}}^1 F(t)dt,$$

since with $k=1$ and

$$g(n) = \begin{cases} n/\phi(n), & \text{if } (n,r)=1, \\ 0, & \text{otherwise}, \end{cases} \tag{4.23}$$

the product $G_1(g(n),0) = \phi(r)/r$. Using this in (4.12) and summing by parts yields

$$\mathfrak{S}^*(\mathcal{H})S_2 \sim (\log R)^{k+1}\int_0^1\left(\int_t^1 F(u)du\right)^2\frac{t^{k-2}}{(k-2)!}dt, \tag{4.24}$$

with

$$
\begin{aligned}
\mathfrak{S}^*(\mathcal{H}) :&= G_{k-1}\left(\frac{\nu^*_{\mathcal{H}}(n)n}{\omega(n)}, 0\right)^{-1} \\
&= \prod_p \left(1 + \frac{\nu_{\mathcal{H}}(p) - 1}{p - \nu_{\mathcal{H}}(p)}\right)^{-1} \left(1 - \frac{1}{p}\right)^{-(k-1)} \\
&= \prod_p \left(\frac{p-1}{p - \nu_{\mathcal{H}}(p)}\right)^{-1} \left(\frac{p-1}{p}\right) \left(1 - \frac{1}{p}\right)^{-k} \\
&= \prod_p \left(1 - \frac{\nu_{\mathcal{H}}(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-k} = \mathfrak{S}(\mathcal{H}).
\end{aligned} \tag{4.25}
$$

With these, we find that (4.6) is asymptotically

$$
\begin{aligned}
\frac{X(\log R)^k}{\mathfrak{S}(\mathcal{H})} \Bigg( k(\log R) \int_0^1 \left(\int_t^1 F(u)du\right)^2 \frac{t^{k-2}}{(k-2)!} dt& \\
- (\log 3X) \int_0^1 F(t)^2 \frac{t^{k-1}}{(k-1)!} dt \Bigg)&.
\end{aligned} \tag{4.26}
$$

In other words, recalling that we can take $R$ as large as $X^{\vartheta/2}$, the positivity of $S$ hinges on the positivity of

$$
\frac{\vartheta}{2}\rho_k(F) - 1, \tag{4.27}
$$

where $\rho_k(F)$ is the ratio

$$
k \int_0^1 \left(\int_t^1 F(u)du\right)^2 \frac{t^{k-2}}{(k-2)!} dt \Bigg/ \int_0^1 F(t)^2 \frac{t^{k-1}}{(k-1)!} dt. \tag{4.28}
$$

In fact it is easy to see that we can more generally find $m+1$ primes if we can show $\frac{\vartheta}{2}\rho_k(F) > m$. Choosing $F(t) = (1-t)^{k+\ell}/(k+\ell)!$ here yields $\rho_k(F) = 4 - o(1)$ as $k, \ell \to \infty$ with $\ell = o(k)$, corresponding to the results discussed in Chapter 2, i.e., we only just fail to produce bounded gaps with level of distribution $\vartheta = 1/2$, but any level $> 1/2$ suffices. Moreover, it can be shown that for any polynomial $F(t)$, the inequality $\rho_k(F) < 4$ holds, so that the results obtained are essentially the limit of the method.

The novelty introduced by the Maynard-Tao method is that instead of weights of type

$$\left(\sum_{d|P(n;\mathcal{H})} \lambda(d)\right)^2, \tag{4.29}$$

they consider

$$\left(\sum_{d_i|n+h_i,\, i=1,\ldots,k} \lambda_{d_1,\ldots,d_k}\right)^2, \tag{4.30}$$

where $d_1 \ldots d_k \leq R$. With these "higher dimensional" weights, all the sieve manipulations go through in an analogous fashion, and one can again use a change of variables $y_{r_1,\ldots,r_k}$ to diagonalize the sums over $\lambda$'s, and in turn put

$$y_{r_1,\ldots,r_k} = F\left(\frac{\log r_1}{\log R}, \ldots, \frac{\log r_k}{\log R}\right), \tag{4.31}$$

with $F(t_1, \ldots, t_i) \in \mathcal{R}$ where $\mathcal{R}$ is the set of Riemann-integrable functions supported on the set $\{(t_1, \ldots, t_k) \in [0,1]^k : \sum t_i \leq 1\}$. Skipping all the technicalities (the method will be sufficiently exposed through application in Chapter 6), the effect of this modification is to replace $\rho_k(F)$ by

$$\rho'_k(F) = \frac{k \int \left(\int F(t_1, \ldots, t_k) dt_k\right)^2 dt_{k-1} \ldots dt_1}{\int F(t_1, \ldots, t_k)^2 dt_k \ldots dt_1} \tag{4.32}$$

The freedom to optimise over a much larger space of functions has a very dramatic effect. It is possible to choose $F$ to satisfy not only $\rho'_k(F) > 4$, but in fact Maynard demonstrates the existence of $F = F_k$ such that $\rho'_k(F) \xrightarrow{k\to\infty} \infty$, so that one can find not just two but any number of primes within a bounded distance of each other, and that with any positive level of distribution $\vartheta > 0$, a consequence much stronger than the previously available results.

CHAPTER 5

# A Bombieri-Vinogradov type theorem

In this chapter we prove a Bombieri-Vinogradov type theorem tailored to the problem of finding prime tuples in an arithmetic progression. In order to find bounds in terms of the moduli that are uniform over as large a range of the moduli as possible, we will have to restrict ourselves to arithmetic progressions in which primes are reasonably well-distributed, i.e. progressions to moduli whose associated Dirichlet $L$-functions don't vanish too close to $s = 1$. We first recall relevant facts about the zeros of Dirichlet $L$-functions and set some notation.

For the imaginary part $\gamma$ of a zero of an $L$-function, we shall denote $|\gamma|+1$ by $\widetilde{\gamma}$ for the sake of brevity. We first recall some basic facts concerning zero-free regions of $L$-functions [3, §14]. There is a constant $c_0$ (the bounds cited below are known in fact for different constants, but we take $c_0$ to be the minimum of those to simplify notation) such that an $L$-function $L(s, \chi)$ to the modulus $q$ has no zero $\beta + i\gamma$ in the region

$$\beta \geq 1 - \frac{c_0}{\log q\widetilde{\gamma}}, \tag{5.1}$$

except possibly a single real zero, which can exist for at most one real character $\chi \pmod q$. We call a modulus to which there's such a *primitive* character an *exceptional modulus*, and the corresponding zero an *exceptional zero*. Exceptional moduli are of the form $q = 2^{\nu} p_1 \ldots p_m$, where $\nu \leq 3$ and $p_1 < p_2 < \ldots < p_m$ are distinct odd primes, whence, by the Prime Number Theorem, we have $p_m \gg \sum_{p \leq p_m} \log p \gg \log q$. On the other hand we have, for the real zeros, the unconditional bound

$$\beta < 1 - \frac{c_0}{q^{1/2}(\log q)^2}. \tag{5.2}$$

Also, if $\chi_1$ and $\chi_2$ are distinct real primitive characters to moduli $q_1$ and $q_2$ respectively and the corresponding $L$-functions have real zeros $\beta_1$ and $\beta_2$, then the Landau-Page theorem states that these zeros must satisfy

$$\min(\beta_1, \beta_2) < 1 - \frac{c_0}{\log q_1 q_2}. \tag{5.3}$$

We shall have to confine ourselves to $L$-functions which don't have a zero in the region

$$\beta \geq 1 - \frac{c^* \log\log X}{\log X}, \quad \widetilde{\gamma} \leq \exp\left(c^\sharp \sqrt{\log X}\right) \tag{5.4}$$

for a parameter $X$ and given constants $c^*$ and $c^\sharp$. This is a consequence of (5.2) when $q \ll \left(\frac{\log X}{(\log\log X)^2}\right)^2$. We suppose that $X$ is large enough in terms of $c^\sharp$ and $c^*$ such that

$$c^\sharp \leq \frac{c_0}{4c^*} \frac{\sqrt{\log X}}{\log\log X} \tag{5.5}$$

holds, and argue that there's at most one modulus $\leq \exp\left(2c^\sharp \sqrt{\log X}\right)$ to which there's a primitive character whose $L$-function vanishes in the region (5.4). By (5.1), no non-exceptional zeros exist in the region stated, so we only need to consider real zeros. Suppose there are two such moduli $q_1$ and $q_2$, with corresponding real zeros $\beta_1$ and $\beta_2$. Then using (5.3) we have,

$$\begin{aligned} 1 - \frac{c^* \log\log X}{\log X} &< 1 - \frac{c_0}{\log q_1 q_2} \\ &\leq 1 - \frac{c_0}{4c^\sharp \sqrt{\log X}}, \end{aligned} \tag{5.6}$$

which is impossible by (5.5). We denote this possibly existing unique modulus by $q_0$ and the greatest prime dividing $q_0$ by $p_0$, or set $p_0 = 1$ in case $q_0$ does not exist. We note that $q_0 \gg \left(\frac{\log X}{(\log\log X)^2}\right)^2$, whence $p_0 \gg \log\log X$. We put

$$P_f = \begin{cases} 1, & \text{if } f(X) \ll (\log X)^C, \\ p_0, & \text{otherwise.} \end{cases} \tag{5.7}$$

Our main parameter $X$ is large enough and $f(X)$ is a given increasing function of $X$ with $f(X) \ll X^{\frac{5}{12} - \frac{5}{6}\vartheta}$ for some positive number $\vartheta < 1/2$. The modulus $M$ of the arithmetic progression does not exceed $f(X)$ and is not a multiple of any number in a set $\mathcal{Z}$ of exceptions whose size $Z_f$ satisfies

$$Z_f = \begin{cases} 0, & \text{if } f(X) \ll (\log X)^C, \\ 1, & \text{if } f(X) \ll \exp\left(c\sqrt{\log X}\right), \\ O\left((\log\log X)^C\right), & \text{otherwise.} \end{cases} \tag{5.8}$$

We denote characters modulo $q$, $M$, and $qM$ by $\psi$, $\xi$, and $\chi$ respectively. A summation $\sum_\chi^*$ over characters with an asterisk in the superscript denotes that the summation is over primitive characters only.

We first quote here a zero-density result [18, Theorem 10.4 and the following remark] which we will need in our proof.

Theorem 5.1. *Let $m$ be given and $N(1-\delta, T, \chi)$ be the number of zeros $\beta + i\gamma$ of $L(s,\chi)$ in the region $1-\delta \leq \beta$, $|\gamma| \leq T$. Put*

$$N(1-\delta, m, Q, T) = \sum_{\substack{q \leq Q \\ (q,m)=1}} \sideset{}{^*}\sum_{\psi \pmod q} \sum_{\xi \pmod m} N(1-\delta, T, \psi\xi). \tag{5.9}$$

*Then for $\delta < 1/2$ and any $\varepsilon > 0$, we have*

$$N(1-\delta, m, Q, T) \ll \left((mQT)^{2\delta} + (mQ^2T)^{c(\delta)\delta}\right)(\log mQT)^A, \tag{5.10}$$

*for some constant $A$, where*

$$c(\delta) = \min\left(\frac{3}{1+\delta}, \frac{3}{2-3\delta}\right). \tag{5.11}$$

We estimate the number of the moduli we will have to exclude in the following proposition.

Proposition 5.2. *Let $c^*$ and $c^\sharp$ be given constants. There is a set $\mathcal{Z}$ of exceptions with $|\mathcal{Z}| \ll (\log X)^C$ such that if $X$ is large enough, then for all $M \leq f(X)$ that is not a multiple of any number in $\mathcal{Z}$ and all $q \leq \exp\left(c^\sharp\sqrt{\log X}\right)$ with $(q, Mp_0) = 1$, the $L$-functions $L(s, \psi\xi)$, where $\psi \pmod q$ is primitive and $\xi$ is any character $\pmod M$, have no zeros in the region $1 - \frac{c^*\log\log X}{\log X} \leq \beta \leq 1$, $|\gamma| \leq \exp\left(c^\sharp\sqrt{\log X}\right)$. The set $\mathcal{Z}$ can have* *elements* $\leq \exp\left(c^\sharp\sqrt{\log X}\right)$ *only if* $q_0$ *exists, in which case those elements are all multiples of* $p_0$.

PROOF. Suppose that $M$ is a modulus such that for some character $\xi$ (mod $M$) and a primitive character $\psi$ (mod $q$), $L(s,\psi\xi)$ has a zero in the region indicated. Then $\psi\xi$ must be induced by a character of the form $\psi\xi^*$, where $\xi^*$ (mod $m$) is a primitive character modulo $m \mid M$. We estimate the number of such $m$. Let

(5.12)
$$\begin{aligned}\mathcal{Z} = \{m \leq f(X) : &\text{there exist } q \leq \exp\left(c^\sharp\sqrt{\log X}\right) \text{ and } \chi \pmod{mq}\\ &\text{with } (q, mp_0) = 1 \text{ and } \chi \text{ primitive, such that } L(\beta + i\gamma, \chi) = 0\\ &\qquad\qquad\text{for some } \beta > 1 - \tfrac{c^* \log\log X}{\log X}\}\end{aligned}$$

be the set of exceptions whose size we wish to bound. We divide the ranges $1 \leq m \leq f(X)$, $1 \leq q \leq \exp\left(c^\sharp\sqrt{\log X}\right)$, and $\widetilde{\gamma} \leq \exp\left(c^\sharp\sqrt{\log X}\right)$ into dyadic segments $[M_\lambda/2, M_\lambda)$, $[Q_\mu/2, Q_\mu)$ and $[T_\nu/2, T_\nu)$ respectively. Then,

$$\#\mathcal{Z} \leq \sum_{\lambda,\mu,\nu} \sum_{m \leq M_\lambda} \sum_{\substack{q \leq Q_\mu \\ (q, mp_0)=1}} \sideset{}{^*}\sum_{\chi \pmod{qm}} N\left(1 - \tfrac{c^* \log\log X}{\log X}, T_\nu, \chi\right). \tag{5.13}$$

Using Theorem 5.1 with $m = 1$ and $M_\lambda Q_\mu$ in place of $Q$, the above is

$$\ll \sum_{\lambda,\mu,\nu} (M_\lambda^2 Q_\mu^2 T_\nu)^{\frac{12c^* \log\log X}{5 \log X}} (\log Q_\mu T_\nu)^C \ll (\log X)^C, \tag{5.14}$$

where $C$ and the implicit constant depend on $c^*$ and $c^\sharp$. Now suppose that $X$ is large enough to satisfy (5.5). Then if $m \in \mathcal{Z}$ with $m \leq \exp\left(c^\sharp\sqrt{\log X}\right)$, so that $mq \leq \exp\left(2c^\sharp\sqrt{\log X}\right)$, then by the discussion in Section 1, $L(s,\chi) = 0$ with primitive $\chi$ (mod $mq$) implies $mq = q_0$, and since $p_0 \nmid q$, we have $p_0 \mid m$. $\square$

REMARK. If $f(X) \leq \exp\left(c\sqrt{\log X}\right)$ for some constant $c$, then choosing $c^\sharp$ to be such a constant, one sees that $\mathcal{Z}$ can be taken to be at most a singleton.

REMARK. We see that asymptotically almost all moduli remain after exceptions, because the excluded moduli number at most $\ll \frac{f(X)}{(\log\log X)}$, since $p_0 \geq \log\log X$.

Using this proposition, we prove the following theorem. We indulge in a slight notational conflict between the Chebyshev function $\psi(X)$ and the character $\psi(n) \pmod q$, which should be tolerable given the context. The functions $\psi(X; q, a)$ and $\psi(X, \chi)$ are distinguished by their arguments.

THEOREM 5.3. *Let $A$ be a given positive number. There exists a positive number $B$ such that for all $M \leq f(X)$, except those that are multiples of numbers in a set of size $Z_f$, we have*

$$\sum_{\substack{q \leq \frac{X^{1/2}}{M^{6/5}}(\log X)^{-B} \\ (q, Mp_0)=1}} \max_{(a,qM)=1} \left| \psi(X; qM, a) - \frac{\psi(X)}{\varphi(qM)} \right| \ll \frac{X}{\varphi(M)}(\log X)^{-A}, \tag{5.15}$$

*where the implicit constants depend on $A$.*

PROOF. Let $c^*$ be a constant to be specified later in terms of $A$, and pick $c^\sharp$ arbitrarily (or, in case $f(X) \leq \exp\left(c\sqrt{\log X}\right)$ for some $c$, pick $c^\sharp$ according to the first remark following Proposition 5.2), so that Proposition 5.2 furnishes us with a set $\mathcal{Z}$ of size $Z_f$. Then if $M$ is not a multiple of any number in $\mathcal{Z}$, $q \leq \exp\left(c^\sharp\sqrt{\log X}\right)$ and $(q, Mp_0) = 1$ then no $L(s, \psi\xi)$ with $\psi$ primitive has a zero in the region $\beta \geq 1 - \frac{c^* \log\log X}{\log X}$, $\widetilde{\gamma} \leq \exp\left(c^\sharp\sqrt{\log X}\right)$. We put $\Omega = X^{1/2}M^{-6/5}(\log X)^{-B}$ for the sake of brevity. We have

$$\psi(X; qM, a) = \frac{1}{\varphi(qM)} \sum_{\chi \pmod{qM}} \overline{\chi}(a)\psi(X, \chi) \tag{5.16}$$

and

$$|\psi(X, \chi_0) - \psi(X)| \leq \sum_{\substack{n \leq X \\ (n, qM) > 1}} \Lambda(n) \ll (\log qM)(\log X), \tag{5.17}$$

so it suffices to consider, within acceptable error,

$$\sum_{\substack{q\le\Omega\\(q,Mp_0)=1}} \max_{(a,qM)=1} \Big|\frac{1}{\varphi(qM)} \sum_{\substack{\chi \,(\mathrm{mod}\, qM)\\ \chi\neq\chi_0}} \overline{\chi}(a)\psi(X,\chi)\Big|. \tag{5.18}$$

Since $(M,q)=1$, we can factorize $\chi$ as $\psi\xi$, where $\psi$ and $\xi$ are characters to the moduli $q$ and $M$ respectively, so that (5.18) is

$$\sum_{\substack{q\le\Omega\\(q,Mp_0)=1}} \max_{(a,qM)=1} \Big|\frac{1}{\varphi(qM)} \sum_{\substack{\psi \,(\mathrm{mod}\, q)\\ \xi \,(\mathrm{mod}\, M)\\ \psi\xi\neq\chi_0}} \overline{\psi\xi}(a)\psi(X,\psi\xi)\Big|. \tag{5.19}$$

We replace each character $\psi$ with the primitive character $\psi^*$ inducing it. This leads to an error of

$$\sum_{\substack{q\le\Omega\\(q,Mp_0)=1}} \frac{1}{\varphi(qM)} \sum_{\substack{\psi \,(\mathrm{mod}\, q)\\ \xi \,(\mathrm{mod}\, M)}} \sum_{\substack{n\le X\\(n,q)>1}} \Lambda(n) \ll \frac{X^{1/2}}{M^{6/5}} \exp\left(-c^\sharp\sqrt{\log X}\right)(\log X)^2, \tag{5.20}$$

and this is acceptable. Using the explicit formula for $\psi(X,\chi)$ in the form

$$\psi(X,\chi) = -\sum_{\substack{|\gamma_\chi|\le X^{1/2}\\ \beta_\chi>1/2}} \frac{X^{\rho_\chi}}{\rho_\chi} + O(X^{1/2}(\log X)^2), \tag{5.21}$$

we are left to bound

$$\frac{1}{\varphi(M)} \sum_{\substack{q\le\Omega\\(q,Mp_0)=1}} \frac{1}{\varphi(q)} \sum_{\xi \,(\mathrm{mod}\, M)} \sum_{\psi \,(\mathrm{mod}\, q)} \sum_{\substack{|\gamma_{\psi^*\xi}|\le X^{1/2}\\ \beta_{\psi^*\xi}>1/2}} \frac{X^{\beta_{\psi^*\xi}}}{|\rho_{\psi^*\xi}|}. \tag{5.22}$$

We rearrange the sum according to the moduli of the primitive characters $\psi^*$ that occur, hence after relabelling the dummy variables so that $q$ is now the modulus of $\psi^*$, we have

$$\begin{aligned} &\frac{X}{\varphi(M)} \sum_{\substack{q\le\Omega\\(q,Mp_0)=1}} \sum_{\xi \,(\mathrm{mod}\, M)} \sideset{}{^*}\sum_{\psi \,(\mathrm{mod}\, q)} \sum_{\substack{|\gamma_{\psi\xi}|\le X^{1/2}\\ \beta_{\psi\xi}>1/2}} \frac{X^{-(1-\beta_{\psi\xi})}}{|\rho_{\psi\xi}|} \sum_{\substack{k\le\Omega/q\\(k,Mp_0)=1}} \frac{1}{\varphi(kq)} \\ \ll &\frac{X(\log X)^2}{\varphi(M)} \sum_{\substack{q\le\Omega\\(q,Mp_0)=1}} \sum_{\xi \,(\mathrm{mod}\, M)} \sideset{}{^*}\sum_{\psi \,(\mathrm{mod}\, q)} \sum_{\substack{|\gamma_{\psi\xi}|\le X^{1/2}\\ \beta_{\psi\xi}>1/2}} \frac{X^{-(1-\beta_{\psi\xi})}}{q|\rho_{\psi\xi}|}. \end{aligned} \tag{5.23}$$

We divide the ranges for $q$ and $\widetilde{\gamma}$ into dyadic segments, and the range for $\beta$ into segments of length $(\log X)^{-1}$ as follows.

$$q \in [Q_\mu/2, Q_\mu), \qquad \widetilde{\gamma} \in [T_\nu/2, T_\nu), \qquad 1-\beta \in [\delta_\lambda - (\log X)^{-1}, \delta_\lambda), \tag{5.24}$$

where $2 \leq Q_\mu = 2^\mu < 2\Omega$, $2 \leq T_\nu = 2^\nu < 2X^{1/2}$ and $(\log X)^{-1} \leq \delta_\lambda = \lambda(\log X)^{-1} \leq 1/2$. So our expression is

$$\ll \frac{X(\log X)^5}{\varphi(M)} \sup_{(\lambda,\mu,\nu)} \frac{N^*(1-\delta_\lambda, M, Q_\mu, T_\nu)}{Q_\mu T_\nu} X^{-\delta_\lambda}, \tag{5.25}$$

where

(5.26)
$$N^*(1-\delta_\lambda, M, Q_\mu, T_\nu) = \sum_{\substack{Q_\mu/2<q\leq Q_\mu \\ (q,Mp_0)=1}} \sideset{}{^*}\sum_{\psi \ (\mathrm{mod}\ q)} \sum_{\xi \ (\mathrm{mod}\ M)} N(1-\delta_\lambda, T_\nu, \psi\xi).$$

Thus we need to show, for all triples $(\lambda, \mu, \nu)$, dropping the subscripts for economy of notation, the upper bound

$$N^*(1-\delta, M, Q, T) \ll QTX^\delta(\log X)^{-A-5}. \tag{5.27}$$

To this end we use the Theorem 5.1, which for our ranges of $Q$ and $T$ yields,

$$N^*(1-\delta, M, R, T) \ll \left((MQT)^{2\delta} + (MQ^2T)^{c(\delta)\delta}\right)(\log X)^{C'}, \tag{5.28}$$

where $C'$ is an absolute constant.

Since for $0 \leq \delta \leq 1/2$, we have

$$\frac{(MQT)^{2\delta}}{QT}(\log X)^{C'} \ll M^{2\delta}(\log X)^{C'}, \tag{5.29}$$

the contribution of the first term on the right hand side of (5.28) is acceptable if $\delta \geq \frac{2}{15}$, say. So we only need to show

$$\frac{(MQ^2T)^{c(\delta)\delta}}{QT} \ll X^\delta(\log X)^{-(A+C'+5)} \tag{5.30}$$

for $0 \leq \delta \leq 1/2$, and

$$\frac{(MQT)^{2\delta}}{QT} \ll X^\delta(\log X)^{-(A+C'+5)} \tag{5.31}$$

for $0 \leq \delta \leq \frac{2}{15}$.

If $\frac{1}{4} \le \delta \le \frac{1}{2}$, we have $c(\delta) = \frac{3}{1+\delta}$. Here $6\delta/(1+\delta) - 1 \le 2\delta$, $3\delta/(1+\delta) - 1 \le 0$ and $3\delta/(1+\delta) \le \frac{12}{5}\delta$, so

$$\begin{aligned} \frac{(MQ^2T)^{\frac{3\delta}{1+\delta}}}{QT} &\ll \frac{(MQ^2)^{\frac{3\delta}{1+\delta}}}{Q} \\ &\ll M^{\frac{12}{5}\delta}\left(\frac{X^{1/2}}{M^{\frac{6}{5}}(\log X)^B}\right)^{2\delta} \\ &\ll X^{\delta}(\log X)^{-2\delta B} \\ &\ll X^{\delta}(\log X)^{-(A+C'+5)}, \end{aligned} \tag{5.32}$$

if $B \ge 2(A+C'+5)$.

If $\frac{2}{15} \le \delta \le \frac{1}{4}$, we have $c(\delta) = 3/(2-3\delta)$. Here also $3\delta/(2-3\delta) \le \frac{12}{5}\delta$, $0 \le 6\delta/(2-3\delta) - 1 \le \frac{4}{5}\delta$ and $3\delta/(2-3\delta) - 1 \le 0$, so

$$\begin{aligned} \frac{(MQ^2T)^{\frac{3\delta}{2-3\delta}}}{QT} &\ll \frac{(MQ^2)^{\frac{3\delta}{2-3\delta}}}{Q} \\ &\ll M^{\frac{12}{5}\delta}\left(\frac{X^{1/2}}{M^{\frac{6}{5}}(\log X)^B}\right)^{\frac{4}{5}\delta} \\ &\ll M^{\frac{36}{25}\delta}X^{\frac{2}{5}\delta}(\log X)^{-\frac{4}{5}\delta B}, \end{aligned} \tag{5.33}$$

and this is $\ll X^{\delta}(\log X)^{-(A+C'+5)}$ if $M \ll X^{5/12}$ and $B \ge \frac{75}{8}(A+C'+5)$.

Now suppose $\delta \le \frac{2}{15}$. Then $6\delta/(2-3\delta) - 1 \le -\frac{1}{2}$ and $3\delta/(2-3\delta) \le 2\delta$, so

$$\frac{(MQ^2T)^{\frac{3\delta}{2-3\delta}}}{QT} \ll M^{2\delta}(QT)^{-1/2}, \tag{5.34}$$

as well as

$$\frac{(MQT)^{2\delta}}{QT} \ll M^{2\delta}(QT)^{-1/2}. \tag{5.35}$$

Now if $M \ll X^{\frac{5}{12}}$ and $QT \ge \exp\left(c^{\sharp}\sqrt{\log X}\right)$, the right hand side is

$$\le M^{2\delta}\exp\left(-\frac{c^{\sharp}}{2}\sqrt{\log X}\right) \ll X^{\delta}(\log X)^{-(A+C'+5)}. \tag{5.36}$$

Otherwise, if $QT \le \exp\left(c^{\sharp}\sqrt{\log X}\right)$, we use the fact that $\delta \ge \frac{c^* \log\log X}{\log X}$ by our assumption on $M$, and we have

$$\begin{aligned} &\le \left(\frac{M^2}{X}\right)^{\delta} \le \exp\left(-\frac{c^*}{5}\log\log X\right) \\ &\qquad \le (\log X)^{-(A+C'+5)}, \end{aligned} \tag{5.37}$$

provided $c^* \ge 5(A + C' + 5)$. □

Remark. Note that when $M$ indeed reaches $X^{5/12}$, the sum is vacuous and the theorem is trivial. We will apply it with $M \ll X^{\frac{5}{12}-\frac{5}{6}\vartheta}$ for some positive $\vartheta$ to get "level of distribution" $\vartheta$.

For the shorter range $M \le (\log X)^C$, we can simply use the classical Bombieri-Vinogradov theorem (see, for instance, [3, §28]) with $A + C$ in place of $A$, and gain a factor of $\phi(M)$ without any further modifications.

Theorem 5.4. *Let $A$ be a given positive number and let $M \ll (\log X)^C$ be an integer. Then there is a positive number $B$ such that*

$$\sum_{\substack{q \le X^{1/2}(\log X)^{-B} \\ (q,M)=1}} \max_{(a,qM)=1} \left|\psi(X; qM, a) - \frac{\psi(X)}{\varphi(qM)}\right| \ll \frac{X}{\varphi(M)}(\log X)^{-A}, \tag{5.38}$$

*where the implicit constant depends on $A$ and $C$.*

We would like to express these results in a unified fashion. To that end, given an increasing function $f(X)$ of $X$ such that $f(X) \ll X^{\frac{5}{12}-\frac{5}{6}\vartheta}$ with $\vartheta > 0$, we introduce the following notation.

$$e_f = \begin{cases} \frac{1}{2}, & \text{if } f(X) \le \exp\left(C\sqrt{\log X}\right), \\ \vartheta, & \text{otherwise.} \end{cases} \tag{5.39}$$

With this we have

Theorem 5.5. *Let $A$ be given positive numbers and $f(X)$ an increasing function of $X$ satisfying $f(X) \ll X^C$ with $C < 5/12$. Then for all $M \le$*

*$f(X)$, except multiples of numbers in a set of size at most $Z_f$, and all $\delta > 0$, we have*

$$\sum_{\substack{q \le X^{e_f - \delta} \\ (q, MP_f)=1}} \max_{(a,qM)=1} \left| \psi(X; qM, a) - \frac{\psi(X)}{\varphi(qM)} \right| \ll \frac{X}{\varphi(M)} (\log X)^{-A}. \tag{5.40}$$

Now we are in a position to prove our main results.

CHAPTER 6

# Bounded gaps between primes in arithmetic progressions

In this chapter we apply the Maynard-Tao method to the problem of finding primes in arithmetic progressions in tuples with diameter that is bounded uniformly in terms of the moduli. We first fix some notation and describe our set-up.

## 1. Notation and setup

We retain the notation introduced in Chapter 5, so that $X$ is our main parameter, and $f(X)$, $Z_f$, $P_f$ and $e_f$ are as described. Put $x = X/M$ and let $W = \prod_{p \le D_0} p$ be the product of primes not exceeding $D_0 = \log\log\log X$, and in turn put $W' = W/(W, P_f M)$ and $V = W'M$. Also put $R = N^{\frac{1}{2}\vartheta - \delta}$ for some small positive $\delta$. Let $\mathcal{H} = \{h_1, \ldots, h_k\}$ be an admissible $k$-tuple with $\operatorname{diam}(\mathcal{H}) < D_0 M$ such that $h_i \equiv a \pmod{M}$, $i = 1, \ldots, k$ for a given residue class $a \pmod{M}$ coprime to $M$. The weights $\lambda_{d_1,\ldots,d_k}$ are supported on $(\prod_i d_i, VP_f) = 1$, $\prod_i d_i < R$, and $\mu(\prod_i d_i)^2 = 1$ (the last condition implies, of course, that $(d_i, d_j) = 1$). We also choose $\nu_0$ such that $(M\nu_0 + h_i, W') = 1$ for $i = 1, \ldots, k$ (this is possible because $\mathcal{H}$ is admissible).

With these, we will consider the sum

$$S^{(\rho)} = \sum_{\substack{x \le n < 2x \\ n \equiv \nu_0 \ (\mathrm{mod}\ W')}} \left( \sum_{i=1}^{k} \chi_{\mathbb{P}}(nM + h_i) - \rho \right) \left( \sum_{d_i \mid nM + h_i} \lambda_{d_1,\ldots,d_k} \right)^2, \tag{6.1}$$

where $\chi_{\mathbb{P}}$ is the characteristic function of primes. Clearly, the positivity of $S^{(\rho)}$ implies that for at least one $n \in [x, 2x)$, the inner sum is positive, and this establishes the existence of at least $\lfloor \rho + 1 \rfloor$ primes among the numbers

$nM+h_i$, $i=1,\dots,k$, but $nM$ lies in $[X,2X)$ and each $nM+h_i$ is congruent to $a \pmod M$ by the condition on $\mathcal{H}$.

## 2. Results

Our main theorem is the following.

THEOREM 6.1. *Let $k$ be a given integer, $\vartheta < 1/2$, and $f(X) \ll X^{\frac{5}{12}-\frac{5}{6}\vartheta}$ an increasing function of $X$. Further, let $\mathcal{S}_k$ be the set of all piecewise differentiable functions $\mathbb{R}^k \to \mathbb{R}$ supported on the set $\mathcal{R}_k = \{(x_1,\dots,x_k) \in [0,1]^k : \sum_{i=1}^k x_i = 1\}$, and put*

$$M_k = \sup_{F\in\mathcal{S}_k} \frac{\sum_{m=1}^k J_k^{(m)}(F)}{I_k(F)}, \tag{6.2}$$

*where*

$$I_k(F) = \int_0^1 \cdots \int_0^1 F(t_1,\dots,t_k)^2 dt_1 \dots dt_k, \tag{6.3}$$

$$J_k^{(m)}(F) = \int_0^1 \cdots \int_0^1 \left( \int_0^1 F(t_1,\dots,t_k) dt_m \right)^2 dt_1 \dots dt_{m-1} dt_{m+1} dt_k. \tag{6.4}$$

*Then, if $X$ is large enough, then for all $M \le f(X)$, except those which are multiples of numbers in a set of size $Z_f$, all residue classes $a \pmod M$ coprime to $M$, and all admissible $k$-tuples $\mathcal{H} = \{h_1,\dots,h_k\}$ such that $h_i \equiv a \pmod M$, $i=1,\dots,k$, there is a multiple $nM$ of $M$ with $nM \in [X,2X]$ such that at least $r_k = \lceil \vartheta M_k/2 \rceil$ of the numbers $nM+h_i$, $i=1,\dots,k$ are primes.*

We can instantiate this to some concrete cases to deduce certain facts. We denote by $p_n'$ the $n$-th prime that is congruent to $a \pmod M$. First note that if $f(X) \le \exp\left(c\sqrt{\log X}\right)$ for some $c$, we can apply the theorem with $\vartheta$ as close to $1/2$ as we like, and the set of exceptions will be empty or a singleton according as $f(X) \ll (\log X)^C$ for some $C$ or not. In either case taking $k = 105$ suffices to produce two primes, by Proposition 4.3 of Maynard [20], and if we use the refinement $M_{54} > 4.002$ from the Polymath project [27, Theorem 23], an admissible 54-tuple, which exists with diameter

270 [27, Theorem 17], is sufficient. To produce $r$ primes with arbitrary $r$, we use the bound $M_k > \log k + O(1)$ [27, Theorem 23] to see that a $\lceil e^{4r+C} \rceil$-tuple suffices. From any admissible tuple $\{h_i\}_i$ we can obtain a tuple $\{Mh_i + a\}_i$ whose members are all congruent to $a \pmod{M}$, with diameter dilated by $M$. Using the admissible tuple $\{Mp_{\pi(k)+1}+a, \ldots, Mp_{\pi(k)+k}+a\}$ of diameter $\ll Mk \log k$ when $r$ is large, we have the following theorems.

THEOREM 6.2. *Let $C$ be a given positive constant. Then if $X$ is sufficiently large, for all $M \ll (\log X)^C$ and all $a$ with $(a, M) = 1$, there is a $p'_n \in [X, 2X]$ such that*

$$p'_{n+1} - p'_n \leq 270M. \tag{6.5}$$

THEOREM 6.3. *Let $c$ be a given positive constant. Then if $X$ is sufficiently large, for all $M \ll \exp\left(c\sqrt{\log X}\right)$ except those that are a multiple of a single number, and all $a$ with $(a, M) = 1$, there is a $p'_n \in [X, 2X]$ such that*

$$p'_{n+1} - p'_n \leq 270M. \tag{6.6}$$

THEOREM 6.4. *Let $r$ be a positive integer and $C$ be a given positive constant. Then if $X$ is sufficiently large, for all $M \ll (\log X)^C$ and all $a$ with $(a, M) = 1$, there is a $p'_n \in [X, 2X]$ such that*

$$p'_{n+r} - p'_n \ll re^{4r}M. \tag{6.7}$$

THEOREM 6.5. *Let $r$ be a positive integer and $c$ be a given positive constant. Then if $X$ is sufficiently large, for all $M \ll \exp\left(c\sqrt{\log X}\right)$ except those that are a multiple of a single number, and all $a$ with $(a, M) = 1$, there is a $p'_n \in [X, 2X]$ such that*

$$p'_{n+r} - p'_n \ll re^{4r}M. \tag{6.8}$$

When $M$ is allowed to grow as large as a power of $X$ our tuple lengths have to grow and our bounds get much weaker. Suppose $M \ll X^{\frac{5}{12}-\eta}$ for some positive $\eta$. In that case Theorem 6.1 applies with $\vartheta = 6\eta/5$, so that to find $r+1$ primes we need $k$ such that

$$\frac{3\eta M_k}{5} > r. \tag{6.9}$$

We again use the fact that

$$M_k > \log k + O(1) \tag{6.10}$$

when $k$ is sufficiently large to see that if $k \geq Ce^{\frac{5r}{3\eta}}$ for some absolute constant $C$, (6.9) is satisfied. We take $k = \lceil Ce^{\frac{5r}{3\eta}} \rceil$, take the admissible tuple $\{Mp_{\pi(k)+1} + a, \ldots, Mp_{\pi(k)+k} + a\}$ of diameter $Mk \log k$, and obtain

THEOREM 6.6. *Let $\eta$ be given with $0 < \eta < 5/12$, and let $r$ be a positive integer. Then if $X$ is sufficiently large, for all $M \ll X^{\frac{5}{12}-\eta}$ except those that are multiples of numbers in a set of size $\ll (\log X)^C$ , and all $a$ with $(a, M) = 1$, there is a $p'_n \in [X, 2X]$ such that*

$$p'_{n+r} - p'_n \ll \left(\frac{r}{\eta}\right) e^{\frac{5r}{3\eta}} M. \tag{6.11}$$

In order to prove Theorem 6.1, we write

$$S^{(\rho)} = S_2 - \rho S_1, \tag{6.12}$$

where

$$S_1 = \sum_{\substack{x \leq n < 2x \\ n \equiv \nu_0 \ (\mathrm{mod}\ W')}} \left( \sum_{d_i | nM + h_i} \lambda_{d_1,\ldots,d_k} \right)^2, \tag{6.13}$$

and

$$\begin{aligned} S_2 &= \sum_{m=1}^{k} S_2^{(m)} \\ &= \sum_{m=1}^{k} \sum_{\substack{x \leq n < 2x \\ n \equiv \nu_0 \ (\mathrm{mod}\ W')}} \chi_{\mathbb{P}}(nM + h_m) \left( \sum_{d_i | nM + h_i} \lambda_{d_1,\ldots,d_k} \right)^2, \end{aligned} \tag{6.14}$$

so that we can estimate $S^{(\rho)}$ by the following proposition.

PROPOSITION 6.7. *Let $k$ be a given integer and let $X$ be a parameter that is large enough. Let $\lambda_{d_1,\ldots,d_k}$ be defined in terms of a fixed piecewise differentiable function $F$ by*

(6.15)
$$\lambda_{d_1,\ldots,d_k} = \Big(\prod_{i=1}^{k}\mu(d_i)d_i\Big) \sum_{\substack{r_1,\ldots,r_k \\ d_i|r_i\forall i \\ (r_i,V)=1\forall i}} \frac{\mu\big(\prod_{i=1}^{k} r_i\big)^2}{\prod_{i=1}^{k}\varphi(r_i)} F\big(\frac{\log r_1}{\log R},\ldots,\frac{\log r_k}{\log R}\big),$$

*whenever $(\prod_{i=1}^{k} d_i, VP_f) = 1$, and let $\lambda_{d_1,\ldots,d_k} = 0$ otherwise. Moreover, let $F$ be supported on $\mathcal{R}_k = \{(x_1,\ldots,x_k) \in [0,1]^k : \sum_{i=1}^{k} x_i \leq 1\}$. Then we have*

$$S_1 = (1+o(1))\,\frac{\varphi(VP_f)^k X(\log R)^k}{V(VP_f)^k} I_k(F), \tag{6.16}$$

$$S_2 = (1+o(1))\,\frac{\varphi(VP_f)^k X(\log R)^{k+1}}{V(VP_f)^k \log X} \sum_{m=1}^{k} J_k^{(m)}(F), \tag{6.17}$$

*provided $I_k(F) \neq 0$ and $J_k^{(m)}(F) \neq 0$ for each $m$, where $I_k(F)$ and $J_k^{(m)}(F)$ are given by* (6.3) *and* (6.4) *respectively.*

From this, Theorem 6.1 immediately follows.

PROOF OF THEOREM 6.1. Let $\mathcal{S}_k$ and $M_k$ be as in Theorem 6.1. Then for any $\delta > 0$, we can find $F_0 \in \mathcal{S}_k$ such that $\sum_{m=1}^{k} J_k^{(m)}(F_0) > (M_k - \delta)I_k(F_0)$. With this $F_0$, we have, by (6.12) and Proposition 6.7

$$\begin{aligned} S^{(\rho)} =& \frac{\varphi(VP_f)^k X(\log R)^k}{V(VP_f)^k}\Big(\frac{\log R}{\log N}\sum_{m=1}^{k} J_k^{(m)}(F_0) - \rho I_k(F_0) + o(1)\Big) \\ \geq& \frac{\varphi(VP_f)^k X(\log R)^k}{V(VP_f)^k} I_k(F)\Big(\big(\frac{\vartheta}{2}-\delta\big)\big(M_k-\delta\big) - \rho + o(1)\Big). \end{aligned}$$

If $\rho = \vartheta M_k/2 - \delta'$, then with $\delta$ sufficiently small, we have $S^{(\rho)} > 0$ for all large enough $X$, implying that at least $\lfloor \rho+1 \rfloor$ of the $nM + h_i$ are prime. Since $\lfloor \rho+1 \rfloor = \lceil \vartheta M_k/2 \rceil$ for $\delta'$ small enough, we obtain our result. □

## 3. Proof of Proposition 6.7

This section consists of lemmata that establish Proposition 6.7. They follow the corresponding results in [20] *mutatis mutandis.* In [20], the parameter $W$ features in a dual role: first in that the weights $\lambda_{d_1,\dots,d_k}$ are supported for $(\prod d_i, W) = 1$, and second in the "$W$-trick", i.e. in restricting $n$ to $n \equiv \nu_0 \pmod{W}$. In our case we have $VP_f$ in the first role and $W'$ in the second.

LEMMA 6.8. *Let*

$$y_{r_1,\dots,r_k} = \left(\prod_{i=1}^{k} \mu(r_i)\varphi(r_i)\right) \sum_{\substack{d_1,\dots,d_k \\ r_i \mid d_i \forall i}} \frac{\lambda_{d_1,\dots,d_k}}{\prod_{i=1}^{k} d_i}, \tag{6.18}$$

*and let* $y_{max} = \sup_{r_1,\dots,r_k} |y_{r_1,\dots,r_k}|$. *Then we have*

$$S_1 = \frac{X}{V} \sum_{u_1,\dots,u_k} \frac{y_{u_1,\dots,u_k}^2}{\prod_{i=1}^{k} \varphi(u_i)} + O\left(\frac{y_{max}^2 \varphi(VP_f)^k X (\log X)^k}{V (VP_f)^k D_0}\right). \tag{6.19}$$

PROOF. We start by rearranging the sum on the right hand side of (6.13) to obtain

$$S_1 = \sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k}} \lambda_{d_1,\dots,d_k} \lambda_{e_1,\dots,e_k} \sum_{\substack{x \le n < 2x \\ n \equiv \nu_0 \ (\mathrm{mod}\ W') \\ [d_i,e_i] \mid nM + h_i}} 1. \tag{6.20}$$

Now when $W', [d_1, e_1], \dots [d_k, e_k]$ are pairwise coprime, the inner sum is over a single residue class modulo $q = W' \prod_i [d_i, e_i]$ by the Chinese Remainder Theorem, otherwise it is empty, in the case $p \mid (W', [d_i, e_i])$ because of the condition $(W', M\nu_0 + h_i) = 1$, and in the case $p \mid ([d_i, e_i], [d_j, e_j])$ because it would imply $p \mid h_i - h_j$, but $h_i - h_j = fM$ for some $f < D_0$ since $h_i$ and $h_j$ lie in the same residue class modulo $M$, but $p \nmid M$ and $p$ can't be a prime less than $D_0$ by the support of $\lambda$. Since $f < D_0$ by the diameter of $\mathcal{H}$, we deduce that there's no contribution when $([d_i, e_i], [d_j, e_j]) > 1$.

Thus the inner sum is $x/q + O(1)$, and we have

$$S_1 = \frac{X}{V} \sideset{}{'}\sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k}} \frac{\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}}{\prod_{i=1}^k [d_i,e_i]} + O\Big( \sideset{}{'}\sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k}} |\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}| \Big), \tag{6.21}$$

where $\sum'$ denotes the coprimality restrictions. The error term is plainly

$$\ll \lambda_{max}^2 \left( \sum_{d<R} \tau_k(d) \right)^2 \ll \lambda_{max}^2 R^2 (\log X)^{2k}, \tag{6.22}$$

where $\lambda_{max} = \sup_{d_1,\ldots,d_k} \lambda_{d_1,\ldots,d_k}$. To deal with the main term, we use the identity

$$\frac{1}{[d_i,e_i]} = \frac{1}{d_i e_i} \sum_{u_i | d_i, e_i} \varphi(u_i) \tag{6.23}$$

and rewrite it as

$$\frac{X}{V} \sum_{u_1,\ldots,u_k} \left( \prod_{i=1}^k \varphi(u_i) \right) \sideset{}{'}\sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k \\ u_i | d_i, e_i \forall i}} \frac{\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}}{\prod_{i=1}^k d_i e_i}. \tag{6.24}$$

By the support of $\lambda$, we may drop the requirement that $W'$ is coprime to $[d_i, e_i]$. Also by the support of $\lambda$, terms with $(d_i, d_j) > 1$ with $i \neq j$ have no contribution. Thus our restrictions boil down to $(d_i, e_j) = 1$ for $i \neq j$. We may remove this requirement by multiplying our expression with $\sum_{s_{i,j} | d_i, e_j} \mu(s_{i,j})$ for all $i, j$. Then our main term becomes

$$\frac{X}{V} \sum_{u_1,\ldots,u_k} \left( \prod_{i=1}^k \varphi(u_i) \right) \sum_{s_{1,2},\ldots,s_{k-1,k}} \left( \prod_{\substack{1 \le i,j \le k \\ i \neq j}} \mu(s_{i,j}) \right) \sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k \\ u_i | d_i, e_i \forall i \\ s_{i,j} | d_i, e_j \forall i \neq j}} \frac{\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}}{\prod_{i=1}^k d_i e_i}. \tag{6.25}$$

We may restrict $s_{i,j}$ to be coprime to $u_i$, $u_j$, $s_{i,a}$ and $s_{b,j}$ for all $a \neq i$ and $b \neq j$ since these have no contribution by the support of $\lambda$. We denote the summation with these restrictions by $\sum^*$. We introduce the change of variable

$$y_{r_1,\dots,r_k} = \left(\prod_{i=1}^{k} \mu(r_i)\varphi(r_i)\right) \sum_{\substack{d_1,\dots,d_k \\ r_i|d_i \forall i}} \frac{\lambda_{d_1,\dots,d_k}}{\prod_{i=1}^{k} d_i}. \tag{6.26}$$

Thus $y_{r_1,\dots,r_k}$ is supported on $r = \prod_i r_i < R$, $(r, VP_f) = 1$ and $\mu(r)^2 = 1$. This change is invertible and we have

$$\sum_{\substack{r_1,\dots,r_k \\ d_i|r_i \forall i}} \frac{y_{r_1,\dots,r_k}}{\prod_{i=1}^{k} \varphi(r_i)} = \frac{\lambda_{d_1,\dots,d_k}}{\prod_{i=1}^{k} \mu(d_i) d_i}. \tag{6.27}$$

Hence any choice of $y_{r_1,\dots,r_k}$ with the above mentioned support will yield a choice of $\lambda_{d_1,\dots,d_k}$. We note here that Maynard's estimate of $\lambda_{max}$ in terms of $y_{max} = \sup_{r_1,\dots,r_k} y_{r_1,\dots,r_k}$ holds verbatim and we have

$$\lambda_{max} \ll y_{max} (\log X)^k. \tag{6.28}$$

So our error term (6.22) is $O(y_{max}^2 R^2 (\log X)^{4k})$. Using our change of variables we obtain

$$\begin{aligned} S_1 = \frac{X}{V} \sum_{u_1,\dots,u_k} \left(\prod_{i=1}^{k} \varphi(u_i)\right) \sideset{}{^*}\sum_{s_{1,2},\dots,s_{k-1,k}} \left(\prod_{\substack{1\le i,j\le k \\ i\neq j}} \mu(s_{i,j})\right) \\ \times \left(\prod_{i=1}^{k} \frac{\mu(a_i)\mu(b_i)}{\varphi(a_i)\varphi(b_i)}\right) y_{a_1,\dots,a_k} y_{b_1,\dots,b_k} + O\left(y_{max}^2 R^2 (\log X)^{4k}\right), \end{aligned} \tag{6.29}$$

where $a_j = u_j \prod_{i\neq j} s_{j,i}$ and $b_j = u_j \prod_{i\neq j} s_{i,j}$. Since there's no contribution when $a_j$ or $b_j$ are not squarefree, we may rewrite $\mu(a_j)$ as $\mu(u_j)\prod_{i\neq j}\mu(s_{j,i})$, and similarly for $\varphi(a_j)$, $\mu(b_j)$ and $\varphi(b_j)$. This gives us

(6.30)

$$\begin{aligned} S_1 = \frac{X}{V} \sum_{u_1,\dots,u_k} \left(\prod_{i=1}^{k} \frac{\mu(u_i)^2}{\varphi(u_i)}\right) \sideset{}{^*}\sum_{s_{1,2},\dots,s_{k,k-1}} \left(\prod_{\substack{1\le i,j\le k \\ i\neq j}} \frac{\mu(s_{i,j})}{\varphi(s_{i,j})^2}\right) y_{a_1,\dots,a_k} y_{b_1,\dots,b_k} \\ + O\left(y_{max}^2 R^2 (\log X)^{4k}\right). \end{aligned}$$

There is no contribution from $s_{i,j}$ with $1 < s_{i,j} < D_0$ because of the restricted support of $y$. The contribution when $s_{i,j} > D_0$ is

(6.31)
$$\ll \frac{y_{max}^2 X}{V}\Bigg(\sum_{\substack{u<R\\(u,VP_f)=1}}\frac{\mu(u_i)^2}{\varphi(u_i)}\Bigg)^k\Bigg(\sum_{s_{i,j}>D_0}\frac{\mu(s_{i,j})^2}{\varphi(s_{i,j})^2}\Bigg)\Bigg(\sum_{s>1}\frac{\mu(s)^2}{\varphi(s)^2}\Bigg)^{k^2-k-1}$$
$$\ll \frac{y_{max}^2\varphi(VP_f)^k X(\log X)^k}{V(VP_f)^k D_0}.$$

Our previous error of $y_{max}^2 R^2(\log X)^{4k}$ can be absorbed into this error, and the terms with $s_{i,j} = 1$ give us our desired main term. □

LEMMA 6.9. *Let $S_2^{(m)}$ be as defined in* (6.14), *and let*

$$y_{r_1,\dots,r_k}^{(m)} = \Big(\prod_{i=1}^k \mu(r_i)g(r_i)\Big)\sum_{\substack{d_1,\dots,d_k\\ r_i|d_i\forall i\\ d_m=1}}\frac{\lambda_{d_1,\dots,d_k}}{\prod_i\varphi(d_i)}, \tag{6.32}$$

*where $g$ is the totally multiplicative function defined on primes by $g(p) = p-2$. Let $y_{max}^{(m)} = \sup_{r_1,\dots,r_k}|y_{r_1,\dots,r_k}^{(m)}|$. Then for any fixed $A > 0$, we have*

(6.33)
$$S_2^{(m)} = \frac{X}{\varphi(V)\log X}\sum_{u_1,\dots,u_k}\frac{(y_{u_1,\dots,u_k}^{(m)})^2}{\prod_{i=1}^k g(u_i)}$$
$$+ O\left(\frac{(y_{max}^{(m)})^2\varphi(VP_f)^{k-1}X(\log X)^{k-2}}{\varphi(V)(VP_f)^{k-1}D_0}\right) + O\left(\frac{y_{max}^2 X}{\varphi(M)(\log X)^A}\right).$$

PROOF. We first rearrange the sum to obtain

$$S_2^{(m)} = \sum_{\substack{d_1,\dots,d_k\\ e_1,\dots,e_k}}\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}\sum_{\substack{x\le n<2x\\ n\equiv\nu_0 \pmod{W'}\\ [d_i,e_i]|nM+h_i}}\chi_{\mathbb{P}}(nM+h_m). \tag{6.34}$$

In the inner sum, when $W', [d_1, e_1], \dots, [d_k, e_k]$ are pairwise relatively prime, the conditions determine $n$ modulo $q = W'\prod_i[d_i, e_i]$, because $(M, [d_i, e_i]) = 1$ by the support of $\lambda$. In turn, $nM + h_m$ is determined modulo $qM = V\prod_i[d_i, e_i]$. Note that here $(q, P_f) = 1$. Also, if $([d_i, e_i], nM + h_m) > 1$ with $i \neq m$, then $p \mid |h_i - h_m| = fM$ for some $p \mid [d_i, e_i]$ and $f < D_0$ by the diameter of $\mathcal{H}$, and since $d_i$ and $e_i$ are relatively prime to both $M$ and $W$ by the support of $\lambda$, this is not possible. So $nM + h_m$ is relatively prime to the modulus if and only if $d_m = e_m = 1$. Thus we can write

(6.35)
$$\sum_{\substack{x\leq n<2x \\ n\equiv \nu_0 \pmod{W'} \\ [d_i,e_i]|nM+h_i}} \chi_{\mathbb{P}}(nM+h_m) = \sum_{\substack{X+h_m\leq n<2X+h_m \\ n\equiv b \pmod{qM}}} \chi_{\mathbb{P}}(n)$$
$$= \frac{\mathcal{P}_X}{\varphi(V)\prod_i \varphi([d_i,e_i])} + E(X,qM) + O(1),$$

where

$$E(X,qM) = \Big| \sum_{\substack{X\leq n<2X \\ n\equiv b \pmod{qM}}} \chi_{\mathbb{P}}(n) - \frac{\mathcal{P}_X}{\varphi(qM)} \Big|, \tag{6.36}$$

$\mathcal{P}_X$ is the number of primes in $[X, 2X]$, and the $O(1)$ term arises from ignoring the shift by $h_m$ in the sum. Thus the main term becomes

$$\frac{\mathcal{P}_X}{\varphi(V)} \sideset{}{'}\sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k}} \frac{\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}}{\prod_i \varphi([d_i,e_i])} \tag{6.37}$$

where $\sum'$ denotes that $W', [d_1, e_1], \ldots, [d_k, e_k]$ are pairwise relatively prime. As before, there is no contribution when $(W', [d_i, e_i]) > 1$ or $(d_i, d_j) > 1$, and we remove the conditions $(d_i, e_j) = 1$ by multiplying our expression by $\sum_{s_{i,j}|d_i,e_j} \mu(s_{i,j})$. We also use the identity (valid for squarefree $d_i$ and $e_i$),

$$\frac{1}{\varphi([d_i,e_i])} = \frac{1}{\varphi(d_i)\varphi(e_i)} \sum_{u_i|d_i,e_i} g(u_i), \tag{6.38}$$

where $g$ is the totally multiplicative function defined on primes by $g(p) = p - 2$. The main term then becomes

(6.39)
$$\frac{\mathcal{P}_X}{\varphi(V)} \sum_{u_1,\ldots,u_k} \Big(\prod_{i=1}^{k} g(u_i)\Big) \sum_{s_{1,2},\ldots,s_{k-1,k}} \Big(\prod_{\substack{1\leq i,j\leq k \\ i\neq j}} \mu(s_{i,j})\Big) \sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k \\ u_i|d_i,e_i \forall i \\ s_{i,j}|d_i,e_j \forall i\neq j \\ d_m=e_m=1}} \frac{\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}}{\prod_i \varphi(d_i)\varphi(e_i)}.$$

We again restrict $s_{i,j}$ to be coprime to $u_i$, $u_j$, $s_{i,a}$ and $s_{b,j}$ for all $a \neq i$ and $b \neq j$ as before, and make the change of variable

$$y^{(m)}_{r_1,\dots,r_k} = \Big(\prod_{i=1}^{k} \mu(r_i)g(r_i)\Big) \sum_{\substack{d_1,\dots,d_k \\ r_i|d_i \forall i \\ d_m=1}} \frac{\lambda_{d_1,\dots,d_k}}{\prod_i \varphi(d_i)}. \tag{6.40}$$

This is invertible, and $y^{(m)}_{r_1,\dots,r_k}$ is supported on $(\prod_i r_i, VP_f) = 1$, $\prod_i r_i < R$, $\mu(\prod_i r_i)^2 = 1$ and $r_m = 1$. Then the main term becomes

(6.41)
$$\frac{\mathcal{P}_X}{\varphi(V)} \sum_{u_1,\dots,u_k} \Big(\prod_{i=1}^{k} \frac{\mu(u_i)^2}{g(u_i)}\Big) \sum_{s_{1,2},\dots,s_{k-1,k}}^{*} \Big(\prod_{\substack{1\le i,j\le k \\ i\neq j}} \frac{\mu(s_{i,j})}{g(s_{i,j})^2}\Big) y^{(m)}_{a_1,\dots,a_k} y^{(m)}_{b_1,\dots,b_k},$$

where $a_j = u_j \prod_{i\neq j} s_{j,i}$ and $b_j = u_j \prod_{i \neq j} s_{i,j}$ for each $1 \le j \le k$. Because of the restricted support of $y$, there is no contribution from terms with $(s_{i,j}, VP_f) > 1$. So we only need to consider $s_{i,j} = 1$ or $s_{i,j} > D_0$. The contribution when $s_{i,j} > D_0$ is

(6.42)
$$\ll \frac{(y^{(m)}_{max})^2 X}{\varphi(V)\log X} \Big( \sum_{\substack{u<R \\ (u,VP_f)=1}} \frac{\mu(u)^2}{g(u)}\Big)^{k-1} \Big(\sum_s \frac{\mu(s)^2}{g(s)^2}\Big)^{k(k-1)-1} \sum_{s_{i,j}>D_0} \frac{\mu(s_{i,j})^2}{g(s_{i,j})^2}$$
$$\ll \frac{(y^{(m)}_{max})^2 \varphi(VP_f)^{k-1} X (\log X)^{k-2}}{\varphi(V)(VP_f)^{k-1} D_0}.$$

The contribution from $s_{i,j} = 1$ gives us the main term which is

$$\frac{\mathcal{P}_X}{\varphi(V)} \sum_{u_1,\dots,u_k} \frac{(y^{(m)}_{u_1,\dots,u_k})^2}{\prod_{i=1}^{k} g(u_i)}. \tag{6.43}$$

By the prime number theorem, $\mathcal{P}_X = X/\log X + O(X/(\log X)^2))$, and the error here contributes

(6.44)
$$\frac{(y^{(m)}_{max})^2 X}{\varphi(V)(\log X)^2} \Big( \sum_{\substack{u<R \\ (u,VP_f)=1}} \frac{\mu(u)^2}{\varphi(u)}\Big)^{k-1} \ll \frac{(y^{(m)}_{max})^2 \varphi(VP_f)^{k-1} X (\log X)^{k-3}}{\varphi(V)(VP_f)^{k-1}},$$

which can be absorbed in the error term from (6.42).

Now we turn to the contribution of the error terms in (6.35), which is

$$\ll \sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k}} |\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}| \left(E(X,qM)+1\right). \tag{6.45}$$

From the support of $\lambda$, we see that we only need to consider square-free $q$ with $q < W'R^2$ and $(q, MP_f) = 1$. Since for a square-free integer $q$ there are at most $\tau_{3k}(q)$ choices of $d_1, \dots, d_k, e_1, \dots, e_k$ for which $q = W' \prod_i [d_i, e_i]$, we see that the error is

(6.46)

$$\ll \lambda_{max}^2 \sum_{\substack{q<W'R^2 \\ (q,MP_f)=1}} \mu(q)^2 \tau_{3k}(q) E(X,qM) + \lambda_{max}^2 \sum_{\substack{q<W'R^2 \\ (q,MP_f)=1}} \mu(q)^2 \tau_{3k}(q).$$

Now the second term is $\ll \lambda_{max}^2 W'R^2 \log(W'R^2)^{3k-1}$. For the first term we use the Cauchy-Schwarz inequality and the trivial bound $E(X,qM) \ll X/\varphi(qM)$ to see that it is

(6.47)

$$\ll \frac{\lambda_{max}^2}{\varphi(M)^{1/2}} \Bigg( \sum_{\substack{q<W'R^2 \\ (q,MP_f)=1}} \mu(q)^2 \tau_{3k}^2(q) \frac{X}{\varphi(q)} \Bigg)^{1/2} \Bigg( \sum_{\substack{q<W'R^2 \\ (q,MP_f)=1}} \mu(q)^2 E(X,qM) \Bigg)^{1/2}.$$

The first sum is $\ll X \log(W'R^2)^{3k}$. Now for $X$ large enough, $W'R^2 \leq X^{e_f - \delta}$, so that Theorem 5.5 applies to yield that the second sum is $\ll \frac{X}{\varphi(M)} (\log X)^{-A}$ for $A$ arbitrarily large. Thus the total contribution is

$$\ll \frac{y_{max}^2 X}{\varphi(M)(\log X)^A}, \tag{6.48}$$

and this completes the proof. $\square$

LEMMA 6.10. *If* $r_m = 1$,

$$y^{(m)}_{r_1,\dots,r_k} = \sum_{a_m} \frac{y_{r_1,\dots,r_{m-1},a_m,r_{m+1},\dots,r_k}}{\varphi(a_m)} + O\left( \frac{y_{max}\varphi(VP_f)\log X}{VP_f D_0} \right). \tag{6.49}$$

PROOF. We assume that $r_m = 1$. We substitute (6.27) into (6.40) and obtain

$$y^{(m)}_{r_1,\dots,r_k} = \Big(\prod_{i=1}^{k} \mu(r_i) g(r_i)\Big) \sum_{\substack{d_1,\dots,d_k \\ r_i | d_i \forall i \\ d_m = 1}} \Big(\prod_{i=1}^{k} \frac{\mu(d_i) d_i}{\varphi(d_i)}\Big) \sum_{\substack{a_1,\dots,a_k \\ d_i | a_i \forall i}} \frac{y_{a_1,\dots,a_k}}{\prod_{i=1}^{k} \varphi(a_i)}. \tag{6.50}$$

Swapping summations over $d$ and $a$, we have

$$y^{(m)}_{r_1,\dots,r_k} = \Big(\prod_{i=1}^{k} \mu(r_i) g(r_i)\Big) \sum_{\substack{a_1,\dots,a_k \\ r_i | a_i \forall i}} \frac{y_{a_1,\dots,a_k}}{\prod_{i=1}^{k} \varphi(a_i)} \sum_{\substack{d_1,\dots,d_k \\ d_i | a_i, r_i | d_i \forall i \\ d_m = 1}} \prod_{i=1}^{k} \frac{\mu(d_i) d_i}{\varphi(d_i)}. \tag{6.51}$$

The inner sum can be directly computed when $a_i$ is squarefree, which is the only case that matters by the support of $y$. We have

$$\begin{aligned} \sum_{d_i | a_i, r_i | d_i} \frac{\mu(d_i) d_i}{\varphi(d_i)} &= \frac{\mu(r_i) r_i}{\varphi(r_i)} \sum_{d_i | \frac{a_i}{r_i}} \frac{\mu(d_i) d_i}{\varphi(d_i)} \\ &= \frac{\mu(r_i) r_i}{\varphi(r_i)} \prod_{p | \frac{a_i}{r_i}} \frac{-1}{p-1} \\ &= \frac{\mu(r_i) r_i}{\varphi(r_i)} \frac{\mu(a_i/r_i)}{\varphi(a_i/r_i)} = \frac{\mu(a_i) r_i}{\varphi(a_i)}. \end{aligned} \tag{6.52}$$

Hence

$$y^{(m)}_{r_1,\dots,r_k} = \Big(\prod_{i=1}^{k} \mu(r_i) g(r_i)\Big) \sum_{\substack{a_1,\dots,a_k \\ r_i | a_i \forall i}} \frac{y_{a_1,\dots,a_k}}{\prod_{i=1}^{k} \varphi(a_i)} \prod_{i \neq m} \frac{\mu(a_i) r_i}{\varphi(a_i)}. \tag{6.53}$$

By the support of $y$, we need only consider $a_j$ with $(a_j, VP_f) = 1$. This implies $a_j = r_j$ or $a_j > D_0 r_j$. The total contribution from $a_j \neq r_j$ when $j \neq m$ is

$$\begin{aligned} &\ll y_{max} \left(\prod_{i=1}^{k} g(r_i) r_i\right) \left(\sum_{a_j > D_0 r_j} \frac{\mu(a_j)^2}{\varphi(a_j)^2}\right) \\ &\quad \times \left(\sum_{\substack{a_m < R \\ (a_m, VP_f) = 1}} \frac{\mu(a_j)^2}{\varphi(a_j)}\right) \prod_{\substack{1 \le i \le k \\ i \neq j, m}} \left(\sum_{r_i | a_i} \frac{\mu(a_i)^2}{\varphi(a_i)^2}\right) \\ &\ll \left(\prod_{i=1}^{k} \frac{g(r_i) r_i}{\varphi(r_i)^2}\right) \frac{y_{max} \varphi(VP_f) \log R}{VP_f D_0} \ll \frac{y_{max} \varphi(VP_f) \log X}{VP_f D_0}. \end{aligned} \tag{6.54}$$

Thus we find that

$$(6.55)\quad y^{(m)}_{r_1,\dots,r_k} = \left(\prod_{i=1}^{k} \frac{g(r_i)r_i}{\varphi(r_i)^2}\right) \sum_{a_m} \frac{y_{r_1,\dots,r_{m-1},a_m,r_{m+1},\dots,r_k}}{\varphi(a_m)} + O\left(\frac{y_{max}\varphi(VP_f)\log X}{VP_f D_0}\right).$$

Since the product is $1 + O(D_0^{-1})$, we have the result. $\square$

LEMMA 6.11. *Let* $y_{r_1,\dots,r_k}$ *be given in terms of a piecewise differentiable function* $F$ *supported on* $\mathcal{R}_k = \{(x_1,\dots,x_k) \in [0,1]^k : \sum_{i=1}^k x_i = 1\}$ *by*

$$(6.56)\qquad y_{r_1,\dots,r_k} = F\big(\frac{\log r_1}{\log R},\dots,\frac{\log r_k}{\log R}\big)$$

*whenever* $r = \prod_i r_i$ *is squarefree and satisfies* $(r, VP_f) = 1$. *Put*

$$(6.57)\qquad F_{max} = \sup_{(t_1,\dots,t_k)\in[0,1]^k} |F(t_1,\dots,t_k)| + \sum_{i=1}^{k} |\frac{\partial F}{\partial t_i}(t_1,\dots,t_k)|.$$

*Then*

$$(6.58)\quad S_1 = \frac{\varphi(VP_f)^k X(\log R)^k}{V(VP_f)^k} I_k(F) + O\big(\frac{F_{max}^2\varphi(VP_f)^k X(\log X)^{k-1}\log\log X}{V(VP_f)^k D_0}\big),$$

*where*

$$(6.59)\qquad I_k(F) = \int_0^1 \cdots \int_0^1 F(t_1,\dots,t_k)^2 dt_1 \dots dt_k.$$

PROOF. We use (6.56) in our expression for $S_1$ from Lemma 6.8 and obtain

$$(6.60)\quad S_1 = \frac{X}{V} \sum_{\substack{u_1,\dots,u_k \\ (u_i,u_j)=1, \forall i\neq j \\ (u_i, VP_f)=1 \forall i}} \left(\prod_{i=1}^{k} \frac{\mu(u_i)^2}{\varphi(u_i)}\right) F\big(\frac{\log u_1}{\log R},\dots,\frac{\log u_k}{\log R}\big)^2 + O\left(\frac{F_{max}^2\varphi(VP_f)^k X(\log X)^k}{V(VP_f)^k D_0}\right).$$

Now if $(u_i, u_j) > 1$ for some $i \neq j$ and $(u_i, VP_f) = (u_j, VP_f) = 1$, then there is a prime $p \mid (u_i, u_j)$ with $p \nmid VP_f$, so *a fortiori* $p \nmid W$ and $p > D_0$. Thus the cost of dropping the condition $(u_i, u_j) = 1$ is an error of size

(6.61)

$$\begin{aligned} &\ll \frac{F_{max}^2 X}{V} \sum_{p > D_0} \sum_{\substack{u_1, \ldots, u_k < R \\ p \mid u_i, u_j \\ (u_i, VP_f) = 1 \forall i}} \prod_{i=1}^{k} \frac{\mu(u_i)^2}{\varphi(u_i)} \\ &\ll \frac{F_{max}^2 X}{V} \sum_{p > D_0} \frac{1}{(p-1)^2} \Big( \sum_{\substack{u < R \\ (u, VP_f) = 1}} \frac{\mu(u)^2}{\varphi(u)} \Big)^k \ll \frac{F_{max}^2 \varphi(VP_f)^k X (\log X)^k}{V (VP_f)^k D_0}. \end{aligned}$$

Thus we are left to evaluate

$$\sum_{\substack{u_1, \ldots, u_k \\ (u_i, VP_f) = 1 \forall i}} \Big( \prod_{i=1}^{k} \frac{\mu(u_i)^2}{\varphi(u_i)} \Big) F\big( \frac{\log u_1}{\log R}, \ldots, \frac{\log u_k}{\log R} \big)^2. \tag{6.62}$$

This differs from the corresponding sum in Maynard's work only in that we have a $VP_f$, which does not have as small prime factors, in place of $W$. We put,

$$\gamma(p) = \begin{cases} 1, & \text{if } p \nmid VP_f, \\ 0, & \text{otherwise.} \end{cases} \tag{6.63}$$

Then we can use Lemma 6.1 of [20] with $\kappa = 1$,

$$\begin{aligned} L &\ll 1 + \sum_{p \mid VP_f} \frac{\log p}{p} \ll \Big( \sum_{p \leq \log R} + \sum_{\substack{p \mid MP_f \\ p > \log R}} \Big) \frac{\log p}{p} \\ &\ll \log \log R + \frac{\log MP_f}{\log R} \ll \log \log X, \end{aligned} \tag{6.64}$$

and $A_1$ and $A_2$ suitable constants. The lemma then yields

(6.65)

$$\begin{aligned} &\sum\nolimits_{\substack{u_1, \ldots, u_k \\ (u_i, VP_f) = 1 \forall i}} \Big( \prod_{i=1}^{k} \frac{\mu(u_i)^2}{\varphi(u_i)} \Big) F\big( \frac{\log u_1}{\log R}, \ldots, \frac{\log u_k}{\log R} \big)^2 \\ &\quad = \frac{\varphi(VP_f)^k (\log R)^k}{(VP_f)^k} I_k(F) + O\Big( \frac{F_{max}^2 \varphi(VP_f)^k (\log X)^{k-1} \log \log X}{(VP_f)^k D_0} \Big) \end{aligned}$$

with

$$I_k(F) = \int_0^1 \cdots \int_0^1 F(t_1, \ldots, t_k)^2 dt_1 \ldots dt_k, \tag{6.66}$$

and the proof is complete. □

LEMMA 6.12. *Let $y_{r_1,\ldots,r_k}$, $F$, and $F_{max}$ be as in Lemma 6.11. Then*

$$S_2^{(m)} = \frac{\varphi(VP_f)^k X(\log R)^{k+1}}{V(VP_f)^k \log X} J_k^{(m)}(F) + O\left(\frac{F_{max}^2 \varphi(VP_f)^k X(\log X)^k}{V(VP_f)^k D_0}\right), \tag{6.67}$$

*where*

$$J_k^{(m)}(F) = \int_0^1 \cdots \int_0^1 \left(\int_0^1 F(t_1, \ldots, t_k) dt_m\right)^2 dt_1 \ldots dt_{m-1} dt_{m+1} \ldots dt_k. \tag{6.68}$$

PROOF. From Lemma 6.9, we want to evaluate the sum

$$\sum_{u_1,\ldots,u_k} \frac{(y_{u_1,\ldots,u_k}^{(m)})^2}{\prod_{i=1}^k g(u_i)}. \tag{6.69}$$

First we estimate $y_{r_1,\ldots,r_k}^{(m)}$. Recall that the weights $y_{r_1,\ldots,r_k}^{(m)}$ are supported on $(\prod_i r_i, VP_f) = 1$, $\mu(\prod_i r_i)^2 = 1$, $(r_i, r_j) = 1$ when $i \neq j$ and $r_m = 1$. Then substituting (6.56) into our expression for $y_{r_1,\ldots,r_k}^{(m)}$ from Lemma 6.10, we obtain

$$\begin{aligned} y_{r_1,\ldots,r_k}^{(m)} = &\sum_{(u, VP_f \prod_i r_i)=1} \frac{\mu(u)^2}{\varphi(u)} F\big(\tfrac{\log r_1}{\log R}, \ldots, \tfrac{\log r_{m-1}}{\log R}, \tfrac{\log u}{\log R}, \tfrac{\log r_{m+1}}{\log R}, \ldots, \tfrac{\log r_k}{\log R}\big) \\ &+ O\left(\frac{F_{max}\varphi(VP_f) \log X}{VP_f D_0}\right). \end{aligned} \tag{6.70}$$

From this it is plain that

$$y_{max}^{(m)} \ll \frac{\varphi(VP_f)}{VP_f} F_{max} \log X. \tag{6.71}$$

Now we use Lemma 6.1 of [20] again, with $\kappa = 1$,

$$\gamma(p) = \begin{cases} 1, & \text{if } p \nmid VP_f \prod_{i=1}^k r_i, \\ 0, & \text{otherwise.} \end{cases} \tag{6.72}$$

$$L \ll 1 + \sum_{p | V \prod_i r_i} \frac{\log p}{p} \ll \Big( \sum_{p \leq \log R} + \sum_{\substack{p | M P_f \prod_i r_i \\ p > \log R}} \Big) \frac{\log p}{p} \ll \log\log X, \tag{6.73}$$

and $A_1$, $A_2$ suitable constants to obtain

(6.74)

$$y^{(m)}_{r_1,\dots,r_k} = (\log R) \frac{\varphi(VP_f)}{VP_f} \Big( \prod_{i=1}^{k} \frac{\varphi(r_i)}{r_i} \Big) F^{(m)}_{r_1,\dots,r_k} + O\left( \frac{F_{max} \varphi(VP_f) \log X}{VP_f D_0} \right),$$

where

$$F^{(m)}_{r_1,\dots,r_k} = \int_0^1 F\Big( \frac{\log r_1}{\log R}, \dots, \frac{\log r_{m-1}}{\log R}, t_m, \frac{\log r_{m+1}}{\log R}, \dots, \frac{\log r_k}{\log R} \Big) dt_m. \tag{6.75}$$

This is valid if $r_m = 1$, and $r = \prod_{i=1}^k r_i$ satisfies $(r, VP_f) = 1$ and $\mu(r)^2 = 1$, otherwise $y^{(m)}_{r_1,\dots,r_k} = 0$. Squared, (6.74) gives

$$\begin{aligned} (y^{(m)}_{r_1,\dots,r_k})^2 =& (\log R)^2 \frac{\varphi(VP_f)^2}{(VP_f)^2} \Big( \prod_{i=1}^{k} \frac{\varphi(r_i)^2}{r_i^2} \Big) (F^{(m)}_{r_1,\dots,r_k})^2 \\ &+ O\left( \frac{(F_{max})^2 \varphi(VP_f)^2 (\log X)^2}{(VP_f)^2 D_0} \right). \end{aligned} \tag{6.76}$$

Using this in the expression for $S_2^{(m)}$ from Lemma 6.9, we have

$$\begin{aligned} S_2^{(m)} = \frac{\varphi(VP_f)^2 X (\log R)^2}{\varphi(V)(VP_f)^2 \log X} & \sum_{\substack{r_1,\dots,r_k \\ (r_i, VP_f)=1 \\ (r_i,r_j)=1 \forall i \neq j \\ r_m = 1}} \Big( \prod_{i=1}^{k} \frac{\mu(r_i)^2 \varphi(r_i)^2}{g(r_i) r_i^2} \Big) (F^{(m)}_{r_1,\dots,r_k})^2 \\ &+ O\left( \frac{F_{max}^2 \varphi(VP_f)^k X (\log X)^k}{V (VP_f)^k D_0} \right). \end{aligned} \tag{6.77}$$

We drop the condition $(r_i, r_j) = 1$ as before, this time introducing an error of size

$$\begin{aligned} &\ll \frac{F_{max}^2 \varphi(VP_f)^2 X (\log R)^2}{\varphi(V)(VP_f)^2 \log X} \Big( \sum_{p > D_0} \frac{\varphi(p)^4}{g(p)^2 p^4} \Big) \Big( \sum_{\substack{r < R \\ (r, VP_f)=1}} \frac{\varphi(r)^2}{g(r) r^2} \Big)^{k-1} \\ &\ll \frac{F_{max}^2 \varphi(VP_f)^{k+1} X (\log X)^k}{\varphi(V)(VP_f)^{k+1} D_0}. \end{aligned} \tag{6.78}$$

Thus we are left to evaluate

$$\sum_{\substack{r_1,\ldots,r_{m-1},r_{m+1},\ldots,r_k \\ (r_i,VP_f)=1}} \Big(\prod_{i=1}^{k} \frac{\mu(r_i)^2\varphi(r_i)^2}{g(r_i)r_i^2}\Big)(F^{(m)}_{r_1,\ldots,r_k})^2. \tag{6.79}$$

Again we apply Lemma 6.1 from Maynard with $\kappa = 1$, with

$$\gamma(p) = \begin{cases} 1 - \frac{p^2-3p+1}{p^3-p^2-2p+1}, & \text{if } p \nmid VP_f, \\ 0, & \text{otherwise,} \end{cases} \tag{6.80}$$

$$L \ll 1 + \sum_{p \mid VP_f} \frac{\log p}{p} \ll \log\log X, \tag{6.81}$$

and $A_1$, $A_2$ suitable constants. The singular series in this case is

$$\mathfrak{S} = \frac{\varphi(VP_f)}{VP_f}\big(1 + O\big(\frac{1}{D_0}\big)\big), \tag{6.82}$$

and we obtain

$$S_2^{(m)} = \frac{\varphi(VP_f)^{k+1}X(\log R)^{k+1}}{\varphi(V)(VP_f)^{k+1}\log X} J_k^{(m)}(F) + O\left(\frac{F_{max}^2\varphi(VP_f)^{k+1}X(\log X)^k}{\varphi(V)(VP_f)^{k+1}D_0}\right). \tag{6.83}$$

Now in the main term we have

$$\begin{aligned} \frac{\varphi(VP_f)}{\varphi(V)(VP_f)} &= \frac{1}{V}\cdot\frac{V}{\varphi(V)}\cdot\frac{\varphi(VP_f)}{(VP_f)} \\ &= \frac{1}{V}\prod_{p\mid V}\frac{p}{p-1}\prod_{p\mid VP_f}\frac{p-1}{p} \\ &= \frac{1}{V}\prod_{\substack{p\mid P_f\\ p\nmid V}}\frac{p-1}{p}. \end{aligned} \tag{6.84}$$

This last product is either vacuous, or consists of a single factor $(1-p_0^{-1})$, which is $1+O\left((\log\log X)^{-1}\right)$. Thus we may replace (6.83), within acceptable error, with

$$S_2^{(m)} = \frac{\varphi(VP_f)^k X(\log R)^{k+1}}{V(VP_f)^k\log X} J_k^{(m)}(F) + O\left(\frac{F_{max}^2\varphi(VP_f)^k X(\log X)^k}{V(VP_f)^k D_0}\right), \tag{6.85}$$

where we have replaced $\frac{\varphi(VP_f)}{\varphi(V)(VP_f)}$ with $1/V$ in the error term as well. $\square$

## 4. Discussion

There is a result due to Baker and Zhao [2] where they also consider primes in arithmetic progressions, except they prove their result for certain smooth moduli (recall that a number is called $y$-smooth if it has no prime factor exceeding $y$). The techniques they employ involve estimating Dirichlet polynomials and appealing to a zero-free region described in terms of the largest prime and the squarefree kernel of $M$ to obtain the required Bombieri-Vinogradov type theorem. Their result [2, Theorem 1] reads as follows (with the notation adapted where applicable to avoid confusion).

THEOREM (BAKER-ZHAO). *Let* $\eta > 0$, $r \geq 1$, *and let* $M = X^\theta$ *with* $0 < \theta \leq 5/12 - \eta$, $(a, M) = 1$. *Let*

$$K(\theta) = \begin{cases} \frac{4}{1-2\theta} & \text{if } \theta < 2/5 - \varepsilon, \\ \frac{40}{9-20\theta} & \text{if } \theta \geq 2/5 - \varepsilon,. \end{cases}$$

*Suppose that* $M$ *satisfies*

$$\max\{p : p \mid M\} < \exp\left(\frac{\log X}{B \log\log X}\right), \quad \prod_{p \mid M} p < X^\delta, \quad w \nmid M$$

*with*

$$B = \frac{C_1}{\eta} \exp\left(\frac{4(r+1)}{K(\theta)}\right), \quad \delta = \frac{C_3 \eta}{r + \log(1/\eta)} \exp\left(-\frac{4(r+1)}{K(\theta)}\right)$$

*for suitable absolute positive constants* $C_1$ *and* $C_3$, *and* $w$ *denotes the possibly existing unique exceptional modulus to which there's a Dirichlet* $L$*-function with a zero in the region* $\beta > c_1/\log X$. *There are primes* $p_n < \ldots < p_{n+r}$ *in* $(X/2, X]$, *with* $p_i \equiv a \pmod{M}$ *such that*

$$p_{n+r} - p_n < C_2 M r \exp\left(K(\theta) r\right).$$

*Here* $C_2$ *is a positive absolute constant.*

Recalling our Theorem 6.6,

$$p_{n+r} - p_n \ll \left(\frac{r}{\eta}\right) \exp\left(\frac{5r}{3\eta}\right) M,$$

one immediately sees that the Baker-Zhao bound is stronger as $r$ grows, and also has the advantage of describing the moduli for which it holds (apart from the possibility of being a multiple of the exceptional modulus if it exists). On the other hand, as per the second remark following Proposition 5.2, the results of the present work holds for at least $X^{5/12-\eta}(1 - c/\log\log X)$ moduli up to $X^{5/12-\eta}$, while by Dickman's Theorem (see, for instance, [23, Theorem 7.2]), there are $o(X^{5/12-\eta})$ integers with no prime divisors exceeding $\exp\left(\frac{\log X}{B\log\log X}\right)$ for which the Baker-Zhao result holds. Hence the present result is valid for a much larger class of arithmetic progressions. With these considerations the two can be regarded as complementary results concerning uniform small gaps between primes in arithmetic progressions over a range of moduli.

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