Decomposing ω-fold coverings

by

Máté Vizer

Submitted to

Central European University

Department of Mathematics and its applications

In partial fulfilment of the requirements for the degree of

Doctor of Philosophy

Supervisor: Professor István Juhász

Budapest, Hungary

2012

I, the undersigned [Máté Vizer], candidate for the degree of Doctor of Philosophy at the Central European University Department of Mathematics and its applications, declare herewith that the present thesis is exclusively my own work, based on my research and only such external information as properly credited in notes and bibliography. I declare that no unidentified and illegitimate use was made of works of others, and no part the thesis infringes on any person's or institution's copyright. I also declare that no part of the thesis has been submitted in this form to any other institution of higher education for an academic degree.

Budapest, 21 December 2012

Signature

Acknowledgments

First, I would like to express my gratitude to my supervisor István Juhász for his constant support and patience during the writing of this thesis.

I would like also thank the numerous remarks and comments received from Lajos Soukup, Márton Elekes and Zoltán Szentmiklóssy and I would like to thank Dömötör Pálvölgyi to pose me some questions concerning covering decomposition.

At last but not the least I would also like to thank my family for the support they provided me through my entire life.

Abstract

We prove several results concerning covering decomposition of ω-fold covers of $\mathbf{R}$ by intervals. Using these results we prove the following: if $\mathcal{F}$, a family of translates of the open or closed unit square is an ω-fold cover of A, a subset of the plane then $\mathcal{F}$ is the disjoint union of ω many (ω-fold) subcovers of A. This answers a generalization of a question posed by Elekes, Matrai and Soukup in [2]. After this we analyze the sharpness of this theorem with providing some constructions.

Table of Contents

1 Introduction

1.1 General introduction

Let X be a set, λ, κ be cardinals and let $\mathcal{F}$ be a family of subsets of X, such that it covers every point of X at least κ times. Can we split $\mathcal{F}$ into λ disjoint covers of X?

In this thesis, we deal with a version of this question. As anybody can formulate his favourite variant of the above mentioned splitting question (with choosing the parameters: X, $\mathcal{F}$, λ, κ), this problem has a long-standing history. Let us briefly summarize the results in some directions:

Finite covers:

A well understood question in this context if X is a finite set and $\mathcal{F}$ is a family of hyperedges of a graph on X. Almost optimal solutions of the relevant problems have already been found long ago.

However if the cardinality of X is not finite, then the answer is not so clear. Pach in [4] posed the above mentioned question in the following form: let X be the plane, and let P be a convex planar set, then can we find a finite number $c(P)(=\kappa)$ such that if $\mathcal{F}$ is the set of translates of P then we can split $\mathcal{F}$ into two covers of the plane?

Note that if P is a polygon, then solution has been found (see [5], and [3]). However the related question for e.g. circles is a much harder one. We do not know the solution, but we note that a positive answer may be hidden in a more than 100 page-long manuscript of Mani-Levitska and Pach.

Infinite covers:

In this direction - κ can be infinite - the first result where the underlying set X had geometric properties (and so would be relevant to us) appeared in [1]:

Theorem 1.1.1. *(Aharoni, Hajnal, Milner) If κ is a cardinal, X is a linearly ordered set and $\mathcal{F}$ is a set of intervals such that each point of X is covered by (at least) κ many elements of $\mathcal{F}$, then $\mathcal{F}$ is the disjoint union of κ many covers.*

After this result, a question of Pach, whether any infinite-fold cover of the plane by axis-parallel rectangles can be decomposed into two disjoint subcovers inspired the authors of [2] to start a systematic study of 'infinite-fold covering' problems and achieved numeoruos results, e.g.:

Theorem 1.1.2 ([2], Theorem 7.4)**.** *Let $\kappa > \omega$ be a cardinal, and $\mathcal{F}$ be a family of closed polygons of the plane such that each point of the plane is covered by at least κ-many elements of $\mathcal{F}$. Then $\mathcal{F}$ can be decomposed into κ many disjoint covers of the plane.*

This theorem is not true for $\kappa = \omega$. With similar techniques the authors of [2] developed, we can construct $\mathcal{F}$, a family of closed unit squares, covering each point of the plane ω times, such that $\mathcal{F}$ can not be split into two covers of the plane (see Theorem 1.2.6). Then it is natural to ask the following question:

Question 1.1.3 ([2], Problem 8.6.1.)**.** *Is it true that every ω-fold cover of $\mathbf{R}^n$ by translates or homothets of the closed unit cube can be decomposed into two disjoint subcovers?*

To answer for this question was the starting point of our investigations.

1.2 Notation, easy facts

1.2.1 Notation

• Let κ be an infinite cardinal, X be a set and $\mathcal{F} \subseteq \mathcal{P}(X)$, a subset of the power set of X. We say that $x \in X$ is *κ-fold covered* by $\mathcal{F}$, if

$$|\{F : x \in F \wedge F \in \mathcal{F}\}| \geq \kappa,$$

and $Y \subseteq X$ is *κ-fold covered* by $\mathcal{F}$ if all $x \in Y$ is κ-fold covered.

• Let X be a set and λ be a cardinal. Then $[X]^{\lambda}, [X]^{\leq\lambda}, [X]^{<\lambda}$ stand for the set of subsets of X which have cardinalities $\lambda, \leq \lambda, < \lambda$ respectively.

• For $\mathcal{F} \subseteq \mathcal{P}(X)$ and $x \in X$ let $ord(x, \mathcal{F}) = |\{F : F \in \mathcal{F} \wedge x \in F\}|$.

• Let X be a set, $\mathcal{F} \subseteq \mathcal{P}(X)$ and κ be a cardinal. Then let $(\mathcal{F})_\kappa = \{x \in X : ord(x, \mathcal{F}) \geq \kappa\}$. (Using the notation just introduced $Y \subseteq (\mathcal{F})_\kappa$ means that Y is κ-fold covered by $\mathcal{F}$.)

• If (X, τ) is a topological space, $A \subseteq X$ then let ∂A denotes the boundary of A.

• For a function $f : X \to Y$ and $\mathcal{F} \subseteq \mathcal{P}(X)$ let $f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\}$.

• For $C \subseteq \mathbf{R}^2$ let us denote by $\mathcal{T}_C$ the set of the translates of C.

Note that given $\mathcal{F} \subseteq \mathcal{P}(X)$, a family of subsets of X, there is 1-1 correspondence between $\cup^*_{i<\kappa}\mathcal{F}_i = \mathcal{F}$, partitions of $\mathcal{F}$ into κ many pieces and functions $c : \mathcal{F} \to \kappa$, mapping $\mathcal{F}$ into κ. When we write during a proof that *coloring* of $\mathcal{F}$ or the *color* of an object F, we mean the function $c : \mathcal{F} \to \kappa$ and $c(F)$ respectively.

1.2.2 Easy facts

Lemma 1.2.1. *For every topological space (X, τ), if $Y \subseteq X$ is σ-compact, $\mathcal{F} \subseteq \tau$ and $Y \subseteq (\mathcal{F})_\omega$, then $\mathcal{F}$ is the union of ω-many disjoint ω-covers of Y.*

Proof. Let $\{K_i : i \in \omega\}$ be an ω-abundant enumeration of compact sets with $\cup_{i \in \omega} K_i = X$.

We define $\mathcal{F}_s$ inductively: for $s \in \omega$ let

$$\mathcal{F}_s \in [\mathcal{F} \setminus \cup_{i<s} \mathcal{F}_i]^{<\omega} \text{ with } K_s \subseteq \cup_{F \in \mathcal{F}_s} F.$$

Choose $\varphi : \omega \to \omega \times \omega$ arbitrary bijection, and then for $Z \in \mathcal{F}$ let

$$c(Z) = \begin{cases} m & \text{if } Z \in \mathcal{F}_a \text{ , } u\text{th appearance of } K_a \text{ in the enumeration and } \varphi(u) = \langle m, w \rangle, \\ 0 & \text{otherwise.} \end{cases}$$

This coloring proves the lemma.

□

Lemma 1.2.2. *For all $Y \subseteq X$ and $\mathcal{F} \subseteq \mathcal{P}(X)$ with $Y \subseteq (\mathcal{F})_\omega$ the following is true:*

If $\mathcal{G}$ is the union of two ω-fold covers of Y for all $\mathcal{G} \subseteq \mathcal{F}$ with $Y \subseteq (\mathcal{G})_\omega$, then

$\mathcal{F}$ is the disjoint union of ω-many ω-fold covers of Y.

Proof. Let $c_0 : \mathcal{F} \to 2$ proving the assumption for $\mathcal{G} = \mathcal{F}$, and for $0 < i$ let

$$c_i : (c_{i-1})^{-1}(\{1\}) \to 2$$

witnessing that $(c_{i-1})^{-1}(\{1\})$ is the union of 2 ω-fold covers of Y.

Then for $Z \in \mathcal{F}$ let:

$$c(Z) = \begin{cases} i & \text{if } Z \in c_i^{-1}(0), \\ 0 & \text{otherwise.} \end{cases}$$

This coloring proves the lemma.

□

1.2.3 Results

We prove several results on decomposing ω-fold covers and discuss the results.

One dimensional case

As the proven theorems are too technical, we do not state them here, however we mention that we prove a kind of 'simultaneous' covering decomposition in this case (Theorem 2.1.5) and a statement about families containing union of 2 disjoint intervals (Lemma 2.1.33).

Two dimensional case

In the this case, our main theorems are:

Theorem 1.2.3. *Let C be the open unit square. For all $\mathcal{F} \subseteq \mathcal{T}_C$ there is*

$$\mathcal{F} = \cup^*_{i<\omega}\mathcal{F}_i \textit{ with } (\mathcal{F})_\omega = (\mathcal{F}_i)_\omega.$$

Theorem 1.2.4. *Let C be the closed unit square. For all $\mathcal{F} \subseteq \mathcal{T}_C$ there is*

$$\mathcal{F} = \cup^*_{i<\omega}\mathcal{F}_i \textit{ with } (\mathcal{F})_\omega = (\mathcal{F}_i)_\omega.$$

Remark. We note that after we mentioned him, Géza Tóth with one of his student could also prove some similar results like we proved in the two dimensional case. They used the earlier developed wedge technique (see [3]).

Constructions

We will also provide 3 constructions showing the sharpness of Theorem 1.2.3 and Theorem 1.2.4:

Construction 1:

Let $\mathcal{R}_\varepsilon$ be the set of axis-parallel closed rectangles with side length between $1-\varepsilon$ and 1; and $\mathcal{Q}_\varepsilon$ be the set of axis-parallel open rectangles with side length between $1-\varepsilon$ and 1.

Theorem 1.2.5.

(1) *For any* $\varepsilon > 0$ *we can find* $A \subseteq \mathbf{R}^2$ *and* $\mathcal{R} \in [\mathcal{R}_\varepsilon]^\omega$ *with:*

(1.1) $\mathcal{R}$ *is an* ω*-fold cover of* A *(i.e.* $(\mathcal{R})_\omega \supseteq A$*),*

(1.2) $\mathcal{R}$ *can not be partitioned into two disjoint parts such that each part covers* A

(i.e. for all partition $\mathcal{R}_1 \cup^* \mathcal{R}_2 = \mathcal{R}$ *either* $\cup\mathcal{R}_1 \not\supseteq A$ *or* $\cup\mathcal{R}_2 \not\supseteq A$*).*

(2) *For any* $\varepsilon > 0$ *we can find* $B \subseteq \mathbf{R}^2$ *and* $\mathcal{Q} \in [\mathcal{Q}_\varepsilon]^\omega$ *with:*

(2.1) $\mathcal{Q}$ *is an* ω*-fold cover of* B *(i.e.* $(\mathcal{Q})_\omega \supseteq B$*),*

(2.2) $\mathcal{Q}$ *can not be partitioned into two disjoint parts such that each part covers* B

(i.e. for all partition $\mathcal{Q}_1 \cup^* \mathcal{Q}_2 = \mathcal{Q}$ *either* $\cup\mathcal{Q}_1 \not\supseteq B$ *or* $\cup\mathcal{Q}_2 \not\supseteq B$*).*

Construction 2:

Let C be the closed unit square and $\mathcal{S}_\varepsilon$ be the set of $t(C)'s$, where t is a transformation of $\mathbf{R}^2$: the composition of an arbitrary translation and a rotation with at most ε.

Theorem 1.2.6. *For all* $\varepsilon > 0$ *there is* $\mathcal{S} \in [\mathcal{S}_\varepsilon]^\omega$ *with:*

(i) $\mathcal{S}$ *is an* ω*-fold cover of* $\mathbf{R}^2$ *(i.e.* $(\mathcal{S})_\omega \supseteq \mathbf{R}^2$*),*

(ii) $\mathcal{S}$ *can not be decomposed into two disjoint parts such that each part covers* $\mathbf{R}^2$

(i.e. for any partition $\mathcal{S}_1 \cup^* \mathcal{S}_2 = \mathcal{R}$ *either* $\cup\mathcal{S}_1 \not\supseteq \mathbf{R}^2$ *or* $\cup\mathcal{S}_2 \not\supseteq \mathbf{R}^2$*).*

Construction 3:

Let $\mathcal{U}_\varepsilon$ be the set of axis-parallel closed squares with side length between $1-\varepsilon$ and 1.

Theorem 1.2.7. *For all* $\frac{1}{4} > \varepsilon > 0$ *there is* $\mathcal{U} \in [\mathcal{U}_\varepsilon]^\omega$ *and* $A \subseteq \mathbf{R}^2$ *with:*

(i) $\mathcal{U}$ *is an* ω*-fold cover of* A *(i.e.* $(\mathcal{U})_\omega \supseteq A$*),*

(ii) $\mathcal{U}$ *can not be decomposed into* ω *parts such that each part covers* A

(i.e. for any partition $\cup^*_{i\in\omega}\mathcal{U}_i = \mathcal{U}$ *there is* $j \in \omega$ *with* $\cup\mathcal{U}_j \not\supseteq A$*).*

1.3 Structure of the thesis

In Chapter 1, we give a general introduction to the topic.

In Chapter 2, we prove different cover decomposition results for families of intervals.

In Chapter 3, we prove cover decompositions for $\mathfrak{S}$, a family of translates of the open unit square. First we understand the structure of $(\mathfrak{S})_\omega$ and $\partial(\mathfrak{S})_\omega$, then the structure of $S \cap \partial(\mathfrak{S})_\omega$ and finally using the results obtained in Chapter 2, we provide a cover decomposition for $\mathfrak{S}$.

In Chapter 4, using the results obtained in Chapter 3, we prove cover decompositions for $\mathfrak{C}$, a family of translates of the closed unit square.

In Chapter 5, we examine the sharpness of the theorems we achieved in the previous chapters.

Finally we state some remarks in Chapter 6.

2 One dimensional case

2.1 Results

Let $\mathfrak{I}$ denote the set of open, nonempty (finite or infinite) intervals in $\mathbf{R}$. The following is true:

Theorem 2.1.1. *([2], Thm 5.1.)*

Let $\mathfrak{R} \in [\mathfrak{I}]^\omega$. Then there is $c : \mathfrak{R} \to \omega$ with

$$(\mathfrak{R})_\omega = \cap_{j\in\omega}(\mathfrak{R} \cap c^{-1}(\{j\}))_\omega.$$

We will prove the following strengthening of Theorem 2.1.1:

Theorem 2.1.2.

Let $\mathfrak{R} \in [\mathfrak{I}]^\omega$ and $\mathfrak{R} = \cup_{n\in\omega}\mathfrak{R}_n$. Then there is $c : \mathfrak{R} \to \omega$ satisfying

$$(\mathfrak{R}_n)_\omega = \cap_{j\in\omega}(\mathfrak{R}_n \cap c^{-1}(\{j\}))_\omega$$

for all $n \in \omega$.

In the two dimensional case we will need a bit strange but crucial extra property for colorings: if a system of intervals we want to color contains ω-many intervals, then it must contain ω-many intervals colored with zero. (Note that it can happen that the set of the ω-fold covered points by that system of intervals is empty.) However if we simply put this extra condition to the requirements of Theorem 2.1.2, it does not remain true:

Theorem 2.1.3.

There are $\mathfrak{R} \in [\mathfrak{I}]^\omega$ *and* $\mathfrak{R} = \cup_{n\in\omega}\mathfrak{R}_n$ *such that there is no* $c : \mathfrak{R} \to \omega$ *with:*

(i) if $|\mathfrak{R}_n| = \omega$ *then* $|c^{-1}(\{0\}) \cap \mathfrak{R}_n| = \omega$ *for all* $n \in \omega$*, and*

(ii) $(\mathfrak{R}_n)_\omega = \cap_{j\in\omega}(\mathfrak{R}_n \cap c^{-1}(\{j\}))_\omega$ *for all* $n \in \omega$*.*

Proof of Theorem 2.1.3.

Let $\varphi : \omega^{<\omega} \to \omega \setminus \{0\}$ be a bijection and for $s \in \omega^{<\omega}$ let $I_s \subseteq (0, 1)$ be open interval satisfying:

$\bullet_1$ $I_t \subseteq I_s$ if $t \supseteq s$, and

$\bullet_2$ $I_t \cap I_s = \emptyset$ if $s \not\subseteq t$ and $t \not\subseteq s$.

Let $\mathfrak{R}_0 = \{I_s : s \in \omega^{<\omega}\}$ and $\mathfrak{R}_n = \{I_{\varphi^{-1}(\{n\})^\frown i} : i \in \omega\}$ for $n \in \omega \setminus \{0\}$.

This easily satisfies the requirements of Theorem 2.1.3.

□

However for $\mathfrak{R}_n's$ with special structure, Theorem 2.1.3 is 'true':

Definition 2.1.4.

Let $\mathfrak{R} \in [\mathfrak{I}]^{\leq\omega}$, $D \in [\mathbf{R} \cup \{-\infty, +\infty\}]^\omega$ with $D \cap \cup\{\partial R : R \in \mathfrak{R}\} = \emptyset$.

For $p, q \in D$ and $\varepsilon \in 4$ let us define the following sets:

$$\mathfrak{R}_{p,q,0} = \{(a, b) \in \mathfrak{R} : a < p, b < q\},$$

$$\mathfrak{R}_{p,q,1} = \{(a, b) \in \mathfrak{R} : a > p, b < q\},$$

$$\mathfrak{R}_{p,q,2} = \{(a, b) \in \mathfrak{R} : a < p, b > q\},$$

$$\mathfrak{R}_{p,q,3} = \{(a, b) \in \mathfrak{R} : a > p, b > q\}.$$

Theorem 2.1.5.

Let $\mathfrak{R} \in [\mathfrak{I}]^\omega$ *and* $D \in [\mathbf{R} \cup \{-\infty, +\infty\}]^\omega$ *with* $D \cap \cup\{\partial R : R \in \mathfrak{R}\} = \emptyset$*.*

There is $c : \mathfrak{R} \to \omega$ *such that for all* $p, q \in D$ *and* $\varepsilon \in 4$ *the followings are true:*

(i) if $|\mathfrak{R}_{p,q,\varepsilon}| = \omega$ *then* $|c^{-1}(\{0\}) \cap \mathfrak{R}_{p,q,\varepsilon}| = \omega$*, and*

(ii) $\bigcap_{j\in\omega}(c^{-1}(\{j\})\cap\mathfrak{R}_{p,q,\varepsilon})_\omega = (\mathfrak{R}_{p,q,\varepsilon})_\omega$.

Note that if we apply a homeomorphism to $\mathbf{R}$ and define the appropriate sets and coloring as in the conditions of the previous theorem, the consequence will be the same. We write it without proof as a corollary of Theorem 2.1.5 and will use it in the next section:

Corollary 2.1.6.

Let $\mathfrak{R}$, D be satisfying the conditions and $c:\mathfrak{R}\to\omega$ be satisfying (i) and (ii) of Theorem 2.1.5 and let $\varphi:\mathbf{R}\to X$ be a homeomorphism. Let

$\bullet_1$ $\mathfrak{Q}=\varphi(\mathfrak{R})$,

$\bullet_3$ $\mathfrak{Q}_{p,q,\varepsilon}=\varphi(\mathfrak{R}_{p,q,\varepsilon})$ *for all* $p,q\in D$ *and* $\varepsilon\in 4$, *and*

$\bullet_2$ $c':\mathfrak{Q}\to\omega$ *satisfying* $c'(\varphi(R))=c(R)$ *for all* $R\in\mathfrak{R}$.

Then the followings are true:

(i) *if* $|\mathfrak{Q}_{p,q,\varepsilon}|=\omega$ *then* $|(c')^{-1}(\{0\})\cap\mathfrak{Q}_{p,q,\varepsilon}|=\omega$, *and*

(ii) $\bigcap_{j\in\omega}((c')^{-1}(\{j\})\cap\mathfrak{Q}_{p,q,\varepsilon})_\omega=(\mathfrak{Q}_{p,q,\varepsilon})_\omega$.

There is no cover decomposition for families of two intervals, however in the two dimensional case we need a decomposition for certain such families. To do this, we also prove a technical lemma in this chapter (see Lemma 2.1.33).

2.1.1 Choosing a subcover

Now we start the proof of Theorem 2.1.5.

Definition 2.1.7. Let $I = \{<, >, =\}$. For $R_1, R_2 \in I$ we denote by $T(R_1, R_2)$ the set of $\mathfrak{A} \in [\mathfrak{I}]^\omega$ which has an enumeration: $\mathfrak{A} = \{(a_n, b_n) : n \in \omega\}$ such that $a_n R_1 a_m \wedge b_n R_2 b_m$ for all $n < m$. For $R \in I$ let $T(R, .) = \cup_{Q \in I} T(R, Q)$, $T(., R) = \cup_{Q \in I} T(Q, R)$ and $T(., .) = \cup_{R \in I} T(R, .)$.

Remark. If $\mathfrak{A} \in T(., .)$ then there is exactly one enumeration of it, witnessing this.

We mention 3 easy claims without proof:

Claim 2.1.8. For all $\mathfrak{A} \in [\mathfrak{I}]^\omega$ there is $\mathfrak{B} \in [\mathfrak{A}]^\omega$ with $\mathfrak{B} \in T(., .)$.

Claim 2.1.9. For all $\mathfrak{A} \in T(., .)$ and $\mathfrak{B} \in [\mathfrak{A}]^\omega$ we have $(\mathfrak{A})_\omega = (\mathfrak{B})_\omega$.

Claim 2.1.10. For all $\mathfrak{A} \in T(., <)$ and $a, b \in \mathbf{R} \cup \{+\infty\}$ with $a \in \cap\mathfrak{A}$ and $[a, b) \subseteq \cup\mathfrak{A}$ we have $[a, b) \subseteq \cup\mathfrak{B}$ for all $\mathfrak{B} \in [\mathfrak{A}]^\omega$.

Lemma 2.1.11. *For $\mathfrak{R} \in [\mathfrak{I}]^{\leq\omega}$ with $(a, b) = \cup\mathfrak{R} \in \mathfrak{I}$ there exist pairwise disjoint*

$$\mathfrak{R}^0, \mathfrak{R}^1, \mathfrak{R}^2 \subseteq \mathfrak{R}$$

satisfying:

(i) *if there is $c \in (a, b)$ with $(a, c) \in \mathfrak{R}$ then $\mathfrak{R}^0$ is empty, and if $\mathfrak{R}^0$ is not empty then $\mathfrak{R}^0 \in T(>, .)$,*

(ii) *if there is $c \in (a, b)$ with $(c, b) \in \mathfrak{R}$ then $\mathfrak{R}^1$ is empty, and if $\mathfrak{R}^1$ is not empty then $\mathfrak{R}^1 \in T(., <)$,*

(iii) *$\mathfrak{R}^2$ is not empty and $(\mathfrak{R}^2)_{10} = \emptyset$,*

(iv) *$(\cap\mathfrak{R}^i) \cap \cup\mathfrak{R}^2 \neq \emptyset$ for $i \in 2$,*

(v) *$\cup(\mathfrak{Q}^0 \cup \mathfrak{Q}^1 \cup \mathfrak{R}^2) = (a, b)$ for all $\mathfrak{Q}^0 \in [\mathfrak{R}^0]^{|\mathfrak{R}^0|}$ and $\mathfrak{Q}^1 \in [\mathfrak{R}^1]^{|\mathfrak{R}^1|}$,*

(vi) *for all $R \in \mathfrak{R}$ we have $|\{Q \in \mathfrak{R}^2 : Q \subseteq R\}| \leq 4$.*

Proof. For $x \in (a,b)$ let:

$$f(x) = sup\{d : (c,d) \in \mathfrak{R} \wedge x \in (c,d)\}, \ \ and$$

$$g(x) = inf\{c : (c,d) \in \mathfrak{R} \wedge x \in (c,d)\}.$$

$$(f^{(0)}(x) = x \text{ and } f^{(n)}(x) = \underbrace{f(f(...(}_{n}x))).)$$

Claim 2.1.12. For all $x \in (a,b)$ the followings are true:

(f) there is $k \in \omega$ with $f^{(k)}(x) = b$ or $\lim_{n\to\infty} f^{(n)}(x) = b$, and

(g) there is $k \in \omega$ with $g^{(k)}(x) = a$ or $\lim_{n\to\infty} g^{(n)}(x) = a$.

Proof of the claim. We prove (f), the proof of (g) is similar.

Suppose there is no $k \in \omega$ with $f^{(k)}(x) = b$. Then, since

$$f^{(0)}(x) < f^{(1)}(x) < ... < b,$$

there is $y \in (a,b]$ with $\lim_{n\to\infty} f^{(n)}(x) = y$.

If $y < b$ then there is $(c,d) \in \mathfrak{R}$ with $y \in (c,d)$, so $y < d$. But there is $m \in \omega$ satisfying $f^{(m)}(x) > c$, so we have $y < d \le f^{(m+1)}(x) < y$. Contradiction, hence $y = b$. □

Claim 2.1.13. The following statements are true:

(f) If $f^{(n)}(x) \in (a,b)$ for all $n \in \omega$, then there is $\mathfrak{R}^{2,r} \subseteq \mathfrak{R}$ with $(\mathfrak{R}^{2,r})_5 = \emptyset$ and $\cup\mathfrak{R}^{2,r} \supseteq [x,b)$,

(g) if $g^{(n)}(x) \in (a,b)$ for all $n \in \omega$ then there is $\mathfrak{R}^{2,l} \subseteq \mathfrak{R}$ with $(\mathfrak{R}^{2,l})_5 = \emptyset$ and $\cup\mathfrak{R}^{2,l} \supseteq (a,x]$.

Remark. Note that in Claim 2.1.13 the r (l resp.) superscript means that we choose these intervals going toward the right (left resp.) endpoint of (a,b).

Proof of the claim. We prove (f), the proof of (g) is similar.

For $i \in \omega$ choose $(a_i^0, b_i^0), (a_i^1, b_i^1) \in \mathfrak{R}$ satisfying the following conditions:

(1) $(a_i^0, b_i^0) \ni f^{(i)}(x)$,

(2) $(a_i^1, b_i^1) \ni f^{(i+1)}(x)$,

(3) $(a_i^0, b_i^0) \cap (a_i^1, b_i^1) \neq \emptyset$.

We can choose (a_i^1, b_i^1) $(i \in \omega)$ satisfying (2) (since $f^{(i+1)}(x) \in (a, b)$) and after this (a_i^0, b_i^0) satisfying (1) such that $(a_i^1, b_i^1) \cap (a_i^0, b_i^0) \neq \emptyset$ for all $i \in \omega$ because of the definition of $f^{(i+1)}(x)$ and the assumption that $f^{(i)}(x)$ is defined for $i \in \omega$. So the chosen interval system will satisfy (3).

If $f^{(i)}(x) \in (c, d) \in \mathfrak{R}$ for $i > 0$ then $c \geq f^{(i-1)}(x)$ and $d \leq f^{(i+1)}(x)$ and if $x \in (c, d) \in \mathfrak{R}$ then $d \leq f^{(1)}(x)$ by the definition of f. Let $\mathfrak{R}^{2,r} = \{(a_i^0, b_i^0) : i \in \omega\} \cup \{(a_{i+1}^1, b_{i+1}^1) : i \in \omega\}$.

If $i > 0$ then $[f^{(i)}(x), f^{(i+1)}(x)]$ meets only $(a_{i-1}^1, b_{i-1}^1), (a_i^0, b_i^0), (a_i^1, b_i^1), (a_{i+1}^0, b_{i+1}^0)$ and $(a_i^0, b_i^0), (a_i^1, b_i^1), (a_{i+1}^0, b_{i+1}^0)$ if $i = 0$, so $(\mathfrak{R}^{2,r})_5 = \emptyset$.

$\cup\mathfrak{R}^{2,r} \supseteq [x, b)$ is true by the fact that $lim_{n\to\infty} f^{(n)}(x) = b$ by Claim 2.1.12 and that $[f^{(i)}(x), f^{(i+1)}(x)] \subseteq (a_i^0, b_i^0) \cup (a_i^1, b_i^1)$ for all $i \in \omega$.

□

Claim 2.1.14. If $x \in (a, b)$ and $k \in \omega$ is such that $f^{(k)}(x) = b$ then we can find

$$\mathfrak{R}^{2,r}, \mathfrak{R}^{1,r} \subseteq \mathfrak{R}$$

with the following properties:

(A) $\mathfrak{R}^{2,r} \cap \mathfrak{R}^{1,r} = \emptyset$,

(B) $(\mathfrak{R}^{2,r})_5 = \emptyset$,

(C) if there is $c \in (a, b)$ with $(c, b) \in \mathfrak{R}$ then $\mathfrak{R}^{1,r}$ is empty, and
if $\mathfrak{R}^{1,r}$ is not empty then $\mathfrak{R}^{1,r} \in T(., <)$,

(D) $\mathfrak{R}^{2,r}$ is not empty and $\cup\mathfrak{R}^{2,r} \bigcap \cap\mathfrak{R}^{1,r} \neq \emptyset$,

(E) $(\cup\mathfrak{B}) \cup (\cup\mathfrak{R}^{2,r}) \supseteq [x, b)$ for all $\mathfrak{B} \in [\mathfrak{R}^{1,r}]^{|\mathfrak{R}^{1,r}|}$,

(F) $|\{Q \in \mathfrak{R}^{2,r} : Q \subseteq R\}| \leq 2$ for all $R \in \mathfrak{R}$.

Proof of the claim.

Case 1:

There is $c \in (a,b)$ with $(c,b) \in \mathfrak{R}$.

In this case $f^{(k-1)}(x) \in (c,b)$ and we can choose intervals $\{(a_i^j, b_i^j) : i \in k-1, j \in 2\}$ for $\{f^{(i)}(x) : i \in k-1\}$ as in the proof of Claim 2.1.13. Let $\mathfrak{R}^{2,r} = \{(a_i^j, b_i^j) : i \in k-1, j \in 2\} \cup (c,b)$ and $\mathfrak{R}^{1,r} = \emptyset$. $(A)-(E)$ are trivially satisfied. (F) is true by the fact that each interval in $\mathfrak{R}^{2,r}$ contains $f^{(i)}(x)$ for some $i \in k$. So if $R \in \mathfrak{R}$ contains an interval from $\mathfrak{R}^{2,r}$, it must contain $f^{(i)}(x)$ for some $i \in k$. But $|\{i \in \omega : f^{(i)}(x) \in R\}| \leq 1$ for all $R \in \mathfrak{R}$ and $x \in (a,b)$ by the definition of f. So $(a_0^0, b_0^0), (c,b)$ if $i = 0$, (a_{i-1}^1, b_{i-1}^1) and (a_i^0, b_i^0) if $0 < i < k-2$ and (a_{k-2}^1, b_{k-2}^1), (c,b) if $i = k-1$ are the only intervals from $\mathfrak{R}^{2,r}$ which can be contained in R.

Case 2:

There is no interval $(c,b) \in \mathfrak{R}$ with $c \in (a,b)$.

We can choose intervals $\{(a_i^j, b_i^j) : i \in k-1, j \in 2\}$ for $\{f^{(i)}(x) : 0 \leq i \leq k-1\}$ as in Claim 2.1.13 and let $\mathfrak{R}^{2,r} = \{(a_i^j, b_i^j) : i \in k-1, j \in 2\}$. Since $f^{(k)}(x) = b$ and we are not in *Case* 1, there exists $\mathfrak{A} = \{(c_n, d_n) : n \in \omega\} \subseteq \mathfrak{R}$ with:

- $\mathfrak{A} \cap \mathfrak{R}^{2,r} = \emptyset$,
- $\lim_{n \to \infty} d_n = b$,
- $f^{(k-1)}(x) \in (c_n, d_n)$ for all $n \in \omega$.

By Claim 2.1.8 there exists $\mathfrak{R}^{1,r} \subseteq \mathfrak{A}$ with $\mathfrak{R}^{1,r} \in T(., <)$. Then

$(A)-(C)$ of the lemma are trivially satisfied,

(D) is true, since $f^{(k-1)}(x) \in \cap \mathfrak{R}^{1,r}$,

(E) is true by Claim 2.1.10,

(F) is true similarly as in *Case* 1.

We are done with the proof of Claim 2.1.14.

□

Claim 2.1.15. If $x \in (a,b)$ and $k \in \omega$ is such that $g^{(k)}(x) = b$ then we can find

$$\mathfrak{R}^{2,l}, \mathfrak{R}^{0,l} \subseteq \mathfrak{R}$$

with the following properties:

(A) $\mathfrak{R}^{2,l} \cap \mathfrak{R}^{0,l} = \emptyset$,

(B) $(\mathfrak{R}^{2,l})_5 = \emptyset$,

(C) if there is $d \in (a,b)$ with $(a,d) \in \mathfrak{R}$ then $\mathfrak{R}^{0,l}$ is empty, and if $\mathfrak{R}^{0,l}$ is not empty then $\mathfrak{R}^{0,l} \in T(>,.)$,

(D) $\mathfrak{R}^{2,l}$ is not empty and $\cup\mathfrak{R}^{2,l} \bigcap \cap\mathfrak{R}^{0,l} \neq \emptyset$,

(E) $(\cup\mathfrak{B}) \cup (\cup\mathfrak{R}^{2,l}) \supseteq (a,x]$ for all $\mathfrak{B} \in [\mathfrak{R}^{0,l}]^{|\mathfrak{R}^{0,l}|}$,

(F) $|\{Q \in \mathfrak{R}^{2,l} : Q \subseteq R\}| \leq 2$ for all $R \in \mathfrak{R}$.

Proof. The proof is similar to the proof of Claim 2.1.14.

□

Let's continue the proof of Lemma 2.1.11 by choosing an arbitrary $x \in (a,b)$ and $\mathfrak{R}^{2,r}$, $\mathfrak{R}^{1,r}$, $\mathfrak{R}^{2,l}$ and $\mathfrak{R}^{0,l}$ by the previous claims. Let

$$\mathfrak{R}^2 = \mathfrak{R}^{2,l} \cup \mathfrak{R}^{2,r},$$

and let

$$\mathfrak{R}^0 \in [\mathfrak{R}^{0,l}]^\omega, \mathfrak{R}^1 \in [\mathfrak{R}^{1,r}]^\omega$$

with $\mathfrak{R}^0 \cap \mathfrak{R}^1 = \emptyset$ if $|\mathfrak{R}^{0,l}| = |\mathfrak{R}^{1,r}| = \omega$ and let $\mathfrak{R}^0 = \mathfrak{R}^{0,l}, \mathfrak{R}^1 = \mathfrak{R}^{1,r}$ otherwise.

Now we want to prove that $(i) - (vi)$ of Lemma 2.1.11 are satisfied:

- $(i), (ii)$ of Lemma 2.1.11 are satisfied by Claim 2.1.14 (C), Claim 2.1.15 (C) and the fact that if $\mathfrak{A}$ is in $T(.,.)$ then any subset of cardinality ω is in the same class,
- (iii) of Lemma 2.1.11 is true by Claim 2.1.14 (B), (D) and Claim 2.1.15 (B), (D) and Claim 2.1.13,
- (iv) of Lemma 2.1.11 is true by Claim 2.1.14 (D) and Claim 2.1.15 (D),
- (v) of Lemma 2.1.11 is true by Claim 2.1.14 (E) and Claim 2.1.15 (E),

- (vi) of Lemma 2.1.11 is true by Claim 2.1.14 (F) and Claim 2.1.15 (F).

We are with the proof of Lemma 2.1.11.

□

Claim 2.1.16. For all $\mathfrak{R} \in [\mathfrak{I}]^{\leq\omega}$ there is $\mathfrak{Q} \subseteq \mathfrak{R}$ with $\cup\mathfrak{Q} = \cup\mathfrak{R}$ and $(\mathfrak{R} \setminus \mathfrak{Q})_\omega = (\mathfrak{R})_\omega$.

Proof. $\mathfrak{Q}$ can be chosen with the help of Lemma 2.1.11. $\cup\mathfrak{R}$ is open so its components are open intervals. Then it is enough to choose $\mathfrak{Q}$ for $\mathfrak{R}$ when $\cup\mathfrak{R}$ is an interval. With the notation of Lemma 2.1.11 let $\mathfrak{R}^0 = \{R_n^0 : n \in \omega\}$ witnessing that $\mathfrak{R}^0 \in T(>,.)$ if it is not empty and $\mathfrak{R}^1 = \{R_n^1 : n \in \omega\}$ witnessing that $\mathfrak{R}^1 \in T(.,<)$ if it is not empty. Let $\mathfrak{Q} = \mathfrak{R}^2 \cup \{R_{2i}^1, R_{2i}^0 : i \in \omega\}$.

$\cup\mathfrak{Q} = \cup\mathfrak{R}$ is true by Lemma 2.1.11 (v), and

$(\mathfrak{R} \setminus \mathfrak{Q})_\omega = (\mathfrak{R})_\omega$ is true by Claim 2.1.9 and by Lemma 2.1.11 (iii).

□

Corollary 2.1.17. *For $\mathfrak{R} \subseteq \mathfrak{I}$ there exist $\mathfrak{R}_i \subseteq \mathfrak{R}$ pairwise disjoint for $i \in \omega$ with:*

(*i*) $\cup\mathfrak{R} = \cup\mathfrak{R}_0$,

(*ii*) $(\mathfrak{R})_\omega = (\mathfrak{R}_i)_\omega$ *for* $i > 0$.

Proof. By induction choose $\mathfrak{Q}_j \subseteq \mathfrak{R} \setminus \cup_{i<j}\mathfrak{Q}_i$ satisfying $\cup\mathfrak{Q}_j = \cup(\mathfrak{R} \setminus \cup_{i<j}\mathfrak{Q}_i)$ and $(\mathfrak{Q}_j)_\omega = (\mathfrak{R} \setminus \cup_{i<j}\mathfrak{Q}_i)_\omega$ using Claim 2.1.16 and let $\mathfrak{Q}_0 = \mathfrak{R}_0$, $\mathfrak{R}_i = \cup\mathfrak{T}_i$, where $\mathfrak{T}_i \in [\{\mathfrak{Q}_j : j > 0\}]^\omega$ pairwise disjoint.

□

2.1.2 Proof of Theorem 2.1.2

As a warm-up before the proof of Theorem 2.1.5, we prove Theorem 2.1.2:

Lemma 2.1.18. *Let $\mathfrak{Q}, \mathfrak{Q}_n \in [\mathfrak{I}]^\omega$ for $n \in \omega$. We can find $\mathfrak{K} \subseteq \mathfrak{Q}$ satisfying the following properties:*

(1) $\cup\mathfrak{K} = \mathfrak{Q}$ *and* $(\mathfrak{Q} \setminus \mathfrak{K})_\omega = (\mathfrak{Q})_\omega$,

(2) $(\mathfrak{Q}_n \setminus \mathfrak{K})_\omega = (\mathfrak{Q}_n)_\omega$ *for all* $n \in \omega$.

Proof. Let $\mathcal{D}$ be the set of components of $\cup\mathfrak{Q}$. Let $\mathfrak{Q}^{k,j}$ be the set provided by Lemma 2.1.11 for $k \in |\mathcal{D}|$, $j \in 3$. With a countable diagonalization argument we can easily find $\mathfrak{L}^{k,j} \subseteq \mathfrak{Q}^{k,j}$ for $k \in |\mathcal{D}|$, $j \in 2$ with:

(1) $(\mathfrak{Q}_n \setminus \mathfrak{L}^{k,j})_\omega = (\mathfrak{Q}_n)_\omega$ for $n \in \omega$,

(2) $\cup_{k\in|\mathcal{D}|,j\in 2}\ \mathfrak{L}^{k,j} \bigcup \cup_{k\in|\mathcal{D}|}\mathfrak{Q}^{k,2} = \cup\mathfrak{Q}$.

Note that since for $k \neq l \in |\mathcal{D}|$, $j \in 2$ $(\cup\mathfrak{L}^{k,j}) \cap (\cup\mathfrak{L}^{l,j}) = \emptyset$,

$$\cup_{k\in|\mathcal{D}|,j\in 2}\mathfrak{L}^{k,j} \bigcup \cup_{k\in|\mathcal{D}|}\mathfrak{Q}^{k,2} = \mathfrak{K}$$

fulfills the requirements of the statement.

□

Proof of Theorem 2.1.2. Let $\{\mathfrak{R}'_n : n \in \omega\}$ be an ω-abundant enumeration of $\{\mathfrak{R}_n : n \in \omega\}$. In the jth step apply Lemma 2.1.18 for

$$(\mathfrak{R}'_j \cup \bigcup_{i<j} \mathfrak{K}_i) = \mathfrak{Q} \ \textit{and}\ (\mathfrak{R}'_n \cup \bigcup_{i<j} \mathfrak{K}_i) = \mathfrak{Q}_n.$$

Fix $\varphi : \omega \to \omega \times \omega$ a bijection and let:

$$c(K) = \begin{cases} m & \text{if } \varphi(l) = \langle m, s\rangle,\ K \in \mathfrak{K}_j \text{ and } \mathfrak{R}'_j \text{ is the } l\text{th appearance of some } \mathfrak{R}_n, \\ 0 & \text{otherwise .} \end{cases}$$

This coloring proves the theorem.

□

2.1.3 Proof of Theorem 2.1.5

Claim 2.1.19. If $\mathfrak{R} \in [\mathfrak{I}]^\omega$ and $(\mathfrak{R})_\omega = \emptyset$ then we can find $\mathfrak{R}^+ \in [\mathfrak{R}]^\omega$ that is either disjoint or nested.

Proof. Let $\{R_t : t \in \omega\}$ be an enumeration of $\mathfrak{R}$. Let $J_{R_0} = \{t \in \omega : \partial R_0 \cap R_t \neq \emptyset\}$. $|J_{R_0}| < \omega$ since $(\mathfrak{R})_\omega = \emptyset$. Let $t_1 = min\ \{\omega \setminus (J_{R_0} \cup \{0\})\}$. Define $J_{R_{t_1}}$ similarly and continue this process in ω steps.

Let $\mathfrak{R}^- = \{R_{t_s} : s \in \omega\}$. $\mathfrak{R}^- \in [\mathfrak{I}]^\omega$ and by our choice for all $A, B \in \mathfrak{R}^-$ the following is true: $A \subseteq B$ or $B \subseteq A$ or $A \cap B = \emptyset$. Then by $\omega \to (\omega)^2_2$ we are done.

□

Definition 2.1.20. For $\mathfrak{R} \in [\mathfrak{I}]^\omega$ let

$$(\mathfrak{R})' = \begin{cases} \{R \in \mathfrak{R} : R \cap (\mathfrak{R})_\omega \neq \emptyset\} & \text{if } (\mathfrak{R})_\omega \neq \emptyset, \\ \mathfrak{R} & \text{otherwise.} \end{cases}$$

We mention the following two claims without proof:

Claim 2.1.21. $|(\mathfrak{R})'| = \omega$ and $((\mathfrak{R})')_\omega = (\mathfrak{R})_\omega$ for all $\mathfrak{R} \in [\mathfrak{I}]^\omega$.

Claim 2.1.22. $\mathfrak{R}_{p,q,\varepsilon} \setminus \mathfrak{K} = \mathfrak{R}_{p,q,\varepsilon} \setminus \mathfrak{K}_{p,q,\varepsilon}$ for all $\mathfrak{R} \in [\mathfrak{I}]^\omega$, $\mathfrak{K} \in [\mathfrak{I}]^{\leq\omega}$, $p, q \in D$, $\varepsilon \in 4$ with $D \in [\mathbf{R} \cup \{-\infty, +\infty\}]^\omega$ satisfying $D \cap \cup\{\partial R : R \in \mathfrak{R} \cup \mathfrak{K}\} = \emptyset$.

Lemma 2.1.23. *Let $\mathfrak{R} \in [\mathfrak{I}]^\omega$, $D \in [\mathbf{R} \cup \{-\infty, +\infty\}]^\omega$ with $D \cap \cup\{\partial R : R \in \mathfrak{R}\} = \emptyset$. Let $\mathfrak{L} \in \{(\mathfrak{R}_{p,q,\varepsilon})' : p, q \in D, \varepsilon \in 4\}$, $\{\mathfrak{R}_i : i \in \omega\} \in [\{(\mathfrak{R}_{p,q,\varepsilon})' : p, q \in D, \varepsilon \in 4\}]^\omega$.*

Then we can choose $\mathfrak{K} \subseteq \mathfrak{L}$ with $\cup\mathfrak{K} \supseteq (\mathfrak{L})_\omega$, if $(\mathfrak{L})_\omega = \emptyset$ then $|\mathfrak{K}| = \omega$ and the followings are true for all $i \in \omega$:

(i) $|\mathfrak{R}_i \setminus \mathfrak{K}| = \omega$,

(ii) $(\mathfrak{R}_i \setminus \mathfrak{K})_\omega = (\mathfrak{R}_i)_\omega$.

Proof.

$\cup\mathfrak{L}$ is an open subset of $\mathbf{R}$ so is the union of countably many open intervals. Let us denote by $\mathcal{C}$ the set of components of $\cup\mathfrak{L}$ and let $\gamma = |\mathcal{C}|$. Fix $\{C_k : k \in \gamma\}$, an enumeration of the components. Then for $m \in 3, k \in \gamma$ let us denote by $\mathfrak{L}^{m,k}$ the subset of $\{R \in \mathfrak{L} : R \subseteq C_k\}$ indexed by m in Lemma 2.1.11.

(an 'index dictionary' for the proof: $m \in 3$ will always refer to subsets come from Lemma 2.1.11. $i \in \omega$ refer to the enumeration $\{\mathfrak{R}_i : i \in \omega\}$ and k denotes the index of a component of $\cup\mathfrak{L}$)

Case 1: $(\mathfrak{L})_\omega \neq \emptyset$

Claim 2.1.24. For all $m \in 2, i \in \omega, k \in \gamma$ there are

$$\mathfrak{Q}_i^{m,k} \subseteq \mathfrak{L}^{m,k} \cap \mathfrak{R}_i \text{ and } \mathfrak{L}_0^{m,k} \subseteq \mathfrak{L}^{m,k}$$

satisfying:

(1) if $(m,i,k) \neq (m^{'},i^{'},k^{'})$ then $\mathfrak{Q}_i^{m,k} \cap \mathfrak{Q}_{i^{'}}^{m^{'},k^{'}} = \emptyset$,

(2) $\mathfrak{L}_0^{m,k} \cap \mathfrak{Q}_i^{m,k} = \emptyset$,

(3) $(\mathfrak{Q}_i^{m,k})_\omega = (\mathfrak{L}^{m,k} \cap \mathfrak{R}_i)_\omega$,

(4) $(\mathfrak{L}_0^{m,k})_\omega = (\mathfrak{L}^{m,k})_\omega$.

Proof. $\mathfrak{L}^{m,k} \in T(.,.)$ or empty for all $m \in 2$, $k \in \gamma$ so by an easy diagonalization argument and by Claim 2.1.9 we are done.

□

In *Case* 1 let

$$\mathfrak{K} = \bigcup_{m \in 2, k \in \omega} \mathfrak{L}_0^{m,k} \cup \bigcup_{k \in \omega} \mathfrak{L}^{2,k}.$$

Now we want to prove that $\mathfrak{K}$ fulfills the requirements:

- the proof of $\cup\mathfrak{K} \supseteq (\mathfrak{L})_\omega$:

Claim 2.1.25. $\cup\mathfrak{K} = \cup\mathfrak{L}$.

Proof. $\cup(\cup_{m \in 2}\mathfrak{L}_0^{m,k} \cup \mathfrak{L}^{2,k}) = \cup(\cup_{m \in 3}\mathfrak{L}^{m,k})$ for all $k \in \gamma$ by Lemma 2.1.11 (v).

□

As $\cup\mathfrak{L} \supseteq (\mathfrak{L})_\omega$, we are done.

- the proof of (ii):

Claim 2.1.26. $(\mathfrak{R}_i \setminus \mathfrak{K})_\omega = (\mathfrak{R}_i)_\omega$ for all $i \in \omega$.

Proof. $(\mathfrak{R}_i \setminus \mathfrak{K})_\omega \subseteq (\mathfrak{R}_i)_\omega$ is trivial, we want to prove the other direction. Let $x \in (\mathfrak{R}_i)_\omega$ arbitrary. If $x \notin (\mathfrak{R}_i \setminus \mathfrak{K})_\omega$ then $x \in (\mathfrak{R}_i \cap \mathfrak{K})_\omega$.

$(\mathfrak{R}_i \cap \bigcup_{k\in\gamma} \mathfrak{L}^{2,k})_\omega = \emptyset$ since $(\bigcup_{k\in\gamma} \mathfrak{L}^{2,k})_\omega = \emptyset$. So $x \in (\mathfrak{R}_i \cap \cup_{m\in 2, k\in\gamma} \mathfrak{L}_0^{m,k})$. But $\cup(\mathfrak{R}_i \cap \mathfrak{L}_0^{m_1,k_1}) \cap \cup(\mathfrak{R}_i \cap \mathfrak{L}_0^{m_2,k_2}) = \emptyset$ for $k_1 \neq k_2 \in \gamma$, $m_1, m_2 \in 2$, as they are in different components of $\cup\mathfrak{L}$, so there exists $k \in \gamma$ and $m \in 2$ with $x \in (\mathfrak{R}_i \cap \mathfrak{L}_0^{m,k})_\omega$. Then $(\mathfrak{R}_i \setminus \mathfrak{K})_\omega \supseteq (\mathfrak{Q}_i^{m,k})_\omega = (\mathfrak{L}^{m,k} \cap \mathfrak{R}_i)_\omega \supseteq (\mathfrak{L}_0^{m,k} \cap \mathfrak{R}_i)_\omega \ni x$ by Claim 2.1.24 (3).

□

- the proof of (i):

Claim 2.1.27. $|\mathfrak{R}_i \setminus \mathfrak{K}| = \omega$ for all $i \in \omega$.

Proof of the claim.

Case A: $(\mathfrak{R}_i)_\omega \neq \emptyset$.

Similarly as in the proof of Claim 2.1.26, if $|\mathfrak{R}_i \setminus \mathfrak{K}| < \omega$ then there are $m \in 2$ and $k \in \gamma$ with $(\mathfrak{R}_i \cap \mathfrak{L}_0^{m,k})_\omega \neq \emptyset$. By Claim 2.1.24 (3) $(\mathfrak{R}_i \cap \mathfrak{L}_0^{m,k})_\omega \subseteq (\mathfrak{R}_i \cap \mathfrak{L}^{m,k})_\omega = (\mathfrak{Q}_i^{m,k})_\omega \neq \emptyset$ so $|\mathfrak{Q}_i^{m,k}| = \omega$. But $\mathfrak{K} \cap \mathfrak{Q}_i^{m,k} = \emptyset$ implies $|\mathfrak{R}_i \setminus \mathfrak{K}| = \omega$.

Case B: $(\mathfrak{R}_i)_\omega = \emptyset$.

Apply Claim 2.1.19 to $\mathfrak{R}_i$ to obtain $\mathfrak{R}_i^+$ that is nested or disjoint.

Subcase B1:

$\mathfrak{R}_i^+ \in [\mathfrak{R}_i]^\omega$ is nested.

In this subcase $|\mathfrak{R}_i^+ \setminus \cup_{k\in\gamma}\mathfrak{L}^{2,k}| = \omega$, since $(\cup_{k\in\gamma}\mathfrak{L}^{2,k})_{10} = \emptyset$ and $\mathfrak{R}_i^+$ is nested. So if $|\mathfrak{R}_i^+ \setminus \mathfrak{K}| < \omega$ then $|\mathfrak{R}_i^+ \cap \cup_{m\in 2, k\in\gamma}\mathfrak{L}_0^{m,k}| = \omega$. Since $\mathfrak{R}_i^+$ is nested and $\cup \bigcup_{m\in 2} \mathfrak{L}_0^{m,k}$ are pairwise disjoint for $k \in \gamma$, there are $m \in 2$ and $k \in \gamma$ with $|\mathfrak{R}_i^+ \cap \mathfrak{L}_0^{m,k}| = \omega$. But this is impossible since $(\mathfrak{R}_i^+ \cap \mathfrak{L}_0^{m,k})_\omega \subseteq (\mathfrak{R}_i)_\omega = \emptyset$ but $\mathfrak{R}_i^+ \cap \mathfrak{L}_0^{m,k} \in [\mathfrak{L}^{m,k}]^\omega$, so by Claim 2.1.9 $(\mathfrak{R}_i^+ \cap \mathfrak{L}_0^{m,k})_\omega \neq \emptyset$. Contradiction, hence $|\mathfrak{R}_i \setminus \mathfrak{K}| = \omega$ in this subcase.

Subcase B2:

$\mathfrak{R}_i^+ \in [\mathfrak{R}_i]^\omega$ is disjoint.

There is $\mathfrak{X} \in [\mathfrak{R_i}^+]^\omega$ such that the order type of the left endpoints of the intervals in $\mathfrak{X}$ is either ω or ω^*. By symmetry we can assume that this order type is ω. As it is enough to prove that $|\mathfrak{X} \setminus \mathfrak{K}| = \omega$, arguing indirectly, we can assume that $\mathfrak{X} \subseteq \mathfrak{K}$.

Let $\{(a_n, b_n) : n \in \omega\}$ be the enumeration of $\mathfrak{X}$ such that $n < v \in \omega$ implies $b_n \leq a_v$. Since $(a_n, b_n) \in \mathfrak{L} = (\mathfrak{R}_{p_\mathfrak{L}, q_\mathfrak{L}, \varepsilon_\mathfrak{L}})'$ where $(\mathfrak{L})_\omega \neq \emptyset$, there is $x_n \in (a_n, b_n) \cap (\mathfrak{L})_\omega$ for all $n \in \omega$. So we can find $\{(x_u^n, y_u^n) : n \in \omega, u \in \omega\} \in [\mathfrak{L} \setminus \mathfrak{R}_i]^\omega$ such that $x_n \in (x_u^n, y_u^n)$ for all $n, u \in \omega$.

Let $A = sup_{n\in\omega} a_n = sup_{n\in\omega} b_n$.

Claim 2.1.28. $(x_u^n, y_u^n) \nsubseteq (b_0, A)$ for all $n \in \omega \setminus \{0\}$, $u \in \omega$.

(equivalently: $x_u^n < b_0$ or $A < y_u^n$)

Proof of the claim.

By contradiction. Suppose $(x_u^n, y_u^n) \subseteq (b_0, A)$ for some $n \in \omega \setminus \{0\}$, $u \in \omega$ and $\mathfrak{R}_i = (\mathfrak{R}_{p,q,\varepsilon})'$ for some $p, q \in D$ and $\varepsilon \in 4$. Note that $y_n^u \notin D$ for all $u, n \in \omega$.

(1) if $\varepsilon = 0$ then $A \leq p, A \leq q$. So if $(x_u^n, y_u^n) \subseteq (b_0, A)$ then $(x_u^n, y_u^n) \in \mathfrak{R}_i$, which is impossible.

(2) if $\varepsilon = 1$ then $p < a_0, A \leq q$. So if $(x_u^n, y_u^n) \subseteq (b_0, A)$ then $(x_u^n, y_u^n) \in \mathfrak{R}_i$, which is impossible.

(3) if $\varepsilon = 2$ then $A \leq p, q < b_0$. So if $(x_u^n, y_u^n) \subseteq (b_0, A)$ then $(x_u^n, y_u^n) \in \mathfrak{R}_i$ $(n \in \omega \setminus \{0\})$, which is impossible.

(4) if $\varepsilon = 3$ then $p < a_0$, $q < b_0$. So if $(x_u^n, y_u^n) \subseteq (b_0, A)$ then $(x_u^n, y_u^n) \in \mathfrak{R}_i$, which is impossible.

We are done with Claim 2.1.28.

□

Let $\mathfrak{C} = \{(x_u^n, y_u^n) : A < y_u^n\}$ and $\mathfrak{D} = \{(x_u^n, y_u^n) : x_u^n < b_0\}$, $\{C_k : k \in \gamma\}$ was an enumeration of the components of $\cup\mathfrak{L}$. We know that $I \cap J \neq \emptyset$ for all $I, J \in \mathfrak{C}$ and that $I \cap J \neq \emptyset$ for all $I, J \in \mathfrak{D}$. It is also true that $(a_n, b_n) \cap (x_u^n, y_u^n) \neq \emptyset$ for all $n, u \in \omega$.

By these facts there are $k_1, k_2 \in \gamma$ with $\cup((\mathfrak{X} \setminus \{(a_0,b_0)\}) \cup \mathfrak{C} \cup \mathfrak{D}) \subseteq C_{k_1} \cup C_{k_2}$. Suppose $|\{I \in \mathfrak{X} \setminus \{(a_0,b_0)\} : I \subseteq C_{k_1}\}| = \omega$. Since C_{k_1} is an open interval and the order type of the left endpoints of the intervals in $\mathfrak{X}$ is ω and they are disjoint, there is $N \in \omega$ such that $(a_n, b_n) \subseteq C_{k_1}$ for all $n > N$. $|(\mathfrak{L}^{0,k_1} \cup \mathfrak{L}^{1,k_1}) \cap \mathfrak{X}| \leq 2$ by Lemma 2.1.11 (i) and (ii), so as $\mathfrak{X} \subseteq \mathfrak{K}$, the remaining intervals are in $\mathfrak{L}^{2,k_1}$. Consider the $S = \{(x_u^{N+10}, y_u^{N+10}) : u \in \omega\}$ intervals. Each interval in S contains the x_{N+10} point and either $x_u^{N+10} < b_0$ or $A < y_u^{N+10}$. By this each contains at least 7 intervals from $\mathfrak{L}^{2,k_1}$ which is impossible by Lemma 2.1.11 (vi) since the elements of S are in $\mathfrak{L}$.

□

Case 2: $(\mathfrak{L})_\omega = \emptyset$.

In this case we use without proof the following claim:

Claim 2.1.29. We can find $\mathfrak{K} \in [\mathfrak{L}]^\omega$ and $\{\mathfrak{U}_i : i \in \omega\}$ such that for all $i \in \omega$ the followings are true:

(1) $\mathfrak{U}_i \subseteq \mathfrak{L} \cap \mathfrak{R}_i$,

(2) $\mathfrak{U}_i \cap \mathfrak{U}_j = \emptyset$ for $i \neq j \in \omega$,

(3) $\mathfrak{K} \cap \mathfrak{U}_i = \emptyset$,

(4) If $(\mathfrak{R}_i)_\omega = \emptyset$ and $|\mathfrak{R}_i \cap \mathfrak{L}| = \omega$ then $|\mathfrak{U}_i| = \omega$.

To continue the proof of Lemma 2.1.23 by choosing $\mathfrak{K}$ as in Claim 2.1.29 in *Case* 2, (i) is the only not trivially satisfied requirement:

if $(\mathfrak{R}_i)_\omega \neq \emptyset$, then $(\mathfrak{R}_i \setminus \mathfrak{L})_\omega \neq \emptyset$ so $(\mathfrak{R}_i \setminus \mathfrak{K})_\omega \neq \emptyset$ so $|\mathfrak{R}_i \setminus \mathfrak{K}_0| = \omega$.
If $(\mathfrak{R}_i)_\omega = \emptyset$ then if $|\mathfrak{R}_i \cap \mathfrak{L}| = \omega$, $\mathfrak{U}_i \in [\mathfrak{R}_i]^\omega$ and $|\mathfrak{U}_i \cap \mathfrak{K}| = \emptyset$.

We are done with the proof of Lemma 2.1.23.

□

Lemma 2.1.30. *Let* $\mathfrak{R} \in [\mathfrak{I}]^\omega$ *and* $D \in [\mathbf{R} \cup \{-\infty, +\infty\}]^\omega$ *with* $D \cap \cup\{\partial R : R \in \mathfrak{R}\} = \emptyset$. *Let* $\{\mathfrak{R}_i : i \in \omega\}$ *be an* ω*-abundant enumeration of* $\{(\mathfrak{R}_{p,q,\varepsilon})' : p, q \in D,\ \varepsilon \in 4\}$. *Then there exists*

$\{\mathfrak{K}_j : j \in \omega\}$ *pairwise disjoint such that for all* $i \in \omega$ *and for all* $j \in \omega$ *the followings hold:*

(*i*) $\mathfrak{K}_j \in [\mathfrak{R}_j]^{\leq\omega}$,

(*ii*) $|\mathfrak{R}_i \setminus \cup_{l\leq i}\mathfrak{K}_l| = \omega$,

(*iii*) $\cup\mathfrak{K}_i \supseteq (\mathfrak{R}_i)_\omega$ *and* $(\mathfrak{R}_i \setminus \cup_{l\leq i}\mathfrak{K}_l)_\omega = (\mathfrak{R}_i)_\omega$,

(*iv*) *if* $(\mathfrak{R}_i)_\omega = \emptyset$ *then* $|\mathfrak{K}_i| = \omega$.

Proof. For all $i \in \omega$ let $p_i, q_i \in D$ and $\varepsilon_i \in 4$ be such that $\mathfrak{R}_i = (\mathfrak{R}_{p_i,q_i,\varepsilon_i})'$.

We choose $\mathfrak{K}_j$ by induction on j ($j \in \omega$), in the jth step:

Let $\mathfrak{R}^{(\mathrm{j})}$ be $\mathfrak{R} \setminus \cup_{l<j}\mathfrak{K}_l$, $\mathfrak{L} = ((\mathfrak{R}^{(\mathrm{j})})_{p_j,q_j,\varepsilon_j})' \in \{((\mathfrak{R}^{(j)})_{p,q,\varepsilon})' : p, q \in D, \varepsilon \in 4\}$ and $\{((\mathfrak{R}^{(j)})_{p_i,q_i,\varepsilon_i})' : i \in \omega\} \in [\{((\mathfrak{R}^{(j)})_{p,q,\varepsilon})' : p, q \in D, \varepsilon \in 4\}]^\omega$. Apply Lemma 2.1.23 with $\mathfrak{R}^{(j)}$, $\mathfrak{L}$ and $\{((\mathfrak{R}^{(j)})_{p_i,q_i,\varepsilon_i})' : i \in \omega\}$. Let $\mathfrak{K}_j$ be the $\mathfrak{K}$ provided by Lemma 2.1.23 with these settings.

We want to prove that $(i) - (iv)$ are satisfied with $\{\mathfrak{K}_j : j \in \omega\}$.

Claim 2.1.31. For all $s, i \in \omega$ we have:

(1) $((\mathfrak{R}^{(s)})_{p_i,q_i,\varepsilon_i})' \in [\{((\mathfrak{R}^{(s)})_{p,q,\varepsilon})' : p, q \in D, \varepsilon \in 4\}]^\omega$,

(2) $|\mathfrak{R}_i \setminus \cup_{l\leq s}\mathfrak{K}_l| = \omega$.

Proof. Induction on s.

$s = 0$:

(1) is an assumption of the lemma

(2) is true by Lemma 2.1.23 (*i*)

$s > 0$:

(1) and (2) are true for $s - 1$:

$\omega = |\mathfrak{R}_i \setminus \cup_{l<s}\mathfrak{K}_l|$ by induction hypothesis and $\mathfrak{R}_i \setminus \cup_{l<s}\mathfrak{K}_l = (\mathfrak{R}_{p_i,q_i,\varepsilon_i})' \setminus \cup_{l<s}\mathfrak{K}_l \subseteq \mathfrak{R}_{p_i,q_i,\varepsilon_i} \setminus \cup_{l<s}\mathfrak{K}_l = (\mathfrak{R} \setminus \cup_{l<s}\mathfrak{K}_l)_{p_i,q_i,\varepsilon_i} = (\mathfrak{R}^{(s)})_{p_i,q_i,\varepsilon_i}$, so $|(\mathfrak{R}^{(s)})_{p_i,q_i,\varepsilon_i}| = \omega$, then (1) is true for s.

$\omega = |((\mathfrak{R}^{(s)})_{p_i,q_i,\varepsilon_i})' \setminus \mathfrak{K}_s|$ by Lemma 2.1.23 (*i*).

$((\mathfrak{R}^{(s)})_{p_i,q_i,\varepsilon_i})' \setminus \mathfrak{K}_s \subseteq (\mathfrak{R}^{(s)})_{p_i,q_i,\varepsilon_i} \setminus \mathfrak{K}_s = (\mathfrak{R} \setminus \cup_{l<s}\mathfrak{K}_l)_{p_i,q_i,\varepsilon_i} \setminus \mathfrak{K}_s =$

$((\mathfrak{R})_{p_i,q_i,\varepsilon_i} \setminus (\cup_{l<s}\mathfrak{K}_l)_{p_i,q_i,\varepsilon_i}) \setminus \mathfrak{K}_s = ((\mathfrak{R})_{p_i,q_i,\varepsilon_i} \setminus \cup_{l<s}\mathfrak{K}_l) \setminus \mathfrak{K}_s$ by Claim 2.1.22.

So $|\mathfrak{R}_i \setminus \cup_{l<(s+1)}\mathfrak{K}_l| = \omega$, thus we are done with (2) and with the proof of Claim 2.1.31.

□

Claim 2.1.32. $(\mathfrak{R}_i \setminus \cup_{l<s}\mathfrak{K}_l)_\omega = (\mathfrak{R}_i)_\omega$ for all $i, s \in \omega$.

Proof. Exactly the same as the proof of Claim 2.1.31 using Lemma 2.1.23 (ii) instead of Lemma 2.1.23 (i). □

- $\{\mathfrak{K}_j : j \in \omega\}$ are pairwise disjoint by the construction.
- (i) of Lemma 2.1.30 is true again by the construction.
- (ii) of Lemma 2.1.30 is true by Claim 2.1.31 (2).
- (iii) of Lemma 2.1.30 is true by the construction and Claim 2.1.32.
- (iv) of Lemma 2.1.30 is true by the construction.

We are done with the proof of Lemma 2.1.30.

□

Proof of Theorem 2.1.5. We use the Lemma 2.1.30 to construct such colorings. Let $\{\mathfrak{R}_i : i \in \omega\}$ be an ω-abundant enumeration of $\{(\mathfrak{R}_{p,q,\varepsilon})' : p, q \in D,\ \varepsilon \in 4\}$ and $\{\mathfrak{K}_j : j \in \omega\}$ be the sets provided in Lemma 2.1.30.

Fix $\varphi : \omega \to \omega \times \omega$ a bijection:

$$c(K) = \begin{cases} m & \text{if } \varphi(l) = \langle m, s\rangle,\ K \in \mathfrak{K}_j,\ (\mathfrak{R}_j)_\omega \neq \emptyset \text{ and} \\ & \mathfrak{R}_j \text{ is the } l\text{th appearance of } (\mathfrak{R}_{p,q,\varepsilon})' \text{ in } \{\mathfrak{R}_i : i \in \omega\} \text{ for } p, q \in D, \varepsilon \in 4, \\ 0 & \text{otherwise.} \end{cases}$$

Now Theorem 2.1.5 (i) holds by Lemma 2.1.30 (iii), and

Theorem 2.1.5 (ii) holds by Lemma 2.1.30 (ii).

□

2.1.4 A technical lemma

Lemma 2.1.33. *Let* $\{a_n, b_n : n \in \omega\}, \{A_n, B_n : n \in \omega\} \subseteq \mathbf{R}$ *such that:*

(i) $a_n \neq a_m$, $b_n \neq b_m$, $A_n \neq A_m$, $B_n \neq B_m$ *for all* $n, m \in \omega$ *different,*

(ii) ($a_n < a_m < b_m < b_n$ *or* $A_n < A_m < B_m < B_n$) *is false for all* $n, m \in \omega$,

(iii) $a_n < a_m \leq b_n < b_m$ *iff* $A_n < A_m \leq B_n < B_m$ *for all* $n, m \in \omega$,

(iv) $a_n < a_m$ *iff* $A_n < A_m$ *for all* $n, m \in \omega$.

Suppose that $\bigcap_{n\in J}(a_n, b_n) \cap \bigcap_{n\in\omega\setminus J}(\mathbf{R} \setminus (a_n, b_n)) \neq \emptyset$ *for some* $J \subseteq \omega$.

Then there exists $J \supseteq J'$ *with* $|J \setminus J'| < 3$ *such that*

$\bigcap_{n\in J'}(A_n, B_n) \cap \bigcap_{n\in\omega\setminus J}(\mathbf{R} \setminus (A_n, B_n)) \neq \emptyset$.

Proof. Let $J \subseteq \omega$ be such that there is $x \in \bigcap_{n\in J}(a_n, b_n) \cap \bigcap_{n\in\omega\setminus J}(\mathbf{R} \setminus (a_n, b_n))$. Let $J_1 = \{n \in \omega \setminus J : b_n \leq x\}$ and $J_2 = \{n \in \omega \setminus J, x \leq a_n\}$. So $\omega = J \cup^* J_1 \cup^* J_2$ and we know that:

(1) $sup_{n\in J} a_n (\leq x) \leq inf_{n\in J} b_n$,

(2) $sup_{n\in J_1} b_n (\leq x) \leq inf_{n\in J_2} a_n$,

(3) $sup_{n\in J_1} b_n (\leq x) \leq inf_{n\in J} b_n$,

(4) $sup_{n\in J} a_n (\leq x) \leq inf_{n\in J_2} a_n$.

using the conditions the same (certainly without x) follows for $A_n's$ and $B_n's$:

$(1)'$ $sup_{n\in J} A_n \leq inf_{n\in J} B_n$ is true since otherwise there would be $n(\neq)m \in J$ such that $B_n < A_m$. But then by *(ii)* and *(iii)* $b_n < a_m$ would be true contradicting (1).

$(2)'$, $(3)'$, $(4)'$ are true similarly.

Let $m = max\{sup_{n\in J_1} B_n, sup_{n\in J} A_n\}$ and $M = min\{inf_{n\in J} B_n, inf_{n\in J_2} A_n\}$

By $(1)' - (4)'$ we know that $m \leq M$ and let $y \in [m, M]$ arbitrary.

Let $I = \{n \in \omega : A_n = y$ or $B_n = y\}$. By *(i)* $|I| \leq 2$. Let $J' = J \setminus I$.

Then $y \in \bigcap_{n\in J'}(A_n, B_n) \cap \bigcap_{n\in\omega\setminus J}(\mathbf{R} \setminus (A_n, B_n))$.

□

Let $\varphi : \mathbf{R} \to X$ homeomorphism. Let us define $<_\varphi$ on X in the following way: for $x, y \in X$ let

$$x <_\varphi y \text{ iff } \varphi^{-1}(x) < \varphi^{-1}(y)$$

The following lemma is an immediate consequence of Lemma 2.1.33.

Lemma 2.1.34. *Let* $\varphi_1 : \mathbf{R} \to X$, $\varphi_2 : \mathbf{R} \to Y$ *arbitrary homeomorphisms. Suppose that* $\{(a_n, b_n) : n \in \omega\} \subseteq X, \{(A_n, B_n) : n \in \omega\} \subseteq Y$ *are satisfying:*

(i) $a_n \neq a_m$, $b_n \neq b_m$, $A_n \neq A_m$, $B_n \neq B_m$ *for all* $n, m \in \omega$ *different,*

(ii) ($a_n <_{\varphi_1} a_m <_{\varphi_1} b_m <_{\varphi_1} b_n$ *or* $A_n <_{\varphi_2} A_m <_{\varphi_2} B_m <_{\varphi_2} B_n$) *is false for all* $n, m \in \omega$,

(iii) $a_n <_{\varphi_1} a_m \leq_{\varphi_1} b_n <_{\varphi_1} b_m$ *iff* $A_n <_{\varphi_2} A_m \leq_{\varphi_2} B_n <_{\varphi_2} B_m$ *for all* $n, m \in \omega$,

(iv) $a_n <_{\varphi_1} a_m$ *iff* $A_n < A_m$ *for all* $n, m \in \omega$.

Suppose that $\bigcap_{n \in J}(a_n, b_n) \cap \bigcap_{n \in \omega \setminus J}(X \setminus (a_n, b_n)) \neq \emptyset$ *for some* $J \subseteq \omega$.

Then there exists $J \supseteq J'$ *with* $|J \setminus J'| < 3$ *such that* $\bigcap_{n \in J'}(A_n, B_n) \cap \bigcap_{n \in \omega \setminus J}(Y \setminus (A_n, B_n)) \neq \emptyset$.

3 Two dimensional case

Now we turn our attention to coloring of sets of translates of the open unit square.

In this chapter we color $\mathfrak{S}$, a set of translates of the open unit square whose intersection contains a small open set.

1. First we understand the geometric structure of $(\mathfrak{S})_\omega$.

2. We understand the geometric structure of the intersection of a square with $\partial(\mathfrak{S})_\omega$.

3. We understand the structure of $\partial(\mathfrak{S})_\omega \cap S$ for $S \in \mathfrak{S}$.

4. Using the structure of $S \in \mathfrak{S}$, we partition $\mathfrak{S}$ and provide a coloring for each part.

5. Finally we put together the colorings we define in **4**. and prove that it is a good coloring for $\mathfrak{S}$.

3.1 Notation

Let us introduce some notation for this section.

• We will work with subsets of $\mathbf{R}^2$, and use boldface letters to denote points of the plane, e.g. $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$. The closed (open, half closed) segment between $\mathbf{x}$ and $\mathbf{y}$ ($\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$) will be denoted by $[\mathbf{x}, \mathbf{y}]$ ($(\mathbf{x}, \mathbf{y})$, $[\mathbf{x}, \mathbf{y})$, $(\mathbf{x}, \mathbf{y}]$ respectively).

• The open unit square will be denoted by E, its vertices by

$$\mathbf{v_1}(E) = (0,1), \mathbf{v_2}(E) = (1,1), \mathbf{v_3}(E) = (1,0), \mathbf{v_4}(E) = (0,0).$$

(the labeling of the vertices is clockwise oriented and it is *mod 4* (meaning that e.g. *4+1=1*)).

• We recall that we denote set of the translates of E by $\mathcal{T}_E$, and for $S \in \mathcal{T}_E$ let us denote by $\mathbf{v_i}(S)$ the *corresponding* vertex of S.

• We use the following notation for the quadrants of $\mathbf{R}^2$:

$$\mathbf{R}^2_1 = \{(x,y) \in \mathbf{R}^2 : x \leq 0, 0 \leq y\}, \mathbf{R}^2_2 = \{(x,y) \in \mathbf{R}^2 : 0 \leq x, 0 \leq y\},$$

$$\mathbf{R}^2_3 = \{(x,y) \in \mathbf{R}^2 : 0 \leq x, y \leq 0\}, \mathbf{R}^2_4 = \{(x,y) \in \mathbf{R}^2 : x \leq 0, y \leq 0\}.$$

• We fix for the rest of this chapter

$$\mathfrak{S} \in [\mathcal{T}_E]^\omega$$

with $\cap\mathfrak{S}$ contains a small open set, T around the origin. We denote by S the elements of $\mathfrak{S}$.

3.2 Limit squares

In this introductory technical part we introduce a new notion: the *limit square*. The definition and the main properties of the *limit squares* will help us to understand the geometry of $(\mathfrak{S})_\omega$.

Definition 3.2.1.

- Let $\{Q_i : i \in \omega\}$ be with $Q_i \in \mathcal{T}_E$ $(i \in \omega)$. Then let

$$R = \{Q_i : i \in \omega\}_{lim} \text{ iff } \mathbf{v_4}(R) = lim_{i\to\infty}\mathbf{v_4}(Q_i) \text{ for } R \in \mathcal{T}_E.$$

- For $\mathfrak{Q} \subseteq \mathcal{T}_E$ let $(\mathfrak{Q})_{lim} = \{R \in \mathcal{T}_E : R = \{Q_i : i \in \omega\}_{lim}$ for some $\{Q_i : i \in \omega\} \in [\mathfrak{Q}]^\omega\}$, the set of *limit squares* of $\mathfrak{Q}$.

Remark. In the sequel we will denote by R the elements of $(\mathfrak{S})_{lim}$.

Let us start with some basic properties of the limit squares:

Claim 3.2.2.

(1) $\mathfrak{Q}_{lim} \neq \emptyset$ for all $\mathfrak{Q} \in [\mathfrak{S}]^\omega$,

(2) $(\mathfrak{Q}_{lim})_{lim} = \mathfrak{Q}_{lim}$ for all $\mathfrak{Q} \in [\mathfrak{S}]^\omega$,

(3) $\cap\mathfrak{S} \subseteq \cap\{\overline{R} : R \in (\mathfrak{S})_{lim}\}$,

(4) $int(\cap\mathfrak{S}) \subseteq \cap\{R : R \in (\mathfrak{S})_{lim}\}$.

Proof.

(1) follows by the fact that $\{\mathbf{v_4}(S) : S \in \mathfrak{Q}\}$ lies in a bounded part of the plane since each S contains the origin.

(2) follows by the obvious analogue lemma for points (for the vertices).

(3) is true by an easy convergence argument.

(4) follows by (3).

□

The most important properties of the *limit squares* of $\mathfrak{S}$ are:

$\bullet_1$ the union of them covers the interior of the closure of the ω-fold covered points (see (1) of Theorem 3.2.3),

$\bullet_2$ the union of their closure covers the closure of the ω-fold covered points (see (2) of Theorem 3.2.3).

Theorem 3.2.3.

(1) $\cup_{R\in(\mathfrak{S})_{lim}} R = int\ \overline{(\mathfrak{S})_\omega}$,

(2) $\cup_{R\in(\mathfrak{S})_{lim}} \overline{R} = \overline{(\mathfrak{S})_\omega}$.

Proof. First we prove $\subseteq$ in (1) and (2).

Lemma 3.2.4. *$R \subseteq (\mathfrak{S})_\omega$ for all $R \in (\mathfrak{S})_{lim}$.*

Proof. Let $\mathbf{z} \in R \in \mathfrak{S}_{lim}$. By the definition of *limit square* there is $\{Q_i : i \in \omega\} \in [\mathfrak{S}]^\omega$ with $\mathbf{v_4}(R) = lim_{i\to\infty}\mathbf{v_4}(Q_i)$. As R is an open set (translate of the open unit square) there is $N(\mathbf{z})$ satisfying $\mathbf{z} \in Q_n$ for all $n > N(\mathbf{z})$, which means $\mathbf{z} \in (\mathfrak{S})_\omega$. □

Again using the fact that R is open, we have the following corollary of Lemma 3.2.4 which proves $\subseteq$ in (1) and (2):

Corollary 3.2.5. *$R \subseteq int(\mathfrak{S})_\omega$ and $\overline{R} \subseteq \overline{(\mathfrak{S})_\omega}$ for all $R \in (\mathfrak{S})_{lim}$.*

Now we prove $\supseteq$ in (2):

Lemma 3.2.6. *For all $\mathbf{x} \in \overline{(\mathfrak{S})_\omega}$ there is $R \in (\mathfrak{S})_{lim}$ with $\mathbf{x} \in \overline{R}$.*

Proof. Let $\mathbf{x} \in \overline{(\mathfrak{S})_\omega}$. By the definition of $\overline{(\mathfrak{S})_\omega}$ there are $\{\mathbf{x_i} : i \in \omega\} \subseteq (\mathfrak{S})_\omega$ and $\{Q_{i,j} : i,j \in \omega\} \in [\mathfrak{S}]^\omega$ satisfying $lim_{j\to\infty}\mathbf{x_i} = \mathbf{x}$ and $\mathbf{x_i} \in Q_{i,j}$ for all $i,j \in \omega$. Since $Q_{i,j}$ contains the origin for all $i,j \in \omega$, we can find $A \in [\omega]^\omega$ and $s \in A^\omega$ with $lim_{i\to\infty,i\in A}\mathbf{v_4}(Q_{i,s(i)})$ exists and let R be a *limit square* of $\{Q_{i,s(i)} : i \in A\}$. By assumptions $R \in (\mathfrak{S})_{lim}$ and $\mathbf{x} \in \overline{R}$. □

Finally the proof of $\supseteq$ in (1):

Lemma 3.2.7. *Let $\mathbf{o}$ be the origin. Then $(\mathbf{x},\mathbf{o}] \subseteq int(\mathfrak{S})_\omega$ for all $\mathbf{x} \in \overline{(\mathfrak{S})_\omega}$.*

Proof. By Lemma 3.2.6 there is $R \in (\mathfrak{S})_{lim}$ with $\mathbf{x} \in \overline{R}$. By the fact that $\emptyset \neq int(\cap\mathfrak{S}) \subseteq R$ (Claim 3.2.2 (3)) it is true that $(\mathbf{x}, \mathbf{o}] \subseteq R$, so by Corollary 3.2.5 we are done.

□

Lemma 3.2.8. *For all* $\mathbf{x} \in int(\mathfrak{S})_\omega$ *there is* $R \in (\mathfrak{S})_{lim}$ *with* $\mathbf{x} \in R$.

Proof. Let $\mathbf{x} \in int(\mathfrak{S})_\omega$. Then there is $\lambda > 1$ such that the endpoint of the $\lambda\vec{\mathbf{ox}}$ vector is in $\overline{(\mathfrak{S})_\omega}$. Let us denote this endpoint by $\lambda\mathbf{x}$. By Lemma 3.2.7 and the fact that $\mathbf{x} \in (\lambda\mathbf{x}, \mathbf{o}]$ we are done.

□

We are done with the proof of Theorem 3.2.3.

□

Another important fact is that $int(\mathfrak{S})_\omega$ is regular open (see Lemma 3.2.9 (1)).

Lemma 3.2.9.

(1) $int\overline{(\mathfrak{S})_\omega} = int(\mathfrak{S})_\omega$,

(2) $\partial\overline{(\mathfrak{S})_\omega} = \partial(\mathfrak{S})_\omega$.

Proof. (2) is a trivial corollary of (1). So let us prove (1).

$\supseteq$ is trivial, so pick any $\mathbf{x} \in int\overline{(\mathfrak{S})_\omega}$. Let $\mathbf{o}$ be the origin. Then there is $\lambda > 1$ such that the endpoint of the $\lambda\vec{\mathbf{ox}}$ vector is in $\overline{(\mathfrak{S})_\omega}$. Let us denote this endpoint by $\lambda\mathbf{x}$. By Lemma 3.2.7 and the fact that $\mathbf{x} \in (\lambda\mathbf{x}, \mathbf{o}]$ we are done. □

After this lemma which proved that the boundary of the closure of the ω-fold covered points and the boundary of the ω-fold covered points are the same, we will prove a structure theorem for the boundary.

3.3 The structure of $\partial(\mathfrak{S})_\omega$

The aim of this section is understanding the structure of the boundary of the ω-fold covered points. We prove that it is the union of 8 parts: 2 vertical segments, 2 horizontal segments and 4 *monotone* (see Definition 3.3.1) parts lying in different quadrants and from this we derive that $\partial(\mathfrak{S})_\omega$ is homeomorphic to S^1.

Definition 3.3.1.

- $A \subseteq \mathbf{R}^2$ is *monotone increasing* if $x_1 < x_2 \Rightarrow y_1 \leq y_2$ for all $(x_1, y_1), (x_2, y_2) \in A$.
- Let $A \subseteq \mathbf{R}^2$ be monotone increasing. Then let
  - $b_1(A) = \{(x, y) \in A : x \leq z \textit{ and } y \leq v \text{ for all } (z, v) \in A\}$,
  - $b_2(A) = \{(x, y) \in A : z \leq x \textit{ and } v \leq y \text{ for all } (z, v) \in A\}$.
- $A \subseteq \mathbf{R}^2$ is *monotone decreasing* if $x_1 < x_2 \Rightarrow y_1 \geq y_2$ for all $(x_1, y_1), (x_2, y_2) \in A$.
- Let $A \subseteq \mathbf{R}^2$ be monotone decreasing. Then let
  - $b_1(A) = \{(x, y) \in A : x \leq z \textit{ and } v \leq y \text{ for all } (z, v) \in A\}$,
  - $b_2(A) = \{(x, y) \in A : z \leq x \textit{ and } y \leq v \text{ for all } (z, v) \in A\}$.

Let us mention the following fact without proof:

Fact.

If A is closed and monotone (increasing or decreasing) then both $b_0(A)$ and $b_1(A)$ exist.

Lemma 3.3.2. *There exist* $\{\mathbf{x_{i,j}} : 1 \leq i \leq 4, j = 1, 2\} \subseteq \mathbf{R}^2$ *and* $\{B_i : 1 \leq i \leq 4\} \subseteq \mathcal{P}(\mathbf{R}^2)$ *with:*

(1) $\mathbf{x_{i,j}} \in \mathbf{R}^2_i$ *for* $1 \leq i \leq 4,\ j = 1, 2$,

$(2)_1$ $\mathbf{x_{i,j}} = b_j(B_i)$ *for* $i = 1, 2,\ j = 1, 2$,

$(2)_2$ $\mathbf{x_{i,j}} = b_{3-j}(B_i)$ *for* $i = 3, 4,\ j = 1, 2$,

(3) B_1, B_3 *are monotone increasing sets,* B_2, B_4 *are monotone decreasing sets,*

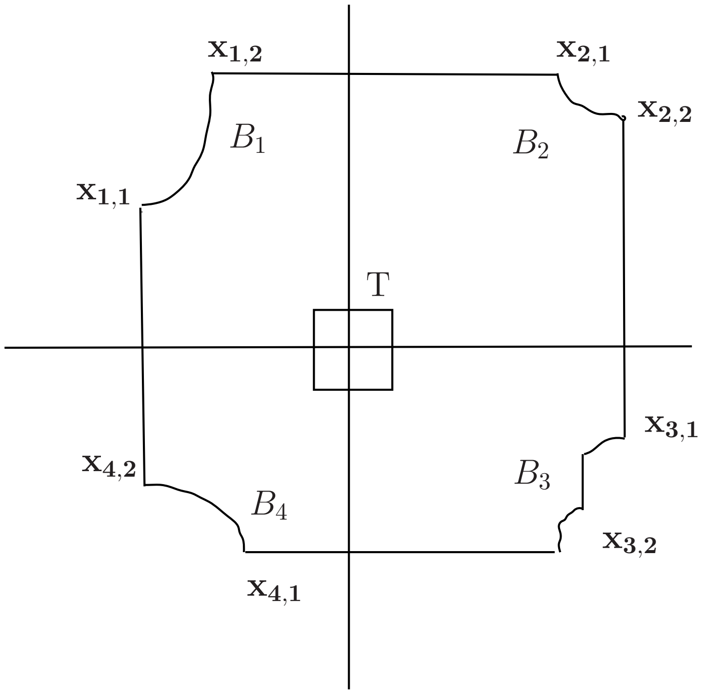

Figure 3.1: The structure of $\partial(\mathfrak{S})_\omega$

$(4)_1$ $[\mathbf{x_{1,2}}, \mathbf{x_{2,1}}], [\mathbf{x_{3,2}}, \mathbf{x_{4,1}}]$ *are horizontal segments of length at least 1,*

$(4)_2$ $[\mathbf{x_{2,2}}, \mathbf{x_{3,1}}], [\mathbf{x_{4,2}}, \mathbf{x_{1,1}}]$ *are vertical segments of length at least 1,*

(5) $\partial(\mathfrak{S})_\omega = \cup_{1\le i\le 4} B_i \bigcup \cup_{1\le i\le 4}[\mathbf{x_{i,2}}, \mathbf{x_{i+1,1}}]$.

Proof. Let

$$Y_M = max\{y : (x,y) \in \overline{(\mathfrak{S})_\omega}\}.$$

The maximum exists, as $\overline{(\mathfrak{S})_\omega}$ is compact. Let

$$X_M^m = min\{x : (x, Y_M) \in \overline{(\mathfrak{S})_\omega}\} \textit{ and } X_M^M = max\{x : (x, Y_M) \in \overline{(\mathfrak{S})_\omega}\}.$$

The maximum and the minimum exist, again by the compactness of $\overline{(\mathfrak{S})_\omega}$. Let

$$\mathbf{x_{1,2}} = (X_M^m, Y_M) \textit{ and } \mathbf{x_{2,1}} = (X_M^M, Y_M).$$

Claim 3.3.3. $[\mathbf{x_{1,2}}, \mathbf{x_{2,1}}] \subseteq \partial(\mathfrak{S})_\omega$ is a horizontal segment of length at least 1.

Proof. There are $R^{\mathbf{x_{1,2}}}, R^{\mathbf{x_{2,1}}} \in (\mathfrak{S})_{lim}$ with $\mathbf{x_{1,2}} \in \overline{R^{\mathbf{x_{1,2}}}}$ and $\mathbf{x_{2,1}} \in \overline{R^{\mathbf{x_{2,1}}}}$ by Theorem 3.2.3. We know that $\mathbf{o} \in R^{\mathbf{x_{1,2}}} \cap R^{\mathbf{x_{2,1}}}$, the y coordinate of $\mathbf{x_{1,2}}$ and $\mathbf{x_{2,1}}$ are equal and that $\overline{R^{\mathbf{x_{2,1}}}}$ and $\overline{R^{\mathbf{x_{1,2}}}}$ are axis-parallel, so $[\mathbf{x_{1,2}}, \mathbf{x_{2,1}}] \subseteq \overline{R^{\mathbf{x_{1,2}}}} \cup \overline{R^{\mathbf{x_{2,1}}}} \subseteq \overline{(\mathfrak{S})_\omega}$. As Y_M is maximal $[\mathbf{x_{1,2}}, \mathbf{x_{2,1}}] \subseteq \partial\overline{(\mathfrak{S})_\omega} = \partial(\mathfrak{S})_\omega$.

□

Define other $\mathbf{x_{i,j}}$'s similarly, clockwise, following the labeling of the quadrants:

$$\mathbf{x_{2,2}} = (X_M, Y_M^M), \mathbf{x_{3,1}} = (X_M, Y_M^m), \mathbf{x_{3,2}} = (Y_m, X_m^M),$$

$$\mathbf{x_{4,1}} = (Y_m, X_m^m), \mathbf{x_{4,2}} = (X_m, Y_m^m), \mathbf{x_{1,1}} = (X_m, Y_m^M).$$

Let Rl_i be the closed rectangle determined by the points $\mathbf{x_{i,1}}$ and $\mathbf{x_{i,2}}$ as opposite vertices for $i = 1, 2, 3, 4$ and let

$$B_i = Rl_i \cap \partial(\mathfrak{S})_\omega.$$

(by this we know that $Rl_i \subseteq \mathbf{R}_i^2$)

Claim 3.3.4. $\partial(\mathfrak{S})_\omega = \cup_{1\le i\le 4} B_i \bigcup \cup_{1\le i\le 4}[\mathbf{x_{i,2}}, \mathbf{x_{i+1,1}}]$.

Proof. $\supseteq$ is trivial. To verify $\subseteq$, by symmetry it is enough to prove that if

$$(z_1, z_2) = \mathbf{z} \in (\partial(\mathfrak{S})_\omega \setminus \cup_{1\le i\le 4}[\mathbf{x_{i,2}}, \mathbf{x_{i+1,1}}]) \cap \mathbf{R}_1^2$$

then $\mathbf{z} \in Rl_1$. Choose $R^{\mathbf{x_{1,2}}}, R^{\mathbf{x_{1,1}}} \in (\mathfrak{S})_{lim}$ with $\mathbf{x_{1,2}} \in \overline{R^{\mathbf{x_{1,2}}}}$ and $\mathbf{x_{1,1}} \in \overline{R^{\mathbf{x_{1,1}}}}$ by Theorem 3.2.3. We know that $\mathbf{z} \notin R^{\mathbf{x_{1,2}}} \cup R^{\mathbf{x_{1,1}}}$ by Corollary 3.2.5 as $\mathbf{z} \in \partial(\mathfrak{S})_\omega$. This means that $z_1 \le X_M^m$ and $Y_m^M \le z_2$. But then $Y_m \le z_2$ and $z_1 \le X_M$ holds by the definition of Y_m, X_M so $\mathbf{z} \in Rl_1$. □

Claim 3.3.5.

1) B_i is monotone increasing for $i = 1, 3$,

2) B_i is monotone decreasing for $i = 2, 4$.

Proof. By symmetry it is enough to prove that B_1 is monotone increasing. Consider $(z_1, v_1), (z_2, v_2) \in B_1$ with $z_1 < z_2$ and choose $R \in (\mathfrak{S})_{lim}$ with $(z_1, v_1) \in \overline{R}$. Then $(z_2, v_2) \notin R$ by Corollary 3.2.5. R is axis parallel, hence $v_1 \leq v_2$ meaning that B_1 is monotone increasing.

□

Then we are done with the proof of Lemma 3.3.2.

□

We described the structure of $\partial(\mathfrak{S})_\omega$ and in the remaining part of this section we would like to prove that $\partial(\mathfrak{S})_\omega$ is homeomorphic to S^1 using Lemma 3.3.2.

Definition 3.3.6. For $A \subseteq \mathbf{R}^2$ let $proj_x(A)/proj_y(A)$ be the projection of A onto the x/y coordinate.

Theorem 3.3.7. *$\partial(\mathfrak{S})_\omega$ is homeomorphic to S^1.*

Proof. By Lemma 3.3.2 $(2)_1 - (4)_2$ and symmetry it is enough to verify that B_1 is homeomorphic to an interval.

Lemma 3.3.8. *For all $A \subseteq \mathbf{R}^2$ monotone increasing (decreasing) there are functions:*

- *$f_A : proj_x(A) \to \mathbf{R}$ monotone increasing (decreasing, resp.), and*
- *$g_A : proj_y(A) \to \mathbf{R}$ monotone decreasing (increasing, resp.)*

with $graph(f_A) \cup graph(g_A) \subseteq A$.

Proof. For any point $z \in proj_x(A)(proj_y(A))$, just pick a point in A with first (second resp.) coordinate z, and let it be $f_A(g_A$ resp.). These functions easily satisfy the requirements of the lemma.

□

Claim 3.3.9. $proj_x(B_1) = proj_x(Rl_1)$ and $proj_y(B_1) = proj_y(Rl_1)$.

Proof. We prove only $proj_x(B_1) = proj_x(Rl_1)$.

By the definition of Rl_1 we know that $proj_x(Rl_1) = [Y_m, X_M^m]$. We also know that $\mathbf{x_{1,1}}, \mathbf{x_{1,2}} \in B_1$, so $Y_m, X_M^m \in proj_x(B_1)$. Consider any $s \in proj_x(Rl_1) \setminus \{Y_m, X_M^m\}$ and let $l_s = \{(s, y) : 0 \leq y\}$, the vertical halfline starting from $(s, 0)$. It is true, that $l_s \cap \partial(\mathfrak{S})_\omega \neq \emptyset$ since $(s, 0) \in \overline{(\mathfrak{S})_\omega} \cap l_s$, but $\overline{(\mathfrak{S})_\omega}$ can not contain a whole halfline since it is a compact subset of the plane. So $l_s \cap B_1 \neq \emptyset$ as $Y_m < s < X_M^m$. We are done.

□

Claim 3.3.10. There is a homeomorphism $\varphi : B_1 \to [0, 1]$.

Proof. Choose (by Lemma 3.3.8) a monotone function $h : proj_x(B_1) \to \mathbf{R}$ with $graph(h) \subseteq B_1$. We know that h is continuous with a countable exceptional set $B \subseteq proj_x(B_1)$, and that $lim_{y \to z-} h(y)$ and $lim_{y \to z+} h(y)$ exist for all $z \in proj_x(B_1)$ and $lim_{y \to z-} h(y) \leq lim_{y \to z+} h(y)$. For all $z \in B$ it is true that $proj_y(proj_x^{-1}(z) \cap B_1)) = [lim_{y \to z-} h(y), lim_{y \to z+} h(y)]$ since B_1 is *monotone* and $proj_y(B_1) = proj_y(Rl_1)$. From this it is clear that $\psi((x, y)) = x + y$ ($(x, y) \in B_1$) provides a homeomorphism between B_1 and a closed interval of $\mathbf{R}$. Which proves our claim.

□

We proved Theorem 3.3.7.

□

By Lemma 3.2.9 we know that $\partial(\mathfrak{S})_\omega = \partial\overline{(\mathfrak{S})_\omega}$. We will denote

$$\partial(\mathfrak{S})_\omega \ by \ \partial$$

in the followings.

We know by Theorem 3.3.7 that ∂ is homeomorphic to S^1 and $int(\mathfrak{S})_\omega$ contains the origin. By Lemma 3.2.7 we also know that in each direction (from the origin) there is only one point on ∂. By this we can talk about the *clockwise orientation* of ∂.

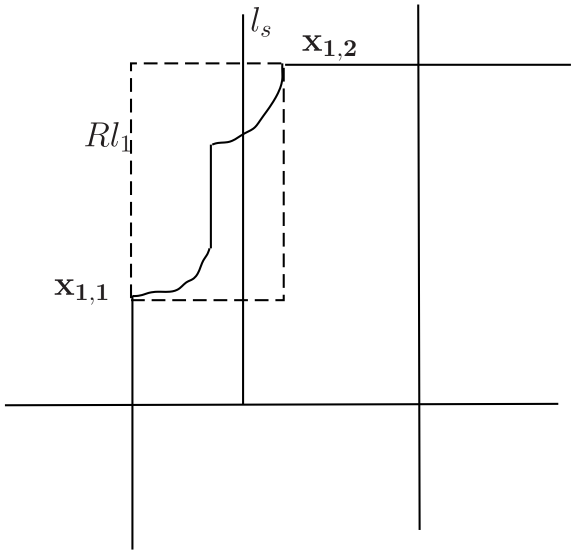

Figure 3.2: l_s and Rl_1

Let us introduce a notation for the *clockwise arc* of ∂. Using the *clockwise orientation* of ∂ for $\mathbf{x}, \mathbf{y} \in \partial$ let us denote by

$$\partial(\mathbf{x}, \mathbf{y}), \partial[\mathbf{x}, \mathbf{y}], \partial[\mathbf{x}, \mathbf{y}), \partial(\mathbf{x}, \mathbf{y}]$$

the open, closed, half-closed *clockwise arc* of ∂ between $\mathbf{x}$ and $\mathbf{y}$. Let us note that $\partial(\mathbf{x}, \mathbf{y}), \partial[\mathbf{x}, \mathbf{y}], \partial[\mathbf{x}, \mathbf{y})$ and $\partial(\mathbf{x}, \mathbf{y}]$ are homeomorphic image of $(0, 1), [0, 1], [0, 1)$ and $(0, 1]$ respectively.

3.4 Intersection of ∂ with $S \in \mathfrak{S}$

In this section we would like to analyze the intersection of ∂ and $S \in \mathfrak{S}$. The aim of this section is to prove that this intersection $(S \cap \partial)$ is homeomorphic to one open interval or to the union of 2 open intervals.

First we analyze the intersection of a (closed) side of $S \in \mathfrak{S}$ and $\overline{(\mathfrak{S})_\omega}$. The followings are true:

$\bullet_1$ a side of $S \in \mathfrak{S}$ is totally inside or totally outside of $\overline{(\mathfrak{S})_\omega}$ *iff* both of its endpoints are inside or outside of $\overline{(\mathfrak{S})_\omega}$ respectively (see Claim 3.4.1).

$\bullet_2$ if one endpoint of a side of $S \in \mathfrak{S}$ is inside of $\overline{(\mathfrak{S})_\omega}$, the other endpoint is outside of $\overline{(\mathfrak{S})_\omega}$ then we can find a point inside that side (which lies in the same quadrant as the endpoint which is outside of $\overline{(\mathfrak{S})_\omega}$), dividing it into two parts such that one part (the half open, half closed) is outside of $\overline{(\mathfrak{S})_\omega}$, the other side (the closed part) is inside of $\overline{(\mathfrak{S})_\omega}$ (see Claim 3.4.2).

Claim 3.4.1. For all $S \in \mathfrak{S}$ and $1 \le i \le 4$ the followings hold:

(1) If $\mathbf{v_i}(S), \mathbf{v_{i+1}}(S) \in \overline{(\mathfrak{S})_\omega}$ then $[\mathbf{v_i}(S), \mathbf{v_{i+1}}(S)] \subseteq \overline{(\mathfrak{S})_\omega}$,

(2) If $\mathbf{v_i}(S), \mathbf{v_{i+1}}(S) \notin \overline{(\mathfrak{S})_\omega}$ then $[\mathbf{v_i}(S), \mathbf{v_{i+1}}(S)] \subseteq \mathbf{R}^2 \setminus \overline{(\mathfrak{S})_\omega}$.

Proof of Claim 3.4.1. The right-to-left implication is trivial in both (1) and (2).

Proof of left-to-right implication in (1):

by Lemma 3.2.6 there are $R^{\mathbf{v_i}(S)}, R^{\mathbf{v_{i+1}}(S)} \in (\mathfrak{S})_{lim}$ with $\mathbf{v_i}(S) \in \overline{R^{\mathbf{v_i}(S)}}$ and $\mathbf{v_{i+1}}(S) \in \overline{R^{\mathbf{v_{i+1}}(S)}}$.

As $T \subseteq \overline{R^{\mathbf{v_i}(S)}} \cap \overline{R^{\mathbf{v_{i+1}}(S)}}$, $[\mathbf{v_i}(S), \mathbf{v_{i+1}}(S)] \subseteq \overline{R^{\mathbf{v_i}(S)}} \cup \overline{R^{\mathbf{v_{i+1}}(S)}}$ and by Corollary 3.2.5 we are done.

Proof of left-to-right implication in (2):

by contradiction, if there is $\mathbf{x} \in [\mathbf{v_i}(S), \mathbf{v_{i+1}}(S)] \cap \overline{(\mathfrak{S})_\omega}$ then by Lemma 3.2.6 there is $R^{\mathbf{x}} \in (\mathfrak{S})_{lim}$ with $\mathbf{x} \in \overline{R^{\mathbf{x}}}$. Since a limit square is axis-parallel and the length of a side of it is at least 1, either $\mathbf{v_i}(R) \in \overline{R^{\mathbf{x}}}$ or $\mathbf{v_{i+1}}(R) \in \overline{R^{\mathbf{x}}}$ and by Corollary 3.2.5 we are done.

□

Claim 3.4.2. For all $S \in \mathfrak{S}$ and $1 \leq i \leq 4$ the following holds:

If $\mathbf{v_i}(S) \notin \overline{(\mathfrak{S})_\omega}$ and $\mathbf{v_{i+1}}(S) \in \overline{(\mathfrak{S})_\omega}$ then $\exists!\ \mathbf{y} \in (\mathbf{v_i}(S), \mathbf{v_{i+1}}(S)) \cap \partial(\mathfrak{S})_\omega \cap \mathbf{R}^2_i$ with:

$\bullet_1$ $[\mathbf{y}, \mathbf{v_{i+1}}(S)] \subseteq \overline{(\mathfrak{S})_\omega}$,

$\bullet_2$ $[\mathbf{v_i}(S), \mathbf{y}) \subseteq \mathbf{R}^2 \setminus \overline{(\mathfrak{S})_\omega}$.

Proof of Claim 3.4.2.

Let d be the 2 dimensional distance function. Consider $f(\mathbf{x}) = d(\mathbf{x}, \mathbf{v_i}(S))$ on $[\mathbf{v_i}(S), \mathbf{v_{i+1}}(S)] \cap \overline{(\mathfrak{S})_\omega}$. By the fact that $\overline{(\mathfrak{S})_\omega} \cap [\mathbf{v_i}(S), \mathbf{v_{i+1}}(S)]$ is a non-empty (since $\mathbf{v_{i+1}}(S) \in \overline{(\mathfrak{S})_\omega}$), compact set and $f(\mathbf{x})$ ($\mathbf{x} \in \mathbf{R}^2$) is a continuous function, $min\{f(\mathbf{x}) : \mathbf{x} \in [\mathbf{v_i}(S), \mathbf{v_{i+1}}(S)] \cap \overline{(\mathfrak{S})_\omega}\}$ exists. $[\mathbf{v_i}(S), \mathbf{v_{i+1}}(S)]$ is a segment, so there is exactly one point of $[\mathbf{v_i}(S), \mathbf{v_{i+1}}(S)]$ where $f(\mathbf{x})$ takes its minimum. Let it be $\mathbf{y}$. Now we want to prove that it satisfies $\bullet_1$ and $\bullet_2$:

$\mathbf{y} \neq \mathbf{v_i}(S)$ since $\mathbf{v_i}(S) \notin \overline{(\mathfrak{S})_\omega}$. Choose $R^{\mathbf{y}} \in (\mathfrak{S})_{lim}$ with $\mathbf{y} \in \overline{R^{\mathbf{y}}}$. Since $[\mathbf{v_i}(S), \mathbf{v_{i+1}}(S)]$ is axis parallel, $R^{\mathbf{y}}$ is axis parallel, the length of a side of $R^{\mathbf{y}}$ is at least 1 and $\mathbf{v_i}(S) \notin \overline{R^{\mathbf{y}}}$ (by Corollary 3.2.5) we know that $[\mathbf{y}, \mathbf{v_{i+1}}(S)] \subseteq \overline{R^{\mathbf{y}}} \subseteq \overline{(\mathfrak{S})_\omega}$ (by Corollary 3.2.5). We proved $\bullet_1$.

$[\mathbf{v_i}(S), \mathbf{y}) \subseteq \mathbf{R}^2 \setminus \overline{(\mathfrak{S})_\omega}$ is true by the definition of $\mathbf{y}$, we proved $\bullet_2$.

$T \subseteq R^{\mathbf{y}}$ so $\mathbf{y} \in (\mathbf{v_i}(S), \mathbf{v_{i+1}}(S)) \cap \overline{(\mathfrak{S})_\omega} \cap \mathbf{R}^2_i$. By $[\mathbf{v_i}(S), \mathbf{y}) \subseteq \mathbf{R}^2 \setminus \overline{(\mathfrak{S})_\omega}$ it is true that $\mathbf{y} \in \partial$.

We finished the proof of Claim 3.4.2.

□

Now we divide $\mathfrak{S}$ into finitely many parts concerning the vertices outside of $\overline{(\mathfrak{S})_\omega}$.

Definition 3.4.3. For $a \in \mathcal{P}(\{1,2,3,4\})$ let $\mathfrak{S}^a = \{S \in \mathfrak{S} : i \in a \Leftrightarrow \mathbf{v_i}(S) \notin \overline{(\mathfrak{S})_\omega}\}$.

We will use the following notation: $\mathcal{A}_2 = \{\{1,3\},\{2,4\}\}$, $\mathcal{A}_1 = \mathcal{P}(\{1,2,3,4\}) \setminus \mathcal{A}_2$.

Claim 3.4.4. $\mathfrak{S}^{\{1,2,3,4\}} = \emptyset$.

Proof. Consider any $R \in (\mathfrak{S})_{lim}$ and $S \in \mathfrak{S}^{\{1,2,3,4\}}$. Since both S and R contains T, a vertex of S must be contained in $\overline{R}$, hence $\mathfrak{S}^{\{1,2,3,4\}} = \emptyset$.

□

The structure of the intersection of $S \in \mathfrak{S}$ and ∂ can be easily derived by applying Claim 3.4.2 and Claim 3.4.1 on each side.

Theorem 3.4.5.

(1) *For all* $a \in \mathcal{A}_1 \setminus \{\emptyset\}$ *and* $S \in \mathfrak{S}^a$ *there exist* $\mathbf{p_1}(S), \mathbf{p_2}(S) \in \partial$ *with*

$$S \cap \partial = \partial(\mathbf{p_1}(S), \mathbf{p_2}(S)) \subseteq \partial \cap (\cup_{i \in a} \mathbf{R}^2_i),$$

(2) *For all* $\{a_1, a_2\} \in \mathcal{A}_2$ *and* $S \in \mathfrak{S}^a$ *there exist* $\mathbf{p_1}(S), \mathbf{p_2}(S), \mathbf{p_3}(S), \mathbf{p_4}(S) \in \partial$ *with:*

$$S \cap \partial = \cup_{i \in 2} \partial(\mathbf{p_{a_i}}(S), \mathbf{p_{a_i+1}}(S)) \subseteq \partial \cap \cup_{i \in a} \mathbf{R}^2_i.$$

Proof. We will prove (1) for $a = \{1\}$, the proof of other cases and the proof of (2) are similar:

- If $a = \{1\}$, then $\mathbf{v_1}(S) \notin \overline{(\mathfrak{S})_\omega}$, $\mathbf{v_2}(S), \mathbf{v_3}(S), \mathbf{v_4}(S) \in \overline{(\mathfrak{S})_\omega}$.

1) By Claim 3.4.1 $[\mathbf{v_2}(S), \mathbf{v_3}(S)], [\mathbf{v_3}(S), \mathbf{v_4}(S)] \subseteq \overline{(\mathfrak{S})_\omega}$.

2) By Claim 3.4.2 there are:

∘ $\mathbf{p_1}(S) \in (\mathbf{v_4}(S), \mathbf{v_1}(S)) \cap \mathbf{R}^2_1$ and $\mathbf{p_2}(S) \in (\mathbf{v_1}(S), \mathbf{v_2}(S)) \cap \mathbf{R}^2_1$ with:

(a) $(\mathbf{p_1}(S), \mathbf{v_1}(S)], [\mathbf{v_1}(S), \mathbf{p_2}(S)) \subseteq \mathbf{R}^2 \setminus \overline{(\mathfrak{S})_\omega}$,

(b) $[\mathbf{p_2}(S), \mathbf{v_2}(S)], [\mathbf{v_4}(S), \mathbf{p_1}(S)] \subseteq \overline{(\mathfrak{S})_\omega}$.

By this it is immediate that $\partial(\mathbf{p_2}(S), \mathbf{p_1}(S)) \subseteq \mathbf{R}^2 \setminus S$. The fact that $\partial(\mathbf{p_1}(S), \mathbf{p_2}(S)) \subseteq S$ comes by the definition of $\mathbf{p_1}(S)$ and $\mathbf{p_2}(S)$ and that $\partial \cap \mathbf{R}^2_1$ is a monotone increasing subset of the plane.

We are done.

□

As an end of this section let us state when do

$$\partial(\mathbf{p_1}(S), \mathbf{p_2}(S))$$

for $a \in \mathcal{A}_1, S \in \mathfrak{S}^a$ or

$$\partial(\mathbf{p_{a_1}}(S), \mathbf{p_{a_1+1}}(S))$$

for $\{a_1, a_2\} \in \mathcal{A}_2, S \in \mathfrak{S}^a$ determine S.

Lemma 3.4.6.

(1) *If* $a \in \mathcal{A}_1$ *and* $|a| = 1$ *or 3, then* $\mathbf{p_1}(S_1) \neq \mathbf{p_1}(S_2)$ *or* $\mathbf{p_2}(S_1) \neq \mathbf{p_2}(S_2)$ *for any two* $S_1, S_2 \in \mathfrak{S}^a$.

(2) *If* $\{a_1, a_2\} = a \in \mathcal{A}_1$*, then* $\mathbf{p_{a_1}}(S_1) \neq \mathbf{p_{a_1}}(S_2)$ *or* $\mathbf{p_{a_1+1}}(S_1) \neq \mathbf{p_{a_1+1}}(S_2)$ *for any two* $S_1, S_2 \in \mathfrak{S}^a$.

Proof. $\mathbf{p_1}(S)$ and $\mathbf{p_2}(S)$ in (1) ($\mathbf{p_{a_1}}(S)$ and $\mathbf{p_{a_1+1}}(S)$ in (2)) are elements of neighboring sides of S, then since they lie inside the of the sides that determine S. □

Remark. The importance of the previous lemma lies in that if we give a color to an arc and that arc determines the square, then we can automatically give the color of the arc to the square.

3.5 Notation, definitions, strategy for coloring

In this subsection, we summarize that we have achieved so far, define arc systems for which we will use the one dimensional coloring theorems and at the end we give a strategy for the two dimensional coloring.

3.5.1 Notation, definitions

(1) For $a \subseteq \{1,2,3,4\}$ let

$$\partial_a = \partial \cap int(\cup_{i\in a}\mathbf{R}^2_i).$$

Note that by Theorem 3.4.5 for all $a \subseteq \{1,2,3,4\}$

$$S \cap \partial = S \cap \partial_a \subseteq int(\cup_{i\in a}\mathbf{R}^2_i) \text{ for } S \in \mathfrak{S}^a.$$

(2)

(2.1.) For all $a \in \mathcal{A}_1$ let

- $\bullet_1$ $f_a : \mathbf{R} \to \partial_a$ homeomorphism satisfying that

$$f_a^{-1}(\mathbf{x}) < f_a^{-1}(\mathbf{y}) \text{ iff } \partial(\mathbf{x},\mathbf{y}) \subseteq \partial_a,$$

  where $\partial(\mathbf{x},\mathbf{y})$ is the clockwise oriented arc between $\mathbf{x}$ and $\mathbf{y}$ in ∂,

- $\bullet_2$ $<_a$ be the pushforward ordering on ∂_a, (thus f_a is an order preserving homeomorphism between $(\mathbf{R},<)$ and $(\partial_a,<_a)$), and

- $\bullet_3$ $\mathcal{T}(a) = \{S \cap \partial_a : S \in \mathfrak{S}^a\} = \{\partial(\mathbf{p_1}(S),\mathbf{p_2}(S)) : S \in \mathfrak{S}^a\}$.

(2.2.) For $i = 3,4$ let

$\bullet_1$ $f'_i : \mathbf{R} \to \partial_i$ homeomorphism satisfying that

$$f_i^{-1}(\mathbf{y}) < f_i^{-1}(\mathbf{x}) \text{ iff } \partial(\mathbf{x}, \mathbf{y}) \subseteq \partial_i,$$

$\bullet_2$ $<'_i$ be the pushforward (by f'_i) ordering on ∂_i

(so $<'_i$ follows the counterclockwise orientation on ∂),

$\bullet_{3.1.}$ $(\partial_{\{1,3\}}, <_{\{1,3\}}) = (\partial_1, <_1) \cup (\partial_3, <'_3)$,

$\bullet_{3.2.}$ $(\partial_{\{2,4\}}, <_{\{2,4\}}) = (\partial_2, <_2) \cup (\partial_4, <'_4)$.

$\bullet_4$ We will give the definition of

$$\mathcal{T}(\{1,3\}) \subseteq \{S \cap \partial_1 : S \in \mathfrak{S}^{\{1,3\}}\} \text{ and } \mathcal{T}(\{2,4\}) \subseteq \{S \cap \partial_2 : S \in \mathfrak{S}^{\{2,4\}}\}$$

in the *Preparing for coloring I.* section.

Remark.

1.) As f_a is an order preserving homeomorphism between $(\mathbf{R}, <)$ and $(\partial_a, <_a)$ for $a \in \mathcal{A}_1$,

1.1.) $(\mathcal{T}(a))_{\mathbf{x},\mathbf{y},\varepsilon}$ is meaningful for $a \in \mathcal{A}_1$ and for all $\mathbf{x}, \mathbf{y} \in D$, $\varepsilon \in 4$ with countable D satisfying $\overline{D} = \overline{\partial_a}$ and $\{\mathbf{p_1}(S), \mathbf{p_2}(S) : S \in \mathfrak{S}^a\} \cap D = \emptyset$ ($\{-\infty, +\infty\}$ as endpoints of ∂_a),

1.2.) $(\mathcal{T}(\{1,3\}))_{\mathbf{x},\mathbf{y},\varepsilon}$ is meaningful for all $\mathbf{x}, \mathbf{y} \in D$, $\varepsilon \in 4$ with countable D satisfying $\overline{D} = \overline{\partial_1}$ and $\{\mathbf{p_1}(S), \mathbf{p_2}(S) : S \in \mathfrak{S}^{\{1,3\}}\} \cap D = \emptyset$ ($\{-\infty, +\infty\}$= endpoints of ∂_1), and

1.3.) $(\mathcal{T}(\{2,4\}))_{\mathbf{x},\mathbf{y},\varepsilon}$ is meaningful for all $\mathbf{x}, \mathbf{y} \in D$, $\varepsilon \in 4$ with countable D satisfying $\overline{D} = \overline{\partial_2}$ and $\{\mathbf{p_2}(S), \mathbf{p_3}(S) : S \in \mathfrak{S}^{\{2,4\}}\} \cap D = \emptyset$ ($\{-\infty, +\infty\}$= endpoints of ∂_2).

3.5.2 Strategy for coloring the squares

- In the following informal description of our plan we say that a coloring $d : \mathfrak{Q} \to \omega$ ($\mathfrak{Q} \subseteq \mathfrak{S}$) is *ok in a point* $\mathbf{x} \in (\mathfrak{Q})_\omega$ if $\mathbf{x} \in \cap_{j\in\omega}(d^{-1}(\{j\}))_\omega$.

1.) We choose $D \subseteq \partial$ with later defined properties.

2.) For $a \in (\mathcal{A}_2 \cup \mathcal{A}_1 \setminus \{a \in \mathcal{A}_1 : |a| = 2\})$ choose

$$c^a : \mathcal{T}(a) \to \omega$$

satisfying conditions of Theorem 2.1.5 with $D \cap \overline{\partial_a}$ as dense subset. As by Lemma 3.4.6 $S \cap \partial$ determines S for $S \in \mathfrak{S}^a$, the following definition is meaningful:

$$d^a(S) = c^a(S \cap \partial_a) \textit{ (for } S \in \mathfrak{S}^a).$$

The question that emerges here is that for $a = \{1, 3\}$ (a ={2,4}) if d^a is a ok in points of ∂_1 (∂_2 resp.) then why will it be ok in points of ∂_3 (∂_4 resp.)? We will deal with this question in *Preparing for coloring I.* section.

3.) For $a \in \{a \in \mathcal{A}_1 : |a| = 2\}$ we provide a coloring $d^a : \mathfrak{S}^a \to \omega$ in *Preparing for coloring II.* section.

4.) Let $d^{\{\emptyset\}}(S) = 0$ for $S \in \mathfrak{S}^{\{\emptyset\}}$ and $d = \cup_{a\in\mathcal{A}_1\cup\mathcal{A}_2\cup\{\emptyset\}} d^a : \mathfrak{S} \to \omega$.

5.) Let $e(S) = d(S)$ if $d(S) > 0$ for $S \in \mathfrak{S}$. We provide $e : d^{-1}(\{0\}) \to \omega$ such that $e : \mathfrak{S} \to \omega$ is ok in every points of $(\mathfrak{S})_\omega$. We will deal with 5.) in *Preparing for coloring III.* and *Coloring* sections.

3.6 Preparing for coloring I.

In this section we choose

$$\mathfrak{S}^{a\bullet} \subseteq \mathfrak{S}^{a}$$

for $a \in \mathcal{A}_2$ which is:

$\bullet_1$ regular enough to be able to apply Lemma 2.1.34, (see Lemma 3.6.1 $\bullet_1$),

$\bullet_2$ big enough that the intersections of the squares with the boundary cover ω-fold exactly the points which are covered by intersections of the squares in $\mathfrak{S}^a$. (see Lemma 3.6.1 $\bullet_2$).

Now we start the construction of $\mathfrak{S}^{a\bullet}$:

Lemma 3.6.1.

Let $\{a_1, a_2\} = a \in \mathcal{A}_2$ with $|\mathfrak{S}^a| = \omega$. There is $\mathfrak{S}^{a\bullet} \in [\mathfrak{S}^a]^\omega$ such that if $\{S_n : n \in \omega\}$ is an enumeration of $\mathfrak{S}^{a\bullet}$ then

$\bullet_1$ *the point systems $\{\mathbf{p_{a_1}}(S_n), \mathbf{p_{a_1+1}}(S_n) : n \in \omega\}$, $\{\mathbf{p_{a_2}}(S_n), \mathbf{p_{a_2+1}}(S_n) : n \in \omega\}$ fulfill conditions $(i) - (iv)$ of Lemma 2.1.34 with:*

- $\varphi_1 = f_{1,a}$, $\varphi_2 = f_{2,a}$,
- $a_n = \mathbf{p_{a_1}}(S_n)$, $b_n = \mathbf{p_{a_1+1}}(S_n)$,
- $A_n = \mathbf{p_{a_2}}(S_n)$, $B_n = \mathbf{p_{a_2+1}}(S_n)$.

$\bullet_2$ $(\mathfrak{S}^{a\bullet})_\omega \cap \partial = (\mathfrak{S}^a)_\omega \cap \partial$.

Proof. By symmetry it is enough to prove our statement for $a = \{1, 3\}$, $a_1 = 1$, $a_2 = 3$.

Claim 3.6.2. There are $\subseteq$-maximal elements in every subset of $\{(\mathbf{p_1}(S), \mathbf{p_2}(S)) : S \in \mathfrak{S}^{\{1,3\}}\}$.

Proof of the claim. Observe first that since $a = \{1, 3\}$, if $(\mathbf{p_1}(S_1), \mathbf{p_2}(S_1)) \subseteq (\mathbf{p_1}(S_2), \mathbf{p_2}(S_2))$ for $S_1, S_2 \in \mathfrak{S}^{\{1,3\}}$ then $\mathbf{v_1}(S_1) \in \overline{S_2}$.

We argue by contradiction. If there would be $(\mathbf{p_1}(S_1), \mathbf{p_2}(S_1)) \subseteq (\mathbf{p_1}(S_2), \mathbf{p_2}(S_2)) \subseteq$ ..., then $\mathbf{v_1}(S_1) \in \overline{S_n}$ for all $n \in \omega$, therefore $\mathbf{v_1}(S_1) \in \overline{(\mathfrak{S})_\omega}$. Since $S_1 \in \mathfrak{S}^{\{1,3\}}$ it is a contradiction, hence we are done.

□

Now let

$$\mathfrak{S}^{\{1,3\}\bullet} = \{S \in \mathfrak{S}^{\{1,3\}} : (\mathbf{p_1}(S), \mathbf{p_2}(S)) \textit{ is } \subseteq\textit{-maximal in } \{(\mathbf{p_1}(S), \mathbf{p_2}(S)) : S \in \mathfrak{S}^{\{1,3\}}\}\}.$$

Claim 3.6.3. $(\mathfrak{S}^{\{1,3\}\bullet})_\omega \cap \partial = (\mathfrak{S}^{\{1,3\}})_\omega \cap \partial$.

Proof of the claim.

Assume on the contrary that there is $\mathbf{x} \in ((\mathfrak{S}^{\{1,3\}})_\omega \cap \partial) \setminus ((\mathfrak{S}^{\{1,3\}\bullet})_\omega \cap \partial)$, i.e. $|\{S \in \mathfrak{S}^{\{1,3\}\bullet} : \mathbf{x} \in S\}| < \omega$ and $|\{S \in \mathfrak{S}^{\{1,3\}} : \mathbf{x} \in S\}| = \omega$. By Claim 3.6.2 there is $S \in \mathfrak{S}^{\{1,3\}\bullet}$ with $|\{Q \in \mathfrak{S}^{\{1,3\}} : (\mathbf{p_1}(Q), \mathbf{p_2}(Q)) \subseteq (\mathbf{p_1}(S), \mathbf{p_2}(S))\}| = \omega$. But $(\mathbf{p_1}(Q), \mathbf{p_2}(Q)) \subseteq (\mathbf{p_1}(S), \mathbf{p_2}(S))$ implies $\mathbf{v_3}(S) \in \overline{Q}$ meaning $\mathbf{v_3}(S) \in \overline{(\mathfrak{S})_\omega}$. Which is a contradiction by $S \in \mathfrak{S}^{\{1,3\}}$.

□

So we verified $\bullet_2$ of Lemma 3.6.1.

To continue our proof, let $S_1, S_2 \in \mathfrak{S}^{\{1,3\}\bullet}$. Note that by $T \subseteq S_1, S_2$ and the definition of $\mathfrak{S}^{\{1,3\}\bullet}$, the boundary of S_1 and S_2 intersect in 2 points. Let us denote them by $\mathbf{q_1}$ and $\mathbf{q_2}$. Let $(x_1, y_1) = \mathbf{v_1}(S_1)$ and $(x_2, y_2) = \mathbf{v_1}(S_2)$. We know that $y_1 = y_2$ can not stand by the definition of $\mathfrak{S}^{\{1,3\}\bullet}$ so by symmetry we can assume $y_1 > y_2$. From this $x_1 > x_2$ holds again by the definition of $\mathfrak{S}^{\{1,3\}\bullet}$, so

$$\mathbf{q_1} = [\mathbf{v_1}(S_1), \mathbf{v_4}(S_1)] \cap [\mathbf{v_1}(S_2), \mathbf{v_2}(S_2)] \text{ and } \mathbf{q_2} = [\mathbf{v_3}(S_1), \mathbf{v_4}(S_1)] \cap [\mathbf{v_3}(S_2), \mathbf{v_2}(S_2)].$$

Claim 3.6.4. The following statements are true:

(a) $\mathbf{q_1} \in \overline{(\mathfrak{S})_\omega}$ iff $\mathbf{q_2} \in \overline{(\mathfrak{S})_\omega}$,

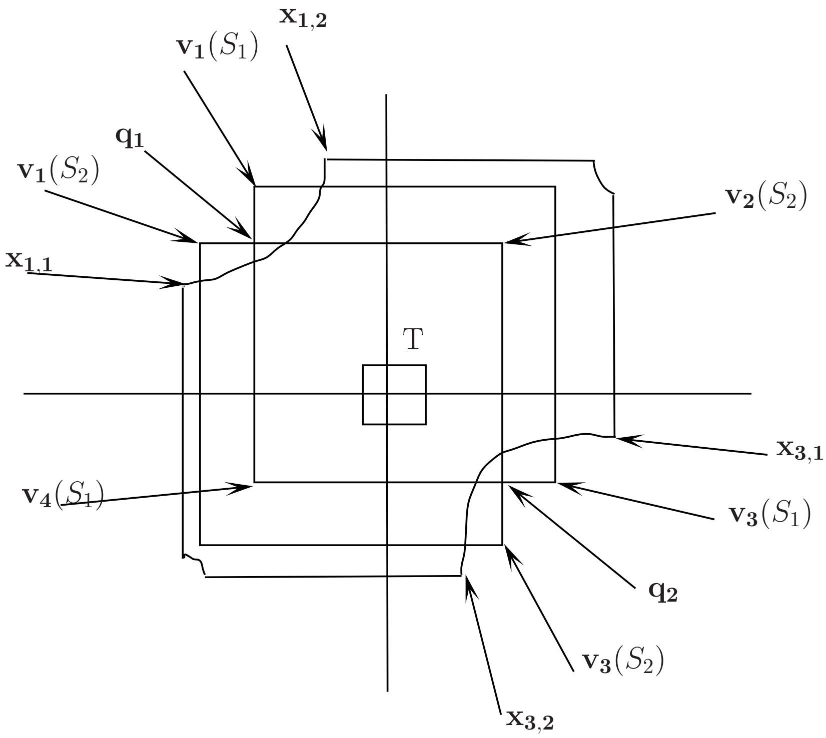

Figure 3.3: The definition of $\mathbf{q_1}$ and $\mathbf{q_2}$

(b) $\mathbf{q_1}, \mathbf{q_2} \notin \partial$.

Proof.

(a): If $\mathbf{q_1} \in \overline{(\mathfrak{S})_\omega}$ then by Lemma 3.2.6 there is $R \in \mathfrak{S}_{lim}$ with $\mathbf{q_1} \in \overline{R}$. Since $\mathbf{v_1}(S_1), \mathbf{v_1}(S_2) \notin \overline{R}$ (as they are $\notin \overline{(\mathfrak{S})_\omega}$), $\mathbf{q_2} \in \overline{R}$ so by Corollary 3.2.5 $\mathbf{q_2} \in \overline{(\mathfrak{S})_\omega}$. And vice versa.

(b): By contradiction. If e.g. $\mathbf{q_1} \in \partial$ then there is $R \in \mathfrak{S}_{lim}$ with $\mathbf{q_1} \in \partial R$. But then either $\mathbf{v_3}(S_1) \in \overline{R}$ or $\mathbf{v_3}(S_2) \in \overline{R}$. Contradiction.

We are done.

□

Claim 3.6.5. For $S_1, S_2 \in \mathfrak{S}^{\{1,3\}\bullet}$ the following statements are equivalent:

(1) $(\mathbf{p_1}(S_1), \mathbf{p_2}(S_1)) \cap (\mathbf{p_1}(S_2), \mathbf{p_2}(S_2)) = \emptyset$,

(2) $(\mathbf{p_3}(S_1), \mathbf{p_4}(S_1)) \cap (\mathbf{p_3}(S_2), \mathbf{p_4}(S_2)) = \emptyset$.

Proof of the claim.

$(1) \Leftrightarrow \mathbf{q_1} \in \overline{(\mathfrak{S})_\omega} \Leftrightarrow \mathbf{q_2} \in \overline{(\mathfrak{S})_\omega} \Leftrightarrow (2)$.

□

- (i) and (ii) of Lemma 2.1.34 is true by the definition of $\mathfrak{S}^{\{1,3\}\bullet}$.
- (iii) is true by Claim 3.6.5.
- (iv) is true by Lemma 3.3.2 (∂ is monotone in each quadrant).

We finished the proof of Lemma 3.6.1.

□

After this lemma we give the definition of the arc systems for $a \in \mathcal{A}_2$:

Definition 3.6.6.

For $a = \{1, 3\}$ let

$$\mathcal{T}(a) = \{\partial(\mathbf{p_1}(S), \mathbf{p_2}(S)) : S \in \mathfrak{S}^{a\bullet}\} \text{ and } \mathcal{T}_2(a) = \{\partial(\mathbf{p_3}(S), \mathbf{p_4}(S)) : S \in \mathfrak{S}^{a\bullet}\},$$

for $a = \{2, 4\}$ let

$$\mathcal{T}(a) = \{\partial(\mathbf{p_2}(S), \mathbf{p_3}(S)) : S \in \mathfrak{S}^{a\bullet}\} \text{ and } \mathcal{T}_2(a) = \{\partial(\mathbf{p_4}(S), \mathbf{p_1}(S)) : S \in \mathfrak{S}^{a\bullet}\}.$$

So as we mentioned in $\bullet_2$ we chose $\mathfrak{S}^{a\bullet}$ to be able to make a connection between the coloring of $\mathcal{T}(a)$ and $\mathcal{T}_2(a)$ for $a \in \mathcal{A}_2$. We state it in the following lemma:

Lemma 3.6.7.

Let $\{a_1, a_2\} = a \in \mathcal{A}_2$ and suppose

$$c : \mathcal{T}(a) \to \omega$$

is satisfying $(\mathcal{T}(a))_\omega \subseteq \cap_{j\in\omega}(c^{-1}(\{j\}))_\omega$.

Let

$$c' : \mathcal{T}_2(a) \to \omega$$

be defined by

$$c'(\partial(\mathbf{p_3}(S), \mathbf{p_4}(S))) = c(\partial(\mathbf{p_1}(S), \mathbf{p_2}(S))) \text{ if } a = \{1,3\},$$

$$c'(\partial(\mathbf{p_4}(S), \mathbf{p_1}(S))) = c(\partial(\mathbf{p_2}(S), \mathbf{p_3}(S))) \text{ if } a = \{2,4\}$$

for $S \in \mathfrak{S}^{a\bullet}$. *Then*

$$(\mathcal{T}_2(a))_\omega \subseteq \cap_{j\in\omega}((c')^{-1}(\{j\}))_\omega.$$

Proof. Pick any $\mathbf{x} \in (\mathcal{T}_2(a))_\omega$. Let $\{S_n : n \in \omega\}$ be an enumeration of $\mathfrak{S}^{a\bullet}$ and let $J = \{n : \mathbf{x} \in S_n\}$. By Lemma 3.6.1 and Lemma 2.1.34 there is $\mathbf{y}(\mathbf{x}) \in \cup\mathcal{T}(a)$, $J \supseteq J'$ with $|J \setminus J'| < 3$ and $\mathbf{y}(\mathbf{x}) \in \bigcap_{n\in J'} S_n \cap \bigcap_{n\in\omega\setminus J}(S_n)^c$, where $(S_n)^c$ means the complement of S_n. So $\mathbf{y}(\mathbf{x}) \in (\mathcal{T}(a))_\omega$. Using the assumption on c we are done with the lemma.

□

3.7 Preparing for coloring II.

Let $a \in \mathcal{A}_1$, $|a| = 2$, the case when two neighboring vertices lie in the complement of $\overline{(\mathfrak{S})_\omega}$. As we mentioned above (in the strategy for coloring the squares subsection), in this section we provide

$$d^a : \mathfrak{S}^a \to \omega$$

coloring. We construct the coloring only for $a = \{1, 2\}$, the other cases are similar.

We will use the notation and results of *Lemma* 3.3.2 in the followings. Let us remind that

$$Y_M = max\{y : (x, y) \in \overline{(\mathfrak{S})_\omega}\},$$

$(0, Y_M) \in \overline{(\mathfrak{S})_\omega}$, $\mathbf{v_1}(S) \in \mathbf{R}_1^2$, $\mathbf{v_2}(S) \in \mathbf{R}_2^2$ and the length of $[\mathbf{x_{1,2}}, \mathbf{x_{2,1}}]$ is at least 1, so

($\bullet$) the x coordinate of both $\mathbf{v_1}(S)$ and $\mathbf{v_2}(S)$ is greater that Y_M for all $S \in \mathfrak{S}^{\{1,2\}}$.

The following lemma states that a limit square of $\mathfrak{S}^{\{1,2\}}$ is very specific:

Lemma 3.7.1. *Let $R \in (\mathfrak{S}^{\{1,2\}})_{lim}$. Then a side of R is part of $[\mathbf{x_{1,2}}, \mathbf{x_{2,1}}]$.*

Proof. By ($\bullet$) the x coordinate of both $\mathbf{v_1}(R)$ and $\mathbf{v_2}(R)$ are greater or equal than Y_M, resulting in $[\mathbf{v_1}(R), \mathbf{v_2}(R)] \subseteq [\mathbf{x_{1,2}}, \mathbf{x_{2,1}}]$.

□

The next statement says that if $\mathbf{x} \in (\mathfrak{S}^{\{1,2\}})_\omega$ then we can find a point $\mathbf{y}(\mathbf{x}) \in [\mathbf{x_{1,2}}, \mathbf{x_{2,1}}]$ satisfying that modulo a finite set the same squares of $\mathfrak{S}^{\{1,2\}}$ contain $\mathbf{x}$ and $\mathbf{y}(\mathbf{x})$.

Lemma 3.7.2. *Let $\mathbf{x} \in (\mathfrak{S}^{\{1,2\}})_\omega$. There is $\mathbf{y}(\mathbf{x}) \in [\mathbf{x_{1,2}}, \mathbf{x_{2,1}}]$ satisfying:*

(1) *if $\mathbf{x} \in S$ then $\mathbf{y}(\mathbf{x}) \in S$ for $S \in \mathfrak{S}^{\{1,2\}}$,*

(2) $|\{S \in \mathfrak{S}^{\{1,2\}} : \mathbf{y}(\mathbf{x}) \in S\} \setminus \{S \in \mathfrak{S}^{\{1,2\}} : \mathbf{x} \in S\}| < \omega$.

Proof. Let $\mathbf{x} = (x_1, x_2)$. Let us define $\mathbf{y}(\mathbf{x})$ as (x_1, Y_M).

Using the fact that each $S \in \mathfrak{S}^{\{1,2\}}$ is axis-parallel, we know that $\mathbf{y}(\mathbf{x}) \in S$ is true for all $S \in \mathfrak{S}^{\{1,2\}}$ with $\mathbf{x} \in S$ by ($\bullet$), proving (1).

To see (2) observe that $|x_2 - Y_M| < 1$ by ($\bullet$). As $\mathbf{y}(\mathbf{x}) \in \partial$ and $S \in \mathfrak{S}^{\{1,2\}}$ are axis-parallel, we are done.

□

Fix $\mathfrak{S}^{\{1,2\}+} \subseteq \mathfrak{S}^{\{1,2\}}$ satisfying:

(1) $\mathcal{T}(\{1,2\}) = \{S \cap \partial : S \in \mathfrak{S}^{\{1,2\}}\} = \{S \cap \partial : S \in \mathfrak{S}^{\{1,2\}+}\}$,

(2) $S_1 \cap \partial \neq S_2 \cap \partial$ for all $S_1 \neq S_2 \in \mathfrak{S}^{\{1,2\}+}$.

Lemma 3.7.3. *Let*

$$c^{\{1,2\}} : \mathcal{T}(\{1,2\}) \to \omega$$

satisfying $(\mathcal{T}(\{1,2\}))_\omega \subseteq \cap_{j\in\omega}((c^{\{1,2\}})^{-1}(\{j\}))_\omega$, *and for* $S \in \mathfrak{S}^{\{1,2\}+}$ *let*

$$d^{\{1,2\}}(S) = c^{\{1,2\}}(S \cap \partial).$$

The following is true for $\mathbf{x} \in int(\mathfrak{S})_\omega$*:*

$$\mathbf{x} \in \cap_{j\in\omega}((d^{\{1,2\}})^{-1}(\{j\}))_\omega \text{ or } \mathbf{x} \notin (\mathfrak{S}^{\{1,2\}+})_\omega.$$

Proof. For $\mathbf{x} \in int(\mathfrak{S})_\omega$ we can choose $\mathbf{y}(\mathbf{x}) \in \partial$ satisfying Lemma 3.7.2. Then we are done by the assumption on $c^{\{1,2\}}$.

□

Let $I \in \mathcal{T}(\{1,2\})$ be satisfying that $S(I) = \{S \in (\mathfrak{S} \setminus \mathfrak{S}^{\{1,2\}+}) : S \cap \partial = I\}$ has cardinality ω. For each such I we color the squares in $S(I)$ the following way:

For $R \in (S(I))_\omega$ a side of R is a subset of $[\mathbf{x_{1,2}}, \mathbf{x_{2,1}}]$ by Lemma 3.7.1, we can choose $\{Q_n : n \in \omega\} \subseteq S(I)$ with the property that the y coordinate of $\mathbf{v_4}(Q_n)$ is not increasing and

$(\{Q_n : n \in \omega\})_\omega = (S(I))_\omega$. Then any $\varphi : \omega \to \omega \times \omega$ bijection with $\varphi(n) = \langle \varphi_1(n), \varphi_2(n) \rangle$ provide a coloring

$$C_I : \{Q_n : n \in \omega\} \to \omega \quad \textit{defined by} \quad C_I(Q_n) = \varphi_2(n),$$

which is easily satisfying $\cap_{j \in \omega}((C_I)^{-1}(\{j\}))_\omega = (\{Q_n : n \in \omega\})_\omega$. Let

$$c_I : S(I) \to \omega$$

be arbitrary extension of C_I, and

$$d^{\{1,2\}}(S) = \begin{cases} c^{\{1,2\}}(S \cap \partial) & \text{where } S \in \mathfrak{S}^{\{1,2\}+} \text{ and } c \text{ is satisfying } \textit{Lemma } 3.7.3, \\ c_I(S) & \text{if } S \in S(I),\ I \in \mathcal{T}(\{1,2\}),\ |S(I)| = \omega, \\ 0 & \text{otherwise.} \end{cases}$$

Then for this coloring, the following is true:

Lemma 3.7.4. $(\mathfrak{S}^{\{1,2\}})_\omega = \cap_{j \in \omega}((d^{\{1,2\}})^{-1}(\{j\}))_\omega$.

Proof. For $\mathbf{x}$ by Lemma 3.7.2 choose $\mathbf{y}(\mathbf{x})$. Then either there is $I \in \mathcal{T}(\{1,2\})$ with $\mathbf{y}(\mathbf{x}) \in (S(I))_\omega$ or $\mathbf{y}(\mathbf{x}) \in (\mathcal{T}(\{1,2\}))_\omega$. In both cases we are easily done by the assumption on $\mathbf{y}(\mathbf{x})$. $\square$

3.8 Preparing for coloring III.

For $\mathbf{x} \in int(\mathfrak{S})_\omega$ and $a \in (\mathcal{A}_2 \cup \mathcal{A}_1 \setminus \{a \in \mathcal{A}_1 : |a| = 2\})$ let

$$\mathfrak{S}^a_{\mathbf{x}} = \begin{cases} \{S \in \mathfrak{S}^a : \mathbf{x} \in S\} & \text{if } a \in \mathcal{A}_1 \setminus \{a \in \mathcal{A}_1 : |a| = 2\}, \\ \{S \in \mathfrak{S}^{a\bullet} : \mathbf{x} \in S\} & \text{if } a \in \mathcal{A}_2. \end{cases}$$

In this section our main aim is to give a characterization for the 'trace' of $\mathfrak{S}^a_{\mathbf{x}}$ on ∂_a in the case of $a \in \mathcal{A}_1 \setminus \{a \in \mathcal{A}_1 : |a| = 2\}$, on ∂_1 in the case of $a = \{1, 3\}$ and on ∂_2 in the case of $a = \{2, 4\}$.

For the sake of simplicity we deal with the case of $a = \{1\}$, the other cases are similar:

Theorem 3.8.1. *Let $\mathbf{x} \in int(\mathfrak{S})_\omega$. There are*

$$\mathbf{p}(\mathbf{x}, \{1\}), \mathbf{q}(\mathbf{x}, \{1\}) \in \partial_1 \textit{ and } \varepsilon(\mathbf{x}, \{1\}) \in 4 \textit{ with:}$$

$$\{S \cap \partial_{\{1\}} : S \in \mathfrak{S}^{\{1\}}_{\mathbf{x}}\} = (\{S \cap \partial_{\{1\}} : S \in \mathfrak{S}^{\{1\}}\})_{\mathbf{p}(\mathbf{x},\{1\}),\mathbf{q}(\mathbf{x},\{1\}),\varepsilon(\mathbf{x},\{1\})}.$$

Proof. By symmetry it is enough to prove the theorem for $\mathbf{x} \in int(\mathfrak{S})_\omega \cap \mathbf{R}^2_3$.

We prove the statement in 3 steps:

Step1:

Observe that by $T \subseteq S$ for $S \in \mathfrak{S}$ and by the definition of $\mathbf{v_4}(S)$, the following fact $(\diamond_1)$ is true:

$(\diamond_1)$ For any $(x_1, x_2) = \mathbf{x} \in int(\mathfrak{S})_\omega \cap \mathbf{R}^2_3$ and $(v, w) = \mathbf{v_4}(S)$ $(S \in \mathfrak{S})$ the followings are equivalent:

(1) $\mathbf{x} \in S$,

(2) $v < x_1$ and $x_2 - 1 < w$.

Step2:

We prove a lemma about ∂_1, which actually gives the translation of inequalities in $\diamond_1(2)$ to inequalities on $(\partial_{\{1\}}, <_{\{1\}})$. We prove using the monotonicity of ∂_1 that if we choose an arbitrary $p \in \mathbf{R}$ and we are interested in the points of ∂_1 whose first (or second) coordinate is less (or bigger) than p, then there is $\mathbf{x}(p,i,I)$ (see the notation of Lemma 3.8.2) such that this subset is exactly the '$<_{\{1\}} \mathbf{x}(p,i,I)$' (or $>_{\{1\}} \mathbf{x}(p,i,I)$) subset of ∂_1.

Lemma 3.8.2. *The followings are true:*

(1) *For all* $p \in \mathbf{R}$, $i = 1,2$ *and* $I \in \{<,>\}$ *there is* $\mathbf{x}(p,i,I) \in \overline{\partial_1}$ *and* $I_{\{1\}} \in \{<_{\{1\}}, >_{\{1\}}\}$ *with:*

$$\{\mathbf{x} \in \partial_1 : \mathbf{x}(p,i,I)\ I_{\{1\}}\ \mathbf{x}\} = \{(x_1,x_2) = \mathbf{x} \in \partial_1 : p\ I\ x_i\}.$$

Proof. We know that ∂_1 is monotone increasing, and w.l.o.g. we can assume that $i = 1$ and I is $<$. Let $\partial_1 \cap (x\text{-}axis) = \mathbf{f_0} = (f_0^0, f_0^1)$ and $\partial_1 \cap (y\text{-}axis) = \mathbf{f_1} = (f_1^0, f_1^1)$

- If $\{x : (p,x) \in \partial_1\} \neq \emptyset$ then let $\mathbf{x}(p) = (p,y)$, where $y = max\{x : (p,x) \in \partial_1\}$. The maximum exists by compactness. The lemma follows by the fact that ∂_1 is monotone increasing.

- If $\{x : (p,x) \in \partial_1\} = \emptyset$ then let $\mathbf{x}(p,i,I) = \mathbf{f_0}$ if $p \leq f_1^0$ and let $\mathbf{x}(p,i,I) = \mathbf{f_1}$ if $f_1^1 \leq p$.

□

Step3:

As a last step we would like to 'translate' the inequalities in $\diamond_1(2)$ to inequalities on $(\partial_{\{1\}}, <_{\{1\}})$ for $\mathbf{p_1}(S)$ and $\mathbf{p_2}(S)$ using Lemma 3.8.2. Note that $\mathbf{p_1}(S), \mathbf{p_2}(S) \in \partial_1$ since $S \in \mathfrak{S}^{\{1,2\}}$ and $\mathbf{p_1}(S)$ and $\mathbf{p_2}(S)$ are lying inside neighboring sides of S, so $\diamond_1(2)$ gives an inequality for a coordinate of both $\mathbf{p_1}(S)$ and $\mathbf{p_2}(S)$. (actually for different coordinates.) So by Lemma 3.8.2 we are done with Theorem 3.8.1.

□

In the sequel we would like to construct a <u>countable dense set</u> $D \subseteq \partial$ such that the followings are true:

- $\bullet_1$ D is disjoint from the set of endpoints of the intersections of the squares with the boundary (see Lemma 3.8.3 (1)),
- $\bullet_2$ let $a \in \mathcal{A}_1 \setminus \{a \in \mathcal{A}_1 : |a| = 2\}$ and $\mathbf{x} \in R$ with $R \in (\mathfrak{S}^a)_{lim}$. Then there are $\mathbf{s}(\mathbf{x}, a), \mathbf{t}(\mathbf{x}, a) \in D$ and $\varepsilon(\mathbf{x}, a) \in 4$ satisfying:
  - $\bullet_{2.1}$ if $S_1 \in \mathfrak{S}^a$ and $S_1 \cap \partial_a \in (\{S \cap \partial_a : S \in \mathfrak{S}^a\})_{\mathbf{s}(\mathbf{x},a),\mathbf{t}(\mathbf{x},a),\varepsilon(\mathbf{x},a)}$ then $\mathbf{x} \in S_1$,
  - $\bullet_{2.2}$ $|(\{S \cap \partial_a : S \in \mathfrak{S}^a\})_{\mathbf{s(x,a)},\mathbf{t(x,a)},\varepsilon(\mathbf{x},\mathbf{a})}| = \omega$, (see Lemma 3.8.3 (2)).
- $\bullet_3$ let $a \in \mathcal{A}_2$ and $\mathbf{x} \in R$ with $R \in (\mathfrak{S}^a)_{lim}$. Then exactly the same holds as in $\bullet_{2.1}$ and $\bullet_{2.2}$, but for ∂_1 instead of ∂_a if $a = \{1, 3\}$ and for ∂_2 instead of ∂_a if $a = \{2, 4\}$, (see Lemma 3.8.3 (3)).

Lemma 3.8.3. *We can choose $D \subseteq \partial$ countable dense satisfying:*

(1) $D \cap \{\mathbf{p_1}(S), \mathbf{p_2}(S) : S \in \mathfrak{S}\} = \emptyset$,

(2) *for all $a \in \mathcal{A}_1 \setminus \{a \in \mathcal{A}_1 : |a| = 2\}$ and $\mathbf{x} \in R$ with $R \in (\mathfrak{S}^a)_{lim}$ there are* $\mathbf{s}(\mathbf{x}, a), \mathbf{t}(\mathbf{x}, a) \in D$ *and* $\varepsilon(\mathbf{x}, a) \in 4$ *with:*

(2.1) *if $S_1 \cap \partial_a \in (\{S \cap \partial_a : S \in \mathfrak{S}^a\})_{\mathbf{s}(\mathbf{x},a),\mathbf{t}(\mathbf{x},a),\varepsilon(\mathbf{x},a)}$ for $S_1 \in \mathfrak{S}^a$ then $\mathbf{x} \in S_1$,*

(2.2) $|(\{S \cap \partial_a : S \in \mathfrak{S}^a\})_{\mathbf{s}(\mathbf{x},a),\mathbf{t}(\mathbf{x},a),\varepsilon(\mathbf{x},a)}| = \omega$.

(3) *For $a \in \mathcal{A}_2$ exactly the same holds as in* (3)*, but for ∂_1 instead of ∂_a if $a = \{1, 3\}$ and for ∂_2 instead of ∂_a if $a = \{2, 4\}$.*

Proof. For $E \subseteq int(\mathfrak{S})_\omega$ countable dense and let $E = \{\mathbf{e_j} = (e^0_j, e^1_j) : j \in \omega\}$ be arbitrary enumeration of E. Using the notation of Lemma 3.8.2 let $\partial_{E,1}$ be the point set $\{\mathbf{x}(p, i, I) : p \in \{e^k_l, e^k_l + 1, e^k_l - 1 : k \in 2, l \in \omega\}, i \in 2, j \in \omega, I \in \{<, >\}\} \subseteq \partial_1$. Define $\partial_{E,2}$, $\partial_{E,3}$, $\partial_{E,4}$ similarly and let ∂_E be the union of them.

Choose $E \subseteq int(\mathfrak{S})_\omega$ countable dense and $\partial_E \subseteq D$ countable dense (in ∂) with

$$\{\mathbf{p_1}(S), \mathbf{p_2}(S) : S \in \mathfrak{S}\} \cap D = \emptyset.$$

As $\mathfrak{S}$ is a countable set, and from the construction given in Lemma 3.8.2 we know that either the *ith* coordinate of $\mathbf{x}(p,i,I)$ is $p-1$, p or $p+1$, or $\mathbf{x}(p,i,I) \in \partial \cap (x\text{-}axis \cup y\text{-}axis)$, we can choose E and then D with the desired properties. Now we want to prove that D satisfies (1) – (3) of Lemma 3.8.3.

- (1) is trivially satisfied with this definition.
- By symmetry it is enough to prove (2) and (3) for $a = \{1\}$ and $\mathbf{x} \in int(\mathfrak{S})_\omega \cap \mathbf{R}^2_3$.

Let δ_1, δ_2 be less than the distance of $\mathbf{x}$ and ∂R with $(x_1 + \delta_1, x_2 - \delta_2) = \mathbf{x}' \in E$. We can choose such δ_1, δ_2 by the fact that E is dense. Then (2.1) is true by the definition of $\mathbf{x}'$ and (2.2) is true by the fact that $\mathbf{x}' \in R \in (\mathfrak{S}^{\{1\}})_{lim}$.

We are done with Lemma 3.8.3.

□

3.8.1 A coloring lemma

Lemma 3.8.4. *Let $D \subseteq \partial$ satisfying Lemma 3.8.3* (1), (2) *and* (3). *Then the following is true:*

For $a \in (\mathcal{A}_2 \cup \mathcal{A}_1 \setminus \{a \in \mathcal{A}_1 : |a| = 2\})$ let

$$c^a : \mathcal{T}(a) \to \omega$$

satisfying that for all $\mathbf{x}, \mathbf{y} \in D \cap \partial_a$ and $\varepsilon \in 4$:

(i) $((\mathcal{T}(a))_{\mathbf{x},\mathbf{y},\varepsilon})_\omega \subseteq \cap_{j\in\omega}((c^a)^{-1}(\{j\}))_\omega$, *and*

(ii) $|(\mathcal{T}(a))_{\mathbf{x},\mathbf{y},\varepsilon}| = |(\mathcal{T}(a))_{\mathbf{x},\mathbf{y},\varepsilon} \cap c^{-1}(\{0\})|$.

Let

$$d^a : \mathfrak{S}^a \to \omega$$

be defined by $d^a(S) = c^a(S \cap \partial_a)$ for $S \in \mathfrak{S}^a$.

Then for all $\mathbf{x} \in R$ *with* $R \in (\mathfrak{S}^a)_{lim}$ *the following is true:*

$$\mathbf{x} \in ((d^a)^{-1}(\{0\}))_\omega.$$

Proof. By Lemma 3.8.3.

□

3.9 The coloring

Lemma 3.9.1. *There exists* $c : \mathfrak{S} \to \omega$ *satisfying:*

(*i*) $\partial \cap (\mathfrak{S})_\omega \subseteq \bigcap_{j>0}(c^{-1}(\{j\}))_\omega \cap \partial$, *and*

(*ii*) $int(\mathfrak{S})_\omega \subseteq (c^{-1}(\{0\}))_\omega$.

Proof. Let $D \subseteq \partial$ be dense satisfying the condition and (1), (2) and (3) of Lemma 3.8.3.

By *Theorem* 2.1.5 we can choose:

$\bullet_1$ $c^a : \mathcal{T}(a) \to \omega$ satisfying Lemma 3.8.4 (1) for $a \in (\mathcal{A}_2 \cup \mathcal{A}_1 \setminus \{a \in \mathcal{A}_1 : |a| \neq 2\})$,

$\bullet_2$ $d^a : \mathfrak{S}^a \to \omega$ satisfying Lemma 3.7.4 for $a \in \mathcal{A}_1, |a| = 2$.

Let $c : \mathfrak{S} \to \omega$ be defined in the following way:

$$c(S) = \begin{cases} c^a(S \cap \partial_a) & \text{if } a \in (\mathcal{A}_2 \cup \mathcal{A}_1 \setminus \{a \in \mathcal{A}_1 : |a| \neq 2\}), \\ d^a(S) & \text{if } a \in \mathcal{A}_1, |a| = 2, \\ 0 & \text{otherwise .} \end{cases}$$

The next table verifies (*i*) and (*ii*) of the lemma:

a	$\mathbf{x} \in \partial \cap (\mathfrak{S})_\omega$	$\mathbf{x} \in int(\mathfrak{S})_\omega$
$a \in (\mathcal{A}_1 \setminus \{a \in \mathcal{A}_1 : \lvert a\rvert \neq 2\})$	by the definition of c	Lemma 3.8.4 (1)
$a \in \mathcal{A}_2$	by the definition of c and Lemma 3.6.7	Lemma 3.8.4 (2)
$a \in \mathcal{A}_1, \lvert a\rvert = 2$	Lemma 3.7.4	Lemma 3.7.4

□

Theorem 3.9.2. *There exists* $c : \mathfrak{S} \to \omega$ *such that* $(\mathfrak{S})_\omega = \cap_{j\in\omega}(c^{-1}(\{j\}))_\omega$.

Proof. There exists $d : \mathfrak{S} \to \omega$ satisfying (i) and (ii) of Lemma 3.9.1.

So

$$\partial \cap (\mathfrak{S})_\omega = \cap_{j>1}(d^{-1}(\{j\}))_\omega \ \textit{and}\ int(\mathfrak{S})_\omega = int(d^{-1}(0) \cup d^{-1}(\{1\}))_\omega$$

by (ii) of Lemma 3.9.1.

Let

$$\mathfrak{S}_1 = \{int(\mathfrak{S})_\omega \cap R : R \in \mathfrak{S} \setminus \cup_{j>1} d^{-1}(j)\}.$$

By the previous remark and the fact that an open subset of the plane is a σ-compact space by Lemma 1.2.1 there exists $d_1 : \mathfrak{S}_1 \to \omega$ with $int(\mathfrak{S})_\omega = \cap_{j\in\omega}(d_1^{-1}(j))_\omega$.

Let

$$c(R) = \begin{cases} d_1(R \cap int(\mathfrak{S})_\omega) & \text{if } R \in (d^{-1}(0) \cup d^{-1}(1)) \\ d(R) - 2 & \text{if } R \in \cup_{j>1} d^{-1}(j) \end{cases}$$

which fulfills the requirements of the theorem.

□

Theorem 3.9.3. *Let C be the open unit square and let $\mathfrak{S} \in [\mathcal{T}_C]^\omega$. Then there exists $c : \mathfrak{S} \to \omega$ with $(\mathfrak{S})_\omega \subseteq \cap_{j\in\omega}(c^{-1}(\{j\}))_\omega$.*

Proof. Consider a grid with distance 1/2 and put an open square with length of side 1/10 onto each point of the grid. Let $\{Q_i : i \in \omega\}$ be the set of these small squares, let $\mathfrak{S}_i' = \{S \in \mathfrak{S} : Q_i \subseteq S\}$, and consider $\mathfrak{S}_i \subseteq \mathfrak{S}_i'$ which are disjoint, and $\cup_{i\in\omega}\mathfrak{S}_i = \cup_{i\in\omega}\mathfrak{S}_i'$. By elementary geometry the following are true:

(1) for all $S \in \mathfrak{S}$ there is $i \in \omega$ such that $S \in \mathfrak{S}_i$.

(2) for all $x \in (\mathfrak{S})_\omega$ there is $i \in \omega$ such that $x \in (\mathfrak{S}_i)_\omega$.

By Theorem 3.9.2 there is $c_i : \mathfrak{S}_i \to \omega$ with $(\mathfrak{S}_i)_\omega = \cap_{j\in\omega}(c_i^{-1}(\{j\}))_\omega$ for all $i \in \omega$.

By (1) and (2) $c = \cup_{i\in\omega} c_i$ will satisfy the requirement of the theorem.

□

4 The closed square case

4.1 Back to the open case

Theorem 4.1.1. *Let C be the closed unit square and let $\mathfrak{C} \in [\mathcal{T}_C]^\omega$. There exists $c : \mathfrak{C} \to \omega$ with*

$$(\mathfrak{C})_\omega = \cap_{j\in\omega}(c^{-1}(\{j\}))_\omega.$$

Proof. We will denote by U the elements of $\mathfrak{C}$, for $U \in \mathfrak{C}$ let us denote by $v(U)$ the set of vertices of U and let $U^{-v} = U \setminus v(U)$. For $\mathfrak{D} \in [\mathcal{T}_C]^\omega$ let $\mathfrak{D}^{-v} = \{U^{-v} : U \in \mathfrak{D}\}$ and $int(\mathfrak{D}) = \{int(U) : U \in \mathfrak{D}\}$. First note that $(\mathfrak{D}^{-v})_\omega = (\mathfrak{D})_\omega$ for $\mathfrak{D} \in [\mathcal{T}_C]^\omega$, since the multiplicity of each translate is 1.

- Let

$$X = (\mathfrak{C}^{-v})_\omega \setminus (int(\mathfrak{C}))_\omega.$$

For $x \in X$ pick $l(x)$, a line with $x \in l(x)$ and $\mathcal{U}(x) \in [\mathfrak{C}^{-v}]^\omega$ with $x \in \cap(\mathcal{U}(x)) \setminus int(\mathcal{U}(x))$ and with the property that the side of the elements of $\mathcal{U}(x)$ which contains x is subset of $l(x)$. Let $L = \{l(x) : x \in X\}$ and write $L = \{l_u : u \in |L|\}$ (we know that $|L| \leq \omega$ as $|\mathfrak{C}| = \omega$).

- First we give a partition $\{L_i : i \in \omega\}$ of L in the following way: for $i \in \omega$

1.) if $L \setminus \cup_{k<i} L_k$ is not empty then let

$$j(i) = min\{j \in |L| : l_j \in L \setminus \cup_{k<i} L_k\},$$

$\overrightarrow{\mathbf{e}}_i$ be a vector with length of 1 and transversal to $l_{j(i)}$, and

$$L_i = \{l_{j(i)} + n\overrightarrow{\mathbf{e}}_i : n \in \mathbf{Z}\} \cap L.$$

2.) if $L \setminus \cup_{k<i} L_k$ is empty then we do not define $j(i)$ and $\overrightarrow{\mathbf{e}}_i$, and let L_i be the emptyset.

• Then let us denote by

$$[l_{j(i)} + n\overrightarrow{\mathbf{e}}_i, l_{j(i)} + (n+1)\overrightarrow{\mathbf{e}}_i]$$

the closed *strip*, whose boundary consists of the two lines: $l_{j(i)} + n\overrightarrow{\mathbf{e}}_i$ and $l_{j(i)} + (n+1)\overrightarrow{\mathbf{e}}_i$.

For $n \in \mathbf{Z}$ and $i \in \omega$ let

$$\mathfrak{C}_i^n = \cup\{\mathcal{U}(x) \setminus \cup_{m\in\omega, j<i} \mathfrak{C}_j^m : V \subseteq [l_{j(i)} + n\overrightarrow{\mathbf{e}}_i, l_{j(i)} + (n+1)\overrightarrow{\mathbf{e}}_i] \text{ for all } V \in \mathcal{U}(x), x \in X\}, \text{ and}$$

$$tr_i^n = \{V \cap (l_{j(i)} + n\overrightarrow{\mathbf{e}}_i) : V \in \mathfrak{C}_i^n\}.$$

By applying Corollary 2.1.17 on each components of $\cup tr_i^n$, we know that there exists a partition

$$\cup^*_{s\in\omega} tr_{i,s}^n = tr_i^n$$

with:

(1) $(tr_{i,u}^n)_\omega = (tr_i^n)_\omega$ for $u \geq 1$, and

(2) $\cup tr_{i,0}^n = \cup tr_i^n$.

For $s \in \omega$ let

$$\mathfrak{C}_{i,s}^n = \{V \in \mathfrak{C}_i^n : V \cap (l_{j(i)} + n\overrightarrow{\mathbf{e}}_i) \in tr_{i,s}^n\} \text{ and } \mathfrak{C}_{i,s} = \cup_{n\in\mathbf{Z}} \mathfrak{C}_{i,s}^n.$$

• Now we give a coloring $c : \mathfrak{C} \to \omega$ and prove that it satisfies Theorem 4.1.1.

By Theorem 3.9.3 we know that there exists

$$d : int(\mathfrak{C}^{-v} \setminus \cup_{i\in\omega,s\geq 1}\mathfrak{C}_{i,s}) \to \omega$$

with

$$(int(\mathfrak{C}^{-v} \setminus \cup_{i\in\omega,s\geq 1}\mathfrak{C}_{i,s}))_\omega = \cap_{v\in\omega}(d^{-1}(\{v\}))_\omega.$$

Let us define $c : \mathfrak{C} \to \omega$ the following way:

$$c(U) = \begin{cases} s-1 & \text{if } U^{-v} \in \mathfrak{C}_{i,s} \text{ for some } i \in \omega, s \geq 1 \\ d(int(U)) & \text{if } U^{-v} \in \mathfrak{C}^{-v} \setminus \cup_{i\in\omega,s\geq 1}\mathfrak{C}^{-v}_{i,s} \end{cases}$$

We prove that this coloring satisfies the requirement of Theorem 4.1.1.

Claim 4.1.2. One of the following two statements is true for all $x \in (\mathfrak{C})_\omega$:

(a) there exists $i \in \omega$ with $x \in (\mathfrak{C}_{i,s})_\omega$ for all $s \geq 1$,

(b) $x \in (int(\mathfrak{C} \setminus \cup_{i\in\omega,s\geq 1}\mathfrak{C}_{i,s}))_\omega$.

Proof of the claim.

$x \in (\mathfrak{C})_\omega$, so by the definition of $\mathfrak{C}_{i,s}$ either there is $i \in \omega$ with $x \in (\cup_{s\in\omega}\mathfrak{C}_{i,s})_\omega$ for all $s \geq 1$ or $x \in (\mathfrak{C} \setminus \cup_{i\in\omega,s\geq 1}\mathfrak{C}_{i,s})_\omega$.

Case 1: if there is $i \in \omega$ with $x \in (\cup_{s\in\omega}\mathfrak{C}_{i,s})_\omega$ for all $s \geq 1$, then using that for all $x \in X$ and $i \in \omega$ there are at most 2 different superscripts $n, m \in \omega$ with $x \in \cup_{s\in\omega}\mathfrak{C}^n_{i,s}$ and $x \in \cup_{s\in\omega}\mathfrak{C}^m_{i,s}$, we have that in this case there are $i, n \in \omega$ with $x \in (\cup_{s\in\omega}\mathfrak{C}^n_{i,s})_\omega$. As we know by the definition of $\mathfrak{C}^n_{i,s}$ and by requirement (2) on $tr^n_{i,s}$ that $(\cup_{s\in\omega}\mathfrak{C}^n_{i,s})_\omega = (\mathfrak{C}^n_{i,s})_\omega$ for all $s \geq 1$, (a) is true in this case.

Case 2: Suppose $x \in (\mathfrak{C} \setminus \cup_{i\in\omega,s\geq 1}\mathfrak{C}_{i,s})_\omega$ and (a) is not true. We prove that (b) holds. Suppose by contradiction that $x \in (\mathfrak{C} \setminus \cup_{i\in\omega,s\geq 1}\mathfrak{C}_{i,s})_\omega \setminus (int(\mathfrak{C} \setminus \cup_{i\in|L|,s\geq 1}\mathfrak{C}_{i,s}))_\omega$. Then using the fact that the elements of $\mathfrak{C}$ are axis-parallel, there is $\mathcal{U}(x) \in [\mathfrak{C} \setminus \cup_{i\in\omega,s\geq 1}\mathfrak{C}_{i,s}]^\omega$ and $l_i \in L$

with the property that the side of the elements of $\mathcal{U}(x)$ which contains x is subset of l_i. This contradicts to the fact that (a) is not true.

So we proved the claim.

□

By Claim 4.1.2 we are done by the definition of c.

□

5 Constructions

In this section we describe constructions, showing the sharpness of Theorem 1.2.3 and Theorem 1.2.4.

Each construction works similarly:

- first we describe an elementary statement, then
- using the elementary statement we construct $A \subseteq \mathbf{R}^2$ in ω steps and a covering of A, which can not be decomposed.

5.1 Axis-parallel rectangles with side length between $1-\varepsilon$ and 1

Before the first construction (Theorem 1.2.5) let us mention a construction from [2]:

Theorem 5.1.1. *([2], Theorem 7.2)*

There exists $\mathcal{R}$, a countable family of axis-parallel closed rectangles with:

(i) $(\mathcal{R})_\omega \supseteq \mathbf{R}^2$, and

(ii) $\mathcal{R}$ can not be decomposed into two

(i.e. for any partition $\mathcal{R}_1 \cup^* \mathcal{R}_2 = \mathcal{R}$ either $\cup\mathcal{R}_1 \not\supseteq \mathbf{R}^2$ or $\cup\mathcal{R}_2 \not\supseteq \mathbf{R}^2$).

$\bullet_1$ In (Theorem 1.2.5) we construct families similar to $\mathcal{R}$ satisfying (i) and (ii) of Theorem 5.1.1 with the minor strengthening that the length of the sides of elements of $\mathcal{R}$ are between $1-\varepsilon$ and 1.

$\bullet_2$ Note that we can not expect a variant of Theorem 5.1.1 with open rectangles instead of closed ones, since $\mathbf{R}^2$ is σ-compact. However we prove that there is $A \subseteq \mathbf{R}^2$ and $\mathcal{Q}$, a countable set of axis-parallel open rectangles with:

(i) $(\mathcal{Q})_\omega \supseteq A$, and

(ii) for any partition $\mathcal{Q}_1 \cup^* \mathcal{Q}_2 = \mathcal{Q}$ either $\cup\mathcal{Q}_1 \not\supseteq A$ or $\cup\mathcal{Q}_2 \not\supseteq A$.

We start to prepare for the proof of Theorem 1.2.5.

Let l be the $x = -y$ line and $\overrightarrow{\mathbf{v}}$ be the vector from the origin to $(1,1)$, and for $A \subseteq \mathbf{R}^2$, $\lambda \in \mathbf{R}$ let $A + \lambda\overrightarrow{\mathbf{v}}$ be the translation of A with $\lambda\overrightarrow{\mathbf{v}}$.

We will use the following elementary geometric statement repeatedly:

Lemma 5.1.2. *For all $\varepsilon > 0$ there are ε_1 and ε_2 with the following property:*

For all $I \subseteq l$ interval with $|I| < \varepsilon_1$ and for all $I_1 \subseteq I$, $I_2 \subseteq I + (1-\varepsilon_2)\overrightarrow{\mathbf{v}}$ closed (open) intervals there is $R \in \mathcal{R}_\varepsilon$ ($R \in \mathcal{Q}_\varepsilon$) with

$$R \cap (l \cup (l + (1-\varepsilon_2)\overrightarrow{\mathbf{v}})) = I_1 \cup I_2.$$

Proof. The proof is immediate by Figure 5.1 and left to the reader.

□

Proof of Theorem 1.2.5. We construct A and $\mathcal{R}$ satisfying (1). (In a similar way one can construct B and $\mathcal{Q}$ for (2).)

1.) Fix $\varepsilon_1, \varepsilon_2$ for ε satisfying Lemma 5.1.2.

2.) Let $I \subseteq l$ with $|I| < \varepsilon_1$ arbitrary and let $I_0 = I$, $J_0 = I + (1-\varepsilon_2)\overrightarrow{\mathbf{v}}$.

3.) Let $\omega_0^{<\omega} = \{\langle s_0, s_1, ..., s_j\rangle \in \omega^{<\omega} : s_0 = 0\}$.

4.) For $j \geq 1$ and $s = \langle s_0, s_1, ..., s_j\rangle \in \omega_0^{<\omega}$ let $s^- = \langle s_0, s_1, ..., s_{j-1}\rangle$ and $|s| = j$.

In the *jth step* ($j \geq 1$) for all $s \in \omega_0^{<\omega}$ with $|s| = j$ choose (see Figure 5.2) :

$\bullet_1$ $I_s \subseteq (I_0 \setminus \cup_{1\leq|s'|\leq j, s'\neq s} I_{s'})$ closed intervals with with $\sum_{1\leq|s|\leq j} |I_s| < \frac{|I_0|}{2}$, and

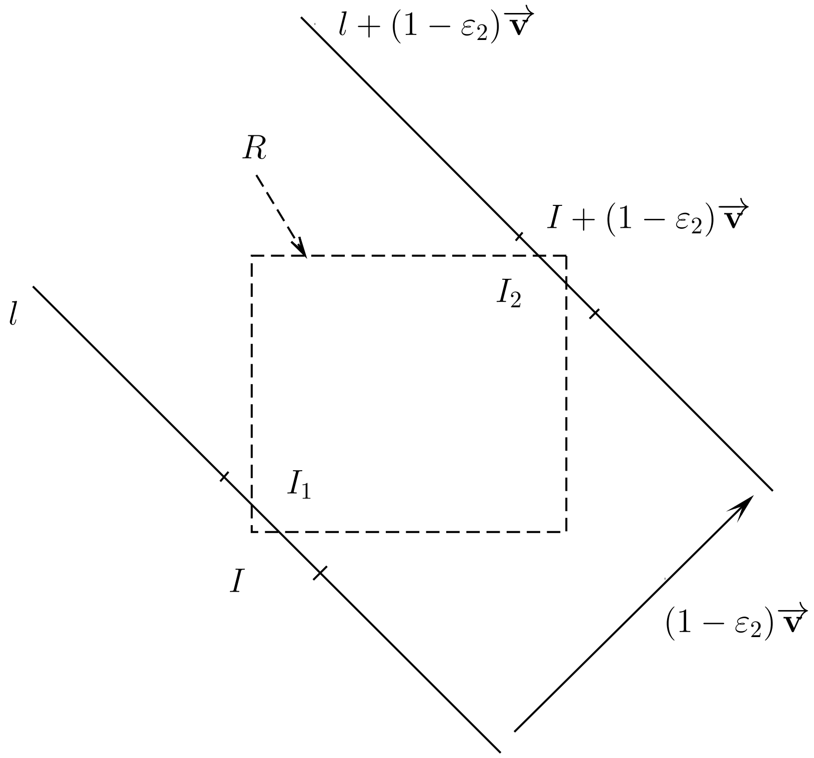

Figure 5.1: Lemma 5.1.2

- $\bullet_2$ $J_s \subseteq J_{s^-}$ closed intervals with $J_{\langle s^-,i\rangle} \cap J_{\langle s^-,j\rangle} = \emptyset$ for all $i \neq j \in \omega$.

In the *jth step* ($j \geq 2$) for all $s \in \omega_0^{<\omega}$ with $|s| = j$ choose:

- $\bullet_3$ choose $R_s \in \mathcal{R}_\varepsilon$ with $R_s \cap (l \cup (l + (1-\varepsilon_2)\overrightarrow{\mathbf{v}})) = I_{s^-} \cup J_s$ by Lemma 5.1.2.

And finally let

- $\bullet_4$ $A = \cap_{j\geq 2} \cup_{|s|=j} J_s \bigcup \cup_{|s|\geq 1} I_s$ and $\mathcal{R} = \{R_s : s \in \omega_0^{<\omega}, |s| \geq 2\}$.

First we prove that A and $\mathcal{R}$ satisfies (1.1):

Claim 5.1.3. $A \subseteq (\mathcal{R})_\omega$.

Proof.

- $I_s \subseteq R_t$ if $t = \langle s, i\rangle$ for $i \in \omega$.
- for $\mathbf{x} \in \cap_{j\geq 2} \cup_{|s|=j} J_s$ one can choose $\langle t_0(\mathbf{x}), t_1(\mathbf{x}), ...\rangle \in \omega^\omega$ with $\mathbf{x} \in J_{\langle t_0(\mathbf{x}), t_1(\mathbf{x}), ..., t_j(\mathbf{x})\rangle}$

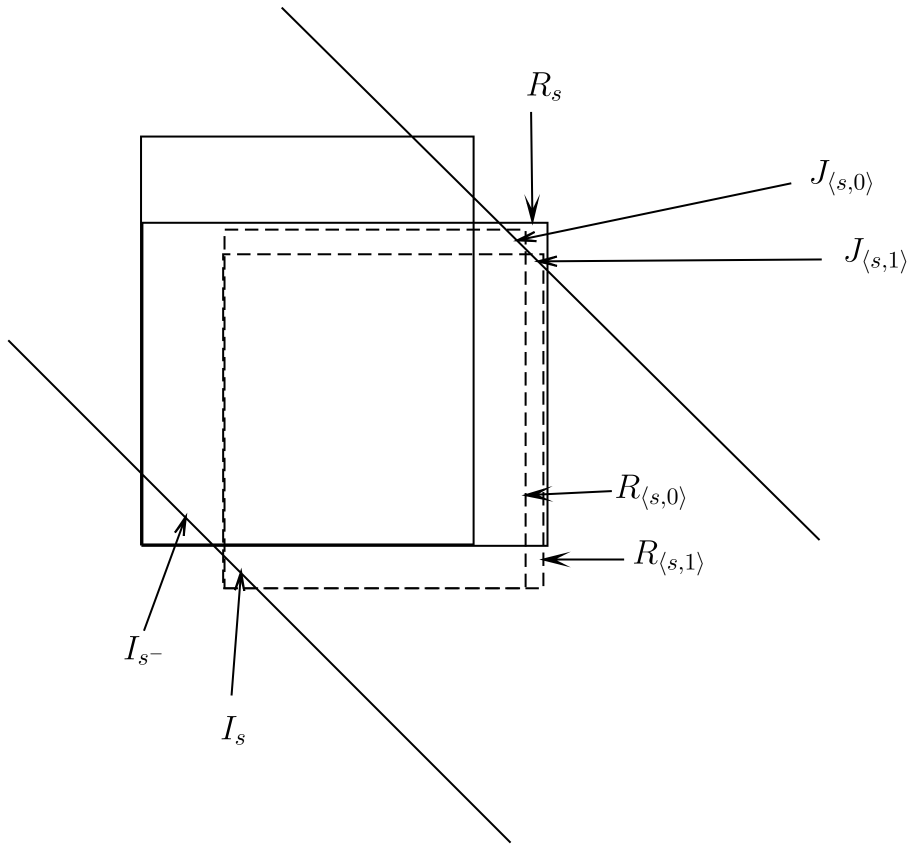

Figure 5.2: The jth step

for all $j \geq 1$ meaning $\mathbf{x} \in R_{\langle t_0(\mathbf{x}), t_1(\mathbf{x}), \ldots, t_j(\mathbf{x})\rangle}$ by $\bullet_3$.

□

Now we want to prove that A and $\mathcal{R}$ satisfies (1.2):

Claim 5.1.4. For all partition $\mathcal{R}_1 \cup^* \mathcal{R}_2 = \mathcal{R}$ either $\cup\mathcal{R}_1 \not\supseteq A$ or $\cup\mathcal{R}_2 \not\supseteq A$.

Proof. By $\bullet_1$ and $\bullet_{3.1}$ for $s, t \in \omega_0^{<\omega}$ with $|s|, |t| \geq 1$ the following is true:

a) $I_s \subseteq R_t$ if $t = \langle s, i\rangle$ for $i \in \omega$ and $I_s \cap R_t = \emptyset$ if $t \neq \langle s, i\rangle$ for $i \in \omega$.

So by the fact that $(\cap_{j\geq 2} \cup_{|s|=j} J_s) \cap (\cup_{|s|\geq 1} I_s) = \emptyset$ for all $s \in \omega_0^{<\omega}, |s| \geq 1$ and for any $c : \mathcal{R} \to 2$ with $\cup c^{-1}(\{k\}) \supseteq \cup_{|s|\geq 1} I_s$ $(k \in 2)$ one can find $n(s) \in \omega$ with $I_s \subseteq R_{\langle s, n(s)\rangle}$ and $c(R_{\langle s,n(s)\rangle}) = 0$. Let $\{t_i \in \omega^i : i \geq 1 \textit{ with } t_{i+1} = \langle t_i, n(t_i)\rangle\}$.

By $\bullet_2$ for $\mathbf{x} \in \cap_{i\geq 1} J_{t_i}$ if $\mathbf{x} \in J_s$ for some $s \in \omega^v$ then $s = t_v$, meaning, that $A \ni \mathbf{x} \notin \cup c^{-1}(\{1\})$.

□

We are done with Theorem 1.2.5.

□

Note that in the construction of A in the proof of Theorem 1.2.5 we can choose I_s ($|s| \geq 1$) and J_s ($|s| \geq 1$) with:

$\circ_1$ $\cap_{j\geq 2} \cup_{|s|=j} J_s \subseteq (l + (1-\varepsilon_2)\overrightarrow{\mathbf{v}})$ is a closed set minus countably many points.

$\circ_2$ $\cup_{|s|\geq 1} I_s \subseteq l$ is also a closed set minus one point.

Choosing ε_2 small enough we can choose $\mathcal{R}_1 \in [\mathcal{R}_\varepsilon]^\omega$ with $\cup\mathcal{R}_1 \subseteq \mathbf{R}^2 \setminus A \subseteq (\mathcal{R}_1)_\omega$, resulting in a bit strengthening of Theorem 5.1.1:

Theorem 5.1.5. *For all $\varepsilon > 0$ there is $\mathcal{R} \in [\mathcal{R}_\varepsilon]^\omega$ with:*

(i) $(\mathcal{R})_\omega \supseteq \mathbf{R}^2$,

(ii) for any partition $\mathcal{R}_1 \cup^* \mathcal{R}_2 = \mathcal{R}$ either $\cup\mathcal{R}_1 \not\supseteq \mathbf{R}^2$ or $\cup\mathcal{R}_2 \not\supseteq \mathbf{R}^2$.

5.2 Closed unit squares with small rotation

The proof of Theorem 1.2.6 is similar to the previous one, we need to use points instead of the I_s intervals and use the following elementary statement instead of the one, described above:

Let l be the $x = -y$ line, $\overrightarrow{\mathbf{v}}$ be the vector from the origin to $(1,1)$.

Lemma 5.2.1. *For any $\varepsilon > 0$ we can choose $\varepsilon_1 > 0$ such that for any $I \subseteq l + (1-\varepsilon_1)\overrightarrow{\mathbf{v}}$ and $A \subseteq l$ finite set, we can find $\{I_i : i \in \omega\} \subseteq I$ disjoint intervals, $\{S_i : i \in \omega\} \subseteq \mathcal{S}_\varepsilon$ and $\mathbf{p} \in (l \setminus A)$ such that for all $i \in \omega$:*

$\bullet_1$ $\mathbf{v_4}(S_i) = \mathbf{p}$, *and*

$\bullet_2$ $S_i \cap (l + (1-\varepsilon_1)\overrightarrow{\mathbf{v}}) = I_i$.

5.3 Axis-parallel closed squares with side length between $1-\varepsilon$ and 1

Proof of Theorem 1.2.7. Let C be the closed unit square, $\frac{1}{4} > \varepsilon > 0$ and let

$$\mathcal{T}(C, \varepsilon) = \{C_i : i \in \omega\}$$

a set of axis-parallel closed squares with:

- $\bullet_1$ $C_0 = C$,
- $\bullet_2$ the side length of C_i is less than the side length of C_{i-1} with ε^{i+1} $(i \geq 1)$,
- $\bullet_3$ the $\overrightarrow{\mathbf{v_4}(C_{i-1})\mathbf{v_4}(C_i)}$ vector is the $\overrightarrow{(0,0)(0,\varepsilon^i)}$ vector.

Using the construction $\mathcal{T}(C, \varepsilon)$, let us introduce some notation:

$\circ_1$ Let $\mathbf{p}(\mathcal{T}(C, \varepsilon)) = (0, \frac{\varepsilon}{1-\varepsilon})$. Note that $\mathbf{p}(\mathcal{T}(C, \varepsilon)) \in C_i$ for all $i \in \omega$, since $\varepsilon < \frac{1}{4}$.

$\circ_2$ Let $\mathcal{A}(\mathcal{T}(C, \varepsilon)) = \{A_i : i \in \omega\}$, the following set of open rectangles:

$\circ_{2.1}$ let A_0 be the open square with $(1, 1)$ and $(1 - \varepsilon^2, 1 - \varepsilon^2)$ as opposite vertices.

Note that $A_0 \subseteq C_0 \setminus (\cup_{j \in (\omega \setminus \{0\})} C_j)$,

$\circ_{2.2}$ let $A_i = int(C_i \setminus (\cup_{j \in (\omega \setminus \{i\})} C_j))$ for $i \geq 1$.

$\circ_3$ For any transformation t of $\mathbf{R}^2$ let

$\circ_{3.1}$ $\mathcal{T}(t(C), \varepsilon) = \{t(C_i) : i \in \omega\}$,

$\circ_{3.2}$ $\mathbf{p}(\mathcal{T}(t(C), \varepsilon)) = t(\mathbf{p}(\mathcal{T}(C, \varepsilon)))$,

$\circ_{3.3}$ $\mathcal{A}(\mathcal{T}(t(C), \varepsilon)) = \{t(A_i) : i \in \omega\}$.

Let us denote by $B(\mathbf{x}, r)$ the 2 dimensional ball around $\mathbf{x}$ with radius r. Let us mention the following easy fact witout proof:

Fact. For any t, a transformation of $\mathbf{R}^2$ with $t(C)$ axis-parallel and $\varepsilon > 0$ there is t_1, a transformation of $\mathbf{R}^2$ and $\varepsilon_1 > 0$ with:

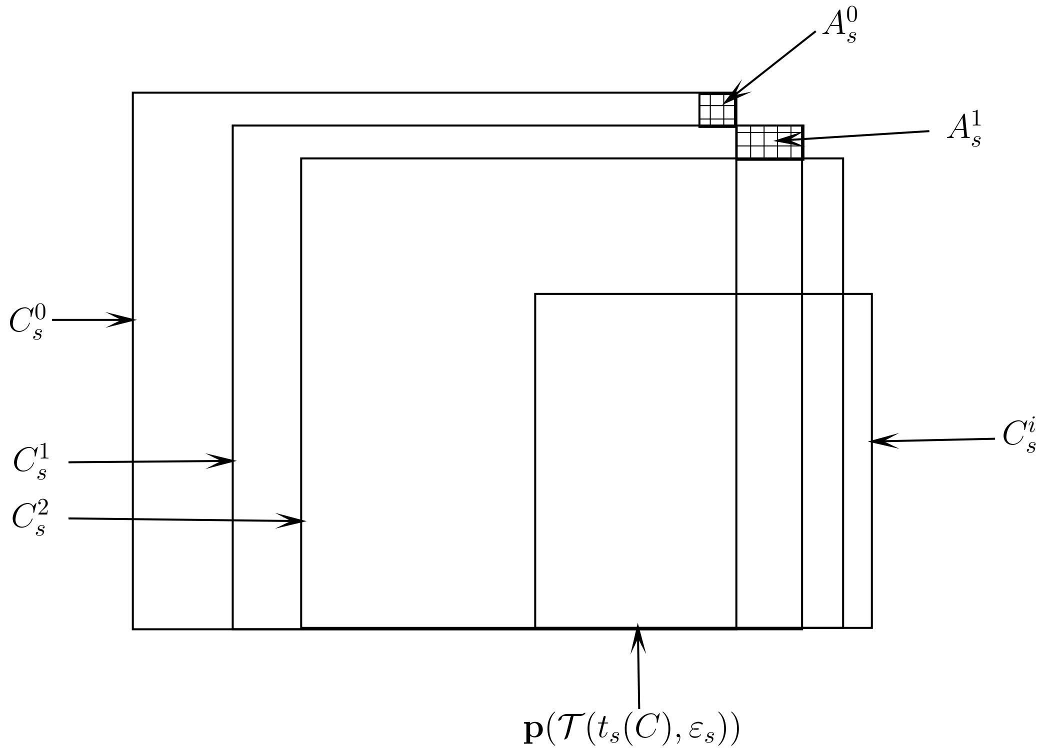

Figure 5.3: The construction of $\mathcal{T}(t_s(C),\varepsilon_s)$

(i) $t_1(C)$ is axis-parallel,

(ii) $int(t(C)) \supseteq \mathcal{T}(t_1(C),\varepsilon_1)$,

(iii) $B(\mathbf{v_4}(t(C_i)),\varepsilon) \supseteq \mathbf{p}(\mathcal{T}(t_1(C),\varepsilon_1))$,

(iv) $B(\mathbf{v_2}(t(C_i)),\varepsilon) \supseteq \mathcal{A}(\mathcal{T}(t_1(C),\varepsilon_1))$.

Using tha fact above we start to write down our construction:

- Let $\omega_0^{<\omega} = \{\langle s_0, s_1, ..., s_j\rangle \in \omega^{<\omega} : s_0 = 0\}$, and
- For $j \geq 1$ and $s = \langle s_0, s_1, ..., s_j\rangle \in \omega_0^{<\omega}$ let $s^- = \langle s_0, s_1, ..., s_{j-1}\rangle$, $|s| = j$ and let $s|i = \langle s_0, s_1, ..., s_{i-1}\rangle$ for $i < |s|$.

In the *jth* step $(j \geq 1)$ for $s \in \omega_0^{<\omega}$, $|s| = j$ we define

$$\mathcal{T}(t_s(C),\varepsilon_s),$$

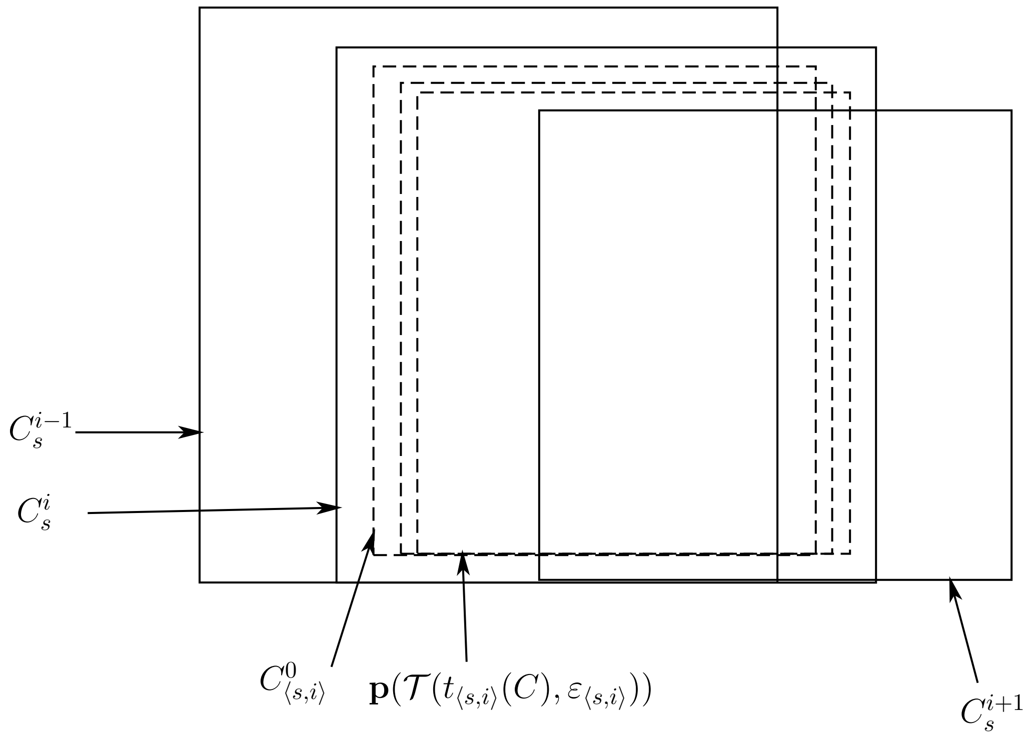

Figure 5.4: A step

where t_s is a transformation of $\mathbf{R}^2$. To do this for $s \in \omega_0^{<\omega}$, $|s| = j$, $(j \geq 1)$ we define:

1.) x_s and y_s, for

$$\mathbf{v_4}(t_s(C)) = (\sum_{j \leq |s|} x_{s|j}, \sum_{j \leq |s|} y_{s|j}),$$

2.) $|t_s(C)|$, and

3.) ε_s.

Now we write down what assumptions do we need on 1.)- 3.):

For $s \in \omega_0^{<\omega}$ and $i \in \omega$ let $A_{i,s} = t_s(A_i)$ and $C_{i,s} = t_s(C_i)$.

If $s = \langle s^-, i\rangle = \langle s_0, s_1, ..., s_w, 0, 0, ..., 0, i\rangle$ with $s_w \neq 0$ let:

a0) $\mathcal{T}(t_s(C), \varepsilon_s) \subseteq \mathcal{U}_\varepsilon$

a1) $C_{i,s^-} \supseteq t_s(C, \varepsilon_s)$, a2) $A_{i,s^-} \supseteq \mathcal{A}(\mathcal{T}(t_s(C), \varepsilon_s))$,

b) $y_s + \frac{\varepsilon_s}{1-\varepsilon_s} < \varepsilon_{s^-}^{i+1}$, and

c) $x_{\langle s^-,0\rangle} > x_{\langle s^-,1\rangle} > ...$ with

$$\sum_{j\le|s|} x_{\langle s^-,0\rangle|j} < \sum_{j\le w-1} x_{\langle s_0,s_1,...,s_w-1,0...0\rangle|j}.$$

Using the fact above we can easily choose $x_s, y_s, |t_s(C)|$ and ε_s satisfying a1)-c) by the fact above and by induction.

Note that

Claim 5.3.1. $\mathbf{p}(\mathcal{T}(t_s(C), \varepsilon_s)) \in C_{j,r} \Leftrightarrow$

1.) $r = s|k$ for some $k < |s|$ and $j \le s_k$, or

2.) $r = s$ and $i \in \omega$.

Proof. It can be easily seen by induction using b) and c).

□

Let

$$A = \cup_{s\in\omega_0^{<\omega}} \mathbf{p}(\mathcal{T}(t_s(C), \varepsilon_s)) \bigcup \cap_{j\ge1} \cup_{|s|=j, s\in\omega_0^{<\omega}} \mathcal{A}(\mathcal{T}(t_s(C), \varepsilon_s)), and$$

$$\mathcal{U} = \cup_{s\in\omega_0^{<\omega}} \mathcal{T}(t_s(C), \varepsilon_s).$$

Lemma 5.3.2. *The followings are true:*

a) $A \subseteq (\cup_{s\in\omega_0^{<\omega}} \mathcal{T}(t_s(C), \varepsilon_s))_\omega$,

b) $A \not\subseteq \cap_{j\in2} c^{-1}(\{j\})_\omega$ *for all* $c : \cup_{s\in\omega_0^{<\omega}} \mathcal{T}(t_s(C), \varepsilon_s) \to 2$.

Proof. a) is immediate by the construction.

b) by Claim 5.3.1 for all $s \in \omega_0^{<\omega}$ there are infinitely many $i \in \omega$ with $c(C_{i,s}) = 0$, so we can choose $t \in \omega^\omega$ with $c(C_{t|i}) = 0$ for all $i \in \omega$. But then $\cap_{i\in\omega} A_{t|i} \not\subseteq c^{-1}(\{1\})_\omega$.

□

□

6 Concluding remarks

6.1 Remarks

In this section first mention that we can use covers with larger cardinality:

Theorem 6.1.1. *Let κ be an infinite cardinal, C be the (open or closed) unit square, and $\mathfrak{C} \in [\mathcal{T}_C]^\kappa$. Then there is $c : \mathfrak{C} \to \omega$ with:*

$$(\mathfrak{C})_\omega = \cap_{j \in \omega} c^{-1}(\{j\}).$$

Proof. If C is the open unit square then this theorem is just the the easy consequence of the hereditarily Lindelöfness of the plane and Theorem 1.2.3.

If C is the closed then the statement is the consequence of Theorem 1.2.4 and the following folklore fact:

Fact. Let $\mathcal{C}$ be a set of closed polygons without vertices. Then there is $\mathcal{C}' \in [\mathcal{C}]^{\leq\omega}$ with

$$\cup\mathcal{C} = \cup\mathcal{C}'.$$

□

Then recall that we defined for $x \in X$ (where X is a set) whether it is κ-fold covered by $\mathcal{F}$, a set of subsets of X. However it is quite natural to define a similar notion with indexed subsets of X. The next theorem says, that the same covering decomposition is true for this notion:

Theorem 6.1.2. *Let $|I| \geq \omega$ and $\mathfrak{C} = \{C_i : i \in I\}$ be such that C_i is the translate of the open or closed unit square for all $i \in I$ and let $[\mathfrak{C}]_\omega = \{x : |\{i : x \in C_i\}| \geq \omega\}$. There is $c : I \to \omega$ with*

$$[\mathfrak{C}]_\omega = \cap_{j\in\omega}[\{C_k : k \in c^{-1}(\{j\})\}]_\omega.$$

Proof. For $C_i \in \mathfrak{C}$ let $(C_i) = \{j \in I : C_j = C_i\}$. If we consider all $(C_i)'s$, this gives a partition of I. Pick an element from each part of this partition and let A be the set of the picked elements.

We know by Theorem 6.1.1 that there is $d : A \to \omega$ with $[\{C_i : i \in A\}]_\omega = \cap_{j\in\omega}[\{C_i : i \in c^{-1}(\{j\})\}]_\omega$. Let $B = \{i \in I : |(C_i)| = \omega\}$, and choose $e : B \setminus A \to \omega$ with $[\{C_i : i \in B \setminus A\}]_\omega = \cap_{j\in\omega}[\{C_i : e^{-1}(\{j\})\}]_\omega$. This can be easily done. Let $c : I \to \omega$ be defined with:

$$c(i) = \begin{cases} d(i) & \text{if } i \in A, \\ e(i) & \text{if } i \in B \setminus A, \\ 0 & \text{otherwise.} \end{cases}$$

Now we want to prove that $[\mathfrak{C}]_\omega \subseteq \cap_{j\in\omega}[\{C_k : k \in c^{-1}(\{j\})\}]_\omega$. If there is $i \in I$ with $x \in [(C_i)]_\omega$, then we are done by the definition of e. If there is no such i, then since $x \in [\mathfrak{C}]_\omega$, we know that $x \in [\{C_i : i \in A\}]_\omega$ and then we are done by the definition of d.

□

At last we mention that our proofs for all open or closed symmetric polygons so actually we've got the following:

Theorem 6.1.3. *Let $\mathfrak{C} = \{C_i : i \in I\}$ be such that C_i is the translate of the given open or closed convex symmetric polygon C for all $i \in I$ and let $[\mathfrak{C}]_\omega = \{x : |\{i : x \in C_i\}| \geq \omega\}$. There is $c : I \to \omega$ with*

$$[\mathfrak{C}]_\omega = \cap_{j\in\omega}[\{C_k : k \in c^{-1}(\{j\})\}]_\omega.$$

6.2 An open question

It worth to pose as a question the ω-fold and generalized version of Pach's conjecture:

Question 6.2.1. *Is it true that for any convex planar set C and any $\mathcal{F}$ family of its translates, there is a partition $\mathcal{F} = \cup^* \mathcal{F}_i$ with $(\mathcal{F})_\omega = (\mathcal{F}_i)_\omega$?*

Bibliography

[1] R. Aharoni, A. Hajnal, E. C. Milner: *Interval covers of a linearly ordered set*, Boise ID (1992-1994), Contemp. Math. 192., Amer. Math. Soc., Providence, RI, 1996, 1-13.

[2] M. Elekes, T. Matrai, L. Soukup: *On splitting infinite-fold covers*, Fund. Math. **212** (2011), 95-127.

[3] I. Kovács: Többszörös fedések zárt sokszögekkel, manuscript

[4] J. Pach: *Decompsition of multiple packing and covering* , Diskrete Geometrie, 2. Kolloq. Math. Univ Salzburg, (1986), 169-178.

[5] D. Pálvölgyi: *Decomposition of Geometric Set Systems and Graphs*, Ph.D. thesis, Ecole Polytechnique Federale de Lausanne, 2010.