Combinatorial and computational problems about points in the plane

By

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Abstract

We study three problems in combinatorial geometry. The problems investigated are conflict-free colorings of point sets in the plane with few colors, polychromatic colorings of the vertices of rectangular partitions in the plane and in higher dimensions and polygonalizations of point sets with few reflex points. These problems are problems of discrete point sets, the proofs are of combinatorial flavour with computational aspects and give efficient algorithms. First we give a historical introduction to the topics and place our results in this context. We also investigate the similarities between the proving methods of the three topics. In the rest of the Thesis we present the results in all detail including proofs.

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Contents

1 Introduction

In this thesis we study several problems in **combinatorial geometry**. The problems investigated are conflict-free colorings of point sets in the plane with few colors, polychromatic colorings of the vertices of rectangular partitions in the plane and in higher dimensions and polygonalizations of point sets with few reflex points. The connections between these problems might not be obvious at first sight as the historical relation between them is not very strong. Instead, many similarities between the proving methods can be observed. All problems are **problems of discrete point sets**, the proofs are of combinatorial flavour and are problems in **computational geometry** with **algorithmic proofs**. First we overview the topics and the results appearing in the Thesis. Finally, we give a more detailed overview of the methods appearing in the proofs.

1.1 Weak conflict-free colorings

The following problem is the meeting point of two areas of combinatorial geometry. The first and older of these two is the problem of **decomposability of multiple coverings** (and the closely related problem of multiple packings which we do not discuss here), the other one is **conflict-free coloring** of points and regions. We start with the introduction of the first one. Multiple coverings (and packings) were introduced independently by Davenport and László Fejes Tóth. Given a system $\mathcal{F}$ of subsets of an underlying set P, we say that they form a k-fold covering if every point of P belongs to at least k members of $\mathcal{F}$. A family $\mathcal{F}$ of regions in the plane is called *cover-decomposable* if there exists a positive integer $k = k(\mathcal{F})$ such that any k-fold covering of the plane with members from $\mathcal{F}$ can be decomposed into two coverings (i.e. into two 1-fold coverings). In an unpublished manuscript, P. Mani-Levitska and Pach [35] showed that every 33-

fold covering of the plane with congruent open disks splits into two coverings, thus the set of the translates of an open disc is cover-decomposable. Pach [37] also showed that the set of the translates of any open centrally symmetric convex polygon is cover-decomposable. Tardos and Tóth [47] extended this result to any open triangle. Recently Pálvölgyi and Tóth [43] extended this result further to any open convex polygon. Pach et al. [40] proved several negative results, among others they proved that the set of translates of any concave quadrilateral is not cover-decomposable.

One important observation is that to prove cover-decomposability, it is enough to solve a finite problem, as for open bounded sets it implies the cover-decomposability. This *finite-cover-decomposability* means that having an arbitrary finite subset of the given set of regions we need a two-coloring of these regions such that any k-fold covered point is covered by regions of both colors. For example the set of translates of an arbitrary open convex polygon is finite-cover-decomposable as well [43]. Recently Pálvölgyi [42] showed that the set of translates of any general (having no parallel edges) concave polygon is not finite-cover-decomposabe. He also characterizes which non-general concave polygons are finite-cover-decomposable and which are not thus answering the finite question for every open polygon.

The other important observation used throughout these proofs is that it is possible to consider the following dual problem. Distinguish an arbitrary point in S as a center (if S is centrally symmetric let it be its real center) and then for a given system $\mathcal{S}$ of translates of S, let $C(\mathcal{S})$ denote the set of centers of all members of $\mathcal{S}$. Clearly, $\mathcal{S}$ forms a k-fold covering of the plane if and only if every translate of $-S$ contains at least k elements of $C(\mathcal{S})$ (where $-S$ denotes the reflection of S to its center, thus for centrally symmetric S we have $-S = S$). I.e. cover-decomposability is equivalent to the existence of a bipartition (two-coloring) of these points such that every translate of $-S$ contains at least one element from both parts (colors).

The dual of the finite-cover-decomposability problem is that having a finite set of points we need a two-coloring of the points such that any translate of $-S$ covering at least k of these points covers points of both colors. This finite dual version can be phrased for any set of regions, not just for the set of translates, in this case the two versions are not equivalent (for example for all discs of the plane it does not hold).

Given a family of regions and a finite subfamily $\mathcal{F}$ of it, following the notation of Smorodinsky [46] one can define the *geometric hypergraph* induced by $\mathcal{F}$. The base set is the set of regions $\mathcal{F}$ and for any point p covered by at least k regions there is a hyperedge r_p containing the regions covering p. A proper coloring of a hypergraph is a coloring of the points such that there are no monochromatic edges. Clearly, determining whether there is a proper two-coloring for any finite subfamily $\mathcal{F}$ is equivalent to the finite-cover-decomposability defined above. The dual problem can be phrased similarly as a hypergraph-coloring problem. Then it is natural to ask what is the minimal number of colors for which a proper coloring exists. This is where we arrive to the notion of weak conflict-free colorings but before defining it in whole generality we have to talk about the other area it relates to, the area of **conflict-free colorings**, which serves as the main motivation to investigate these questions.

Motivated by a frequency assignment problem in cellular telephone networks, Even, Lotker, Ron and Smorodinsky [23] studied the following problem. Cellular networks facilitate communication between fixed *base stations* and moving *clients*. Fixed frequencies are assigned to base-stations to enable links to clients. Each client continuously scans frequencies in search of a base-station within its range with good reception. The fundamental problem of frequency assignment in cellular networks is to assign frequencies to base-stations such that every client is served by some base-station, i.e. it can communicate with a station such that the frequency of that station is not assigned to any other station it could also communicate with (to avoid interference). Given a fixed set of base-stations we want to minimize the number of assigned frequencies.

First we assume that the ranges are determined by the clients, i.e. if a base-station is in the range of some client, then they can communicate. Let P be the set of base-stations and $\mathcal{F}$ the family of all possible ranges of any client. Given some family $\mathcal{F}$ of planar regions and a finite set of points P we define $cf(\mathcal{F}, P)$ as the smallest number of colors which are enough to color the points of P such that in every region of $\mathcal{F}$ containing at least one point, there is a point whose color is unique among the points in that region. The maximum over all point sets of size n is the so called **conflict-free coloring number** (cf-coloring in short), denoted by $cf(\mathcal{F}, n)$ (for a summary of the definitions of the different versions of cf-colorings see Definition 1.1). Determining the cf-coloring number for different types of regions $\mathcal{F}$ is the main aim in this topic. Regions for which the problems has been studied include discs ([23], [46], etc.) and axis-parallel rectangles ([16], [41], [7], etc), we give a more detailed overview after introducing the definitions needed.

It is natural to define a dual version for conflict-free colorings as well. It is natural to assume that the ranges are determined by the base-stations, i.e. if a client is in the range of some base-station, then they can communicate. For a *finite* family of planar regions $\mathcal{F}$ we define $\overline{cf}(\mathcal{F})$ as the smallest number of colors which is enough for coloring the regions of $\mathcal{F}$ such that for every point in $\cup\mathcal{F}$ there is a region whose color is unique among the colors of the regions covering it. For a (not necessarily finite) family $\mathcal{F}$ of planar regions let $\overline{cf}(\mathcal{F}, n)$, the **conflict-free region-coloring number of $\mathcal{F}$** be the maximum of $\overline{cf}(\mathcal{F}')$ for $\mathcal{F}' \subseteq \mathcal{F}$, $|\mathcal{F}'| = n$. Smorodinsky [45] and then Har-Peled et al. [27] defined generalized versions of these notions. A *cf_k-coloring* of a point set is a coloring such that for each region F of $\mathcal{F}$ containing at least one point, there is a color which is assigned to at most

k points covered by F. Note that a cf-coloring is actually a cf_1-coloring. The region-coloring version is defined similarly.

Definition 1.1. *Given a family $\mathcal{F}$ of planar regions and a finite set of points P,*

- $cf_k(\mathcal{F}, P) = \min c$*: $\exists$ c-coloring f of P s.t. $\forall F \in \mathcal{F}$ $\exists x$ s.t.* $1 \leq |\{p : p \in F, f(p) = x\}| \leq k$
- $cf_k(\mathcal{F}, n) = \max_{|P|=n} cf_k(\mathcal{F}, P)$*;* $cf(\mathcal{F}, n) = cf_1(\mathcal{F}, n)$
- *if $\mathcal{F}$ is finite,* $\overline{cf_k}(\mathcal{F}) = \min c$*: $\exists$ c-coloring f of $\mathcal{F}$ s.t. $\forall p \in \cup\mathcal{F}$ $\exists x$ s.t.* $1 \leq |\{F, p \in F, f(F) = x\}| \leq k$
- $\overline{cf_k}(\mathcal{F}, n) = \max_{|\mathcal{F}'|=n, \mathcal{F}' \subset \mathcal{F}} \overline{cf_k}(\mathcal{F}')$*;* $\overline{cf}(\mathcal{F}, n) = \overline{cf_1}(\mathcal{F}, n)$

Now we arrived to the point where we can introduce the notion of **weak conflict-free colorings**. For a summary of the forthcoming definitions of the different versions of wcf-colorings see Definition 1.5. Modifying the definition of a conflict-free coloring, $wcf(\mathcal{F}, P)$ equals to the minimum number of colors needed to color the points of P such that whenever a region covers at least 2 points of P, it covers 2 of different colors (the region is *not monochromatic*). The maximum over all n element point set of size n is the **weak conflict-free coloring number** (wcf-coloring in short), denoted by $wcf(\mathcal{F}, n)$.

Observation 1.2. $cf(\mathcal{F}, n) \geq wcf(\mathcal{F}, n)$*.*

Generalizing our definition we can define $wcf_k(\mathcal{F}, P)$ as the minimum number of colors needed to color the points of P such that whenever a region covers at least k points of P, then it covers 2 points of different colors. The maximum of this value over all point sets of size n is denoted by $wcf_k(\mathcal{F}, n)$. Note that $wcf(\mathcal{F}, n) = wcf_2(\mathcal{F}, n)$.

Observation 1.3. $wcf_k(\mathcal{F}, n) \leq wcf_l(\mathcal{F}, n)$ *if* $k \geq l$*.*

Even et al. [23] presented a general algorithmic framework on conflict-free colorings, refined version of this approach appeared in [27] (and later in [46]) where they summarized it in a lemma showing essentially that the weak conflict-free coloring number gives a good upper bound to the conflict-free coloring number (they do not use the notation wcf-coloring and they deal only with the case $k = 2$). The algorithm giving a conflict-free coloring from weak conflict-free colorings is the following. In each step take a biggest color class in a weak conflict-free coloring of the point set. After coloring it to a new color, delete it and do the same for the new (smaller) point set. This framework also works for $k > 2$ as it is easy to see that taking in each step a color class of a wcf_k-coloring we get a cf_{k-1}-coloring. The generalized version of the lemma stated in [27] is as follows.

Lemma 1.4. *For any fixed* $k \geq 2$

(i) if $wcf_k(\mathcal{F}, n) \leq c$ *for some constant* c*, then* $cf_{k-1}(\mathcal{F}, n) \leq \frac{\log n}{\log(c/(c-1))} = O(\log n)$*,*

(ii) if $wcf_k(\mathcal{F}, n) = O(n^\epsilon)$ *for some* $\epsilon > 0$*, then* $cf_{k-1}(\mathcal{F}, n) = O(n^\epsilon)$*.*

Observation 1.2 and Lemma 1.4 show that wcf and cf are usually close to each other. Often the best known bound for cf is obtained from Lemma 1.4. This is the main motivation why we want to determine the weak conflict-free coloring number for different types of regions. As this lemma gives bound for cf_{k-1} using wcf_k, it motivates the investigation of wcf_k-colorings for $k > 2$.

Again we can define the dual version. For a finite $\mathcal{F}$, $\overline{wcf}(\mathcal{F})$ equals to the smallest number of colors which are enough to assign colors to the regions in $\mathcal{F}$ such that for every point covered by at least 2 regions in $\mathcal{F}$, there are two differently colored regions among the regions covering it (the point is *not monochromatic*). For a not necessarily finite $\mathcal{F}$ the maximum of this value over all n element subfamilies of $\mathcal{F}$ is the **weak conflict-free region-coloring number**, denoted by $\overline{wcf}(\mathcal{F}, n)$. Finally, we can again define $\overline{wcf_k}(\mathcal{F}, n)$ by restricting the condition only for points covered by at least k regions in $\mathcal{F}$. The dual version of Lemma 1.4 holds as well. Let us recall the geometric hypergraph defined earlier. Given a finite family of regions $\mathcal{F}$, the base set of the hypergraph is the family of regions $\mathcal{F}$ and for any point p covered by at least k regions there is a hyperedge r_p containing the regions covering p. We were interested in determining its chromatic number, which is clearly equivalent to determining $\overline{wcf_k}(\mathcal{F})$. Further, the existence of a k for which $\overline{wcf_k(\mathcal{F})} = 2$ is equivalent to the statement that $\mathcal{F}$ is finite-cover-decomposable. Indeed, a good weak conflict-free region-coloring of $\mathcal{F}$ gives a good partition and vice versa. In the dual version such equivalence holds as well.

For most regions we study, the weak conflict-free coloring number can always be bounded from above by a constant not depending on n. Thus, for infinite $\mathcal{F}$ we define $wcf_k(\mathcal{F}) = \max_n wcf_k(\mathcal{F}, n)$ and similarly $\overline{wcf_k}(\mathcal{F}) = \max_n \overline{wcf_k}(\mathcal{F}, n)$ if they exist. Our main aim is to determine these numbers and give coloring algorithms using this minimal number of colors. Before doing so, let us collect here the definitions of the different versions of wcf-colorings.

Definition 1.5. *Given a family $\mathcal{F}$ of planar regions and a finite set of points P,*

- $wcf_k(\mathcal{F}, P) = \min c$*: $\exists$ c-coloring f of P s.t. $\forall F \in \mathcal{F}$ if $|\{p : p \in F\}| \geq k$ then $\exists p, q \in P, p \neq q : p, q \in F$ and $f(p) \neq f(q)$*
- $wcf_k(\mathcal{F}, n) = \max_{|P|=n} wcf_k(\mathcal{F}, P)$*; $wcf(\mathcal{F}, n) = wcf_2(\mathcal{F}, n)$*
- *if $\mathcal{F}$ is infinite, $wcf_k(\mathcal{F}) = \max_n wcf_k(\mathcal{F}, n)$, if it exists*
- *if $\mathcal{F}$ is finite, $\overline{wcf_k}(\mathcal{F}) = \min c$: $\exists$ c-coloring f of $\mathcal{F}$ s.t. $\forall p \in \cup\mathcal{F}$ if $|\{F : p \in F\}| \geq k$ then $\exists F, G \in \mathcal{F}, F \neq G : p \in F, G$ and $f(F) \neq f(G)$*
- $\overline{wcf_k}(\mathcal{F}, n) = \max_{|\mathcal{F}'|=n, \mathcal{F}' \subset \mathcal{F}} \overline{wcf_k}(\mathcal{F}')$*; $\overline{wcf}(\mathcal{F}, n) = \overline{wcf_2}(\mathcal{F}, n)$*
- *if $\mathcal{F}$ is infinite, $\overline{wcf_k}(\mathcal{F}) = \max_n \overline{wcf_k}(\mathcal{F}, n)$, if it exists*

Slightly modifying the notation of [27] we call a family of regions $\mathcal{F}$ **monotone** if for any finite $P, F \in \mathcal{F}$ and l positive integer if F covers at least l points of F then there exists $F' \in \mathcal{F}$ covering exactly l points of P, all covered by F as well.

Observation 1.6. *For monotone families of regions in the definition of wcf-coloring it is enough to restrict our condition to regions covering exactly* 2 *points of the point set. Similarly, for the definition of* wcf_k *it is enough to restrict the condition to regions covering exactly* k *points.*

Note that monotonicity could be defined in the dual version as well, but none of the types of regions we study are monotone in that dual sense.

Let us now summarize the results known about axis-parallel rectangles and discs (the two most widely examined region types) and pose open problems regarding the unsolved cases. The general case of axis-parallel rectangles (denoted by $\mathcal{R}$) is still far from being solved, the best bounds are $wcf(\mathcal{R}, n) = \Omega(\frac{\log n}{(\log\log n)^2})$ by Chen et al. [16] from below and recently by Ajwani et al. [7] $wcf(\mathcal{R}, n) = \tilde{O}(n^{.382+\epsilon})$ from above, improving the previous bound $wcf(\mathcal{R}, n) = O(\sqrt{\frac{n \log\log n}{\log n}})$ by Pach et al. [41]. So probably one of the most interesting problems is still to give better bounds for $wcf(\mathcal{R}, n)$, i.e. the lowest number of colors needed to color any set of n points, such that if an axis-parallel rectangle covers at least two of them then not all of those covered by it have the same color.

For the dual case of coloring axis-parallel rectangles the proof of the upper bound $\overline{cf}(\mathcal{R}, n) = O(\log^2 n)$ ([27]) can be modified easily to give the upper bound $\overline{wcf}(\mathcal{R}, n) = O(\log n)$. There is a matching lower bound $\overline{wcf}(\mathcal{R}, n) = \Omega(\log n)$ by Pach et al. [39]. This implies the same lower bound for $\overline{cf}(\mathcal{R}, n)$, thus in this case there is still a slight gap between the lower and upper bounds.

The case of discs (denoted by $\mathcal{D}$) in the plane is only partially solved. It is easy to see that a proper coloring of the Delaunay-triangulation of a point set gives a weak conflict-free coloring. From the four-color theorem we conclude that $wcf_2(\mathcal{D}) = 4$. Further, Pach et al. [40] showed that $wcf_k(\mathcal{D}) > 2$ for any k.

Problem 1.7. *Is it true for some* k *that* $wcf_k(\mathcal{D}) = 3$*? If yes, find the smallest such* k.

Answering the question if $\overline{wcf_k}(\mathcal{D})$ exists for some k Smorodinsky [46] showed

k	2	3	≥ 4
$wcf_k(\mathcal{B})$	*3*	**3**	**2**
$\overline{wcf_k}(\mathcal{B})$	**3**	**2**	*2*
$wcf_k(\mathcal{H})$	**4**	**2**	*2*
$\overline{wcf_k}(\mathcal{H})$	**3**	*2 or 3*	**2**

Table 1: table of results about $\mathcal{B}$ and $\mathcal{H}$

that $\overline{wcf_2}(\mathcal{D}) = 4$.

Problem 1.8. *Give better bounds for* $\overline{wcf_k}(\mathcal{D})$ *when* $k > 2$.

Now we can summarize the new results presented in Chapter 2. In Section 2.1 we solve all cases for *bottomless rectangles*, a special case of axis-parallel rectangles. A **bottomless rectangle** is the set of points (x, y) such that $a < x < b, y < c$ for some a, b and c. The set of all bottomless rectangles is denoted by $\mathcal{B}$. For our coloring purposes the family of bottomless rectangles is equivalent with the family of (ordinary) axis-parallel rectangles having their lower edge on a common horizontal *base-line*. At the end of the section we deduce results for another very similar special case, the set of rectangles intersecting a common base-line. In Section 2.2 we prove theorems which give an almost complete answer for *half-planes* (the set of all half-planes is denoted by $\mathcal{H}$). The only case not completely solved is to determine $\overline{wcf_3}(\mathcal{H})$. In Table 1 we summarize these results, the bold ones are proved in Chapter 2, others come from monotonicity except for $wcf_2(\mathcal{B}) = 3$ which is folklore.

1.2 Reflexivity of point sets in the plane

In the following problem again we need to find some optimal structure for a given set of points. Here instead of a coloring we have to find a polygonalization of the points. Given a set S of $n \geq 3$ points in the plane, a *polygonalization* of S is a simple polygon P whose vertices are the points of S. We always assume that the points are in *general position*, that is, no three of them are collinear. A vertex of a simple polygon is *reflex* if the (interior) angle of the polygon at that vertex is greater than π. We denote by $\rho(P)$ the number of reflex vertices of a polygon P. The *reflexivity* of a set of points S, $\rho(S)$, is the smallest number of reflex vertices any polygonalization of S must have. Further, we denote by $\rho(n)$ the maximum value $\rho(S)$, such that S is a set of n points.

The notion of reflexivity was suggested by Arkin et al. [9] as a measure for the "goodness" of a polygonalization of a set of points. According to their motivation, the reflexivity quantifies, in a combinatorial sense, the degree to which the set of points is in convex position. There are several other such functions, e.g., the minimum number of points to delete from S such that the remaining point set is in convex position or the number of convex layers. There are several applications in computational geometry in which the number of reflex vertices of a polygon can play an important role in the complexity of algorithms. If one or more polygons are given to us, there are many problems for which more efficient algorithms can be written with complexity depending on the number of reflex vertices, instead of the total number of vertices. The number of reflex vertices also plays an important role in convex decomposition problems for polygons (see for example Keil [34] for a recent survey).

From a slightly different point of view, reflexivity has strong connections with the convex cover number $\kappa_c(S)$, the minimal number of convex set of points (convex chains) covering the whole point set S. These kind of questions can be originated from the classical problems of Erdős and Szekeres ([21], [22]). They studied convex chains in finite planar point sets and showed that any point set of size n has a convex subset of size $\Omega(logn)$. This is closely related to the convex cover number κ_c, since it implies an asymptotically tight bound on the worst-case value of κ_c for sets of size n (denoted by $\kappa_c(n)$) as Urabe ([48],[49])

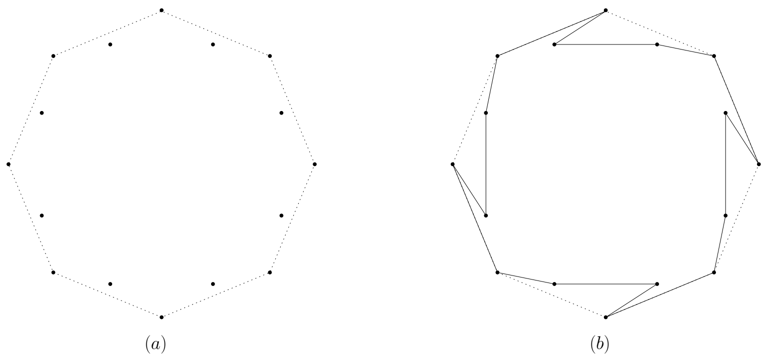


Figure 1: (a) configuration of points $S_0(n)$ having reflexivity $\rho(S_0(n)) = n/4$ and (b) a polygonalization of S_0 with $n/4$ reflex vertices

showed that $\kappa_c(n) = \Theta(n/\log n)$. There are still a number of open problems regarding the exact relationship between the size of the biggest convex chain and n (see for example [38] for recent developments). Another variant of this measure is the convex partition number κ_p, the minimal number of convex set of points covering the whole point set S such that their convex hull are pair-wise disjoint. Urabe ([48],[49]) and Urabe and Hosono [31] has shown that $\lceil (n-1)/4 \rceil \leq \kappa_p(n) \leq \lceil 5n/8 \rceil$. Clearly, $\kappa_c \leq \kappa_p$. The ratio $\kappa_p(S)/\kappa_c(S)$ for a set S may be as large as $\Theta(n)$. By an observation of Chazelle [15] one can partition an optimal polygonalization to $\rho(S) + 1$ convex parts by adding $\rho(S)$ segments, bisecting each reflex angle. This gives also a good partition of the point set, proving $\kappa_p \leq \rho(n) + 1$. Arkin et al. believe that the ratio between these two notions cannot be big, i.e. they conjectured that $\rho(S) = O(\kappa_p(S))$. They could not prove this, and the worst example they found has reflexivity twice as big as κ_c.

Taking into account that the points on the convex hull of S are always convex points of any polygonalization, it is natural to regard $\rho(S)$ with respect to n_I, the points of S which are interior to the convex hull $\mathcal{CH}(S)$. Arkin et al. proved that $\rho(S) \leq \lceil n_I/2 \rceil$ and that this bound is tight. The example $S_0(n)$

(see Figure 1) having $n \geq 6$ points and showing tightness has $n_I = \lfloor n/2 \rfloor$, thus implying $\rho(S_0(n)) \geq \lceil n_I/2 \rceil \geq \lfloor n/4 \rfloor$. As $n_I \leq n$ these bounds imply $\lfloor n/4 \rfloor \leq \rho(n) \leq \lceil n/2 \rceil$. Arkin et al. conjectured that the lower bound is tight – see also Conjecture 7 in Chapter 8.5 of [12]. The case $n = 6$ already shows that the lower bound is not tight but with ceiling instead of floor it may be still a tight bound. Settling this conjecture is one of the open problems listed in *The Open Problems Project* [17]. We refer to [9] and [12] for further discussions on the notion of reflexivity, its applications, and related problems.

Table 2 lists $\rho(n)$ for $n \leq 10$. These values were verified using a computer [5, 9].

n	3	4	5	6	7	8	9	10
$\rho(n)$	0	1	1	2	2	2	3	3

Table 2: $\rho(n)$ for $n \leq 10$

The main result of Chapter 3 is the following improvement for the upper bound of $\rho(n)$.

Theorem 1.9. $\rho(n) \leq 3\lfloor \frac{n-2}{7} \rfloor + 2$.

The result is obtained by considering a slightly modified version of reflexivity, namely to force a given convex hull edge to be part of the polygonalization. The main ingredient of the proof is an iterative subdivision of the point set, together with a good polygonalization of sets of constant size.

Utilizing a computer-aided abstract order type extension [6] the upper bound is further improved to

Theorem 1.10. $\rho(n) \leq 5\lfloor \frac{n-2}{12} \rfloor + 4$.

At the end of Chapter 3 we consider the setting when Steiner points are allowed. Following the notation of [9], a *Steiner point* is a point $q \notin S$ that may be added to S in order to improve some structure. Moreover, we pose several questions regarding the relation between the reflexivity and the above mentioned modified reflexivity of a point set. The results about reflexivity are joint work with Eyal Ackerman and Oswin Aichholzer.

1.3 Polychromatic colorings of rectangular partitions

In the following set of problems we have a plane graph on a set of points and just like in the case of weak conflict-free colorings, we want a coloring of the points satisfying some properties.

Definition 1.11. *A k-coloring of the vertices of a plane graph is* polychromatic *(or* face-respecting*) if on all its faces all k colors appear at least once (with the possible exception of the outer face).*

Definition 1.12. *The polychromatic number of a plane graph G is the maximum number k such that G admits a polychromatic k-coloring, we denote this number by $\chi_f(G)$.*

This problem is closely related to the **vertex-guard** problem, where we want to place a set of guards in the points such that every face has one on its boundary (i.e. a vertex-guard guards all faces incident to it and we want to guard all faces). Indeed, any color class of a polychromatic coloring guards all faces. We define the *length* of a face as the number of vertices on its boundary. Alon et al. [8] showed that if g is the length of a shortest face of a plane graph G, then $\chi_f(G) \geq \lfloor (3g-5)/4 \rfloor$ (clearly $\chi_f(G) \leq g$), and showed that this bound is sufficiently tight. Mohar et al. [36] proved using the four-color theorem that every simple plane graph admits a polychromatic 2-coloring, later Bose et al. [11] proved that without using the four-color theorem. Horev et al. [30] proved that every plane graph of maximum degree at most 3, other than K_4 admits a polychromatic 3-coloring. Horev et al. [28] proved that every 2-connected three-

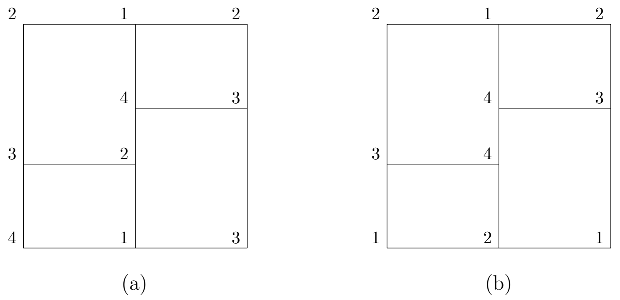


Figure 2: (a) polychromatic 4-coloring and (b) strong rectangle-respecting 4-coloring of a guillotine partition

regular bipartite plane graph admits a polychromatic 4-coloring. This result is tight, since any such graph must contain a face of size four.
Getting closer to the problems investigated in Chapter 4, we define a *rectangular partition* as a partition of an axis-parallel rectangle into an arbitrary number of non-overlapping axis-parallel rectangles, such that no four rectangles meet at a common point. One may view a rectangular partition as a plane graph whose vertices are the corners of the rectangles and edges are the line segments connecting these corners. Dinitz et al. [20] proved that every rectangular partition admits a polychromatic 3-coloring.

A *guillotine-partition* is obtained by recursively cutting a rectangle into two subrectangles by either a vertical or a horizontal line (for an illustration see Figure 3. For this subclass of rectangular partitions Horev et al. [29] proved that they admit a polychromatic 4-coloring.

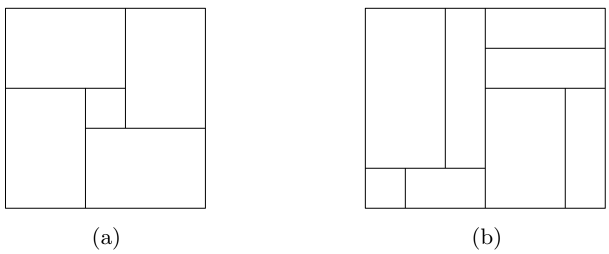


Figure 3: (a) a non-guillotine and (b) a guillotine partition

Actually, they prove a stronger statement. We define a *strong rectangle-respecting k-coloring* ($k \leq 4$) of a rectangular partition R as a vertex coloring of R with k colors such that every rectangle of R has all k colors among the four corners defining it. A strong rectangle-respecting 4-coloring is clearly a polychromatic 4-coloring as well. For examples see Figure 2. They proved that such coloring exists for any guillotine-partition. With Eyal Ackerman [1] we observed that a conjecture of Seymour about planar graphs implies that any rectangular partition admits a strong rectangle-respecting 4-coloring. Independently, Dimitrov et al. [18] made the same observation and also noticed that Guenin [26] proved the conjecture of Seymour. In Chapter 4 first we show how Seymour's conjecture (i.e. Guenin's theorem) implies the existence of a strong rectangle-respecting 4-coloring for every rectangular partition.

All the above mentioned results are restricted to partitions where no more than 2 rectangles meet at a common corner. We will say that a partition is a *general rectangular partition* if more than 2 rectangles are allowed to meet at a common corner. In the second part of Chapter 4 we solve several problems about general rectangular partitions.
Before continuing, let us generalize these colorings to more then 4 colors to have a more general overview of the set of remaining problems.

Definition 1.13. *A* weak (resp. strong) rectangle-respecting k-coloring *of a rectangular partition is a k-coloring of the vertices of the partition such that every rectangle has at least* $\min(k, 4)$ *different colors on its boundary (resp. on its corners).*

We remark that the definition of a weak rectangle-respecting k-coloring and of a polychromatic coloring is the same for $k \leq 4$. Note that a strong rectangle-respecting coloring is necessarily also a weak rectangle-respecting coloring as the boundary of a rectangle includes its four corners. For $k \geq 4$, it is clear that the

k	$\leq$2	3	4	5	$\geq$6
weak	yes	yes	not always	yes	yes
strong	yes	not known	not always	not known	yes

Table 3: the existence of rectangle-respecting k-colorings for general partitions

existence of a weak (resp. strong) rectangle-respecting k-coloring implies the existence of a weak (resp. strong) rectangle-respecting $(k+1)$-coloring (just ignore additional colors). Furthermore, for $k \leq 4$ the existence of a weak (resp. strong) rectangle-respecting k-coloring implies the existence of a weak (resp. strong) rectangle-respecting $(k-1)$-coloring (just merge two colors). Thus, we should focus our attention on finding weak and strong rectangle-respecting k-colorings for k as close to 4 as possible.

In [18] Dimitrov et al. present a simple example showing that not all general rectangular partitions have a strong rectangle-respecting 4-coloring and they ask if every general partition has a polychromatic 4-coloring. We answer this question in the negative.

Theorem 1.14. *There exists a general partition with no weak rectangle-respecting* 4*-coloring (i.e. a polychromatic* 4*-coloring).*

Our construction is also a guillotine-partition. Furthermore, a simple characterization of polychromatic 4-colorability is unlikely according to the following theorem, which we mention without proof as it is rather technical. It can be found in [24].

Theorem 1.15. *Deciding whether a general partition admits a polychromatic* 4*-coloring is* **NP***-complete.*

By Theorem 1.14 we know that not every general partition admits a weak rectangle-respecting 4-coloring. We show that such a coloring with 3 colors always exists. Note that the result of Dinitz et al. [20] follows. We also show that such a coloring with 5 colors exists thus giving a complete answer in the weak rectangle-respecting coloring case.

Proposition 1.16. *Every general partition admits a weak rectangle-respecting* 3*-coloring (i.e. a polychromatic* 3*-coloring).*

Proposition 1.17. *Every general partition admits a weak rectangle-respecting* 5*-coloring.*

Now let us turn our attention to the remaining questions for strong rectangle-respecting k-colorings. We show the existence of strong rectangle-respecting 2- and 6-colorings.

Proposition 1.18. *Every general partition admits a strong rectangle-respecting* 2*-coloring.*

Proposition 1.19. *Every general partition admits a strong rectangle-respecting* 6*-coloring.*

Note that simple coloring algorithms will follow from the proofs of the above propositions. These results are collected in Table 3. The existence of strong rectangle-respecting 3- and 5-colorings for every partition remains unknown.

Problem 1.20. *Does every general partition admit a*

1. *strong rectangle-respecting* 3*-coloring?*
2. *strong rectangle-respecting* 5*-coloring?*

The last (and main) result of Chapter 4 is a generalization of the result for guillotine-partitions for n dimensions. An *n-dimensional hyperbox* is an n-dimensional axis-parallel hyperbox. For us a *partition* of an n-dimensional hypercube or hyperbox is a partition to hyperboxes such that each corner vertex is a corner of 2 hyperboxes, except the corners of the original hypercube. Note that this definition differs again a bit from the natural definition (i.e. it is not a general partition), where we would allow a vertex to be the corner of more than 2 hyperboxes. This is needed, as using the more natural definition even in the plane there are simple counterexamples for our main theorem. The hyperboxes of the partition are called the *basic hyperboxes*. A *guillotine-partition* is obtained by starting with a partition containing only one basic hyperbox and recursively cutting a basic hyperbox into two hyperboxes by a hyperplane orthogonal to one of the n axes. The structure of such partitions is widely investigated, used in the area of integrated circuit layouts and other areas. Guillotine-partitions are also the underlying structure of orthogonal *binary space partitions* (BSPs) which are widely used in computer graphics. In [2] Ackerman et al. determine the asymptotic number of structurally different guillotine-partitions. We refer to the introduction of the same paper for more on this topic.

A *strong hyperbox-respecting k-coloring* of a partition is a coloring of the corners of its basic hyperboxes with k colors such that any basic hyperbox has all the colors appearing on its corners. Note that a corner belongs to two basic hyperboxes except the 2^n corners of the partitioned big hyperbox, which belong to only one basic hyperbox. The natural extension to n dimensions of a polychromatic k-coloring would be a coloring of the corners of its basic hyperboxes with k colors such that any basic hyperbox has all the colors appearing on its boundary. Clearly, every strong hyperbox-respecting k-coloring has this property. For simplicity we define a *strong hyperbox-respecting coloring* as a strong hyperbox-respecting 2^n-coloring. Note that 2^n is the maximum number of colors for which such colorings may exist for every n-dimensional partition.

Theorem 1.21. *There is a strong hyperbox-respecting coloring of any n-dimensional guillotine-partition.*

We note that all results of the Thesis except the one about polychromatic 4-colorings of partitions give efficient algorithms as well. Finally, we would like to summarize the methods which were used in the proofs of the problems discussed and emphasize the strong connections between them.

First, in several problems discussed, the horizontal and vertical order of the points (and regions) served as the important combinatorial structure. Such were the weak conflict-free colorings of bottomless rectangles and points wrt. bottomless rectangles. Here, another important tool we used was that when we wanted to find a coloring with some properties, we rather gave a coloring satisfying some additional properties (for details see Chapter 2). Using this stronger assumption we were able to color inductively. The same idea appears in a different setting when searching for good polygonalizations of a set of points. Here, we searched for a polygonalization having the additional property that it contains a fixed edge on the convex hull of the points. This stronger assumption again made possible to prove inductively. Whereas the stronger requirement in the first case can always be met when the original requirement can, in the second case the relation between the stronger assumption and the original problem is not clear and various conjectures phrasing this lack of understanding where stated at the end of Chapter 3. Back to orderings, in Chapter 4 when dealing with general partitions we gave algorithms considering the vertices in the lexicographical ordering. Further, to give a strong hyperbox-respecting coloring for an n-dimensional partition we used the natural order of the guillotine-cuts to give again a recursive proof which implied an efficient algorithm as well.

In the same chapter when dealing with 2-dimensional partitions, instead of understanding the structure of the partitions, we reduced the problem to another problem dealing with graphs. This idea of "dualizing" the problem to a much more investigated and understood topic of combinatorics (in this case the theory of planar graphs) is a standard method and saved us a lot of energy (a similar ap-

proach led to the investigation of coloring of point sets from covering problems). Still, it would be desirable to understand whether the deep theorem of Guenin (implying the four-color theorem) which we used is necessary or the problem has a simpler (probably straightforward) proof.
The second general tool we used many times was the understanding of the (geometric and combinatorial) structure of a convex hull of a set of points. This was crucial when giving weak conflict-free colorings of half-planes and points wrt. half-planes. Further, when building the polygonalization in Chapter 3 the convex hull had a central role as well. Giving efficient algorithms for these problems depended mainly on the ability to build dynamically a convex hull efficiently.
The third important tool, introduced in Chapter 3 was a partition of the set of points which has some nice separating properties as well, making it possible to give results for arbitrary point sets by solving the problem just for small set of points (by case analysis and then with computer aid). This partition is an interesting technique (discovered independently by others (see [25], [14], [10]) with applications to straight-line drawings of outerplanar graphs), which is not widely known and may be useful to solve different problems as well where such arguments can be helpful.

The results of Chapter 2 about weak conflict-free colorings appeared in [32] without proofs and the paper was invited to the CCCG 2007 conference issue of CGTA. The results of Chapter 3 about the reflexivity of point sets are joint work with Eyal Ackerman and Oswin Aichholzer and appeared in [3] and will be soon published in CGTA. The results in Chapter 4 about general rectangular partitions are joint work with Dániel Gerbner, Nathan Lemons, Cory Palmer, Balázs Patkós and Dömötör Pálvölgyi and can be found in [24]. The main result of Chapter 4 about n-dimensional polychromatic colorings appeared in [33].

2 Weak conflict-free colorings

2.1 Bottomless rectangles

Recall that $\mathcal{B}$ is the family of all bottomless rectangles. We prove exact bounds for $wcf_k(\mathcal{B})$ and $\overline{wcf_k}(\mathcal{B})$ for all k.

2.1.1 Coloring points

From now on we assume that there are no two points with the same x- or y-coordinate. It is easy to show that if this is not the case, then coloring the point set after a small perturbation gives a needed coloring for the original point set as well. In this section *upwards order* means the ordering of points according to their y-coordinate starting with the point having the smallest y-coordinate (the *lowest* point).

The proof of the following, rather trivial result is just presented for the sake of completeness.

Claim 2.1. *(folklore) $wcf_2(\mathcal{B}) = 3$ i.e. any set of points can be colored with 3 colors such that any bottomless rectangle covering at least 2 of them is not monochromatic.*

Proof. First we prove that $wcf_2(\mathcal{B}) \leq 3$. We want to color the points with 3 colors such that any bottomless rectangle covering at least 2 points covers two differently colored points.

First we color the lowest point of P arbitrarily with one of the three colors then we color the points one by one in upwards order. In each step we color the next point p with a color maintaining that in the x-coordinate order of the points already colored there are no two consecutive points with the same color.

In this way any bottomless rectangle B covering at least two points covers two differently colored ones. Indeed, when the highest point p in $P \cap B$ is considered,

$B \cap P$ is an interval in the left to right order of the points considered so far. By the property maintained any such interval contains points of at least two colors. The lower bound $wcf_2(\mathcal{B}) \geq 3$ follows from the fact that for example the points with coordinates (0,0), (1,1) and (2,0) cannot be colored with 2 colors in a proper way, since any two of them can be cut off by a bottomless rectangle. □

The following theorem shows that the smallest k for which $wcf_k(\mathcal{B}) = 2$ is 4 and so $wcf_k(\mathcal{B})$ is determined for every k as trivially $wcf_k(\mathcal{B}) \geq 2$ for any k.

Theorem 2.1.

(i) $wcf_3(\mathcal{B}) = 3$.

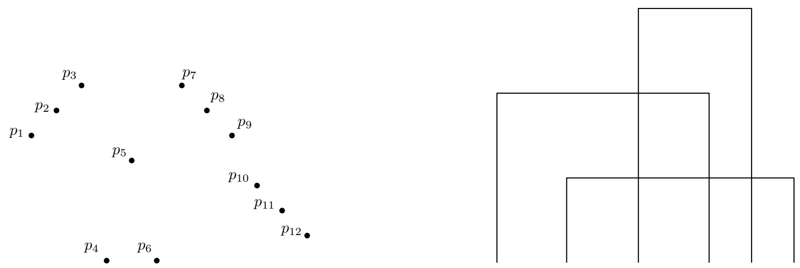
(ii) $wcf_4(\mathcal{B}) = 2$ *i.e. any set of points can be colored with* 2 *colors such that any bottomless rectangle covering at least* 4 *of them is not monochromatic.*

(iii) Such colorings can be found in $O(n \log n)$ *time.*

Proof. (i) Using Observation 1.3 with Claim 2.1 we got that $wcf_3(\mathcal{B}) \leq wcf_2(\mathcal{B}) = 3$. Thus, it is enough to prove that $wcf_3(\mathcal{B}) > 2$. For that we show that the 12 point construction on Figure 4(a) cannot be colored with 2 colors such that any bottomless rectangle covering at least 3 points covers two differently colored points. Suppose on the contrary that there is such a coloring. Denote the points ordered by their x-coordinate from left to right by $p_1, p_2, \ldots, p_{12}$. Among the points p_4, p_5, p_6 there are two with the same color, wlog. assume that this color is red. If p_4 and p_5 are red, then all of p_1, p_2, p_3 are blue as there is a bottomless rectangle covering only p_4, p_5 and any one of these 3 points. This is a contradiction as there is a bottomless rectangle covering only these 3 points, all blue. If p_4 and p_6 are red then similar argument for the points p_{10}, p_{11}, p_{12}, if p_5 and p_6 are red then similar argument for the points p_7, p_8, p_9 yields to a contradiction.

(ii) We want to color the points red and blue such that any bottomless rectangle covering at least 4 points covers two differently colored points.

First we color the lowest point of P red then we consider the points in upwards



(a) Theorem 2.1(i) (b) Theorem 2.2(i)

Figure 4: Lower bound constructions for bottomless rectangles

order. We do not color every vertex as soon as it is considered. We maintain that in the x-coordinate order of the points considered so far there are no two consecutive uncolored points and the colored points alternate in color. When a new point is considered we keep it uncolored unless it has an uncolored left or right neighbor (note that it cannot have both) in the x-coordinate order. In that case we color the new point and its uncolored neighbor in a way that keeps the alternation. At the end we arbitrarily color the remaining points in P. Now we only need to prove that this coloring is good. Consider a bottomless rectangle B covering at least 4 points. Let p be the highest point covered by B. When p is considered $B \cap P$ is an interval in the left to right order of the points considered so far. By the properties maintained any such interval of at least 4 vertices contains both red and blue points as needed.

(*iii*) We need to prove that the algorithms presented in the proof of Claim 2.1 and Theorem 2.1 (*ii*) run in time $O(n \log n)$. Computing the upwards order of the points takes $O(n \log n)$ time, the rest of the algorithm has n steps in both cases, each computable in $O(\log n)$ time, in the latter algorithm there is a final coloring step that takes at most linear time, so the whole algorithm runs in $O(n \log n)$ time in both case. □

2.1.2 Coloring bottomless rectangles

In [46] a very similar version is considered, namely, weak conflict-free coloring of axis-parallel rectangles intersecting a common base-line. The proof of their result with a slight modification gives $\overline{wcf_2}(\mathcal{B}) \leq 4$. The following theorem determines $\overline{wcf_k}(\mathcal{B})$ for every k, also improving this bound to 3 colors, which is optimal.

From now on we assume that there are no two bottomless rectangles with overlapping sides. It is easy to show that if this is not the case, then coloring the rectangles after perturbing them such that afterwards there are no overlappings, gives a needed coloring for the original family of rectangles as well.

Theorem 2.2.

(i) $\overline{wcf_2}(\mathcal{B}) = 3$ *i.e. any family of bottomless rectangles can be colored with* 3 *colors such that any point covered by at least* 2 *of them is not monochromatic.*

(ii) $\overline{wcf_3}(\mathcal{B}) = 2$ *i.e. any family of bottomless rectangles can be colored with* 2 *colors such that any point covered by at least* 3 *of them is not monochromatic.*

(iii) Such colorings can be found in $O(n^2)$ *time.*

Proof. (i) For the lower bound, the arrangement of 3 rectangles on Figure 4(b) shows that 3 colors are sometimes needed. For the upper bound, given a family of rectangles with a common base line we want to color the rectangles red, blue and green such that any point covered by at least 2 rectangles is covered by two differently colored rectangles. We color the rectangles in downwards order according to their top edge's y-coordinate. We start with the empty family and reinsert the rectangles in this order. We color the first, i.e. the highest rectangle blue. After each step we have a proper coloring and we preserve the following additional assumption. *If a point on the base-line is covered by exactly* 1 *rectangle, then it is not red.*

In each step we insert the next rectangle B in downwards order, so its top edge is below the top edge of all the rectangles already inserted. We color B red. We

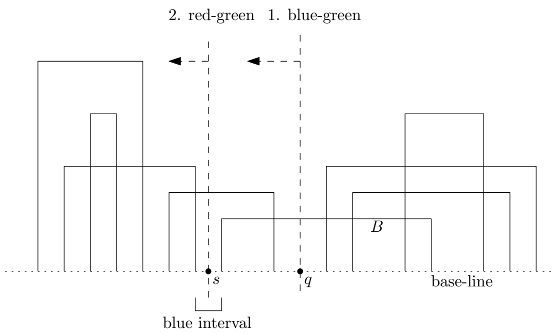


Figure 5: The color switches of the ‘divide and color’ method in Theorem 2.2(i).

claim that this is again a proper coloring. Indeed, the condition for points not in B already holds. For any point covered by B the base-line point with the same x-coordinate is covered by the same rectangles as B is the lowest rectangle. Thus, it is enough to check the condition for base-line points in B. If a point is covered by at least 2 rectangles besides B then it is good by induction. Otherwise it is covered by B, which is red and exactly one more rectangle, which is not red by the assumption.

If there is no base-line point covered by only B, then the additional assumption holds too. If q is such a point then we need to do something else to maintain the validity of our assumption. If a base-line point is covered by only 1 rectangle then we say that the color of this point is the color of the rectangle covering it. It is easy to see that if there is such a point p and we switch the other two colors on the rectangles completely to the left (or to the right) to p, the coloring remains good. With only such ‘divide and color’ steps we will change the coloring such that there will be no point on the base-line covered by exactly 1 green rectangle. Finally we will switch the colors green and red on all the rectangles to have a good coloring satisfying the assumption. For an illustration of the rest of the proof see Figure 5.

In the current coloring all green base-line points are left or right to B as B is red.

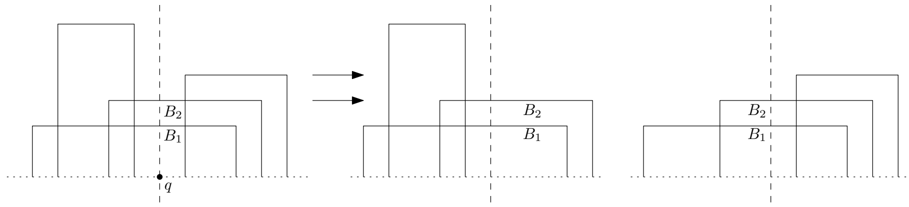


Figure 6: The division of the 'divide and color' method in Theorem 2.2(ii).

We will deal with the left side first, changing the colors only of rectangles strictly left from q and making a good coloring satisfying the condition for any base-line point left to q. On the right side we proceed analogously, changing the colors only of rectangles strictly right from q and making a good coloring satisfying the condition for any base-line point right to q. This way we get a good coloring satisfying the assumption for all base-line points.

On the base-line to the left of B there are some intervals of single colored points, all of them green or blue. If there is no green one then we are done. Otherwise, we can suppose that the closest such interval to B is blue, otherwise switch colors blue and green on the rectangles strictly left from q, still having a good coloring. Now switch colors red and green on the rectangles strictly left from any point s of this blue interval, this way we got rid of all green points, making the assumption true for all points left to B. This concludes the proof.

(ii) Given a family of rectangles with a common base line we want to color them red and blue such that any point covered by at least 3 rectangles is covered by two differently colored rectangles. Moreover, the coloring we got will always satisfy the following additional assumption. *Any point on the base-line covered by exactly* 2 *rectangles is covered by a red and a blue rectangle.*

Now we need a different version of the 'divide and color' method. We will proceed by induction on the number of rectangles. A single rectangle is colored red. In a general step first assume that there is a point q on the base-line not covered by any rectangle and there are some rectangles strictly left and strictly right from

that point too. Color the rectangles to the left of q and the ones to the right of q separately by induction, putting these together this is clearly a good coloring.
Now assume that there is a point q on the base-line covered by exactly 1 rectangle B and there are some rectangles strictly left and strictly right from that point too. Color first the rectangles to the left of q together with B then the rectangles to the right together with B. As there were some rectangles on both sides, this can be done by induction. By a possible switching of the colors in the left and right parts, B is red in both colorings. Putting together the two half-colorings we get a good coloring.
The next case is when we have a point q on the base-line covered by exactly two rectangles, B_1 and B_2 (see Figure 6 for an illustration)and there is at least one rectangle both to the right and to the left of q. In this case color first by induction the rectangles strictly left from q together with these two rectangles. Using the assumption on q we see that B_1 and B_2 have different colors, after a possible switch of the two colors we can assume that B_1 is red and B_2 is blue. The same way we color the rectangles strictly right from q together with these two rectangles. This way the two rectangles are colored with the same colors in both colorings and so we can put together these two half-colorings (if none of them is empty) to have a coloring of the whole family of rectangles. This coloring is good by induction.
In the remaining case for any base-line point covered by exactly 1 or 2 rectangles, there is no rectangle strictly to the left or to the right to that point. The left and right sides of the rectangles divide the base-line into 2 half-lines and $2n-1$ intervals. It is easy to see that in this case the only base-line points covered by exactly 1 rectangle are the points of the leftmost L_1 and rightmost R_1 interval and the 2-covered points are the points of the second leftmost L_2 and second rightmost R_2 interval. Consider rectangle B, the one with the lowest top edge. It is easy to see that if it does not cover 1- or 2-covered base-line points then we

can color the rest of the rectangles by induction and then color B with arbitrary color not ruining the coloring and the additional assumption. Otherwise B covers some intervals from L_1, L_2, R_1, R_2.

If it covers some of L_1 and L_2 and does not cover R_1 and R_2 then color the rest of the rectangles by induction and then color B to a different color from the other rectangle covering L_2, this way we obtain a good coloring satisfying our additional assumption. If it covers some of R_1 and R_2 and does not cover L_1 and L_2 then a symmetrical argument gives a good coloring.

It remains to deal with the case when B covers L_2 and R_2 as well. Consider now the rectangle B_2 with the second lowest top edge. First assume that B_2 does not cover any of L_1, L_2, R_1, R_2. In this case color the rest of the rectangles (including B) by induction and then color B_2 with the same color as B. We claim that this coloring is good. Any point ruining the condition must be in B_2. Assume on the contrary that there is a point p covered by at least 3 rectangles all having the same color.

If p is covered by 3 rectangles besides B_2 then by induction there are differently colored rectangles among these, a contradiction.

If p is covered by exactly 2 rectangles besides B_2 then take the base-line point p' having the same x-coordinate. If it is covered by the same rectangles as p, then by the additional assumption it is covered by two differently colored rectangles besides B_2. This holds for p as well, a contradiction. If p' is not covered by the same rectangles as p, the only possibility is that it is covered by B too (as only B is lower then B_2). By induction this point was covered by red and blue rectangles as well without considering B_2. As B has the same color as B_2, the same holds for p, a contradiction.

The additional assumption holds as well as it is enough to check the points of L_2 and R_2 and here the coloring is good by induction.

By dealing with the case when B_2 covers some of L_2, R_2 we exhaust all possibilities. By symmetry we can assume that B_2 covers L_2 (and maybe R_2 too). In this final case delete both B and B_2 and color the rest of the rectangles by induction. Now put back these two rectangles. If R_2 is covered by some rectangle besides B and B_2 then color B differently from the color of this rectangle. Otherwise color B arbitrarily. Finally, color B_2 differently from B. Any point ruining the condition must be in B or B_2. Again, suppose there is such a point p covered by at least 3 rectangles all having the same color.

If p is covered by both B and B_2 then its a contradiction as they are differently colored. If it is covered by at least 3 rectangles besides B and B_2 then again its a contradiction by induction.

If p is covered by one of B and B_2 and only two other rectangles then the base-line point p' with the same x-coordinate was covered by exactly two rectangles in the coloring without B and B_2. Thus, these rectangles have different colors by the assumption. As B and B_2 are the lowest rectangles, the point p is covered by these differently colored rectangles as well, a contradiction. The additional assumption holds as well as it is enough to check the points of L_2 and R_2 and here the coloring is clearly good.

(iii) Finding the upwards order of the rectangles takes $O(n \log n)$ time. In each step we maintain an array of the intervals of the base line. If an interval is covered only by one rectangle, we keep its color as well.

In the algorithm of (i) in each step we search for some colored interval constant times and recolor some rectangles with a given property (left from a given interval, etc.) constant times. This takes $c \cdot k$ time if we have k rectangles at that step. We have n such steps and $k \leq n$ always, so the running time is $O(n^2)$.

For the algorithm of (ii) we will prove by induction on the number of bottomless rectangles that $c_0 \cdot n^2$ is an upper bound on the number of steps needed to color any family of n bottomless rectangles for some c_0 large enough. Except the last case we always do the 'divide and color' step by cutting the family into two nontrivial parts and color separately. Finding whether there is such a cut, doing the cut (and maintaining the upwards order in the two parts) and the possible recolorings after the recursional colorings take $c_1 \cdot n$ time for n rectangles. By induction, this and the two recursional algorithms together take at most $c_0 \cdot a^2 + c_0 \cdot b^2 + c_1 \cdot n$ time where $a + b \leq n + 2$. When we do a recursional step by deleting B or B_2 or both we can decide which kind of step is needed and color B and B_2 in $c_2 \cdot n$ steps, and we can do the recursion in $c_0 \cdot (n-1)^2$ steps. Thus we need at most $c_0 \cdot (n-1)^2 + c_2 n$ time in this case. It is easy to see that in both of these cases the time can be bounded from above by $c_0 \cdot n^2$ if c_0 have been chosen to be large enough (depending on c_1 and c_2). □

Consider now the case of axis-parallel rectangles intersecting a common base-line (denoted by $\mathcal{B}'$). We start with the case of region coloring, that is, estimating $\overline{wcf_k}(\mathcal{B}')$. For this the best upper bound is due to [46], proving $\overline{wcf_2}(\mathcal{B}') \leq 8$, and for the case of $k > 2$ we can separately color the upper and lower parts (divided by the base-line) of the rectangles with 2 colors by Theorem 2.2 (*ii*) and then for a rectangle colored by a in the upper part and b in the lower part, we give the ordered pair (a, b) as a color. It is easy to see that this is a good $\overline{wcf_3}$-coloring of the rectangles, thus proving $\overline{wcf_3}(\mathcal{B}') \leq 4$.

The case of coloring points seems less natural for axis-parallel rectangles intersecting a common base-line, still it can be considered. Coloring the points in the lower and upper parts with different colors ensures that any rectangle covering one from both sides is not monochromatic. The two sides can be colored by Claim 2.1 with 3-3 colors, thus proving $wcf_2(\mathcal{B}') \leq 6$ (a rectangle either covers points from both sides or covers at least 2 points on one side). Further, the same claim implies $wcf_3(\mathcal{B}') \leq 3$. Indeed, color both sides with the same 3 colors according to Claim 2.1, then any rectangle covering at least 3 points covers 2 point on one side, thus covering two differently colored ones as well. Finally, $wcf_7(\mathcal{B}') = 2$ as if we color both sides with the same two colors according to Theorem 2.1 (*ii*),

k	2	3...6	≥7
$wcf_k(\mathcal{B}')$	**≥3, ≤6**	**3**	**2**
$\overline{wcf_k}(\mathcal{B}')$	**≥4**, *≤8*	**≤4**	*≤4*

Table 4: table of results about $\mathcal{B}'$

then any rectangle covering at least 7 points covers 4 point on one side, thus covering a red and blue one as well. The lower bounds for bottomless rectangles trivially hold for the case of $\mathcal{B}'$ as well, further a simple construction shows that $\overline{wcf_2}(\mathcal{B}') \geq 4$. Summarizing, the best known results are collected in Table 4.

Problem 2.2. *Give better bounds for* $\overline{wcf_k}(\mathcal{B}', n)$ *and* $wcf_k(\mathcal{B}', n)$*.*

2.2 Half-planes

The family of all half-planes is denoted by $\mathcal{H}$. We prove exact bounds for $wcf_k(\mathcal{H})$ and almost exact bounds for $\overline{wcf_k}(\mathcal{H})$.

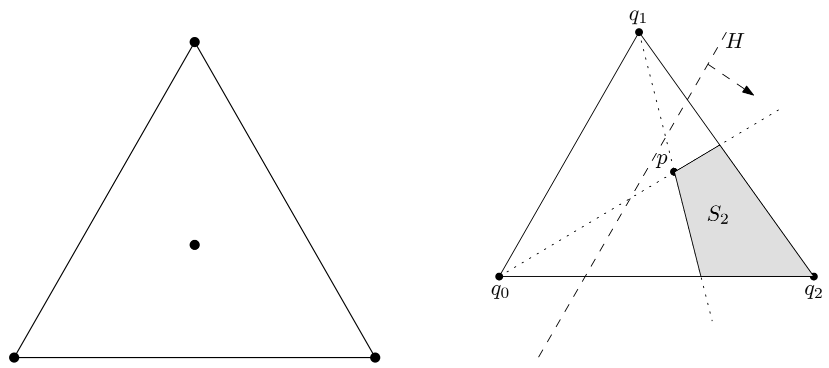
From now on we assume that there are no 3 points on one line. It is easy to show that if this is not the case, then coloring the point set after a small perturbation gives a needed coloring for the original point set as well. This way the vertices of the convex hull of a point set P are exactly the points of P being on the boundary of this convex hull.

2.2.1 Coloring points

The following lemma follows easily from the definition of the convex hull.

Lemma 2.3. *Any half-plane H covering at least one point of P covers some vertex of the convex hull of P too. Moreover, the vertices of the convex hull of P covered by H are consecutive on the hull.*

Theorem 2.3.



(a) The exceptional case of Theorem 2.3(ii)

(b) The proof of Theorem 2.3(ii)

Figure 7: Theorem 2.3(ii)

(i) $wcf_2(\mathcal{H}) = 4$ *i.e. any set of points can be colored with* 4 *colors such that any half-plane covering at least* 2 *of them is not monochromatic, and* 4 *colors might be needed.*

(ii) $wcf_2(\mathcal{H}, P) \leq 3$*, except when* P *has* 4 *points, with one of them inside the triangle determined by the other* 3 *points (see Figure 7(a)), in which case* $wcf_2(\mathcal{H}, P) = 4$.

(iii) $wcf_3(\mathcal{H}) = 2$ *i.e. any set of points can be colored with* 2 *colors such that any half-plane covering at least* 3 *of them is not monochromatic.*

(iv) Such colorings can be found in $O(n \log n)$ *time.*

Proof. (i) This follows from (ii), yet we give a short proof for the upper bound. Color the vertices of the convex hull of P with 3 colors such that there are no 2 vertices next to each other on the hull with the same color. Color all the remaining points with the 4th color. This coloring is good as by Lemma 2.3 any half-plane covering at least two points covers two neighboring vertices on the hull or one vertex on the hull and one point inside.

(ii) Clearly, in the case mentioned in the lemma we need four colors to have a good coloring as any two points can be covered by some half-plane not covering the rest of the points.

As $\mathcal{H}$ is monotone, by Observation 1.6 it is enough to consider half-planes covering exactly 2 points of P. We color the vertices of the convex hull with 3 colors as in (i). At this time the assumption already holds for every half-plane covering two vertices of the convex hull as by Lemma 2.3 they cover two neighboring ones which don't have the same color, as needed. Now we color the points inside the hull in a more clever way then in (i). Take an arbitrary point p inside the hull. The only case when the color of this inside point can ruin the coloring, is when there is a half-plane covering only this point and one vertex of the convex hull. If this can happen only with two vertices of the hull, then coloring p different from these, the coloring will be good for all half-planes covering p. Doing the same for every inside point we get a good coloring.

Denote the vertices of the convex hull by $q_0, ..., q_{k-1}$ in clockwise order (indexes are mod k). It is enough to prove that except the case mentioned in the lemma, there are no 3 such vertices on the hull corresponding to some p. For this, notice that if q_i and p can be covered by a half-plane not covering any other point, then p is inside $q_{i-1}qq_{i+1}\Delta$. It is easy to see that if the hull has more then 3 vertices, then there are no 3 such triangles having a common inner point. For the rest of the proof see Figure 7(b). If the hull has 3 vertices and p can be covered with any of these 3 vertices by some half-plane not covering any other point of P then regard the lines going through some q_i and p partitioning the triangle into 6 triangles. For each vertex q_i, denote the union of the two triangles having it as a vertex by S_i. Thus, we have three quadrilaterals, all of which must be empty. Indeed, for example by assumption there is some half-plane H covering p and q_2 not covering any other point of P. This half-plane always covers the quadrilateral S_2 and so it must be empty. The same argument for the other two quadrilaterals shows that all of them are empty and so p is the only point in the triangle, which is the excluded case.

(*iii*) As $\mathcal{H}$ is monotone, it is enough to consider half-planes covering exactly 3 points of P. We color the points with colors red and blue. The points inside the convex hull of P are colored blue. The vertices of the convex hull of P are denoted by $q_0, \ldots, q_{k-1}$ in clockwise order. For each q_i we assign $T_i = q_{i-1}q_iq_{i+1}\Delta$, where indexes are modulo k. If T_i has some point of P inside it, then color q_i red.
If there are no nonempty T_i's then color the vertices of the convex hull with alternating colors, if its size is odd, then with the exception of two neighboring red points. If there is at least one nonempty T_i then these red points cut the boundary of the convex hull into chains. For each chain color its vertices with alternating colors, a chain with size one is colored blue.
Now we need to prove that this coloring is good. First observe that there are no 2 consecutive blue vertices on the convex hull. Take again an arbitrary half-plane H covering exactly 3 points. By Lemma 2.3 it covers some consecutive vertices of the convex hull of P. If it covers at least two consecutive vertices on the hull then it covers at least one red point. If it covers at least one point inside the hull, then it covers at least one blue point. If it covers three vertices of the convex hull but no points inside then it is easy to see that the triangle corresponding to the middle point in the ordering must be empty. So it belongs to some alternatingly colored chain. If any of its neighbors corresponds to the same chain, then H covers a red and a blue point too, if this point is a chain of size 1 then it is blue and its neighbors are red, again good. The only case remaining when H covers one vertex of the convex hull, q_i and two points of P inside the convex hull. The latter points are blue and they must be in T_i, that is q_i is red, as needed.
(*iv*) The algorithm in (*i*) clearly works in $O(n \log n)$, the same as building the convex hull. For the other two algorithm we need the dynamic convex hull algorithm presented in [13].
For the algorithm in (*iii*) we first compute a convex hull in $O(n \log n)$ amortized time and then we take its points one by one and do the following. Delete tem-

porarily the convex hull vertex p, compute the new convex hull temporarily, if it has some new vertices on it, then the triangle corresponding to p is not empty. As any inner point has been added and deleted from the set of vertices of the hull at most two times and the convex hull algorithm makes a step in $O(\log n)$ amortized time, we could decide in $O(n \log n)$ time which vertices of the hull have empty triangles. After that the coloring of the vertices of the hull and the inside points takes $O(n)$ time, $O(n \log n)$ altogether.

For the algorithm in (ii) we do the same just when we temporarily delete p we assign to any additional convex hull vertex the point p, as this vertex can be cut out by a half-plane together with p. After these we simply color the vertices of the convex hull as needed and all the inner points with a color different from the color of the at most two convex hull vertices assigned to it. Altogether this is again $O(n \log n)$ time. □

Observation 2.4. *The algorithm in the proof of Theorem 2.3* (iii) *gives a coloring which additionally guarantees that there are no half-planes covering exactly two points, both of them blue.*

2.2.2 Coloring half-planes

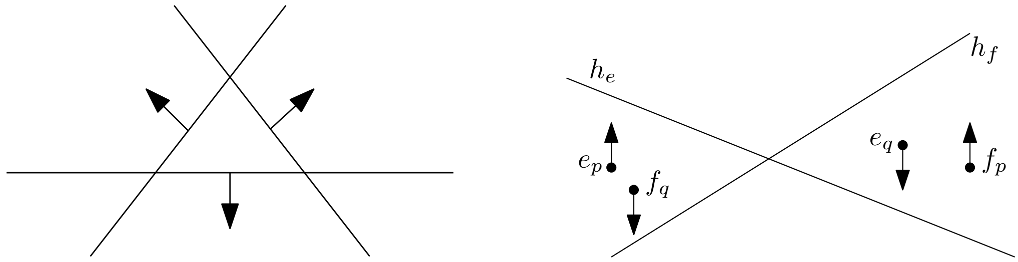
Theorem 2.4.

(i) $\overline{wcf_2}(\mathcal{H}) = 3$ *i.e. any family of half-planes can be colored with* 3 *colors such that any point covered by at least* 2 *of them is not monochromatic.*

(ii) $\overline{wcf_4}(\mathcal{H}) = 2$ *i.e. any family of half-planes can be colored with* 2 *colors such that any point covered by at least* 4 *of them is not monochromatic.*

(iii) Such colorings can be found in $O(n \log n)$ *time for* (ii) *and in* $O(n^2)$ *time for* (i).

Proof. We can assume that there are no half-planes with vertical boundary line. We dualize the half-planes and points of the plane S with the points (with an



(a) Construction for Theorem 2.4(i)

(b) Proof of Theorem 2.4(i)

Figure 8: Theorem 2.4(i)

additional orientation) and lines of plane S', then we color the set of directed points corresponding to the half-planes which will give a good coloring of the original family of half-planes. The dualization is as follows. For a half-plane H with a boundary line given by the equality $y = ax + b$ the corresponding dual point h has coordinates (a, b). If this line is a lower boundary, then h has *orientation north*, otherwise it has orientation *south*. For an arbitrary point p with coordinates (c, d) the corresponding line P is given by $y = -cx + d$. Now it is easy to see that H contains p on the primal plane if and only if the vertical ray starting in h and going into its orientation meets line P (we say that h and P *see each other* or h is *looking at* P). Indeed, for a half-plane with lower boundary both hold if and only if $d > ac+b$, for a plane with an upper boundary both hold if and only if $d < ac+b$. From this it follows that the wcf_k-coloring of half-planes is equivalent to a coloring of the dual set of oriented points such that any line with at least k points looking at it, there are at least two with different colors among these points.

All the proofs give colorings for directed points and from now on we assume that there are no 3 directed points on one line. It is easy to show that if this is not the case, then coloring the set of directed points after a small perturbation gives a needed coloring for the original set of directed points as well.

(i) For a construction proving that 3 colors might be needed, see Figure 8(a).

For the upper bound given a set of directed points we will color them with 3 colors such that for any line seeing at least 2 points, not all of these points have the same color.

Take the lower boundary of the convex hull of the set of north-directed points and denote the vertices of it by $p_1, p_2, \ldots, p_k$ ordered by their x-coordinate. Take the upper boundary of the convex hull of the set of south-directed points and denote the vertices of it by $q_1, q_2, \ldots, q_l$ ordered by their x-coordinate. The rest of the points we call *inner* points. Similarly to Lemma 2.3 any line seeing at least one north-directed point sees one p_i as well and any line seeing at least one south-directed point sees one q_j as well and the p_i's and q_i's seen by a line are consecutive. First we give a coloring of the p_i's and q_j's with 3 colors such that no two consecutive points have the same color and if for some p_i and q_j there is a line which sees exactly these two points, then these points have different colors. As a line seeing at least two points which does not see inner points sees either exactly one p_i and q_j or at least two consecutive ones of the same type, the coloring will be good for all such lines.

We define a graph on the points p_i and q_j. The consecutive points are connected forming a path of p's and a path of q's. Moreover, p_i and q_j are connected if there is a line which sees exactly these two points. Clearly, we need a proper 3-coloring of this graph. For algorithmic reasons we take a graph with more edges and prove that it can be 3-colored as well. In this graph p_i and q_j are connected if there is a line which sees no other points of the p-path and q-path. We claim that drawing the p-path and the q-path on two parallel straight lines, the q-path being on the higher line and in reverse order, and drawing all the edges with straight lines, we have a graph without intersecting edges. In other words the graph is a caterpillar-tree between two paths. Such a graph is outer-planar and thus three-colorable.

So it is enough to prove that there are no intersecting edges. Without loss of generality such two edges e and f would correspond to points e_p, e_q, f_p and f_q with x-coordinates $e_p^x < f_p^x$ and $e_q^x < f_q^x$ (the points with index p are from the p-path and the points with index q are from the q-path). The line seeing only e_p and e_q is denoted by h_e, the line seeing only f_p and f_q is denoted by h_f. These two lines divide the plane into four parts, which can be defined as the north, south, west and east part. Clearly from e_p and f_p one must be in the west part and one in the east part. By $e_p^x < f_p^x$, e_p is in the west and f_p is in the east part. This means that h_e must be the line above the east and south parts and so e_q must be in the east part and f_q in the west, a contradiction together with $e_q^x < f_q^x$ (see Figure 8(b)).

Now we finish the coloring such that the condition will hold also for lines seeing inner points. As in Theorem 2.3 (*ii*) for any other north-directed point p there are two points p_i and p_{i+1} (the unique ones for which p_i has smaller and p_{i+1} has bigger x-coordinate then p) such that whenever a line h sees p then it sees p_i or p_{i+1} as well. Then coloring p differently from these points, guarantees that any h seeing p sees two differently colored points. Doing the same for the south-directed points we finished the coloring such that whenever a line sees some point which is not a p_i or q_j then it sees points with both colors.

(*ii*) Given a set of directed points we will color them with 2 colors such that for any line seeing at least 4 points, not all of these points have the same color. We color the north-directed points with the same algorithm as in Theorem 2.3 (*iii*). We color the south-directed points with the same algorithm as in Theorem 2.3 (*iii*) just with inverted colors. This guarantees that any line which sees at least 3 north-directed points, sees red and blue points as well. If a line sees exactly 2 points of each kind, then sees red and blue points as well of one kind or sees 2 red north-directed points and 2 blue south-directed points by Observation 2.4, again seeing points with both colors. There are no more cases for a line seeing at least 4 points, so the proof is complete.

(iii) The algorithm in (ii) clearly runs in time $O(n \log n)$ using Theorem 2.3 (iv). The algorithm in (i) can be made similarly to work in this time, only the building of the caterpillar tree might need $O(n^2)$ steps. Indeed, we just need to prove that deciding whether there is an edge between some p_i and q_j can be done in constant time. For that we just need to check whether the linear equations for a line assuring that it goes above q_{j-1}, q_{j+1} and below q_j, below p_{i-1}, p_{i+1} and above p_i have a solution. □

Problem 2.5. *Determine the value of $\overline{wcf_3}(\mathcal{H})$, i.e. the lowest number of colors needed to color any finite family of half-planes such that if a point of the plane is covered by at least 3 of them then not all of the covering half-planes have the same color.*

Note that $\overline{wcf_3}(\mathcal{H})$ is either 2 or 3.

3 Reflexivity of point sets

3.1 Modified Reflexivity and Iterative Subdivision

Let us recall the theorems which we would like to prove in this chapter. First we will prove that:

Theorem 3.1. $\rho(n) \leq 3\lfloor \frac{n-2}{7} \rfloor + 2$.

And then improve this to:

Theorem 3.2. $\rho(n) \leq 5\lfloor \frac{n-2}{12} \rfloor + 4$.

Recall that the convex hull of a finite set S of points, $\mathcal{CH}(S)$, is composed of the boundary and the interior of a convex polygon. A *boundary edge* of $\mathcal{CH}(S)$ is an edge of that polygon. To prove a stronger variant of Theorem 3.1 we first introduce some notation. Let S be a set of points and let e be a boundary edge of $\mathcal{CH}(S)$. We denote by $\rho_e(S)$ the minimum number of reflex vertices in any polygonalization P of S, such that e is an edge of P. Similarly, let $\bar{\rho}(S)$ be the maximum value of $\rho_e(S)$ taken over all the edges e of the boundary of $\mathcal{CH}(S)$, and let $\bar{\rho}(n)$ be the maximum value of $\bar{\rho}(S)$ taken over all sets S of size n. The definition of $\bar{\rho}(\cdot)$ is perhaps a bit counter-intuitive (one might expect to take the minimum over all edges), however, it is crucial for our purposes.

Obviously $\rho(n) \leq \bar{\rho}(n)$, so our goal is to derive good upper bounds for $\bar{\rho}(n)$. To this end we first provide a central lemma, which allows us to subdivide a point set in a way that we can consider the polygonalizations of the subsets rather independently.

Lemma 3.3. *Given an integer $k > 2$, a set S of $n > k$ points, and two points $p, q \in S$, such that pq is a boundary edge of $\mathcal{CH}(S)$, then there exists a point $t \in S \setminus \{p, q\}$ and two sets $L, R \subset S$ such that:*

1. *$L \cup R = S$, $L \cap R = \{t\}$, $q \in R$, and $p \in L$;*

2. *The triangle* $\triangle pqt$ *contains no other points from* S*;*

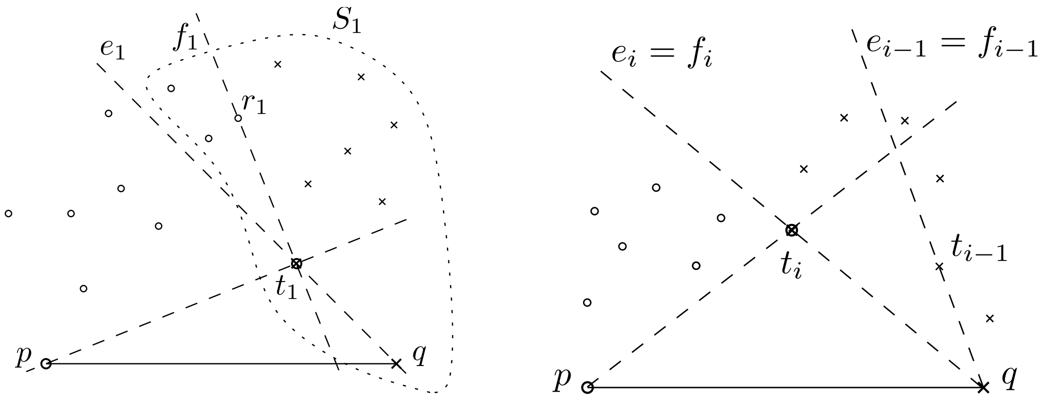
3. $\mathcal{CH}(R) \cap \mathcal{CH}(L) = \{t\}$*; and*

4. $|R| = k$.

Proof. Let us first give a sketch of the proof. We will define sets L_1 and R_1 satisfying properties (1)–(3) of the lemma and $|R_1| \leq k$. Then, we will define step by step pairs of sets L_i and R_i for $i \geq 1$ all satisfying properties (1)–(3) and $|R_i| \leq k$. Further, $|R_i|$ will strictly increase in each step until for some j we will have $|R_j| = k$, thus L_j and R_j will be a pair of sets satisfying all properties of the lemma, as needed.

Assume, w.l.o.g., that p and q lie on the x-axis, such that p is to the left of q and all the remaining points are above the x-axis. Let t_1 be the point of S such that the angle $\angle t_1pq$ is the smallest. Let e_1 be the line determined by q and t_1, and let H_1 be the closed half-plane to right of e_1. Let S_1 be the subset of points of S contained in H_1. If $|S_1| > k$, then define $r_1 \in S_1 \backslash \{q, t_1\}$ to be the point creating the $(k-1)$st smallest angle $\angle r_1t_1q$, and denote by f_1 the line through t_1 and r_1. Otherwise, if $|S_1| \leq k$ let $f_1 = e_1$. Set $R_1 = \{q, t_1\} \cup \{p' \in S_1 \mid p' \text{ is to the right of } f_1\}$ and $L_1 = (S \setminus R_1) \cup \{t_1\}$. Note that r_1, if defined, is in L_1. We claim that t_1, R_1, and L_1 satisfy properties (1)–(3) of the lemma: (1) This property holds by the definition of R_1 and L_1; (2) By the choice of t_1 the triangle $\triangle pqt_1$ is empty; (3) All the points in R_1 are to the right of f_1, except for t_1 and possibly q. All the points in L_1 are to the left of f_1, except for t_1 and possibly r_1. However q and r_1 cannot both lie on f_1. If $|S_1| \geq k$, then we also have that $|R_1| = k$ (either by the choice of r_1 or because $|S_1| = k$, see Figure 9(a) for an illustration of the former case).

Suppose now that $|S_1| < k$. We define t_i, e_i, and S_i for $i > 1$ and $|S_{i-1}| < k$ recursively. Let t_i be the point that minimizes the angle $\angle t_ipq$ among the points in $L_{i-1} \setminus \{t_{i-1}\}$ (note that this set of points is not empty since $|S_{i-1}| < k$ and we show below that $S_1 \subset \cdots \subset S_{i-1}$). Let e_i be the line through q and t_i, let H_i be



(a) More than $k = 8$ points on or to the right of e_1. The points in R_1 and L_1 are marked by crosses and circles, respectively.

(b) Less than k points on or to the right of e_i. The points in R_i and L_i are marked by crosses and circles, respectively.

Figure 9: Illustrations for the proof of Lemma 3.3

the closed half-plane to the right of e_i, and let S_i be the set of points contained in H_i. Next, we define r_i, f_i, R_i, and L_i. If $|S_i| > k$ define $r_i \in S_i \setminus \{q, t_i\}$ to be the point creating the $(k-1)$st smallest angle $\angle r_i t_i q$, and denote by f_i the line through t_i and r_i. Otherwise, if $|S_i| \leq k$ set $f_i = e_i$. Set $R_i = \{q, t_i\} \cup \{p' \in S_i \mid p' \text{ is to the right of } f_i\}$ and $L_i = (S \setminus R_i) \cup \{t_i\}$. See Figure 9(b) for an example where $|S_i| < k$. The existence of a point t and sets $R, L \subset S$ as required, will follow from the next claim.

Proposition 3.4. *Set $S_0 = \emptyset$. Then, for every $i \geq 1$ such that $|S_{i-1}| < k$, t_i, R_i, and L_i satisfy properties (1)–(3) of Lemma 3.3, and $S_{i-1} \subsetneq S_i$.*

Proof. By induction on i. For $i = 1$ the claim holds by the discussion above. Assume that $i > 1$ and $|S_{i-1}| < k$. Property (1) holds by the definition of R_i and L_i. The triangle $\triangle t_i pq$ is empty since: $\triangle t_{i-1} pq$ is empty; t_i is to the left of f_{i-1} and therefore $\triangle t_i pq$ does not contain any point from R_{i-1}; and by the choice of t_i. Thus, Property (2) holds. Property (3) clearly holds if $f_i = e_i$. Otherwise, if r_i is defined, denote by C_i the cone whose apex is at p and is bounded by the line through p and t_{i-1} and the line through p and t_i. By the choice of t_i all the points in C_i are in S_{i-1}. Since $|S_{i-1}| < k$ it follows that r_i is to the left of the line through p and t_i. Recall that r_i is to the right of e_i, since $r_i \in S_i$. Therefore, f_i is tangent to both $\mathcal{CH}(R_i)$ and $\mathcal{CH}(L_i)$ and separates them, except for the point t_i. Thus, Property (3) holds. Finally, since t_i is to the left of e_{i-1} we have $S_{i-1} \subseteq S_i$. However $t_i \in S_i \setminus S_{i-1}$, thus, $S_{i-1} \subsetneq S_i$. □

Since $|S_i| > |S_{i-1}|$ there is an integer j such that $|S_{j-1}| < k$ and $|S_j| \geq k$. It follows from Proposition 3.4 and the definition of R_j that t_j, R_j, and L_j satisfy the required properties. □

Note that Lemma 3.3 implies that pt is a boundary edge of $\mathcal{CH}(L)$ and tq is a boundary edge of $\mathcal{CH}(R)$, respectively. Using this fact we will apply the suggested subdivision in the next section in order to obtain our first main result.

3.2 A New Upper Bound

Figure 10 illustrates the subdivision obtained in the previous section. The idea to prove an upper bound on $\bar{\rho}(n)$ is to iteratively split a set into subsets of constant size, to obtain good polygonalizations for these sets, and then to combine them based on Lemma 3.3. The base case is covered by the following result.

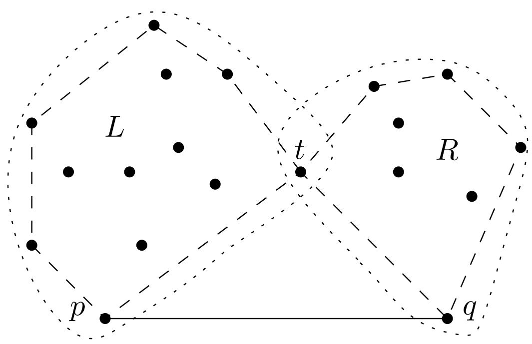


Figure 10: The subdivision of S guaranteed by Lemma 3.3

Lemma 3.5. *Let S be a set of at most 8 points in the plane. Then $\bar{\rho}(S) \leq 2$.*

Proof. The claim is clearly true for $n \leq 5$ since any vertex on the boundary of $\mathcal{CH}(S)$ is a convex vertex of any polygonalization of S. For $6 \leq n \leq 8$ we prove the statement by a case-analysis over the size of the convex layers of S; see Appendix A for details. The correctness of the statement was also verified using a computer by checking all possible configurations of at most 8 points in general position. □

We are now ready for the first upper bound on $\bar{\rho}(n)$.

Theorem 3.6. $\bar{\rho}(n) \leq 3\lfloor \frac{n-2}{7} \rfloor + 2$.

Proof. We prove the claim by induction on n. For $n \leq 8$ we directly get the result from Lemma 3.5.

For $n > 8$ we apply Lemma 3.3 on the set S with $k = 8$ and some edge pq of the boundary of $\mathcal{CH}(S)$, and obtain the point t and the subsets L and R. Now according to Lemma 3.5 there is a polygonalization of R containing the edge qt with at most two reflex vertices (note that qt is a boundary edge of $\mathcal{CH}(R)$). By induction, L has a polygonalization containing the edge pt (which is a boundary edge of $\mathcal{CH}(L)$) with at most $3\lfloor \frac{n-9}{7} \rfloor + 2 = 3\lfloor \frac{n-2}{7} \rfloor - 1$ reflex vertices. By removing the edge qt from the first polygonalization and the edge pt from the second, the remaining polygonal chains, along with the edge pq, form a proper polygonalization of S with at most $2 + 3\lfloor \frac{n-2}{7} \rfloor - 1 + 1 = 3\lfloor \frac{n-2}{7} \rfloor + 2$ reflex vertices (note that t may be a reflex vertex in the resulting polygon). □

3.3 Improving the Constant

Generalizing the approach used to prove Theorem 3.6 to arbitrary $k \geq 2$ we get

Corollary 3.7. *If for some $k \geq 2$ we have $\bar{\rho}(k) \leq l$, then $\rho(n) \leq (l+1)\lfloor \frac{n-2}{k-1} \rfloor + k \leq \frac{l+1}{k-1}n + k$. If, additionally, for any $k' \leq k$ we have $\bar{\rho}(k') \leq l$, then $\rho(n) \leq \frac{l+1}{k-1}n + l$.*

Improved bounds for $\bar{\rho}(n)$ for small, constant values of n thus yield a better bound on the reflexivity of arbitrarily large sets of points. From Lemma 3.5 together with an extension to $n = 9, 10$ by using the point set order type data base [5] we observe that $\bar{\rho}(n) = \rho(n)$ for $n \leq 10$, see Table 2. Therefore our next goal is to determine good bounds on $\bar{\rho}(n)$ for $n \geq 11$. To this end, we use the following observation which is implied by Lemma 3.3 and the discussion in the previous section.

Observation 3.8. *For any integers $2 < k < n$, we have $\bar{\rho}(n) \leq \bar{\rho}(n-k+1) + \bar{\rho}(k) + 1$. Moreover, for every set of n points, S, there is a subset $L \subset S$, such that $|L| = n - k + 1$ and $\bar{\rho}(S) \leq \bar{\rho}(L) + \bar{\rho}(k) + 1$.*

Using the values of Table 2 for $k = 3$ and $k = 8$ we get

$$\bar{\rho}(n) \leq \bar{\rho}(n-2) + 1 \tag{1}$$

$$\bar{\rho}(n) \leq \bar{\rho}(n-7) + 3$$

Applying these two relations we obtain the upper bounds on $\bar{\rho}(n)$ shown in Table 5 with an exception for $n = 13$.

n	11	12	13	14	15	16
$\rho(n)$	3	3..4	3..4	4..5	4..5	4..6
$\bar{\rho}(n)$	4	4	4	4..5	4..5	4..6

Table 5: $\rho(n)$ and $\bar{\rho}(n)$ for $n = 11 \ldots 15$

By using the point set order type data base for $n = 11$ points it turned out that $\rho(11) = 3$ whereas $\bar{\rho}(11) = 4$. Interestingly, only for 36 of the 2 334 512 907 existing order types, the best polygonalization required 4 reflex vertices. In all these sets the boundary of the convex hull was a triangle, and only for one (out of three) convex hull edge e we obtained $\rho_e(S) = 4$. This has to be seen in contrast to the worst case examples for $\rho(S)$ obtained in [9], which are so-called double circles. There half of the vertices are on the convex hull, and the remaining vertices form a second onion layer, each point lying close to the middle of one edge of the convex hull.

We have extended examples providing $\bar{\rho}(11) = 4$ to verify that 4 reflex vertices are necessary for polygonalizing certain point sets of size $n = 12, \ldots, 16$, as is listed in Table 5. Thus, for $n = 12$, together with Equation 1 we have $\bar{\rho}(12) = 4$. So we will have to look for values of $k > 12$ in order to benefit from Corollary 3.7. Thus we aim to show that $\bar{\rho}(13) = 4$.

From Equation 1 we already know that $\bar{\rho}(13) \leq 5$. So assume that there exists a set S, $|S| = 13$, with $\bar{\rho}(S) = 5$. By Observation 3.8 S contains a subset L of 11 points with $\bar{\rho}(L) = 4$. We now apply abstract order type extension, which is a tool that can be used to generate all (abstract) point sets containing a given class of sets of smaller cardinality, see [6] for details. Applying this method to the 36 sets of $n = 11$ points which require 4 reflex vertices, we obtain all sets for $n = 13$ which might require 5 reflex vertices. Our computations show that all obtained sets contain a polygonalization with at most 4 reflex vertices, contradicting our assumption, and we conclude that $\bar{\rho}(13) = 4$.

By Corollary 3.7 we therefore get

Corollary 3.9. $\bar{\rho}(n) \leq 5\lfloor\frac{n-2}{12}\rfloor + 4$

which implies Theorem 3.2. Obviously determining $\bar{\rho}(n)$ for $n \geq 14$ could further improve the constant of Corollary 3.9, and we leave this for future research.

3.4 An Algorithm

After establishing the existence of a polygonalization with few reflex vertices we describe an efficient way to find one.

Theorem 3.10. *Given a set of n points S and two points $p, q \in S$ such that pq is a boundary edge of $\mathcal{CH}(S)$, a polygonalization P of S such that pq is an edge*

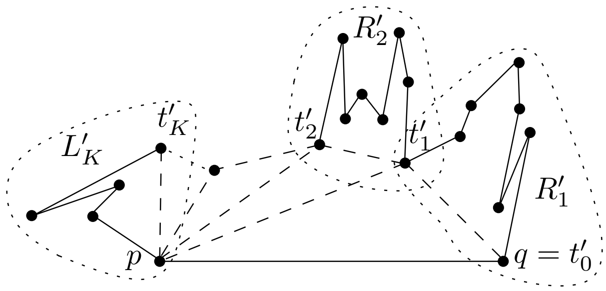


Figure 11: The subdivision of the point set into constant-size subsets

of P and $\rho(P) \leq 5\lfloor \frac{n-2}{12} \rfloor + 4$ can be found in $O(n \log n)$ time.

Proof. The proof of Theorem 3.6 and the discussion in Section 3.3 yield an algorithm for computing a polygonalization with at most $5\lfloor \frac{n-2}{12} \rfloor + 4$ reflex vertices, based on the subdivision of the set S into (not necessarily disjoint) subsets of size 13 (apart from one subset of size at most 13). Set $t'_0 = q$, $L'_0 = S$, and $M = \lfloor (n-2)/12 \rfloor$. Define t'_l, R'_l, and L'_l, recursively, to be the point t and the sets R and L, respectively, guaranteed by applying Lemma 3.3 on the set L'_{l-1} and the edge pt'_{l-1}, $1 \leq l \leq M$. Once the points t'_l and the sets R'_l have been computed it is easy to compute in linear time the polygonalization of S with the stated number of reflex vertices: For each $1 \leq l \leq M$ we computed (in constant time) a polygonalization P_l of R'_l, such that P_l contains the edge $t'_l t'_{l-1}$ and has at most four reflex vertices. By removing the edge $t'_l t'_{l-1}$ from every polygon P_l, we get a polygonal chain P starting at $t'_0 = q$ and ending at t'_M. If $|L'_M| = 2$, that is $L'_M = \{p, t'_M\}$, then we obtain the desired polygonalization of S by concatenating to P the edges pt'_M and pq. Otherwise, one can compute in constant time a polygonalization P_{M+1} of L'_M containing the edge pt'_M and having at most four reflex vertices (note that $|L'_M| \leq 13$). By removing the edge pt'_M from P_{M+1} and concatenating the resulting chain to P and the edge pq we obtain the desired polygonalization of S (see Figure 11 for an illustration).

Therefore, it remains to depict the details of the subdivision described in

Lemma 3.3. Algorithm 3.4 describes the implementation of this subdivision. It uses the dynamic planar convex hull of Brodal and Jacob [13]. This data structure, denoted by `DCH`, maintains the convex hull of a set of points and supports, among other things, the following operations:

- `DCH.INSERT(`v`)`: insert a new point v;
- `DCH.DELETE(`v`)`: remove the point v; and
- `DCH.CCW(`v`)`: get the counter-clockwise neighbor in the convex hull of the point v, where v is a vertex of the convex hull;

Insertions and deletions are performed in $O(\log n)$ amortized time, while the counter-clockwise neighbor query takes $O(\log n)$ worst-case time.

Next, we explain Algorithm 3.4, using the notation of Lemma 3.3. The algorithm begins by removing the vertex q (line 5). Now, CCW(p) is t_1. Then, we remove CCW(t_1) repeatedly at most $k-2$ times while it is to the right of e_1 (lines 15–23). If $k-2$ times CCW(t_1) was to the right of e_1, then all those points that were removed along with t_1 constitute the set R, with $t = t_1$ (lines 24–27). Otherwise, by deleting t_1 (line 5), CCW(p) is the point t_2 that forms the smallest angle $\angle t_2pq$ among the points to the left of e_1. The algorithm proceeds by re-inserting all the points that were deleted in the previous iteration and are to the left of the line determined by p and t_2 (lines 9–12). This step is performed since it is possible that these points will not be among the set of points $p' \in S_2$ creating the $(k-2)$nd smallest angles $p't_iq$ (whereas the points that are to the right of the line through p and t_2 must be in this set, and thus, remain in R). Next, we remove CCW(t_2) repeatedly (as long as it is to the right of e_2), this time $k-1-|R|$ times (lines 15–23). As before, if $k-1-|R|$ times CCW(t_2) was to the right of e_2, then we are done. Otherwise, we proceed to the next point t_3.

For the same arguments used to claim that $|S_i| > |S_{i-1}|$ in the proof of Lemma 3.3 it follows that the size of R at, say, line 24 grows along the iterations

```
Require: A set of n points S; DCH(S); p, q ∈ S s.t. pq is a boundary edge of CH(S);
  an integer k > 2
Ensure: The set R and the point t as described in Lemma 3.3.
  R ← ∅;
  i ← 0;
  t_i ← q;
  while |R| < k do
    DCH.DELETE(t_i);
    R ← R ∪ {t_i};
    i ← i + 1;
    t_i ← DCH.CCW(p);
    for all {r ∈ R : r is to the left of the line through p and t_i} do
      R ← R \ {r};
      DCH.INSERT(r);
    end for
    e_i ← the line through q and t_i;
    m ← (k − 1 − |R|); /* The number of points missing in R */
    for j = 1 to m do
      s ← DCH.CCW(t_i);
      if s is to the right of e_i then
        DCH.DELETE(s);
        R ← R ∪ {s};
      else
        quit the for-loop;
      end if
    end for
    if s is to the right of e_i then
      R ← R ∪ {t_i}; /* |R| = k */
      t ← t_i;
    end if
  end while
```

Algorithm 1: Generating the subdivision of Lemma 3.3

of the main loop (lines 4–28). Therefore, the main loop is executed $O(k)$ times, and thus, the run-time of the procedure described in Algorithm 3.4 is $O(k^2 \log n)$ amortized time. The number of times this procedure is executed is $O(n/k)$. Thus, the overall run-time, including the initialization of `DCH`, is $O(nk \log n)$. As in our case $k = 13$, the run-time is $O(n \log n)$. □

3.5 Discussion and Open Problems

We showed that for every set S of n points in general position in the plane there is a polygonalization of S with at most $5\lfloor \frac{n-2}{12} \rfloor + 4$ reflex vertices, and such a polygonalization can be found in $O(n \log n)$ time. The basic idea of the proof is that by Lemma 3.3 we can subdivide S into some fixed-size parts and use a stronger result on each of these parts. It would be interesting to find other applications of the subdivision suggested in Lemma 3.3. Recently Günter Rote informed us that this kind of subdivision was utilized in papers about straight line embeddings of outerplanar graphs (see [25], [14], [10]). Moreover, for achieving the same subdivision they use a simpler algorithm which - using the `DCH` data structure - works in amortized time $O(k \log n)$, whereas ours works in $O(k^2 \log n)$ amortized time. Yet, in our application k is a small constant, thus utilizing their algorithm only simplifies things, does not improve the result significantly.

Conjecture 3.4 in [9] states that $\rho(n) = \lfloor \frac{n}{4} \rfloor$. As already mentioned in the Introduction, considering the values for $\rho(n)$ in Tables 2 and 5 the conjecture has to be modified to

Conjecture 3.11. $\lfloor \frac{n}{4} \rfloor \leq \rho(n) \leq \lceil \frac{n}{4} \rceil$.

It is a challenging open problem to determine the structure of sets maximizing the reflexivity for fixed cardinality. On the one hand we have the sets used in [9] to provide the bound of $\rho(n) \geq \lfloor \frac{n}{4} \rfloor$, which have half of their vertices on the boundary of the convex hull. This so-called double circle configuration is also conjectured to minimize the number of triangulations [4], and therefore seems to be a promising extremal example, supporting Conjecture 3.11. On the other hand all maximizing examples for $\bar{\rho}(11)$ have a triangular convex hull, so it could be that for larger cardinality $\bar{\rho}(n)$ is more than a constant additive factor larger than $\rho(n)$, contradicting Conjecture 3.12.

It would be interesting to bound $\bar{\rho}(n)$ in terms of $\rho(n)$.

Conjecture 3.12. *There is a constant c_0 such that $\bar{\rho}(n) \leq \rho(n) + c_0$.*

Note that the stronger statement that $\bar{\rho}(S) \leq \rho(S) + O(1)$ for any set S might also hold.

Conjecture 3.12, if true, would mean that it is possible (although not necessarily practical) to get arbitrarily close to the best possible linear upper bound by checking only finitely many small cases. In other words, suppose the conjecture holds and c is a constant such that $\rho(n) \leq cn$. Then, for any $\epsilon > 0$ there is $k = k(\epsilon)$ such that if we verify that $\rho(k) \leq ck$, then for $n > k$ we have $\rho(n) \leq (c + \epsilon)n + O(1)$. Indeed, k large enough such that $\frac{ck+c_0+1}{k-1} \leq (c + \epsilon)$ holds, would do. Moreover, the discussion above is still valid if we replace c_0 in Conjecture 3.12 by some function $f(n)$ such that $f(n) \in o(n)$.

Conjecture 3.12 is true when we consider reflexivity in the presence of *Steiner points*. Following the notation of [9], a *Steiner point* is a point $q \notin S$ that may be added to S in order to improve some structure. For example, we define the *Steiner reflexivity* of S, $\rho'(S)$, to be the minimum number of reflex vertices of any simple polygon with vertex set $V \supseteq S$. Similarly, $\rho'(n) = \max_{|S|=n} \rho'(S)$. The (stronger statement) of Conjecture 3.12 can be easily proved if we allow Steiner points.

Lemma 3.13. *Let S be a set of n points and let pq be a boundary edge of $\mathcal{CH}(S)$. Then, there are points p', q' (inside $\mathcal{CH}(S)$) such that $S \cup \{p', q'\}$ has a polygonalization containing the edge pq and having at most $\rho(S) + 1$ reflex vertices.*

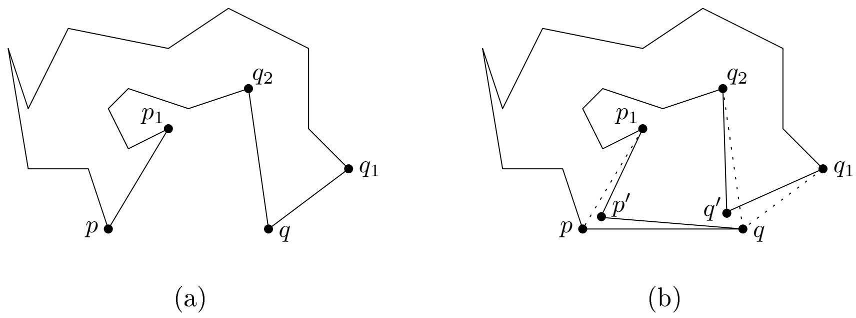


Figure 12: An illustration for the proof of Lemma 3.13

Proof. We assume, w.l.o.g., that the fixed edge pq is horizontal, p is left to q, and the remaining points $S \setminus \{p, q\}$ are above the line through p and q. Let P be a polygonalization of S, such that P does not contain the edge pq. We show that P can be modified into a polygonalization P' of a set $V \supset S$ such that P' contains the edge pq and $\rho(P') \leq \rho(P) + 1$.

Let p_1 be the counter-clockwise neighbor of p in P, and let q_1 and q_2 be the counter-clockwise and clockwise neighbors of q in P, respectively. Fix p' slightly to the right and above p, and q' slightly to the left and above q. Now by replacing the chain $q_1\,q\,q_2$ with the chain $q_1\,q'\,q_2$, and the edge $p\,p_1$ with the chain $p\,q\,p'\,p_1$, one obtains the desired polygonalization P' (see Figure 12 for an illustration). Note that the only reflex vertex that might be introduced in these steps is p'. □

As before, this implies that we can get arbitrarily close to any linear upper bound on $\rho'(n)$ by checking only finitely many small cases. Note that it is important here that the Steiner points we add lie inside the convex hull of the original set of points.

Adding a (Steiner) point to a set of points might result in a set of points whose reflexivity is smaller than this of the original set (see [9] for examples). However, we are confident, although we were not able to prove, that for every set of points one can add some point that will not reduce the reflexivity. This would imply

Conjecture 3.14. $\rho(n+1) \geq \rho(n)$

A similar statement should hold for restricted reflexivity.

Conjecture 3.15. $\bar{\rho}(n+1) \geq \bar{\rho}(n)$

If this conjecture is true, then the last inequality of Corollary 3.7 always holds.

4 Polychromatic colorings of rectangular partitions

First we regard the problem of polychromatic colorings and rectangle-respecting colorings for rectangular partitions, and then for guillotine-partitions in higher dimensions.

4.1 Polychromatic colorings of planar rectangular partitions

Polychromatic 4-colorings of rectangular partitions

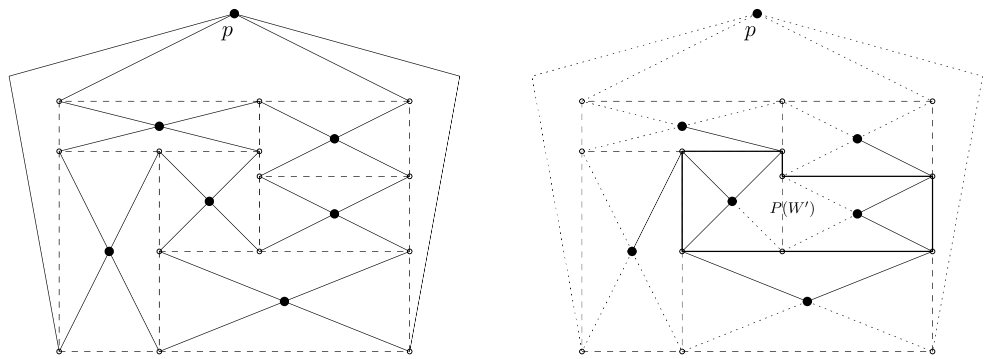
We start with the definition of r-graphs, then state the conjecture of Seymour [44] about 4-graphs, which was proved recently by Guenin [26]. Note that this theorem implies the four-color theorem. An *r-graph* is an r-regular (multi)-graph G on an even number of vertices with the property that every edge-cut separating $V(G)$ into two sets of odd cardinality has size at least r.

Theorem 4.1 (Guenin). *Every planar 4-graph is 4-edge-colorable.*

In the rest of this section we will prove that this theorem implies the following.

Theorem 4.2. *There is a strong rectangle-respecting 4-coloring of any rectangular partition.*

Proof. Take an arbitrary rectangular partition. We will give a strong rectangle-respecting 4-coloring of it. First, if it has an even number of rectangles then add one rectangle to the left side of the partition for example. Coloring this new partition (which has an odd number of rectangles) also gives a coloring of the original one. Now we contruct the planar graph on which we can apply Theorem 4.1. First put a point in the center of all the rectangles. Connect two such points with an edge whenever the corresponding rectangles have a common corner. Draw



(a) The graph constructed from the rectangular partition

(b) The edges of the edge-cut guaranteed by the proof

Figure 13: Example for the proof of Theorem 4.2

this edge in the plane by two segments, one going from the center point of the first rectangle to the common corner and the second going from here to the center point of the other. This way we clearly built a planar graph on an odd number of points. Further, add a point p outside the partitioned rectangle. For each corner c of the partitioned rectangle take the rectangle R in the partition that has c as a corner. Then connect p to the center r of R, with a polygon going through c as well. This can easily be done for all four corners of the partitioned rectangle in a way that the final graph is still a plane graph. Now our graph has an even number of points. The new point p has 4 outgoing edges and an arbitrary point corresponding to a rectangle has 4 outgoing edges as well, one going through each of its corners. Thus, this graph G has an even number of points and is 4-regular. For an example see Figure 13(a). Note that the resulting graph can be a multigraph and that we needed that no 4 rectangles meet at a common corner.

To be able to apply Theorem 4.1 we only need to prove that every edge-cut separating $V(G)$ into two sets of odd cardinality has size at least r. Take an arbitrary cut of this kind. The two point sets of the cut are W and $V \setminus W$ where p is in $V \setminus W$. Take a maximal connected component W' of W. If two points are

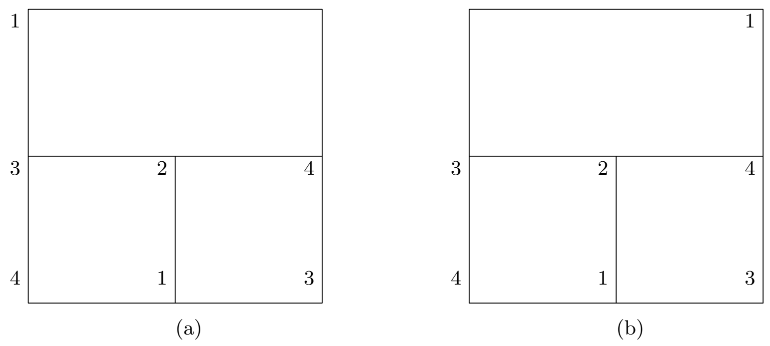


Figure 14: The partition T and two different colorings

connected in the graph then the corresponding rectangles have a common segment on their borders, thus the rectangles corresponding to W' form a polygon $P(W')$ whose every edge is horizontal or vertical. Thus, W' has at least four convex corners. It is enough to prove that the edge corresponding to such a corner c is part of the edge-cut. Indeed one end of such an edge is the point corresponding to the rectangle in $P(W')$ (and so it is in W) whose corner is c and the other end is a point corresponding to a rectangle outside $P(W')$ and by the maximality of W' this cannot be in W. For an example see Figure 13(b), where the 5 edges of G corresponding to the convex corners of $P(W')$ are solid, the rest are dotted. Now, applying Theorem 4.1 we get a 4-coloring of the edges. Give the same colors for the corresponding corners of the partition, this is clearly a strong rectangle-respecting 4-coloring as we needed. □

Polychromatic 4-colorings of general rectangular partitions

In the rest of this section we deal with general rectangular partitions, and so for simplicity, in the remainder of this section we will use the terms partition and general partition interchangeably. The previous theorem does not hold for general rectangular partitions as it was mentioned in the Introduction. Now we prove that even a polychromatic 4-coloring does not always exist for general partitions.

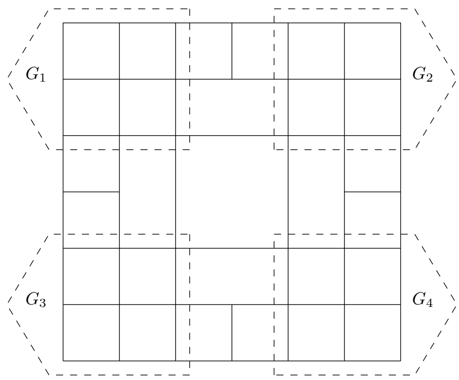


Figure 15: The partition $\mathcal{Q}$ and subpartitions G_1, G_2, G_3, G_4

We want to find a 4-coloring of a given partition such that for any rectangle all four colors appear on the vertices on the boundary of that rectangle. Denote by $\mathcal{G}$ the 3×3 grid (i.e. four squares) and by T the partition obtained from a 3×3 grid by merging the upper two squares (see Figure 14). When referring to a *side* of the partition $\mathcal{G}$ or T we mean the set of vertices on the corresponding vertical or horizontal boundary of the partition (e.g. left, right, upper, lower). We begin with some simple observations.

Observation 4.3. *If a weak rectangle-respecting* 4*-coloring of the* 3×3 *grid* $\mathcal{G}$ *assigns three colors to some side of* $\mathcal{G}$*, then the same three colors appear on the opposite side of* $\mathcal{G}$*.*

Observation 4.4. *A weak rectangle-respecting* 4*-coloring of a* 3×3 *grid* $\mathcal{G}$ *cannot simultaneously assign three colors to the left (or right) and lower (or upper) sides.*

Observation 4.5. *A weak rectangle-respecting* 4*-coloring of* T *assigns three colors to the left side or right side of* T*.* (See Figure 14.)

Let us define a new partition $\mathcal{Q}$ as follows: start with a 7×7 grid, first merge the four central squares, then for each side of this new center square merge the two smaller squares adjacent to that side. In this way we obtain a partition that contains four copies of $\mathcal{G}$ and four rotations of T. See Figure 15 for an illustration

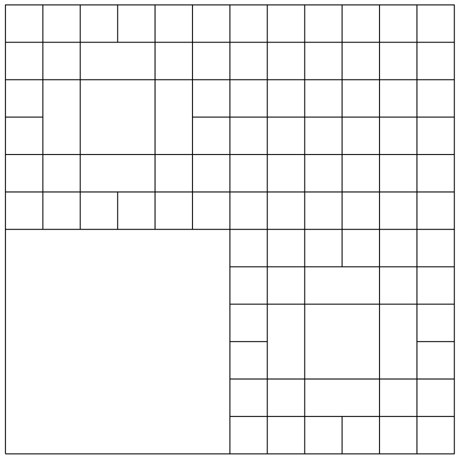


Figure 16: Proof of Theorem 1.14, $\mathcal{C}$, the counterexample

of $\mathcal{Q}$.

Claim 4.6. *Let G_1 (resp. G_2) be the 3×3 grid in the upper-left (resp. upper-right) corner of $\mathcal{Q}$. A weak rectangle-respecting 4-coloring of the partition $\mathcal{Q}$ must assign three colors to either the upper side of G_1 or the upper side of G_2.*

Proof. Let G_3 (resp. G_4) be the 3×3 grid in the lower-left (resp. lower-right) corner of $\mathcal{Q}$ (see Figure 15). Assume that neither the upper side of G_1 nor the upper side of G_2 are assigned three colors. By Observation 4.3 neither the lower side of G_1 nor the lower side of G_2 are assigned three colors. By Observation 4.5 the upper sides of G_3 and G_4 are each assigned three colors. Finally, by Observation 4.4 the right side of G_3 has exactly two colors and the left side of G_4 has exactly two color. However, now we have colored the right and left sides of the partition T on the bottom of $\mathcal{Q}$ in a way that contradicts Observation 4.5. □

Note that a similar claim holds for each side of $\mathcal{Q}$ as we can simply rotate $\mathcal{Q}$ and follow the proof of Claim 4.6. Let us define a new partition $\mathcal{C}$ as follows: start with a 3×3 grid and embed a 7×7 grid in the upper-right square, a copy of $\mathcal{Q}$ in the upper-left square and a copy of $\mathcal{Q}$ in the lower-right square. See Figure 16 for an illustration. We will show that partition $\mathcal{C}$ has no weak rectangle-respecting

4-coloring thus proving Theorem 1.14.

Proof of Theorem 1.14. Let Q_1 be the upper-left copy of $\mathcal{Q}$ and let Q_2 be the lower-right copy of $\mathcal{Q}$. By applying Claim 4.6 to Q_2 we find 3 consecutive vertices on the lower side of the 7×7 grid with three colors. By applying Claim 4.6 to a -90^o rotation of Q_1 we find 3 consecutive vertices on the right side of the 7×7 grid with three colors. By application of Observation 4.3 it is easy to see that we can find a 3×3 grid in the 7×7 grid that has three colors on both the lower side and right side. This contradicts Observation 4.4. □

The elements of this construction are the main building parts to prove that it is NP-complete to decide if a general rectangular partition admits a polychromatic 4-coloring. We know several smaller partitions that have no weak rectangle-respecting 4-coloring, the smallest known construction has 65 vertices (whereas the counterexample presented here has 124 vertices).

Weak rectangle-respecting 3- and 5-colorings of general rectangular partitions

In this part we want to prove that any partition admits a weak rectangle-respecting 3- and 5-coloring.

Consider a partition in the coordinate plane. Let us arrange vertices from smallest to largest x coordinate then from largest to smallest y coordinate i.e. from left to right then top to bottom. We refer to this ordering as the *upper-left order* of the vertices. Two vertices are neighbors if there is a segment between them containing no other vertex.

Proposition 4.7. *Every general partition admits a weak rectangle-respecting* 3-*coloring (i.e. a polychromatic* 3-*coloring).*

Proof. Let R be a partition. We will greedily 3-color the vertices of R. Note that any v vertex has at most one neighbor above it and at most one left to it (the neighbor with smaller x-coordinate and the neighbor with larger y-coordinate). We will always maintain a coloring such that any vertex has a different color from these at most 2 neighbors of it.

Next, consider the vertices of R arranged in upper-left order. For a given rectangle we will ensure that it has at least 3 colors on its boundary in the step when we consider its lower right corner.

Let v be the pending vertex to be colored. If v has exactly one previously-colored neighbor w then color v with a color different from w. If v has two previously-colored neighbors, say x and y, then v, x and y are on the boundary of a common rectangle and v is the bottom-right corner of this rectangle.

If x and y are colored with different colors then color v with the third color. Thus, the rectangle r containing v, x and y will have 3 different colors on its boundary as we wanted.

If y and x are colored with the same color then we will color v with a color different from that used on x and y. These two points must have at least one already colored neighbor w on the boundary of rectangle r (this may be a common neighbor of x and y, in which case it is the top left corner of r). By induction w has different color from the color of x and y. Color v with the color unused by x and w. Thus, rectangle r will have 3 different colors on its boundary as needed.

In this way every rectangle of R will include three different colors among the vertices on its boundary. □

Proposition 4.8. *Every general partition admits a weak rectangle-respecting* 5-*coloring.*

Proof. Let R be a partition. We will greedily 5-color the vertices of R. Note that any v vertex has at most one neighbor above it and at most one left to it (the neighbor with smaller x-coordinate and the neighbor with larger y-coordinate). Again, we will always maintain a coloring such that any vertex has a different

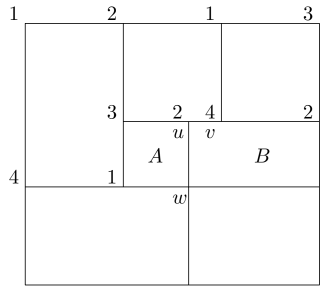


Figure 17: w cannot be colored to $1, 2, 3$ because of A, neither to 4 because of v in B

color from these at most 2 neighbors of it.

Next, consider the vertices of R arranged in upper-left order. For a given rectangle B we will ensure that it has at least 2 colors on its boundary in the step when we consider the vertex v right to its top left vertex u (v and u are neighbors and so we color v differently from u). We will ensure that B has at least 3 colors on its boundary in the step when we consider the vertex w which is the neighbor of u below u (we color it differently from u and u's right neighbor v). We will ensure that B has at least 4 colors on its boundary in the step when we consider its lower right corner x (in case there are only 3 colors present on the boundary of B, we color x differently from these).

Following these rules we will get a needed coloring. Thus, it is enough to prove that when coloring a vertex there are at most 4 colors we need to avoid, thus using 5 colors we can color greedily all the vertices. If a vertex does not need to ensure a third or fourth color on some rectangle, then we need to color it differently only from its left and upper neighbor, that is we need to avoid at most 2 colors. If a vertex w ensures a third color on some B (but it does not need to ensure a fourth color on any of the rectangles) then when coloring it we need to avoid the color of its left neighbor (if it exists), its upper neighbor u and the right neighbor of u, thus at most 3 colors. If a vertex needs to ensure a fourth color on some rectangle A, then we need to avoid at most 3 colors (including the colors of its left and upper neighbor). If at the same time this vertex ensures a third color on some rectangle B, then one more color needs to be avoided (the color of the right neighbor of its upper neighbor). Together this is again at most 4 colors to be avoided (see Figure 17 for an illustration of this worst case). □

Strong rectangle-respecting 2- and 6-colorings of general rectangular partitions

In this section we either want to find the minimal number $k \geq 4$ or the maximal number $k < 4$ of colors such that every partition can be colored with k colors such that $\min\{k, 4\}$ colors appear on the 4 corners of every rectangle of the partition. First we show the existence of a strong rectangle-respecting 2-coloring.

Proposition 4.9. *Every general partition admits a strong rectangle-respecting* 2-*coloring.*

Proof. For a given partition R let us color the vertices in upper-left order with exactly two colors. Let v be the pending vertex to be colored. Only vertices above and to the left of v are already colored. Thus, only the rectangle that has v in its lower right corner may have three previously colored corners. If the three previously colored corners have the same color, then choose the other color for v. Otherwise choose any color for v. After coloring all vertices in this way clearly no rectangle will have a single color among its four corners. □

A simple greedy algorithm similar to those used for weak rectangle-respecting 5-colorings shows that for a partition there is always a strong rectangle-respecting 7-coloring. We present a proof that every partition has a strong rectangle-respecting 6-coloring.

Proposition 4.10. *Every general partition admits a strong rectangle-respecting* 6-*coloring.*

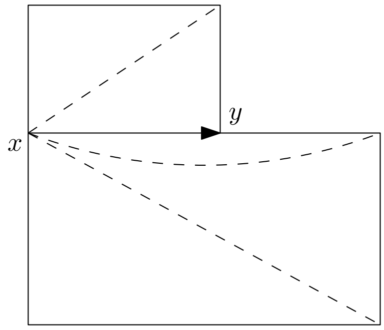


Figure 18: orientation of edge xy

Proof. For a given partition R, let G be the graph with the vertex set of R, where xy is an edge of G if and only if x and y are corners of the same rectangle in R. Clearly, the proposition is proved if we can find a proper 6-coloring of G. First we will color the vertices where four rectangles meet. We use a greedy algorithm with the vertices in upper-left order. Every vertex has at most four previously colored neighbors, hence six colors are (more than) enough to properly color such vertices.

Denote by W the set of all so-far uncolored vertices of R. A vertex in W is the corner of at most two rectangles, thus has degree at most six in the graph G. Let $W' \subset W$ be the set of vertices of degree 6. A vertex $x \in W'$ must be the corner of exactly two rectangles that do not share a second corner. Hence x has two neighbors lying on a common line segment starting from x (see Figure 18). Denote the closer one of these two neighbors by y. Observe that y must be the corner of exactly two rectangles. Now for every such pair x and y direct the edge xy in G from x to y. Note that this procedure will never direct y to x. Thus all vertices in W' have outdegree exactly one.

Let us first color the vertices of W' with indegree zero. Let $x \in W'$ be a vertex with indegree zero. The vertex x has outdegree one, so x has an uncolored neighbor, thus there is an available color for x. Color x with an available color and remove x from W'. Repeat this step until no vertex with indegree zero remains in W'. Now every vertex in W' has outdegree and indegree equal to 1. Therefore the remaining vertices of W' can be partitioned into directed cycles. The vertices on a directed cycle in W' have at most 4 previously colored neighbors, so each vertex has a list of at least two available colors.

If we examine these cycles in R it is clear that the edges must alternate between vertical and horizontal orientation. Thus these directed cycles are of even length. The list-chromatic number of an even cycle is 2, hence each cycle can be colored properly. Now all vertices in W' are colored. The remaining uncolored vertices in G are the vertices of $W - W'$. These vertices have degree at most 5 and thus all have an available color. $\square$

4.2 Polychromatic colorings of n-dimensional guillotine-partitions

In this section our aim is to prove the following theorem.

Theorem 4.11. *There is a strong hyperbox-respecting coloring of any n-dimensional guillotine-partition.*

Note that the theorem is about partitions where no more than 2 hyperboxes are allowed to meet at a common corner. First we start with some definitions to be able to phrase the theorem we will actually prove, implying Theorem 4.11. We begin by introducing some notations. From now on $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y}, \mathbf{z}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, etc. always refer to some n-long 0-1 vector. We define the sum of two such vectors (denoted simply by $+$) as summing independently all coordinates mod 2. The $(0, 0, \ldots, 0)$ vector is denoted by $\mathbf{0}$ and the vector $(1, 0, 0, \ldots, 0)$ by $\mathbf{e_1}$. A *face* is always an $(n-1)$-dimensional face of a hyperbox. Fix now the unit hyperbox, the 0-1 vectors being its corners. First we define the forthcoming notions only for this hyperbox. For some fixed $\mathbf{x} \neq \mathbf{0}$ we define the *reflection* $R_{\mathbf{x}}$ being the function on the set of corners for which $R_{\mathbf{x}}(\mathbf{y}) = \mathbf{x} + \mathbf{y}$ for all $\mathbf{y}$. It is indeed a reflection on an $(n-k)$-dimensional hyper-plane where k is the number of 1's in $\mathbf{x}$. Observe that $R_{\mathbf{x}}(R_{\mathbf{x}}(\mathbf{y})) = \mathbf{y}$ for any $\mathbf{y}$.

We say that a coloring of the corners of the hyperbox is an $R_{\mathbf{x}}$-*coloring* ($\mathbf{x} \neq \mathbf{0}$) if two corners $\mathbf{y}$ and $\mathbf{z}$ have the same colors if and only if $R_{\mathbf{x}}(\mathbf{y}) = \mathbf{z}$ (or equivalently $R_{\mathbf{x}}(\mathbf{z}) = \mathbf{y}$). Observe that such a coloring will have 2^{n-1} different colors appearing on the corners of the hyperbox, each occuring twice. Further, we say that the coloring of the corners is an $R_{\mathbf{0}}$-*coloring* if all corners are colored differently. Note that for any $\mathbf{x}$, permuting the colors of an $R_{\mathbf{x}}$-coloring gives another $R_{\mathbf{x}}$-coloring.

Given now an arbitrary axis-parallel hyperbox B, the unit hypercube can be uniquely scaled (with positive coefficients) and translated into B. Denote this transformation by C (depends on B). The function C maps the 0-1 vectors (i.e. the corners of the unit hypercube) onto the corners of B, thus the set of corners of B are the set of $C(\mathbf{y})$'s (B and $\mathbf{y}$ given, $C(\mathbf{y})$ uniquely denotes one of B's corners). By this bijection an $R_{\mathbf{x}}$-coloring can be analogously defined on this hyperbox, thus on an arbitrary hyperbox.

From now on we will always restrict ourselves to these kinds of colorings. Further, when we speak about a coloring of some partitioned hyperbox then it will be always a hyperbox-respecting coloring. If any pair of such colorings could be put together along any axes to form another such coloring then it would already imply a recursive proof for the main theorem. As this is not the case we have to be more precise about our freedom of how to color a partition, making necessary to define sets of such colorings.

For any $\mathbf{x} \neq \mathbf{0}$ $S_{\mathbf{x}}$ is defined as the union of all $R_{\mathbf{y}}$'s for which $\mathbf{x} \cdot \mathbf{y} = 1$ (the scalar product of $\mathbf{x}$ and $\mathbf{y}$ mod 2). $S_{\mathbf{0}}$ is the one element set of $R_{\mathbf{0}}$. If for some $\mathbf{x} \neq \mathbf{0}$ for all $\mathbf{y} \in S_{\mathbf{x}}$ the hyperbox partition has a strong hyperbox-respecting coloring which is an $R_{\mathbf{y}}$-coloring on its corners, we say that the hyperbox partition can be colored by the *color-range* $S_{\mathbf{x}}$. If it has a strong hyperbox-respecting coloring which is an $R_{\mathbf{0}}$-coloring on its corners, we say that the hyperbox partiton can be colored by the (one element) *color-range* $S_{\mathbf{0}}$.

We will prove the following theorem, which implies Theorem 4.11.

Theorem 4.12. *Any n-dimensional guillotine-partition can be colored by some color-range.*

Proof. We proceed by induction on the number of guillotine-cuts of the partiton. The corners of a hyperbox containing only one basic hyperbox (i.e. the partition has 0 cuts) can be colored trivially with all different colors, thus colorable by color-range $S_{\mathbf{0}}$. In the general step we take a cut of the hyperbox B splitting it into two hyperboxes B_1 and B_2 with smaller number of cuts in them. Thus, by induction they can be colored by some color-ranges $S_{\mathbf{x}}$ and $S_{\mathbf{y}}$ for some $\mathbf{x}$ and $\mathbf{y}$. We need to prove that there exists a $\mathbf{z}$ for which our hyperbox partition can be colored by $S_{\mathbf{z}}$. First we prove this for the case when the cut is orthogonal to the first axis. Finally, we will prove that as the definition of R's and S's is symmetrical on every pair of axes, the claim follows for any kind of cut.
We regard the first axis (the one which corresponds to the first coordinate of points) as the usual x-axis, and so we can say that an object (corner, face, hyperbox etc.) is *left* from another if its first coordinates are smaller or equal than the other's (B_1 is left from B_2 for example). Similarly we can say *right* when its coordinates are bigger or equal than the other's.
We always do the following. Take an $R_{\mathbf{a}} \in S_{\mathbf{x}}$ and $R_{\mathbf{b}} \in S_{\mathbf{y}}$ and take a coloring of B_1 which is an $R_{\mathbf{a}}$-coloring on its corners and a coloring of B_2 which is an $R_{\mathbf{b}}$-coloring on its corners by induction such that the colors of the corners which should fit together (the right face of B_1 and the left face of B_2) have the same colors at the corners which will be identified. This is not always possible but when it is, it gives a coloring of B (the corners on the left face of B_1 and on the right face of B_2 are the corners of B). Note that we can permute the colors on the two hyperboxes in order to achieve such a fit of the colors. Clearly, the resulting coloring of B is a hyperbox-respecting coloring by induction. If

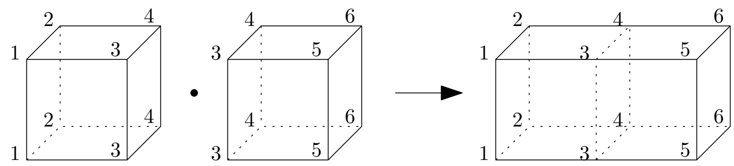


Figure 19: Example to Lemma 4.13(c): $R_{010} \cdot R_{010} \to R_{010}$

the resulting coloring can be an $R_{\mathbf{c}}$-coloring on the corners for some $\mathbf{c}$ then we write $R_{\mathbf{a}} \cdot R_{\mathbf{b}} \to R_{\mathbf{c}}$. See Figure 19 and 20 for examples for 3 dimensions. The definition of $\to$ is good as the existence of such a fit depends only on the color of the corners. Observe that this operation is not commutative by definition and can hold for more than one $\mathbf{c}$ and has the hidden parameter that we put them together along the first axis (i.e. the two partitions are put together by the face which is orthogonal to the first axis). As we remarked earlier, if for any $\mathbf{a}$ and $\mathbf{b}$ there would be a $\mathbf{c}$ with $R_{\mathbf{a}} \cdot R_{\mathbf{b}} \to R_{\mathbf{c}}$ then it would be enough to prove the main theorem by induction without defining color-ranges. As this is not the case we need to deal with color-ranges and define the function $\to$ on them as well.

We write $S_{\mathbf{x}} \cdot S_{\mathbf{y}} \to S_{\mathbf{z}}$ if $\forall R_{\mathbf{c}} \in S_{\mathbf{z}}$ $\exists R_{\mathbf{a}} \in S_{\mathbf{x}}$ and $R_{\mathbf{b}} \in S_{\mathbf{y}}$ such that $R_{\mathbf{a}} \cdot R_{\mathbf{b}} \to R_{\mathbf{c}}$. Clearly, we need to prove that there exists such a $\mathbf{z}$ for any choice of $\mathbf{x}$ and $\mathbf{y}$ that $S_{\mathbf{x}} \cdot S_{\mathbf{y}} \to S_{\mathbf{z}}$. Lemma 4.14 states this. For the proof of this lemma we will first need to prove Lemma 4.13 about the behaviour of $\to$ for R's.

Finally, we need to prove that we can put together color-ranges along any axis. One can argue that we can obviously do that as the definition of color-ranges is symmetrical on any pair of coordinates and because of that analogs of Lemma 4.13 and Lemma 4.14 are true for an arbitrary axis. For a more rigorous argument see Lemma 5.7 and its proof in the Appendix.

Lemma 4.13 (fitting together colorings). *For* $\mathbf{a}, \mathbf{b}, \mathbf{c} \neq \mathbf{0}$ *we have*

(a) $R_{\mathbf{0}} \cdot R_{\mathbf{0}} \to R_{\mathbf{c}}$, *if the first coordinate of* $\mathbf{c}$ *is 1,*

(b) $R_{\mathbf{a}} \cdot R_{\mathbf{0}} \to R_{\mathbf{0}}$ *and* $R_{\mathbf{0}} \cdot R_{\mathbf{a}} \to R_{\mathbf{0}}$, *if the first coordinate of* $\mathbf{a}$ *is 1,*

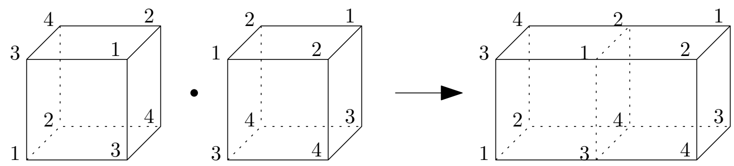


Figure 20: Example to Lemma 4.13(d): $R_{110} \cdot R_{101} \to R_{111}$

(c) $R_{\mathbf{a}} \cdot R_{\mathbf{a}} \to R_{\mathbf{a}}$*, if the first coordinate of* $\mathbf{a}$ *is 0,*

(d) $R_{\mathbf{a}} \cdot R_{\mathbf{b}} \to R_{\mathbf{c}}$*, if the first coordinate of* $\mathbf{a}$ *and* $\mathbf{b}$ *is 1 and* $\mathbf{c} = \mathbf{a} + \mathbf{b} + \mathbf{e_1}$*.*

Proof. (a) Take an arbitrary $\mathbf{c}$ with its first coordinate being 1. For an $R_{\mathbf{c}}$-coloring each color appearing on the corners of the hyperbox appears once on its left and once on its right face. We want to fit together two $R_{\mathbf{0}}$-colorings to have an $R_{\mathbf{c}}$-coloring. Take an arbitrary $R_{\mathbf{0}}$-coloring of B_1. Take an $R_{\mathbf{0}}$-coloring of B_2 and permute its colors such that the corners on its left face fit together with the corners on the right face of B_1. Now the set of colors on the right face of B_2 is the same set of colors as on the left face of B_1. After a possible permutation of these colors on B_2 we can get an $R_{\mathbf{c}}$-coloring on B.
(b) Take an $R_{\mathbf{a}}$-coloring of B_1 and an $R_{\mathbf{0}}$-coloring of B_2, permute the colors on B_2 such that the needed faces fit together. This can be done as $R_{\mathbf{a}}$ does not have a color appearing twice on its right face. As on B_1's left face the same set of colors appear as on its right face, B has all the colors of B_2's coloring appearing on its corners, thus it is an $R_{\mathbf{0}}$-coloring of B. The proof for the other claim is similar.
(c) Take an $R_{\mathbf{a}}$-coloring of B_1. Take an $R_{\mathbf{a}}$-coloring of B_2 and permute the colors on it such that the corners on its left face fit together with the corners on the right face of B_1 and on its right face all colors are different from the ones we used to color the corners of B_1. This can be done as both are $R_{\mathbf{a}}$-colorings where the first coordinate of $\mathbf{a}$ is 0, so on the common face the same pair of corners need to have the same color. Similarly, we see that these fit together to form an

$R_{\mathbf{a}}$-coloring of B. For an illustration for 3 dimensions see Figure 19.

(d) Take an $R_{\mathbf{a}}$-coloring of B_1. Take an $R_{\mathbf{b}}$-coloring of B_2 and permute again the colors such that the corners on its left face fit together with the corners on the right face of B_1. This can be done as the corners on the right face of B_1 all have different colors and this is what we need on the left face of B_2 to make an $R_{\mathbf{b}}$ coloring (as the first coordinate of $\mathbf{a}$ and $\mathbf{b}$ is 1). Now it is enough to see that the resulting coloring of B is an $R_{\mathbf{c}}$ coloring with $\mathbf{c} = \mathbf{a} + \mathbf{b} + \mathbf{e_1}$ (recall that $\mathbf{e_1}$ is the vector with all-0 coordinates except the first coordinate being 1). Take an arbitrary corner on its left face, $C(\mathbf{d})$ (thus $\mathbf{d}$ has first coordinate 0). In the coloring of B_1 its pair (the corner with the same color) is $C(\mathbf{d} + \mathbf{a})$. This is on the right face of B_1, and so it is fitted together with the corner $C(\mathbf{d} + \mathbf{a} + \mathbf{e_1})$ of B_2 on B_2's left face. By the $R_{\mathbf{b}}$-coloring of B_2 the corner $C(\mathbf{d} + \mathbf{a} + \mathbf{e_1} + \mathbf{b})$ has the same color. This is also the $C(\mathbf{d} + \mathbf{a} + \mathbf{e_1} + \mathbf{b})$ corner of B. This holds for any corner of B on its left side and symmetrically on its right side as well, and so this is indeed an $R_{\mathbf{c}}$-coloring of B. For an illustration for 3 dimensions see Figure 20. □

Lemma 4.14 (fitting together color-ranges). *For* $\mathbf{x}, \mathbf{x}', \mathbf{y} \neq \mathbf{0}$ *we have*

(a) $S_{\mathbf{0}} \cdot S_{\mathbf{0}} \rightarrow S_{\mathbf{e_1}}$*,*

(b) $S_{\mathbf{x}} \cdot S_{\mathbf{0}} \rightarrow S_{\mathbf{0}}$ *and* $S_{\mathbf{0}} \cdot S_{\mathbf{x}} \rightarrow S_{\mathbf{0}}$*,*

(c) $S_{\mathbf{x}} \cdot S_{\mathbf{y}} \rightarrow S_{\mathbf{e_1}}$*, if* $\mathbf{x}$ *and* $\mathbf{y}$ *differ somewhere which is not the first coordinate,*

(d) $S_{\mathbf{x}} \cdot S_{\mathbf{x}} \rightarrow S_{\mathbf{x}'}$*, if* $\mathbf{x}'$ *is the same as* $\mathbf{x}$ *with the possible exception at the first coordinate, which is 1 in* $\mathbf{x}'$*.*

(e) $S_{\mathbf{x}} \cdot S_{\mathbf{x}'} \rightarrow S_{\mathbf{x}}$ *and* $S_{\mathbf{x}'} \cdot S_{\mathbf{x}} \rightarrow S_{\mathbf{x}}$*, if* $\mathbf{x}'$ *is the same as* $\mathbf{x}$ *except at the first coordinate, which is 0 in* $\mathbf{x}$ *and 1 in* $\mathbf{x}'$*.*

Proof. Let us recall first that $S_{\mathbf{0}}$ is the one element set of $R_{\mathbf{0}}$ and for any $\mathbf{x} \neq \mathbf{0}$ $S_{\mathbf{x}}$ is defined as the union of all $R_{\mathbf{y}}$ for which $\mathbf{x} \cdot \mathbf{y} = 1$.

(a) by Lemma 4.13(a) $R_{\mathbf{0}} \cdot R_{\mathbf{0}} \rightarrow R_{\mathbf{c}}$ for any $\mathbf{c} \cdot \mathbf{e_1} = 1$.

(b) In $S_{\mathbf{x}}$ ($\mathbf{x} \neq \mathbf{0}$) there is always an $R_{\mathbf{a}}$ where the first coordinate of $\mathbf{a}$ is 1. By Lemma 4.13(b) $R_{\mathbf{a}} \cdot R_{\mathbf{0}} \rightarrow R_{\mathbf{0}}$. The proof for the other claim is similar.

(c) We need to prove that for any $R_{\mathbf{c}} \in S_{\mathbf{e_1}}$ ($\mathbf{c} \cdot \mathbf{e_1} = 1$) there is an $R_{\mathbf{a}} \in S_{\mathbf{x}}$ and $R_{\mathbf{b}} \in S_{\mathbf{y}}$ such that $R_{\mathbf{a}} \cdot R_{\mathbf{b}} \rightarrow R_{\mathbf{c}}$.

Suppose $\mathbf{x}$ and $\mathbf{y}$ differ in the kth coordinate ($k \neq 1$). Define X as the set of coordinates l where $x_l = 1$, and Y the set of coordinates l where $y_l = 1$. We want to apply Lemma 4.13(d) which is symmetric on $\mathbf{a}$ and $\mathbf{b}$ and so we can suppose that $k \notin X$ and $k \in Y$. The first coordinate of $\mathbf{c}$ is 1, so we choose $\mathbf{a}$ and $\mathbf{b}$ having the first coordinate 1 as well. We need that $\mathbf{a} + \mathbf{b} + \mathbf{e_1} = \mathbf{c}$ to be able to apply Lemma 4.13(d). First define the coordinates of $\mathbf{a}$ being in X all zero except one (this is the first if $1 \in X$, some other otherwise), thus by any choice of the other coordinates we will have $R_{\mathbf{a}} \in S_{\mathbf{x}}$. Now define the coordinates of $\mathbf{b}$ being in $X \setminus \{1\}$ such that $a_l + b_l = c_l$ for all $l \in X$. Define the rest of the coordinates of $\mathbf{b}$ such that $\mathbf{b} \cdot \mathbf{y} = 1$, this can be done as we can choose the kth coordinate as we want. Thus, $R_{\mathbf{b}} \in S_{\mathbf{y}}$ as well. Finally, choose the coordinates of $\mathbf{a}$ not in X such that $a_l + b_l = c_l$ for all $l \notin X$. This way $\mathbf{a} + \mathbf{b} + \mathbf{e_1} = \mathbf{c}$ as needed.

(d) We need to prove that for any $R_{\mathbf{c}} \in S_{\mathbf{x'}}$ there is an $R_{\mathbf{a}} \in S_{\mathbf{x}}$ and $R_{\mathbf{b}} \in S_{\mathbf{x}}$ such that $R_{\mathbf{a}} \cdot R_{\mathbf{b}} \rightarrow R_{\mathbf{c}}$.

First we prove the case when the first coordinate of $\mathbf{x}$ is 1 and so $\mathbf{x'} = \mathbf{x}$. For a $\mathbf{c}$ with first coordinate 0 by Lemma 4.13(c) we have $R_{\mathbf{c}} \cdot R_{\mathbf{c}} \rightarrow R_{\mathbf{c}}$, all in $S_{\mathbf{x}}$ as needed. For a $\mathbf{c}$ with first coordinate 1 take an arbitrary $\mathbf{a}$ with first coordinate 1 and $R_{\mathbf{a}} \in S_{\mathbf{x}}$. Choose $\mathbf{b}$ such that $\mathbf{a}+\mathbf{b}+\mathbf{e_1} = \mathbf{c}$ and so by Lemma 4.13(d) $R_{\mathbf{a}} \cdot R_{\mathbf{b}} \rightarrow R_{\mathbf{c}}$ holds. We need that $R_{\mathbf{b}}$ is in $S_{\mathbf{x}}$, which is true as $\mathbf{b} \cdot \mathbf{x} = (\mathbf{a} + \mathbf{c} + \mathbf{e_1}) \cdot \mathbf{x} = 1 + 1 + 1 = 1$.

Now we prove the case when the first coordinate of $\mathbf{x}$ is 0 and so $\mathbf{x'} = \mathbf{x}+\mathbf{e_1}$. For a $\mathbf{c}$ with first coordinate 0 by Lemma 4.13(c) we have $R_{\mathbf{c}} \cdot R_{\mathbf{c}} \rightarrow R_{\mathbf{c}}$, all in $S_{\mathbf{x}}$ and in $S'_{\mathbf{x}}$

too (as for such a $\mathbf{c}$ we have $\mathbf{c}\cdot\mathbf{x} = \mathbf{c}\cdot\mathbf{x}' = 1$). For a $\mathbf{c}$ with first coordinate 1 take an arbitrary $\mathbf{a}$ with first coordinate 1 and $R_{\mathbf{a}} \in S_{\mathbf{x}}$. Choose $\mathbf{b}$ such that $\mathbf{a}+\mathbf{b}+\mathbf{e_1} = \mathbf{c}$ and so by Lemma 4.13(d) $R_{\mathbf{a}} \cdot R_{\mathbf{b}} \to R_{\mathbf{c}}$ holds. We need that $R_{\mathbf{b}}$ is in $S_{\mathbf{x}}$, which is true as $\mathbf{c}\cdot\mathbf{x}' = 1$, $\mathbf{c}\cdot\mathbf{e_1} = 1$ and so $\mathbf{b}\cdot\mathbf{x} = (\mathbf{a}+\mathbf{c}+\mathbf{e_1})\cdot\mathbf{x} = 1+\mathbf{c}\cdot(\mathbf{x}'+\mathbf{e_1})+0 = 1$.
(e) For $S_{\mathbf{x}} \cdot S_{\mathbf{x}'} \to S_{\mathbf{x}}$ we need to prove that for any $R_{\mathbf{c}} \in S_{\mathbf{x}}$ there is an $R_{\mathbf{a}} \in S_{\mathbf{x}}$ and $R_{\mathbf{b}} \in S_{\mathbf{x}'}$ such that $R_{\mathbf{a}} \cdot R_{\mathbf{b}} \to R_{\mathbf{c}}$.
For a $\mathbf{c}$ with first coordinate 0 by Lemma 4.13(c) we have $R_{\mathbf{c}}\cdot R_{\mathbf{c}} \to R_{\mathbf{c}}$, all in $S_{\mathbf{x}}$ and in $S'_{\mathbf{x}}$ too (as for such a $\mathbf{c}$ we have $\mathbf{c}\cdot\mathbf{x} = \mathbf{c}\cdot\mathbf{x}' = 1$). For a $\mathbf{c}$ with first coordinate 1 take an arbitrary $\mathbf{a}$ with first coordinate 1 and $R_{\mathbf{a}} \in S_{\mathbf{x}}$. Again, choose $\mathbf{b}$ such that $\mathbf{a}+\mathbf{b}+\mathbf{e_1} = \mathbf{c}$ and so by Lemma 4.13(d) $R_{\mathbf{a}} \cdot R_{\mathbf{b}} \to R_{\mathbf{c}}$ holds. We need that $R_{\mathbf{b}}$ is in $S'_{\mathbf{x}}$, which is true as $\mathbf{b}\cdot\mathbf{x}' = (\mathbf{a}+\mathbf{c}+\mathbf{e_1})\cdot\mathbf{x}' = (\mathbf{a}+\mathbf{c}+\mathbf{e_1})\cdot(\mathbf{x}+\mathbf{e_1}) = \mathbf{a}\cdot\mathbf{x}+\mathbf{c}\cdot\mathbf{x}+\mathbf{e_1}\cdot\mathbf{x}+\mathbf{a}\cdot\mathbf{e_1}+\mathbf{c}\cdot\mathbf{e_1}+\mathbf{e_1}\cdot\mathbf{e_1} = 1+1+0+1+1+1 = 1$.
As Lemma 4.13(d) is symmetric on $\mathbf{a}$ and $\mathbf{b}$, $S_{\mathbf{x}'}\cdot S_{\mathbf{x}} \to S_{\mathbf{x}}$ follows the same way. □

The Lemmas above conclude the proof of Theorem 4.12. □

Assuming we know the cut-structure of the partition, the proof yields a simple linear time algorithm (in the number of cuts, regarding the dimension n as a fixed constant) to give a strong hyperbox-respecting coloring. First we determine the color-ranges and then the colorings of the hyperboxes using the lemmas. We will sketch how to do that.
First we construct the rooted binary tree with its root on the top representing our guillotine-cuts (each node corresponds to a hyperbox, the leaves are the basic boxes, the root is the original hyperbox). From bottom to top we can determine for each node v the unique $\mathbf{s}(v)$ for which the corresponding hyperbox will have color-range $S_{\mathbf{s}(v)}$ (leaves have color-range $S_{\mathbf{0}}$, then it is easy to determine the rest going upwards using Lemma 4.14). Now from top to bottom we can give appropriate $R_{\mathbf{y}}$-colorings to the hyperboxes. For the root w give arbitrary $R_{\mathbf{r}(w)}$-coloring with $\mathbf{r}(w) \in S_{\mathbf{s}(w)}$. Then by induction if we gave an $R_{\mathbf{r}(w)}$-coloring

($\mathbf{r}_w \in S_{\mathbf{s}(w)}$) to some hyperbox corresponding to the node w with children u and v then by Lemma 4.14 there exists $\mathbf{r}(u) \in S_{\mathbf{s}(u)}$ and $\mathbf{r}(v) \in S_{\mathbf{s}(v)}$ such that an $R_{\mathbf{r}(u)}$ and an $R_{\mathbf{r}(v)}$ can be put together (at the appropriate face) to form an $R_{\mathbf{r}(w)}$-coloring. Such colorings can be found in the same way as in the proof of Lemma 4.14. Thus, we can give such colorings to the hyperboxes corresponding to u and v. Finishing the coloring this way the basic boxes will have $R_{\mathbf{0}}$-colorings, i.e. the coloring will be a strong hyperbox-respecting coloring.

It is easy to see that using this algorithm any $S_{\mathbf{x}}$ color-range can appear with appropriate cuts.

It was observed by D. Dimitrov and R. Skrekovski [19] using a double-counting argument that when a (not necessary guillotine) partition contains an odd number of basic hyperboxes then a coloring of it must have all the corners colored differently. From Lemma 4.13 one can easily deduce that when the partition contains an odd number of basic hyperboxes then our algorithm will give an $R_{\mathbf{0}}$-coloring thus having all corners colored differently indeed. Further it was also observed that when a partition contains an even number of basic hyperboxes then all the colors appear pair times on the corners of the hyperbox. In the even case our algorithm will give an $R_{\mathbf{a}}$-coloring with $\mathbf{a} \neq \mathbf{0}$ thus having all colors appearing zero times or twice on the corners.

As mentioned in the Introduction, the general case is solved in 2-dimensions, but it is still unknown for which other dimensions can it hold.

Problem 4.15. *For which $n > 2$ do exist a strong hyperbox-respecting coloring of any n-dimensional partition.*

5 Concluding remarks

In this thesis we regarded three problems of combinatorial geometry, these are conflict-free colorings of point sets in the plane with few colors, polygonalizations of point sets with few reflex points and polychromatic colorings of points sets in the plane and in higher dimensions. The first has roots in the topic of multiple coverings of the plane and their decomposability, the second originates from the classical problem of Erdős and Szekeres about finding big convex chains in planar point sets and the last regards a natural coloring number of point sets, thus having roots in the theory of chromatic numbers. Historically these topics are only loosely connected, yet there are many similarities between the methods applied during the proofs. The majority of the proofs gave efficient algorithms.

In Chapter 2 by extending earlier definitions about cover-decomposability, conflict-free coloring number and geometric hypergraph coloring we defined the weak conflict-free coloring number and then solved many questions about it. We gave complete answer about the weak conflict-free coloring number of points wrt. bottomless rectangles, solving the dual problem also. Further, we gave almost complete answer for half-planes, again in the dual case as well. The cases of discs and axis-parallel rectangles have still many open problems to answer.

In Chapter 3 we improved the earlier bound on the reflexivity of point sets significantly utilizing a subdivision technique which may be used in other areas as well. Besides the problem of improving further the upper or lower bounds, several other relevant questions were posed.

In Chapter 4 we regarded the natural notion of polychromatic colorings of plane graphs for the special case of rectangular partitions. First we proved that a rectangular partition always admits a polychromatic 4-coloring proving a stronger statement, i.e. that it admits a strong rectangle-respecting 4-coloring as well. Further, we regarded general rectangular partitions and among others we proved that not all general partitions admit a polychromatic 4-coloring, yet they always admit a polychromatic 3-coloring. Generalizing these notions to higher dimensions, we gave colorings in the strong sense with the highest possible number of colors for n-dimensional guillotine partitions.

Appendix

The reflexivity of point sets with at most 8 points

Lemma 5.4. *Let S be a set of at most 8 points in the plane. Then $\bar{\rho}(S) \leq 2$.*

Proof. Given a set of n points S', the algorithm in the proof of Theorem 3.1 in [9] generates a polygonalization of S' with at most $\lceil n_I/2 \rceil$ reflex vertices, where n_I is the number of points in S' that are internal points of $\mathcal{CH}(S')$. Moreover, this algorithm begins with fixing one edge of the boundary of $\mathcal{CH}(S')$ (the edge p_0p_1 in [9]'s notation), and one can observe that this edge is an edge of the resulting polygonalization when the algorithm terminates. Therefore, it is enough to consider the case in which $|S| = 8$ and the boundary of $\mathcal{CH}(S)$ is a triangle.

Let $\mathcal{CH}_i(S)$ denote the ith layer in the "onion peeling" of S. More precisely, set $\mathcal{CH}_0(S) = \mathcal{CH}(S)$, and let $\mathcal{CH}_i(S)$ be the convex hull of $S \setminus \{p \in S \mid p \text{ is a vertex of the boundary of } \mathcal{CH}_j(S), 0 \leq j < i\}$. We say that a point p outside of $\mathcal{CH}_i(S)$ *sees* a vertex q of the boundary of $\mathcal{CH}_i(S)$ if the segment pq does not cross $\mathcal{CH}_i(S)$.

Assume that the fixed edge of the boundary of $\mathcal{CH}(S)$ is $e = (A, B)$, such that e is on the x-axis, A is left of B, and let C be the third vertex of the boundary of $\mathcal{CH}(S)$. Consider the lines determined by C and each of the internal points. Let p_0 be the point that determines the line with smallest slope. Clearly, p_0 is a vertex of the boundary of $\mathcal{CH}_1(S)$. Let $p_1, p_2, \ldots, p_k$ be the remaining vertices of the boundary of $\mathcal{CH}_1(S)$ in a clockwise order around $\mathcal{CH}_1(S)$. Denote by p_i the point that determines (along with C) the line with the largest slope, and by p_l the lowest point in $\mathcal{CH}_1(S)$. (Note that it is possible that $p_l = p_0$ or $p_l = p_i$.) The following two observations are easy.

Observation 5.5. *The point A (resp., B) sees all the vertices on the boundary of $\mathcal{CH}_1(S)$ along the clockwise (resp., counter-clockwise) chain from p_l to p_i (resp.,*

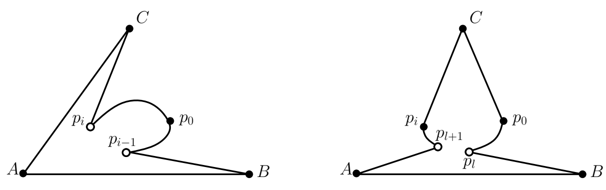


Figure 21: The case of 5 vertices on the boundary of $\mathcal{CH}_1(S)$

p_0).

Proof. Follows from convexity. □

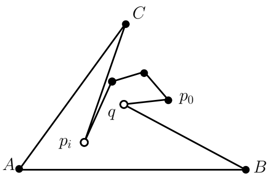
Observation 5.6. *If $i \leq 3$ then A sees p_1 or B sees p_{i-1}.*

Proof. Since $i \leq 3$ we have $l \in \{0, 1, i-1, i\}$. If $l \in \{1, i-1\}$, then, since both A and B see p_l we are done. Otherwise, suppose that $l = 0$. Then, by the previous observation, A sees all the vertices on the clockwise chain from p_0 to p_i. Similarly, B sees this chain in case $l = i$. □

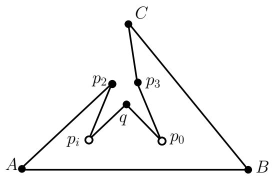
We proceed proving Lemma 3.5 by case analysis, based on the number of vertices on the boundary of $\mathcal{CH}_1(S)$.

Case 1: There are 5 vertices on the boundary of $\mathcal{CH}_1(S)$. We consider two subcases: (a) Suppose that $i \leq 3$. Then by Observation 5.6 A sees p_1 or B sees p_{i-1}. Assume, w.l.o.g., that B sees p_{i-1}. Then we draw the desired polygon as in Figure 21(a). (b) Suppose that $i = 4$. Then $p_l \neq p_0$ or $p_l \neq p_i$. Assume, w.l.o.g., that $p_l \neq p_i$. Then by Observation 5.5 A sees p_{l+1}. The polygon $A\,p_{l+1}\,\ldots\,p_i\,C\,p_0\,\ldots\,p_l\,B\,A$ is the desired polygon (see Figure 21(b)).

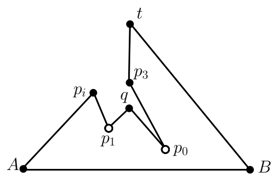
Case 2: There are 4 vertices on the boundary of $\mathcal{CH}_1(S)$. Let q be the single vertex on the boundary of $\mathcal{CH}_2(S)$. We consider the different subcases, based on the value of i. (a) Suppose $i = 1$. If Aq or Bq cross p_0p_1, then we can draw the desired polygon as in Figure 22(a). Otherwise, A sees the vertex p_2 of



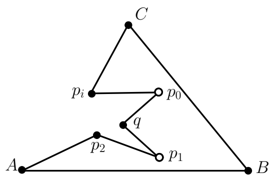
(a) $i = 1$, Bq crosses p_0p_1



(b) $i = 1$, p_0p_1 is not crossed by Aq or Bq



(c) $i = 2$



(d) $i = 3$

Figure 22: The case of 4 vertices on the boundary of $\mathcal{CH}_1(S)$

the boundary of $\mathcal{CH}_1(S)$ (and B sees p_3), and we can draw the polygon as in Figure 22(b). (b) Suppose $i = 2$. The vertex p_1 of the pentagon $A\,B\,p_0\,p_1\,p_2$ is reflex. Thus at most one of the vertices p_0 and p_2 of this pentagon is reflex (a pentagon has at most two reflex vertices). Assume, w.l.o.g., $\angle p_1p_2A$ is less than π, then we can draw the polygon as in Figure 22(c). (c) Suppose $i = 3$. Then at most one of vertices p_1 and p_2 of the quadrangle $B\,p_1\,p_2\,A$ is reflex. Assume, w.l.o.g., that $\angle p_1p_2A$ is less than π. Then we draw the polygon as in Figure 22(d). (Note that if A does not see p_2, then p_3 is below the segment Ap_2 and therefore $\angle p_1p_2A$ is greater than π.)

Case 3: There are 3 vertices on the boundary of $\mathcal{CH}_1(S)$. If $i = 2$ then by Observation 5.6 A or B sees p_1. If $i = 1$, then A and B sees p_0 or p_1. Hence, A sees p_{i-1} or B sees p_1. We assume, w.l.o.g., that A sees p_{i-1}. Then we have the chain $p_i\,C\,B\,A\,p_{i-1}$. It remains to connect p_{i-1} to p_i through p_{i+1} (addition is modulo 3) and the two vertices q_1, q_2 of the boundary of $\mathcal{CH}_2(S)$. Let q_1 be the point such that $\angle p_{i-1}p_{i+1}q_1 < \angle p_{i-1}p_{i+1}q_2$. Then the chain $p_{i-1}\,q_1\,p_{i+1}\,q_2\,p_i$

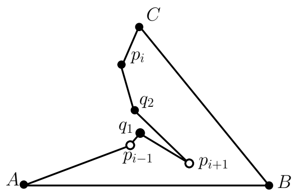


Figure 23: The case of 3 vertices on the boundary of $\mathcal{CH}_1(S)$

completes the desired polygon (see Figure 23). □

Putting together color-ranges along a general axis

Define $\circ^i$ as the function on the 0-1 vectors which exchanges the first and the ith coordinates, i.e. for a vector $\mathbf{x}$ the vector $\mathbf{x}^i$ has the same coordinates except that $x_1^i = x_i$ and $x_i^i = x_1$ (thus $\circ^{ii}$ is the identity and $\circ^i$ is a bijection). For vectors corresponding to corners of a hyperbox this is a reflection on an $(n-1)$-dimensional hyper-plane going through the corners having the same first and ith coordinate. Clearly, applying $\circ^i$ on an $R_\mathbf{x}$-coloring of the corners we get an $R_{\mathbf{x}^i}$-coloring of the corners. Lemma 5.7 states that the color-ranges $S_\mathbf{x}$ and $S_\mathbf{y}$ can be put together along the ith axis to give the color-range $S_\mathbf{z}$ if the color-ranges $S_{\mathbf{x}^i}$ and $S_{\mathbf{y}^i}$ can be put together along the first axis to give the color range $S_{\mathbf{z}^i}$. We have seen this can be done for any $\mathbf{x}^i$ and $\mathbf{y}^i$ with some $\mathbf{z}^i$, thus fitting along any other axis is also possible.

Lemma 5.7 (fitting together along a general axis). *If the color-ranges $S_{\mathbf{x}^i}$ and $S_{\mathbf{y}^i}$ can be put together along the first axis to give the color range $S_{\mathbf{z}^i}$ then the color-ranges $S_\mathbf{x}$ and $S_\mathbf{y}$ can be put together along the ith axis to give the color-range $S_\mathbf{x}$.*

Proof. First we prove that if $R_{\mathbf{a}^i} \cdot R_{\mathbf{b}^i} \to R_{\mathbf{c}^i}$ for some $\mathbf{c}$ then an appropriate $R_\mathbf{a}$-coloring and $R_\mathbf{b}$-coloring can be put together by the ith axis to form an $R_\mathbf{c}$-coloring. For that take an $R_{\mathbf{a}^i}$-coloring and an $R_{\mathbf{b}^i}$-coloring which fit together along the first axis to form an $R_{\mathbf{c}^i}$-coloring. Apply $\circ^i$ on these colorings. The original ones had the same colors on the pair of corners $C(\mathbf{v})$ on the first one and $C(\mathbf{v}+\mathbf{e_1})$ on the second one for arbitrary $\mathbf{v}$ having first coordinate 1. Thus after applying $\circ^i$ their images, the pair of corners $C(\mathbf{w})$ and $C(\mathbf{w}+\mathbf{e}_i)$ ($\mathbf{e}_i$ is the vector with all-0 coordinates except the ith coordinate being 1), will have the same colors for arbitrary $\mathbf{w}$ with ith coordinate 1 and so we can put together the two colorings along the ith axis.

By assumption when putting together along the first axis, the result was an $R_{\mathbf{c}^i}$-coloring. If $\mathbf{c} = \mathbf{c}^i = \mathbf{0}$ then it had all different colors on its corners, thus the same is true after applying $\circ^i$ and putting together along the ith axis, so the result is indeed an $R_{\mathbf{c}}$-coloring.

Otherwise if $\mathbf{c}^i$ has first coordinate 0 then on the $R_{\mathbf{a}^i}$-coloring the corners $C(\mathbf{v})$ and $C(\mathbf{v}+\mathbf{c}^i)$ had the same colors for any $\mathbf{v}$ with first coordinate 0 and on the $R_{\mathbf{b}^i}$-coloring the corners $C(\mathbf{w})$ and $C(\mathbf{w}+\mathbf{c}^i)$ had the same colors for any $\mathbf{w}$ with first coordinate 1. Thus after applying $\circ^i$, the corners $C(\mathbf{v})$ and $C(\mathbf{v}+\mathbf{c})$ of the $R_{\mathbf{a}}$-coloring have the same colors for any $\mathbf{v}$ with ith coordinate 0 and the corners $C(\mathbf{w})$ and $C(\mathbf{w}+\mathbf{c})$ of the $R_{\mathbf{b}}$-coloring have the same colors for any $\mathbf{w}$ with ith coordinate 1. As in this case the ith coordinate of $\mathbf{c}$ is 0, the resulting coloring after fitting these two together along the ith axis is indeed an $R_{\mathbf{c}}$-coloring.

If $\mathbf{c}^i$ has first coordinate 1 then the corner $C(\mathbf{v})$ of the $R_{\mathbf{a}^i}$-coloring and the corner $C(\mathbf{v}+\mathbf{c}^i)$ of the $R_{\mathbf{b}^i}$-coloring had the same color for any $\mathbf{v}$ with first coordinate 0. Thus after applying $\circ^i$, the corners $C(\mathbf{v})$ of the $R_{\mathbf{a}}$-coloring and the corner $C(\mathbf{v}+\mathbf{c})$ of the $R_{\mathbf{b}}$-coloring have the same colors for any $\mathbf{v}$ with ith coordinate 0. Putting these together along the ith axis gives indeed an $R_{\mathbf{c}}$-coloring.

Finally, back to the hyperboxes colorable with color-ranges $S_{\mathbf{x}}$ and $S_{\mathbf{y}}$ which need to be put together along the ith axis, applying $\mathbf{x}^i$ on all the colorings of $S_{\mathbf{x}}$ we get $S_{\mathbf{x}^i}$ and similarly from $S_{\mathbf{y}}$ we get the color-range $S_{\mathbf{y}^i}$ and we can put these together by the first axis to get the color-range $S_{\mathbf{z}^i}$ for some $\mathbf{z}$ and so $S_{\mathbf{x}}$ and $S_{\mathbf{y}}$ can be put together by the ith axis to get the color-range $S_{\mathbf{z}}$. □

References

[1] E. Ackerman, personal communication (2007)

[2] E. Ackerman, G. Barequet, R. Y. Pinter and D. Romik, The Number of Guillotine Partitions in d Dimensions, *Inf. Process. Lett.* **98(4)** (2006), 162–167.

[3] E. Ackerman, O. Aichholzer, B. Keszegh, Improved Upper Bounds on the Reflexivity of Point Sets, *Proc. 19th Canadian Conf. on Computational Geometry, Carleton University, Ottawa, Canada* (2007), 29–32.

[4] O. Aichholzer, F. Hurtado, and M. Noy, A lower bound on the number of triangulations of planar point sets. *Computational Geometry: Theory and Applications*, **29(2)** (2004), 135–145.

[5] O. Aichholzer and H. Krasser, The point set order type data base: A collection of applications and results, *Proc. 13th Canadian Conf. on Computational Geometry*, Waterloo, Ontario, Canada (August 2001), 17–20.

[6] O. Aichholzer and H. Krasser, Abstract order type extension and new results on the rectilinear crossing number, *Computational Geometry: Theory and Applications*, (2006) **36(1)**, 2–15.

[7] D. Ajwani, K. Elbassioni, S. Govindarajan, S. Ray, Conflict-Free Coloring for Rectangle Ranges Using $\tilde{O}(n^{.382+\epsilon})$ Colors, *Proceedings of the 19th annual ACM symposium on Parallel algorithms and architectures* (2007), 181–187.

[8] N. Alon, R. Berke, K. Buchin, M. Buchin, P. Csorba, S. Shannigrahi, B. Speckmann, Ph. Zumstein, Polychromatic Colorings of Plane Graphs, *24th Annual ACM Symposium on Computational Geometry* (2008)

[9] E. M. Arkin, S. P. Fekete, F. Hurtado, J. S. B. Mitchell, M. Noy, V. Sacristán, and S. Sethia, On the reflexivity of point sets, In B. Aronov, S. Basu, J. Pach, and M. Sharir, editors, *Discrete and Computational Geometry: The Goodman-Pollack Festschrift*, Springer (2003), 139–156.

[10] P. Bose, On embedding an outer-planar graph in a point set, *Comput. Geom. Theory Appl.* **23** (2002), 303–312.

[11] P. Bose, D. Kirkpatrick, Z. Li, Worst-case-optimal algorithms for guarding planar graphs and polyhedral surfaces, *Comput. Geom. Theory Appl.* **26(3)** (2003), 209–219.

[12] P. Brass, W. Moser, J. Pach, *Research Problems in Discrete Geometry*, Springer (2005)

[13] G. Brodal and R. Jacob, Dynamic planar convex hull, *Proc. 43rd IEEE Symp. on Foundations of Computer Science*, Vancouver, BC, Canada (November 2002), 617–626.

[14] N. Castañeda, J. Urrutia, Straight line embeddings of planar graphs on point sets, *Proc. Eighth Canadian Conf. on Comp. Geom.* (1996), 312–318.

[15] B. Chazelle Computational geometry and convexity, Ph.D. thesis, Dept. Comput. Sci., Yale Univ., New Haven, CT, Carnegie-Mellon Univ. Report CS-80-150. (1979)

[16] X. Chen, J. Pach, M. Szegedy, G. Tardos, Delaunay graphs of point sets in the plane with respect to axis-parallel rectangles, *SODA '08: Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms* (2008), 94–101.

[17] E. Demaine, J. S. B. Mitchell, and J. O'Rourke, Problem 66: Reflexivity of Point Sets, *The Open Problems Project*, `http://maven.smith.edu/~orourke/TOPP/P66.html`

[18] D. Dimitrov, E. Horev and R. Krakovski, Polychromatic 4-coloring of Rectangular Partitions, *24th European Workshop on Computational Geometry* (2008)

[19] D. Dimitrov, personal communication (2007)

[20] Y. Dinitz, M. J. Katz, R. Krakovski, Guarding rectangular partitions, *23rd European Workshop on Computational Geometry*, (2007)

[21] P. Erdős, G. Szekeres, A combinatorial problem in geometry, *Compositio Math.*, **2** (1935), 463–470.

[22] P. Erdős, G. Szekeres, On some extremum problems in elementary geometry, *Ann. Univ. Sci. Budapest*, **3-4** (1960), 53–62.

[23] G. Even, Z. Lotker, D. Ron, S. Smorodinsky, Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks, *SIAM J. Comput.* **33(1)** (2003), 94–136.

[24] D. Gerbner, B. Keszegh, N. Lemons, C. Palmer, B. Patkós, D. Pálvölgyi, Polychromatic colorings of arbitrary rectangular partitions, manuscript

[25] P. Gritzmann, B. Mohar, J. Pach, R. Pollack, Embedding a planar triangulation with vertices at specified points (solution to problem e3341), *Amer. Math. Monthly* **98** (1991) 165–166.

[26] B. Guenin, Edge coloring plane regular multigraphs, manuscript

[27] S. Har-Peled, S. Smorodinsky, On Conflict-free Coloring of Points and Simple Regions in the Plane, *Discrete & Comput. Geom (DCG)* **34** (2005), 47–70.

[28] E. Horev, M. J. Katz, R. Krakovski, Polychromatic colorings of cubic bipartite plane graphs, submitted (2007)

[29] E. Horev, M. J. Katz, R. Krakovski, M. Löffler, Polychromatic 4-colorings of guillotine subdivision, submitted (2007)

[30] E. Horev, R. Krakovski, Polychromatic colorings of bounded degree plane graphs, submitted (2007)

[31] K. Hosono, M. Urabe On the number of disjoint convex quadrilaterals for a plannar point set, *Comp. Geom. Theory Appl.* **20** (2001), 97–104.

[32] B. Keszegh, Weak conflict-free colorings of point sets and simple regions, *Proc. 19th Canadian Conf. on Computational Geometry, Carleton University, Ottawa, Canada* (2007), 97–100.

[33] B. Keszegh, Polychromatic Colorings of n-dimensional Guillotine-Partitions, *Computing and Combinatorics, 14th Annual International Conference, Proceedings, Lecture Notes in Computer Science*, Springer **5092** (2008), 110–118.

[34] J. M. Keil, Polygon decomposition, *Handbook of Computational Geometry*, J.-R. Sack and J. Urrutia, editors, Elsevier Science Publishers B.V. North-Holland, Amsterdam (2000), 491–518.

[35] P. Mani-Levitska, J. Pach, Decomposition problems for multiple coverings with unit balls, manuscript (1987)

[36] B. Mohar, R. Škrekovski, The Grötzsch theorem for the hypergraph of maximal cliques, *Electr. J. Comb.* **R26** (1999) 1–13.

[37] J. Pach, Covering the Plane with Convex Polygons, *Discrete and Computational Geometry* **1** (1986), 73–81.

[38] J. Pach, editor Special Issue Dedicated to Paul Erdős, *Discrete Comput. Geom.* volume 19 (1998)

[39] J. Pach, G. Tardos, Coloring axis-parallel rectangles, *Computational Geometry and Graph Theory (KyotoCGGT2007)* (2007)

[40] J. Pach, G. Tardos and G. Tóth, Indecomposable coverings, *Discrete Geometry, Combinatorics and Graph Theory, The China–Japan Joint Conference (CJCDGCGT 2005), Lecture Notes in Computer Science*, Springer **4381** (2007), 135–148.

[41] J. Pach, G. Tóth, Conflict free colorings, *Discrete and Computational Geometry - The Goodman-Pollack Festschrift (S. Basu et al, eds.)*, Springer Verlag, Berlin (2003), 665–671.

[42] D. Pálvölgyi, Indecomposable coverings with concave polygons, manuscript (2008)

[43] D. Pálvölgyi, G. Tóth, Decomposable coverings with convex polygons, manuscript (2008)

[44] P. D. Seymour, On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte, *London Math Soc.* **38** (1979), 423–460.

[45] S. Smorodinsky, Combinatorial Problems in Computational Geometry, *PhD thesis, School of Computer Science, Tel-Aviv University, Tel-Aviv, Israel* (2003)

[46] S. Smorodinsky, On The Chromatic Number of Some Geometric Hypergraphs, *Proc. of 17th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)* (2006)

[47] G. Tardos, G. Tóth, Multiple coverings of the plane with triangles, *Discrete and Computational Geometry* **38(2)** (2007), 443–450.

[48] M. Urabe, On a partition into convex polygons, *Discrete Appl. Math.*, **64** (1996), 179–191.

[49] M. Urabe, On a partition of point sets into convex polygons, *Proc. 9th Canad. Conf. Comput. Geom.* (1997), 21–24.