Economic Optimization Problems
via
Riemann-Finsler Geometry

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Submitted to
Department of Mathematics and its Applications
Central European University

In partial fulfilment of the requirements for the degree of
Doctor of Philosophy in Mathematics and its Applications

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Budapest, Hungary
2010

Abstract

The purpose of the present thesis is to study economic optimization problems within a non-linear, geometrical framework, motivated by various examples occuring in our daily life. In order to describe and explain such phenomena, we exploit elements from Riemann-Finsler geometry.

In the first chapter we recall notions and results from Riemann-Finsler geometry which will be used in the thesis. We also present new material on Busemann NPC spaces on Finsler manifolds which solves partially a fifty-year-old open problem of Busemann and Pedersen.

In the second chapter we study a general Weber location problem with unit weights on Finsler manifolds, i.e., to minimize the sum of distances (the cost of transportation) to the sample points situated on a not necessarily reversible Finsler manifold. Some existence, uniqueness and multiplicity results are presented in various geometrical contexts together with some practical examples.

In the third chapter existence and location of Nash-type equilibrium points are studied for a large class of finite families of payoff functions whose domains are not necessarily convex in the usual sense. The geometric idea is to embed these non-convex domains into suitable Riemannian manifolds, thus regaining certain geodesic convexity properties of them. By using variational inequalities, set-valued analysis, dynamical systems, and non-smooth calculus on Riemannian manifolds, various existence, location and stability results of Nash-type equilibrium points are derived. Some of the results can be obtained only on Hadamard manifolds as a curvature rigidity result shows.

Preface

It is common knowledge that the problem of minimization of functionals has always been present in the real world. Much attention has been paid over centuries to understanding and resolving such problems. The interest is to find those objects/points that make a functional to attain a local (or global) minimum value. One of the most known examples of a variational problem is to find the curve of minimal length that connects two given points among all curves which lie on a given surface and connect the two points; such a curve is called a geodesic. A physical phenomenon is the so-called Fermat's principle which says that light follows the shortest optical length between two given points.

The main purpose of the present thesis is to study economic optimization/minimization problems within a non-linear, geometrical framework. More explicitly, we are focusing to two classes of variational economics problems which are treated via Riemann-Finsler geometry: (a) Weber-type problems, and (b) Nash equilibrium problems. Although Riemann-Finsler geometry has not been and is still not as popular as Euclidean geometry, Riemann-Finsler geometry describes our world more precisely than any other reasonable geometry. Therefore, such developments are highly motivated from a practical point of view supported by various examples coming from our daily life.

The thesis contains three chapters; we roughly present them.

Chapter 1 (Foundational Material). We recall those notions and results from Riemann-Finsler geometry which will be used throughout the thesis, such as geodesics, flag curvature, properties of the metric projection operator, elements of non-smooth analysis on manifolds, and variational inequalities. Most of these results are well-known for specialists; however, we also present new material on Busemann NPC spaces on Finsler manifolds (see §1.2) which solves partially a more than fifty years old open problem formulated by H. Busemann and F. Pedersen. These results are based on the paper of Kristály and Kozma [22].

Chapter 2 (Weber-type problems on manifolds). We deal with a general Weber location problem with unit weights on Finsler manifolds, i.e., to minimize the sum of distances (the cost of transportation) to the sample points situated on a not necessarily reversible Finsler manifold. Therefore, Finsler distances are used to measure the cost of travel between some fixed markets and a central deposit which is going to be constructed. Some existence, uniqueness and multiplicity results are proved in various geometrical contexts (non-positively curved Berwald spaces, Minkowski spaces, Hadamard manifolds). We also present some concrete examples involving the slope metric of a hillside (thus, the transportation cost depends on the direction of travel due to the gravity) as well as the 'gravitational' Finslerian-Poincaré disc. The results of this chapter are based on the paper of Kristály, Moroşanu and Róth [24].

Chapter 3 (Nash-type equilibria on manifolds). The Nash equilibrium problem involves n players such that each player knows the equilibrium strategies of the partners, but moving away from his/her own strategy alone a player has nothing to gain. In this chapter we study the existence and location of Nash equilibrium points in a non-linear setting where the strategy sets of the players are not necessarily convex in the usual sense. The geometric idea is to embed these non-convex domains into suitable Riemannian manifolds regaining certain geodesic convexity properties of them. First, we guarantee the existence of Nash equilibrium points via McClendon-type variational inequalities in ANRs. Then, we introduce the concept of Nash-Stampacchia equilibrium points for a finite family of non-smooth functions defined on geodesic convex sets of certain Riemannian manifolds. Characterization, existence, and stability of Nash-Stampacchia equilibria are studied when the strategy sets are subsets of certain Hadamard manifolds, exploiting two well-known geometrical features of these spaces; namely, the obtuse-angle property and non-expansiveness of the metric projection operator for geodesic convex sets. These two properties actually characterize the non-positivity of the sectional curvature of complete and simply connected Riemannian spaces, delimiting the Hadamard manifolds as the optimal geometrical framework of Nash-Stampacchia equilibrium problems. Our analytical approach exploits various elements from set-valued analysis, dynamical systems, and non-smooth calculus on Riemannian manifolds. The results of this chapter are based on the author's papers [20], [21].

Acknowledgements.

I am extremely grateful to my advisor, Professor Gheorghe Moroşanu, for his kind support and professional advice during my PhD studies at the Central European University. I have also benefited from his kind assistance in solving real life problems.

I am indebted to Ágoston Róth and Csaba Varga who have provided generous support over the years.

I am also grateful to Elvira Kadvány for her constant assistance in solving various administrative problems.

Last, but not least, I want to express my appreciation to my wife, Tünde, for her different sort of support and patience, as well as to my children, Bora and Marót, for their extremely different sort of support.

Contents

Chapter 1

Foundational Material

What is now proved was once
only imagined.

William Blake (1757-1827)

1.1 Metrics, geodesics, flag curvature

1.1.1 Finsler manifolds

Let M be a connected m-dimensional C^∞ manifold and let $TM = \bigcup_{p\in M} T_pM$ be its tangent bundle. If the continuous function $F : TM \to [0,\infty)$ satisfies the conditions that it is C^∞ on $TM \setminus \{0\}$; $F(tu) = tF(u)$ for all $t \geq 0$ and $u \in TM$, i.e., F is positively homogeneous of degree one; and the matrix $g_{ij}(u) := [\frac{1}{2}F^2]_{y^iy^j}(u)$ is positive definite for all $u \in TM \setminus \{0\}$, then we say that (M, F) is a *Finsler manifold.* If F is absolutely homogeneous, then (M, F) is said to be *reversible.*

Let π^*TM be the pull-back of the tangent bundle TM by $\pi : TM \setminus \{0\} \to M$. Unlike the Levi-Civita connection in Riemann geometry, there is no unique natural connection in the Finsler case. Among these connections on π^*TM, we choose the *Chern connection* whose coefficients are denoted by Γ^k_{ij}; see Bao, Chern and Shen [3, p.38]). This connection induces the *curvature tensor*, denoted by R; see [3, Chapter 3]. The

Chern connection defines the *covariant derivative* $D_V U$ of a vector field U in the direction $V \in T_pM$. Since, in general, the Chern connection coefficients Γ^i_{jk} in natural coordinates have a directional dependence, we must say explicitly that $D_V U$ is defined with a fixed reference vector. In particular, let $\sigma : [0, r] \to M$ be a smooth curve with velocity field $T = T(t) = \dot\sigma(t)$. Suppose that U and W are vector fields defined along σ. We define $D_T U$ with *reference vector* W as

$$D_T U = \left[\frac{dU^i}{dt} + U^j T^k (\Gamma^i_{jk})_{(\sigma, W)} \right] \frac{\partial}{\partial p^i}_{|\sigma(t)},$$

where $\left\{ \frac{\partial}{\partial p^i}_{|\sigma(t)} \right\}_{i=1,m}$ is a basis of $T_{\sigma(t)}M$. A C^∞ curve $\sigma : [0, r] \to M$, with velocity $T = \dot\sigma$ is a (Finslerian) *geodesic* if

$$D_T \left[\frac{T}{F(T)} \right] = 0 \text{ with reference vector } T. \tag{1.1}$$

If the Finslerian velocity of the geodesic σ is constant, then (1.1) becomes

$$\frac{d^2\sigma^i}{dt^2} + \frac{d\sigma^j}{dt}\frac{d\sigma^k}{dt} (\Gamma^i_{jk})_{(\sigma, T)} = 0, \quad i = 1, ..., m = \dim M. \tag{1.2}$$

For any $p \in M$ and $y \in T_pM$ we may define the *exponential map* $\exp_p : T_pM \to M$, $\exp_p(y) = \sigma(1, p, y)$, where $\sigma(t, p, y)$ is the unique solution (geodesic) of the second order differential equation (1.1) (or, (1.2)) which passes through p at $t = 0$ with velocity y. Moreover,

$$d \exp_p(0) = \mathrm{id}_{T_pM}. \tag{1.3}$$

If U, V and W are vector fields along a curve σ, which has velocity $T = \dot\sigma$, we have the *derivative rule*

$$\frac{d}{dt} g_W(U, V) = g_W(D_T U, V) + g_W(U, D_T V) \tag{1.4}$$

whenever $D_T U$ and $D_T V$ are with reference vector W and *one* of the following conditions holds:

- U or V is proportional to W, or

• $W = T$ and σ is a geodesic.

A vector field J along a geodesic $\sigma : [0,r] \to M$ (with velocity field T) is said to be a *Jacobi field* if it satisfies the equation

$$D_T D_T J + R(J,T)T = 0, \tag{1.5}$$

where R is the curvature tensor. Here, the covariant derivative D_T is defined with reference vector T.

We say that q *is conjugate to* p along the geodesic σ if there exists a nonzero Jacobi field J along σ which vanishes at p and q.

Let $\gamma : [0,r] \to M$ be a piecewise C^∞ curve. Its *integral length* is defined as

$$L_F(\gamma) = \int_0^r F(\gamma(t), \dot\gamma(t))\, dt.$$

Let $\Sigma : [0,r] \times [-\varepsilon, \varepsilon] \to M$ ($\varepsilon > 0$) be a piecewise C^1 variation of a geodesic $\gamma : [0,r] \to M$ with $\Sigma(\cdot, 0) = \gamma$. Let $T = T(t,u) = \frac{\partial \Sigma}{\partial t}$, $U = U(t,u) = \frac{\partial \Sigma}{\partial u}$ the velocities of the t-*curves* and u-*curves*, respectively.

For $p, q \in M$, denote by $\Gamma(p,q)$ the set of all piecewise C^∞ curves $\gamma : [0,r] \to M$ such that $\gamma(0) = p$ and $\gamma(r) = q$. Define the map $d_F : M \times M \to [0, \infty)$ by

$$d_F(p,q) = \inf_{\gamma \in \Gamma(p,q)} L_F(\gamma). \tag{1.6}$$

Of course, we have $d_F(p,q) \geq 0$, where equality holds if and only if $p = q$, and the *triangle inequality* holds, i.e., $d_F(p_0,p_2) \leq d_F(p_0,p_1) + d_F(p_1,p_2)$ for every $p_0, p_1, p_2 \in M$. In general, since F is only a positive homogeneous function, $d_F(p,q) \neq d_F(q,p)$; thus, (M, d_F) is only a quasi-metric space. The geodesic segment $\gamma : [0,r] \to M$ is called minimizing if its integral length $L_F(\gamma)$ is not larger than the integral length of any other piecewise differentiable curve joining $\gamma(0)$ and $\gamma(r)$, i.e., $L_F(\gamma) = d_F(\gamma(0), \gamma(r))$.

If (M,g) is a Riemannian manifold, we will use the notation d_g instead of d_F which becomes a usual metric function.

For $p \in M$, $r > 0$, we define the *forward* and *backward Finsler-metric ball*s, respectively, with center $p \in M$ and radius $r > 0$, by

$$\mathcal{B}_F^+(p,r) = \{q \in M : d_F(p,q) < r\} \text{ and } \mathcal{B}_F^-(p,r) = \{q \in M : d_F(q,p) < r\}.$$

We denote by $B(p,r) := \{y \in T_pM : F(p,y) < r\}$ the open *tangent ball* at $p \in M$ with radius $r > 0$. It is well-known that the topology generated by the forward (resp. backward) metric balls coincide with the underlying manifold topology, respectively.

By Whitehead's theorem (see [3, Exercise 6.4.3, p. 164]) and [3, Lemma 6.2.1, p. 146] we can derive the following useful local result (see also [23]).

Proposition 1.1.1 *Let* (M,F) *be a Finsler manifold, where* F *is positively* (*but perhaps not absolutely*) *homogeneous of degree one. For every point* $p \in M$ *there exist a small* $\rho_p > 0$ *and* $c_p > 1$ (*depending only on* p) *such that for every pair of points* q_0, q_1 *in* $\mathcal{B}_F^+(p,\rho_p)$ *we have*

$$\frac{1}{c_p} d_F(q_1,q_0) \leq d_F(q_0,q_1) \leq c_p d_F(q_1,q_0). \tag{1.7}$$

Moreover, for every real number $k \geq 1$ *and* $q \in \mathcal{B}_F^+(p,\rho_p/k)$ *the mapping* $\exp_q$ *is* C^1*-diffeomorphism from* $B(q,2\rho_p/k)$ *onto* $\mathcal{B}_F^+(q,2\rho_p/k)$ *and every pair of points* q_0, q_1 *in* $\mathcal{B}_F^+(p,\rho_p/k)$ *can be joined by a unique minimal geodesic from* q_0 *to* q_1 *lying entirely in* $\mathcal{B}_F^+(p,\rho_p/k)$.

A set $M_0 \subseteq M$ is *forward bounded* if there exist $p \in M$ and $r > 0$ such that $M_0 \subseteq \mathcal{B}_F^+(p,r)$. Similarly, $M_0 \subseteq M$ is *backward bounded* if there exist $p \in M$ and $r > 0$ such that $M_0 \subseteq \mathcal{B}_F^-(p,r)$.

A set $M_0 \subseteq M$ is *geodesic convex* if for any two points of M_0 there exists a unique geodesic joining them which belongs entirely to M_0.

Let $(p,y) \in TM \setminus 0$ and let V be a section of the pulled-back bundle π^*TM. Then,

$$K_p(y,V) = \frac{g_{(p,y)}(R(V,y)y,V)}{g_{(p,y)}(y,y)g_{(p,y)}(V,V) - [g_{(p,y)}(y,V)]^2} \tag{1.8}$$

is the *flag curvature* with flag y and transverse edge V. Here,

$$g_{(p,y)} := g_{ij(p,y)} dp^i \otimes dp^j := [\frac{1}{2}F^2]_{y^iy^j} dp^i \otimes dp^j, \quad p \in M, \ y \in T_pM, \tag{1.9}$$

is the Riemannian metric on the pulled-back bundle π^*TM. In particular, when the Finsler structure F arises from a Riemannian metric g (i.e., the fundamental tensor $g_{ij} = [\frac{1}{2}F^2]_{y_i y_j}$ does not depend on the direction y), the flag curvature coincides with the usual *sectional curvature*.

If $K_p(V,W) \leq 0$ for every $0 \neq V, W \in T_pM$, and $p \in M$, with V and W not collinear, we say that the flag curvature of (M,F) is *non-positive*.

A Finsler manifold (M,F) is said to be *forward (resp. backward) geodesically complete* if every geodesic $\sigma : [0,1] \to M$ parameterized to have constant Finslerian speed, can be extended to a geodesic defined on $[0,\infty)$ (resp. $(-\infty,1]$). (M,F) is *geodesically complete* if every geodesic $\sigma : [0,1] \to M$ can be extended to a geodesic defined on $(-\infty,\infty)$. In the Riemannian case, instead of geodesically complete we simply say *complete Riemannian manifold*.

Theorem 1.1.1 (Theorem of Hopf-Rinow, [3, p. 168]) *Let (M,F) be a connected Finsler manifold, where F is positively (but perhaps not absolutely) homogeneous of degree one. The following two criteria are equivalent:*

(a) *(M,F) is forward (backward) geodesically complete;*

(b) *Every closed and forward (backward) bounded subset of (M,d_F) is compact.*

Moreover, if any of the above holds, then every pair of points in M can be joined by a minimizing geodesic.

Theorem 1.1.2 (Theorem of Cartan-Hadamard, [3, p. 238]) *Let (M,F) be a forward/backward geodesically complete, simply connected Finsler manifold of non-positive flag curvature. Then:*

(a) *Geodesics in (M,F) do not contain conjugate points.*

(b) *The exponential map $\exp_p : T_pM \to M$ is a C^1 diffeomorphism from the tangent space T_pM onto the manifold M.*

A Finsler manifold (M,F) is a *Minkowski space* if M is a vector space and F is a Minkowski norm inducing a Finsler structure on M by translation; its flag curvature is identically zero, the geodesics are straight lines, and for any two points $p, q \in M$, we have $F(q-p) = d_F(p,q)$, see Bao, Chern and Shen [3, Chapter 14]. In particular, (M, F) is both forward and backward geodesically complete.

A Finsler manifold is of *Berwald type* if the Chern connection coefficients Γ_{ij}^k in natural coordinates depend only on the base point. Special Berwald spaces are the (*locally*) *Minkowski space*s and the *Riemannian manifold*s. In the latter case, the Chern connection coefficients Γ_{ij}^k coincide the usual Christofel symbols

$$\overline{\Gamma}_{ij}^k(p) = \frac{1}{2}\left[\left(\frac{\partial g_{mj}}{\partial p_j}\right)_p + \left(\frac{\partial g_{mi}}{\partial p_j}\right)_p - \left(\frac{\partial g_{ij}}{\partial p_m}\right)_p\right] g^{mk}(p)$$

where the g^{ij}'s are such that $g_{im}g^{mj} = \delta_{ij}$.

1.1.2 Riemannian manifolds

Since every Riemannian manifold is a Finsler manifold, the results from the previous subsection are valid also for Riemannian manifolds. In this subsection we recall further elements from Riemannian geometry which will be used in the sequel and they are mainly typical features of Riemannian manifolds. We follow Cartan [9] and do Carmo [15].

Let (M, g) be a connected m-dimensional Riemannian manifold, $TM = \cup_{p\in M}(p, T_pM)$ and $T^*M = \cup_{p\in M}(p, T_p^*M)$ be the tangent and cotangent bundles to M. For every $p \in M$, the Riemannian metric induces a natural Riesz-type isomorphism between the tangent space T_pM and its dual T_p^*M; in particular, if $\xi \in T_p^*M$ then there exists a unique $W_\xi \in T_pM$ such that

$$\langle \xi, V\rangle_{g,p} = g_p(W_\xi, V) \text{ for all } V \in T_pM. \tag{1.10}$$

Instead of $g_p(W_\xi, V)$ and $\langle \xi, V\rangle_{g,p}$ we shall write simply $g(W_\xi, V)$ and $\langle \xi, V\rangle_g$ when no confusion arises. Due to (1.10), the elements ξ and W_ξ are identified. With the above notations, the norms on T_pM and T_p^*M are defined by

$$\|\xi\|_g = \|W_\xi\|_g = \sqrt{g(W_\xi, W_\xi)}.$$

Moreover, the generalized Cauchy-Schwartz inequality is also valid, saying that for every $V \in T_pM$ and $\xi \in T_p^*M$,

$$|\langle \xi, V\rangle_g| \leq \|\xi\|_g \|V\|_g. \tag{1.11}$$

Let $\xi_k \in T_{p_k}^*M$, $k \in \mathbb{N}$, and $\xi \in T_p^*M$. The sequence $\{\xi_k\}$ converges to ξ, denoted by $\lim_k \xi_k = \xi$, when $p_k \to p$ and $\langle \xi_k, W(p_k)\rangle_g \to \langle \xi, W(p)\rangle_g$ as $k \to \infty$, for every C^∞ vector field W on M.

Let $h : M \to \mathbb{R}$ be a C^1 functional at $p \in M$; the differential of h at p, denoted by $dh(p)$, belongs to T_p^*M and is defined by

$$\langle dh(p), V\rangle_g = g(\text{grad}h(p), V) \text{ for all } V \in T_pM.$$

If $(x^1, ..., x^m)$ is the local coordinate system on a coordinate neighborhood (U_p, ψ) of $p \in M$, and the local components of dh are denoted $h_i = \frac{\partial h}{\partial x_i}$, then the local components of grad$h$ are $h^i = g^{ij}h_j$. Here, g^{ij} are the local components of g^{-1}.

For every $p \in M$ and $r > 0$, we define the open ball of center $p \in M$ and radius $r > 0$ by $\mathcal{B}_g(p, r) = \{q \in M : d_g(p, q) < r\}$.

Let us denote by ∇ the unique natural covariant derivative on (M, g), also called the Levi-Civita connection. A vector field W along a C^1 path γ is called parallel when $\nabla_{\dot\gamma}W = 0$. A C^∞ parameterized path γ is a geodesic in (M, g) if its tangent $\dot\gamma$ is parallel along itself, i.e., $\nabla_{\dot\gamma}\dot\gamma = 0$.

A similar reason as in Proposition 1.1.1 shows that there exists an open (starlike) neighborhood $\mathcal{U}$ of the zero vectors in TM and an open neighborhood $\mathcal{V}$ of the diagonal $M \times M$ such that the exponential map $V \mapsto \exp_{\pi(V)}(V)$ is smooth and the map $\pi \times \exp : \mathcal{U} \to \mathcal{V}$ is a diffeomorphism, where π is the canonical projection of TM onto M. Moreover, for every $p \in M$ there exists a number $r_p > 0$ and a neighborhood $\tilde{U}_p$ such that for every $q \in \tilde{U}_p$, the map $\exp_q$ is a C^∞ diffeomorphism on $B(0, r_p) \subset T_qM$ and $\tilde{U}_p \subset \exp_q(B(0, r_p))$; the set $\tilde{U}_p$ is called a *totally normal neighborhood* of $p \in M$. In particular, it follows that every two points $q_1, q_2 \in \tilde{U}_p$ can be joined by a minimizing geodesic of length less than r_p. Moreover, for every $q_1, q_2 \in \tilde{U}_p$ we have

$$\| \exp_{q_1}^{-1}(q_2)\|_g = d_g(q_1, q_2). \tag{1.12}$$

We conclude this subsection by recalling a less used form of the *sectional curvature* by the so-called Levi-Civita parallelogramoid. Let $p \in M$ and $V_0, W_0 \in T_pM$ two vectors with $g(V_0, W_0) = 0$. Let $\sigma : [-\delta, 2\delta] \to M$ be the geodesic segment $\sigma(t) = \exp_p(tV_0)$ and W be the unique parallel vector field along σ with the initial data $W(0) = W_0$, the number $\delta > 0$ being small enough. For any $t \in [0, \delta]$, let $\gamma_t : [0, \delta] \to M$ be the geodesic $\gamma_t(u) = \exp_{\sigma(t)}(uW(t))$. Then, the sectional curvature of the two-dimensional subspace $S =$span$\{W_0, V_0\} \subset T_pM$ at the point $p \in M$ is given by

$$K_p(S) = \lim_{u,t\to 0} \frac{d_g^2(p, \sigma(t)) - d_g^2(\gamma_0(u), \gamma_t(u))}{d_g(p, \gamma_0(u)) \cdot d_g(p, \sigma(t))},$$

see Cartan [9, p. 244-245]. The infinitesimal geometrical object determined by the four points p, $\sigma(t)$, $\gamma_0(u)$, $\gamma_t(u)$ (with t, u small enough) is called the *parallelogramoid of Levi-Civita.*

A Riemannian manifold (M, g) is a *Hadamard manifold* if it is complete, simply connected and its sectional curvature is non-positive. In particular, on every Hadamard manifold (M, g), relation (1.12) is fulfilled for every $q_1, q_2 \in M$.

Theorem 1.1.3 [15, Lemma 3.1] *Let (M, g) be a Hadamard manifold and consider the geodesic triangle determined by vertices $a, b, c \in M$. If $\widehat{c}$ is the angle belonging to vertex c and if $A = d_g(b, c)$, $B = d_g(a, c)$, $C = d_g(a, b)$, then*

$$A^2 + B^2 - 2AB\cos\widehat{c} \le C^2.$$

1.2 Metric relations on NPC spaces: Busemann-type inequalities on Finsler manifolds

In the fourties, Busemann developed a synthetic geometry on metric spaces. In particular, he axiomatically elaborated a whole theory of non-positively curved metric spaces which have no differential structure a priori and they possess the essential qualitative geometric properties of Finsler manifolds. This notion of non-positive curvature requires that in small geodesic triangles the length of a side is at least the twice of the geodesic distance of the mid-points of the other two sides, see Busemann [7, p. 237].

To formulate in a precise way this notion, let (M, d) be a quasi-metric space and for every $p \in M$ and radius $r > 0$, we introduce the *forward* and *backward metric balls*

$$B_d^+(p,r) = \{q \in M : d(p,q) < r\} \text{ and } B_d^-(p,r) = \{q \in M : d(q,p) < r\}.$$

A continuous curve $\gamma : [a,b] \to M$ with $\gamma(a) = x$, $\gamma(b) = y$ is a *shortest geodesic*, if $l(\gamma) = d(x,y)$, where $l(\gamma)$ denotes the *generalized length* of γ and it is defined by

$$l(\gamma) = \sup\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < ... < t_n = b, \ n \in \mathbb{N}\}.$$

In the sequel, we always assume that the shortest geodesics are parametrized proportionally to arclength, i.e., $l(\gamma|_{[0,t]}) = tl(\gamma)$.

Remark 1.2.1 A famous result of Busemann and Meyer [8, Theorem 2, p. 186] from Calculus of Variations shows that the generalized length $l(\gamma)$ and the integral length $L_F(\gamma)$ of any (piecewise) C^1 curves coincide for Finsler manifolds. Therefore, the minimal Finsler geodesic and shortest geodesic notions coincide.

We say that (M, d) is a *locally geodesic* (*length*) *space* if for every point $p \in M$ there is a $\rho_p > 0$ such that for every two points $x, y \in B_d^+(p, \rho_p)$ there exists a shortest geodesic joining them.

Definition 1.2.1 *A locally geodesic space* (M, d) *is said to be a* Busemann non-positive curvature space (*shortly,* Busemann NPC space)*, if for every* $p \in M$ *there exists* $\rho_p > 0$ *such that for any two shortest geodesics* $\gamma_1, \gamma_2 : [0, 1] \to M$ *with* $\gamma_1(0) = \gamma_2(0) = x \in B_d^+(p, \rho_p)$ *and with endpoints* $\gamma_1(1), \gamma_2(1) \in B_d^+(p, \rho_p)$ *we have*

$$2d(\gamma_1(\frac{1}{2}), \gamma_2(\frac{1}{2})) \leq d(\gamma_1(1), \gamma_2(1)).$$

(*We shall say that* γ_1 *and* γ_2 *satisfy the* Busemann NPC inequality).

Let (M, g) be a Riemannian manifold and (M, d_g) the metric space induced by itself. In this context, the Busemann NPC inequality is well-known. Namely, we have

Proposition 1.2.1 [7, Theorem (41.6)] (M, d_g) *is a Busemann non-positive curvature space if and only if the sectional curvature of* (M, g) *is non-positive.*

However, the picture for Finsler spaces is not so nice as in Proposition 1.2.1 for Riemannian manifolds. To see this, we consider the Hilbert metric of the interior of a simple, closed and convex curve C in the Euclidean plane. In order to describe this metric, let $M_C \subset \mathbb{R}^2$ be the region defined by the interior of the curve C and fix $x_1, x_2 \in \mathrm{Int}(M_C)$. Assume first that $x_1 \neq x_2$. Since C is a convex curve, the straight line passing to the points x_1, x_2 intersects the curve C in two point; denote them by $u_1, u_2 \in C$. Then, there are $\tau_1, \tau_2 \in (0, 1)$ such that $x_i = \tau_i u_1 + (1 - \tau_i)u_2$ $(i = 1, 2)$. The *Hilbert distance* between x_1 and x_2 is

$$d_H(x_1, x_2) = \left| \log \left(\frac{1 - \tau_1}{1 - \tau_2} \cdot \frac{\tau_2}{\tau_1} \right) \right|.$$

We complete this definition by $d_H(x, x) = 0$ for every $x \in \mathrm{Int}(M_C)$. One can easily prove that $(\mathrm{Int}(M_C), d_H)$ is a metric space and it is a projective Finsler metric with constant flag curvature -1. However, due to Kelly and Straus, we have

Proposition 1.2.2 [19] *The metric space* $(\mathrm{Int}(M_C), d_H)$ *is a Busemann non-positive curvature space if and only if the curve* $C \subset \mathbb{R}^2$ *is an ellipse.*

This means that, although for Riemannian spaces the non-positivity of the sectional curvature and Busemann's curvature conditions are mutually equivalent, the non-positivity of the flag curvature of a *generic* Finsler manifold is not enough to guarantee Busemann's property. See also Figure 1.1.

Therefore, in order to obtain a characterization of Busemann's curvature condition for Finsler spaces, we have two possibilities:

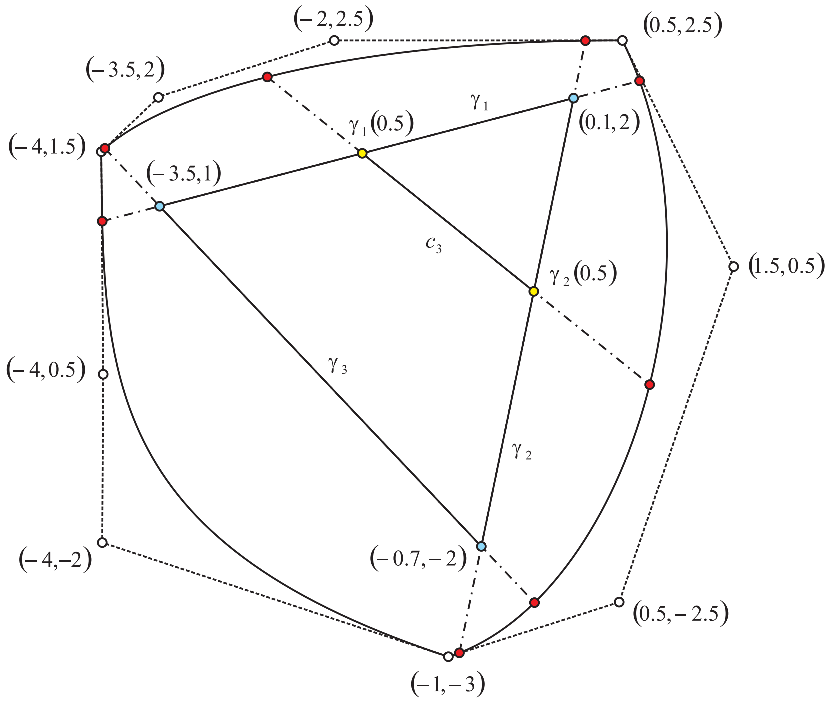

Figure 1.1: The closed convex curve is constructed by Bézier curves of degree 3, using 4 control points for each of its differentiable portions. In the geodesic triangle we have $L(\gamma_3) = d_H(\gamma_1(1), \gamma_2(1)) \approx 3.8692$ while $L(c_3) = d_H(\gamma_1(1/2), \gamma_2(1/2)) \approx 1.9432$. Thus, Busemann NPC inequality for γ_1 and γ_2 is not valid.

(I) To find a *new* notion of curvature in Finsler geometry such that for an arbitrary Finsler manifold the non-positivity of this curvature is equivalent with the Busemann non-positive curvature condition, as it was proposed by Shen [34, Open Problem 41]; or,

(II) To keep the flag curvature, but put some restrictive condition on the Finsler metric.

In spite of the fact that (reversible) Finsler manifolds are included in G-spaces, only few results are known which establish a link between the *differential invariants* of a Finsler manifold and the *metric properties* of the induced metric space. The main result of this section is due Kristály and Kozma [22] (see also Kristály, Kozma and Varga [23]), which makes a strong connection between an analytical property and a synthetic concept of non-positively curved metric spaces. Namely, we have

Theorem 1.2.1 ([22], [23]) *Let (M, F) be a Berwald space with non-positive flag curvature, where F is positively (but perhaps not absolutely) homogeneous of degree one. Then (M, d_F) is a Busemann NPC space.*

In view of Theorem 1.2.1, Berwald spaces seem to be the first class of Finsler metrics that are non-positively curved in the sense of Busemann and which are neither flat nor Riemannian. Moreover, the above result suggests a full characterization of the Busemann curvature notion for Berwald spaces. Indeed, we refer the reader to Kristály and Kozma [22] where the converse of Theorem 1.2.1 is also proved; here we omit this technical part since only the above result is applied for Economical problems.

Note that Theorem 1.2.1 includes a partial answer to the open question of Busemann [7] (see also Pederden [33, p. 87]), i.e., every reversible Berwald space of non-positive flag curvature has convex capsules (i.e., the loci equidistant to geodesic segments).

In the fifties, Aleksandrov introduced independently another notion of curvature in metric spaces, based on the convexity of the distance function. It is well-known that the condition of Busemann curvature is weaker than the Aleksandrov one, see Jost [17, Corollary 2.3.1]. Nevertheless, in Riemannian spaces the Aleksandrov curvature condition holds if and only if the sectional curvature is non-positive, see Bridson and Haefliger [6, Theorem 1A.6]), but in the Finsler case the picture is quite rigid. Namely, if on a reversible Finsler manifold (M, F) the Aleksandrov curvature condition holds (on the induced metric space by (M, F)) then (M, F) it must be Riemannian, see [6, Proposition 1.14].

It would be interesting to examine whether or not Theorem 1.2.1 works for *a larger class* of Finsler spaces than the Berwald ones, working in the (II) context. As far as the Busemann curvature condition is concerned, we believe that, as in the Aleksandrov case, we face a rigidity result. Namely:

Conjecture. *Let (M, F) be a Finsler manifold such that (M, d_F) is a Busemann NPC space. Then (M, F) is a Berwald space.*

Proof of Theorem 1.2.1. Let us fix $p \in M$ and consider $\rho_p > 0$, $c_p > 1$ from Proposition 1.1.1. We will prove that $\rho'_p = \frac{\rho_p}{c_p}$ is an appropriate choice in Definition 1.2.1. To do this, let $\gamma_1, \gamma_2 : [0, 1] \to M$ be two (minimal) geodesics with $\gamma_1(0) = \gamma_2(0) = x \in \mathcal{B}_F^+(p, \rho'_p)$ and $\gamma_1(1), \gamma_2(1) \in \mathcal{B}_F^+(p, \rho'_p)$. By Proposition 1.1.1, we can construct a unique geodesic $\gamma : [0, 1] \to M$ joining $\gamma_1(1)$ with $\gamma_2(1)$ and $d_F(\gamma_1(1), \gamma_2(1)) = L(\gamma)$. Clearly, $\gamma(s) \in \mathcal{B}_F^+(p, \rho'_p)$ for all $s \in [0, 1]$ (we applied Proposition 1.1.1 for $k = c_p$). Moreover, $x \in \mathcal{B}_F^+(\gamma(s), 2\rho_p)$. Indeed, by (1.7), we obtain

$$d_F(\gamma(s), x) \leq d_F(\gamma(s), p) + d_F(p, x) \leq c_p d_F(p, \gamma(s)) + \rho'_p \leq (c_p + 1)\rho'_p < 2\rho_p.$$

Therefore, we can define $\Sigma : [0, 1] \times [0, 1] \to M$ by

$$\Sigma(t, s) = \exp_{\gamma(s)}((1 - t) \cdot \exp^{-1}_{\gamma(s)}(x)).$$

The curve $t \mapsto \Sigma(1 - t, s)$ is a radial geodesic which joins $\gamma(s)$ with x. Taking into account that (M, F) is of Berwald type, the reverse of $t \mapsto \Sigma(1 - t, s)$, i.e. $t \mapsto \Sigma(t, s)$ is a geodesic too (see [3, Exercise 5.3.3, p. 128]) for all $s \in [0, 1]$. Moreover, $\Sigma(0, 0) = x = \gamma_1(0)$, $\Sigma(1, 0) = \gamma(0) = \gamma_1(1)$. From the uniqueness of the geodesic between x and $\gamma_1(1)$, we have $\Sigma(\cdot\,, 0) = \gamma_1$. Analogously, we have $\Sigma(\cdot\,, 1) = \gamma_2$. Since Σ is a geodesic variation (of the curves γ_1 and γ_2), the vector field J_s, defined by

$$J_s(t) = \frac{\partial}{\partial s}\Sigma(t, s) \in T_{\Sigma(t,s)}M$$

is a Jacobi field along $\Sigma(\cdot\,, s)$, $s \in [0, 1]$ (see [3, p. 130]). In particular, we have $\Sigma(1, s) = \gamma(s)$, $J_s(0) = 0$, $J_s(1) = \frac{\partial}{\partial s}\Sigma(1, s) = \frac{d\gamma}{ds}$ and $J_s(\frac{1}{2}) = \frac{\partial}{\partial s}\Sigma(\frac{1}{2}, s)$.

Now, we fix $s \in [0, 1]$. Since $J_s(0) = 0$ and the flag curvature in non-positive, then the geodesic $\Sigma(\cdot, s)$ has no conjugated points, see Theorem 1.1.2. Therefore,

$$J_s(t) \neq 0 \quad \text{for all} \quad t \in (0, 1].$$

Hence $g_{J_s}(J_s, J_s)(t)$ is well defined for every $t \in (0,1]$. Moreover,

$$F(J_s)(t) := F(\Sigma(t,s), J_s(t)) = [g_{J_s}(J_s, J_s)]^{\frac{1}{2}}(t) \neq 0 \qquad \forall\, t \in (0,1]. \quad (1.13)$$

Let T_s the velocity field of $\Sigma(\cdot, s)$. Applying twice formula (1.4), we obtain

$$\frac{d^2}{dt^2}[g_{J_s}(J_s, J_s)]^{\frac{1}{2}}(t) = \frac{d^2}{dt^2}F(J_s)(t) = \frac{d}{dt}\left[\frac{g_{J_s}(D_{T_s}J_s, J_s)}{F(J_s)}\right](t)$$

$$= \frac{[g_{J_s}(D_{T_s}D_{T_s}J_s, J_s) + g_{J_s}(D_{T_s}J_s, D_{T_s}J_s)] \cdot F(J_s) - g^2_{J_s}(D_{T_s}J_s, J_s) \cdot F(J_s)^{-1}}{F^2(J_s)}(t)$$

$$= \frac{g_{J_s}(D_{T_s}D_{T_s}J_s, J_s) \cdot F^2(J_s) + g_{J_s}(D_{T_s}J_s, D_{T_s}J_s) \cdot F^2(J_s) - g^2_{J_s}(D_{T_s}J_s, J_s)}{F^3(J_s)}(t),$$

where the covariant derivatives (for generic Finsler manifolds) are with reference vector J_s. Since (M, F) is a Berwald space, the Chern connection coefficients do not depend on the direction, i.e., the notion of reference vector becomes irrelevant. Therefore, we can use the Jacobi equation (1.5), concluding that

$$g_{J_s}(D_{T_s}D_{T_s}J_s, J_s) = -g_{J_s}(R(J_s, T_s)T_s, J_s).$$

Using the symmetry property of the curvature tensor, the formula of the flag curvature, and the Schwarz inequality we have

$$\begin{aligned} -g_{J_s}(R(J_s, T_s)T_s, J_s) &= -g_{J_s}(R(T_s, J_s)J_s, T_s) \\ &= -K(J_s, T_s) \cdot [g_{J_s}(J_s, J_s)g_{J_s}(T_s, T_s) - g^2_{J_s}(J_s, T_s)] \geq 0. \end{aligned}$$

For the last two terms of the numerator we apply again the Schwarz inequality and we conclude that

$$\frac{d^2}{dt^2}F(J_s)(t) \geq 0 \ \text{ for all } t \in (0,1].$$

Since $J_s(t) \neq 0$ for $t \in (0,1]$, the mapping $t \mapsto F(J_s)(t)$ is C^∞ on $(0,1]$. From the above inequality and the second order Taylor expansion about $v \in (0,1]$, we obtain

$$F(J_s)(v) + (t - v)\frac{d}{dt}F(J_s)(v) \leq F(J_s)(t) \ \text{ for all } t \in (0,1]. \quad (1.14)$$

Letting $t \to 0$ and $v = 1/2$ in (1.14), by the continuity of F, we obtain

$$F(J_s)(\frac{1}{2}) - \frac{1}{2}\frac{d}{dt}F(J_s)(\frac{1}{2}) \leq 0.$$

Let $v = 1/2$ and $t = 1$ in (1.14), and adding the obtained inequality with the above one, we conclude that

$$2F(\Sigma(\frac{1}{2}, s), \frac{\partial}{\partial s}\Sigma(\frac{1}{2}, s)) = 2F(J_s)(\frac{1}{2}) \leq F(J_s)(1) = F(\gamma(s), \frac{d\gamma}{ds}).$$

Integrating the last inequality with respect to s from 0 to 1, we obtain

$$\begin{aligned} 2L_F(\Sigma(\frac{1}{2}, \cdot)) &= 2\int_0^1 F(\Sigma(\frac{1}{2}, s), \frac{\partial}{\partial s}\Sigma(\frac{1}{2}, s))\, ds \\ &\leq \int_0^1 F(\gamma(s), \frac{d\gamma}{ds})\, ds \\ &= L_F(\gamma) \\ &= d_F(\gamma_1(1), \gamma_2(1)). \end{aligned}$$

Since $\Sigma(\frac{1}{2}, 0) = \gamma_1(\frac{1}{2}), \Sigma(\frac{1}{2}, 1) = \gamma_2(\frac{1}{2})$ and $\Sigma(\frac{1}{2}, \cdot)$ is a C^∞ curve, by the definition of the metric function d_F, we conclude that γ_1 and γ_2 satisfy the Busemann NPC inequality. This concludes the proof of Theorem 1.2.1. □

A direct consequence of Theorem 1.2.1 is

Corollary 1.2.1 *Let (M, F) be a forward/backward geodesically complete, simply connected Berwald space with non-positive flag curvature, where F is positively (but perhaps not absolutely) homogeneous of degree one. Then (M, d_F) is a global Busemann NPC space, i.e., the Busemann NPC inequality holds for any pair of geodesics.*

The following result is crucial in Chapter 2.

Proposition 1.2.3 *Let (M, F) be a forward/backward geodesically complete, simply connected Berwald space with non-positive flag curvature, where F is positively (but perhaps not absolutely) homogeneous of degree one. Fix two geodesics $\gamma_1, \gamma_2 : [0, 1] \to M$. Then, the function $t \mapsto d_F(\gamma_1(t), \gamma_2(t))$ is convex.*

Proof. Due to Hopf-Rinow theorem (see Theorem 1.1.1), there exists a geodesic $\gamma_3 : [0,1] \to M$ joining $\gamma_1(0)$ and $\gamma_2(1)$. Moreover, due to Cartan-Hadamard theorem (see Theorem 1.1.2), γ_3 is unique. Applying Corollary 1.2.1 first to the pair γ_1, γ_3 and then to the pair γ_3, γ_2 (with opposite orientation), we obtain

$$d_F(\gamma_1(\frac{1}{2}), \gamma_3(\frac{1}{2})) \leq \frac{1}{2} d_F(\gamma_1(1), \gamma_3(1));$$

$$d_F(\gamma_3(\frac{1}{2}), \gamma_2(\frac{1}{2})) \leq \frac{1}{2} d_F(\gamma_3(0), \gamma_2(0)).$$

Note that the opposite of γ_3 and γ_2 are also geodesics, since (M, F) is of Berwald type, see [3, Example 5.3.3]. Now, using the triangle inequality, we obtain

$$d_F(\gamma_1(\frac{1}{2}), \gamma_2(\frac{1}{2})) \leq \frac{1}{2} d_F(\gamma_1(1), \gamma_2(1)) + \frac{1}{2} d_F(\gamma_1(0), \gamma_2(0)),$$

which means actually the $\frac{1}{2}$-convexity of the function $t \mapsto d_F(\gamma_1(t), \gamma_2(t))$. The continuity and $\frac{1}{2}$-convexity of the function $t \mapsto d_F(\gamma_1(t), \gamma_2(t))$ yield its convexity on $[0, 1]$. □

1.3 Metric projections on Riemannian manifolds

Let (M, g) be an m-dimensional Riemannian manifold ($m \geq 2$), $K \subset M$ be a non-empty set. Let

$$P_K(q) = \{p \in K : d_g(q, p) = \inf_{z \in K} d_g(q, z)\}$$

be the set of *metric projections* of the point $q \in M$ to the set K. Due to the theorem of Hopf-Rinow, if (M, g) is complete, then any closed set $K \subset M$ is *proximinal*, i.e., $P_K(q) \neq \emptyset$ for all $q \in M$. In general, P_K is a set-valued map. When $P_K(q)$ is a singleton for every $q \in M$, we say that K is a *Chebyshev set*. The map P_K is *non-expansive* if

$$d_g(P_K(q_1), P_K(q_2)) \leq d_g(q_1, q_2) \text{ for all } q_1, q_2 \in M.$$

In particular, K is a Chebyshev set whenever the map P_K is non-expansive.

The set $K \subset M$ is *geodesic convex* if every two points $q_1, q_2 \in K$ can be joined by a unique geodesic whose image belongs to K. Note that (1.12) is also valid for every $q_1, q_2 \in K$ in a geodesic convex set K since $\exp_{q_i}^{-1}$ is well-defined on K, $i \in \{1, 2\}$. The function $f : K \to \mathbb{R}$ is *convex*, if $f \circ \gamma : [0, 1] \to \mathbb{R}$ is convex in the usual sense for every geodesic $\gamma : [0, 1] \to K$ provided that $K \subset M$ is a geodesic convex set.

A non-empty closed set $K \subset M$ verifies the *obtuse-angle property* if for fixed $q \in M$ and $p \in K$ the following two statements are equivalent:

(OA_1) $p \in P_K(q)$;

(OA_2) If $\gamma : [0, 1] \to M$ is the unique minimal geodesic from $\gamma(0) = p \in K$ to $\gamma(1) = q$, then for every geodesic $\sigma : [0, \delta] \to K$ $(\delta \geq 0)$ emanating from the point p, we have $g(\dot\gamma(0), \dot\sigma(0)) \leq 0$.

Remark 1.3.1 (a) The first variational formula of Riemannian geometry shows that (OA_1) implies (OA_2) for every closed set $K \subset M$ in a complete Riemannian manifold (M, g).

(b) In the Euclidean case $(\mathbb{R}^m, \langle\cdot,\cdot\rangle_{\mathbb{R}^m})$, (here, $\langle\cdot,\cdot\rangle_{\mathbb{R}^m}$ is the standard inner product in $\mathbb{R}^m$), every non-empty closed convex set $K \subset \mathbb{R}^m$ verifies the obtuse-angle property, see Moskovitz and Dines [28], which reduces to the well-known geometric form:

$$p \in P_K(q) \Leftrightarrow \langle q - p, z - p\rangle_{\mathbb{R}^m} \leq 0 \ \text{ for all } z \in K.$$

A Riemannian manifold (M, g) is a *Hadamard manifold* if it is complete, simply connected and its sectional curvature is non-positive. It is well-known that on a Hadamard manifold (M, g) every geodesic convex set is a Chebyshev set. Moreover, we have

Proposition 1.3.1 *Let (M, g) be a finite-dimensional Hadamard manifold, $K \subset M$ be a closed set. The following statements hold true:*

(i) (Walter [38]) *If $K \subset M$ is geodesic convex, it verifies the obtuse-angle property;*

(ii) (Grognet [16]) *P_K is non-expansive if and only if $K \subset M$ is geodesic convex.*

1.4 Non-smooth calculus on manifolds

We first recall some basic notions and results from the subdifferential calculus on Riemannian manifolds, developed by Azagra, Ferrera and López-Mesas [1], Ledyaev and Zhu [25]. Then, we establish an analytical characterization of the limiting/Fréchet normal cone on Riemannian manifolds (see Theorem 1.4.1) which plays a crucial role in the study of Nash-Stampacchia equilibrium points.

Let (M, g) be an m-dimensional Riemannian manifold and let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function with $\text{dom}(f) \neq \emptyset$. The *Fréchet-subdifferential* of f at $p \in \text{dom}(f)$ is the set

$$\partial_F f(p) = \{dh(p) : h \in C^1(M) \text{ and } f - h \text{ attains a local minimum at } p\}.$$

Proposition 1.4.1 [1, Theorem 4.3] *Let (M, g) be an m-dimensional Riemannian manifold and let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, $p \in \text{dom}(f) \neq \emptyset$ and $\xi \in T_p^* M$. The following statements are equivalent:*

(i) $\xi \in \partial_F f(p)$;

(ii) *For every chart $\psi : U_p \subset M \to \mathbb{R}^m$ with $p \in U_p$, if $\zeta = \xi \circ d\psi^{-1}(\psi(p))$, we have that*

$$\liminf_{v \to 0} \frac{(f \circ \psi^{-1})(\psi(p) + v) - f(p) - \langle \zeta, v \rangle_g}{\|v\|} \geq 0;$$

(iii) *There exists a chart $\psi : U_p \subset M \to \mathbb{R}^m$ with $p \in U_p$, if $\zeta = \xi \circ d\psi^{-1}(\psi(p))$, then*

$$\liminf_{v \to 0} \frac{(f \circ \psi^{-1})(\psi(p) + v) - f(p) - \langle \zeta, v \rangle_g}{\|v\|} \geq 0.$$

In addition, if f is locally bounded from below, i.e., for every $q \in M$ there exists a neighborhood U_q of q such that f is bounded from below on U_q, the above conditions are also equivalent to

(iv) *There exists a function $h \in C^1(M)$ such that $f - h$ attains a global minimum at p and $\xi = dh(p)$.*

Now, we recall two further notions of subdifferential. Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function; the *limiting subdifferential* and *singular subdifferential of f at* $p \in M$ are the sets

$$\partial_L f(p) = \{\lim_k \xi_k : \xi_k \in \partial_F f(p_k),\ (p_k, f(p_k)) \to (p, f(p))\}$$

and

$$\partial_\infty f(p) = \{\lim_k t_k \xi_k : \xi_k \in \partial_F f(p_k),\ (p_k, f(p_k)) \to (p, f(p)), t_k \to 0^+\}.$$

Proposition 1.4.2 [25] *Let (M, g) be a finite-dimensional Riemannian manifold and let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Then, we have*

(i) $\partial_F f(p) \subset \partial_L f(p),\ p \in \mathrm{dom}(f)$;

(ii) $0 \in \partial_\infty f(p)$, $p \in M$;

(iii) *If $p \in \mathrm{dom}(f)$ is a local minimum of f, then* $0 \in \partial_F f(p) \subset \partial_L f(p)$.

Let $K \subset M$ be a closed set. Following [25], the *Fréchet-normal cone* and *limiting normal cone* of K at $p \in K$ are the sets

$$N_F(p; K) = \partial_F \delta_K(p)$$

and

$$N_L(p; K) = \partial_L \delta_K(p),$$

where δ_K is the indicator function of the set K, i.e., $\delta_K(q) = 0$ if $q \in K$ and $\delta_K(q) = +\infty$ if $q \notin K$. The following result - which is one of our key tools to study Nash-Stampacchia equilibrium points on manifolds - is probably known, but since we have not found an explicit reference, we give its complete proof.

Theorem 1.4.1 *Let (M,g) be an m-dimensional Riemannian manifold. For any closed, geodesic convex set $K \subset M$ and $p \in K$, we have*

$$N_F(p;K) = N_L(p;K) = \{\xi \in T_p^*M : \langle \xi, \exp_p^{-1}(q)\rangle_g \leq 0 \textit{ for all } q \in K\}.$$

Proof. We first prove that

$$N_F(p;K) \subset \{\xi \in T_p^*M : \langle \xi, \exp_p^{-1}(q)\rangle_g \leq 0 \text{ for all } q \in K\}. \qquad (1.15)$$

To see this, let us fix $\xi \in N_F(p;K) = \partial_F \delta_K(p)$, i.e., on account of Proposition 1.4.1 (i) $\Leftrightarrow$ (iv), there exists $h \in C^1(M)$ such that $\xi = dh(p)$ and $\delta_K - h$ attains a global minimum at p. In particular, the latter fact implies that

$$h(q) \leq h(p) \text{ for all } q \in K. \qquad (1.16)$$

Fix $q \in K$. Since K is geodesic convex, the unique geodesic $\gamma : [0,1] \to M$ joining the points p and q, defined by $\gamma(t) = \exp_p(t \exp_p^{-1}(q))$, belongs entirely to K. Therefore, in view of (1.16), we have that $(h \circ \gamma)(t) \leq (h \circ \gamma)(0) = h(p)$ for every $t \in [0,1]$. Consequently,

$$(h \circ \gamma)'(0) = \lim_{t \to 0^+} \frac{(h \circ \gamma)(t) - (h \circ \gamma)(0)}{t} \leq 0.$$

On the other hand, we have that

$$(h \circ \gamma)'(0) = \langle dh(\gamma(0)), \dot{\gamma}(0)\rangle_g = \langle \xi, \exp_p^{-1}(q)\rangle_g,$$

which concludes the proof of relation (1.15).

Now, we prove that

$$N_L(p;K) \subset \{\xi \in T_p^*M : \langle \xi, \exp_p^{-1}(q)\rangle_g \leq 0 \text{ for all } q \in K\}. \qquad (1.17)$$

Indeed, let $\xi \in N_L(p;K) = \partial_L \delta_K(p)$. Thus, there exists a sequence $\{p_k\} \subset M$ such that $(p_k, \delta_K(p_k)) \to (p, \delta_K(p))$ with $\xi_k \in \partial_F \delta_K(p_k)$ and $\lim_k \xi_k = \xi$. Note that $\delta_K(p) = 0$, thus we necessarily have $\{p_k\} \subset K$. By relation (1.15) and $\xi_k \in \partial_F \delta_K(p_k) = N_F(p_k;K)$ we have that $\langle \xi_k, \exp_{p_k}^{-1}(q)\rangle_g \leq 0$ for all $q \in K$ and $k \in \mathbb{N}$. Letting $k \to \infty$ in the last inequality and taking into account that $\lim_k \xi_k = \xi$, we conclude that $\langle \xi, \exp_p^{-1}(q)\rangle_g \leq 0$ for all

$q \in K$, i.e., (1.17) is proved. Now, according to Proposition 1.4.2 (i) and relation (1.17), we have that

$$N_F(p;K) \subset N_L(p;K) \subset \{\xi \in T_p^*M : \langle \xi, \exp_p^{-1}(q)\rangle_g \leq 0 \text{ for all } q \in K\}.$$

To conclude the proof, it remains to show that

$$\{\xi \in T_p^*M : \langle \xi, \exp_p^{-1}(q)\rangle_g \leq 0 \text{ for all } q \in K\} \subset N_F(p;K).$$

Let us fix $\xi \in T_p^*M$ with

$$\langle \xi, \exp_p^{-1}(q)\rangle_g \leq 0 \text{ for all } q \in K. \tag{1.18}$$

We show that (iii) from Proposition 1.4.1 holds true with the choices $f = \delta_K$ and $\psi = \exp_p^{-1} : \tilde{U}_p \to T_pM = \mathbb{R}^m$ where $\tilde{U}_p \subset M$ is a totally normal ball centered at p. Due to these choices, the inequality from Proposition 1.4.1 (iii) reduces to

$$\liminf_{v \to 0} \frac{\delta_K(\exp_p(v)) - \langle \xi, v\rangle_g}{\|v\|} \geq 0, \tag{1.19}$$

since we have $\delta_K(p) = 0$, $\psi(p) = 0$ and $d\psi^{-1}(\psi(p)) = d\exp_p(0) = \mathrm{id}_{T_pM}$, see (1.3). To verify (1.19), two subcases are considered ($\|v\|$ is assumed to be small enough):

(a) $\exp_p(v) \notin K$. Then $\delta_K(\exp_p(v)) = +\infty$, thus the inequality (1.19) is proved.

(b) $\exp_p(v) \in K$. Then $\delta_K(\exp_p(v)) = 0$ and there exists a unique $q \in K \cap \tilde{U}_p$ such that $v = \exp_p^{-1}(q)$. Thus, (1.19) follows at once from (1.18).

Consequently, from Proposition 1.4.1 (i) $\Leftrightarrow$ (iii), we have that $\xi \in \partial_F \delta_K(p)$, i.e., $\xi \in N_F(p;K)$. $\square$

Proposition 1.4.3 [25, Theorem 4.13 (Sum rule)] *Let (M,g) be an m-dimensional Riemannian manifold and let $f_1, ..., f_H : M \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous functions. Then, for every $p \in M$ we have either $\partial_L(\sum_{l=1}^H f_l)(p) \subset \sum_{l=1}^H \partial_L f_l(p)$, or there exist $\xi_l^\infty \in \partial_\infty f_l(p)$, $l = 1, ..., H$, not all zero such that $\sum_{l=1}^H \xi_l^\infty = 0$.*

Let $U \subset M$ be an open subset of the Riemannian manifold (M, g). We say that a function $f : U \to \mathbb{R}$ is *locally Lipschitz at* $p \in U$ if there exist an open neighborhood $U_p \subset U$ of p and a number $C_p > 0$ such that for every $q_1, q_2 \in U_p$,

$$|f(q_1) - f(q_2)| \leq C_p d_g(q_1, q_2).$$

The function $f : U \to \mathbb{R}$ is *locally Lipschitz* on (U, g) if it is locally Lipschitz at every $p \in U$.

Fix $p \in U$, $v \in T_pM$, and let $\tilde{U}_p \subset U$ be a totally normal neighborhood of p. If $q \in \tilde{U}_p$, following [1, Section 5], for small values of $|t|$, we may introduce

$$\sigma_{q,v}(t) = \exp_q(tw),\ w = d(\exp_q^{-1} \circ \exp_p)_{\exp_p^{-1}(q)} v.$$

If the function $f : U \to \mathbb{R}$ is locally Lipschitz on (U, g), then

$$f^0(p, v) = \limsup_{q \to p,\ t \to 0^+} \frac{f(\sigma_{q,v}(t)) - f(q)}{t}$$

is called the *Clarke generalized derivative of* f *at* $p \in U$ *in direction* $v \in T_pM$, and

$$\partial_C f(p) = \mathrm{co}(\partial_L f(p))$$

is the *Clarke subdifferential of* f *at* $p \in U$, where 'co' stands for the convex hull. When $f : U \to \mathbb{R}$ is a C^1 functional at $p \in U$ then $\partial_C f(p) = \partial_L f(p) = \partial_F f(p) = \{df(p)\}$, see [1, Proposition 4.6]. Moreover, when (M, g) is the standard Euclidean space, the Clarke subdifferential and the Clarke generalized gradient do coincide, see Clarke [14].

One can easily prove that the function $f^0(\cdot, \cdot)$ is upper-semicontinuous on $TU = \cup_{p \in U} T_pM$ and $f^0(p, \cdot)$ is positive homogeneous. In addition, if $U \subset M$ is geodesic convex and $f : U \to \mathbb{R}$ is convex, then

$$f^0(p, v) = \lim_{t \to 0^+} \frac{f(\exp_p(tv)) - f(p)}{t}, \tag{1.20}$$

see Claim 5.4 and the first relation on p. 341 of [1].

Proposition 1.4.4 [25, Corollary 5.3] *Let (M, g) be a complete Riemannian manifold and let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Then the following statements are equivalent:*

(i) *f is locally Lipschitz at $p \in M$;*

(ii) *$\partial_C f$ is bounded in a neighborhood of $p \in M$;*

(iii) *$\partial_\infty f(p) = \{0\}$.*

1.5 Dynamical systems on manifolds

In this subsection we recall the existence of a local solution for a Cauchy-type problem defined on Riemannian manifolds and its viability relative to a closed set.

Let (M, g) be a finite-dimensional Riemannian manifold and $G : M \to TM$ be a vector field on M, i.e., $G(p) \in T_pM$ for every $p \in M$. We assume in the sequel that $G : M \to TM$ is a C^{1-0} vector field (i.e., locally Lipschitz); then the dynamical system

$$(DS)_G \qquad \begin{cases} \dot\eta(t) = G(\eta(t)), \\ \eta(0) = p_0, \end{cases}$$

has a unique maximal semiflow $\eta : [0, T) \to M$, see Chang [11, p. 15]. In particular, η is an absolutely continuous function such that $[0, T) \ni t \mapsto \dot\eta(t) \in T_{\eta(t)}M$ and it verifies $(DS)_G$ for a.e. $t \in [0, T)$.

A set $K \subset M$ is *invariant with respect to the solutions of* $(DS)_G$ if for every initial point $p_0 \in K$ the unique maximal semiflow/orbit $\eta : [0, T) \to M$ of $(DS)_G$ fulfills the property that $\eta(t) \in K$ for every $t \in [0, T)$. We introduce the Hamiltonian function as

$$H_G(p, \xi) = \langle \xi, G(p) \rangle_g, \ (p, \xi) \in M \times T_p^*M.$$

Note that $H_G(p, dh(p)) < \infty$ for every $p \in M$ and $h \in C^1(M)$. Therefore, after a suitable adaptation of the results from Ledyaev and Zhu [25, Subsection 6.2] we may state

Proposition 1.5.1 *Let $G : M \to TM$ be a C^{1-0} vector field and $K \subset M$ be a non-empty closed set. The following statements are equivalent:*

(i) *K is invariant with respect to the solutions of $(DS)_G$;*

(ii) *$H_G(p, \xi) \leq 0$ for any $p \in K$ and $\xi \in N_F(p; K)$.*

1.6 Variational inequalities on ANRs

Existence results for Nash equilibria are often derived from intersection theorems (KKM theorems) or fixed point theorems. For instance, the original proof of Nash concerning equilibrium point is based on the Brouwer fixed point theorem. These theorems are actually equivalent to minimax theorems or variational inequalities, as Ky Fan minimax theorem, etc. In this section we recall some results which will be used in Chapter 3.

A nonempty set X is *acyclic* if it is connected and its Čech homology (coefficients in a fixed field) is zero in dimensions greater than zero. Note that every contractible set is acyclic (but the converse need not holds in general).

The following two results are the main tools in the proof of the existence of Nash-type equilibria in compact settings; the first being a McClendon-type minimax result while the second is the so-called Begle's fixed point theorem for set-valued maps.

Proposition 1.6.1 [27, Theorem 3.1] *Suppose that X is a compact acyclic finite-dimensional ANR. Suppose $h : X \times X \to \mathbb{R}$ is a function such that $\{(x, y) : h(y, y) > h(x, y)\}$ is open and $\{x : h(y, y) > h(x, y)\}$ is contractible or empty for all $y \in X$. Then there is a $y_0 \in X$ with $h(y_0, y_0) \leq h(x, y_0)$ for all $x \in X$.*

Proposition 1.6.2 [27, Proposition 1.1] *Let X be a compact acyclic finite-dimensional ANR. Suppose that $F : X \to 2^X$ is a set-valued map with closed graph having nonempty and acyclic values. The F has a fixed point, i.e., there exists $x \in X$ such that $x \in F(x)$.*

The following result is probably know, but since we have not found an explicit reference, we give its proof.

Proposition 1.6.3 *Let (M, g) be a complete, finite-dimensional Riemannian manifold. Then any geodesic convex set $K \subset M$ is contractible.*

Proof. Let us fix $p \in K$ arbitrarily. Since K is geodesic convex, every point $q \in K$ can be connected to p uniquely by the geodesic segment $\gamma_q : [0,1] \to K$, i.e., $\gamma_q(0) = p$, $\gamma_q(1) = q$. Moreover, the map $K \ni q \mapsto \exp_p^{-1}(q) \in T_pM$ is well-defined and continuous. Note actually that $\gamma_q(t) = \exp_p(t \exp_p^{-1}(q))$. We define the map $G : [0,1] \times K \to K$ by $G(t, q) = \gamma_q(t)$. It is clear that G is continuous, $G(1, q) = q$ and $G(0, q) = p$ for all $q \in K$, i.e., the identity map id_K is homotopic to the constant map p. □

1.7 Comments

In this chapter we have recalled those elements from Finsler and Riemannian geometry which will be used throughout the thesis: geodesics, flag curvature, metric projections, non-smooth analysis and dynamical systems on manifolds, and variational inequalities. The main part of this chapter is standard and can be found in any reasonable textbook on Riemann-Finsler geometry, for example in Bao, Chern and Shen [3]. However, based on the paper of Kristály and Kozma [22], we also presented new material on non-positively curved Berwald spaces as well as a conjecture regarding the rigidity of Finsler manifolds under the Busemann curvature condition (see §1.2) which could be of interest for the community of Geometers.

Chapter 2

Weber-type problems: Minimization of cost-functions on manifolds

> Geography has made us neighbors. History has made us friends. Economics has made us partners, and necessity has made us allies. Those whom God has so joined together, let no man put asunder.
>
> John F. Kennedy (1917–1963)

2.1 Introduction

Let us consider three markets P_1, P_2, P_3 placed on an inclined plane (slope) with an angle α to the horizontal plane, denoted by (S_α). Assume that three cars transport products from (resp. to) deposit $P \in (S_\alpha)$ to (resp. from) markets $P_1, P_2, P_3 \in (S_\alpha)$ such that

- they move always in (S_α) along straight roads;

- the Earth gravity acts on them (we omit other physical perturbations such as friction, air resistance, etc.);
- the transport costs coincide with the *distance* (measuring actually the *time* elapsed to arrive) from (resp. to) deposit P to (resp. from) markets P_i $(i = 1, 2, 3)$.

We emphasize that usually the two distances, i.e., from the deposit to the markets and conversely, are *not* the same. The point here is that the travel speed depends heavily on both the slope of the terrain and the direction of travel. More precisely, if a car moves with a constant speed v $[m/s]$ on a horizontal plane, it goes $l_t = vt + \frac{g}{2}t^2 \sin\alpha \cos\theta$ meters in t seconds on (S_α), where θ is the angle between the straight road and the direct downhill road (θ is measured in clockwise direction). The law of the above phenomenon can be described relatively to the horizontal plane by means of the parametrized function

$$F_\alpha(y_1, y_2) = \frac{y_1^2 + y_2^2}{v\sqrt{y_1^2 + y_2^2} + \frac{g}{2} y_1 \sin\alpha}, \qquad (y_1, y_2) \in \mathbb{R}^2 \setminus \{(0,0)\}. \tag{2.1}$$

Here, $g \approx 9.81 m/s^2$. Since (M, F_α) is a (non-reversible) *Minkowski space*, the *distance* (measuring the *time* to arrive) from $P = (P^1, P^2)$ to $P_i = (P_i^1, P_i^2)$ is

$$d_\alpha(P, P_i) = F_\alpha(P_i^1 - P^1, P_i^2 - P^2),$$

and for the converse it is

$$d_\alpha(P_i, P) = F_\alpha(P^1 - P_i^1, P^2 - P_i^2).$$

Consequently, we have to minimize the functions

$$C_f(P) = \sum_{i=1}^{3} d_\alpha(P, P_i) \ \text{ and } \ C_b(P) = \sum_{i=1}^{3} d_\alpha(P_i, P), \tag{2.2}$$

when P moves on (S_α), i.e., we have two *Weber-type problem*s defined in a highly non-reversible setting. The function C_f (resp. C_b) denotes the *total forward* (*resp. backward*) *cost* between the deposit $P \in (S_\alpha)$ and

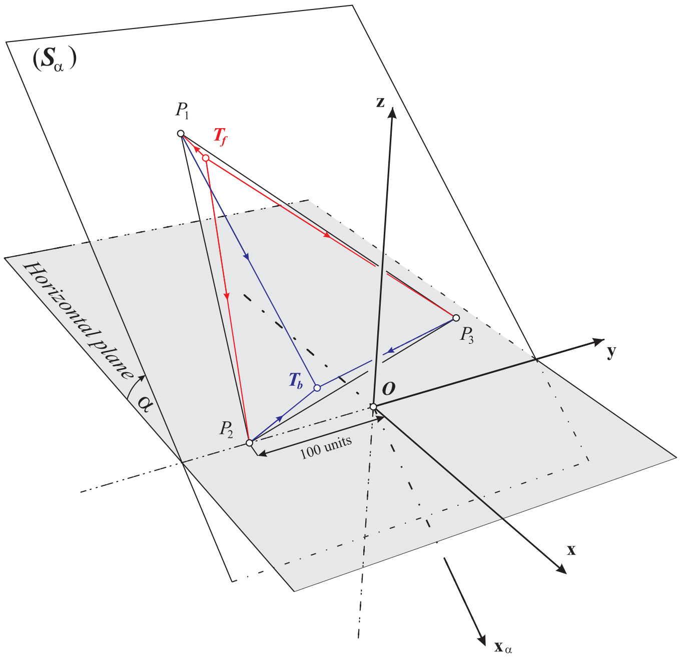

Figure 2.1: We fix $P_1 = (-250, -50)$, $P_2 = (0, -100)$ and $P_3 = (-50, 100)$ on the slope (S_α) with angle $\alpha = 35^o$. If $v = 10$, the minimum of the total forward cost on the slope is $C_f \approx 40.3265$; the corresponding deposit is located at $T_f \approx (-226.11, -39.4995) \in (S_\alpha)$. However, the minimum of the total backward cost on the slope is $C_b \approx 38.4143$; the corresponding deposit has the coordinates $T_b \approx (-25.1332, -35.097) \in (S_\alpha)$.

markets $P_1, P_2, P_3 \in (S_\alpha)$. The minimum points of C_f and C_b, respectively, may be far from each other (see Figure 2.1), due to the fact that F_α (and d_α) is not symmetric unless $\alpha = 0$, i.e., $F_\alpha(-y_1, -y_2) \neq F_\alpha(y_1, y_2)$ for each $(y_1, y_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

We will use in general T_f (resp. T_b) to denote a minimum point of C_f (resp. C_b), which corresponds to the position of a deposit when we measure costs in forward (resp. backward) manner, see (2.2).

In the case $\alpha = 0$ (when (S_α) is a horizontal plane), the functions C_f and C_b coincide (the same is true for T_f and T_b). The minimum point $T = T_f = T_b$ is the well-known *Torricelli point* corresponding to the triangle $P_1P_2P_{3\Delta}$. Note that $F_0(y_1, y_2) = \sqrt{y_1^2 + y_2^2}/v$ corresponds to the standard Euclidean metric; indeed,

$$d_0(P, P_i) = d_0(P_i, P) = \sqrt{(P_i^1 - P^1)^2 + (P_i^2 - P^2)^2}/v$$

measures the time, which is needed to arrive from P to P_i (and vice-versa) with constant velocity v.

Unfortunately, finding critical points as possible minima does not yield any result: either the minimization function is not smooth enough (usually, it is only a locally Lipschitz function) or the system, which would give the critical points, becomes very complicated even in quite simple cases (see (2.5) below). Consequently, the main purpose of the present chapter is to study the set of these minima (existence, location) in various geometrical settings.

The chapter is divided as follows. In §2.2 we deal with a necessary condition while in §2.3 some existence, uniqueness and multiplicity results are demonstrated for a general Weber problem on non-positively curved Berwald space which model various real life phenomena. In §2.4, relevant numerical examples and counterexamples are constructed by means of evolutionary methods and computational geometry tools, emphasizing the applicability and sharpness of our results.

2.2 A necessary condition

Let (M, F) be an m-dimensional connected Finsler manifold, where F is positively (but perhaps not absolutely) homogeneous of degree one.

In this section we prove some results concerning the set of minima for functions

$$C_f(P_i, n, s)(P) = \sum_{i=1}^{n} d_F^s(P, P_i) \quad \text{and} \quad C_b(P_i, n, s)(P) = \sum_{i=1}^{n} d_F^s(P_i, P),$$

where $s \geq 1$ and $P_i \in M$, $i = 1, \ldots, n$, correspond to $n \in \mathbb{N}$ markets. The value $C_f(P_i, n, s)(P)$ (resp. $C_b(P_i, n, s)(P)$) denotes the *total s-forward* (*resp. s-backward*) *cost* between the deposit $P \in M$ and the markets

$P_i \in M$, $i = 1, ..., n$. When $s = 1$, we simply say *total forward* (*resp. backward*) *cost,* and the minimization problems are the non-reversible counterparts of the well-known Weber problem.

By using the triangle inequality, for every $x_0, x_1, x_2 \in M$ we have

$$|d_F(x_1, x_0) - d_F(x_2, x_0)| \leq \max\{d_F(x_1, x_2), d_F(x_2, x_1)\}. \tag{2.3}$$

Given any point $P \in M$, there exists a coordinate map φ_P defined on the closure of some precompact open subset U containing P such that φ_P maps the set U diffeomorphically onto the open Euclidean ball $B^m(r)$, $r > 0$, with $\varphi_P(P) = 0_{\mathbb{R}^m}$. Moreover, there is a constant $c > 1$, depending only on P and U such that

$$c^{-1}\|\varphi_P(x_1) - \varphi_P(x_2)\| \leq d_F(x_1, x_2) \leq c\|\varphi_P(x_1) - \varphi_P(x_2)\| \tag{2.4}$$

for every $x_1, x_2 \in U$; see [3, p. 149]. Here, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^m$. We claim that for every $Q \in M$, the function $d_F(\varphi_P^{-1}(\cdot), Q)$ is a Lipschitz function on $\varphi_P(U) = B^m(r)$. Indeed, for every $y_i = \varphi_P(x_i) \in \varphi_P(U)$, $i = 1, 2$, due to (2.3) and (2.4), one has

$$|d_F(\varphi_P^{-1}(y_1), Q) - d_F(\varphi_P^{-1}(y_2), Q)| = |d_F(x_1, Q) - d_F(x_2, Q)| \leq$$

$$\leq \max\{d_F(x_1, x_2), d_F(x_2, x_1)\} \leq c\|y_1 - y_2\|.$$

Consequently, for every $Q \in M$, there exists the *generalized gradient* of the locally Lipschitz function $d_F(\varphi_P^{-1}(\cdot), Q)$ on $\varphi_P(U) = B^m(r)$, see Clarke [14, p. 27], i.e., for every $y \in \varphi_P(U) = B^m(r)$ we have

$$\partial d_F(\varphi_P^{-1}(\cdot), Q)(y) = \{\xi \in \mathbb{R}^m : d_F^0(\varphi_P^{-1}(\cdot), Q)(y; h) \geq \langle \xi, h\rangle \ \text{ for all } h \in \mathbb{R}^m\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^m$ and

$$d_F^0(\varphi_P^{-1}(\cdot), Q)(y; h) = \limsup_{z \to y,\ t \to 0^+} \frac{d_F(\varphi_P^{-1}(z + th), Q) - d_F(\varphi_P^{-1}(z), Q)}{t}$$

is the *generalized directional derivative.*

Theorem 2.2.1 [24] *Assume that* $T_f \in M$ *is a minimum point for* $C_f(P_i, n, s)$ *and* φ_{T_f} *is a map as above. Then*

$$0_{\mathbb{R}^m} \in \sum_{i=1}^{n} d_F^{s-1}(T_f, P_i)\partial d_F(\varphi_{T_f}^{-1}(\cdot), P_i)(\varphi_{T_f}(T_f)). \tag{2.5}$$

Proof. Since $T_f \in M$ is a minimum point of the locally Lipschitz function $C_f(P_i, n, s)$, then

$$0_{\mathbb{R}^m} \in \partial\left(\sum_{i=1}^{n} d_F^s(\varphi_{T_f}^{-1}(\cdot), P_i)\right)(\varphi_{T_f}(T_f)),$$

see [14, Proposition 2.3.2]. Now, using the basic properties of the generalized gradient, see [14, Proposition 2.3.3] and [14, Theorem 2.3.10], we conclude the proof. □

Remark 2.2.1 A result similar to Theorem 2.2.1 can also be obtained for $C_b(P_i, n, s)$.

Example 2.2.1 Let $M = \mathbb{R}^m$, $m \geq 2$, be endowed with the natural Euclidean metric. Taking into account (2.5), a simple computation shows that the unique minimum point $T_f = T_b$ (i.e., the place of the deposit) for $C_f(P_i, n, 2) = C_b(P_i, n, 2)$ is the centre of gravity of markets $\{P_1, ..., P_n\}$, i.e., $\frac{1}{n}\sum_{i=1}^{n} P_i$. In this case, φ_{T_f} can be the identity map on $\mathbb{R}^m$.

Remark 2.2.2 The system (2.5) may become very complicated even for simple cases; it is enough to consider the Matsumoto metric given by (2.1). In such cases, we are not able to give an explicit formula for minimal points. In fact, even in the Euclidean case when the points are not situated on a straight line, Bajaj [4] has shown via Galois theory that the Torricelli point(s) in the Weber problem cannot be expressed using radicals (arithmetical operations and k^{th} roots).

2.3 Existence and uniqueness results

The next result gives an alternative concerning the number of minimum points of the function $C_f(P_i, n, s)$ in a general geometrical framework. (Similar result can be obtained for $C_b(P_i, n, s)$.) Namely, we have the following theorem.

Theorem 2.3.1 [24] *Let (M, F) be a simply connected, geodesically complete Berwald manifold of non-positive flag curvature, where F is positively (but perhaps not absolutely) homogeneous of degree one. Then*

(a) *there exists either a unique or infinitely many minimum points for $C_f(P_i, n, 1)$;*

(b) *there exists a unique minimum point for $C_f(P_i, n, s)$ whenever $s > 1$.*

Proof. First of all, we observe that M is not a backward bounded set. Indeed, if we assume that it is, then M is compact due to Hopf-Rinow theorem, see Theorem 1.1.1. On the other hand, due Cartan-Hadamard theorem, see Theorem 1.1.2, the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism for every $p \in M$. Thus, the tangent space $T_pM = \exp_p^{-1}(M)$ is compact, a contradiction. Since M is not backward bounded, in particular, for every $i = 1, ..., n$, we have that

$$\sup_{P \in M} d_F(P, P_i) = \infty.$$

Consequently, outside of a large backward bounded subset of M, denoted by M_0, the value of $C_f(P_i, n, s)$ is large. But, M_0 being compact, the continuous function $C_f(P_i, n, s)$ attains its infimum, i.e., the set of the minima for $C_f(P_i, n, s)$ is always nonempty.

On the other hand, due to Proposition 1.2.3 for every nonconstant geodesic $\sigma : [0, 1] \to M$ and $p \in M$, the function $t \mapsto d_F(\sigma(t), p)$ is convex and $t \mapsto d_F^s(\sigma(t), p)$ is strictly convex, whenever $s > 1$ (see also Jost [17, Corollary 2.2.6]).

(a) Let us assume that there are at least two minimum points for $C_f(P_i, n, 1)$, denoting them by T_f^0 and T_f^1. Let $\sigma : [0, 1] \to M$ be a geodesic with constant Finslerian speed such that $\sigma(0) = T_f^0$ and $\sigma(1) = T_f^1$. Then, for every $t \in (0,1)$ we have

$$\begin{aligned} C_f(P_i, n, 1)(\sigma(t)) &= \sum_{i=1}^{n} d_F(\sigma(t), P_i) \\ &\leq (1-t)\sum_{i=1}^{n} d_F(\sigma(0), P_i) + t\sum_{i=1}^{n} d_F(\sigma(1), P_i) \quad (2.6) \\ &= (1-t)\min C_f(P_i, n, 1) + t\min C_f(P_i, n, 1) \\ &= \min C_f(P_i, n, 1). \end{aligned}$$

Consequently, for every $t \in [0,1]$, $\sigma(t) \in M$ is a minimum point for $C_f(P_i, n, 1)$.

(b) It follows directly from the strict convexity of the function $t \mapsto d_F^s(\sigma(t), p)$, whenever $s > 1$; indeed, in (2.6) we have strict inequality instead of " $\leq$ " which shows we cannot have more then one minimum point for $C_f(P_i, n, s)$. $\square$

Example 2.3.1 Let F be the Finsler metric introduced in (2.1). One can see that $(\mathbb{R}^2, F)$ is a typical non-reversible Finsler manifold. Actually, it is a (locally) Minkowski space, so a Berwald space as well; its Chern connection vanishes, see [3, p. 384]. According to (1.2) and (1.8), the geodesics are straight lines (hence $(\mathbb{R}^2, F)$ is geodesically complete in both sense) and the flag curvature is identically 0. Thus, we can apply Theorem 2.3.1. For instance, if we consider the points $P_1 = (a, -b) \in \mathbb{R}^2$ and $P_2 = (a, b) \in \mathbb{R}^2$ with $b \neq 0$, the minimum points of the function $C_f(P_i, 2, 1)$ form the segment $[P_1, P_2]$, *independently* of the value of α. The same is true for $C_b(P_i, 2, 1)$. However, considering more complicated constellations, the situation changes dramatically, see Figure 2.2.

It would be interesting to study in similar cases the precise orbit of the (Torricelli) points T_f^α and T_b^α when α varies from 0 to $\pi/2$. Several numerical experiments show that T_f^α tends to a top point of the convex polygon (as in the Figure 2.2).

In the sequel, we want to study our problem in a special constellation: we assume the markets are situated on a common "straight line", i.e.,

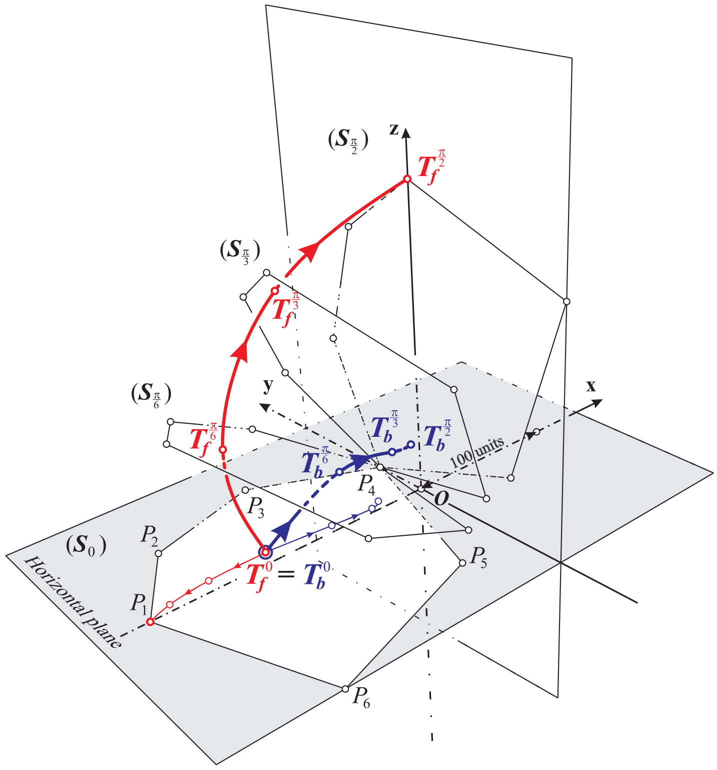

Figure 2.2: A hexagon with vertices $P_1, P_2, ..., P_6$ in the Matsumoto space. Increasing the slope's angle α from 0 to $\pi/2$, points T_f^α and T_b^α are wandering in the presented directions. Orbits of points T_f^α and T_b^α were generated by natural cubic spline curve interpolation.

on a geodesic which is in a Riemannian manifold. Note that, in the Riemannian context, the forward and backward costs coincide, i.e.,

$$C_f(P_i, n, 1) = C_b(P_i, n, 1),$$

while finding the minimum point(s) of the above function leads to the well-known Weber problem on Riemannian manifolds. We denote this common value by $C(P_i, n, 1)$.

Theorem 2.3.2 [24] *Let (M, g) be a Hadamard manifold. Assume the points $P_i \in M$, $i = 1, ..., n$, $(n \geq 2)$, belong to a geodesic $\sigma : [0, 1] \to M$ such that $P_i = \sigma(t_i)$ with $0 \leq t_1 < ... < t_n \leq 1$. Then*

(a) *the unique minimum point for $C(P_i, n, 1)$ is $P_{[n/2]}$ whenever n is odd;*

(b) *the minimum points for $C(P_i, n, 1)$ is the whole geodesic segment situated on σ between $P_{n/2}$ and $P_{n/2+1}$ whenever n is even.*

Proof. Since (M, g) is complete, we extend σ to $(-\infty, \infty)$, keeping the same notation. First, we prove that the minimum point(s) for $C(P_i, n, 1)$ belong to the geodesic σ. We assume the contrary, i.e., let $T \in M \setminus \text{Image}(\sigma)$ be a minimum point of $C(P_i, n, 1)$. Let $T_\perp \in \text{Image}(\sigma)$ be the projection of T on the geodesic σ, i.e.

$$d_g(T, T_\perp) = \min_{t \in \mathbb{R}} d_g(T, \sigma(t)).$$

It is clear that the (unique) geodesic lying between T and $T_\perp$ is perpendicular to σ with respect to the Riemannian metric g.

Let $i_0 \in \{1, ..., n\}$ such that $P_{i_0} \neq T_\perp$. Applying the cosine inequality, see Theorem 1.1.3, for the triangle with vertices P_{i_0}, T and $T_\perp$ (so, $\widehat{T_\perp} = \pi/2$), we have

$$d_g^2(T_\perp, T) + d_g^2(T_\perp, P_{i_0}) \leq d_g^2(T, P_{i_0}).$$

Since

$$d_g(T_\perp, T) > 0,$$

we have

$$d_g(T_\perp, P_{i_0}) < d_g(T, P_{i_0}).$$

Consequently,

$$C(P_i, n, 1)(T_\perp) = \sum_{i=1}^{n} d_g(T_\perp, P_i) < \sum_{i=1}^{n} d_g(T, P_i) = \min C(P_i, n, 1),$$

a contradiction. Now, conclusions (a) and (b) follow easily by using simple arithmetical reasons. $\square$

2.4 Examples

We emphasize that Theorem 2.3.2 is sharp in the following sense: neither the non-positivity of the sectional curvature (see Example 2.4.1) nor the Riemannian structure (see Example 2.4.2) can be omitted.

Example 2.4.1 (Sphere) Let us consider the 2-dimensional unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ endowed with its natural Riemannian metric h inherited by $\mathbb{R}^3$. We know that it has constant curvature 1. Let us fix $P_1, P_2 \in \mathbb{S}^2$ $(P_1 \neq P_2)$ and their antipodals $P_3 = -P_1$, $P_4 = -P_2$. There exists a unique great circle (geodesic) connecting P_i, $i = 1, ..., 4$. However, we observe that the function $C(P_i, 4, 1)$ is *constant* on $\mathbb{S}^2$; its value is 2π. Consequently, *every* point on $\mathbb{S}^2$ is a minimum for the function $C(P_i, 4, 1)$.

Example 2.4.2 (Finslerian-Poincaré disc) Let us consider the disc

$$M = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}.$$

Introducing the polar coordinates (r, θ) on M, i.e., $x = r\cos\theta$, $y = r\sin\theta$, we define the non-reversible Finsler metric on M by

$$F((r, \theta), V) = \frac{1}{1 - \frac{r^2}{4}}\sqrt{p^2 + r^2q^2} + \frac{pr}{1 - \frac{r^4}{16}},$$

where

$$V = p\frac{\partial}{\partial r} + q\frac{\partial}{\partial \theta} \in T_{(r,\theta)}M.$$

The pair (M, F) is the so-called *Finslerian-Poincaré disc.* Within the classification of Finsler manifolds, (M, F) is a *Randers space,* see [3, Section 12.6], which has the following properties:

(p1) it has constant negative flag curvature $-1/4$;

(p2) the geodesics have the following trajectories: Euclidean circular arcs that intersect the boundary ∂M of M at Euclidean right angles; Euclidean straight rays that emanate from the origin; and Euclidean straight rays that aim to the origin;

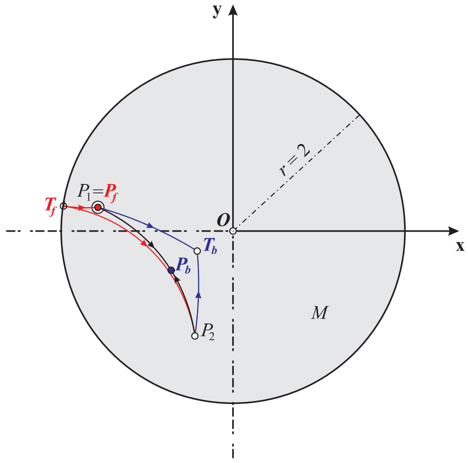

Figure 2.3:
Step 1: The minimum of the total backward (resp. forward) cost function $C_b(P_i, 2, 1)$ (resp. $C_f(P_i, 2, 1)$) is *restricted to the geodesic* determined by $P_1(1.6, 170°)$ and $P_2(1.3, 250°)$. The point which minimizes $C_b(P_i, 2, 1)$ (resp. $C_f(P_i, 2, 1)$) is approximated by $P_b(0.8541, 212.2545°)$ (resp. $P_f = P_1$); in this case $C_b(P_i, 2, 1)(P_b) \approx 1.26$ (resp. $C_f(P_i, 2, 1)(P_f) \approx 2.32507$).
Step 2: The minimum of the total backward (resp. forward) cost function $C_b(P_i, 2, 1)$ (resp. $C_f(P_i, 2, 1)$) is *on the whole Randers space* M. The minimum point of total backward (resp. forward) cost function is approximated by $T_b(0.4472, 212.5589°)$ (resp. $T_f(1.9999, 171.5237°)$), which gives $C_b(P_i, 2, 1)(T_b) \approx 0.950825 < C_b(P_i, 2, 1)(P_b)$ (resp. $C_f(P_i, 2, 1)(T_f) \approx 2.32079 < C_f(P_i, 2, 1)(P_f)$).

(p3) $\text{dist}_F((0,0), \partial M) = \infty$, while $\text{dist}_F(\partial M, (0,0)) = \log 2$.

Although (M, F) is forward geodesically complete (but *not* backward geodesically complete), it has constant negative flag curvature $-\frac{1}{4}$ and it is contractible (thus, simply connected), the conclusion of Theorem 2.3.2 may be false. Indeed, one can find points in M (belonging to the same geodesic) such that the minimum point for the total forward (resp. backward) cost function is *not* situated on the geodesic, see Figure 2.3.

Remark 2.4.1 Note that Example 2.4.2 (Finslerian-Poincaré disc) may give a model of a *gravitational field* whose centre of gravity is located at the origin $O = (0, 0)$, while the boundary ∂M means the "infinity".

Suppose that in this gravitational field, we have several spaceships, which are delivering some cargo to certain bases or to another spacecraft. Also, assume that these spaceships are of the same type and they consume k liter/second fuel ($k > 0$). Note that the expression $F(d\sigma)$ denotes the physical *time* elapsed to traverse a short portion $d\sigma$ of the spaceship orbit. Consequently, traversing a short path $d\sigma$, a spaceship consumes $kF(d\sigma)$ liter of fuel. In this way, the number $k\int_0^1 F(\sigma(t), d\sigma(t))dt$ expresses the quantity of fuel used up by a spaceship traversing an orbit $\sigma : [0,1] \to M$.

Suppose that two spaceships have to meet each other (for logistical reasons) starting their trip from bases P_1 and P_2, respectively. Consuming as low total quantity of fuel as possible, they will choose T_b as a meeting point and *not* P_b on the geodesic determined by P_1 and P_2. Thus, the point T_b could be a position for an optimal deposit-base.

Now, suppose that we have two damaged spacecraft (e.g., without fuel) at positions P_1 and P_2. Two rescue spaceships consuming as low total quantity of fuel as possible, will blastoff from base T_f and not from $P_f = P_1$ on the geodesic determined by P_1 and P_2. In this case, the point T_f is the position for an optimal rescue-base. If the spaceships in trouble are close to the center of the gravitational field M, then any rescue-base located closely also to the center O, implies the consumption of a great amount of energy (fuel) by the rescue spaceships in order to reach their destinations (namely, P_1 and P_2). Indeed, they have to overcome the strong gravitational force near the centre O. Consequently, this is the reason why the point T_f is so far from O, as Figure 2.3 shows. Note that further numerical experiments support this observation. However, there are certain special cases when the position of the optimal rescue-base is either P_1 or P_2: from these two points, the farthest one from the gravitational center O will be the position of the rescue-base. In such case, the orbit of the (single) rescue spaceship is exactly the geodesic determined by points P_1 and P_2.

2.5 Comments

The results of this chapter are based on the paper of Kristály and Kozma [22], and Kristály, Moroşanu and Róth [24]. As we emphasized in the text, real life phenomena may be well described by using Finsler metrics representing external force as current or gravitation (as the slope metric of a hillside or the gravitational Finslerian-Poincaré ball). In this chapter we studied not necessarily reversible Weber-type problems within the framework of Riemann-Finsler geometry taking into account the curvature and geometric structure of the ambient space.

Chapter 3

Nash-type equilibria on manifolds

I did have strange ideas during
certain periods of time.

John F. Nash (b. 1928)

3.1 Introduction

After the seminal papers of Nash [29], [30] there has been considerable interest in the theory of Nash equilibria due to its applicability in various real-life phenomena (game theory, price theory, networks, etc). Appreciating Nash's contributions, R. B. Myerson states that "*Nash's theory of noncooperative games should now be recognized as one of the outstanding intellectual advances of the twentieth century*". The Nash equilibrium problem involves n players such that each player knows the equilibrium strategies of the partners, but moving away from his/her own strategy alone a player has nothing to gain. Formally, if the sets K_i denote the strategies of the players and $f_i : K_1 \times ... \times K_n \to \mathbb{R}$ are their loss-functions, $i \in \{1, ..., n\}$, the problem is to find an n-tuple $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{K} = K_1 \times ... \times K_n$ such that $f_i(\mathbf{p}) \leq f_i(\mathbf{p}; q_i)$ for every $q_i \in K_i$ and $i \in \{1, ..., n\}$, where $(\mathbf{p}; q_i) = (p_1, ..., p_{i-1}, q_i, p_{i+1}, ..., p_n) \in \mathbf{K}$. Such point $\mathbf{p}$ is called a *Nash equilibrium point for* $(\mathbf{f}, \mathbf{K}) = (f_1, ..., f_n; K_1, ..., K_n)$, the set of these points being denoted in the sequel by $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$.

While most of the known developments in the Nash equilibrium theory deeply exploit the usual convexity of the sets K_i together with the vector space structure of their ambient spaces M_i (i.e., $K_i \subset M_i$), it is nevertheless true that these results are in large part *geometrical* in nature. The main purpose of this chapter is to enhance those geometrical and analytical structures which serve as a basis of a systematic study of Nash-type equilibrium problems in a general setting as possible. In the light of these facts our contribution to the Nash equilibrium theory should be considered rather intrinsical and analytical than game-theoretical. For the sake of completeness, we mention some works where Nash equilibrium problems were studied in a non-standard case; however, these results are weakly connected only to our results; the convexity/regularity of the payoff functions f_i are relaxed (see for instance Kassay, Kolumbán and Páles [18], and Ziad [36]), or the convexity of the strategy sets K_i are weakened (see Nessah and Kerstens [31], and Tala and Marchi [35]).

We assume *a priori* that the strategy sets K_i are *geodesic convex* subsets of certain finite-dimensional Riemannian manifolds (M_i, g_i). This approach can be widely applied when the strategy sets are 'curved'; note that the choice of such Riemannian structures does not influence the Nash equilibrium points for $(\mathbf{f}, \mathbf{K})$. The first step into this direction was made recently in [20], guaranteeing the existence of at least one Nash equilibrium point for $(\mathbf{f}, \mathbf{K})$ whenever $K_i \subset M_i$ are compact and geodesic convex sets of certain finite-dimensional Riemannian manifolds (M_i, g_i) while the functions f_i have certain regularity properties, $i \in \{1, ..., n\}$.

In [20] we introduced and studied for a wide class of non-smooth functions the set of *Nash-Clarke points for* $(\mathbf{f}, \mathbf{K})$, denoted in the sequel as $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$; for details, see §3.2. Note that $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ is larger than $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$; thus, a promising way to find the elements of $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$ is to determine the set $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$. In spite of the naturalness of this approach, we already pointed out its limited applicability due to the involved structure of $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$, conjecturing a more appropriate concept in order to locate the elements of $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$.

Motivated by the latter problem, we observe that the Fréchet and limiting subdifferential calculus of lower semicontinuous functions on Riemannian manifolds developed by Ledyaev and Zhu [25] and Azagra, Ferrera and López-Mesas [1] provides a very satisfactory approach. The idea is to consider the following system of variational inequalities: find $\mathbf{p} \in \mathbf{K}$ and $\xi_C^i \in \partial_C^i f_i(\mathbf{p})$ such that

$$\langle \xi_C^i, \exp_{p_i}^{-1}(q_i) \rangle_{g_i} \geq 0 \text{ for all } q_i \in K_i,\ i \in \{1, ..., n\},$$

where $\partial_C^i f_i(\mathbf{p})$ denotes the Clarke subdifferential of the locally Lipschitz function $f_i(\mathbf{p}; \cdot)$ at the point $p_i \in K_i$; for details, see §3.2. The solutions of this system form the set of *Nash-Stampacchia equilibrium points for* $(\mathbf{f}, \mathbf{K})$, denoted by $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$, which is the main concept of the third chapter.

One of the advantages of the new concept is that the set $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ is 'closer' to $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$ than $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$. More precisely, we state that $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ for the same class of non-smooth functions $\mathbf{f} = (f_1, ..., f_n)$ as in [20] (see Theorem 3.2.3 (i)-(ii)). Moreover, if $\mathbf{f} = (f_1, ..., f_n)$ verifies a suitable convexity assumption then the three Nash-type equilibria coincide (see Theorem 3.2.3 (iii)).

The main purpose of this chapter is to establish existence, location and stability of Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ in different settings. While a Nash equilibrium point is obtained precisely as the fixed point of a suitable function (see Nash's original proof via Kakutani fixed-point theorem), Nash-Stampacchia equilibrium points are expected to be characterized in a similar way as fixed points of a special map defined on the product Riemannian manifold $\mathbf{M} = M_1 \times ... \times M_n$ endowed with its natural Riemannian metric $\mathbf{g}$ inherited from the metrics g_i, $i \in \{1, ..., n\}$. In order to achieve this aim, certain curvature and topological restrictions are needed on the manifolds (M_i, g_i). By assuming that the ambient Riemannian manifolds (M_i, g_i) for the geodesic convex strategy sets K_i are *Hadamard manifolds*, the key observation (see Theorem 3.3.1) is that $\mathbf{p} \in \mathbf{K}$ is a Nash-Stampacchia equilibrium point for $(\mathbf{f}, \mathbf{K})$ if and only if $\mathbf{p}$ is a fixed point of the set-valued map $A_\alpha^{\mathbf{f}} : \mathbf{K} \to 2^{\mathbf{K}}$ defined by

$$A_\alpha^{\mathbf{f}}(\mathbf{p}) = P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha \partial_C^{\Delta} \mathbf{f}(\mathbf{p}))).$$

Here, $P_{\mathbf{K}}$ is the metric projection operator associated to the geodesic convex set $\mathbf{K} \subset \mathbf{M}$, $\alpha > 0$ is a fixed number, and $\partial_C^\Delta \mathbf{f}(\mathbf{p})$ is the diagonal Clarke subdifferential at point $\mathbf{p}$ of $\mathbf{f} = (f_1, ..., f_n)$; see §3.2.

Within this geometrical framework, two cases are discussed. First, when $\mathbf{K} \subset \mathbf{M}$ is *compact*, one can prove via the Begle's fixed point theorem for set-valued maps the existence of at least one Nash-Stampacchia equilibrium point for $(\mathbf{f}, \mathbf{K})$ (see Theorem 3.3.2). Second, we consider the case when $\mathbf{K} \subset \mathbf{M}$ is *not* necessarily *compact*. By requiring more regularity on $\mathbf{f}$ in order to avoid technicalities, we consider two dynamical systems; a discrete one

$$(DDS)_\alpha \qquad \mathbf{p}_{k+1} = A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\mathbf{p}_k)), \quad \mathbf{p}_0 \in \mathbf{M};,$$

and a continuous one

$$(CDS)_\alpha \qquad \begin{cases} \dot{\eta}(t) = \exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))) \\ \eta(0) = \mathbf{p}_0 \in \mathbf{M}. \end{cases}$$

The main result (see Theorem 3.3.3) proves that the set of Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ is a *singleton* and the orbits of both dynamical systems exponentially converge to this unique point whenever a Lipschitz-type condition holds on $\partial_C^\Delta \mathbf{f}$. It is clear by construction that the orbit of $(DDS)_\alpha$ is viable relative to the set $\mathbf{K}$, i.e., $\mathbf{p}_k \in \mathbf{K}$ for every $k \geq 1$. By using a recent result of Ledyaev and Zhu [25], one can also prove an invariance property of the set $\mathbf{K}$ with respect to the orbit of $(CDS)_\alpha$. Note that the aforementioned results concerning the 'projected' dynamical system $(CDS)_\alpha$ are new even in the Euclidean setting studied by Cavazzuti, Pappalardo and Passacantando [10], Xia and Wang [39].

Since the manifolds (M_i, g_i) are assumed to be of Hadamard type (see Theorems 3.3.1-3.3.3), so is the product manifold $(\mathbf{M}, \mathbf{g})$. Our analytical arguments concerning Nash-Stampacchia equilibrium problems deeply exploit two geometrical features of closed, geodesic convex sets of the product *Hadamard manifold* $(\mathbf{M}, \mathbf{g})$:

(A) *Validity of the obtuse-angle property*, see Proposition 1.3.1 (i). This fact is exploited in the characterization of Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ via the fixed points of the map $A_\alpha^{\mathbf{f}}$, see Theorem 3.3.1.

(B) *Non-expansiveness of the projection operator*, see Proposition 1.3.1 (ii). This property is applied several times in the proof of Theorems 3.3.2-3.3.3.

It is natural to ask to what extent the Riemannian structures of (M_i, g_i) are determined when the properties (A) and (B) simultaneously hold on the product manifold $(\mathbf{M}, \mathbf{g})$. A constructive proof combined with the parallelogramoid of Levi-Civita and a result of Chen [13] shows that if (M_i, g_i) are complete, simply connected Riemannian manifolds then (A) and (B) are both verified on $(\mathbf{M}, \mathbf{g})$ if and only if (M_i, g_i) are Hadamard manifolds (see Theorem 3.4.1). Thus, we may assert that Hadamard manifolds are the optimal geometrical framework to elaborate a fruitful theory of Nash-Stampacchia equilibrium problems on manifolds.

The chapter is divided as follows. In §3.2 we compare the three Nash-type equilibria. In §3.3, we prove the main results of this paper. First, we are dealing with the existence of Nash-Stampacchia points for $(\mathbf{f}, \mathbf{K})$ in the compact case. Then, the uniqueness and exponential stability of Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ is proved whenever $\mathbf{K}$ is not necessarily compact in the Hadamard manifold $(\mathbf{M}, \mathbf{g})$. We present an example in both cases. In §3.4 we characterize the geometric properties (A) and (B) on $(\mathbf{M}, \mathbf{g})$ by the Hadamard structures of the complete and simply connected Riemannian manifolds (M_i, g_i), $i \in \{1, ..., n\}$. Finally, in Section §3.5 some model examples are presented showing the applicability of our results.

In the sequel, the following notations are used:

- $\mathbf{K} = K_1 \times ... \times K_n$;
- $\mathbf{f} = (f_1, ..., f_n)$;
- $(\mathbf{f}, \mathbf{K}) = (f_1, ..., f_n; K_1, ..., K_n)$;
- $\mathbf{p} = (p_1, ..., p_n)$;
- $(\mathbf{p}; q_i) = (p_1, ..., p_{i-1}, q_i, p_{i+1}, ..., p_n)$;
- $(\mathbf{K}; U_i) = K_1 \times ... \times K_{i-1} \times U_i \times K_{i+1} \times ... \times K_n$, for some $U_i \supset K_i$.

3.2 Nash-type equilibria on Riemannian manifolds: basic existence and comparison results

Let $K_1, ..., K_n$ $(n \geq 2)$ be non-empty sets, corresponding to the strategies of n players and $f_i : K_1 \times ... \times K_n \to \mathbb{R}$ $(i \in \{1, ..., n\})$ be the payoff functions, respectively.

Definition 3.2.1 *The set of* Nash equilibrium points for $(\mathbf{f}, \mathbf{K})$ *is*

$$\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = \{\mathbf{p} \in \mathbf{K} : f_i(\mathbf{p}; q_i) \geq f_i(\mathbf{p}) \quad \textit{for all } q_i \in K_i,\ i \in \{1, ..., n\}\}.$$

The main result of the paper [20] states that in a quite general framework the set of Nash equilibrium points for $(\mathbf{f}, \mathbf{K})$ is not empty. More precisely, we have

Theorem 3.2.1 [20] *Let (M_i, g_i) be finite-dimensional Riemannian manifolds; $K_i \subset M_i$ be non-empty, compact, geodesic convex sets; and $f_i : \mathbf{K} \to \mathbb{R}$ be continuous functions such that $K_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$ is convex on K_i for every $\mathbf{p} \in \mathbf{K}$, $i \in \{1, ..., n\}$. Then there exists at least one Nash equilibrium point for $(\mathbf{f}, \mathbf{K})$, i.e., $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) \neq \emptyset$.*

Proof. We apply Proposition 1.6.1 by choosing $X = \mathbf{K} = \Pi_{i=1}^n K_i$ and $h : X \times X \to \mathbb{R}$ defined by $h(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^n [f_i(\mathbf{p}; q_i) - f_i(\mathbf{p})]$. First of all, note that the sets K_i are ANRs, being closed subsets of finite-dimensional manifolds (thus, locally contractible spaces). Moreover, since a product of a finite family of ANRs is an ANR (see Bessage and Pelczyński [5, p. 69]), it follows that X is an ANR. Due to Proposition 1.6.3, X is contractible, thus acyclic.

Note that the function h is continuous, and $h(\mathbf{p}, \mathbf{p}) = 0$ for every $\mathbf{p} \in X$. Consequently, the set $\{(\mathbf{q}, \mathbf{p}) \in X \times X : 0 > h(\mathbf{q}, \mathbf{p})\}$ is open.

It remains to prove that $S_\mathbf{p} = \{\mathbf{q} \in X : 0 > h(\mathbf{q}, \mathbf{p})\}$ is contractible or empty for all $\mathbf{p} \in X$. Assume that $S_\mathbf{p} \neq \emptyset$ for some $\mathbf{p} \in X$. Then, there exists $i_0 \in \{1, ..., n\}$ such that $f_{i_0}(\mathbf{p}; q_{i_0}) - f_{i_0}(\mathbf{p}) < 0$ for some $q_{i_0} \in K_{i_0}$. Therefore, $\mathbf{q} = (\mathbf{p}; q_{i_0}) \in S_\mathbf{p}$, i.e., $\text{pr}_i S_\mathbf{p} \neq \emptyset$ for every $i \in \{1, ..., n\}$. Now, we fix $\mathbf{q}^j = (q_1^j, ..., q_n^j) \in S_\mathbf{p}$, $j \in \{1, 2\}$ and let $\gamma_i : [0, 1] \to K_i$ be the unique geodesic joining the points $q_i^1 \in K_i$ and $q_i^2 \in K_i$ (note that K_i is geodesic convex), $i \in \{1, ..., n\}$. Let $\gamma : [0, 1] \to \mathbf{K}$ defined by $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t))$. Due to the convexity of the function $K_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$, for every $t \in [0, 1]$, we have

$$\begin{aligned} h(\gamma(t), \mathbf{p}) &= \sum_{i=1}^{n} [f_i(\mathbf{p}; \gamma_i(t)) - f_i(\mathbf{p})] \\ &\leq \sum_{i=1}^{n} [t f_i(\mathbf{p}; \gamma_i(1)) + (1-t) f_i(\mathbf{p}; \gamma_i(0)) - f_i(\mathbf{p})] \\ &= t h(\mathbf{q}^2, \mathbf{p}) + (1-t) h(\mathbf{q}^1, \mathbf{p}) \\ &< 0. \end{aligned}$$

Consequently, $\gamma(t) \in S_\mathbf{p}$ for every $t \in [0, 1]$, i.e., $S_\mathbf{p}$ is a geodesic convex set in the product manifold $\mathbf{M} = M_1 \times ... \times M_n$ endowed with its natural (warped-)product metric (with the constant weight functions 1), see O'Neill [32, p. 208]. Now, Proposition 1.6.3 implies that $S_\mathbf{p}$ is contractible. Alternatively, we may exploit the fact that the projections $\mathrm{pr}_i S_\mathbf{p}$ are geodesic convex, thus contractible sets, $i \in \{1, ..., n\}$.

On account of Proposition 1.6.1, there exists $\mathbf{p} \in \mathbf{K}$ such that $0 = h(\mathbf{p}, \mathbf{p}) \leq h(\mathbf{q}, \mathbf{p})$ for every $\mathbf{q} \in \mathbf{K}$. In particular, putting $\mathbf{q} = (\mathbf{p}; q_i)$, $q_i \in K_i$ fixed, we obtain that $f_i(\mathbf{p}; q_i) - f_i(\mathbf{p}) \geq 0$ for every $i \in \{1, ..., n\}$, i.e., $\mathbf{p}$ is a Nash equilibrium point for $(\mathbf{f}, \mathbf{K})$. $\square$

Similarly to Theorem 3.2.1, let us assume that for every $i \in \{1, ..., n\}$, one can find a finite-dimensional Riemannian manifold (M_i, g_i) such that the strategy set K_i is closed and geodesic convex in (M_i, g_i). Let $\mathbf{M} = M_1 \times ... \times M_n$ be the product manifold with its standard Riemannian product metric

$$\mathbf{g}(\mathbf{V}, \mathbf{W}) = \sum_{i=1}^{n} g_i(V_i, W_i)$$

for every $\mathbf{V} = (V_1, ..., V_n), \mathbf{W} = (W_1, ..., W_n) \in T_{p_1}M_1 \times ... \times T_{p_n}M_n = T_\mathbf{p}\mathbf{M}$. Let $\mathbf{U} = U_1 \times ... \times U_n \subset \mathbf{M}$ be an open set such that $\mathbf{K} \subset \mathbf{U}$; we always mean that U_i inherits the Riemannian structure of (M_i, g_i). Let

$$\mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})} = \{\mathbf{f} \in C^0(\mathbf{K}, \mathbb{R}^n) \; : \; \begin{array}{l} f_i : (\mathbf{K}; U_i) \to \mathbb{R} \text{ is continuous and} \\ f_i(\mathbf{p}; \cdot) \text{ is locally Lipschitz on } (U_i, g_i) \\ \text{for all } \mathbf{p} \in \mathbf{K}, \ i \in \{1, ..., n\}\}. \end{array}$$

The next notion has been introduced in [20].

Definition 3.2.2 *Let* $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$. *The set of* Nash-Clarke points for $(\mathbf{f}, \mathbf{K})$ *is*

$$\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K}) = \{\mathbf{p} \in \mathbf{K} : f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i)) \geq 0 \ \textit{for all } q_i \in K_i, \ i \in \{1, ..., n\}\}.$$

Here, $f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i))$ denotes the Clarke generalized derivative of $f_i(\mathbf{p}; \cdot)$ at point $p_i \in K_i$ in direction $\exp_{p_i}^{-1}(q_i) \in T_{p_i}M_i$. More precisely,

$$f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i)) = \limsup_{q \to p_i, q \in U_i, \ t \to 0^+} \frac{f_i(\mathbf{p}; \sigma_{q, \exp_{p_i}^{-1}(q_i)}(t)) - f_i(\mathbf{p}; q)}{t}, \quad (3.1)$$

where $\sigma_{q,v}(t) = \exp_q(tw)$, and $w = d(\exp_q^{-1} \circ \exp_{p_i})_{\exp_{p_i}^{-1}(q)} v$ for $v \in T_{p_i}M_i$, and $|t|$ is small enough. The following existence result is available concerning the Nash-Clarke points for $(\mathbf{f}, \mathbf{K})$.

Theorem 3.2.2 [20] *Let (M_i, g_i) be complete finite-dimensional Riemannian manifolds; $K_i \subset M_i$ be non-empty, compact, geodesic convex sets; and $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$ such that for every $\mathbf{p} \in \mathbf{K}$, $i \in \{1, ..., n\}$, $K_i \ni q_i \mapsto f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i))$ is convex and f_i^0 is upper semicontinuous on its domain of definition. Then there exists at least one Nash-Clarke point for $(\mathbf{f}, \mathbf{K})$, i.e., $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K}) \neq \emptyset$.*

Proof. The proof is similar to that of Theorem 3.2.1; we show only the differences. Let $X = \mathbf{K} = \Pi_{i=1}^n K_i$ and $h : X \times X \to \mathbb{R}$ defined by $h(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^n f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(q_i))$. It is clear that $h(\mathbf{p}, \mathbf{p}) = 0$ for every $\mathbf{p} \in X$.

First of all, the upper-semicontinuity of $h(\cdot, \cdot)$ on $X \times X$ implies the fact that the set $\{(\mathbf{q}, \mathbf{p}) \in X \times X : 0 > h(\mathbf{q}, \mathbf{p})\}$ is open.

Now, let $\mathbf{p} \in X$ such that $S_{\mathbf{p}} = \{\mathbf{q} \in X : 0 > h(\mathbf{q}, \mathbf{p})\}$ is not empty. Then, there exists $i_0 \in \{1, ..., n\}$ such that $f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(q_{i_0})) < 0$ for some $q_{i_0} \in K_{i_0}$. Consequently, $\mathbf{q} = (\mathbf{p}; q_{i_0}) \in S_{\mathbf{p}}$, i.e., $\mathrm{pr}_i S_{\mathbf{p}} \neq \emptyset$ for every $i \in \{1, ..., n\}$. Now, we fix $\mathbf{q}^j = (q_1^j, ..., q_n^j) \in S_{\mathbf{p}}$, $j \in \{1, 2\}$, and let $\gamma_i : [0, 1] \to K_i$ be the unique geodesic joining the points $q_i^1 \in K_i$ and $q_i^2 \in K_i$. Let also $\gamma : [0, 1] \to \mathbf{K}$ defined by $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t))$. Since $K_i \ni q_i \mapsto f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i))$ is convex, the convexity of the function $[0, 1] \ni t \mapsto h(\gamma(t), \mathbf{p})$, $t \in [0, 1]$ easily follows. Therefore, $\gamma(t) \in S_{\mathbf{p}}$ for every $t \in [0, 1]$, i.e., $S_{\mathbf{p}}$ is a geodesic convex set, thus contractible.

Thus, Proposition 1.6.1 implies the existence of $\mathbf{p} \in \mathbf{K}$ such that $0 = h(\mathbf{p}, \mathbf{p}) \leq h(\mathbf{q}, \mathbf{p})$ for every $\mathbf{q} \in \mathbf{K}$. In particular, if $\mathbf{q} = (\mathbf{p}; q_i)$, $q_i \in K_i$ fixed, we obtain that $f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(q_i)) \geq 0$ for every $i \in \{1, ..., n\}$, i.e., $\mathbf{p}$ is a Nash-Clarke point for $(\mathbf{f}, \mathbf{K})$. The proof is complete. □

Remark 3.2.1 Although Theorem 3.2.2 gives a possible approach to locate Nash equilibrium points on Riemannian manifolds, its applicability is quite reduced. As far as we know, only two special cases can be described which imply the convexity of $K_i \ni q_i \mapsto f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i))$; namely,

(a) (M_i, g_i) is Euclidean, $i \in I_1$;

(b) $K_i = \mathrm{Im}\gamma_i$ where $\gamma_i : [0, 1] \to M_i$ is a minimal geodesic and $f_i^0(\mathbf{p}, \dot\gamma_i(t_i)) \geq -f_i^0(\mathbf{p}, -\dot\gamma_i(t_i))$, $i \in I_2$ for every $\mathbf{p} \in \mathbf{K}$ with $p_i = \gamma_i(t_i)$ $(0 \leq t_i \leq 1)$. Note that the sets $I_1, I_2 \subset \{1, ..., n\}$ are such that $I_1 \cup I_2 = \{1, ..., n\}$.

Let us discuss in the sequel these items.

(a) The problem reduces to the property that the Clarke generalized derivative $K_i \ni q_i \mapsto f_i^0(\mathbf{p}; q_i)$ is subadditive and positively homogeneous, thus convex, see Clarke [14, Proposition 2.1.1]. Note that in this case $\exp_{p_i} = p_i + \mathrm{id}_{\mathbb{R}^{\dim M_i}}$.

(b) If $\sigma_i : [0, 1] \to M_i$ is a geodesic segment joining the points $\sigma_i(0) = \gamma_i(\tilde t_0)$ with $\sigma_i(1) = \gamma_i(\tilde t_1)$ $(0 \leq \tilde t_0 < \tilde t_1 \leq 1)$, then $\mathrm{Im}\sigma_i \subseteq \mathrm{Im}\gamma_i = K_i$. Fix $p_i = \gamma_i(t_i) \in K_i$ $(0 \leq \tilde t_i \leq 1)$. Let $a_0, a_1 \in \mathbb{R}$ $(a_0 < a_1)$ such that $\exp_{p_i}(a_0\dot\gamma_i(t_i)) = \gamma_i(\tilde t_0)$ and $\exp_{p_i}(a_1\dot\gamma_i(t_i)) = \gamma_i(\tilde t_1)$. Then, $\sigma_i(t) = \exp_{p_i}((a_0 + (a_1 - a_0)t)\dot\gamma_i(t_i))$. The claim follows if $t \mapsto f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(\sigma_i(t))) = f_i^0(\mathbf{p}, (a_0 + (a_1 - a_0)t)\dot\gamma_i(t_i)) := g(t)$ is convex. If

$a_0 \geq 0$ or $a_1 \leq 0$, then g is affine. If $a_0 < 0 < a_1$, then

$$g(t) = \begin{cases} -(a_0 + (a_1 - a_0)t) f_i^0(\mathbf{p}, -\dot{\gamma}_i(t_i)), & t \in [0, -a_0/(a_1 - a_0)], \\ (a_0 + (a_1 - a_0)t) f_i^0(\mathbf{p}, \dot{\gamma}_i(t_i)), & t \in (-a_0/(a_1 - a_0), 1]. \end{cases}$$

Therefore, g is convex if and only if $-f_i^0(\mathbf{p}, -\dot{\gamma}_i(t_i)) \leq f_i^0(\mathbf{p}, \dot{\gamma}_i(t_i))$.

(c) We finally emphasize that the inequality in the second case holds automatically whenever $D_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$ is either convex (see Udrişte [37, Theorem 4.2, p. 71-72]), or it is of class C^1, for every $\mathbf{p} \in \mathbf{K}$, $i \in I_2$.

The limited applicability of Theorem 3.2.2 motivates the introduction and study of the following concept which plays the central role in the present chapter.

Definition 3.2.3 *Let* $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$. *The set of* Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ *is*

$$\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \{\mathbf{p} \in \mathbf{K} \ : \ \exists \xi_C^i \in \partial_C^i f_i(\mathbf{p}) \ \textit{such that} \ \langle \xi_C^i, \exp_{p_i}^{-1}(q_i) \rangle_{g_i} \geq 0, \ \textit{for all} \ q_i \in K_i, \ i \in \{1, ..., n\}\}.$$

Here, $\partial_C^i f_i(\mathbf{p})$ denotes the Clarke subdifferential of the function $f_i(\mathbf{p}; \cdot)$ at point $p_i \in K_i$, i.e., $\partial_C f_i(\mathbf{p}; \cdot)(p_i) = \mathrm{co}(\partial_L f_i(\mathbf{p}; \cdot)(p_i))$.

Our first aim is to compare the three Nash-type points introduced in Definitions 3.2.1-3.2.3. Before to do that, we introduce another class of functions. If $U_i \subset M_i$ is geodesic convex for every $i \in \{1, ..., n\}$, we may define

$$\mathcal{K}_{(\mathbf{K},\mathbf{U},\mathbf{M})} = \{\mathbf{f} \in C^0(\mathbf{K}, \mathbb{R}^n) \ : \ f_i : (\mathbf{K}; U_i) \to \mathbb{R} \text{ is continuous and } f_i(\mathbf{p}; \cdot) \text{ is convex on } (U_i, g_i) \text{ for all } \mathbf{p} \in \mathbf{K}, \ i \in \{1, ..., n\}\}.$$

Remark 3.2.2 Due to Azagra, Ferrera and López-Mesas [1, Proposition 5.2], one has $\mathcal{K}_{(\mathbf{K},\mathbf{U},\mathbf{M})} \subset \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$ whenever $U_i \subset M_i$ is geodesic convex for every $i \in \{1, ..., n\}$.

The main result of this section reads as follows.

Theorem 3.2.3 [21] *Let (M_i, g_i) be finite-dimensional Riemannian manifolds; $K_i \subset M_i$ be non-empty, closed, geodesic convex sets; $U_i \subset M_i$ be open sets containing K_i; and $f_i : \mathbf{K} \to \mathbb{R}$ be some functions, $i \in \{1, ..., n\}$. Then, we have*

(i) $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ *whenever* $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$;

(ii) $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ *whenever* $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$ *and $U_i \subset M_i$ are geodesic convex for every $i \in \{1, ..., n\}$;*

(iii) $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ *whenever* $\mathbf{f} \in \mathcal{K}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$.

Proof. (i) Let $\mathbf{p} \in \mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$ and fix $i \in \{1, ..., n\}$. Since $f_i(\mathbf{p}; q_i) \geq f_i(\mathbf{p})$ for all $q_i \in K_i$, then

$$f_i(\mathbf{p}; q_i) + \delta_{K_i}(q_i) - f_i(\mathbf{p}) - \delta_{K_i}(p_i) \geq 0 \text{ for all } q_i \in U_i,$$

which means that $p_i \in K_i$ is a global minimum of $f_i(\mathbf{p}; \cdot) + \delta_{K_i}$ on U_i. According to Proposition 1.4.2 (iii), one has $0 \in \partial_L(f_i(\mathbf{p}; \cdot) + \delta_{K_i})(p_i)$. Since $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$, conform Proposition 1.4.4 (i)$\Leftrightarrow$(iii), we have that $\partial_\infty f_i(\mathbf{p}; \cdot)(p_i) = \{0\}$. Thus, considering the functions $f_i(\mathbf{p}; \cdot)$ and δ_{K_i} in Proposition 1.4.3, we may exclude its second alternative, obtaining

$$\begin{aligned} 0 &\in \partial_L f_i(\mathbf{p}; \cdot)(p_i) + \partial_L \delta_{K_i}(p_i) = \partial_L f_i(\mathbf{p}; \cdot)(p_i) + N_L(p_i; K_i) \\ &\subset \partial_C f_i(\mathbf{p}; \cdot)(p_i) + N_L(p_i; K_i) = \partial_C^i f_i(\mathbf{p}) + N_L(p_i; K_i). \end{aligned}$$

Consequently, there exists $\xi_C^i \in \partial_C^i f_i(\mathbf{p})$ with $-\xi_C^i \in N_L(p_i; K_i)$. On account of Theorem 1.4.1, we obtain $\langle \xi_C^i, \exp_{p_i}^{-1}(q_i) \rangle_{g_i} \geq 0$ for all $q_i \in K_i$, i.e., $\mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$.

(ii) Let $\mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$. Fix also arbitrarily $i \in \{1, ..., n\}$ and $q_i \in K_i$. It follows that there exists $\xi_C^i \in \partial_C^i f_i(\mathbf{p}) = \partial_C f_i(\mathbf{p}; \cdot)(p_i)$ such that

$$\langle \xi_C^i, \exp_{p_i}^{-1}(q_i) \rangle_{g_i} \geq 0. \tag{3.2}$$

By definition, there exist some $\lambda_l \geq 0$, $l \in J$, with card$J < \infty$ and $\sum_{l \in J} \lambda_l = 1$ such that $\xi_{L,l}^i \in \partial_L f_i(\mathbf{p}; \cdot)(p_i)$ and $\xi_C^i = \sum_l \lambda_l \xi_{L,l}^i$. Consequently, for each $l \in J$, there exists a sequence $\{p_{i,l}^k\} \subset U_i$ and $\xi_{i,l}^k \in \partial_F f_i(\mathbf{p}; \cdot)(p_{i,l}^k)$ with

$$\lim_k p_{i,l}^k = p_i, \ \lim_k \xi_{i,l}^k = \xi_{L,l}^i. \tag{3.3}$$

We may assume that $p_{i,l}^k \neq q_i$ for each $k \in \mathbb{N}$ and $l \in J$. In view of Proposition 1.4.1 (i)$\Leftrightarrow$(ii), we have in particular that

$$\liminf_{t\to 0^+} \frac{f_i(\mathbf{p}; \exp_{p_{i,l}^k}(t \exp_{p_{i,l}^k}^{-1}(q_i))) - f_i(\mathbf{p}; p_{i,l}^k) - \langle \xi_{i,l}^k, t \exp_{p_{i,l}^k}^{-1}(q_i)\rangle_{g_i}}{t d_{g_i}(p_{i,l}^k, q_i)} \geq 0. \quad (3.4)$$

Indeed, since $U_i \subset M_i$ is convex, we may choose $\psi = \exp_{p_{i,l}^k}^{-1} : U_i \to T_{p_{i,l}^k} M_i = \mathbb{R}^{\dim M_i}$ and $v = t \exp_{p_{i,l}^k}^{-1}(q_i)$ with $t \to 0^+$; consequently, $\psi(p_{i,l}^k) = 0$, $d\psi^{-1}(\psi(p_{i,l}^k)) = d \exp_{p_{i,l}^k}(0) = \mathrm{id}_{T_{p_{i,l}^k} M_i}$, and $\| \exp_{p_{i,l}^k}^{-1}(q_i)\|_{g_i} = d_{g_i}(p_{i,l}^k, q_i)$.

Now, by (3.1) and (3.4) it follows that for every $k \in \mathbb{N}$,

$$f_i^0((\mathbf{p}; p_{i,l}^k), \exp_{p_{i,l}^k}^{-1}(q_i)) \geq \langle \xi_{i,l}^k, \exp_{p_{i,l}^k}^{-1}(q_i)\rangle_{g_i}.$$

By the upper-semicontinuity of $f_i^0((\mathbf{p}; \cdot), \cdot)$ and relation (3.3), we have that

$$\begin{aligned} f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i)) &= f_i^0((\mathbf{p}; p_i), \exp_{p_i}^{-1}(q_i)) \\ &\geq \limsup_k f_i^0((\mathbf{p}; p_{i,l}^k), \exp_{p_{i,l}^k}^{-1}(q_i)) \\ &\geq \limsup_k \langle \xi_{i,l}^k, \exp_{p_{i,l}^k}^{-1}(q_i)\rangle_{g_i} \\ &= \langle \xi_{L,l}^i, \exp_{p_i}^{-1}(q_i)\rangle_{g_i}. \end{aligned}$$

Multiplying by λ_l the above inequality and adding them for each $l \in J$, from relation (3.2) we obtain that

$$f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i)) \geq \langle \sum_{l\in J} \lambda_l \xi_{L,l}^i, \exp_{p_i}^{-1}(q_i)\rangle_{g_i} = \langle \xi_C^i, \exp_{p_i}^{-1}(q_i)\rangle_{g_i} \geq 0.$$

In conclusion, we have that $\mathbf{p} \in \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$.

(iii) Due to (i)-(ii) and Remark 3.2.2, it is enough to prove that $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$. Let $\mathbf{p} \in \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$, i.e., for every $i \in \{1, ..., n\}$ and $q_i \in K_i$,

$$f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i)) \geq 0. \quad (3.5)$$

Fix $i \in \{1, ..., n\}$ and $q_i \in K_i$ arbitrary. Since $f_i(\mathbf{p}; \cdot)$ is convex on (U_i, g_i), on account of (1.20), we have

$$f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i)) = \lim_{t \to 0^+} \frac{f_i(\mathbf{p}; \exp_{p_i}(t \exp_{p_i}^{-1}(q_i))) - f_i(\mathbf{p})}{t}. \tag{3.6}$$

Note that the function

$$R(t) = \frac{f_i(\mathbf{p}; \exp_{p_i}(t \exp_{p_i}^{-1}(q_i))) - f_i(\mathbf{p})}{t}$$

is well-defined on the whole interval $(0, 1]$; indeed, $t \mapsto \exp_{p_i}(t \exp_{p_i}^{-1}(q_i))$ is the minimal geodesic joining the points $p_i \in K_i$ and $q_i \in K_i$ which belongs to $K_i \subset U_i$. Moreover, it is well-known that $t \mapsto R(t)$ is non-decreasing on $(0, 1]$. Consequently,

$$f_i(\mathbf{p}; q_i) - f_i(\mathbf{p}) = f_i(\mathbf{p}; \exp_{p_i}(\exp_{p_i}^{-1}(q_i))) - f_i(\mathbf{p}) = R(1) \geq \lim_{t \to 0^+} R(t).$$

On the other hand, (3.5) and (3.6) give that $\lim_{t \to 0^+} R(t) \geq 0$, which concludes the proof. $\square$

Remark 3.2.3 In [20] we considered the sets $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$ and $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$. Note however that the set of Nash-Stampacchia equilibrium points, i.e. $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$, which is between the former ones, seems to be the most appropriate concept to find Nash equilibrium points in very general contexts: (a) the set of Nash-Stampacchia equilibria is larger than those of Nash equilibrium points; (b) an efficient theory of Nash-Stampacchia equilibria can be developed whenever the sets K_i, $i \in \{1, ..., n\}$, are subsets of certain Hadamard manifolds. In the next section we fully develop this theory.

3.3 Nash-Stampacchia equilibria on Hadamard manifolds: existence, uniqueness and exponential stability

Let (M_i, g_i) be finite-dimensional Hadamard manifolds, $i \in \{1, ..., n\}$. Standard arguments show that $(\mathbf{M}, \mathbf{g})$ is also a Hadamard manifold, see Ballmann [2, Example 4, p.147] and O'Neill [32, Lemma 40, p. 209]. Moreover, on account of the characterization of (warped) product geodesics, see O'Neill [32, Proposition 38, p. 208], if $\exp_{\mathbf{p}}$ denotes the usual exponential map on $(\mathbf{M}, \mathbf{g})$ at $\mathbf{p} \in \mathbf{M}$, then for every $\mathbf{V} = (V_1, ..., V_n) \in T_{\mathbf{p}}\mathbf{M}$, we have

$$\exp_{\mathbf{p}}(\mathbf{V}) = (\exp_{p_1}(V_1), ..., \exp_{p_n}(V_n)).$$

We consider that $K_i \subset M_i$ are non-empty, closed, geodesic convex sets and $U_i \subset M_i$ are open sets containing K_i, $i \in \{1, ..., n\}$.

Let $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$. The *diagonal Clarke subdifferential of* $\mathbf{f} = (f_1, ..., f_n)$ at $\mathbf{p} \in \mathbf{K}$ is

$$\partial_C^{\Delta}\mathbf{f}(\mathbf{p}) = (\partial_C^1 f_1(\mathbf{p}), ..., \partial_C^n f_n(\mathbf{p})).$$

From the definition of the metric $\mathbf{g}$, for every $\mathbf{p} \in \mathbf{K}$ and $\mathbf{q} \in \mathbf{M}$ it turns out that

$$\langle \xi_C^{\Delta}, \exp_{\mathbf{p}}^{-1}(\mathbf{q})\rangle_{\mathbf{g}} = \sum_{i=1}^{n} \langle \xi_C^i, \exp_{p_i}^{-1}(q_i)\rangle_{g_i}, \quad \xi_C^{\Delta} = (\xi_C^1, ..., \xi_C^n) \in \partial_C^{\Delta}\mathbf{f}(\mathbf{p}). \tag{3.7}$$

For each $\alpha > 0$ and $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$, we define the set-valued map $A_\alpha^{\mathbf{f}} : \mathbf{K} \to 2^{\mathbf{K}}$ by

$$A_\alpha^{\mathbf{f}}(\mathbf{p}) = P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha\partial_C^{\Delta}\mathbf{f}(\mathbf{p}))), \quad \mathbf{p} \in \mathbf{K}.$$

Note that for each $\mathbf{p} \in \mathbf{K}$, the set $A_\alpha^{\mathbf{f}}(\mathbf{p})$ is non-empty and compact. The following result plays a crucial role in our further investigations.

Theorem 3.3.1 [21] *Let (M_i, g_i) be finite-dimensional Hadamard manifolds; $K_i \subset M_i$ be non-empty, closed, geodesic convex sets; $U_i \subset M_i$ be open sets containing K_i, $i \in \{1, ..., n\}$; and $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$. Then the following statements are equivalent:*

(i) $\mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$;

(ii) $\mathbf{p} \in A_\alpha^{\mathbf{f}}(\mathbf{p})$ *for all* $\alpha > 0$;

(iii) $\mathbf{p} \in A_\alpha^{\mathbf{f}}(\mathbf{p})$ *for some* $\alpha > 0$.

Proof. In view of relation (3.7) and the identification between $T_{\mathbf{p}}\mathbf{M}$ and $T^*_{\mathbf{p}}\mathbf{M}$, see (1.10), we have that

$$\begin{aligned} \mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \;\Leftrightarrow\; & \exists \xi_C^\Delta = (\xi_C^1, ..., \xi_C^n) \in \partial_C^\Delta \mathbf{f}(\mathbf{p}) \text{ such that} \\ & \langle \xi_C^\Delta, \exp_{\mathbf{p}}^{-1}(\mathbf{q})\rangle_{\mathbf{g}} \geq 0 \;\text{ for all } \mathbf{q} \in \mathbf{K} \\ \Leftrightarrow\; & \exists \xi_C^\Delta = (\xi_C^1, ..., \xi_C^n) \in \partial_C^\Delta \mathbf{f}(\mathbf{p}) \text{ such that} \\ & \mathbf{g}(-\alpha\xi_C^\Delta, \exp_{\mathbf{p}}^{-1}(\mathbf{q})) \leq 0 \;\text{ for all } \mathbf{q} \in \mathbf{K} \text{ and} \\ & \text{for all/some } \alpha > 0. \end{aligned} \tag{3.8}$$

On the other hand, let $\gamma, \sigma : [0, 1] \to \mathbf{M}$ be the unique minimal geodesics defined by $\gamma(t) = \exp_{\mathbf{p}}(-t\alpha\xi_C^\Delta)$ and $\sigma(t) = \exp_{\mathbf{p}}(t \exp_{\mathbf{p}}^{-1}(\mathbf{q}))$ for any fixed $\alpha > 0$ and $\mathbf{q} \in \mathbf{K}$. Since $\mathbf{K}$ is geodesic convex in $(\mathbf{M}, \mathbf{g})$, then $\mathrm{Im}\sigma \subset \mathbf{K}$ and

$$\mathbf{g}(\dot\gamma(0), \dot\sigma(0)) = \mathbf{g}(-\alpha\xi_C^\Delta, \exp_{\mathbf{p}}^{-1}(\mathbf{q})). \tag{3.9}$$

Taking into account relation (3.9) and Proposition 1.3.1 (i), i.e., the validity of the obtuse-angle property on the Hadamard manifold $(\mathbf{M}, \mathbf{g})$, (3.8) is equivalent to

$$\mathbf{p} = \gamma(0) = P_{\mathbf{K}}(\gamma(1)) = P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha\xi_C^\Delta)),$$

which is nothing but $\mathbf{p} \in A_\alpha^{\mathbf{f}}(\mathbf{p})$. □

Remark 3.3.1 Note that the implications (ii) ⇒ (i) and (iii) ⇒ (i) hold for arbitrarily Riemannian manifolds, see Remark 1.3.1 (a). These implications are enough to find Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ via fixed points of the map $A_\alpha^{\mathbf{f}}$. However, in the sequel we exploit further aspects of the Hadamard manifolds as non-expansiveness of the projection operator of geodesic convex sets and a Rauch-type comparison property. Moreover, in the spirit of Nash's original idea that Nash equilibria appear exactly as fixed points of a specific map, Theorem 3.3.1 provides a full characterization of Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ via the fixed points of the set-valued map $A_\alpha^{\mathbf{f}}$ when (M_i, g_i) are Hadamard manifolds.

In the sequel, two cases will be considered to guarantee Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$, depending on the compactness of the strategy sets K_i.

3.3.1 Nash-Stampacchia equilibria; compact case

Our first result guarantees the existence of a Nash-Stampacchia equilibrium point for $(\mathbf{f}, \mathbf{K})$ whenever the sets K_i are compact; the proof is based on Begle's fixed point theorem for set-valued maps. More precisely, we have

Theorem 3.3.2 [21] *Let (M_i, g_i) be finite-dimensional Hadamard manifolds; $K_i \subset M_i$ be non-empty, compact, geodesic convex sets; and $U_i \subset M_i$ be open sets containing K_i, $i \in \{1, ..., n\}$. Assume that $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$ and $\mathbf{K} \ni \mathbf{p} \mapsto \partial_C^\Delta \mathbf{f}(\mathbf{p})$ is upper semicontinuous. Then there exists at least one Nash-Stampacchia equilibrium point for $(\mathbf{f}, \mathbf{K})$, i.e., $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \neq \emptyset$.*

Proof. Fix $\alpha > 0$ arbitrary. We prove that the set-valued map $A_\alpha^{\mathbf{f}}$ has closed graph. Let $(\mathbf{p}, \mathbf{q}) \in \mathbf{K} \times \mathbf{K}$ and the sequences $\{\mathbf{p}_k\}, \{\mathbf{q}_k\} \subset \mathbf{K}$ such that $\mathbf{q}_k \in A_\alpha^{\mathbf{f}}(\mathbf{p}_k)$ and $(\mathbf{p}_k, \mathbf{q}_k) \to (\mathbf{p}, \mathbf{q})$ as $k \to \infty$. Then, for every $k \in \mathbb{N}$, there exists $\xi_{C,k}^\Delta \in \partial_C^\Delta \mathbf{f}(\mathbf{p}_k)$ such that $\mathbf{q}_k = P_{\mathbf{K}}(\exp_{\mathbf{p}_k}(-\alpha \xi_{C,k}^\Delta))$. On account of Proposition 1.4.4 (i)$\Leftrightarrow$(ii), the sequence $\{\xi_{C,k}^\Delta\}$ is bounded on the cotangent bundle $T^*\mathbf{M}$. Using the identification between elements of the tangent and cotangent fibers, up to a subsequence, we may assume that $\{\xi_{C,k}^\Delta\}$ converges to an element $\xi_C^\Delta \in T_{\mathbf{p}}^*\mathbf{M}$. Since the set-valued map $\partial_C^\Delta \mathbf{f}$ is upper semicontinuous on $\mathbf{K}$ and $\mathbf{p}_k \to \mathbf{p}$ as $k \to \infty$, we have that $\xi_C^\Delta \in \partial_C^\Delta \mathbf{f}(\mathbf{p})$. The non-expansiveness of $P_{\mathbf{K}}$ (see Proposition 1.3.1 (ii)) gives that

$$
\begin{aligned}
&\mathbf{d_g}(\mathbf{q}, P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha \xi_C^\Delta))) \leq \\
&\quad \leq \mathbf{d_g}(\mathbf{q}, \mathbf{q}_k) + \mathbf{d_g}(\mathbf{q}_k, P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha \xi_C^\Delta))) \\
&\quad = \mathbf{d_g}(\mathbf{q}, \mathbf{q}_k) + \mathbf{d_g}(P_{\mathbf{K}}(\exp_{\mathbf{p}_k}(-\alpha \xi_{C,k}^\Delta)), P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha \xi_C^\Delta))) \\
&\quad \leq \mathbf{d_g}(\mathbf{q}, \mathbf{q}_k) + \mathbf{d_g}(\exp_{\mathbf{p}_k}(-\alpha \xi_{C,k}^\Delta), \exp_{\mathbf{p}}(-\alpha \xi_C^\Delta))
\end{aligned}
$$

Letting $k \to \infty$, both terms in the last expression tend to zero. Indeed, the former follows from the fact that $\mathbf{q}_k \to \mathbf{q}$ as $k \to \infty$, while the latter is a simple consequence of the local behaviour of the exponential map. Thus,

$$\mathbf{q} = P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha \xi_C^\Delta)) \in P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha \partial_C^\Delta \mathbf{f}(\mathbf{p}))) = A_\alpha^{\mathbf{f}}(\mathbf{p}),$$

i.e., the graph of $A_\alpha^{\mathbf{f}}$ is closed.

By definition, for each $\mathbf{p} \in \mathbf{K}$ the set $\partial_C^\triangle \mathbf{f}(\mathbf{p})$ is convex, so contractible. Since both $P_{\mathbf{K}}$ and the exponential map are continuous, $A_\alpha^{\mathbf{f}}(\mathbf{p})$ is contractible as well for each $\mathbf{p} \in \mathbf{K}$, so acyclic.

Now, we are in position to apply Begle's fixed point theorem, see Proposition 1.6.2. Consequently, there exists $\mathbf{p} \in \mathbf{K}$ such that $\mathbf{p} \in A_\alpha^{\mathbf{f}}(\mathbf{p})$. On account of Theorem 3.3.1, $\mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$. $\square$

3.3.2 Nash-Stampacchia equilibria; non-compact case

In the sequel, we are focusing to the location of Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ in the case when K_i are *not* necessarily compact on the Hadamard manifolds (M_i, g_i). In order to avoid technicalities in our further calculations, we introduce the class of functions

$$\mathcal{C}_{(\mathbf{K},\mathbf{U},\mathbf{M})} = \{\mathbf{f} \in C^0(\mathbf{K}, \mathbb{R}^n) \ : \ f_i : (\mathbf{K}; U_i) \to \mathbb{R} \text{ is continuous and } f_i(\mathbf{p}; \cdot) \text{ is of class } C^1 \text{ on } (U_i, g_i) \text{ for all } \mathbf{p} \in \mathbf{K},\ i \in \{1, ..., n\}\}.$$

If is clear that $\mathcal{C}_{(\mathbf{K},\mathbf{U},\mathbf{M})} \subset \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$. Moreover, when $\mathbf{f} \in \mathcal{C}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$ then $\partial_C^\triangle \mathbf{f}(\mathbf{p})$ and $A_\alpha^{\mathbf{f}}(\mathbf{p})$ are singletons for every $\mathbf{p} \in \mathbf{K}$ and $\alpha > 0$.

Let $\mathbf{f} \in \mathcal{C}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$, $\alpha > 0$ and $0 < \rho < 1$. We assume the Lipschitz-type condition: for all $\mathbf{p}, \mathbf{q} \in \mathbf{K}$ the following is fulfilled

$$(H_{\mathbf{K}}^{\alpha,\rho}) \quad \mathbf{d_g}(\exp_{\mathbf{p}}(-\alpha \partial_C^\triangle \mathbf{f}(\mathbf{p})), \exp_{\mathbf{q}}(-\alpha \partial_C^\triangle \mathbf{f}(\mathbf{q}))) \leq (1-\rho)\mathbf{d_g}(\mathbf{p}, \mathbf{q}).$$

Finding fixed points for $A_\alpha^{\mathbf{f}}$, one could expect to apply *dynamical systems*; we consider both *discrete* and *continuous* ones. First, for some $\alpha > 0$ and $\mathbf{p}_0 \in \mathbf{M}$ fixed, we consider the discrete dynamical system

$$(DDS)_\alpha \qquad \mathbf{p}_{k+1} = A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\mathbf{p}_k)).$$

Second, according to Theorem 3.3.1, we clearly have that

$$\mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \Leftrightarrow 0 = \exp_{\mathbf{p}}^{-1}(A_\alpha^{\mathbf{f}}(\mathbf{p})) \ \text{ for all/some } \alpha > 0.$$

Consequently, for some $\alpha > 0$ and $\mathbf{p}_0 \in \mathbf{M}$ fixed, the above equivalence motivates the study of the continuous dynamical system

$$(CDS)_\alpha \qquad \begin{cases} \dot{\eta}(t) = \exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))) \\ \eta(0) = \mathbf{p}_0. \end{cases}$$

The next result describes the stability of the orbits in both cases.

Theorem 3.3.3 [21] *Let (M_i, g_i) be finite-dimensional Hadamard manifolds; $K_i \subset M_i$ be non-empty, closed geodesics convex sets; $U_i \subset M_i$ be open sets containing K_i; and $f_i : \mathbf{K} \to \mathbb{R}$ be functions, $i \in \{1, ..., n\}$ such that $\mathbf{f} \in \mathcal{C}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$. Assume that $(H_{\mathbf{K}}^{\alpha,\rho})$ holds true for some $\alpha > 0$ and $0 < \rho < 1$. Then the set of Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ is a singleton, i.e., $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \{\tilde{\mathbf{p}}\}$. Moreover, for each $\mathbf{p}_0 \in \mathbf{M}$,*

(i) *the orbit $\{\mathbf{p}_k\}$ of $(DDS)_\alpha$ converges exponentially to $\tilde{\mathbf{p}} \in \mathbf{K}$ and*

$$\mathbf{d_g}(\mathbf{p}_k, \tilde{\mathbf{p}}) \leq \frac{(1-\rho)^k}{\rho} \mathbf{d_g}(\mathbf{p}_1, \mathbf{p}_0) \text{ for all } k \in \mathbb{N};$$

(ii) *the orbit η of $(CDS)_\alpha$ is globally defined on $[0, \infty)$ and it converges exponentially to $\tilde{\mathbf{p}} \in \mathbf{K}$ and*

$$\mathbf{d_g}(\eta(t), \tilde{\mathbf{p}}) \leq e^{-\rho t} \mathbf{d_g}(\mathbf{p}_0, \tilde{\mathbf{p}}) \text{ for all } t \geq 0.$$

Proof. Let $\mathbf{p}, \mathbf{q} \in \mathbf{M}$ be arbitrarily fixed. On account of the non-expansiveness of the projection $P_{\mathbf{K}}$ (see Proposition 1.3.1 (ii)) and hypothesis $(H_{\mathbf{K}}^{\alpha,\rho})$, we have that

$$\begin{aligned}
&\mathbf{d_g}((A_\alpha^{\mathbf{f}} \circ P_{\mathbf{K}})(\mathbf{p}), (A_\alpha^{\mathbf{f}} \circ P_{\mathbf{K}})(\mathbf{q})) \\
&\quad = \mathbf{d_g}(P_{\mathbf{K}}(\exp_{P_{\mathbf{K}}(\mathbf{p})}(-\alpha \partial_C^{\Delta} \mathbf{f}(P_{\mathbf{K}}(\mathbf{p})))), P_{\mathbf{K}}(\exp_{P_{\mathbf{K}}(\mathbf{q})}(-\alpha \partial_C^{\Delta} \mathbf{f}(P_{\mathbf{K}}(\mathbf{q}))))) \\
&\quad \leq \mathbf{d_g}(\exp_{P_{\mathbf{K}}(\mathbf{p})}(-\alpha \partial_C^{\Delta} \mathbf{f}(P_{\mathbf{K}}(\mathbf{p}))), \exp_{P_{\mathbf{K}}(\mathbf{q})}(-\alpha \partial_C^{\Delta} \mathbf{f}(P_{\mathbf{K}}(\mathbf{q})))) \\
&\quad \leq (1-\rho) \mathbf{d_g}(P_{\mathbf{K}}(\mathbf{p}), P_{\mathbf{K}}(\mathbf{q})) \\
&\quad \leq (1-\rho) \mathbf{d_g}(\mathbf{p}, \mathbf{q}),
\end{aligned}$$

which means that $A_\alpha^{\mathbf{f}} \circ P_{\mathbf{K}} : \mathbf{M} \to \mathbf{M}$ is a $(1-\rho)$-contraction on $\mathbf{M}$.

(i) Since $(\mathbf{M}, \mathbf{d_g})$ is a complete metric space, a standard Banach fixed point argument shows that $A_\alpha^{\mathbf{f}} \circ P_{\mathbf{K}}$ has a unique fixed point $\tilde{\mathbf{p}} \in M$. Since $\mathrm{Im} A_\alpha^{\mathbf{f}} \subset \mathbf{K}$, then $\tilde{\mathbf{p}} \in \mathbf{K}$. Therefore, we have that $A_\alpha^{\mathbf{f}}(\tilde{\mathbf{p}}) = \tilde{\mathbf{p}}$. Due to Theorem 3.3.1, $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \{\tilde{\mathbf{p}}\}$ and the estimate for $\mathbf{d_g}(\mathbf{p}_k, \tilde{\mathbf{p}})$ yields in a usual manner.

(ii) Since $A_\alpha^{\mathbf{f}} \circ P_{\mathbf{K}} : \mathbf{M} \to \mathbf{M}$ is a $(1-\rho)$-contraction on $\mathbf{M}$ (thus locally Lipschitz in particular), the map

$$\mathbf{M} \ni \mathbf{p} \mapsto G(\mathbf{p}) := \exp_{\mathbf{p}}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\mathbf{p})))$$

is of class C^{1-0}. Now, due to the arguments from §1.5, we may guarantee the existence of a unique maximal orbit $\eta : [0, T_{\max}) \to M$ of $(CDS)_\alpha$.

We assume that $T_{\max} < \infty$. Let us define the continuous function $h : [0, T_{\max}) \to \mathbb{R}$ by

$$h(t) = \frac{1}{2}\mathbf{d}_{\mathbf{g}}^2(\eta(t), \tilde{\mathbf{p}}).$$

The function h is differentiable for a.e. $t \in [0, T_{\max})$ and in the differentiable points of η we have

$$\begin{aligned}
h'(t) &= -\mathbf{g}(\dot{\eta}(t), \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})) \\
&= -\mathbf{g}(\exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))), \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})) \qquad \text{(see } (CDS)_\alpha) \\
&= -\mathbf{g}(\exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))) - \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}), \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})) \\
&\quad -\mathbf{g}(\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}), \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})) \\
&\leq \| \exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))) - \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})\|_{\mathbf{g}} \cdot \| \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}))\|_{\mathbf{g}} \\
&\quad -\| \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}))\|_{\mathbf{g}}^2.
\end{aligned}$$

In the last estimate we used the Cauchy-Schwartz inequality (1.11). From (1.12) we have that

$$\| \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}))\|_{\mathbf{g}} = \mathbf{d}_{\mathbf{g}}(\eta(t), \tilde{\mathbf{p}}). \tag{3.10}$$

We claim that for every $t \in [0, T_{\max})$ one has

$$\| \exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))) - \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})\|_{\mathbf{g}} \leq \mathbf{d}_{\mathbf{g}}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))), \tilde{\mathbf{p}}). \tag{3.11}$$

To see this, fix a differentiable point $t \in [0, T_{\max})$ of η, and let $\gamma : [0,1] \to \mathbf{M}$, $\tilde{\gamma} : [0,1] \to T_{\eta(t)}\mathbf{M}$ and $\overline{\gamma} : [0,1] \to T_{\eta(t)}\mathbf{M}$ be three curves such that

- γ is the unique minimal geodesic joining the two points $\gamma(0) = \tilde{\mathbf{p}} \in \mathbf{K}$ and $\gamma(1) = A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))$;

- $\tilde{\gamma}(s) = \exp_{\eta(t)}^{-1}(\gamma(s))$, $s \in [0,1]$;
- $\overline{\gamma}(s) = (1-s)\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}) + s\exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))))$, $s \in [0,1]$.

By the definition of γ, we have that

$$L_{\mathbf{g}}(\gamma) = \mathbf{d_g}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))), \tilde{\mathbf{p}}). \tag{3.12}$$

Moreover, since $\overline{\gamma}$ is a segment of the straight line in $T_{\eta(t)}\mathbf{M}$ that joins the endpoints of $\tilde{\gamma}$, we have that

$$l(\overline{\gamma}) \leq l(\tilde{\gamma}). \tag{3.13}$$

Here, l denotes the length function on $T_{\eta(t)}\mathbf{M}$. Moreover, since the curvature of $(\mathbf{M}, \mathbf{g})$ is non-positive, we may apply a Rauch-type comparison result for the lengths of γ and $\tilde{\gamma}$, see do Carmo [15, Proposition 2.5, p.218], obtaining that

$$l(\tilde{\gamma}) \leq L_{\mathbf{g}}(\gamma). \tag{3.14}$$

Combining relations (3.12), (3.13) and (3.14) with the fact that

$$l(\overline{\gamma}) = \| \exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))) - \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})\|_{\mathbf{g}},$$

relation (3.11) holds true.

Coming back to $h'(t)$, in view of (3.10) and (3.11), it turns out that

$$h'(t) \leq \mathbf{d_g}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))), \tilde{\mathbf{p}}) \cdot \mathbf{d_g}(\eta(t), \tilde{\mathbf{p}}) - \mathbf{d}_{\mathbf{g}}^2(\eta(t), \tilde{\mathbf{p}}). \tag{3.15}$$

On the other hand, note that $\tilde{\mathbf{p}} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$, i.e., $A_\alpha^{\mathbf{f}}(\tilde{\mathbf{p}}) = \tilde{\mathbf{p}}$. By exploiting the non-expansiveness of the projection operator $P_{\mathbf{K}}$, see Proposition 1.3.1 (ii) and $(H_{\mathbf{K}}^{\alpha,\rho})$, we have that

$$\begin{aligned}
&\mathbf{d_g}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))), \tilde{\mathbf{p}}) = \\
&\quad = \mathbf{d_g}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))), A_\alpha^{\mathbf{f}}(\tilde{\mathbf{p}})) \\
&\quad = \mathbf{d_g}(P_{\mathbf{K}}(\exp_{P_{\mathbf{K}}(\eta(t))}(-\alpha\partial_C^{\Delta}\mathbf{f}(P_{\mathbf{K}}(\eta(t))))), P_{\mathbf{K}}(\exp_{\tilde{\mathbf{p}}}(-\alpha\partial_C^{\Delta}\mathbf{f}(\tilde{\mathbf{p}})))) \\
&\quad \leq \mathbf{d_g}(\exp_{P_{\mathbf{K}}(\eta(t))}(-\alpha\partial_C^{\Delta}\mathbf{f}(P_{\mathbf{K}}(\eta(t)))), \exp_{\tilde{\mathbf{p}}}(-\alpha\partial_C^{\Delta}\mathbf{f}(\tilde{\mathbf{p}}))) \\
&\quad \leq (1-\rho)\mathbf{d_g}(P_{\mathbf{K}}(\eta(t)), \tilde{\mathbf{p}}) \\
&\quad = (1-\rho)\mathbf{d_g}(P_{\mathbf{K}}(\eta(t)), P_{\mathbf{K}}(\tilde{\mathbf{p}})) \\
&\quad \leq (1-\rho)\mathbf{d_g}(\eta(t), \tilde{\mathbf{p}}).
\end{aligned}$$

Combining the above relation with (3.15), for a.e. $t \in [0, T_{\max})$ it yields

$$h'(t) \quad \leq \quad (1-\rho)\mathbf{d}_{\mathbf{g}}^2(\eta(t), \tilde{\mathbf{p}}) - \mathbf{d}_{\mathbf{g}}^2(\eta(t), \tilde{\mathbf{p}}) = -\rho \mathbf{d}_{\mathbf{g}}^2(\eta(t), \tilde{\mathbf{p}}),$$

which is nothing but

$$h'(t) \leq -2\rho h(t) \quad \text{for a.e. } t \in [0, T_{\max}).$$

Due to the latter inequality, we have that

$$\frac{d}{dt}[h(t)e^{2\rho t}] = [h'(t) + 2\rho h(t)]e^{2\rho t} \leq 0 \quad \text{for a.e. } t \in [0, T_{\max}).$$

After integration, one gets

$$h(t)e^{2\rho t} \leq h(0) \quad \text{for all } t \in [0, T_{\max}). \tag{3.16}$$

According to (3.16), the function h is bounded on $[0, T_{\max})$; thus, there exists $\overline{\mathbf{p}} \in \mathbf{M}$ such that $\lim_{t \nearrow T_{\max}} \eta(t) = \overline{\mathbf{p}}$. The last limit means that η can be extended toward the value $T_{\max}$, which contradicts the maximality of $T_{\max}$. Thus, $T_{\max} = \infty$.

Now, relation (3.16) leads to the required estimate; indeed, we have

$$\mathbf{d}_{\mathbf{g}}(\eta(t), \tilde{\mathbf{p}}) \leq e^{-\rho t}\mathbf{d}_{\mathbf{g}}(\eta(0), \tilde{\mathbf{p}}) = e^{-\rho t}\mathbf{d}_{\mathbf{g}}(\mathbf{p}_0, \tilde{\mathbf{p}}) \quad \text{for all } t \in [0, \infty),$$

which concludes our proof. □

Remark 3.3.2 We assume the hypotheses of Theorem 3.3.3 are still verified and $\mathbf{p}_0 \in \mathbf{K}$.

(i) *Discrete case.* Since $\mathrm{Im}A_\alpha^{\mathbf{f}} \subset \mathbf{K}$, then the orbit of $(DDS)_\alpha$ belongs to the set $\mathbf{K}$, i.e., $\mathbf{p}_k \in \mathbf{K}$ for every $k \in \mathbb{N}$.

(ii) *Continuous case.* We shall prove that $\mathbf{K}$ is invariant with respect to the solutions of $(CDS)_\alpha$, i.e., the image of the global solution $\eta : [0, \infty) \to \mathbf{M}$ of $(CDS)_\alpha$ with $\eta(0) = \mathbf{p}_0 \in \mathbf{K}$, entirely belongs to the set $\mathbf{K}$. To show the latter fact, we are going to apply Proposition 1.5.1 by choosing $M := \mathbf{M}$ and $G : \mathbf{M} \to T\mathbf{M}$ defined by $G(\mathbf{p}) := \exp_{\mathbf{p}}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\mathbf{p})))$.

Fix $\mathbf{p} \in \mathbf{K}$ and $\xi \in N_F(\mathbf{p}; \mathbf{K})$. Since $\mathbf{K}$ is geodesic convex in $(\mathbf{M}, \mathbf{g})$, on account of Theorem 1.4.1, we have that $\langle \xi, \exp_{\mathbf{p}}^{-1}(\mathbf{q})\rangle_{\mathbf{g}} \leq 0$ for all

$\mathbf{q} \in \mathbf{K}$. In particular, if we choose $\mathbf{q}_0 = A^{\mathbf{f}}_{\alpha}(P_{\mathbf{K}}(\mathbf{p})) \in \mathbf{K}$, it turns out that

$$\begin{aligned} H_G(\mathbf{p}, \xi) &= \langle \xi, G(\mathbf{p}) \rangle_{\mathbf{g}} = \langle \xi, \exp_{\mathbf{p}}^{-1}(A^{\mathbf{f}}_{\alpha}(P_{\mathbf{K}}(\mathbf{p}))) \rangle_{\mathbf{g}} \\ &= \langle \xi, \exp_{\mathbf{p}}^{-1}(\mathbf{q}_0) \rangle_{\mathbf{g}} \\ &\leq 0, \end{aligned}$$

thus, our claim is proved by applying Proposition 1.5.1.

3.4 Curvature rigidity: Metric projections vs Hadamard manifolds

The obtuse-angle property and the non-expansiveness of $P_{\mathbf{K}}$ for the closed, geodesic convex set $\mathbf{K} \subset \mathbf{M}$ played indispensable roles in the proof of Theorems 3.3.1-3.3.3, which are well-known features of Hadamard manifolds (see Proposition 1.3.1). In §3.3 the product manifold $(\mathbf{M}, \mathbf{g})$ is considered to be a Hadamard one due to the fact that (M_i, g_i) are Hadamard manifolds themselves for each $i \in \{1, ..., n\}$. We actually have the following characterization which is also of geometric interests in its own right and entitles us to assert that Hadamard manifolds are the natural framework to develop the theory of Nash-Stampacchia equilibria on manifolds.

Theorem 3.4.1 [21] *Let (M_i, g_i) be complete, simply connected Riemannian manifolds, $i \in \{1, ..., n\}$, and $(\mathbf{M}, \mathbf{g})$ their product manifold. The following statements are equivalent:*

(i) *Any non-empty, closed, geodesic convex set $\mathbf{K} \subset \mathbf{M}$ verifies the obtuse-angle property and $P_{\mathbf{K}}$ is non-expansive;*

(ii) *(M_i, g_i) are Hadamard manifolds for every $i \in \{1, ..., n\}$.*

Proof. (ii)⇒(i). As mentioned before, if (M_i, g_i) are Hadamard manifolds for every $i \in \{1, ..., n\}$, then $(\mathbf{M}, \mathbf{g})$ is also a Hadamard manifold, see Ballmann [2, Example 4, p.147] and O'Neill [32, Lemma 40, p. 209]. We apply Proposition 1.3.1 for $(\mathbf{M}, \mathbf{g})$.

(i)⇒(ii). We first prove that $(\mathbf{M},\mathbf{g})$ is a Hadamard manifold. Since (M_i,g_i) are complete and simply connected Riemannian manifolds for every $i\in\{1,...,n\}$, the same is true for $(\mathbf{M},\mathbf{g})$. We now show that the sectional curvature of $(\mathbf{M},\mathbf{g})$ is non-positive. To see this, let $\mathbf{p}\in\mathbf{M}$ and $\mathbf{W}_0,\mathbf{V}_0\in T_{\mathbf{p}}\mathbf{M}\setminus\{\mathbf{0}\}$. We claim that the sectional curvature of the two-dimensional subspace $S=$span$\{\mathbf{W}_0,\mathbf{V}_0\}\subset T_{\mathbf{p}}\mathbf{M}$ at the point $\mathbf{p}$ is non-positive, i.e., $K_{\mathbf{p}}(S)\leq 0$. We assume without loosing the generality that $\mathbf{V_0}$ and $\mathbf{W_0}$ are $\mathbf{g}$-perpendicular, i.e., $\mathbf{g}(\mathbf{W}_0,\mathbf{V}_0)=0$.

Let us fix $r_{\mathbf{p}}>0$ and $\delta>0$ such that $\mathcal{B}_{\mathbf{g}}(\mathbf{p},r_{\mathbf{p}})$ is a totally normal ball of $\mathbf{p}$ and

$$\delta\left(\|\mathbf{W}_0\|_{\mathbf{g}}+2\|\mathbf{V}_0\|_{\mathbf{g}}\right)<r_{\mathbf{p}}. \tag{3.17}$$

Let $\sigma:[-\delta,2\delta]\to\mathbf{M}$ be the geodesic segment $\sigma(t)=\exp_{\mathbf{p}}(t\mathbf{V}_0)$ and $\mathbf{W}$ be the unique parallel vector field along σ with the initial data $\mathbf{W}(0)=\mathbf{W}_0$. For any $t\in[0,\delta]$, let $\gamma_t:[0,\delta]\to\mathbf{M}$ be the geodesic segment $\gamma_t(u)=\exp_{\sigma(t)}(u\mathbf{W}(t))$.

Let us fix $t,u\in[0,\delta]$ arbitrarily, $u\neq 0$. Due to (3.17), the geodesic segment $\gamma_t|_{[0,u]}$ belongs to the totally normal ball $\mathcal{B}_{\mathbf{g}}(\mathbf{p},r_{\mathbf{p}})$ of $\mathbf{p}$; thus, $\gamma_t|_{[0,u]}$ is the unique minimal geodesic joining the point $\gamma_t(0)=\sigma(t)$ to $\gamma_t(u)$. Moreover, since $\mathbf{W}$ is the parallel transport of $\mathbf{W}(0)=\mathbf{W}_0$ along σ, we have $\mathbf{g}(\mathbf{W}(t),\dot\sigma(t))=\mathbf{g}(\mathbf{W}(0),\dot\sigma(0))=\mathbf{g}(\mathbf{W}_0,\mathbf{V}_0)=0$; therefore,

$$\mathbf{g}(\dot\gamma_t(0),\dot\sigma(t))=\mathbf{g}(\mathbf{W}(t),\dot\sigma(t))=0.$$

Consequently, the minimal geodesic segment $\gamma_t|_{[0,u]}$ joining $\gamma_t(0)=\sigma(t)$ to $\gamma_t(u)$, and the set $\mathbf{K}=\mathrm{Im}\sigma=\{\sigma(t):t\in[-\delta,2\delta]\}$ fulfill hypothesis (OA_2). Note that Imσ is a closed, geodesic convex set in $\mathbf{M}$; thus, from hypothesis (i) it follows the set Imσ verifies the obtuse-angle property and the map $P_{\mathrm{Im}\sigma}$ is non-expansive. Therefore, (OA_2) implies (OA_1), i.e., for every $t,u\in[0,\delta]$, we have $\sigma(t)\in P_{\mathrm{Im}\sigma}(\gamma_t(u))$. Since Im$\sigma$ is a Chebyshev set (cf. the non-expansiveness of $P_{\mathrm{Im}\sigma}$), for every $t,u\in[0,\delta]$, we have

$$P_{\mathrm{Im}\sigma}(\gamma_t(u))=\{\sigma(t)\}. \tag{3.18}$$

In particular, for every $t,u\in[0,\delta]$, relation (3.18) and the non-expansiveness of $P_{\mathrm{Im}\sigma}$ imply

$$\mathbf{d_g}(\mathbf{p},\sigma(t)) \;=\; \mathbf{d_g}(\sigma(0),\sigma(t)) \tag{3.19}$$

$$\begin{aligned} &= \mathbf{d_g}(P_{\mathrm{Im}\sigma}(\gamma_0(u)), P_{\mathrm{Im}\sigma}(\gamma_t(u))) \\ &\leq \mathbf{d_g}(\gamma_0(u), \gamma_t(u)). \end{aligned}$$

The above construction (i.e., the parallel transport of $\mathbf{W}(0) = \mathbf{W}_0$ along σ) and the formula of the sectional curvature in the parallelogramoid of Levi-Civita defined by the points p, $\sigma(t)$, $\gamma_0(u)$, $\gamma_t(u)$, see §1.1.2, give

$$K_{\mathbf{p}}(S) = \lim_{u,t\to 0} \frac{\mathbf{d}^2_{\mathbf{g}}(\mathbf{p}, \sigma(t)) - \mathbf{d}^2_{\mathbf{g}}(\gamma_0(u), \gamma_t(u))}{\mathbf{d_g}(\mathbf{p}, \gamma_0(u)) \cdot \mathbf{d_g}(\mathbf{p}, \sigma(t))}.$$

According to (3.19), the latter limit is non-positive, so $K_{\mathbf{p}}(S) \leq 0$, which concludes the first part, namely, $(\mathbf{M}, \mathbf{g})$ is a Hadamard manifold.

Now, the main result of Chen [13, Theorem 1] implies that the metric spaces (M_i, d_{g_i}) are Aleksandrov NPC spaces for every $i \in \{1, ..., n\}$. Consequently, for each $i \in \{1, ..., n\}$, the Riemannian manifolds (M_i, g_i) have non-positive sectional curvature, thus they are Hadamard manifolds. The proof is complete. □

Remark 3.4.1 The last result entitles us to assert that the Hadamard manifolds are the natural framework to develop a powerful theory of Nash-Stampacchia equilibrium points on Riemannian manifolds.

3.5 Examples

In this section we present some examples which show the applicability of the results from the previous section in order to localize Nash-type equilibrium points in a non-convex framework.

Example 3.5.1 Let $K_1 = [-1, 1]$, $K_2 = \{(\cos t, \sin t) : t \in [\pi/4, 3\pi/4]\}$, and $f_1, f_2 : K_1 \times K_2 \to \mathbb{R}$ defined for every $x \in K_1$, $(y_1, y_2) \in K_2$ by

$$f_1(x, (y_1, y_2)) = |x|y_1^2 - y_2, \quad f_2(x, (y_1, y_2)) = (1 - |x|)(y_1^2 - y_2^2).$$

Note that $K_1 \subset \mathbb{R}$ is convex in the usual sense, but $K_2 \subset \mathbb{R}^2$ is not. However, if we consider the Poincaré upper-plane model $(\mathbb{H}^2, g_{\mathbb{H}})$, the set $K_2 \subset \mathbb{H}^2$ is geodesic convex with respect to the metric $g_{\mathbb{H}}$, being the image of a geodesic segment from $(\mathbb{H}^2, g_{\mathbb{H}})$. It is clear that $f_1(\cdot, (y_1, y_2))$ is a convex function on K_1 in the usual sense for every $(y_1, y_2) \in K_2$. Moreover, $f_2(x, \cdot)$ is also a convex function on $K_2 \subset \mathbb{H}^2$ for every $x \in K_1$. Indeed, the latter fact reduces to the convexity of the function $t \mapsto (1 - |x|)\cos(2t)$, $t \in [\pi/4, 3\pi/4]$. Therefore, Theorem 3.2.1 guarantees the existence of at least one Nash equilibrium point for $(\mathbf{f}, \mathbf{K}) = (f_1, f_2; K_1, K_2)$. Using Theorem 3.2.3 (iii), a simple calculation shows that the set of Nash equilibrium (as well as Nash-Clarke equilibrium and Nash-Stampacchia equilibrium) points for $(\mathbf{f}, \mathbf{K})$ is $K_1 \times \{(0, 1)\}$.

Example 3.5.2 Let $K_1 = [-1, 1]^2$, $K_2 = \{(y_1, y_2) : y_2 = y_1^2,\ y_1 \in [0, 1]\}$, and $f_1, f_2 : K_1 \times K_2 \to \mathbb{R}$ defined for every $(x_1, x_2) \in K_1$, $(y_1, y_2) \in K_2$ by

$$f_1((x_1, x_2), (y_1, y_2)) = -x_1^2 y_2 + x_2 y_1, \quad f_2((x_1, x_2), (y_1, y_2)) = x_1 y_2^2 + x_2 y_1^2.$$

The set $K_1 \subset \mathbb{R}^2$ is convex, but $K_2 \subset \mathbb{R}^2$ is not in the usual sense. However, K_2 may be considered as the image of a geodesic segment on the paraboloid of revolution $p_{\rm rev}(u, v) = (v \cos u, v \sin u, v^2)$ endowed with its natural Riemannian structure having the coefficients

$$g_{11}(u, v) = v^2,\ g_{12}(u, v) = g_{21}(u, v) = 0,\ g_{22}(u, v) = 1 + 4v^2. \tag{3.20}$$

More precisely, K_2 becomes geodesic convex on ${\rm Im}p_{\rm rev}$, being actually identified with $\{(y, 0, y^2) : y \in [0, 1]\} \subset {\rm Im}p_{\rm rev}$. Note that neither $f_1(\cdot, (y_1, y_2))$ nor $f_2((x_1, x_2), \cdot)$ is convex (the convexity of the latter function being considered on $K_2 \subset{\rm Im}p_{\rm rev}$); thus, Theorem 3.2.1 is not applicable. In view of Remark 3.2.1 (c), Theorem 3.2.2 can be applied in order to determine the set of Nash-Clarke equilibrium points. This set is nothing but the set of solutions in the form $((\tilde{x}_1, \tilde{x}_2), (\tilde{y}, \tilde{y}^2)) \in K_1 \times K_2 = \mathbf{K}$ of the system

$$\begin{cases} -2\tilde{x}_1 \tilde{y}^2 (x_1 - \tilde{x}_1) + \tilde{y}(x_2 - \tilde{x}_2) \geq 0, & \forall (x_1, x_2) \in K_1, \\ \tilde{y}(2\tilde{y}^2 \tilde{x}_1 + \tilde{x}_2)(y - \tilde{y}) \geq 0, & \forall y \in [0, 1]. \end{cases} \tag{S_1}$$

Note that the second inequality is obtained by (3.20) and relations

$$\partial_2 f_2((x_1,x_2),(y_1,y_2)) = \left(2x_2y_1y_2^{-2}, 2x_1y_2(1+4y_2^2)^{-1}\right),$$

$$\exp^{-1}_{(\tilde{y},0,\tilde{y}^2)}(y,0,y^2) = (y-\tilde{y}, 2\tilde{y}(y-\tilde{y})), \quad \tilde{y}, y \in [0,1].$$

We distinguish three cases: (a) $\tilde{y}=0$; (b) $\tilde{y}=1$; and (c) $0<\tilde{y}<1$.

(a) $\tilde{y}=0$. Then, any $((\tilde{x}_1,\tilde{x}_2),(0,0)) \in \mathbf{K}$ solves (S_1).

(b) $\tilde{y}=1$. After an easy computation, we obtain that the points $((-1,-1),(1,1)) \in \mathbf{K}$ and $((0,-1),(1,1)) \in \mathbf{K}$ solve (S_1).

(c) $0<\tilde{y}<1$. The unique situation when (S_1) is solvable is $\tilde{y}=\sqrt{2}/2$. In this case, (S_1) has a unique solution $((1,-1),(\sqrt{2}/2,1/2)) \in \mathbf{K}$. Consequently, the set of Nash-Clarke points for $(\mathbf{f},\mathbf{K}) = (f_1,f_2;K_1,K_2)$, i.e. $\mathcal{S}_{NC}(\mathbf{f},\mathbf{K})$, is the union of the points from (a), (b) and (c), respectively.

Due to Theorem 3.2.3 (i), we may select the elements of $\mathcal{S}_{NE}(\mathbf{f},\mathbf{K})$ from $\mathcal{S}_{NC}(\mathbf{f},\mathbf{K})$. Therefore, the elements of $\mathcal{S}_{NE}(\mathbf{f},\mathbf{K})$ are the solutions $((\tilde{x}_1,\tilde{x}_2),(\tilde{y},\tilde{y}^2)) \in \mathcal{S}_{NC}(\mathbf{f},\mathbf{K})$ of the system

$$\begin{cases} -x_1^2\tilde{y}^2 + x_2\tilde{y} \geq -\tilde{x}_1^2\tilde{y}^2 + \tilde{x}_2\tilde{y}, & \forall (x_1,x_2) \in K_1, \\ \tilde{x}_1 y^4 + \tilde{x}_2 y^2 \geq \tilde{x}_1\tilde{y}^4 + \tilde{x}_2\tilde{y}^2, & \forall y \in [0,1]. \end{cases} \qquad (S_2)$$

We consider again the above three cases.

(a) $\tilde{y}=0$. Among the elements $((\tilde{x}_1,\tilde{x}_2),(0,0)) \in \mathbf{K}$ which solve (S_1), only those are solutions for (S_2) which fulfill the condition $\tilde{x}_2 \geq \max\{-\tilde{x}_1, 0\}$.

(b) $\tilde{y}=1$. On one hand, we have $((-1,-1),(1,1)) \in \mathcal{S}_{NE}(\mathbf{f},\mathbf{K})$. However, $((0,-1),(1,1)) \notin \mathcal{S}_{NE}(\mathbf{f},\mathbf{K})$.

(c) $0<\tilde{y}<1$. We have $((1,-1),(\sqrt{2}/2,1/2)) \in \mathcal{S}_{NE}(\mathbf{f},\mathbf{K})$.

Example 3.5.3 Let

$$K_1 = \{(x_1,x_2) \in \mathbb{R}^2_+ : x_1^2 + x_2^2 \leq 4 \leq (x_1-1)^2 + x_2^2\}, \ K_2 = [-1,1],$$

and the functions $f_1, f_2 : K_1 \times K_2 \to \mathbb{R}$ defined for $(x_1,x_2) \in K_1$ and $y \in K_2$ by

$$f_1((x_1,x_2),y) = y(x_1^3 + y(1-x_2)^3); \ \ f_2((x_1,x_2),y) = -y^2x_2 + 4|y|(x_1+1).$$

It is clear that $K_1 \subset \mathbb{R}^2$ is not convex in the usual sense while $K_2 \subset \mathbb{R}$ is. However, if we consider the Poincaré upper-plane model $(\mathbb{H}^2, g_{\mathbb{H}})$, the set $K_1 \subset \mathbb{H}^2$ is geodesic convex (and compact) with respect to the metric $g_{\mathbb{H}} = (\frac{\delta_{ij}}{x_2^2})$. Therefore, we embed the set K_1 into the Hadamard manifold $(\mathbb{H}^2, g_{\mathbb{H}})$, and K_2 into the standard Euclidean space $(\mathbb{R}, g_0)$. After natural extensions of $f_1(\cdot, y)$ and $f_2((x_1, x_2), \cdot)$ to the whole $U_1 = \mathbb{H}^2$ and $U_2 = \mathbb{R}$, respectively, we clear have that $f_1(\cdot, y)$ is a C^1 function on $\mathbb{H}^2$ for every $y \in K_2$, while $f_2((x_1, x_2), \cdot)$ is a locally Lipschitz function on $\mathbb{R}$ for every $(x_1, x_2) \in K_1$. Therefore, $\mathbf{f} = (f_1, f_2) \in \mathcal{L}_{(K_1 \times K_2, \mathbb{H}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R})}$ and for every $((x_1, x_2), y) \in \mathbf{K} = K_1 \times K_2$, we have

$$\begin{aligned} \partial_C^1 f_1((x_1, x_2), y) &= \text{grad} f_1(\cdot, y)(x_1, x_2) = (g_{\mathbb{H}}^{ij} \frac{\partial f_1(\cdot, y)}{\partial x_j})_i \\ &= 3yx_2^2(x_1^2, -y(1 - x_2)^2); \end{aligned}$$

$$\partial_C^2 f_2((x_1, x_2), y) = \begin{cases} -2yx_2 - 4(x_1 + 1) & \text{if} \quad y < 0, \\ 4(x_1 + 1)[-1, 1] & \text{if} \quad y = 0, \\ -2yx_2 + 4(x_1 + 1) & \text{if} \quad y > 0. \end{cases}$$

It is now clear that the map $\mathbf{K} \ni ((x_1, x_2), y) \mapsto \partial_C^{\Delta} \mathbf{f}(((x_1, x_2), y))$ is upper semicontinuous. Consequently, on account of Theorem 3.3.2, $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \neq \emptyset$, and its elements are precisely the solutions $((\tilde{x}_1, \tilde{x}_2), \tilde{y}) \in \mathbf{K}$ of the system

$$\begin{cases} \langle \partial_C^1 f_1((\tilde{x}_1, \tilde{x}_2), \tilde{y}), \exp_{(\tilde{x}_1, \tilde{x}_2)}^{-1}(q_1, q_2) \rangle_{g_{\mathbb{H}}} \geq 0 & \forall (q_1, q_2) \in K_1, \\ \xi_C^2 (q - \tilde{y}) \geq 0 \;\; \text{for some } \xi_C^2 \in \partial_C^2 f_2((\tilde{x}_1, \tilde{x}_2), \tilde{y}) & \forall q \in K_2. \end{cases} \tag{S_3}$$

In order to solve (S_3) we first observe that

$$K_1 \subset \{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{3} \leq x_2 \leq 2(x_1 + 1)\}. \tag{3.21}$$

We distinguish some cases:

(a) If $\tilde{y} = 0$ then both inequalities of (S_3) hold for every $(\tilde{x}_1, \tilde{x}_2) \in K_1$ by choosing $\xi_C^2 = 0 \in \partial_C^2 f_2((\tilde{x}_1, \tilde{x}_2), 0)$ in the second relation. Thus, $((\tilde{x}_1, \tilde{x}_2), 0) \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ for every $(\tilde{x}_1, \tilde{x}_2) \in \mathbf{K}$.

(b) Let $0 < \tilde{y} < 1$. The second inequality of (S_3) gives that $-2\tilde{y}\tilde{x}_2 + 4(\tilde{x}_1 + 1) = 0$; together with (3.21) it yields $0 = \tilde{y}\tilde{x}_2 - 2(\tilde{x}_1 + 1) < \tilde{x}_2 - 2(\tilde{x}_1 + 1) \leq 0$, a contradiction.

(c) Let $\tilde{y} = 1$. The second inequality of (S_3) is true if and only if $-2\tilde{x}_2 + 4(\tilde{x}_1 + 1) \le 0$. Due to (3.21), we necessarily have $\tilde{x}_2 = 2(\tilde{x}_1 + 1)$; this Euclidean line intersects the set K_1 in the unique point $(\tilde{x}_1, \tilde{x}_2) = (0, 2) \in K_1$. By the geometrical meaning of the exponential map one can conclude that

$$\{t \exp^{-1}_{(0,2)}(q_1, q_2) : (q_1, q_2) \in K_1, t \ge 0\} = \{(x, -y) \in \mathbb{R}^2 : x, y \ge 0\}.$$

Taking into account this relation and $\partial^1_C f_1((0, 2), 1) = (0, -12)$, the first inequality of (S_3) holds true as well. Therefore, $((0, 2), 1) \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$.

(d) Similar reason as in (b) (for $-1 < \tilde{y} < 0$) and (c) (for $\tilde{y} = -1$) gives that $((0, 2), -1) \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$.

Thus, from (a)-(d) we have that

$$\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = (K_1 \times \{0\}) \cup \{((0, 2), 1), ((0, 2), -1)\}.$$

Now, on account of Theorem 3.2.3 (i) we may choose the Nash equilibrium points for $(\mathbf{f}, \mathbf{K})$ among the elements of $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ obtaining that $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = K_1 \times \{0\}$.

Example 3.5.4 (a) Assume that K_i is closed and convex in the Euclidean space $(M_i, g_i) = (\mathbb{R}^{m_i}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{m_i}})$, $i \in \{1, ..., n\}$, and let $\mathbf{f} \in \mathcal{C}_{(\mathbf{K},\mathbf{U},\mathbb{R}^m)}$ where $m = \sum_{i=1}^n m_i$. If $\partial^\Delta_C \mathbf{f}$ is $L-$globally Lipschitz and κ-strictly monotone on $\mathbf{K} \subset \mathbb{R}^m$, then the function $\mathbf{f}$ verifies $(H^{\alpha,\rho}_{\mathbf{K}})$ with $\alpha = \frac{\kappa}{L^2}$ and $\rho = \frac{\kappa^2}{2L^2}$. (Note that the above facts imply that $\kappa \le L$, thus $0 < \rho < 1$.) Indeed, for every $\mathbf{p}, \mathbf{q} \in \mathbf{K}$ we have that

$$\begin{aligned}
&\mathbf{d}^2_\mathbf{g}(\exp_\mathbf{p}(-\alpha \partial^\Delta_C \mathbf{f}(\mathbf{p})), \exp_\mathbf{q}(-\alpha \partial^\Delta_C \mathbf{f}(\mathbf{q}))) \\
&= \|\mathbf{p} - \alpha \partial^\Delta_C \mathbf{f}(\mathbf{p}) - (\mathbf{q} - \alpha \partial^\Delta_C \mathbf{f}(\mathbf{q}))\|^2_{\mathbb{R}^m} = \|\mathbf{p} - \mathbf{q} - (\alpha \partial^\Delta_C \mathbf{f}(\mathbf{p}) - \alpha \partial^\Delta_C \mathbf{f}(\mathbf{q}))\|^2_{\mathbb{R}^m} \\
&= \|\mathbf{p} - \mathbf{q}\|^2_{\mathbb{R}^m} - 2\alpha \langle \mathbf{p} - \mathbf{q}, \partial^\Delta_C \mathbf{f}(\mathbf{p}) - \partial^\Delta_C \mathbf{f}(\mathbf{q}) \rangle_{\mathbb{R}^m} + \alpha^2 \|\partial^\Delta_C \mathbf{f}(\mathbf{p}) - \alpha \partial^\Delta_C \mathbf{f}(\mathbf{q})\|^2_{\mathbb{R}^m} \\
&\le (1 - 2\alpha\kappa + \alpha^2 L^2)\|\mathbf{p} - \mathbf{q}\|^2_{\mathbb{R}^m} = (1 - \frac{\kappa^2}{L^2})\mathbf{d}^2_\mathbf{g}(\mathbf{p}, \mathbf{q}) \\
&\le (1 - \rho)^2 \mathbf{d}^2_\mathbf{g}(\mathbf{p}, \mathbf{q}).
\end{aligned}$$

(b) Let $K = K_1 = K_2 = \mathbb{R}^2_+$ and for $x = (x_1, x_2) \in K$ and $y = (y_1, y_2) \in K$, we consider the functions $f_1, f_2 : K \times K \to \mathbb{R}$ defined by

$$f_1(x, y) = (c_{11}x_1 - h_{11}(y))^2 + (c_{12}x_2 - h_{12}(y))^2;$$

$$f_2(x, y) = (c_{21}y_1 - h_{21}(x))^2 + (c_{22}y_2 - h_{22}(x))^2,$$

where $c_{ij} > 0$ are fixed numbers and $h_{ij} : K \to \mathbb{R}$ are L_{ij}-globally Lipschitz functions, $i, j \in \{1, 2\}$. Assume that

$$2 \min_{i,j} c_{ij} > 3 \max_{i,j} L_{ij}.$$

We may prove that there exists a unique Nash equilibrium point for $(\mathbf{f}, \mathbf{K}) = (f_1, f_2; K, K)$. Indeed, we first consider Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$. Extending in a natural way $f_1(\cdot, y)$ and $f_2(x, \cdot)$ to the whole $U_1 = U_2 = \mathbb{R}^2$ for every $x, y \in K$, it yields that $\mathbf{f} \in \mathcal{C}_{(\mathbf{K},\mathbb{R}^4,\mathbb{R}^4)}$. Moreover, for every $(x, y) \in \mathbf{K}$, we have

$$\partial_C^\Delta \mathbf{f}(x, y) = 2(c_{11}x_1 - h_{11}(y), c_{12}x_2 - h_{12}(y), c_{21}y_1 - h_{21}(x), c_{22}y_2 - h_{22}(x)).$$

A simple calculation shows that $\partial_C^\Delta \mathbf{f}$ is $L-$globally Lipschitz and κ-strictly monotone on $\mathbf{K} \subset \mathbb{R}^4$ with

$$L = 2\sqrt{3} \max_{i,j} c_{ij} > 0; \quad \kappa = 2 \min_{i,j} c_{ij} - 3 \max_{i,j} L_{ij} > 0.$$

According to (a), $\mathbf{f}$ verifies $(H_{\mathbf{K}}^{\alpha,\rho})$ with $\alpha = \frac{\kappa}{L^2}$ and $\rho = \frac{\kappa^2}{2L^2}$. On account of Theorem 3.3.3, the set of Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ contains exactly one point, i.e., the system

$$\begin{cases} (c_{1i}\tilde{x}_i - h_{1i}(\tilde{y}))(x - \tilde{x}_i) \geq 0 & \text{for all } \; x \in [0, \infty), \; i \in \{1, 2\}, \\ (c_{2j}\tilde{y}_j - h_{2j}(\tilde{x}))(y - \tilde{y}_j) \geq 0 & \text{for all } \; y \in [0, \infty), \; j \in \{1, 2\}, \end{cases} \tag{S_4}$$

has a unique solution $(\tilde{x}, \tilde{y}) \in \mathbf{K}$. Moreover, the orbits of both dynamical systems $(DDS)_\alpha$ and $(CDS)_\alpha$ exponentially converge to $(\tilde{x}, \tilde{y})$. Since $f_1(\cdot, y)$ and $f_2(x, \cdot)$ are convex functions on K for every $x, y \in K$, then $\mathbf{f} \in \mathcal{K}_{(\mathbf{K},\mathbb{R}^4,\mathbb{R}^4)}$ as well. Due to Theorem 3.2.3 (iii), we have that $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \{(\tilde{x}, \tilde{y})\}$.

3.6 Comments

The results of this chapter are based on the author's papers [20], [21] where Nash-type equilibrium points are studied in a non-standard geometric framework. First, existence and location of Nash equilibrium points are obtained via McClendon-type minimax inequalities for a large class of finite families of payoff functions whose domains are not necessarily convex in the usual sense. To overcome this difficulty, we assumed that these domains can be embedded into suitable Riemannian manifolds regaining certain geodesic convexity property of them. Then, characterization, existence, and stability of Nash-Stampacchia equilibria are guaranteed whenever the strategy sets are compact or noncompact subsets of certain Hadamard manifolds. Here, we exploited non-smooth and set-valued analysis on manifolds, and two well-known geometrical features of Hadamard spaces. We have also shown that the latter properties characterize the non-positivity of the sectional curvature of complete and simply connected Riemannian spaces, delimiting the Hadamard manifolds as the optimal geometrical framework of Nash-Stampacchia equilibrium problems.

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