Problems in extremal finite set theory

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Submitted to

Central European University

Department of Mathematics and its Applications

In partial fulfilment of the requirements for the degree of Doctor of Philosophy (PhD) in Mathematics and its Applications

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Budapest, Hungary

May 2007

Abstract

In this thesis, we focus on two types of problems in extremal finite set theory. First we introduce a distance-like concept, the $\mathcal{F}$-free distance of two $\mathcal{F}$-free hypergraphs. For a fixed hypergraph $\mathcal{F}$, we will consider the problem of finding the pairs of hypergraphs with the largest $\mathcal{F}$-free distance. For some hypergraphs we will obtain exact results while for some others we will obtain upper and lower bounds on the largest $\mathcal{F}$-free distance. In the second part of the thesis, we will elaborate on extremal problems of weighted set systems, where the weight of a set depends only on its size. The main tool in our investigation will be the so-called profile vector of a set system and we will determine the convex hull of the profile vectors of set systems with some prescribed properties.

Acknowledgement

First of all, I am indebted to my supervisor Gyula O.H. Katona for proposing me several combinatorial problems and for the fruitful discussions.

The Mathematics PhD-program of the Central European University is a joint program with the Alfréd Rényi Institute of Mathematics. I would like to thank Ervin Győri and Miklós Simonovits (research fellows of the Institute) for their generous support given to my applications for research stipend at Tel Aviv University and at University of Bielefeld and also for giving me the chance to present my results at various workshops.

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1 Introduction

One of the first theorems in extremal finite set theory is that of Sperner [31], stating that if we consider a family $\mathcal{F}$ of subsets of an n-element set (n-*set* for short) S such that no set $F \in \mathcal{F}$ can contain any other $F' \in \mathcal{F}$, then the number of sets in $\mathcal{F}$ is at most $\binom{n}{\lfloor n/2 \rfloor}$. The celebrated theorem of Erdős, Ko and Rado [6] (it was published only in 1961, 23 years after it was proved by the authors!) asserts that for any two positive integers $t \leq k$ there exists a third one $n_0(k,t)$ such that if a family $\mathcal{G}$ consists of k-subsets of an n-set, where $n \geq n_0(k,t)$ and if for any two sets $G, G' \in \mathcal{G}$, we have $|G \cap G'| \geq t$, then the size of $\mathcal{G}$ is at most $\binom{n-t}{k-t}$.

Both theorems deal with a problem of finding the largest size that a family of subsets of a fixed underlying set can have if the family satisfies some prescribed property. Problems of this type are in the focus of extremal finite set theory.

We will use the standard notation 2^X to denote the power set of the set X, and $\binom{X}{k}$ will denote the set of all k-subsets of X. The set of the first n positive integers will be denoted by $[n]$. A *hypergraph* (or *set system*) $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H})$ with $E(\mathcal{H}) \subseteq 2^{V(\mathcal{H})}$. $V(\mathcal{H})$ is the vertex set of the hypergraph and $E(\mathcal{H})$ is the edge set of $\mathcal{H}$ (mostly we will identify hypergraphs with the set (family) of their (hyper)edges). If $E(\mathcal{H}) \subseteq \binom{V(\mathcal{H})}{k}$ then $\mathcal{H}$ is said to be k-uniform. We will say that a hypergraph $\mathcal{H}$ contains a copy of another hypergraph $\mathcal{F}$ if there exists an edge preserving injection f from $V(\mathcal{F})$ to $V(\mathcal{H})$, i.e. whenever $F \in E(\mathcal{F})$, then $f(F) = \{f(x) : x \in F\} \in E(\mathcal{H})$, and $\mathcal{H}$ is said to be $\mathcal{F}$-*free* if it does not contain a copy of $\mathcal{F}$. We will call a mapping f with the properties above an *embedding* of $\mathcal{F}$ to $\mathcal{H}$, and an embedding of $\mathcal{F}$ to itself is an *automorphism* of $\mathcal{F}$.

With the notations above we can formulate the general problem (mentioned in the second paragraph) as follows: given a set of families of sets $\mathbb{A} \subseteq 2^{2^X}$ (i.e. a set of hypergraphs, all with vertex set X), we have to find $\max_{\mathcal{F} \in \mathbb{A}}\{|\mathcal{F}|\}$ (and describe all families with this size, the so called *extremal families*).

One way to define the family $\mathbb{A}$ is to fix a hypergraph $\mathcal{F}$ and let $\mathbb{A}$ be the set of $\mathcal{F}$-free hypergraphs with vertex set X. (Or one may forbid a, possibly infinite, collection $\mathcal{C} = \{\mathcal{F}_1, \mathcal{F}_2, ...\}$ of hypergraphs.) If $\mathcal{F}$ is k-uniform and $\mathbb{A}$ is the family of all k-uniform $\mathcal{F}$-free hypergraphs with vertex set $[n]$, then $ex(n, \mathcal{F}) := \max_{\mathcal{F}\in\mathbb{A}}\{|\mathcal{F}|\}$ is the Turán number of $\mathcal{F}$. By an observation of Katona, Nemetz and Simonovits [25] the sequence $\frac{ex(n,\mathcal{F})}{\binom{n}{k}}$ is non-negative and monotone non-increasing, so its limit, the *Turán density* exists. For ordinary graphs (i.e. when $k = 2$), the Turán density is determined by the Erdős-Stone-Simonovits theorem [10], [9] (even if a collection of graphs is forbidden), but only sporadic results are known if $k \geq 3$ (for a survey on the topic see [17]).

In Section 2 (which is based on results from [27] and [28]) we will consider problems that are also related to $\mathcal{F}$-free hypergraphs. Let us suppose, we are given two maximal $\mathcal{F}$-free hypergraphs $\mathcal{H}_1 = (V, E_1)$ and $\mathcal{H}_2 = (V, E_2)$ with the same vertex set V (here maximality means, that whenever we add a subset of the vertex set to the edge set, the hypergraph obtained will not be $\mathcal{F}$-free). Then their union $\mathcal{H}_1 \cup \mathcal{H}_2 := (V, E_1 \cup E_2)$ cannot be $\mathcal{F}$-free, because of their maximality. So several copies of $\mathcal{F}$ will appear in $\mathcal{H}_1 \cup \mathcal{H}_2$ witnessing that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two different maximal $\mathcal{F}$-free hypergraphs. The more evidence (the more copy of $\mathcal{F}$) we have, the more different they are.

Therefore to measure the difference between two $\mathcal{F}$-free hypergraphs we introduce their *$\mathcal{F}$-free distance* (which is a bit misleading, since the triangle inequality does not hold even if we consider only maximal $\mathcal{F}$-free set systems) as the number of copies of $\mathcal{F}$ that are contained in $\mathcal{H}_1 \cup \mathcal{H}_2$ ($\mathcal{H}_1$ and $\mathcal{H}_2$ need not to be maximal, so if there exists a maximal $\mathcal{F}$-free hypergraph containing both of them, then their distance is 0) and we denote this quantity by $D_{\mathcal{F}}(\mathcal{H}_1, \mathcal{H}_2)$. To be more precise, $D_{\mathcal{F}}(\mathcal{H}_1, \mathcal{H}_2)$ is the number of embeddings of $\mathcal{F}$ into $\mathcal{H}_1 \cup \mathcal{H}_2$ divided by the number of automorphisms of $\mathcal{F}$. For a collection $\mathcal{C}$ of hypergraphs and two $\mathcal{C}$-free set systems (i.e. $\mathcal{F}$-free for all $\mathcal{F} \in \mathcal{C}$) we define their $\mathcal{C}$-free distance $D_{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_2)$ by

$$D_{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_2) = \sum_{\mathcal{F} \in \mathcal{C}} D_{\mathcal{F}}(\mathcal{H}_1, \mathcal{H}_2).$$

Having introduced the definitions above, we can ask the following question: given a hypergraph $\mathcal{F}$ (or a collection of hypergraphs $\mathcal{C}$), what is the maximum $\mathcal{F}$-free ($\mathcal{C}$-free) distance that two $\mathcal{F}$-free ($\mathcal{C}$-free) hypergraphs can have if the vertex set of both hypergraphs is $[n]$. In the introductory part of Section 2. we will show some examples when finding the maximum distance is easy and then we move on to more difficult cases.

Let us turn back to our starting problem: how we can choose the most number of subsets of $[n]$ such that the set system of our chosen sets satisfy some prescribed property. In applications (and from theoretical point of view, as well) it might happen, that we have some preference in picking the subsets, so it is quite natural to consider a weighted version of this problem. If w is a real-valued function (a *weight function*) on all the possible subsets (i.e. $w : 2^{[n]} \to \mathbb{R}$), then we define the weight of a family of sets $\mathcal{F} \subseteq 2^{[n]}$ by

$$w(\mathcal{F}) = \sum_{F \in \mathcal{F}} w(F),$$

and now we are interested in finding the largest weight that a set system (satisfying the prescribed property) may have. Note, that the original problem corresponds to the all-one weight, or if we consider only k-subsets (as in the Erdős-Ko-Rado theorem), then all k-sets should have weight 1, and all other sets should have weight 0.

Dealing with all possible weight functions seems hopeless (and not very interesting), but there are some types of weight functions that are quite well studied. One type of weight functions comes from a probabilistic approach. Let us suppose that we pick a random subset X of $[n]$ in such a way, that for all $i \in [n]$ we put i into X with probability p_i ($0 \le p_i \le 1$) independently from what happens to all other $j \in [n]$.

Then for any subset $A \subseteq [n]$ we have

$$\mathbb{P}(X = A) = \prod_{i \in A} p_i \prod_{i \notin A} (1 - p_i).$$

If we let this probability to be the weight of a subset, we get that the weight of a set system is the probability that a randomly chosen subset will belong to it. Results on this type of weight functions can be found (among others) in [11] or [16].

In Section 3 (which is based on joint results with Dániel Gerbner [18], [19]) we will consider weight functions where the weight of a subset depends only on the size of the set. So formally let $f : \{0, 1, ..., n\} \to \mathbb{R}$ be a real-valued function and for any subset $A \subseteq [n]$ let $w(A) := f(|A|)$. A very natural weight function of this type is defined by taking f to be the identity function (i.e. $w(F) = f(|F|) = |F|$). In this case the weight of a set system is

$$w(\mathcal{F}) = \sum_{F \in \mathcal{F}} |F|$$

the *volume* of $\mathcal{F}$.

When considering this kind of weights, it is very useful to introduce two vectors of length $n + 1$ (the coordinates indexed from 0 to n). The ith coordinate of the *weight vector* is the weight of any set with size i. We will denote the weight vector by $\mathbf{w}$ and its ith coordinate by w_i. The ith coordinate of the *profile vector* of a set system $\mathcal{F} \subseteq 2^{[n]}$ is the number of sets that belong to $\mathcal{F}$ that have size i. The profile vector of $\mathcal{F}$ will be denoted by $f(\mathcal{F})$ and its ith coordinate by $f(\mathcal{F})_i$.

With this notation the weight of a family for a given weight function w is simply the inner product of the weight vector and the profile vector:

$$w(\mathcal{F}) = \sum_{F \in \mathcal{F}} w(F) = \sum_{i=0}^{n} f(\mathcal{F})_i w_i = f(\mathcal{F}) \cdot \mathbf{w}.$$

So we transformed our problem: if for a set $\mathbb{A}$ of set systems we denote by $\mu(\mathbb{A})$ the set of profile vectors of the set systems in $\mathbb{A}$, then we are looking for

$$\max_{f \in \mu(\mathbb{A})} \{\mathbf{w} \cdot f\}.$$

We know from linear programming, that for any weight vector $\mathbf{w}$ the maximum above is taken at one of the extreme points of the convex hull of $\mu(\mathbb{A})$, which we denote by $\langle\mu(\mathbb{A})\rangle$ and which is called the *profile polytope* of $\mathbb{A}$. The set of extreme points is denoted by $E(\mathbb{A})$ and the families having a profile in $E(\mathbb{A})$, the extremal families by $\mathcal{E}(\mathbb{A})$. So, if one determines $E(\mathbb{A})$, then to get the maximum weight for any weight vector $\mathbf{w}$, one just has to compute the weight for the vectors in $\mathcal{E}(\mathbb{A})$. Unfortunately, the size of $E(\mathbb{A})$ might grow exponentially with n (the size of the underlying set) tending to infinity.

However, if the weights are non-negative, then increasing any coordinate of the profile vector increases the weight of the family, so the maximum for these weights is taken at an extreme point which is maximal with respect to the coordinate-wise ordering. We call these vectors *essential* extreme points and denote them by $E^*(\mathbb{A})$ and the corresponding families by $\mathcal{E}^*(\mathbb{A})$. Luckily, in most known results, the size of $E^*(\mathbb{A})$ grows only polynomially. Note that to prove that a set of profiles are the extreme points of the profile polytope one has to express all profiles as a convex combination of these vectors, while to prove that a set of profiles are the essential extreme points of the polytope it is enough to dominate (a vector f dominates g if it is larger in the coordinate-wise ordering) any other profiles.

The systematic investigation of profile vectors and profile polytopes was started by P.L. Erdős, P. Frankl and G.O.H. Katona in [7] and [8], an overview of the topic can be found in the book of K. Engel [4].

The notion of profile vector can be introduced for any ranked partially ordered set (poset) P (a poset P is said to be ranked if there exist a non-negative integer l and a surjective mapping $r : P \to \{0, 1, ..., l\}$ such that for any $p_1, p_2 \in P$ if p_2 covers p_1, we have $r(p_1) + 1 = r(p_2)$). In this case the profile of a family $\mathcal{F} \subseteq P$ is defined by

$$f(\mathcal{F})_i = |\{p \in \mathcal{F} : \text{rank } (p) = i\}| \quad (i = 0, 1, ..., n),$$

where rank (p) denotes the rank of an element p and n is the largest rank in P. Several results are known about profile vectors in the generalized context as well (see e.g. [4], [13], [31]).

One of the most studied ranked poset is $L_n(q)$, the poset of subspaces of an n-dimensional vector space V over the finite field $GF(q)$ with q elements (the ordering is just set-theoretic inclusion). In this case the rank of a subspace is just its dimension, so the profile vector $f(\mathcal{U})$ of a family $\mathcal{U}$ of subspaces is a vector of length $n+1$ (indexed from 0 to n) with $f(\mathcal{U})_i = |\{U \in \mathcal{U} : \dim U = i\}|$, $i = 0, 1, ..., n$. In the thesis, we determine the profile polytope of intersecting families in the poset $L_n(q)$. A family $\mathcal{U}$ of subspaces is called intersecting if for any $U, U' \in \mathcal{U}$ we have $\dim(U \cap U') \geq 1$ (and t-intersecting if for any $U, U' \in \mathcal{U}$ we have $\dim(U \cap U') \geq t$).

In the first subsection of Section 3, we will determine the extreme points of the profile polytope of intersecting families of subspaces, while in the second subsection we will introduce a generalization of the notion of profile vectors and prove some results for the new concept.

2 The distance of $\mathcal{F}$-free families

In this section we consider the following problem: given a hypergraph $\mathcal{F}$, let us find the pair of two $\mathcal{F}$-free hypergraphs that are "the most different" from each other. If $\text{embed}(\mathcal{F},\mathcal{G})$ denotes the number of embeddings of $\mathcal{F}$ into $\mathcal{G}$ and $\text{aut}(\mathcal{F})$ denotes the number of automorphisms of $\mathcal{F}$, then the difference of two $\mathcal{F}$-free families with the same vertex set (from now on, all hypergraphs considered have vertex set $[n]$) is measured by their $\mathcal{F}$-free distance

$$D_{\mathcal{F}}(\mathcal{H}_1,\mathcal{H}_2)=\frac{\text{embed}(\mathcal{F},\mathcal{H}_1\cup\mathcal{H}_2)}{\text{aut}(\mathcal{F})}.$$

For a collection of forbidden subhypergraphs $\mathcal{C}$ and two $\mathcal{C}$-free hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$, the $\mathcal{C}$-free distance is defined by $D_{\mathcal{C}}(\mathcal{H}_1,\mathcal{H}_2)=\sum_{\mathcal{F}\in\mathcal{C}}D_{\mathcal{F}}(\mathcal{H}_1,\mathcal{H}_2)$.

Let us consider two easy examples, before we proceed to the more complicated problems. In our first example we examine hypergraphs of which any pair of hyperedges H_1,H_2 either $H_1\subseteq H_2$ or $H_2\subseteq H_1$ hold. To put this property of hypergraphs in our context, we have to define the collection of forbidden hypergraphs. Obviously, we have to include in the collection all non-isomorphic non-including pairs. Any such pair is determined by a triple: the size of $H_1\cap H_2$, $H_1\setminus H_2$ and $H_2\setminus H_1$, so formally $\mathcal{C}_{\subseteq}=\{\mathcal{F}_{k,l,m}:0\le k,1\le l\le m\}$, where $\mathcal{F}_{k,l,m}=\{\{1,2,...k,k+1,...k+l\},\{1,2,...,k,k+l+1,k+l+2,...,k+l+m\}\}$. Informally, to compute the $\mathcal{C}_{\subseteq}$-free distance of two hypergraphs with the property above we should count the pairs of hyperedges H_1,H_2 in the union for which both $H_1\not\subseteq H_2$ and $H_2\not\subseteq H_1$ hold. Maximal families with this "non-inclusion-free" property are saturated chains. (A chain $\mathbf{C}=\{C_0\subseteq C_1\subseteq ...\subseteq C_n\}$ is saturated if for all j, $|C_j|=j$ holds.) The empty set is a subset of each set, and each set is a subset of the whole underlying set $[n]$, so the maximum number of pairs of sets none of them containing the other, where the sets are taken from two chains $\mathbf{C}_1,\mathbf{C}_2$, is at most $(n-1)^2$. And for any pair of saturated chains of the form

$\mathbf{C} = \{C_0 \subseteq C_1 \subseteq C_2 \subseteq ... \subseteq C_{n-1} \subseteq C_n\}$ and $\mathbf{C}' = \{\overline{C}_n \subseteq \overline{C}_{n-1} \subseteq ... \subseteq \overline{C}_1 \subseteq \overline{C}_0\}$ where $\overline{A}$ denotes the complement of the set A, we have $D_{\mathcal{C}_{\subseteq}}(\mathbf{C}, \mathbf{C}') = (n-1)^2$.

As another example, let us consider the set systems with the property that for every $F_1, F_2 \in \mathcal{F}$ $F_1 \cap F_2 = \emptyset$ (i.e. $\mathcal{F}$ is a family of pairwise disjoint sets). This time, the collection of forbidden configurations is $C_{\not\cap} = \{\mathcal{F}'_{k,l,m} : 1 \leq k, 0 \leq l \leq m\}$, where $\mathcal{F}'_{k,l,m} = \{\{1, ..., k, k+1, ..., k+l\}\{1, ..., k, k+l+1, ..., k+l+m\}\}$ and $D_{\mathcal{C}_{\not\cap}}(\mathcal{H}_1, \mathcal{H}_2)$ is the number of pairs of *intersecting* hyperedges. In the case of this property maximal families are partitions of $[n]$. If for every pair we point out an element of the intersection, then we get an injective mapping from the non-disjoint pairs to the base set. So the number of such pairs can be at most n. For any partition $\mathcal{P}$ we can create another partition $\mathcal{P}'$ by choosing an element from each non-empty set to form a set in $\mathcal{P}'$, then again choosing one element from all remaining non-empty sets, and so on to have $D_{\not\cap}(\mathcal{P}, \mathcal{P}') = n$ (as an explicit example, one can think of the partition $\mathcal{P}_1$ consisting only of the whole underlying set $[n]$ and the partition $\mathcal{P}_2$ consisting of all singletons of $[n]$).

2.1 Intersecting families

In this subsection we will consider intersecting families of sets. Just to remember, $\mathcal{F} \subseteq 2^{[n]}$ is called intersecting if for any two $F, F' \in \mathcal{F}$ we have $|F \cap F'| \geq 1$. This is equivalent to that there is no disjoint pair of sets in $\mathcal{F}$. So, to get into our framework of forbidden configurations, we define $\mathcal{C}_{\cap} = \{\mathcal{F}_{k,l} : 0 \leq k \leq l\}$, where $\mathcal{F}_{k,l} = \{\{1, ..., k\}\{k+1, ..., k+l\}\}$. In this way, the $\mathcal{C}_{\cap}$-free distance of two intersecting families is the number of disjoint pairs in their union.

The precise form of the Erdős-Ko-Rado theorem [6] for intersecting families (not how it is mentioned in the introduction) states that if $k \leq n/2$, then the size of any k-uniform intersecting family $\mathcal{F} \subseteq 2^{[n]}$ is at most $\binom{n-1}{k-1}$ and if $k < n/2$ and $|\mathcal{F}| = \binom{n-1}{k-1}$, then $\mathcal{F}$ must be isomorphic to the family $\mathcal{F}_0 = \{F \in \binom{[n]}{k} : 1 \in F\}$. It is quite natural to conjecture that the pair

$$\mathcal{F}_0 = \{F \in \binom{[n]}{k} : 1 \in F\}, G_0 = \{G \in \binom{[n]}{l} : n \in G\}$$

will have the largest $\mathcal{C}_\cap$-free distance if we restrict ourselves to pairs $\mathcal{F}, \mathcal{G}$, where $\mathcal{F}$ is k-uniform and $\mathcal{G}$ is l-uniform.

Though in the non-uniform case any maximal intersecting family has size 2^{n-1} (not only the family $\mathcal{F}_0' = \{F \subseteq n : 1 \in F\}$), one still expects, that the following pair of intersecting families have the largest $\mathcal{C}_\cap$-free distance:

$$\mathcal{F}_0' = \{F \subseteq [n] : 1 \in F\}, G_0' = \{G \subseteq [n] : n \in G\}.$$

We will refer to the pairs $(\mathcal{F}_0, \mathcal{G}_0$ and $(\mathcal{F}_0', \mathcal{G}_0')$ as the *conjectured hypergraphs/set systems.*

In what follows we prove that the conjectured sets systems are in fact optimal in the non-uniform case and if n is large enough they are optimal in the uniform case as well.

2.1.1 The Uniform Case

Throughout this subsection we will assume that $\mathcal{F}$ is k-uniform and $\mathcal{G}$ is l-uniform. Now if $k + l > n$, then there are no disjoint k and l element subsets.

If $k + l \leq n$, but, say, $l > \frac{n}{2}$, then any two l-element subsets meet each other. For any fixed k-element subset there are $\binom{n-k}{l}$ l-element subsets disjoint from this fixed set. So the best one can do is to let $\mathcal{F}$ be the largest intersecting k-uniform set system, and let $\mathcal{G}$ consist of all l-element subsets disjoint from at least one set in $\mathcal{F}$. The Erdős-Ko-Rado theorem [6] says that $\mathcal{F}$ should be all k-element sets containing a fixed element, so then $\mathcal{G}$ should be all l-element sets not containing this fixed element. Thus in this case the conjectured set systems are not optimal.

If $2k = n$ and $k = l$ then any set has only one disjoint pair (considering now only the k-element sets), its complement. So one can put from each pair one set into $\mathcal{F}$ and one into $\mathcal{G}$, and since in this way subsets containing 1 *and* n together (or containing none of them) will be put into $\mathcal{F}$ or $\mathcal{G}$, these families will have more disjoint pairs, than the conjectured systems (and clearly will be maximal ones).

Despite these failures of the conjectured systems, one can state the following

Theorem 2.1.1 *For any k and l, there exists an $n(k,l)$ such that if $n \geq n(k,l)$ and $\mathcal{F}$, $\mathcal{G}$ are k and l-uniform hypergraphs , then $D_\cap(\mathcal{F},\mathcal{G}) \leq D_\cap(\mathcal{F}_0,\mathcal{G}_0)$ where $\mathcal{F}_0,\mathcal{G}_0$ are the conjectured hypergraphs.*

Proof:

CASE A $\bigcap \mathcal{F} \neq \emptyset$ and $\bigcap \mathcal{G} \neq \emptyset$.

In this case $\bigcap \mathcal{F}$ and $\bigcap \mathcal{G}$ must be disjoint, since otherwise there would be no disjoint sets in $\mathcal{F}$ and $\mathcal{G}$. Let us pick an $i \in \bigcap \mathcal{F}$ and a $j \in \bigcap \mathcal{G}$, and add $\{F \subseteq [n] : i \in F\}$ to $\mathcal{F}$ and $\{G \subseteq [n] : j \in G\}$ to $\mathcal{G}$. In this way we get the conjectured hypergraphs, and clearly $D(\mathcal{F},\mathcal{G})$ cannot decrease.

CASE B $\bigcap \mathcal{F} = \emptyset$ (or similarly $\bigcap \mathcal{G} = \emptyset$).

Observe the following two things:

1, if $n \geq k + 2l$ then again by [6] one gets that for a fixed $F \in \mathcal{F}$ the number of sets in $\mathcal{G}$ from which F is disjoint is at most $\binom{n-k-1}{l-1}$, which is the case in the conjectured hypergraphs for all sets in $\mathcal{F}_{i,j}$. So if $|\mathcal{F}| \leq |\mathcal{F}_{i,j}| = \binom{n-2}{k-1}$ then we are done.

2, Since $\bigcap \mathcal{F} = \emptyset$ then as a special case of Theorem3 of [21] we get that

$$|\mathcal{F}| \leq 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$$

and as for large enough n

$$\binom{n-2}{k-1} > 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$$

holds, by the remark made after the first observation we are done. □

2.1.2 The Non-Uniform Case

Let us first state the main theorem of this subsection.

Theorem 2.1.2 *For any* $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ *and for any* $n \geq 2$, $D_{\mathcal{C}_\cap}(\mathcal{F}, \mathcal{G}) \leq D_{\mathcal{C}_\cap}(\mathcal{F}'_0, \mathcal{G}'_0)$ *holds, where* $\mathcal{F}'_0, \mathcal{G}'_0$ *is the conjectured pair.*

Proof: Without loss of generality one can assume that the pair $(\mathcal{F}, \mathcal{G})$ is maximal with respect to the property that all $F \in \mathcal{F}$ have at least one $G \in \mathcal{G}$ disjoint from it (and the same holds for any $G \in \mathcal{G}$). Our conjectured pair of set systems does not have this property, so if we remove the "negligible" sets (the ones that are not contained in any disjoint pair of sets in the union $\mathcal{F}'_0 \cup \mathcal{G}'_0$) we get the following pair of hypergraphs:

$$\mathcal{F}''_0 = \{F \subseteq [n] : 1 \in F, n \notin F\}, \quad \mathcal{G}''_0 = \{G \subseteq [n] : n \in G, 1 \notin G\},$$

to which we will still refer as *the conjectured pair of hypergraphs* (and for which $D_{\mathcal{C}_\cap}(\mathcal{F}'_0, \mathcal{G}'_0) = D_{\mathcal{C}_\cap}(\mathcal{F}''_0, \mathcal{G}''_0)$ holds).

We begin the proof with the following claim:

Claim 2.1.3 $F \in \mathcal{F} \Leftrightarrow \overline{F} \in \mathcal{G}$.

Proof of Claim: If $F \in \mathcal{F}$ then there is some $G \in \mathcal{G}$ such that $F \cap G = \emptyset$. This means $G \subseteq \overline{F}$, and as G meets all sets in $\mathcal{G}$, $\overline{F}$ meets them, too. So by maximality $\overline{F} \in \mathcal{G}$. The other direction follows, since we can change the role of $\mathcal{F}$ and $\mathcal{G}$. □

By virtue of the above claim, we can “forget about” $\mathcal{G}$. But what should we count, and are there any additional conditions on $\mathcal{F}$? Concerning the first question: as for a fixed F we counted the Gs disjoint from it, and since $F \cap G = \emptyset \Leftrightarrow F \subseteq \overline{G}$, by the claim we get, that now for a fixed F we should count the number of $F' \in \mathcal{F} : F \subseteq F'$. (Note that $F \subseteq F$ also counts, because this is for the pair $(F, \overline{F})$!) Let us denote this number by $\rho_{\mathcal{F}}(F)$ (and we will omit $\mathcal{F}$ from the index, if it is clear from the context), and put $\rho(\mathcal{F}) = \sum_{F\in\mathcal{F}} \rho_{\mathcal{F}}(F)$.

Now to the other question: since by the above claim we know that $\mathcal{G} = \overline{\mathcal{F}} = \{\overline{F} : F \in \mathcal{F}\}$ and the original conditions were that both $\mathcal{F}$ and $\mathcal{G}$ should be intersecting, we get that $\mathcal{F}$ should be intersecting *and* co-intersecting. So we conclude the following

Claim 2.1.4 *$max\{D(\mathcal{F},\mathcal{G}) : \mathcal{F},\mathcal{G}$ are intersecting$\} = \max\{\rho(\mathcal{F}) : \mathcal{F}$ is intersecting and co-intersecting$\}$.* $\square$

By this claim we are left to show that $\rho(\mathcal{F}) \le \rho(\mathcal{F}_0'')$ whenever $\mathcal{F}$ is an intersecting and co-intersecting family (and therefore we will call $\mathcal{F}_0''$ alone *the conjectured hypergraph*).

Now note that, when counting $\rho(\mathcal{F})$ one counts the pairs (F, F') where $F, F' \in \mathcal{F}$ and $F \subseteq F'$. But this can be done from the point of view of F', that is, if we put $\delta_{\mathcal{F}}(F') = |\{F \in \mathcal{F} : F' \supseteq F\}|$ and $\delta(\mathcal{F}) = \sum_{F\in\mathcal{F}} \delta_{\mathcal{F}}(F)$, then $\rho(\mathcal{F}) = \delta(\mathcal{F})$. With this remark we are able to prove

Lemma 2.1.5 *If $\mathcal{F}$ is intersecting and co-intersecting, furthermore $\bigcap \mathcal{F} \neq \emptyset$, then $\delta(\mathcal{F}) \le \delta(\mathcal{F}_0'')$.*

Proof: W.l.o.g. one can assume that $1 \in F$ for all $F \in \mathcal{F}$. Consider the hypergraph $\mathcal{F}^* = \{F \setminus \{1\} : F \in \mathcal{F}\}$. Since we removed 1, this need no longer be intersecting, but it is clearly co-intersecting on $[2, ..., n]$, furthermore $\delta_{\mathcal{F}}(F) = \delta_{\mathcal{F}^*}(F \setminus \{1\})$.

It is well-known, that if a hypergraph is maximal co-intersecting, then it contains one set from any pair of complements, and if $F \subseteq F' \in \mathcal{F}^*$, then $F \in \mathcal{F}^*$. So $\delta_{\mathcal{F}^*}(F \setminus \{1\}) = 2^{|F\setminus\{1\}|}$, hence to obtain the largest $\delta(\mathcal{F}^*)$ one should put the most possible large sets into $\mathcal{F}^*$. Again, by [6], we know that for fixed $k \ge \frac{n-1}{2}$ we can put at most $\binom{n-2}{k}$ k-element sets into $\mathcal{F}^*$, but in the case of the conjectured hypergraph exactly that many sets (now with $k+1$-elements, as we put back 1 to all the sets) are there. So for all k we put the most possible number of large sets into our family when considering the k and $n-1-k$-element complementing pairs. □

So we will be done, if we can prove

Lemma 2.1.6 *For any intersecting and co-intersecting family $\mathcal{F}$, there exists another $\mathcal{F}'$ with $\bigcap \mathcal{F}' \neq \emptyset$ and $\rho(\mathcal{F}) \leq \rho(\mathcal{F}')$.*

Before starting the proof of Lemma 2.1.6, we introduce some notation: the shift operation $\tau_{i,j}$ is defined by

$$\tau_{i,j}(F) = \begin{cases} F \setminus \{j\} \cup \{i\} \text{ if } & j \in F, i \notin F \text{and} F \setminus \{j\} \cup \{i\} \notin \mathcal{F} \\ F & \text{otherwise} \end{cases} \tag{1}$$

Put $\tau_{i,j}(\mathcal{F}) = \{\tau_{i,j}(F) : F \in \mathcal{F}\}$.

The shift operation is a very well-known and very often used technique in extremal finite set theory. It was introduced by Erdős, Ko and Rado in [6] and had numerous applications ever since. For a good (but not recent) survey see Frankl's paper [12]. The proof of the following properties of the shift operation can be found both in [6] and [12]: it preserves the intersecting and co-intersecting property. It is also known, that starting from any family of sets, performing finitely many shift operation, one can obtain a so-called *left-shifted* family, that is a family for which $\tau_{i,j}(\mathcal{F}) = \mathcal{F}$ for all $i < j$. So in what follows, we can assume that $\mathcal{F}$ is left-shifted, if we can prove the following

Claim 2.1.7 $\rho(\mathcal{F}) \leq \rho(\tau_{i,j}(\mathcal{F}))$.

Proof: We will consider how $\rho(F)$ changes when performing the operation $\tau_{i,j}$.

CASE A If $i, j \in F$ or $i, j \notin F$, then $\tau_{i,j}(F) = F$ and for all $F' \in \mathcal{F}$ with $F \subseteq F'$ we have $F \subseteq \tau_{i,j}(F')$. So $\rho_{\mathcal{F}}(F) \leq \rho_{\tau_{i,j}(\mathcal{F})}(F) = \rho_{\tau_{i,j}(\mathcal{F})}(\tau_{i,j}(F))$.

CASE B Let $A \subseteq [n]$ with $i, j \notin A$. Put $F = A \cup \{i\}$ and $F' = A \cup \{j\}$.

SUBCASE B1 $F \in \mathcal{F}, F' \notin \mathcal{F}$

Now for all $G \supseteq F$ $i \in G$, therefore $G = \tau_{i,j}(G) \supseteq \tau_{i,j}(F) = F$, thus $\rho_{\mathcal{F}}(F) \leq \rho_{\tau_{i,j}(\mathcal{F})}(F) = \rho_{\tau_{i,j}(\mathcal{F})}(\tau_{i,j}(F))$.

Subcase B2 $F \notin \mathcal{F}, F' \in \mathcal{F}$

Now $\tau_{i,j}(F') = F$, and if $F' \subset G \in \mathcal{F}$ with $i \notin G$, then $(G \setminus \{j\} \cup \{i\}) = G' \in \tau_{i,j}(\mathcal{F})$ and clearly $F \subseteq G'$. If $F' \subseteq G$ with $i, j \in G$, then $G = \tau_{i,j}(G) \supseteq F$, thus we conclude, that $\rho_{\mathcal{F}}(F') \leq \rho_{\tau_{i,j}(\mathcal{F})}(F) = \rho_{\tau_{i,j}(\mathcal{F})}(\tau_{i,j}(F'))$.

Subcase B3 $F, F' \in \mathcal{F}$ (thus $\tau_{i,j}(F) = F, \tau_{i,j}(F') = F'$)

Now let $G \in \mathcal{F}$ contain at least one of F, F'. If $i \in G$, then $\tau_{i,j}(G) = G$ contains as many of F, F' as before performing the τ-operation. Otherwise $i \notin G, j \in G$ and G contains only F'. So, putting $G' = G \setminus \{j\} \cup \{i\}$, if $G' \notin \mathcal{F}$, then $\tau_{i,j}(G) = G'$ and $G' \supseteq F$, while if $G' \in \mathcal{F}$, then $\tau_{i,j}(G) = G$ and still $F' \subseteq G$. So we get $\rho_{\tau_{i,j}(\mathcal{F})}(F) + \rho_{\tau_{i,j}(\mathcal{F})}(F') \geq \rho_{\mathcal{F}}(F) + \rho_{\mathcal{F}}(F')$.

So for sets of type of the first case $\rho(F)$ does not decrease, and we can partition the sets of type of the second case into “pairs" (of which one may be missing) for which the sum of $\rho(F)$s does not decrease. $\square$

Further notations:

$$\mathcal{F} + \mathcal{G} = \{F \cup G : F \in \mathcal{F}, G \in \mathcal{G}\},\ \mathcal{F} - \mathcal{G} = \{F \setminus G : F \in \mathcal{F}, G \in \mathcal{G}\}$$

$$\Delta\mathcal{F} = \mathcal{F} - \mathcal{F};\ \mathrm{Sub}\mathcal{F} = \{S : S \subseteq F \in \mathcal{F}\}$$

And we will write $\mathbf{1} + \mathcal{F}$ if $\mathcal{G}$ consists of one single set containing only 1.

Now we can return to the proof of Lemma 2.1.6. In the proof we will use the basic ideas of [26].

Proof of Lemma 2.1.6 For arbitrary $\mathcal{F}$ intersecting and co-intersecting family we have to define another one of which each set has an element in common. Now let $\mathcal{F} = \mathcal{F}^0 \cup^* \mathcal{F}^1$, where $\mathcal{F}^1 = \{F \in \mathcal{F} : 1 \in F\}$ and $\mathcal{F}^0 = \{F \in \mathcal{F} : 1 \notin F\}$. Put $\mathcal{F}' = \mathcal{F}^1 \cup (\mathbf{1} + \mathrm{Sub}\mathcal{F}^0)$.

We have to prove that a, $\bigcap \mathcal{F}' \neq \emptyset$ (and therefore it is intersecting), b, $\mathcal{F}'$ is co-intersecting and c, $\rho(\mathcal{F}') \geq \rho(\mathcal{F})$.

a, is clear, as by definition $1 \in F$ for all $F \in \mathcal{F}'$.

To prove b, we will use that $\mathcal{F}$ is left-shifted (and maximal).

Claim 2.1.8 $\mathbf{1} + \mathcal{F}^0 \subset \mathcal{F}^1$

Proof: Since for any $F \in \mathcal{F}^0$ $F' = \{1\} \cup F \supset F$, F' meets all sets in $\mathcal{F}$. We have to show, that there is no $G \in \mathcal{F}$ such that $F' \cup G = [n]$. Suppose to the contrary that such a G exists. Note that $1 \notin G$, because otherwise $G \cup F = [n]$ would hold, contradicting the co-intersecting property of $\mathcal{F}$. Now as $\mathcal{F}$ is intersecting, there is $j \in F \cap G$. But since $\mathcal{F}$ is left-shifted, $G \setminus \{j\} \cup \{1\} = G' \in \mathcal{F}$. But then $G' \cup F = [n]$ would hold - a contradiction. $\square$

By Claim 2.1.8 we know that all new sets in $\mathcal{F}'$ are subsets of one of the old sets (that is a set from $\mathcal{F}$), therefore as $\mathcal{F}$ was co-intersecting, so is $\mathcal{F}'$.

It remains to prove c,. For this purpose we will define an injective mapping $f : \mathcal{F}^0 \rightarrow \mathbf{1} + \Delta\mathcal{F}^0$ (observe that $\Delta\mathcal{F}^0 \subseteq \text{Sub}\mathcal{F}^0$!) such that for all $F \in \mathcal{F}^0$ $\rho_{\mathcal{F}'}(f(F)) \geq \rho_{\mathcal{F}}(F)$. This is clearly enough, because $\mathcal{F}^1 \subseteq \mathcal{F}'$, so $\rho(F)$ cannot decrease for any $F \in \mathcal{F}^1$ (and if $F_1, F_2 \in \mathcal{F}^0$, then $\{1\} \cup F_1 \setminus F_2$ is disjoint from F_2, so, by the intersecting property of $\mathcal{F}$, it is not an element of $\mathcal{F}^1$, so we will not count twice any $\rho(F)$).

To define f (using the notation of [26]) let $k = \min\{|I| : I = F_1 \cap F_2; F_1, F_2 \in \mathcal{F}^0\}$ (note, that I is not empty, as $\mathcal{F}$ is intersecting!) and fix F_1, F_2 with $I = F_1 \cap F_2 : |I| = k$.

Now consider the following partition of $\mathcal{F}^0$:

$$\mathcal{C} = \{F \in \mathcal{F}^0 : I \nsubseteq F\}; \mathcal{A} = \{F \in \mathcal{F}^0 : I \subseteq F, \text{there is } F' \in \mathcal{F}^0 \text{ with } F \cap F' = I\};$$

$$\mathcal{B} = \mathcal{F}^0 \setminus (\mathcal{A} \cup \mathcal{C}).$$

For a better understanding, $F \in \mathcal{B}$ if $I \subset F$ and whenever there is a set $F' \in \mathcal{F}^0$ with $I \subseteq F'$, then F should meet F' outside I, as well. Note that $\mathcal{A}$ is not empty, since

$F_1, F_2 \in \mathcal{A}$. Now for any $A \in \mathcal{A}$ let $f(A) = (A \setminus I) \cup \{1\}$. (Observe that for any $A \in \mathcal{A}$ there is $A' \in \mathcal{A} \subseteq \mathcal{F}^0$ with $A \cap A' = I$, $A \setminus I = A \setminus A'$, so $f(A) \in \mathbf{1} + \Delta\mathcal{F}^0$ as required!) As all $A \in \mathcal{A}$ contain I, f is injective restricted to $\mathcal{A}$.

To show that $\rho_{\mathcal{F}'}(f(A)) \geq \rho_{\mathcal{F}}(A)$, observe that $f(A) \subset A \cup \{1\}$. Therefore if $A \subset F \in \mathcal{F}$ and $1 \in F$ (that is $F \in \mathcal{F}^1$, therefore $F \in \mathcal{F}'$, too) , then $f(A) \subset F$, as well, so the part of $\rho(A)$ which comes from the Fs in $\mathcal{F}^1$ cannot decrease.

We have to handle the sets $A \subset F \in \mathcal{F}^0$. To do this let $(F \setminus I) \cup \{1\} = F'$. Then $F' \in \mathcal{F}'$ and $f(A) \subseteq F'$ by definition. If $F \neq G$ then $F' \neq G'$, because we took the same set I away from both (and $I \subseteq F, G$), and 1 was neither in F nor in G. We still have to point out that F' is not equal to any $G \in F^1$, G containing A for any $F \in \mathcal{F}^0$ (because in that case we would take into account that containing relation twice when counting $\rho(f(A))$). But this is clear, because a G of this form contains I (as $I \subseteq A$), and $F' \cap I = \emptyset$ by definition (and as we pointed out I is not empty).

To finish the proof we need to continue this procedure now considering the remaining sets, that is $\mathcal{B} \cup \mathcal{C}$. So we define a new I' and a new k' now only considering sets in $\mathcal{B} \cup \mathcal{C}$, then get a new partition $\mathcal{A}', \mathcal{B}', \mathcal{C}'$ with respect to this new I' and new k', and define f on $\mathcal{A}'$ with the help of I', and then start again with $\mathcal{B}' \cup \mathcal{C}'$... This procedure ends after finitely many steps, as the $\mathcal{A}$s are never empty, so there is strictly less and less remainder. In each step f is injective, the only difficulty is to assure for sets A, B on which f is defined at different steps $f(A) = f(B)$ cannot happen. This is clearly done by

Claim 2.1.9 $(\mathcal{A} - \{I\}) \cap \Delta(\mathcal{B} \cup \mathcal{C}) = \emptyset$

$\mathcal{A} - \{I\}$ is the set of the f-images defined at a step (if we do not consider 1, which is an element of all images). For a set B on which f is defined later, the image is of the form $B \setminus I' = B \setminus B'$ (again without 1), so it is in $\Delta(\mathcal{B} \cup \mathcal{C})$. Therefore by the claim we will be really done.

Proof: This is in fact the lemma in [26], but to be self-contained we repeat the proof.

CASE 1: $A \in \mathcal{A}, B \in \mathcal{B}, F \in \mathcal{B} \cup \mathcal{C}$.

By the definition of $\mathcal{B}$, B must meet A outside of I, too. Therefore $F \setminus B$ does not contain this (these) element(s), while $A \setminus I$ does.

CASE 2: $A \in \mathcal{A}, C \in \mathcal{C}, F \in \mathcal{B} \cup \mathcal{C}$

By the definition of $\mathcal{C}$, C does not contain I, therefore by the minimality of $|I|$, C must meet A outside of I, too. The rest is as in CASE 1. □ □ □

2.2 Sperner Families

In the introduction of the thesis, we cited Sperner's famous theorem about Sperner systems without using this expression for the concept. Let us define it now explicitly.

Definition: $\mathcal{F}$ is a *Sperner system/ family* if $F_1 \not\subseteq F_2$ for any distinct $F_1, F_2 \in \mathcal{F}$.

Being a Sperner family is a property that can be defined via forbidden configurations, too. Let $\mathcal{C}_{\not\subseteq} = \{\mathcal{G}_{k,l} : k \in \mathbb{N}, l \in \mathbb{N}\}$ where $\mathcal{G}_{k,l} = \{\{1, ..., k\}, \{1, ..., k+l\}\}$ is the collection of forbidden hypergraphs, and for shortness' sake let us write $D_{\not\subseteq}(\mathcal{F}, \mathcal{G}) = D_{\mathcal{C}_{\not\subseteq}}(\mathcal{F}, \mathcal{G})$. So the distance of two Sperner systems is $D_{\not\subseteq}(\mathcal{F}, \mathcal{G}) = |\{\{A_1, A_2\} : A_i \in \mathcal{F} \cup \mathcal{G} \text{ and } A_1 \subseteq A_2\}|$.

Theorem 2.2.1 *If $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ are two Sperner systems, then*

$$D_{\not\subseteq}(\mathcal{F}, \mathcal{G}) \leq D_{\not\subseteq}(\mathcal{F}_0, \mathcal{G}_0)$$

where $\mathcal{F}_0$ is the family of all sets of size k_1, $\mathcal{G}_0$ is the family of all sets of size k_2 ($k_1 < k_2$) with each of $k_1, k_2 - k_1, n - k_2$ differing by at most one. In particular, if 3 divides n, then $\mathcal{F}_0$ is the family of all sets of size $n/3$ and $\mathcal{G}_0$ is the family of all sets of size $2n/3$.

First proof: W.l.o.g. we can assume that both $\mathcal{F}$ and $\mathcal{G}$ are maximal Sperner families, since adding new sets to the families cannot decrease the distance.

Our goal is to show that by starting with any pair of Sperner systems $(\mathcal{F}, \mathcal{G})$, in finitely many steps $(\mathcal{F}^i, \mathcal{G}^i)$ we can reach $(\mathcal{F}_0, \mathcal{G}_0) = (\mathcal{F}^m, \mathcal{G}^m)$ such that

$$D_{\not\subseteq}(\mathcal{F}, \mathcal{G}) \leq D_{\not\subseteq}(\mathcal{F}^1, \mathcal{G}^1) \leq D_{\not\subseteq}(\mathcal{F}^2, \mathcal{G}^2) \leq ... \leq D_{\not\subseteq}(\mathcal{F}^m, \mathcal{G}^m) = D_{\not\subseteq}(\mathcal{F}_0, \mathcal{G}_0).$$

Step 1

Let $\mathcal{C} = \mathcal{F} \cap \mathcal{G}$ and partition $\mathcal{F}$ and $\mathcal{G}$ by

$$\mathcal{F} = \mathcal{C} \cup \mathcal{F}_1 \cup \mathcal{F}_2 \text{ and } \mathcal{G} = \mathcal{C} \cup \mathcal{G}_1 \cup \mathcal{G}_2$$

where $\mathcal{F}_1 = \{F \in \mathcal{F} : \text{there is } G \in \mathcal{G}\ G \subsetneq F\}$, $\mathcal{F}_2 = \{F \in \mathcal{F} : \text{there is } G \in \mathcal{G}\ G \supsetneq F\}$ and $\mathcal{G}_1, \mathcal{G}_2$ defined similarly. Note, that any $F \in \mathcal{F}$ contains or is contained in some $G \in \mathcal{G}$, because otherwise we could add it to $\mathcal{G}$, which would contradict the maximal property of $\mathcal{G}$, and no $F \in \mathcal{F}$ belongs to both $\mathcal{F}_1, \mathcal{F}_2$, otherwise there exist $G_1, G_2 \in \mathcal{G}$ such that $G_1 \subsetneq F \subsetneq G_2$ contradicting the Sperner property of $\mathcal{G}$. So $\mathcal{C}, \mathcal{F}_1, \mathcal{F}_2$ is really a partition of $\mathcal{F}$.

Now let $\mathcal{F}^1 = \mathcal{C} \cup \mathcal{F}_1 \cup \mathcal{G}_1$ and $\mathcal{G}^1 = \mathcal{C} \cup \mathcal{F}_2 \cup \mathcal{G}_2$. It is easy to check that both $\mathcal{F}^1$ and $\mathcal{G}^1$ are Sperner systems. The fact that $D_{\not\subseteq}(\mathcal{F}, \mathcal{G}) = D_{\not\subseteq}(\mathcal{F}^1, \mathcal{G}^1)$ follows from the fact that $E(\mathcal{F}) \cup E(\mathcal{G}) = E(\mathcal{F}^1) \cup E(\mathcal{G}^1)$.

By the above change of the systems there is no $F \in \mathcal{F}^1$ for which there exists a $G \in \mathcal{G}^1$ with $F \subseteq G$, so we can refer to $\mathcal{F}^1$ as the upper Sperner family, and to $\mathcal{G}^1$ as the lower family.

From now on in any even step we replace some of the sets of the upper Sperner system by other sets of larger size, and in any odd step we do the same to some sets of the lower family.

Step 2

Partition $\mathcal{F}^1$ into two subsystems: $\mathcal{F}_1^1 = \{F \in \mathcal{F}^1 : |F| > n/2\}, \mathcal{F}_2^1 = \{F \in \mathcal{F}^1 : |F| \le n/2\}$. Put $\mathcal{F}^2 = \mathcal{F}_1^1 \cup \{F \in \binom{[n]}{\lceil n/2 \rceil} :$ there is $F' \in \mathcal{F}_2^1$ such that $F' \subseteq F\}$ and $\mathcal{G}^2 = \mathcal{G}^1$. It is clear that $\mathcal{F}^2$ is a Sperner family.

$D_{\not\subseteq}(\mathcal{F}^1, \mathcal{G}^1) \le D_{\not\subseteq}(\mathcal{F}^2, \mathcal{G}^2)$ follows from Sperner's lemma [31] stating, that if $\mathcal{G}$ is a k-uniform family with $k \le \frac{n}{2}$, then $|\nabla\mathcal{G}| \ge |\mathcal{G}|$, where $\nabla\mathcal{G} = \{G' \subset [n] : |G'| = k+1$ and there is $G \in \mathcal{G}$ such that $G \subseteq G'\}$.

Step 3

Now we want to "push the lower system up", so we replace the small sets.

$$\mathcal{G}_2^2 = \{G \in \mathcal{G}^2 : |G| < \lceil \frac{\lceil \frac{n}{2} \rceil}{2} \rceil\};\ \mathcal{G}_1^2 = \mathcal{G}^2 \setminus \mathcal{G}_2^2$$

$$\mathcal{G}^3 = \mathcal{G}_1^2 \cup \{G \in \binom{[n]}{\lceil \frac{\lceil \frac{n}{2} \rceil}{2} \rceil} : \text{there is } G' \in \mathcal{G}_2^2 \text{ with } G' \subseteq G\};\ \mathcal{F}^3 = \mathcal{F}^2$$

Just as in the argument in Step 2 $\mathcal{G}^3$ is a Sperner system, and using the original proof of Sperner's theorem one can verify that for any fixed $F \in \mathcal{F}^3 = \mathcal{F}^2$ the number of sets in $\mathcal{G}^3$ contained by F is at least the number of sets in $\mathcal{G}^2$ contained by F.

Suppose we achieved in Step $2k$ that the sets in the upper set system have size at least $c_k n$, and in Step $2k+1$ that all the sets in the lower set system have size at least $d_k n$. Then in Step $2(k+1)$ we will show that that all sets in the upper family have size at least $c_{k+1}n = d_k n + \lceil \frac{1-d_k}{2} n \rceil$, and in Step $2(k+1)+1$ that the sets of the lower family have size at least $d_{k+1}n = \lceil \frac{c_{k+1}n}{2} \rceil$. Formally Step $2(k+1)$ and Step $2(k+1)+1$ are defined as follows:

Step $2(k+1)$

Let $\mathcal{F}^{2k+1} = \mathcal{F}_1^{2k+1} \cup \mathcal{F}_2^{2k+1}$, where $\mathcal{F}_1^{2k+1} = \{F \in \mathcal{F}^{2k+1} : |F| > d_k n + \lceil \frac{1-d_k}{2} n \rceil\}$ and $\mathcal{F}_2^{2k+1} = \mathcal{F}^{2k+1} \setminus \mathcal{F}_1^{2k+1}$. Then let

$$\mathcal{F}^{2(k+1)} = \mathcal{F}_1^{2k+1} \cup \{F' \in \binom{[n]}{d_k n + \lceil \frac{1-d_k}{2} n \rceil} : \exists F \in \mathcal{F}^{2k+1} \text{ such that } F \subseteq F'\}$$

and let

$$\mathcal{G}^{2(k+1)} = \mathcal{G}^{2k+1}.$$

Step $2(k+1)+1$

Let us partition $\mathcal{G}^{2(k+1)}$ into two subfamilies: $\mathcal{G}_1^{2(k+1)} = \{G \in \mathcal{G}^{2(k+1)} : |G| > \lceil \frac{c_{k+1}n}{2} \rceil\}$ and $\mathcal{G}_2^{2(k+1)} = \mathcal{G}^{2(k+1)} \setminus \mathcal{G}_1^{2(k+1)}$. Then put

$$\mathcal{G}^{2(k+1)+1} = \mathcal{G}_1^{2(k+1)} \cup \{G' \in \binom{[n]}{\lceil \frac{c_{k+1}n}{2} \rceil} : \exists G \in \mathcal{G}^{2(k+1)} \text{ such that } G \subseteq G'\}$$

and

$$\mathcal{F}^{2(k+1)+1} = \mathcal{F}^{2(k+1)}.$$

The fact that during Step $2(k+1)$ and Step $2(k+1)+1$ the distance of our families cannot decrease follows just as in the case of Step 2 and Step 3. (Note that in Step $2(k+1)$ we apply Sperner's lemma to the posets $\mathbb{P}_G = \{H \backslash G : H \supseteq G\}$, where G ranges over the sets in $\mathcal{G}^{2k+1}$, while in Step $2(k+1)+1$) to the posets $\mathbb{P}_F = \{H : H \subseteq F\}$, where F ranges over the sets in $\mathcal{F}^{2(k+1)}$.) The statement about c_{k+1} and d_{k+1} is true by definition.

So (forgetting the ceiling signs for a moment) $c_{k+1} = \frac{1}{2}c_k + \frac{1-\frac{1}{2}c_k}{2} = 1/2 + c_k/4$ (and $d_{k+1} = c_{k+1}/2$). As for any $x \in [0; 2/3)$ $x < 1/2 + x/4$, in finitely many steps (by virtue of the ceiling sign) we can achieve that all the sets in the upper family have size at least $\lceil 2n/3 \rceil$, and all the sets in the lower family have size at least $\lceil n/3 \rceil$.

To finish the proof we need the observation that the complement system of a Sperner system is a Sperner system, and that (denoting the complement system of $\mathcal{F}$ by $\overline{\mathcal{F}} = \{[n] \setminus F : F \in \mathcal{F}\}$), we have $D_{\not\subseteq}(\mathcal{F}, \mathcal{G}) = D_{\not\subseteq}(\overline{\mathcal{F}}, \overline{\mathcal{G}})$.

In the complement systems of the above pair, all sets have size at most $\lfloor n/3 \rfloor$ or $\lfloor 2n/3 \rfloor$, and after the same "pushing up procedure" we get one of the optimal pairs. □

Second proof: By Step 1 of the previous proof we reduce the problem to Sperner families $\mathcal{F}, \mathcal{G}$ where for all $F \in \mathcal{F}$ there is a $G \in \mathcal{G}$ with $F \subseteq G$. Then we are done by the following theorem of Katona.

Theorem 2.2.2 [24] *(Iterated Sperner theorem) Let $A_1, ..., A_m$ be subsets of an n element set satisfying $A_j \not\subseteq A_k$ $1 \leq j, k \leq m, j \neq k$. For each $i = 1, ..., m$, suppose $B_{i,1}, ..., B_{i,m_i}$ are subsets of A_i satisfying $B_{i,j} \not\subseteq B_{i,k}$ $1 \leq j, k \leq m_i$. Then*

$$\sum_{i=1}^{m} m_i \leq \binom{n}{\lfloor \frac{2n}{3} \rfloor} \binom{\lfloor \frac{2n}{3} \rfloor}{\lfloor \frac{n}{3} \rfloor}.$$

□

Remark: Theorem 2.2.2 (besides Step 1) is stronger than Theorem 2.2.1 (since in Theorem 2.2.2 we do not require that the Bs form a Sperner family), but Katona's proof of Theorem 2.2.2 uses a generalization of the LYM-inequality, while our first proof uses only Sperner's original idea of his well-known theorem.

2.3 K_r-free Graphs

We denote by K_r the complete graph on r vertices. The K_r-free distance of two K_r-free graphs (G_1, G_2) on the same underlying set V is

$$D_{K_r}(G_1, G_2) = |\{\{x_1, ..., x_r\} : x_i \in V \text{ for all } i, \text{ and any } (x_i, x_j) \in E(G_1 \cup G_2)\}|$$

In all the cases we have already treated (intersecting, pairwise disjoint and Sperner families, chains), the structure of the families in the optimal pair (the pair with the maximum distance) was very similar to that of the optimal family in the original problem (what is the largest family with the desired property). Therefore it is quite natural to conjecture that Turán graphs will come into sight. (Turán's well-known theorem [32] says that a K_r-free graph on n vertices with the most possible number of edges must be isomorphic to the complete $r-1$-partite graph, where the sizes of any two partition classes may differ by at most one. These graphs are called Turán graphs.)

Though it is not true that if $D_{K_r}(G_1, G_2)$ is maximal, then both G_1, G_2 should be Turán graphs, still Turán graphs will play an important role in the proof of the next theorem. First we need to introduce some notation.

$T(n,r)$ is the usual notation for the r-partite Turán graph on n vertices and $t(n,r)$ denotes the number of edges in the graph. Now let $k_s(G)$ denote the number of s-cliques in G. (So $t(n,r) = k_2(T(n,r))$.)

The Ramsey number $R(k)$ denotes the least integer n for which any E_0, E_1 partition of the edges of K_n, there is a sample of K_k either in E_0 or in E_1.

Let us write furthermore $D_r^n := \max\{D_{K_r}(G_1, G_2) : G_1, G_2$ are K_r-free on the same vertex set $[n]\}$ and put $m = R(r) - 1$.

Theorem 2.3.1 $D_r^n = k_r(T(n,m))$

Proof: For the $\geq$ part we need a construction.

Let us fix a partition E_0, E_1 of the edges of K_m such that there is no K_r neither in E_0 nor in E_1. We want to define G_0, G_1 two K_r-free graphs on $[1, .., n]$. So we have to decide which edges we want to put into G_0 and which into G_1. To do this, for any $1 \leq i < j \leq n$ write $i = l_i m + i'$, $j = l_j m + j'$ where $1 \leq i', j' \leq m$.

Now put (i,j) into $E(G_0)$ iff $(i', j') \in E_0$, and into $E(G_1)$ iff $(i', j') \in E_1$. Since (i,j) is an edge if and only if $i \neq j \mod m$, therefore $G_0 \cup G_1$ is just $T(n,m)$ and the classes are just the congruency classes modulo m. We have to check that G_0, G_1 are both K_r-free. If not, then $i_1, i_2, .., i_r$ form a K_r in, say, G_0. But then $i'_1, i'_2, ..., i'_r$ should be all distinct, and should form a K_r in E_0 - a contradiction.

For the $\leq$ part of the proof, note that $G_0 \cup G_1$ cannot contain a $K_{R(r)}$ as otherwise G_0 or G_1 would contain a K_r. So the following result of Sauer (its $s = 2$ case is exactly

Turán's theorem) completes the proof.

Lemma 2.3.2 (Sauer [30] see also [3]) *If $s < p$ and G is a K_p-free graph on n vertices, then the number of K_ss in G is at most $k_s(T(n, p-1))$.* $\square$

Remark: If m divides n, then $k_r(T(n,m)) = \binom{m}{r}(\frac{n}{m})^r$, so the problem of giving the exact value of D_r^n for large enough n is equivalent to giving the exact value of $R(r)$.

2.4 Trees

Trees (and forests) are cycle-free graphs, so $C_{\text{cycle}} = \{C_k : k \geq 3\}$ where C_k is the cycle of length k. Therefore this time the question is, how many cycles we can have in the union of two trees on the same n-element vertex set. $D_{\text{cycle}}(T_1, T_2) = D_{C_{\text{cycle}}}(T_1, T_2)$ will denote the tree-distance (the number of cycles in the union) of two trees T_1 and T_2. D_{cycle}^n will denote the maximum tree-distance of two trees on the same n vertices.

A trivial upper bound on D_{cycle}^n is 4^{n-1}, since the union of two trees may contain at most $2(n-1)$ edges, so the number of subsets of the edge set of the union is clearly an upper bound for the number of cycles.

The following recursive construction (fig.1) shows that D_{cycle}^n does have an exponential growth. Suppose we have T_1^n, T_2^n two trees on n vertices, and an edge e (with endpoints x and y) in their union, through which there are c_n cycles in $T_1^n \cup T_2^n$. Likewise suppose we have T_1^m, T_2^m two trees on m vertices (with vertex set disjoint from that of T_1^n and T_2^n), and an edge f (with endpoints u and v) in their union, through which there are c_m cycles in $T_1^m \cup T_2^m$. Let $T_1^{n+m} = T_1^n \cup T_1^m \cup \{xu\}$ and $T_2^{n+m} = T_2^n \cup T_2^m \cup \{yv\}$. We claim that in $T_1^{n+m} \cup T_2^{n+m}$ there are $(c_n + 1)(c_m + 1)$ cycles through the edge xu. Indeed, there are c_n paths from x to y in $T_1^n \cup T_2^n$ plus the edge $xy = e$, then we have to go through the edge yv, then choose among the c_m paths in $T_1^m \cup T_2^m$ from v to u (or the edge $vu = f$), and then we finish off the cycle using the edge xu. Since we just took the sum of the number of vertices, and the number of cycles got multiplied, this is really of exponential growth. To be more concrete: we can cover K_4 by two (edge-disjoint) paths, so we have $D^4_{cycle} = 7$. By the above recursive bound we get that $D^{4n}_{cycle} \geq 7^n = (7^{1/4})^{4n}$. ($7^{1/4} = 1.625...$)

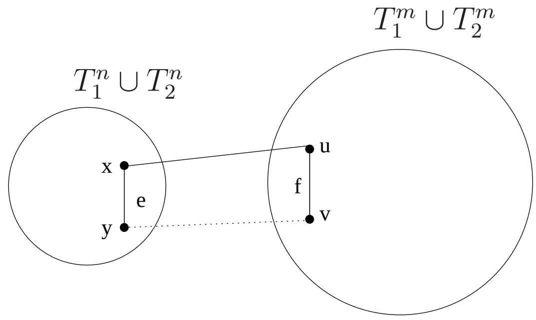

Figure 1: The recursive construction showing the exponential growth of D^n_{tree}

In the next two subsections we prove the following lower and upper bounds on D^n_{cycle}:

Theorem 2.4.1 *There exists a constant c for which the following inequalities hold*

$$cx_0^n \leq D^n_{tree} \leq \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} = 2^{n-1} - 1,$$

where x_0 is the unique real root of the equation $x^3 - x^2 - x - 1 = 0$ ($x_0 = 1.8392...$).

2.4.1 Lower Bound on D^n_{cycle}

In this subsection we will give a "real" construction for the lower bound on Δ (see fig.2). Both of the trees in the construction are paths, and we will refer to them as the blue tree (denoted by B_n) and the red tree (denoted by R_n). The vertices of the trees are the integers from $-k$ up to k if $n = 2k+1$ and the integers from $-k$ to $k-1$ if $n = 2k$. Two integers are adjacent in B_n if and only if they are consecutive. If $n = 2k+1$, then the edge set of R_n is $\{\{-l, l\} : 1 \leq l \leq k\} \cup \{\{l, -(l+1)\} : 1 \leq l < k\} \cup \{\{k, 0\}\}$. If

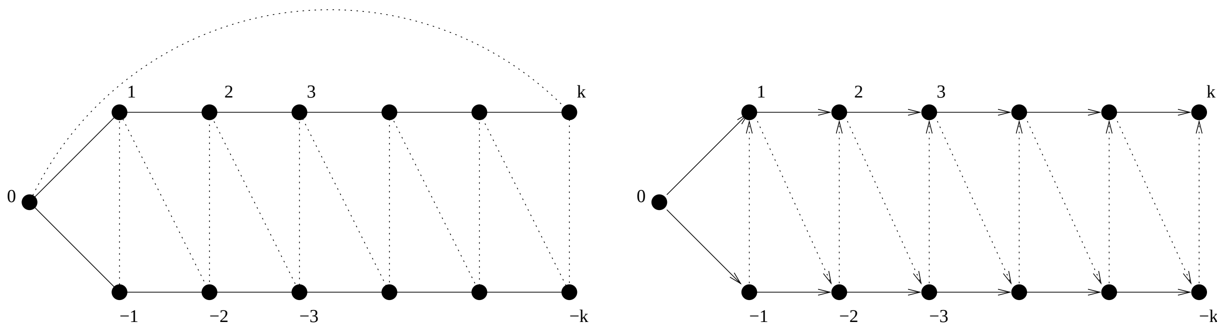

Figure 2: The “real construction” $R_n \cup B_n$ and the orientation of its edges

$n = 2k$ is even, one just drops the vertex k and the edges incident to it, and add the edge $\{-k, 0\}$ to the red tree.

Let $c(n)$ denote the number of cycles through the edge $\{k, 0\}$ (that is the number of paths from the vertex 0 to the vertex k) if $n = 2k + 1$ and the number of cycles through $\{-k, 0\}$ if $n = 2k$. We claim that the following recurrence holds: $c(n) = c(n-1) + c(n-2) + c(n-3)$ where $c(1) = c(2) = 1,\ c(3) = 2$.

To see this let us consider the graph $B_n \cup R_n \setminus \{\{0, k\}\}$ ($B_n \cup R_n \setminus \{\{0, -k\}\}$ if $n = 2k$) as a directed graph with the following orientation of the edges (fig.2): all edges are directed from the vertex of smaller absolute value to the vertex of bigger absolute value. The red edges of type $\{-l, l\}$ are directed from the vertex $-l$ toward the vertex l. In the path $0 = x_0, x_1, ..., x_{l-1}, x_l = k$ the edge $\{x_j, x_{j+1}\}$ is called a *backward edge* if x_j is the endpoint and x_{j+1} is the starting point of the edge in the above orientation. Other edges will be called *forward edges.*

First note, that there can be no blue backward edges in a path from 0 to k. Since if there was, let us take the “rightmost” one $\{x_j, x_{j+1}\}$ (i.e. the one with an endpoint of greatest absolute value). Assume $x_j = -(l+1), x_{j+1} = -l$ (the case $x_j = l+1, x_{j+1} = l$ is similar). Then because this is the rightmost backward blue edge in the path, x_{j-1} cannot be $-(l+2)$ ($\{-(l+2), -(l+1)\}$ would be a backward blue edge "further to the right"). Therefore x_{j-1} is either $l+1$ or l. In both cases the vertex $x_{j+1} = -l$ is

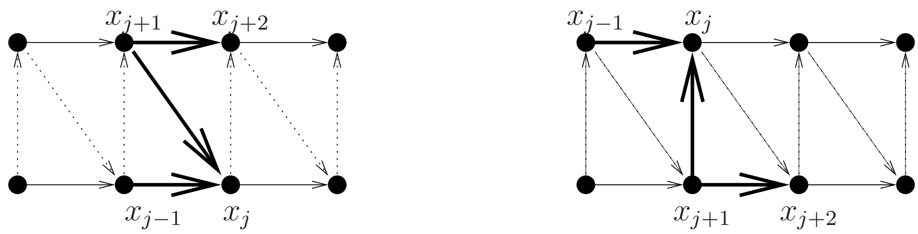

Figure 3: Backward red edges in oriented paths

cut from the vertex k by the edge $\{x_{j-1}, x_j\}$, so we cannot finish the path in this way.

How about backward red edges? (fig.3) If $x_j = -(l+1)$, $x_{j+1} = l$, then x_{j-1} cannot be $-(l+2)$ as $\{-(l+2), -(l+1)\}$ would be a backward blue edge. x_{j-1} cannot be $l+1$ either for the edge $\{x_{j-1}, x_j\}$ would cut x_{j+1} from the vertex k. So x_{j-1} must be $-l$. x_{j+2} cannot be $l-1$ (backward blue edge), so x_{j+2} is $l+1$.

In the same manner one can see, if $\{x_j, x_{j+1}\}$ is a backward red edge with $x_j = l$, $x_{j+1} = -l$, then x_{j-1} should be $l-1$ and x_{j+2} should be $-(l+1)$. So if we add the directed edges $\{\{l, -(l+2)\} : \ 0 \leq l \leq k-2\} \cup \{\{-l, l+1\} : \ 1 \leq l \leq k-1\}$ to the directed graph $R_n \cup B_n$, then in this new graph, the number of directed paths from 0 to k is equal to the number of non-directed paths from 0 to k in the non-directed graph $R_n \cup B_n$. (In fact we constructed a bijection among the non-oriented and the oriented paths of the two graphs: whenever a non-oriented path of the original graph uses a backward edge, the corresponding new edge should be used in the new graph to create an oriented path, and vice versa.)

If we reindex the vertices as in fig.4 the recurrence formula above follows, as any vertex l is adjacent to an incoming edge from $l-3, l-2$ and $l-1$. Solving this formula we get that $\Delta \geq cx_0^n$ for some constant c, where x_0 is the unique real root of the equation $x^3 - x^2 - x - 1 = 0$, $x_0 = 1.8392...$

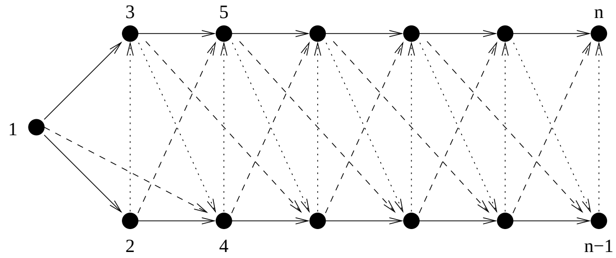

Figure 4: $R_n \cup B_n$ with the added oriented edges

2.4.2 Upper Bound on D^n_{cycle}

Proof of the upper bound in Theorem 2.4.1: To establish the inequality let us consider two trees B_n and R_n (a blue one and a red one) on the same n-element vertex set. In a cycle in $R_n \cup B_n$ there are consecutive red edges, then consecutive blue edges, then red edges again, and so on. (Edges that are both red and blue will be considered as red.) A maximal path of consecutive red edges will be called a *red segment* (a *blue segment* is defined similarly). The number of blue segments in a cycle clearly equals the number of red segments, and since each segment contain at least one edge, the number of red segments is at most $\lfloor \frac{n}{2} \rfloor$ and is at least 1 (for a cycle without red segment is a blue cycle, which is impossible, since B_n is a tree).

We will count the cycles in $R_n \cup B_n$ partitioning them according to the number of red segments. So we have to show that there are at most $\binom{n}{2i}$ cycles having i red segments. To do this first note that in a fixed cycle the set of the endpoints of the red segments and the set of the endpoints of the blue segments are just the same.

Lemma 2.4.2 *Given a tree and $2i$ vertices of its vertex set, then there is at most one way to choose i vertex-disjoint paths in the tree with the given vertices as endpoints.*

Proof: By induction on i. If $i = 1$ then clearly the statement holds, for in a tree there is exactly one path from any vertex to any other vertex.

Let $i > 1$. A set of vertex-disjoint paths defines naturally a matching on the set of endpoints. Notice, that an edge in such a matching corresponds always to the same path (for there is one single path between any two vertices of a tree). Let us suppose to the contrary, that there are two different sets of paths satisfying statement of the lemma. If there exists a common edge in the corresponding matchings, then removing this common edge (and its endpoints) we arrive at a contradiction by induction. If there is no such edge, then the two matchings have together $2i$ edges on $2i$ vertices, so there should be a cycle involving these edges, that is there should be a cycle in the corresponding paths, which contradicts the fact that our graph is a tree. □

To finish the proof of the upper bound, observe that by Lemma 2.4.2. in $R_n \cup B_n$ the mapping where the image of a cycle is the set of endpoints of the segments is injective. The statement of the theorem follows. □

Remark: It is easy to see, that in the statement of Lemma 2.4.2, "at most" is necessary if the tree is not a path. Hence we know that the upper bound for D^n_{cycle} holds with strict inequality for non-path trees. Since the graphs of the construction in the previous subsection were paths, one may conjecture, that trees with maximal distance are paths. But even if this conjecture is false, the question that how many cycles we can have in the union of two paths is a distance-type question. To see this we just have to figure out what the path-distance of two paths is. Since a path is a cycle-free (connected) graph in which all vertices have degree at most 2, the forbidden collection of subgraphs consists of the cycles and the 3-star (i.e. the graph consisting of the edges $\{1,2\},\{1,3\},\{1,4\}$). But since any vertex has degree at most 4 in the union of two paths, any vertex can be the middle vertex of at most $\binom{4}{3} = 4$ 3-stars, therefore there can be at most $4n$ 3-stars in the union of two paths on n vertices. As $4n$ is negligible compared to the exponentially growing number of cycles, $D_{\text{path}}(P_1^n, P_2^n) = \Theta(D_{\text{cycle}}(P_1^n, P_2^n))$ for the sequence of optimal pairs of paths.

3 Profile vectors

In this section we deal with *weighted* problems on set systems. Especially, we are interested in weight functions depending only on the size of the sets (i.e. $w(F) = w(G)$ whenever $|F| = |G|$). As explained in the introduction, we have to determine the convex hull of the profile vectors (the profile polytope) of all set systems having some prescribed property (intersecting, Sperner, etc.).

Determining the profile polytope means, that we have to find its extreme points or at least the essential extreme points. A property $P \subseteq 2^{2^{[n]}}$ is said to be *hereditary* (sometimes the term *monotone* is used) if $\mathcal{G} \subseteq \mathcal{F} \in P$ implies $\mathcal{G} \in P$. Note, that any property that can be defined through forbidden configurations is hereditary, so in particular the intersecting property (when P is the set of all intersecting families) is hereditary. If the examined property is hereditary, then we know (cf. [8]) that all extreme points can be obtained from an essential extreme point by changing some of the non-zero coordinates to zero.

In this section we will present two methods how to determine profile polytopes (both methods were used already in [7], [8] or [14], for a survey on results about profile vectors see Chapter 3 of Engel's book [4]). In the next subsection, we use the *method of inequalities* to determine the essential extreme points of the profile polytope of the set of intersecting families of subspaces, and in the second subsection, we introduce a generalization of the profile vector, which we call *l-chain profile vector* and obtain results on them with the *reduction method.*

3.1 Intersecting families of subspaces

In this subsection we determine the essential extreme points of the profile polytope of the set of intersecting families of subspaces.

We will use the symbol $\left[{n \atop k}\right]_q = \frac{(q^n-1)(q^{n-1}-1)...(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)...(q-1)}$ for the Gaussian (q-nomial) coefficient denoting the number of k-dimensional subspaces of an n-dimensional linear space over $GF(q)$ (and q will be omitted, when it is clear from the context). The set of all k-dimensional subspaces of a vector space V will be denoted by $\begin{bmatrix} V \\ k \end{bmatrix}$.

To simplify our counting arguments we introduce the following

Notation. If $k + d \leq n$, then $\begin{bmatrix} n \\ k \end{bmatrix}_q^{*(d)}$ denotes the number of k-dimensional subspaces of an n-dimensional vector space V over $GF(q)$ that are disjoint from a fixed d-dimensional subspace W of V.

Here are some basic facts about these numbers:

Facts.

$$\begin{bmatrix} n \\ k \end{bmatrix}^{*(d)} = \begin{bmatrix} n-d \\ k \end{bmatrix} q^{dk},$$

$$\frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}}{\begin{bmatrix} n \\ k \end{bmatrix}^{*(d)}} \leq \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(n-k)}}{\begin{bmatrix} n \\ k \end{bmatrix}^{*(n-k)}} = \frac{1}{q^{n-k}} \leq \frac{1}{q^{k+1}} \qquad (\text{if } 2k+1 \leq n),$$

and so inductively

$$\frac{\begin{bmatrix} n-p \\ k-p \end{bmatrix}^{*(d)}}{\begin{bmatrix} n \\ k \end{bmatrix}^{*(d)}} \leq \frac{1}{q^{p(k+1)}} \qquad (\text{if } 2k+1 \leq n).$$

To determine the profile polytope of intersecting families we follow the so-called *method of inequalities*. Briefly it consists of the following steps:

- establish as many linear inequalities valid for the profile of any intersecting family as possible (each inequality determines a halfspace, therefore the profiles must lie in the intersection of all halfspaces determined by the inequalities),
- determine the extreme points of the polytope determined by the above halfspaces,
- for all of the above extreme points find an intersecting family having this extreme point as profile vector.

The last step gives that the extreme points of the polytope determined by the halfspaces are the extreme points of the profile polytope that we are looking for.

The following theorem on intersecting families of subspaces was first proved by Hsieh [22] (only for $n \geq 2k+1$) in 1977, then by Greene and Kleitman [20] (for the cases $k|n$ so especially if $n = 2k$) in 1978.

Theorem 3.1.1. *(Erdős - Ko - Rado for vector spaces, Hsieh's theorem) If* $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ *is an intersecting family of subspaces and* $n \geq 2k$*, then*

$$|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

The above theorem yields to the following inequalities concerning the profile vector of any intersecting family:

- $0 \leq f_i \leq \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}, \quad 0 \leq i \leq n/2$
- $0 \leq f_i \leq \begin{bmatrix} n \\ i \end{bmatrix}, \quad n/2 < i \leq n$

To establish more inequalities we will need the following statement:

Theorem 3.1.2. *The following generalization of Hsieh's theorem holds:*
a, if $2k \leq n \leq 2k+2$ *and* $d = 0$ *or* $d = n-k$
or
b, if $n \geq 2k+3$ *and* $k + d \leq n$

then for any intersecting family $\mathcal{F}$ *of* k*-dimensional subspaces of an* n*-dimensional vector space* V *with all members disjoint from a fixed* d*-dimensional subspace* U *of* V

$$|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Note that the $d = 0$ case is just Hsieh's theorem.

Proof: If $k|d|n$ or $k|n$ and $d = 0$ then the argument of Greene and Kleitman [20] works. One can partition $V \setminus U$ into isomorphic copies of $V_k \setminus \{\underline{0}\}$, and since among the k-dimensional spaces of each such partition $\mathcal{F}$ may contain at most 1, the statement of the theorem follows.

So now we can assume $2k + 1 \leq n$. We follow the argument in [22]. First we verify the validity of the lemmas from [22] in our context. For $x \in V$ ($A \leqslant V$) let $\mathcal{F}_x$ ($\mathcal{F}_A$) denote the set of subspaces in $\mathcal{F}$ containing x (A).

Lemma 3.1.3 (the equivalent of Lemma 4.2. in [22]) *Suppose $n \geq 2k + 1$ and let $\mathcal{F}$ be an intersecting family of k-subspaces of an n-dimensional space V such that all k-subspaces belonging to $\mathcal{F}$ are disjoint from a fixed d-dimensional subspace W of V (where $d \leq n - k$). If for all x we have $|\mathcal{F}_x| \leq \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)}$, then*

$$|\mathcal{F}| < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)} \text{ or } |\mathcal{F}_A| \leq \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p-1}$$

for all 2-dimensional subspaces A, where $1leqp \leq k - 1$.

Proof: First we check the validity of the following consequence of the "facts":

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)} > q^p \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)} \geq \begin{bmatrix} s \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)}, \qquad (2)$$

for $1 \leq s \leq p$. Indeed,

$$\frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}}{\begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)}} \geq \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(n-k)}}{\begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(n-k)}} = q^{p(n-k)} > q^p \left(\frac{q^k - 1}{q - 1} \right)^p = q^p \begin{bmatrix} k \\ 1 \end{bmatrix}^p,$$

where the first inequality follows from the facts and the second one uses the assumption $n \geq 2k + 1$.

Let us take an arbitrary 2-dimensional subspace $\langle x, y \rangle \subset V$. If $U \in \mathcal{F}$ implies

$U \cap \langle x, y\rangle \neq \{\underline{0}\}$, then by (2) (and the assumption of the lemma) we have

$$|\mathcal{F}| \leq \sum_{Z\subset \langle x,y\rangle, Z 1-\dim} |\mathcal{F}_Z| \leq \begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^p \begin{bmatrix}n-1-p\\k-1-p\end{bmatrix}^{*(d)} < \begin{bmatrix}n-1-p\\k-1-p\end{bmatrix}^{*(d)}.$$

Thus we can suppose there is some $U_1 \in \mathcal{F}$ such that $U_1 \cap \langle x, y\rangle = \{\underline{0}\}$. Take $\underline{0} \neq z_1 \in U_1$. If $U \in \mathcal{F}$ implies $U \cap \langle x, y, z_1\rangle \neq \{\underline{0}\}$, then (again using (2))

$$|\mathcal{F}| \leq \begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^p \begin{bmatrix}n-1-p\\k-1-p\end{bmatrix}^{*(d)} < \begin{bmatrix}n-1-p\\k-1-p\end{bmatrix}^{*(d)}.$$

Thus we can suppose that there is some $U_2 \in \mathcal{F}$ such that $U_2 \cap \langle x, y, z_1\rangle = \{\underline{0}\}$. Hence $|\mathcal{F}_{x,y,,z_1}| \leq \left[\begin{smallmatrix}k\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}n-4\\k-4\end{smallmatrix}\right]^{*(d)}$, and so $|\mathcal{F}_{x,y}| \leq \left[\begin{smallmatrix}k\\1\end{smallmatrix}\right]^2\left[\begin{smallmatrix}n-4\\k-4\end{smallmatrix}\right]^{*(d)}$.

Suppose that for $1 \leq j \leq i, \underline{0} \neq z_j \in U_j$ and $\langle x, y, z_1, ..., z_j\rangle \cap U_{j+1} = \{\underline{0}\}$. Take $\underline{0} \neq z_{i+1} \in U_{i+1}$. If $U \in \mathcal{F}$ implies $U \cap \langle x, y, z_1, ..., z_{i+1}\rangle \neq \{\underline{0}\}$, then by (2)

$$|\mathcal{F}| \leq \begin{bmatrix}i+3\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^p \begin{bmatrix}n-1-p\\k-1-p\end{bmatrix}^{*(d)} < \begin{bmatrix}n-1-p\\k-1-p\end{bmatrix}^{*(d)}.$$

Thus we can suppose that there is some $U_{i+2} \in \mathcal{F}$ such that $U_{i+2} \cap \langle x, y, z_1, ..., z_{i+1}\rangle = \{\underline{0}\}$. Hence we have

$$|\mathcal{F}_{x,y,Z_1,...,z_{i+1}}| \leq \begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-i-4\\k-i-4\end{bmatrix}^{*(d)},$$

and by induction we obtain

$$|\mathcal{F}_{x,y}| \leq \begin{bmatrix}k\\i\end{bmatrix}^{*(d)}.$$

Thus for $1 \leq i \leq p$, either we have $|\mathcal{F}| < \left[\begin{smallmatrix}n-1\\k-1\end{smallmatrix}\right]^{*(d)}$ or $|\mathcal{F}_{x,y}| \leq \left[\begin{smallmatrix}k\\1\end{smallmatrix}\right]^{i-1}\left[\begin{smallmatrix}n-1-i\\k-1-i\end{smallmatrix}\right]^{*(d)}$, as a special case with $i = p$ either we have $|\mathcal{F}_{x,y}| \leq \left[\begin{smallmatrix}k\\1\end{smallmatrix}\right]^{p-1}\left[\begin{smallmatrix}n-1-p\\k-1-p\end{smallmatrix}\right]^{*(d)}$.

□

We will need one more lemma from Hsieh's paper (actualized to our context):

Lemma 3.1.4 (the equivalent of Lemma 4.3. in [22]) *Let $\mathcal{F}$ be a family of intersecting k-subspaces of an n-dimensional space V of which all subspaces are disjoint from a fixed d-dimensional subspace W of V. Furthermore if* *a,* $q \geq 3$ *and* $n \geq 2k+1$ *and for all* x *we have* $|\mathcal{F}_x| \leq {k \brack 1}^{k-1}$,

or if

b, $q = 2$ *and*

- $n \geq 2k+1$

- *and for all* x *we have* $|\mathcal{F}_x| \leq {k \brack 1}^{\min\{k-1, n-k-d\}} \prod_{i=1}^{k-1-(n-k-d)} \left({k \brack 1} - {i \brack 1}\right)$ *(if* $k-1 < n-k-d$*, then the product is empty and equals 1),*

then

$$|\mathcal{F}| < {n-1 \brack k-1}^{*(d)}.$$

Proof: In all cases $|\mathcal{F}|$ is at most ${k \brack 1}$ times the bound on $|\mathcal{F}_x|$.

Now if $q \geq 3$, then

$$|\mathcal{F}| \leq {k \brack 1}^k = \left(\frac{q^k - 1}{q-1}\right)^k < q^{k^2-1} \leq q^{(k-1)(n-k)} = {n-1 \brack k-1}^{*(n-k)} \leq {n-1 \brack k-1}^{*(d)}.$$

If $q = 2$, then for any $n \geq 2k+1$ and $d = n-k$ we have

$$|\mathcal{F}| \leq \prod_{i=0}^{k-1}\left({k \brack 1} - {i \brack 1}\right) < {k \brack 1}^{k-1}\left({k \brack 1} - {k-1 \brack 1}\right) < (q^k)^{k-1}q^{k-1} =$$

$$q^{k^2-1} \leq q^{(k-1)(n-k)} = {n-1 \brack k-1}^{*(n-k)}.$$

Since $n \geq 2k+1$, we have $n-2k+1 \geq 2$ holds. This gives

$$|\mathcal{F}| \leq {k \brack 1}^k = \left(\frac{q^k-1}{q-1}\right)^k < q^{2(k-1)}\frac{(q^{2k-2}-1)(q^{2k-3}-1)...(q^k-1)}{(q^{k-1}-1)(q^{k-2}-1)...(q-1)} \leq$$

$$\leq q^{(k-1)(n-2k+1)}{2k-2 \brack k-1} = {n-1 \brack k-1}^{*(n-2k+1)}.$$

This establishes the lemma for $0 \leq d \leq n-2k+1$. For the remaining cases ($n-2k+1 < d < n-k$), one has to observe that the largest value of d for which the bound on $|\mathcal{F}_x|$ in the conditions of the lemma is ${k \brack 1}^{k-1}$ equals $n-2k+1$ (i.e. the largest d for which

$k-1 \leq n-k-d$ holds). It follows, that when moving from d to $d+1$ the known bound on $|F|$ is multiplied by $\frac{\left[{k \atop 1}\right]-\left[{n-k-d \atop 1}\right]}{\left[{k \atop 1}\right]}$, while our targeted bound decreases by a factor of $\frac{\left[{n-1 \atop k-1}\right]^{*(d+1)}}{\left[{n-1 \atop k-1}\right]^{*(d)}}$.

Easy calculations show that this latter ratio is larger up till $n-k-d < (k-1)/2$, and the former ratio is larger when $n-k-d \geq (k-1)/2$. This means that the gap between the bound on $|\mathcal{F}|$ and our targeted bound grows while $n-k-d < (k-1)/2$, from then on this gap decreases, but since it still holds in the end, it must hold in between as well.

This finishes the proof of the lemma. □

Before we get into the details of the proof of Theorem 3.1.2, we just collect its main ideas:

the heart of the proof is the concept of *covering number*. For a family of *subsets* $\mathcal{F} \subseteq 2^{[n]}$ this is the size of the smallest set $S \subseteq [n]$ that intersect all sets in $\mathcal{F}$ (S need not be in $\mathcal{F}$). For a family of subspaces $\mathcal{F} \subseteq \left[{V \atop k}\right]$ its covering number is the smallest number τ such that there is a τ-dimensional subspace U of V that intersects all subspaces that belong to $\mathcal{F}$. Observe that the proof of Lemma 3.1.3 was done by an induction on the covering number. The proof of Theorem 3.1.2 is again based on an induction on the covering number of $\mathcal{F}$. (During the proof, almost all computations will use the "facts" about Gaussian coefficients, all inequalities without any further remarks follow from them.)

If $x \in \cap\mathcal{F}$ for some $\underline{0} \neq x \in V$ then $|\mathcal{F}| \leq \left[{n-1 \atop k-1}\right]^{*(d}$. Thus we can suppose that $\cap\mathcal{F} = \{\underline{0}\}$.

Let $x_1 \neq \underline{0}$ be such that $|\mathcal{F}_{x_1}| = \max_{x \in V} |\mathcal{F}_x|$.

By our assumption, there is some $A_1 \in \mathcal{F}$ not containing x_1. Thus $|\mathcal{F}_{x_1}| \leq \left[{k \atop 1}\right]\left[{n-2 \atop k-2}\right]^{*(d)}$.

Suppose that there are two independent vectors $z_1, z_2 \in A_1$ such that $A \in \mathcal{F} \Rightarrow A \cap \langle x_1, z_i \rangle \neq \{\underline{0}\}$ for $i = 1, 2$. If $u_i \in \langle x_1, z_i \rangle \setminus \langle x_1 \rangle$, then the u_i's are independent. Thus

$$|\mathcal{F}| \leq |\mathcal{F}_{x_1}| + \sum_{U_i \subset (\langle x_1, z_i \rangle \setminus \langle x_1 \rangle) \cup \{\underline{0}\},\ dim(U_i)=1} |\mathcal{F}_{U_1,U_2}|$$
$$\leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}^{*(d)} + \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1 \right)^2 \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}^{*(d)} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{(*(d))}.$$

Thus we can suppose that there is at most one $z \in A_1$ such that $A \in \mathcal{F} \Rightarrow A \cap \langle x_1, z \rangle \neq \{\underline{0}\}$. Suppose that $z \in A_1$ is such. Take $x \in A_1 \setminus \langle z \rangle$, then there is some $A \in \mathcal{F}$ such that $A \cap \langle x_1, x \rangle = \{\underline{0}\}$ and hence $|\mathcal{F}_{x_1,x}| \leq \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} n-3 \\ k-3 \end{smallmatrix}\right]^{*(d)}$. Thus

$$|\mathcal{F}_{x_1}| \leq |\mathcal{F}_{x_1,z}| + \sum_{X \subset (A_1 \setminus \langle z \rangle) \cup \{\underline{0}\},\ \ dim(X)=1} |\mathcal{F}_{x_1,X}| \leq \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}^{*(d)} + \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}^{*(d)}.$$

But then

$$|\mathcal{F}| \leq \sum_{X \subset \langle x_1, z \rangle, dim(X)=1} |\mathcal{F}_X| \leq \begin{bmatrix} 2 \\ 1 \end{bmatrix} \left(\begin{bmatrix} n-2 \\ k-2 \end{bmatrix}^{*d} + \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}^{*(d)} \right) \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Thus we can suppose that for all $x \in A_1$ there is some $A \in \mathcal{F}$ such that $A \cap \langle x_1, x \rangle = \{\underline{0}\}$, and hence $|\mathcal{F}_{x_1,x}| \leq \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} n-3 \\ k-3 \end{smallmatrix}\right]^{*(d)}$. Thus $|\mathcal{F}_{x_1}| \leq \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]^2 \left[\begin{smallmatrix} n-3 \\ k-3 \end{smallmatrix}\right]^{*(d)}$.

In general, suppose that for $1 \leq p \leq k-3$ we have non-zero vectors $y_1, y_2, ..., y_p \in V$ and $A_1, A_2, ..., A_{p+1} \in \mathcal{F}$ such that $y_i \in A$ and $A_{i+1} \cap \langle x_1, y_1, .., y_p \rangle = \{\underline{0}\}$ for $1 \leq i \leq p$. (We have just proved that for any $y_1 \in A_1$ there exists such an $A_2 \in \mathcal{F}$ or the statement of the theorem holds.) Thus

$$|\mathcal{F}_{x_1,y_1,...,y_p}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)},$$

and so inductively we obtain that

$$|\mathcal{F}_{x_1}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}.$$

By Lemma 3.1.3, we have

$$|\mathcal{F}_{x,y}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

for all 2-dimensional $\langle x, y\rangle \subset V$.

Suppose that there are $p+2$ linearly independent vectors $z_1, z_2, ..., z_{p+2}$ in A_{p+2} such that $\langle x_1, y_1, ..., y_p, z_i\rangle \cap A \neq \{\underline{0}\}$ for all $A \in \mathcal{F}$ and $i = 1, 2, ..., p+2$. Let $u_i \in \langle x_1, y_1, ..., y_p, z_i\rangle \setminus \langle x_1, y_1, ..., y_p\rangle$, $i = 1, 2, ..., p+2$, then $u_1, u_2, ..., u_{p+2}$ are independent. Thus

$$|\mathcal{F}| \leq \sum_{X \subset \langle x_1, y_1, ..., y_p\rangle, dim(X)=1} |\mathcal{F}_X| + \sum_{U_i \subset (\langle x_1, y_1, ..., y_p, z_i\rangle \setminus \langle x_1, y_1, ..., y_p\rangle) \cup \{\underline{0}\}, dim(U_i)=1} |\mathcal{F}_{U_1, U_2, ..., U_{p+2}}|$$
$$\leq \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)} + \left(\begin{bmatrix} p+2 \\ 1 \end{bmatrix} - \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \right)^{p+2} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$
$$\leq \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)} + q^{(p+1)(k-1)} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$
$$\leq \left(\begin{bmatrix} p+1 \\ 1 \end{bmatrix} + 1 \right) \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*d} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Thus we can suppose that there are at most $p+1$ such z_i's. Hence

$$|\mathcal{F}_{x_1, y_1, ..., y_p}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)},$$

and so

$$|\mathcal{F}_{x_1}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}.$$

Suppose that we do have independent vectors $z_1, z_2 \in A_{p+2}$ such that $A \in \mathcal{F} \Rightarrow A \cap \langle x_1, y_1, ..., y_p, z_i\rangle \neq \{\underline{0}\}$ for $i = 1, 2$. Then

$$|\mathcal{F}| \leq \sum_{X \subset \langle x_1, y_1, ..., y_p\rangle, dim(X)=1} |\mathcal{F}_X| + \sum_{U_i \subset (\langle x_1, y_1, ..., y_p, z_i\rangle \setminus \langle x_1, y_1, ..., y_p\rangle) \cup \{\underline{0}\}, dim(U_i)=1} |\mathcal{F}_{U_1, U_2}|$$
$$\leq \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)} \right) +$$
$$+ \left(\begin{bmatrix} p+2 \\ 1 \end{bmatrix} - \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \right)^2 \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$
$$= \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \left(\begin{bmatrix} p+2 \\ 1 \end{bmatrix}^2 + q^{2(p+1)} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \right) \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

$$\leq \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + q^p \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

$$\leq \left(\frac{\begin{bmatrix} p+1 \\ 1 \end{bmatrix}}{q^{p+2}} + \frac{1}{q} \right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Thus we can suppose that there is at most one such z. Hence

$$|\mathcal{F}_{x_1}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}.$$

Suppose that $z_1 \in A_{p+1}$ is such a z, then

$$|\mathcal{F}| \leq \sum_{X \subset \langle x_1, y_1, \ldots, y_p, z_1 \rangle, dim(x)=1} |\mathcal{F}_X| \leq \begin{bmatrix} p+2 \\ 1 \end{bmatrix} \left(\begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)} \right)$$

$$< \begin{bmatrix} p+2 \\ 1 \end{bmatrix} (\begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \frac{1}{q} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

$$\leq \left(\frac{\begin{bmatrix} p+2 \\ 1 \end{bmatrix}}{q^{p+2}} + \frac{1}{q^{p+2}} \right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Thus we can suppose that for all $z \in A_{p+1}$, there is some $A \in \mathcal{F}$ such $A \cap \langle x_1, y_1, ..., y_p, z \rangle = \{\underline{0}\}$. Take $y_{p+1} \in A_{p+1}$, and let A_{p+2} be such that $A \cap \langle x_1, y_1, ..., y_p, y_{p+1} \rangle = \{\underline{0}\}$.

We obtained, that either the statement of the theorem holds, or there are linearly independent vectors $x_1, y_1, ..., y_{k-1}$ and $A_i \in \mathcal{F}$ $i = 1, ...k-1$ such that $y_i \in A_i$ and $\langle x_1, y_1, ...y_{i-1} \rangle \cap A_i = \{\underline{0}\}$.

If $q \geq 3$, this means that either $|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}$ or $|\mathcal{F}_x| \leq |\mathcal{F}_{x_1}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{k-1}$ and then we are done by Lemma 3.1.4.

If $q = 2$, we have to sharpen our estimations on $|\mathcal{F}_{x_1}|$. We know that for j independent vectors $x_1, y_1, ..., y_{j-1}$ with $U \cap \langle x_1, y_1, ..., y_{j-1} \rangle = \underline{0}$ there exists a subspace $A_j \in \mathcal{F}$ such that $A_j \cap \langle x_1, y_1 ..., x_{j-1} \rangle = \underline{0}$. Then we would have the following upper bound on the number of subspaces in $\mathcal{F}$ containing all x_i $(1 \leq i \leq j)$: $\begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-j-1 \\ k-j-1 \end{bmatrix}^{*(d)}$. But suppose further that for some positive l $j + k + d = n + l$. Then $\dim(\langle x_1, y_1 ..., y_{j-1}, A_j \rangle \cap U) \geq l$ and so (denoting $\langle x_1, y_1 ..., y_{j-1}, A_j \rangle \cap U$ by U_j) $\dim(\langle x_1, ..., x_j, U_j \rangle \cap A_j) \geq l$ as well, therefore when choosing among the vectors of A_j a subspace of dimension at least l is forbidden. Therefore we have the following better estimate on the number of subspaces in $\mathcal{F}$ containing $x_1, y_1, ..., y_{j-1}$:

$$\left(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} l \\ 1 \end{bmatrix}\right) \begin{bmatrix} n-j-1 \\ k-j-1 \end{bmatrix}^{*(d)}.$$

Hence we have that either the statement of the theorem holds or the degree of any vector x is bounded by the expression given in the conditions of Lemma 3.1.4. So Lemma 3.1.4 establishes our theorem in this case, too. □

Corollary. *For the profile vector f of any family $\mathcal{F}$ of intersecting subspaces of an n-dimensional vector space V, and for any $k < n/2$ and $n/2 < d \leq n-k$, the following holds*

$$c_{k,d} f_k + f_d \leq \begin{bmatrix} n \\ d \end{bmatrix},$$

where $c_{k,d} = q^d \frac{\left[\begin{smallmatrix} n-k \\ d \end{smallmatrix}\right]}{\left[\begin{smallmatrix} n-d-1 \\ k-1 \end{smallmatrix}\right]}$, and equality holds in case of $f_k = 0, f_d = \left[\begin{smallmatrix} n \\ d \end{smallmatrix}\right]$ or $f_k = \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right], f_d = \left[\begin{smallmatrix} n-1 \\ d-1 \end{smallmatrix}\right]$.

Proof: Let us doublecount the disjoint pairs formed by the elements of $\mathcal{F}_k = \{U \in \mathcal{F} : \dim U = k\}$ and $\mathcal{F}'_d = \left[\begin{smallmatrix} V \\ d \end{smallmatrix}\right] \setminus \mathcal{F}_d = \{U \leqslant V, U \notin \mathcal{F} : \dim U = d\}$. On the one hand, for each $U \in \mathcal{F}_k$ there are exactly $q^{dk} \left[\begin{smallmatrix} n-k \\ d \end{smallmatrix}\right]$ such pairs (this uses the first *fact* about q-nomial coefficients), while on the other hand by Theorem 3.1.2 we know, that for any $W \in \mathcal{F}'_d$ there are at most $\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]^{*(d)} = q^{d(k-1)} \left[\begin{smallmatrix} n-d-1 \\ k-1 \end{smallmatrix}\right]$ such pairs. This proves the required inequality and it is easy to see that equality holds in the cases stated in the Corollary. □

Having established these inequalities, we are able to prove the main theorem of this subsection.

Theorem 3.1.5 *The essential extreme points of the profile polytope of the set of intersecting families of subspaces are the vectors v_i $(1 \le i \le n/2)$ for even n and there is an additional essential extreme point v^+ for odd n, where*

$$(v_i)_j = \begin{cases} 0 & \text{if } 0 \le j < i \\ \left[{n-1 \atop j-1}\right] & \text{if } i \le j \le n-i \\ \left[{n \atop j}\right] & \text{if } j > n-i. \end{cases} \tag{3}$$

and

$$(v^+)_j = \begin{cases} 0 & \text{if } 0 \le j < n/2 \\ \left[{n \atop j}\right] & \text{if } j > n/2. \end{cases} \tag{4}$$

Proof: First of all, for any $x \in V$, for the families $\mathcal{G}_i = \{U : x \in U, i \le \dim U \le n-i\} \cup \{U : \dim U > n-i\}$ $(1 \le i \le n/2)$ $f(\mathcal{G}_i) = v_i$ holds, and if n is odd then the profile of the family $\mathcal{G}^+ = \{U : \dim U > n/2\}$ is v^+, and clearly none of these vectors can be dominated by any convex combination of the others.

We want to dominate the profile vector f of any fixed intersecting family $\mathcal{F}$ with a convex combination of the vectors v_j (and possibly v^+ if n is odd). We define the coefficients of the v_js recursively. Let i denote the index of the smallest non-zero coordinate of f. For all $j < i$ let $\alpha_j = 0$. Now if for all $j' < j$ $\alpha_{j'}$ has already been defined, let

$$\alpha_j = \max\left\{ \frac{f_j}{\left[{n-1 \atop j-1}\right]} - \sum_{j'=i}^{j-1} \alpha_{j'}, 0 \right\}.$$

Note, that for all j $(i \le j \le n/2)$ the jth coordinate of $\sum_{j'=i}^{j} \alpha_{j'} v_{j'}$ is at least f_j (and equality holds if when choosing α_j, the first expression is taken as maximum), so these vectors already dominates the "first part" of f.

When all α_js ($i \leq j \leq n/2$) are defined, then let $\alpha^+ = 1 - \sum_{j'=i}^{n/2} \alpha_{j'}$ and let α^+ be the coefficient of v^+ if n is odd or add α^+ to the coefficient of $v_{n/2}$ if n is even. Note also that α^+ is non-negative since for all $i \leq j \leq k \leq n/2$ $(v_j)_k = \left[{n-1 \atop k-1}\right]$ and by Hsieh's theorem $0 \leq f_k \leq \left[{n-1 \atop k-1}\right]$. Therefore this is really a convex combination of the v_js.

The easy observation that this convex combination dominates f in the coordinates larger then $n-i$ follows from the fact that all v_js (and v^+ as well) have $\left[{n \atop d}\right]$ in the dth coordinate, therefore so does the convex combination which is clearly an upper bound for f_d.

All what remains is to prove the domination in the dth coordinates for all $n/2 < d \leq n-i$, that is to prove the inequality

$$f_d \leq \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \sum_{j=i}^{n-d} \alpha_j + \begin{bmatrix} n \\ d \end{bmatrix} \left(1 - \sum_{j=i}^{n-d} \alpha_j\right).$$

Let $k \leq n-d$ be the largest index with $\alpha_k > 0$. Then we have

$$f_d \leq \begin{bmatrix} n \\ d \end{bmatrix} - c_{k,d} f_k = \begin{bmatrix} n \\ d \end{bmatrix} - c_{k,d} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \sum_{j=i}^{k} \alpha_j = \left(1 - \sum_{j=i}^{k} \alpha_j\right) \begin{bmatrix} n \\ d \end{bmatrix} + \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \sum_{j=i}^{k} \alpha_j$$

$$= \left(1 - \sum_{j=i}^{n-d} \alpha_j\right) \begin{bmatrix} n \\ d \end{bmatrix} + \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \sum_{j=i}^{n-d} \alpha_j$$

where the inequality is just the Corollary, the first equality follows from the fact that $\alpha_k > 0$, the second equality uses again the Corollary (the statement about when equality holds) and the last equality uses the defining property of k (for all $k < j \leq n-d$ $\alpha_j = 0$).

This proves the theorem. $\square$

Note that, the (essential) extreme points are 'the same' as in the Boolean case (which was solved in [8]), one just has to change the binomial coefficients to the corresponding q-nomial coefficients and the structure of the extremal families are really the same.

3.2 l-chain profile vectors

Before getting into the details of the topic of this subsection, let us give some motivation. In Section 2, one of the properties we dealt with was Sperner property. This property has a natural generalization: a family $\mathcal{F} \subseteq 2^{[n]}$ is said to be *k-Sperner* if it contains no chains of length $k+1$ (*$k+1$-chains* for short), or equivalently if $\mathcal{F} = \bigcup_{i=1}^{k} \mathcal{F}_i$, where every $\mathcal{F}_i$ is a Sperner family. So the union of two k-Sperner families is a $2k$-Sperner family, and, using the terminology of Section 2, their k-Sperner distance is the number $k+1$-chains in their union.

In general, one may ask for any $r \leq s$, what is the maximum number of r-chains that an s-Sperner family $\mathcal{F}$ may contain (as always, with assumption that $\mathcal{F} \subseteq 2^{[n]}$). (This problem is "somewhat" analogous to the well-known result of Turán/Sauer [32], [30] which gives the maximum number of K_rs that a K_s-free graph can have.) A 1-chain is simply a set in the family, so the case $r = 1$ asks for the maximum size of an s-Sperner family. This was solved by Paul Erdős [5] in 1945. Theorem 2.2.1. settles the case $r = s = 2$ and "we are motivated" by the $r = k+1, s = 2k$ case for any $k \geq 2$.

The original profile vector does not help to deal with this problem: two sets with the same size might be contained in differently many l-chains (if $l > 1$). What is more! The same set may be contained in differently many l-chains depending on which system it takes part of. To overcome this problem we introduce a generalization of the concept of profile vector (which reduces to the ordinary profile if $l = 1$).

Definition: The *l-chain profile vector* $f^l(\mathcal{F})$ of a family $\mathcal{F} \subseteq 2^{[n]}$ is a vector of length $\binom{n+1}{l}$. The coordinates are indexed with l-tuples $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l)$ $(0 \leq \alpha_1 < ... < \alpha_l \leq n)$ and the αth coordinate $f^l(\mathcal{F})_\alpha$ is the number of l-chains contained in $\mathcal{F}$ with the property that the smallest set in the chain has size α_1, the second one has size α_2 and so on.

If we denote the all one vector (of length $\binom{n+1}{l}$) by $\mathbf{1}$, then the number of l-chains contained in a family $\mathcal{F}$ is $f^l(\mathcal{F}) \cdot \mathbf{1}$ and using other weight vectors one can treat weighted problems for l-chains, where the weight of two l-chains must coincide if the ith sets in both chains are taken from the same level for all $1 \leq i \leq l$. Although generally weighted l-chains do not come into picture very often, but containing pairs of sets and disjoint pairs of sets (which could be transformed into containing ones, since $F \cap G = \emptyset \Leftrightarrow F \subseteq \overline{G}$) are much more investigated, so results on 2-chain profiles might have some applications.

Anyhow, after presenting some further definitions and some introductory results on l-chain profiles, we will demonstrate the power of the socalled *reduction method* of Péter L. Erdős, Péter Frankl and Gyula O.H. Katona [8] by applying to some not very complicated sets of families in this new 'l-chain context'.

3.2.1 Definitions and remarks

In this section we give some further definitions and describe some basic connections between the extreme points in the l-chain case and the extreme points in the original (1-chain) case.

Notation. For αs with $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l), 0 \leq \alpha_1 < \alpha_2 < ... < \alpha_l \leq n$ we define the following multinomial coefficient:

$$\binom{n}{\alpha} = \prod_{i=1}^{l-1} \binom{n - \alpha_{i-1}}{\alpha_{i+1} - \alpha_i} = \frac{n!}{\alpha_1!(\alpha_2 - \alpha_1)!...(\alpha_l - \alpha_{l-1})!(n - \alpha_l)!}$$

where $\alpha_0 = 0$ and $0! = 1$ as usual. Note that $\binom{n}{\alpha}$ is the number of l-chains that can be formed from subsets of an n-element set in such a way that the smallest set has size α_1, the second smallest has size α_2 and so on.

Definition: Given an underlying set X and a family $\mathcal{F}$ of its subsets, the *up set* of $\mathcal{F}$ is $\mathcal{U}(\mathcal{F}) = \{G \subseteq X : \exists F \in \mathcal{F} \text{ such that } F \subseteq G\}$ and the *down set* of $\mathcal{F}$ is

$\mathcal{D}(\mathcal{F}) = \{G \subseteq X : \exists F \in \mathcal{F} \text{ such that } F \supseteq G\}$.

Definition: A set $\mathbb{A}$ of families is upward (downward) closed if $\mathcal{F} \in \mathbb{A}$ implies $\mathcal{U}(\mathcal{F}) \in \mathbb{A}$ ($\mathcal{D}(\mathcal{F}) \in \mathbb{A}$).

Examples: Clearly the set of t-intersecting (t-co-intersecting) families is upward (downward) closed. (A family $\mathcal{F}$ is said to be t-intersecting if for any two $F_1, F_2 \in \mathcal{F}$ $|F_1 \cap F_2| \geq t$, and a family $\mathcal{G}$ is said to be t-co-intersecting if $\overline{\mathcal{G}} = \{\overline{G} : G \in \mathcal{G}\}$ is t-intersecting or equivalently if for any two $G_1, G_2 \in \mathcal{G}$ $|\overline{G_1 \cup G_2}| \geq t$.)

Definition: Let $\mu_l(\mathbb{A})$ denote the set of all l-chain profile vectors of families in $\mathbb{A}$, $\langle \mu_l(\mathbb{A}) \rangle$ its convex hull, $\mathcal{E}_l(\mathbb{A})$ the extreme points of $\langle \mu_l(\mathbb{A}) \rangle$ and $E_l(\mathbb{A})$ the families from $\mathbb{A}$ with l-chain profile in $\mathcal{E}_l(\mathbb{A})$. Let furthermore $\mathcal{E}^*_l(\mathbb{A})$ denote the essential extreme points and $E^*_l(\mathbb{A})$ the corresponding families.

Theorem 3.2.1. *For any upward or downward closed set of families $\mathbb{A} \subseteq 2^{2^X}$ and for any $l \geq 1$*

$$\mathcal{E}^*_l(\mathbb{A}) \subseteq \mu_l(E^*_1(\mathbb{A})).$$

Note that equality does not always hold as the set of intersecting families, the family $\mathcal{F} = \{F \subseteq X : |F| > |X|/2\}$ and any $l > |X|/2$ shows.

Proof: The proof is the same for downward and upward closed sets of families, so we assume that $\mathbb{A}$ is upward closed.

Let $E^*_1(\mathbb{A}) = \{\mathcal{F}_1, \mathcal{F}_2, .., \mathcal{F}_m\}$ and let f^i the profile of $\mathcal{F}_i$, $f^{i,l}$ the l-chain profile of $\mathcal{F}^i$ and $f^{i,l}_\alpha$ its αth coordinate.

We have to prove that the l-chain profile f^l of any family $\mathcal{F}$ in $\mathbb{A}$ can be dominated by a convex combination of the $f^{i,l}$s. Denote the profile of $\mathcal{F}$ by f. Clearly we have

$$f^l_\alpha \leq f_{\alpha_1} \binom{n - \alpha_1}{\alpha^*},$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l), \alpha^* = (\alpha_2 - \alpha_1, \alpha_3 - \alpha_1, ..., \alpha_l - \alpha_1)$. Inequality holds with equality for the f^i_αs and the $f^{i,l}_\alpha$s (since $\mathbb{A}$ is upward closed). The fact that the f^is are the essential extreme points of $\langle \mu_l(\mathbb{A}) \rangle$ means that for some convex combination $c_i, i = 1, ..., m$

$$f \leq \sum_{i=1}^{m} c_i f^i.$$

But then

$$f^l_\alpha \leq f_{\alpha_1} \binom{n - \alpha_1}{\alpha^*} \leq \binom{n - \alpha_1}{\alpha^*} \sum_{i=1}^{m} c_i f^i_{\alpha_1} = \sum_{i=1}^{m} c_i f^{i,l}_\alpha,$$

which completes the proof. □

Since the convex hull of the profile polytope of the set of intersecting families were determined by P.L. Erdős, P. Frankl and G.O.H. Katona in [8], Theorem 3.2.1 provides the essential extreme points of the convex hull of the l-chain profile polytopes.

Definition: For any family $\mathcal{F}$ on a base set X let $\text{conv}(\mathcal{F}) = \{G \subseteq X : \exists F, F' \in \mathcal{F}(F \subseteq G \subseteq F')\}$ denote its *convex closure*. $\mathcal{F}$ is said to be convex if $\mathcal{F} = \text{conv}(\mathcal{F})$.

Definition: A set of families $\mathbb{A}$ is said to be convex closed if $\mathcal{F} \in \mathbb{A}$ implies $\text{conv}(\mathcal{F}) \in \mathbb{A}$.

Example: The basic example for a convex closed set is the set of intersecting and co-intersecting families.

Theorem 3.2.2. *For any convex closed set of families $\mathbb{A} \subseteq 2^{2^X}$ and for any $l \geq 2$*

$$\mathcal{E}^*_l(\mathbb{A}) \subseteq \mu_l(E^*_2(\mathbb{A})).$$

Proof: The proof is analogous to that of Theorem 3.2.1., the inequality needed is

$$f^l_\alpha \leq f^2_{\alpha_1, \alpha_l} \binom{\alpha_l - \alpha_1}{\alpha^*}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l)$, $\alpha^* = (\alpha_2 - \alpha_1, \alpha_3 - \alpha_1, ..., \alpha_{l-1} - \alpha_1)$ and for families with essential extreme profile inequality holds with equality. □

Unfortunately neither the extreme points of the 1-chain, nor that of the 2-chain profile polytope are known for the set of intersecting and co-intersecting families.

3.2.2 The reduction method

In this section we describe our main tool in determining the l-chain profile polytope of families of sets with some given property. We call this tool the reduction method. In fact, this is not a new one. Most of the proofs of results already obtained went this way, what we observed that the method works for the l-chain case as well, and - what seems to us more important - in some cases it is enough to reduce the original problem to the chain instead of the cycle (what previous proofs did mostly). For the precise definitions, see below.

Definition: For any l let $T^l_{\mathbf{C}}$ denote the following operator acting on the $\binom{n+1}{l}$-dimensional $\mathbf{R}$-space (coordinates are still indexed by l-tuples of the set $\{0, 1, ..., n\}$)

$$T^l_{\mathbf{C}} : e \mapsto T^l_{\mathbf{C}}(e) \qquad \text{where} \qquad T^l_{\mathbf{C}}(e)_\alpha = \binom{n}{\alpha} e_\alpha.$$

Definition: For a family $\mathcal{F}$ on a base set X and a maximal chain $\mathbf{C}$ in X let $\mathcal{F}(\mathbf{C}) = \{F \in \mathcal{F} \cap \mathbf{C}\}$ and for a set of families $\mathbb{A}$ let $\mathbb{A}(\mathbf{C}) = \{\mathcal{F}(\mathbf{C}) : \mathcal{F} \in \mathbb{A}\}$.

Theorem 3.2.3 *For any set of families $\mathbb{A} \subseteq 2^{2^X}$ if the extreme points $e_1, e_2, ..., e_m$ of $\langle \mu_l(\mathbb{A}(\mathbf{C})) \rangle$ do not depend on the choice of $\mathbf{C}$, then*

$$\langle \mu_l(\mathbb{A}) \rangle \subseteq \langle \{T^l_{\mathbf{C}}(e_1), ..., T^l_{\mathbf{C}}(e_m)\} \rangle.$$

Proof: The modification of the argument in [8] works. Let $\mathcal{F}$ be an element of $\mathbb{A}$ with l-profile $f = (..., f_\alpha, ...)$. For $\mathbf{F} = \{F_1 \subset F_2 \subset ... \subset F_l\}$ with $|F_i| = \alpha_i, i = 1, ..., l$ let $\underline{w}(\mathbf{F})$ be the vector of length $\binom{n+1}{l}$ with $1/n!$ in the αth coordinate and 0 everywhere else (where n is the size of the base set). Consider the sum $\sum \underline{w}(\mathbf{F})$ for all pairs $(\mathbf{C}, \mathbf{F})$, where $\mathbf{C}$ is a maximal chain on X and $\mathbf{F} \subset \mathcal{F} \cap \mathbf{C}$ an l-chain. For a fixed $\mathbf{C}$ we have

$$\sum_{\mathbf{F} \in \mathcal{F}(\mathbf{C})} \underline{w}(\mathbf{F}) = \frac{1}{n!}(\text{profile of } \mathcal{F}(\mathbf{C})).$$

Here the profile of $\mathcal{F}(\mathbf{C})$ is a convex linear combination $\sum_{i=1}^{m} \lambda_i(\mathbf{C}) e_i$ of the e_is. Therefore

$$\sum_{\mathbf{C},\mathbf{F}} \underline{w}(\mathbf{F}) = \sum_{\mathbf{C}} \sum_{\mathbf{F}} \underline{w}(\mathbf{F}) = \sum_{\mathbf{C}} \frac{1}{n!} \sum_{i=1}^{m} \lambda_i(\mathbf{C}) e_i = \sum_{i=1}^{m} \frac{1}{n!} (\sum_{\mathbf{C}} \lambda_i(\mathbf{C})) e_i \tag{5}$$

holds where $\sum_{\mathbf{C}} \frac{1}{n!} \sum_{i=1}^{m} \lambda_i(\mathbf{C}) = 1$. Thus $\sum \underline{w}(\mathbf{F})$ is a convex linear combination of the e_is.

Summing in the other way around, we have

$$\sum_{\mathbf{C},\mathbf{F}} \underline{w}(\mathbf{F}) = \sum_{\mathbf{F}} \sum_{\mathbf{C}} \underline{w}(\mathcal{F}) =$$

$$\sum_{\mathbf{F}} \left(0, 0, ..., \frac{|F_1|!(|F_2| - |F_1|)!...(|F_l| - |F_{l-1}|)!(n - |F_l|)!}{n!}, ..., 0\right) = \left(..., \frac{f_\alpha}{\binom{n}{\alpha}}, ...\right), \tag{6}$$

since for a fixed $\mathbf{F} = \{F_1 \subset F_2 \subset ... \subset F_l\}$ there are exactly $|F_1|!(|F_2| - |F_1|)!...(|F_l| - |F_{l-1}|)!(n - |F_l|)!$ chains containing $\mathbf{F}$. So (5) and (6) give that this last vector is a convex linear combination of the e_is, which implies that f is the linear combination of $T^l_{\mathbf{C}}(e_1), ..., T^l_{\mathbf{C}}(e_m)$. $\square$

The structure of maximal chains are too simple, so using only them is not enough to determine the l-chain profile polytope of more complicated sets of families. But the proof of Theorem 3.2.3. works if we replace the chain by a pair of complement maximal chains (i.e. for $i = 1, 2$ $\mathbf{C}^i = \{C^i_0, C^i_1, ..., C^i_n\}$ with $C^i_j = X \setminus C^{3-i}_{n-j} = \overline{C}^{3-i}_{n-j}$ for all $j = 0, 1, ..., n$) or the cycle (i.e. the family of subsets of consecutive elements with respect to a cyclic permutation of the base set). In the proof one has to write (instead of $\frac{1}{n!}$) $\frac{2}{(n!)}$ and $\frac{1}{(n-1)!}$ (respectively) in the definition of $\underline{w}(\mathbf{F})$, and modify the definition of the T-operator to

$$(T^l_{\mathbf{C}_1,\mathbf{C}_2}(e))_\alpha = \frac{1}{d_\alpha}\binom{n}{\alpha} \qquad (T^l_{\mathcal{C}}(e))_\alpha = \frac{1}{c_\alpha}\binom{n}{\alpha},$$

where d_α (c_α) is the number of α-type l-chains in the pair of complementing chains (in the cycle). For completeness' sake we state these versions of the theorem, too.

Theorem 3.2.4 (a) *For any set of families $\mathbb{A} \subseteq 2^{2^X}$ if the extreme points $e_1, e_2, ..., e_m$ of $\langle \mu_l(\mathbb{A}(\mathbf{C}^1, \mathbf{C}^2)) \rangle$ do not depend on the choice of $\mathbf{C}^1, \mathbf{C}^2$, then*

$$\langle \mu_l(\mathbb{A}) \rangle \subseteq \langle \{ T^l_{\mathbf{C}^1,\mathbf{C}^2}(e_1), ..., T^l_{\mathbf{C}^1,\mathbf{C}^2}(e_m) \} \rangle.$$

(b) *For any set of families $\mathbb{A} \subseteq 2^{2^X}$ if the extreme points $e_1, e_2, ..., e_m$ of $\langle \mu_l(\mathbb{A}(\mathcal{C})) \rangle$ do not depend on the choice of $\mathcal{C}$, then*

$$\langle \mu_l(\mathbb{A}) \rangle \subseteq \langle \{ T^l_{\mathcal{C}}(e_1), ..., T^l_{\mathcal{C}}(e_m) \} \rangle.$$

3.2.3 Applications

In this section we determine the profile polytope of some sets of families using the reduction method. In the first part of this subsection the problem will be reduced to the case of the maximal chain while in the second part we will consider reduction to a pair of complement chains. Using the results obtained by the latter we will give examples when the extreme families of the l-profile polytope can really depend on l.

Reduction to the chain

Theorem 3.2.5 *For all $l \geq 1$ the extreme points of the convex hull of the l-chain profile vectors of convex families are the following:*

the all zero vector

$$\mathbf{0} = (0, ..., 0)$$

and for all $0 \leq i \leq j \leq n$ the vectors $v_{i,j}$

$$(v_{i,j})_\alpha = \begin{cases} \binom{n}{\alpha} \text{ if } & i \le \alpha_1 < \alpha_l \le j \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Proof: The vector $v_{i,j}$ is the l-profile of the family $\mathcal{F}_{i,j} = \{F \subseteq [n] : i \le |F| \le j\}$, which is convex.

On a chain any convex family must consist of some consecutive subsets of the chain. The statement of the theorem follows now from Theorem 3.2.3. □

Note that the set of convex families is not hereditary, therefore the extreme points (for the original profile vectors) need not be the ones obtained from the essential extreme points (in this case there is only one such, the profile of $2^{[n]}$) by changing some of the non zero coordinates to zero - and as Theorem 3.2.4. shows, they are not those vectors, indeed.

Theorem 3.2.6 *For any $l \le k$ the extreme points of the l-chain profile polytope of k-Sperner families are the following:*

the all zero vector

$$\mathbf{0} = (0, ..., 0, ...0)$$

and for all $l \le z \le k$ and $\beta = \{\beta_1, ..., \beta_z\}$ with $0 \le \beta_1 < ... < \beta_z \le n$ the vectors v_β

$$(v_\beta)_\alpha = \begin{cases} \binom{n}{\alpha} & \text{if } \alpha \subseteq \beta \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

The case $l = 1$ is a result of P.L. Erdős, P. Frankl and G.O.H. Katona [8].

Proof: It is trivial to see that these vectors are l-chain profiles of the corresponding levels, and they are convex linearly independent.

A k-Sperner family on a maximal chain consists of at most k sets, therefore its l-chain profile vector have ones in those coordinates $\alpha = (\alpha_1, ..., \alpha_l)$ for which there is an element in the family with size α_i for all $i = 1, ..., l$. All these vectors are convex independent. Therefore they form the convex hull of the profile polytope on the chain, and Theorem 3.2.3 implies now Theorem 3.2.6. □

Applying Theorem 3.2.6 for the constant 1 weight function one gets

Corollary *For any $l \leq k$ if a family $\mathcal{F}$ on an n-element base set X does not contain a chain of length $k+1$, then the number of l-chains in $\mathcal{F}$ is at most*

$$\max_{\beta \subset [0,n]; |\beta| = k} \sum_{\alpha \subseteq \beta; |\alpha| = l} \binom{n}{\alpha}.$$

As a special case we get that the answer to our "motivating problem" is that the maximum distance of two k-Sperner families is

$$\max_{\beta \subset [0,n]; |\beta| = 2k} \sum_{\alpha \subseteq \beta; |\alpha| = k+1} \binom{n}{\alpha}.$$

Remarks.

- In the case $l = k$, even the very simple argument of [24] works. First we need a LYM-type inequality. To get this we double-count the pairs $(\mathbf{C}, \mathbf{F})$ where $\mathbf{C}$ is a maximal chain and $\mathbf{F}$ is an l-chain contained in $\mathbf{C}$. If we decompose the k-Sperner family into k antichains, then all sets of an $\mathbf{F}$ come from different antichains, and any $\mathbf{C}$ can contain at most k sets from our family, so by a standard calculation we obtain

$$\sum_{\alpha} \frac{f_\alpha}{\binom{n}{\alpha}} \leq \binom{k}{l}. \tag{9}$$

If $l = k$, then the RHS is 1, and we can finish the proof as follows

$$\sum_{\alpha} w_\alpha f_\alpha = \sum_{\alpha} \frac{f_\alpha w_\alpha}{\binom{n}{\alpha}} \binom{n}{\alpha} \leq \max_{\alpha} \left\{ w_\alpha \binom{n}{\alpha} \right\} \sum_{\alpha} \frac{f_\alpha}{\binom{n}{\alpha}} \leq \max_{\alpha} \left\{ w_\alpha \binom{n}{\alpha} \right\}, \tag{10}$$

where w_α is any non-negative weight function and the last inequality in (10) uses (9).

If $l = k$, then the Corollary gives the maximum number of k-chains that a k-Sperner family can contain. This is $\binom{n}{\alpha}$ where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k)$ and the numbers $\alpha_1, \alpha_2 - \alpha_1, ...\alpha_k - \alpha_{k-1}$ differ by at most one. If $k+1$ divides n, then we get the uniqueness of the extremal system (take all $F \subseteq X$ with $|F| = \alpha_i$ for some $i = 1, ..., k$) automatically. If $k+1$ does not divide n, then we can lift up (4) to an AZ-type identity (for the original AZ-identity see the paper of Ahlswede and Zhang [2]) which will assure the uniqueness.

- With the notation of Section 3.2.1, Theorem 3.2.6 implies (if $\mathbb{S}_k$ denotes the set of k-Sperner families) $E_1(\mathbb{S}_k) = E_l(\mathbb{S}_k)$. But the bordering faces of the convex hulls $\langle \mu_1(\mathbb{S}_k) \rangle$ and $\langle \mu_l(\mathbb{S}_k) \rangle$ are "not the same". If $l = 1$ the convex hull determined by the faces given by the inequalities $0 \leq f_i \leq \binom{n}{i}$ and the LYM-inequality $\sum_i f_i / \binom{n}{i} \leq k$ (see [8]), while if $l > 1$ the hyperplanes given by $0 \leq f_\alpha \leq \binom{n}{\alpha}$ and the LYM-type inequality of (5) are bordering faces, but there are some additional ones, which can be seen through the following observation. Choosing $\binom{k}{l}$ αs in such a way that their union has size strictly larger than k and putting $f_\alpha = \binom{n}{\alpha}$ for these αs and 0 for the others, we obtain an essential extreme point of the polytope determined by the above inequalities, and which is not an l-chain profile of any k-Sperner families.

Reduction to a pair of complement chains

Theorem 3.2.7 *Let $n = 2m+1$ and $k \leq m+1$. Then the extreme points of the 1-chain profile polytope (i.e. the ordinary profile polytope) of the set of complement-free k-Sperner families are the following vectors (indexed with a z-element ($z \leq k$) subset α of $\{0, 1, 2, ..., n\}$ where $\alpha_i \in \alpha$ implies $n - \alpha_i \notin \alpha$)*

$$v_\alpha = \left(0, ..., 0, \binom{n}{\alpha_1}, 0, ..., 0, \binom{n}{\alpha_2}, 0, ..., 0, ..., 0, \binom{n}{\alpha_z}, 0, ..., 0\right).$$

Proof: By Theorem 3.2.3 **(a)**, it is enough to prove the following

Lemma 3.2.8 *If $n = 2m+1$ and $k \leq m+1$, then the extreme points of the profile polytope of complement-free k-Sperner families on a pair of maximal complement chains are the vectors with at most k non-zero coordinates, where all the non-zero coordinates are 2 (except for the first or the last coordinate, if one of them is non-zero, it equals 1), and if the ith coordinate is non-zero, then the $n - ith$ coordinate is zero.*

Proof of Lemma 3.2.8: If the non-zero coordinates of such a vector are $\alpha_1, \alpha_2, ..., \alpha_z$ (satisfying the condition of the lemma), then the sets in the two chains with cardinality α_i for some $i = 1, ..., z$ form a complement-free k-Sperner family with the vector as profile.

Now let $\mathcal{F}$ be a complement-free k-Sperner family on a pair of complement chains $\mathbf{C}_1, \mathbf{C}_2$ with profile vector f. Let α be the set of indices of the non-zero coordinates of f. Partition α into three subsets. Let CL (complete levels) denote the indices α_i with $f_{\alpha_i} = 2$ (and 0 or n if f_0 or f_1 equals 1). Let furthermore CP (complementing pairs) denote the indices $\alpha_i \in \alpha$ with $n - \alpha_i \in \alpha$, and let $R = \alpha \setminus (CL \cup CP)$. Note that $CP \cap CL = \emptyset$, for otherwise $\mathcal{F}$ would not be complement-free. Now form two subsets α^1, α^2 of α in the following way. Put all indices in CL into both α^1 and α^2. For all pairs of indices $i, n-i$ in CP (note that these are really pairs, for n is odd) put one of the indices into α^1 and the other into α^2. Finally, choose α^1 or α^2 for all indices of R in such a way, that $|\alpha^1| \leq k$ and $|\alpha^2| \leq k$ hold. (This is possible, for $\mathcal{F}$ is k-Sperner, therefore $|\alpha| \leq 2k$.) Now let f^i, $i = 1, 2$ the following vectors.

$$f^i_j = \begin{cases} 2 \quad (1) \text{ if } \quad j \neq 0, n \quad (j = 0, n) \quad j \in \alpha^i \\ 0 \qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{11}$$

By the facts that both f^is are of the form of the statement of the lemma and $f = \frac{1}{2}f^1 + \frac{1}{2}f^2$, the proof is completed. □ □

The case of complement-free families is very analogous (and even simpler), therefore we just sketch the proof.

Theorem 3.2.9 *The extreme points of the convex hull of the 1-chain profile vectors of complement-free families are the vectors corresponding to the families consisting of*

i, a set I of levels with the property that the ith and the $n-i$th levels cannot be both in I, if n is odd,

ii, a set I of levels with the property that the ith and the $n-i$th levels cannot be both in I and possibly half of the sets with size $n/2$ one from each pair of complementary sets, if n is even.

Proof: It is easy to see (with the help of Theorem 3.2.4 **(a)**) that it is enough to solve the problem reduced to a pair of maximal complement chains. There the statement holds, since there a complement-free family can contain at most two sets out of the four with size i or $n-i$, and the vectors $(1,1),(0,1),(1,0)$ are convex combinations of the vectors $(2,0),(0,2),(0,0)$. □

Theorem 3.2.1 and 3.2.2 state that for a certain class of sets of families all candidates for the families with essential extreme l-chain profiles are among the families with essential extreme 1-chain (2-chain) profile. Theorem 3.2.6 states, that for k-Sperner families the above statement is true for all extreme profiles (not only for essential extreme profiles). It seems natural to conjecture (with the notation of Section 2) that for all set of families $\mathbb{A}$ and $l > 1$ $E_l(\mathbb{A}) \subseteq E_1(\mathbb{A})$ and/or $E^*_l(\mathbb{A}) \subseteq E^*_1(\mathbb{A})$. But this is false. Here we present two counterexamples.

The first example is based on Theorem 3.2.7. Note that the families corresponding to the extreme points cannot contain sets of size i and $n-i$ at the same time. Hence all 2-chain profiles of those families have 0 in their coordinates indexed with the sets $\{i, n-i\}$, and therefore all their convex combinations have 0 in those coordinates. But a pair of subsets in inclusion with size i and $n-i$ is of course a complement-free k-

Sperner family (if $k \geq 2$), and its profile is not in the convex hull of the above-mentioned vectors.

The second example is absolutely analogous to the first one. According to Theorem 3.2.9 in the extremal families of the set of complement-free systems there are no pairs of sets in inclusion with size i and $n-i$ (so the corresponding coordinate is 0 in any convex combinations), but there are complement-free families with such pairs.

4 Concluding remarks

In lack of space, we could not survey all areas of extremal set theory in this thesis. We mainly focused on two types of problems. In this last section we would like to summarize the possibilities of future research and place our results in the theory of extremal set systems.

Though the problem of finding $\mathcal{F}$-free families with largest $\mathcal{F}$-free difference seems very natural, it was introduced quite recently in [28] by the author and therefore could not be subject of extensive research yet. This means that we are not aware of any other results of this type, thus considering other forbidden configurations that we did in this dissertation and obtaining theorems on the corresponding largest possible distance could be the first step in future research. Another challenging question could be to establsish connections with other areas of extremal set theory or other topics in combinatorics. For example it would be very interesting to know whether Theorem 2.3.1 could be used to deduce results on Ramsey numbers.

Finding the profile polytope of families with a prescribed property was the other type of problems we considered in the thesis. The first result in this area was mentioned (implicitly) by G.O.H. Katona in [24] but the systematical research was initiated by P.L. Erdős, P. Frankl and G.O.H. Katona in [7] and [8]and many researchers were engaged in the topic ever since. As we mentioned in the introduction, determining the profile polytope enables us to maximize easily any weight function with the property that the weight of a set depends only on its size (or more generally, in ranked posets the weight of an element of the poset depends only on its rank), so after finding the maximum size of a family with some prescribed properties (which is the basic question in the theory of extremal set systems) this seems to be the most natural generalization. However, it is quite useful to restrict ourselves to properties where this type of weight functions come into picture naturally, i.e. applications of the weighted results exists.

(One of) the most important property for which the profile polytope is yet to be determined is the t-intersecting property (if $t \geq 2$, the $t = 1$ was solved in [7]). In Theorem 3.1.5 we determined the extreme points of the profile polytope of intersecting families of subspaces (the $t = 1$ case for the poset $L_n(q)$) and it seems that determining the profile polytope in $L_n(q)$ for $t \geq 2$ could be easier than in the Boolean case. This 'conjecture' is based on the fact (a theorem of Ahlswede and Khatchatrian [1] and another theorem of Frankl and Wilson [15]), that in the Boolean case if t and k are fixed, there are many types of different k-uniform extremal families as n ranges through the integers larger than $2k - t$, while in the case of the poset of subspaces there are just 2.

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