

On Semigroups Of Linear Operators

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Introduction

Consider the Cauchy problem

$$\begin{cases} u'(t) = Au(t), & t \geq 0, \\ u(0) = x, \end{cases}$$

where $A = (a_{ij})$ is an $n \times n$ matrix with $a_{ij} \in \mathbb{C}$ for $i, j = 1, 2, \dots, n$, and x is a given vector in \mathbb{C}^n .

It is well-known that the above Cauchy problem has a unique solution given by

$$u(t) = e^{tA}x, \quad t \geq 0,$$

where e^{tA} represents the fundamental matrix of the linear differential system $u'(t) = Au(t)$ which equals I for $t = 0$. We have

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k,$$

valid for all $t \in \mathbb{R}$. Here A and e^{tA} can be interpreted as linear operators, $A \in \mathcal{L}(X)$, $e^{tA} \in \mathcal{L}(X)$, where $X = \mathbb{C}^n$, equipped with any of its equivalent norms.

Note that the family of matrices (operators) $\{T(t) = e^{tA}, t \geq 0\}$ is a (uniformly continuous) semigroup on $X = \mathbb{C}^n$. Even more, $\{T(t), t \geq 0\}$ extends to a group of linear operators, $\{e^{tA}, t \in \mathbb{R}\}$.

The representation of the solution $u(t)$ as

$$(1) \quad u(t) = T(t)x, \quad t \geq 0,$$

allows the derivation of some properties of the solution from the properties of the family $\{T(t), t \geq 0\}$. This idea extends easily to the case in which X is a general Banach space and A is a bounded linear operator, $A \in \mathcal{L}(X)$.

If A is a linear unbounded operator, $A : D(A) \subset X \rightarrow X$, with some additional conditions, then one can associate with A a so-called C_0 -semigroup of linear operators $\{T(t) \in \mathcal{L}(X), t \geq 0\}$, such that the solution of the Cauchy problem is again

represented by the above formula (1). In this way one can solve various linear PDE problems, where A represents linear unbounded differential operators with respect to the spacial variables, defined on convenient function spaces.

It is worth mentioning that the connection between A and the corresponding semigroup $\{T(t), t \geq 0\} \subset \mathcal{L}(X)$ is established by the well-known Hille-Yosida theorem. Linear semigroup theory received considerable attention in the 1930s as a new approach in the study of linear parabolic and hyperbolic partial differential equations.

Note that the linear semigroup theory has later developed as an independent theory, with applications in some other fields, such as ergodic theory, the theory of Markov processes, etc.

This thesis is intended to present fundamental characterizations of the linear semigroup theory and to illustrate them by some interesting applications.

Structure of the Thesis

In Chapter 1, we give some important examples of semigroups, theorems related to them and their connection to differential equations. First, we present an example about strongly continuous semigroups, second, multiplication semigroups and we conclude with translation semigroups.

In Chapter 2, we start with an introduction of the theory of strongly continuous semigroups of linear operators in Banach spaces, then we associate a generator to them and illustrate their properties by means of some theorems. Hille-Yosida generation theorem characterizes the infinitesimal generators of these strongly continuous one-parameter semigroups, by providing a necessary and sufficient condition for an unbounded operator on a Banach space to be a generator. This theorem will be stated and proved on the last part of the thesis and then proceeded by the Feller-Miyadera-Phillips theorem, which is the general case of it.

In the end of each chapter, a section of notes will be provided, where we identify our sources and suggest further reading.

Chapter 1

Examples of semigroups

In this chapter we are going to describe the matrix valued function $T(\cdot) : \mathbb{R}_+ \rightarrow M_n(\mathbb{C})$ which satisfies the functional equation (1.1) discussed on Section 1.1. We will see that, for $A \in M_n(\mathbb{C})$, the continuous map $\mathbb{R}_+ \ni t \mapsto e^{tA} \in M_n(\mathbb{C})$ satisfies the functional equation and that e^{tA} forms a semigroup of matrices depending on $t \in \mathbb{R}_+$. We call $\{T(t) := e^{tA}; t \geq 0\}$ the (one-parameter) semigroup generated by the matrix $A \in M_n(\mathbb{C})$ and we are done with the first example on semigroups. After this, we will proceed with the next two examples, namely multiplicative semigroups and translation semigroups.

1.1 Motivation

Exponential function e^x , where $x \in \mathbb{C}$, is one of the most important functions in mathematics and can be expressed by power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

or as a limit

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n.$$

Consider the following functional equation

$$(1.1) \quad \begin{cases} u(t+s) = u(t)u(s), & t, s \geq 0, \\ u(0) = 1 \end{cases}$$

where $u(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{C}$. It is easy to see that the exponential functions $t \mapsto e^{ta}$, $a \in \mathbb{C}$, satisfy this equation. Furthermore we pose the problem in finding the solution of

the initial value problem

$$(IVP) \quad \begin{cases} \frac{d}{dt}u(t) = au(t), & t \geq 0 \\ u(0) = 1. \end{cases}$$

Again we see that $u(t) = e^{ta}$, $a \in \mathbb{C}$ satisfies this (IVP) and later we will show that actually this solution is unique.

In the exponential function if we put instead of x the matrix $A \in M_n(\mathbb{C}) = \mathcal{L}(X)$, we can write the exponential of A in the form

$$(1.2) \quad e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

e^A is called the matrix exponential.

Proposition 1.1. *The matrix exponential e^A is convergent.*

Proof. Let $A \in \mathcal{L}(X)$ and $\|A\| = \max_{1 \leq j \leq n} \sum_{i=1}^n |A_{ij}|$. For all $A, B \in \mathcal{L}(X)$ we have

$$\begin{aligned} \|AB\| &= \max_{1 \leq j \leq n} \sum_{i=1}^n |(AB)_{ij}| = \max_{1 \leq j \leq n} \sum_{i=1}^n \left| \sum_{k=1}^n A_{ik} B_{kj} \right| \leq \max_{1 \leq j \leq n} \sum_{i=1}^n \sum_{k=1}^n |A_{ik}| |B_{kj}| \\ &\leq \max_{1 \leq k \leq n} \sum_{i=1}^n |A_{ik}| \max_{1 \leq j \leq n} \sum_{k=1}^n |B_{kj}| = \|A\| \|B\|. \end{aligned}$$

By induction we will have

$$\|A^k\| \leq \|A\|^k, \quad \forall k \in \mathbb{N}.$$

Thus,

$$\|e^A\| = \left\| \sum_{k=0}^{\infty} \frac{A^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{\|A^k\|}{k!} \leq \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!} = e^{\|A\|} < \infty. \quad \blacksquare$$

Below we prove some properties of the matrix exponential.

Properties 1.2. *Let $A, B \in \mathcal{L}(X)$, I and 0 be the identity and 0 matrices respectively,*

i) *If $AB = BA$ then $e^A e^B = e^{A+B}$,*

ii) *$e^{(t+s)A} = e^{tA} e^{sA}$ for all $t, s \geq 0$,*

iii) $e^0 = I$.

Proof.

$$\begin{aligned}
 \text{(i)} \quad e^A e^B &= \left(I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots \right) \left(I + B + \frac{1}{2}B^2 + \frac{1}{3!}B^3 + \dots \right) \\
 &= I + B + \frac{1}{2}B^2 + \frac{1}{3!}B^3 + A + AB + \frac{1}{2}AB^2 + \frac{1}{2}A^2 + \frac{1}{2}A^2B + \frac{1}{3!}A^3 + \dots \\
 &= I + (A + B) + \frac{1}{2}(A + B)^2 + \frac{1}{3!}(A + B)^3 + \dots = e^{A+B}.
 \end{aligned}$$

$$\text{(ii)} \quad \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \sum_{k=0}^{\infty} \frac{s^k A^k}{k!} = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{t^{n-k} A^{n-k} s^k A^k}{(n-k)!k!} = \sum_{n=0}^{\infty} \frac{(t+s)^n A^n}{n!}.$$

Next, we use the fact that matrix $A = 0$ is diagonal and its exponential can be obtained by exponentiating each element of the diagonal, namely,

$$\text{(iii)} \quad e^0 = \text{diag}(e^0, e^0, \dots, e^0) = I.$$

■

By the proceeding theorem we will show the relation between the matrix exponential and the differential equations. Using Picard iteration, we are going to prove that the solution to the initial value problem (1.3) exists and it is of matrix exponential form. Actually, by Proposition 1.5, this solution is unique.

Theorem 1.3. *There exists a solution $u(t)$ of matrix exponential form to the initial value problem*

$$\text{(1.3)} \quad \begin{cases} \frac{d}{dt}u(t) = Au(t), & t \geq 0 \\ u(0) = u_0, \end{cases}$$

where $A \in M_n(\mathbb{C})$ and u_0 is a given number.

Proof. Using Picard iteration,

$$\begin{aligned}
 u_0(t) &= u_0 \\
 u_1(t) &= u_0 + \int_0^t Au_0(s)ds = u_0 + Au_0 \int_0^t ds = u_0 + tAu_0 \\
 u_2(t) &= u_0 + \int_0^t Au_1(s)ds = u_0 + Au_0 \int_0^t ds + A^2u_0 \int_0^t sds = u_0 + tAu_0 + \frac{t^2}{2}A^2u_0 \\
 &\vdots \\
 u_n(t) &= \sum_{j=0}^n \frac{t^j A^j}{j!} u_0.
 \end{aligned}$$

Take $n \rightarrow \infty$, we get

$$(1.4) \quad u(t) = \lim_{n \rightarrow \infty} u_n(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j u_0.$$

As defined in (1.2) we have the matrix valued function to be

$$\sum_{j=0}^{\infty} \frac{t^j A^j}{j!} = e^{tA}.$$

This suggests that $e^{tA}u_0$ is the solution of (1.3). Actually, plugging (1.4) into the differential equation (1.3), we see that the differential equation with initial condition is satisfied. ■

Next we see another proof which shows that e^{tA} is differentiable.

Theorem 1.4. e^{tA} is differentiable and its derivative is Ae^{tA} .

Proof. From the definition of the norm we know that

$$\|A\| = \max\{|A_{ij}| \mid 1 \leq i, j \leq n\}.$$

For any two matrices $A, B \in \mathcal{L}(X)$, denote by $(AB)_{ij}$ the element of the i th row and j th column of the product matrix AB . Note that

$$(1.5) \quad |(AB)_{ij}| = \left| \sum_{k=1}^n A_{ik} B_{kj} \right| \leq \sum_{k=1}^n |A_{ik}| |B_{kj}| \leq n \|A\| \|B\|.$$

For e^{tA} to be differentiable we need to have

$$\frac{d}{dt} e^{tA} = \lim_{h \rightarrow 0} \frac{e^{(t+h)A} - e^{tA}}{h} = \lim_{h \rightarrow 0} \frac{(e^{hA} - I)}{h} e^{tA}.$$

We claim that $\lim_{h \rightarrow 0} \frac{(e^{hA} - I)}{h} = A$.

$$(1.6) \quad \frac{e^{hA} - I}{h} - A = \frac{I + \frac{hA}{1} + \frac{(hA)^2}{2!} + \frac{(hA)^3}{3!} + \dots - I}{h} - A = \frac{h}{2!} A^2 + \frac{h^2}{3!} A^3 + \dots$$

Using the property in (1.5) we will estimate the series (1.6)

$$\begin{aligned}
\left| \left(\frac{h}{2!}A^2 + \frac{h^2}{3!}A^3 + \dots \right)_{ij} \right| &\leq \frac{1}{2}|h(A^2)_{ij}| + \frac{1}{3!}|h^2(A^3)_{ij}| + \dots \\
&\leq \frac{1}{2}|h|n\|A\|^2 + \frac{1}{3!}|h|^2n^2\|A\|^3 + \dots \\
&= \|A\| \left(\frac{1}{2}|h|n\|A\| + \frac{1}{3!}|h|^2n^2\|A\|^2 + \dots \right) \\
&= \frac{\|A\|}{|h|n\|A\|} (e^{|h|n\|A\|} - 1 - |h|n\|A\|) \\
&= \|A\| \left(\frac{e^{|h|n\|A\|} - 1}{|h|n\|A\|} - 1 \right).
\end{aligned}$$

Take $h \rightarrow 0$

$$\lim_{h \rightarrow 0} \frac{e^{|h|n\|A\|} - 1}{|h|n\|A\|} - 1 = \frac{d}{dx} e^x \Big|_{x=0} - 1 = 1 - 1 = 0.$$

So (1.6) is equal to 0, proving the claim that $\lim_{h \rightarrow 0} \frac{(e^{hA} - I)}{h} = A$ and consequently $\frac{d}{dt} e^{At} = A e^{tA}$. ■

As a conclusion to what we discussed above, we showed that applying Picard iteration to the differential equation (1.3) and the differentiability of e^{tA} proved on Theorem 1.4, indicate that the solution of (1.3) exists and is of matrix exponential form.

Proposition 1.5. *The solution to the initial value problem (1.3) is unique.*

Proof. Indeed, suppose there are two solutions satisfying (1.3). The second one is

$$\begin{cases} \frac{d}{dt}v(t) = Av(t), & v \geq 0 \\ v(0) = u_0. \end{cases}$$

We want to show that $u(t) = v(t)$. Define

$$z(t) = e^{-tA}v(t)$$

$$\frac{dz(t)}{dt} = -Ae^{-tA}v(t) + e^{-tA}v'(t) = -Ae^{-tA}v(t) + e^{-tA}Av(t) = 0.$$

So $z(t)$ is constant. For $t \geq 0$

$$z(t) = e^{-tA}v(t) = u_0.$$

It can be seen that

$$v(t) = e^{tA}u_0 = u(t).$$

So the solution is unique. ■

Remark 1.6. We may pose the same problem in a Banach space. Let the vector space X be a Banach space over \mathbb{C} (or \mathbb{R}), so X is equipped with a norm and is complete with respect to the norm, let A be an operator in $\mathcal{L}(X)$, where $\mathcal{L}(X)$ is the space of all bounded linear operators on X and $M_n(\mathbb{C})$ be the space of all $n \times n$ matrices with complex entries. By defining any norm on $M_n(\mathbb{C})$ the matrix exponential

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

is convergent and bounded, namely

$$\|e^{tA}\| \leq e^{t\|A\|}.$$

The solution of (1.3), for A bounded linear operator, exists and is unique, i.e., $u(t) = e^{tA}u_0$. The function e^{tA} in $M_n(\mathbb{C})$ satisfies the functional equation and forms a semigroup of matrices which depends on the parameter $t \in \mathbb{R}_+$. We call $(e^{tA})_{t \geq 0}$ the (one-parameter) semigroup generated by the matrix $A \in M_n(\mathbb{C})$. See Theorem 2.9 where this is proved.

1.1.1 Matrix semigroups

Now let us show how to compute the matrix semigroups e^{tA} when we have $A \in M_n(\mathbb{C})$ given.

We are going to illustrate this by the following cases:

i) Let $A = \text{diag}(a_1, \dots, a_n)$. The semigroup generated by matrix A is

$$e^{tA} = \text{diag}(e^{ta_1}, \dots, e^{ta_n}).$$

So what we do here is exponentiating the diagonal elements.

ii) Let $J_{m \times m}$ be a Jordan block

$$J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}$$

where $\lambda \in \mathbb{C}$. We can decompose J into $J = \lambda I + N$, where I is the identity matrix and N is

$$N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 \\ 0 & \ddots & \ddots & \vdots & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

It is easy to see that $N^m = 0$. From the definition of matrix exponential we know that $e^{tN} = \sum_{n=0}^{m-1} \frac{t^n N^n}{n!}$, namely

$$e^{tN} = \begin{bmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{m-2}}{(m-2)!} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & t \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

Using the fact that λI and N commute,

$$e^{tJ} = e^{t\lambda} e^{tN}.$$

We will consider the above cases and the following theorem to give a full answer to the question on how to find e^{tA} when A is given and conclude that the above cases are sufficient.

Theorem 1.7. *Let A be a complex $n \times n$ matrix. There exists a linear change in coordinates U such that A can be transformed into a block matrix,*

$$U^{-1}AU = \begin{bmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & J_{l-1} & 0 \\ 0 & \cdots & \cdots & 0 & J_l \end{bmatrix}$$

where each J has the form $J = \lambda I + N$ as in the above case ii).

Proof can be found in the book [16], Chap.3, Thm. 3.2.

1.2 Multiplication semigroups

Let $C_0(\mathbb{R})$ be a Banach space equipped with sup-norm (uniform or infinity norm) of all continuous, complex-valued functions $f \in C(\mathbb{R})$ on some bounded or unbounded interval in \mathbb{R} that vanish at infinity, namely

$$\forall \epsilon > 0, \exists \text{ a compact subset } K_\epsilon \subset \mathbb{R}, \text{ such that } |f(s)| < \epsilon, \forall s \in \mathbb{R} \setminus K_\epsilon.$$

Definition 1.8. The multiplication operator M_q induced on $C_0(\mathbb{R})$ by some continuous function $q : \mathbb{R} \rightarrow \mathbb{C}$ is defined by

$$M_q f = q \cdot f, \quad \forall f \in D(M_q)$$

$$D(M_q) = \{f \in C_0(\mathbb{R}) : q \cdot f \in C_0(\mathbb{R})\}.$$

In the following proposition we are going to state and prove some properties of the multiplication operator M_q which are related to the continuous function q .

Proposition 1.9. Let M_q with domain $D(M_q)$ be the multiplication operator induced on $C_0(\mathbb{R})$ by some continuous function q , then the following hold:

i) M_q is closed and densely defined.

ii) M_q is bounded (with $D(M_q) = C_0(\mathbb{R})$) iff q is bounded. In that case one has

$$\|M_q\| = \|q\| := \sup_{s \in \mathbb{R}} |q(s)|.$$

iii) M_q has a bounded inverse iff q has a bounded inverse $\frac{1}{q}$, i.e, $0 \notin \overline{q(\mathbb{R})}$. In that case one has

$$M_q^{-1} = M_{\frac{1}{q}}.$$

iv) The spectrum of M_q is the closed range of q , i.e,

$$\sigma(M_q) = \overline{q(\mathbb{R})}.$$

Proof. i) By definition the operator $M_q : D(M_q) \subseteq C_0(\mathbb{R}) \rightarrow C_0(\mathbb{R})$ is said to be closed if for every sequence $\{f_n\} \subset D(M_q)$ such that $f_n \rightarrow f \in C_0(\mathbb{R})$ and $M_q f_n \rightarrow g \in C_0(\mathbb{R})$ we have $f \in D(M_q)$ and $M_q f = g$. Actually, having f_n converging to f such that $\lim_{n \rightarrow \infty} q f_n := g$, since convergence in norm implies pointwise convergence, we get $g = qf = M_q f$ and $f \in D(M_q)$.

To show that M_q is densely defined it is enough to show that the continuous functions with compact support $C_c(\mathbb{R}) := \{f \in C(\mathbb{R}) | \text{supp } f \text{ is compact}\}$ are dense in $C_0(\mathbb{R})$, because of the fact that $C_c(\mathbb{R}) \subset D(M_q) \subseteq C_0(\mathbb{R})$.

For every compact subset K in \mathbb{R} we can find $h_K \in C(\mathbb{R})$ with compact support where

$$0 \leq h_K \leq 1 \text{ and } h_K = 1, \quad \forall s \in K.$$

So for every $f \in C_0(\mathbb{R})$,

$$\|f - f \cdot h_k\| = \sup_{s \in \mathbb{R} \setminus K} |f(s)(1 - h_K(s))| \leq 2 \sup_{s \in \mathbb{R} \setminus K} |f(s)|$$

meaning that for every $K \subseteq \mathbb{R}$ the functions $f \cdot h_K$ with compact support in \mathbb{R} , are dense in $C_0(\mathbb{R})$, that is what we wanted to prove.

- ii) Let M_q be bounded and f_s be a continuous function with compact support such that $\|f_s\| = 1 = f_s(s)$, $\forall s \in \mathbb{R}$. Then

$$\|M_q\| \geq \|M_q f_s\| \geq |q(s)f_s(s)| = |q(s)|.$$

Let q be bounded and $f \in C_0(\mathbb{R})$, then

$$\|M_q f\| = \sup_{s \in \mathbb{R}} |q(s)f(s)| \leq \|q\| \|f\|.$$

So having $\|M_q\| \geq \|q\|$ and $\|M_q\| \leq \|q\|$ we conclude

$$\|M_q\| = \|q\| = \sup_{s \in \mathbb{R}} |q(s)|.$$

- iii) $0 \notin \overline{q(\mathbb{R})}$ means that the inverse of q which is $\frac{1}{q}$ is also a continuous and bounded function. The inverse of multiplication operator M_q is $M_{\frac{1}{q}}$.

Let M_q have a bounded inverse M_q^{-1} . Then we have

$$\|f\| = \|M_q^{-1} M_q f\| \leq \|M_q^{-1}\| \|M_q f\|, \forall f \in D(M_q).$$

For $\|f\| = 1$

$$\delta := \frac{1}{\|M_q^{-1}\|} \leq \sup_{s \in \mathbb{R}} |q(s)f(s)|.$$

Suppose $\inf_{s \in \mathbb{R}} |q(s)| < \frac{\delta}{2}$, there will exist an open set $A \subset \mathbb{R}$, where $|q(s)| < \frac{\delta}{2}$ for all $s \in A$. There will exist a function $g_0 \in C_0(\mathbb{R})$ such that $g_0(s) = 0$ for all $s \in \mathbb{R} \setminus A$ and $g_0(s) = 1$ for all $s \in B$ where B is closed subset of A . We have

$$\sup_{s \in \mathbb{R}} |q(s)g_0(s)| \leq \sup_{s \in A} |q(s)g_0(s)| \leq \frac{\delta}{2}$$

which is a contradiction to what we supposed. So $q(s)$ must be in $0 < \frac{\delta}{2} \leq |q(s)|$ for all $s \in \mathbb{R}$ and $M_{\frac{1}{q}}$ is bounded and it is the inverse of M_q , namely

$$\left[M_{\frac{1}{q}}(M_q f) \right] (s) = \frac{1}{q(s)}(M_q f)(s) = f(s)$$

$$\left[(M_q f) M_{\frac{1}{q}} \right] (s) = (M_q f)(s) \frac{1}{q(s)} = f(s).$$

- iv) $\lambda \in \sigma(M_q)$ iff $\lambda I - M_q$ is not invertible.

$$M_{\lambda-q} f = (\lambda - q)f = \lambda f - M_q f = (\lambda - M_q)f, \forall f \in C_0(\mathbb{R}).$$

Thus,

$$\lambda I - M_q = M_{\lambda-q}$$

is not invertible.

$\sigma(M_q) \subset \overline{q(\mathbb{R})}$: By *iii*) we know that the operator $M_{\lambda-q}$ has a bounded inverse iff $\lambda - q$ has a bounded inverse $\frac{1}{\lambda-q}$, i.e., $0 \notin \overline{\lambda - q(\mathbb{R})}$, meaning that $\lambda \notin \overline{q(\mathbb{R})}$.

So $M_{\lambda-q}$ does not have a bounded inverse iff $\lambda \in \overline{q(\mathbb{R})}$.

$\sigma(M_q) \supset \overline{q(\mathbb{R})}$: Trivial by the definition. ■

Definition 1.10. Let $q : \mathbb{R} \rightarrow \mathbb{C}$ be a continuous function such that $\sup_{s \in \mathbb{R}} \operatorname{Re} q(s) < \infty$.

Then the semigroup $\{T_q(t); t \geq 0\}$ defined by

$$T_q(t)f := e^{tq}f,$$

for $t \geq 0$ and $f \in C_0(\mathbb{R})$ is called the multiplication semigroup generated by the multiplication operator M_q (or the function q) on $C_0(\mathbb{R})$.

The exponential functions

$$s \mapsto e^{tq(s)}, \quad t \geq 0,$$

where $s \in \mathbb{R}$ and $q : \mathbb{R} \rightarrow \mathbb{C}$ is any continuous function, satisfy the functional equation (1.1). The multiplication operators

$$T_q(t)f = e^{tq}f$$

must be bounded on $C_0(\mathbb{R})$ so one can get one-parameter semigroup on $C_0(\mathbb{R})$, namely $e^{\sup_{s \in \mathbb{R}} t \operatorname{Re} q(s)} < \infty$.

Proposition 1.11. The multiplication semigroup $\{T_q(t); t \geq 0\}$ generated by $q : \mathbb{R} \rightarrow \mathbb{C}$ is uniformly continuous iff q is bounded.

Proof. First, given that q is bounded, by the Proposition 1.9 the operator M_q will be bounded and $T_q(t) = e^{tq} = e^{tM_q}$. As e^{tM_q} is uniformly continuous, then $T_q(t)$ is uniformly continuous.

Suppose q is unbounded. We want to show here that $\|T_q(0) - T_q(t_n)\|$ does not converge to 0. Define $t_n = \frac{1}{|qs(n)|} \rightarrow 0$, where $(s_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ is any sequence such that $|q(s_n)|$ is divergent as $n \rightarrow \infty$. So we will have $|t_n q(s_n)| = 1$. Exponentiating this function we get $e^{t_n q(s_n)} \neq 1$, so $\exists \delta > 0$ such that

$$|1 - e^{t_n q(s_n)}| \geq \delta, \quad \forall n \in \mathbb{N}.$$

Let $\|f_n\| = 1 = f_n(s_n)$, for $f_n \in C_0(\mathbb{R})$, then

$$\|T_q(0) - T_q(t_n)\| \geq \|f_n - e^{t_n q} f_n\| \geq |1 - e^{t_n q(s_n)}| \geq \delta.$$

So $\|T_q(0) - T_q(t_n)\|$ does not converge to 0 as $n \rightarrow \infty$, that is $\{T_q(t); t \geq 0\}$ is not a uniformly continuous semigroup. ■

Theorem 1.12. Let $\{T(t); t \geq 0\}$ be the multiplication semigroup generated by a continuous function $q : \mathbb{R} \rightarrow \mathbb{C}$, satisfying $\sup_{s \in \mathbb{R}} \operatorname{Re} q(s) < \infty$. Then the mappings

$$\mathbb{R}_+ \ni t \mapsto T_q(t)f = e^{tq}f \in C_0(\mathbb{R})$$

are continuous for every $f \in C_0(\mathbb{R})$.

Proof. Let $f \in C_0(\mathbb{R})$ such that $\|f\| \leq 1$. For $\epsilon > 0$, let K be a compact subset of \mathbb{R} , where

$$|f(s)| \leq \frac{\epsilon}{e^{|\sup_{s \in \mathbb{R}} \operatorname{Re} q(s)|} + 1}, \quad \forall s \in \mathbb{R} \setminus K.$$

There exists $t_0 \in (0, 1]$ such that

$$|e^{tq(s)} - 1| \leq \epsilon, \quad \forall s \in K \text{ and } 0 \leq t \leq t_0.$$

Then for all $0 \leq t \leq t_0$,

$$\|e^{tq}f - f\| \leq \sup_{s \in K} (|e^{tq(s)} - 1| |f(s)|) + (e^{|\sup_{s \in \mathbb{R}} \operatorname{Re} q(s)|} + 1) \sup_{s \in \mathbb{R} \setminus K} |f(s)| \leq 2\epsilon.$$

That is $e^{tq}f$ is continuous for every $f \in C_0(\mathbb{R})$. ■

So this theorem means that for the multiplication semigroups $\{T(t); t \geq 0\}$, generated by an unbounded continuous function $q : \mathbb{R} \rightarrow \mathbb{C}$, where $\sup_{s \in \mathbb{R}} \operatorname{Re} q(s) < \infty$, although we can not achieve a one-parameter semigroup that is uniformly continuous, we can achieve continuity property.

In the following proposition we see that if a semigroup is composed of multiplication operators $T(t)f := m_t f$ on $C_0(\mathbb{R})$, for $t \geq 0$, and the mappings $\mathbb{R} \ni t \mapsto T(t)f \in C_0(\mathbb{R})$ are continuous for every $f \in C_0(\mathbb{R})$, then there exists a continuous function $q : \mathbb{R} \rightarrow \mathbb{C}$, whose real part is bounded from above, such that the semigroup defined by $T_q(t)f := e^{tq}f$, for $t \geq 0$ and $f \in C_0(\mathbb{R})$, is a multiplication semigroup.

Proposition 1.13. For $t \geq 0$, let $m_t : \mathbb{R} \rightarrow \mathbb{C}$ be bounded continuous functions and assume that

i) the multiplication operators

$$T(t)f := m_t f$$

form a semigroup $\{T(t); t \geq 0\}$ of bounded operators on $C_0(\mathbb{R})$,

ii) the mappings

$$\mathbb{R} \ni t \mapsto T(t)f \in C_0(\mathbb{R})$$

are continuous for every $f \in C_0(\mathbb{R})$.

Then there exists a continuous function $q : \mathbb{R} \rightarrow \mathbb{C}$, where $\sup_{s \in \mathbb{R}} \operatorname{Re} q(s) < \infty$ such that

$$m_t(s) = e^{tq(s)}, \quad \forall s \in \mathbb{R}, t \geq 0.$$

Proof can be found in book [4], Chap. 1, Prop. 4.6.

1.3 Translation Semigroups

Translation semigroups are another very good example which satisfy the semigroup properties and produce one-parameter operator semigroups whose continuity properties depend on the space on which we are working on. First of all let us define them.

Definition 1.14. For a function $f : \mathbb{R} \rightarrow \mathbb{C}$ and $t \geq 0$, we call

$$(T_l(t)f)(x) := f(x + t), \quad x \in \mathbb{R},$$

the left translation (of f by t), while

$$(T_r(t)f)(x) := f(x - t), \quad x \in \mathbb{R},$$

is the right translation (of f by t).

Let $X := C_0(\mathbb{R})$ be the Banach space of all continuous functions on \mathbb{R} vanishing at infinity. Define the operator $T(t)$

$$(T(t)f)(x) := f(t + x)$$

where $x, t \in \mathbb{R}$ and $f \in X$. The translation of the function f satisfies the semigroup properties, namely $T(t + s) = T(t)T(s)$ and $T(0) = I$. Furthermore since

$$\lim_{t \rightarrow 0^+} \sup_{x \in \mathbb{R}} \|f(t + x) - f(x)\| = 0$$

we have that

$$\lim_{t \rightarrow 0^+} T(t)f = f$$

forming so a translation semigroup $\{T(t); t \geq 0\}$ on \mathbb{R} which is continuous. Actually, since $\|T(t)f\| = \|f\|$, we have to do here with a contraction semigroup. Note that

$$\|T(t) - I\| \leq \sup_{\|f\| \leq 1} \|T(t)f - f\| = \sup_{\|f\| \leq 1} \sup_{s \in \mathbb{R}} |f(s + t) - f(s)|$$

does not converge to 0, so the semigroup is not uniformly continuous.

So the translation operators define strongly continuous semigroup on the space $C_0(\mathbb{R})$.

Notes

The matrix valued exponential functions as solutions of linear differential equations are treated in books about ordinary differential equations (see [Coddington [3], Chap. 1., pg. 1-32],[Hartman [8], Chap.4., pg. 70-92],[Teschl [16], Chap. 3., pg. 59-103]). Also we may find the material about matrix exponentials in texts about matrix analysis (Gantmacher [[6], Chap. 4., pg. 135-198]), or some other books on semigroups like for e.g, [Engel [5], Chap. 1, Sec. 1 and 2], or books about functional analysis (Schechter [[15], Chap. 10., pg. 225-238], [Rudin [14], Prologue, pg. 1-4]).

The multiplication operators and multiplication semigroups can be found in (Engel [[5], Chap. 1, Sec. 4, pg. 24-33], [Nagel [4], Chap. 1, pg. 11-26], [Nagel [12], Chap. C-II., pg. 248-290]). Translation semigroups appear in ([Engel [5], Chap. 1, Sec. 4, pg. 33-36], [Nagel [4], Chap. 1, pg. 30-34], [Nagel [12], Chap. B-II., pg. 122-162]).

Chapter 2

Abstract theory

As described in the examples of the first chapter, we see that although a semigroup may have the continuity property, it does not mean that it will be uniformly continuous, making so the uniform continuity a very strong property for the semigroups. Still we notice that in Theorem 2.17 the strong continuity holds, giving us a good reason to dedicate a chapter to the theory about them. We will start with the definitions, go on with the properties of the strongly continuous semigroups of bounded linear operators and conclude with the generation theorems of strongly continuous semigroups of closed linear operators.

2.1 Basic properties

In this section we are going to describe some definitions, theorems, corollaries, which provide us with some properties of the semigroups.

Definition 2.1. *A semigroup is a pair $(S, *)$, where the set $S \neq \emptyset$ and $*$ is a binary associative operation on S and $*$: $S \times S \rightarrow S$. That is $\forall x, y, z \in S$, we have $(x * y) * z = x * (y * z)$.*

Example 2.2. *Set of all natural numbers $N = \{1, 2, 3, \dots\}$ forms the semigroup $(N, +)$ under addition and (N, \cdot) under multiplication.*

Example 2.3. *Consider the Cauchy problem*

$$\begin{cases} u'(t) = Au(t), & t \geq 0, \\ u(0) = x, \end{cases}$$

where $A \in \mathcal{L}(X)$ and x is a given vector in \mathbb{C}^n .

From Theorem 1.3 and Proposition 1.5 the above Cauchy problem has a unique solution given by

$$u(t) = e^{tA}x, \quad t \geq 0,$$

where e^{tA} represents the fundamental matrix of the linear differential system $u'(t) = Au(t)$ which equals I for $t = 0$.

Let X be a Banach space over \mathbb{C} with norm $\|\cdot\|$ and let $\mathcal{L}(X)$ be the set of all linear bounded (continuous) operators $T : D(T) = X \rightarrow X$. $\mathcal{L}(X)$ is also a Banach space with respect to the operator norm,

$$\|T\| = \sup\{\|Tx\| : x \in X, \|x\| \leq 1\}.$$

Further on we proceed with the definitions of strongly continuous semigroup.

Definition 2.4. A semigroup is called the one parameter family $\{T(t); t \geq 0\} \subset \mathcal{L}(X)$ satisfying:

$$\begin{aligned} T(t+s) &= T(t)T(s), \quad t, s > 0 \\ T(0) &= I, \end{aligned}$$

where I is the identity operator on X .

Definition 2.5. A C_0 -semigroup (or strongly continuous semigroup, or a semigroup of class C_0) is called the one parameter family $\{T(t); t \geq 0\}$ of bounded linear operators from X to X if

$$\begin{aligned} T(t+s) &= T(t)T(s) \quad t, s > 0 \\ T(0) &= I \\ \lim_{t \rightarrow 0^+} \|T(t)x - x\| &= 0, \quad \forall x \in X. \end{aligned}$$

Definition 2.6. The family $\{G(t), t \in \mathbb{R}\} \subset L(X)$ is said to be a group if

$$\begin{aligned} G(t+s) &= G(t)G(s), \quad \forall t, s \in \mathbb{R} \\ G(0) &= I. \end{aligned}$$

$\{G(t), t \in \mathbb{R}\} \subset L(X)$ is said to be a C_0 -group if we add to the above definition the continuity property

$$\lim_{t \rightarrow 0^+} \|G(t)x - x\| = 0, \quad \forall x \in X.$$

Definition 2.7. The one parameter family $\{T(t); t \geq 0\}$ satisfying:

$$\begin{aligned} T(t+s) &= T(t)T(s) \quad t, s > 0 \\ T(0) &= I \\ \lim_{t \rightarrow 0^+} \|T(t) - I\| &= 0 \end{aligned}$$

is called a uniformly continuous semigroup of bounded linear operators.

Note that uniformly continuous semigroups are a subset of strongly continuous semigroups, because of the third condition of the uniformly continuous semigroups which is much stronger than the third condition of the strongly continuous semigroups.

Definition 2.8. *The linear operator $A : D(A) \subset X \rightarrow X$ is called the infinitesimal generator of the semigroup $\{T(t); t \geq 0\}$ if it satisfies*

$$Ax = \lim_{h \rightarrow 0^+} \frac{1}{h} (T(h)x - x), \quad \forall x \in X$$

where $D(A)$ is the set of all $x \in X$ such that the above limit exists.

This definition tells us how to find the (infinitesimal) generator A when the semigroup $\{T(t); t \geq 0\}$ is given, that is by differentiating the semigroup $\{T(t); t \geq 0\}$ as t tends to 0, namely

$$\frac{d}{dt} T(t) \Big|_{t=0} = A e^{tA} \Big|_{t=0} = A.$$

Theorem 2.9. *Let us have a bounded operator A from X to X , then $\{T(t); t \geq 0\}$, where $T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$, is a uniformly continuous semigroup.*

Proof. Given that A is a bounded operator, this means that $\|A\| < \infty$ and

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$$

converges to the bounded linear operator $T(t)$ for every $t \geq 0$. Lets check whether the power series fulfills the conditions to be a semigroup and even more a uniformly continuous semigroup.

First of all check $T(t+s) = T(t)T(s)$:

$$\left(\sum_{i=0}^{\infty} \frac{t^i}{i!} \right) \left(\sum_{j=0}^{\infty} \frac{s^j}{j!} \right) = \sum_{k=0}^{\infty} \frac{(t+s)^k}{k!}.$$

Secondly, $T(0) = I$, where I is the identity operator in X . This is obvious.

The last thing we need to check is the condition of uniformly continuous semigroup $\lim_{t \rightarrow 0^+} \|T(t) - I\| = 0$.

$$\|T(t) - I\| = \left\| \sum_{n=1}^{\infty} \frac{(tA)^n}{n!} \right\| \leq \sum_{n=1}^{\infty} \frac{t^n \|A\|^n}{n!} = e^{t\|A\|} - 1 \xrightarrow{t \rightarrow 0} 0^+.$$

Adapting the proof of Theorem 1.4 we will show that

$$\frac{d}{dt}e^{tA} = \lim_{h \rightarrow 0} \frac{e^{(t+h)A} - e^{tA}}{h} = \lim_{h \rightarrow 0} \frac{(e^{hA} - I)}{h} e^{tA}.$$

We claim that $\lim_{h \rightarrow 0} \frac{(e^{hA} - I)}{h} = A$.

$$(2.1) \quad \frac{e^{hA} - I}{h} - A = \frac{I + \frac{hA}{1} + \frac{(hA)^2}{2!} + \frac{(hA)^3}{3!} + \dots - I}{h} - A = \frac{h}{2!}A^2 + \frac{h^2}{3!}A^3 + \dots.$$

Using the property in (1.5) we will estimate the series (2.1)

$$\begin{aligned} \left| \left(\frac{h}{2!}A^2 + \frac{h^2}{3!}A^3 + \dots \right)_{ij} \right| &\leq \frac{1}{2}|h(A^2)_{ij}| + \frac{1}{3!}|h^2(A^3)_{ij}| + \dots \\ &\leq \frac{1}{2}|h|n\|A\|^2 + \frac{1}{3!}|h|^2n^2\|A\|^3 + \dots \\ &= \|A\| \left(\frac{1}{2}|h|n\|A\| + \frac{1}{3!}|h|^2n^2\|A\|^2 + \dots \right) \\ &= \frac{\|A\|}{|h|n\|A\|} (e^{|h|n\|A\|} - 1 - |h|n\|A\|) \\ &= \|A\| \left(\frac{e^{|h|n\|A\|} - 1}{|h|n\|A\|} - 1 \right). \end{aligned}$$

Take $h \rightarrow 0$

$$\lim_{h \rightarrow 0} \frac{e^{|h|n\|A\|} - 1}{|h|n\|A\|} - 1 = \frac{d}{dx}e^x \Big|_{x=0} - 1 = 1 - 1 = 0.$$

So (2.1) is equal to 0, proving the claim that $\lim_{h \rightarrow 0} \frac{(e^{hA} - I)}{h} = A$ and consequently $\frac{d}{dt}e^{At} = Ae^{tA}$. ■

Now we pass to some theorems related to C_0 semigroups.

Theorem 2.10. *Let $\{T(t), t \geq 0\} \subset L$ be a C_0 -semigroup, then there exist $M \geq 1$, $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0.$$

Proof. First, we prove that there exists $\delta > 0$ such that $\|T(t)\|$ is bounded for $0 \leq t \leq \delta$. Suppose this is not true. So, there must exist a sequence (t_k) , where $t_k \geq 0$, $\lim_{k \rightarrow \infty} t_k = 0$, such that $\|T(t_k)\| \rightarrow \infty$. By the condition of the continuity of the function $t \rightarrow T(t)x$ we may compute

$$\|T(t_k)x\| \leq \|T(t_k)x - x\| + \|x\| \leq M + \|x\|, \quad \forall t \in [0, \delta]$$

where $x \in X$, $k \in \mathbb{N}$, $M \geq 0$. By the Uniform Boundedness Theorem we have that $\|T(t_k)x\|$ is bounded which is a contradiction, so $T(t) \leq M$. We know that $\|T(0)\| = \|I\| = 1$, which means that $M \geq 1$. $\forall t \geq 0$ we can do the following

$$t = n\delta + r, \quad t \geq 0, \quad 0 \leq r < \delta, \quad n \in \mathbb{N}.$$

From the semigroup we use the property that $T(t+s) = T(t)T(s)$, $\forall t, s \geq 0$, namely,

$$\|T(t)\| = \|T(\delta)^n T(r)\| \leq \|T(\delta)\|^n \|T(r)\| \leq M^{n+1} \leq M M^{t/\delta} = M e^{\omega t}$$

where $\omega = \frac{\ln M}{\delta}$. ■

Corollary 2.11. *Let $\{T(t), t \geq 0\} \subset L$ be a C_0 -semigroup, then for each $x \in X$, $t \mapsto T(t)x$ is a continuous function from $[0, \infty)$ to X .*

Proof. The following proof will again use the properties of the C_0 -semigroup and the previous theorem 2.10. We will show that the $t \rightarrow T(t)x$ is continuous from the right and the left.

For $t, h \geq 0$ we have,

$$\|T(t+h)x - T(t)x\| = \|T(t)[T(h)x - x]\| \leq \|T(t)\| \|T(h)x - x\| \leq M e^{\omega t} \|T(h)x - x\|$$

for $0 \leq h \leq t$ we have,

$$\begin{aligned} \|T(t-h)x - T(t)x\| &= \|T(t-h)x - T(t-h+h)x\| = \|T(t-h)x - T(t-h)T(h)x\| \\ &\leq \|T(t-h)\| \|x - T(h)x\| \leq M e^{\omega(t-h)} \|x - T(h)x\| \end{aligned}$$

implying that $t \rightarrow T(t)$ is continuous. ■

Theorem 2.12. *Let $\{T(t); t \geq 0\}$ be a C_0 -semigroup and A its infinitesimal generator. Then*

a) *For each $x \in X$,*

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x.$$

b) *For each $x \in X$,*

$$\int_0^t T(s)x ds \in D(A)$$

and

$$A \left(\int_0^t T(s)x ds \right) = T(t)x - x.$$

c) $\forall t \geq 0, x \in D(A)$ we have,

$$T(t)x \in D(A)$$

and

$$(2.2) \quad \frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

d) For each $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Ax d\tau = \int_s^t AT(\tau)x d\tau.$$

Proof. a) From the Corollary 2.11 above we know that $t \mapsto T(s)x$ is a continuous function. From this follows a).

b) For $x \in X$ and $t > 0$,

$$\begin{aligned} \frac{T(h) - I}{h} \int_0^t T(s)x ds &= \frac{1}{h} \int_0^t [T(s+h)x - T(s)x] ds \\ &= \frac{1}{h} \left[\int_h^{t+h} T(s)x ds - \int_0^t T(s)x ds \right] \\ &= \frac{1}{h} \int_t^{t+h} T(s)x ds - \frac{1}{h} \int_0^t T(s)x ds. \end{aligned}$$

As $h \downarrow 0$ the right hand side goes to $T(t)x - x$, namely

$$\lim_{h \rightarrow 0^+} \frac{1}{h} (T(h) - I) \int_0^t T(s)x ds = T(t)x - x.$$

c) For $t \geq 0, h > 0$ and $x \in D(A)$,

$$T(t)Ax = \lim_{h \rightarrow 0^+} \frac{1}{h} T(t)[T(h)x - x] = \lim_{h \rightarrow 0^+} \frac{1}{h} [T(h)T(t)x - T(t)x].$$

So we have $T(t)x \in D(A)$ and

$$AT(t)x = T(t)Ax.$$

Lets compute the right derivative of $T(t)x$

$$\lim_{h \rightarrow 0^+} \frac{1}{h} [T(t+h)x - T(t)x] = \lim_{h \rightarrow 0^+} T(t) \frac{1}{h} [T(h)x - x] = T(t)Ax.$$

This implies that

$$\frac{d^+}{dt}T(t)x = AT(t)x = T(t)Ax.$$

Lets compute the left derivative of $T(t)x$

$$\begin{aligned} \left\| \frac{T(t)x - T(t-h)x}{h} - T(t)Ax \right\| &= \left\| T(t-h) \left[\frac{T(h)x - x}{h} - T(h)Ax \right] \right\| \\ &\leq Me^{\omega(t-h)} \left\{ \left\| \frac{T(h)x - x}{h} - Ax \right\| + \|Ax - T(h)Ax\| \right\}. \end{aligned}$$

So this implies that

$$\frac{d^-}{dt}T(t)x = T(t)Ax.$$

Concluding, right and left derivatives exist and are equal to $T(t)Ax$.

d) Let's integrate what we have in part c),

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax$$

from s to t , namely,

$$\int_s^t \frac{d}{d\tau}T(\tau)x d\tau = T(t)x - T(s)x = \int_s^t T(\tau)Ax d\tau = \int_s^t AT(\tau)x d\tau$$

and we are done with the proof. ■

Theorem 2.13. *Let $\{T(t); t \geq 0\}$ be a C_0 -semigroup and let A be its infinitesimal generator. Then*

1. $\overline{D(A)} = X$ ($D(A)$ dense in X , where $D(A)$ is the domain of A);
2. A is a closed linear operator;

Proof. 1. We start with proving that $\overline{D(A)} = X$. First, show that $\forall x \in X$ and $x_t = \frac{1}{t} \int_0^t T(s)x ds$ one has $x_t \in D(A)$ and $x_t \rightarrow x$ as $t \rightarrow 0^+$.

We have $x_t \in D(A)$ from part b) of the previous theorem and $x_t \rightarrow x$ as $t \rightarrow 0^+$ follows from part a) of the theorem.

2. Now lets pass to the proof of A is a closed operator. Let $x_n \in D(A)$ such that for $n \rightarrow \infty$, we have $x_n \rightarrow x$ and $Ax_n \rightarrow y$.

$$T(t)x_n - x_n = \lim_{h \rightarrow 0^+} \int_0^t T(s)h^{-1}[T(h)x_n - x_n] ds = \int_0^t T(s)Ax_n ds, \quad \forall t > 0.$$

$$T(t)x - x = \int_0^t T(s)y ds, \quad \forall t > 0.$$

Divide both sides by $t > 0$ and let $t \searrow 0$,

$$Ax = \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} = \lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t T(s)y ds = y,$$

so $x \in D(A)$ and $y = Ax$. ■

Theorem 2.14. *Let $\{T(t); t \geq 0\}$ and $\{S(t); t \geq 0\}$ be C_0 -semigroups of bounded linear operators. If their infinitesimal generators are the same, then the C_0 -semigroups coincide.*

Proof. Let $x \in D(A)$ where A is the common infinitesimal generator. By (2.12) we have

$$\frac{d}{ds}[T(t-s)S(s)x] = T(t-s)AS(s)x - T(t-s)AS(s)x = 0, \quad \forall 0 \leq s < t.$$

From the above equation it can be seen that $\forall t > 0$ and $x \in D(A)$, $s \mapsto T(t-s)S(s)x$ is a constant, furthermore $T(t)x = S(t)x$ and $T(t) = S(t)$ on $D(A)$. $\overline{D(A)} = X$ and we are done with the proof. ■

Now that we got introduced to the generators, we return to the example on translation semigroups from Chap. 1, Sec. 1.3. What we are going to do in the following proposition is to identify the generator $(A, D(A))$ of the translation semigroup on $C_0(\mathbb{R})$.

Proposition 2.15. *The generator of the left translation semigroup $\{T_l(t), t \geq 0\}$ on the space $X := C_0(\mathbb{R})$ is given by*

$$Af = f'$$

with domain

$$D(A) = \{f \in C_0(\mathbb{R}) : f \text{ differentiable and } f' \in C_0(\mathbb{R})\}.$$

Proof. We have to show that the generator $(B, D(B))$ is a restriction of the operator $(A, D(A))$, so what we need is $B \subseteq A$ and $\rho(A) \cap \rho(B) \neq \emptyset$.

For the generator $(B, D(B))$ of the contraction semigroup $\{T_l(t), t \geq 0\}$ on X , we have that if $\operatorname{Re} \lambda > \omega$, $\omega \in \mathbb{R}$, then $\lambda \in \rho(B)$. So for the operator $(A, D(A))$ we have that $\lambda \in \rho(A)$ too.

Let $f \in D(B)$ be fixed, where $(B, D(B))$ is the generator of $\{T_l(t), t \geq 0\}$. For a strongly continuous semigroup $\{T_l(t), t \geq 0\}$ and $x \in X$ we know that for the orbit map $\psi_x : t \mapsto T_l(t)x$ we have $\psi_x(\cdot)$ differentiable in \mathbb{R}_+ . Furthermore $Bx := \psi'_x(0)$. So for a continuous linear form δ_0 on $C_0(\mathbb{R})$ we have

$$\mathbb{R}_+ \ni t \mapsto \delta_0(T_l(t)f) = f(t)$$

and

$$Bf = \frac{d}{dt}T_i(t)f \Big|_{t=0} = \frac{d}{dt}f(t, \cdot) \Big|_{t=0} = f'.$$

So $D(B) \subseteq D(A)$ and the restriction of A in $D(B)$ is B , therefore $A = B$. ■

After introducing the basic definitions and theorems on strongly continuous semigroups of linear operators, we are now ready for the generation theorems. Before going to them we will need to define what is a semigroup of contractions, recall the definition of the resolvent and prove an important lemma.

2.2 Generation Theorems

2.2.1 The Hille-Yosida Theorem

Let $\{T(t), t \geq 0\}$ be a semigroup of linear operators. From Theorem 2.10 we know that there $\exists \omega \in \mathbb{R}$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$. In the case when $\omega = 0$ and $M = 1$ we call $T(t)$ a C_0 semigroup of contractions, namely $\|T(t)\| \leq 1$.

Given A a not necessarily bounded linear operator on X , the resolvent set of A is denoted by $\rho(A) = \{\lambda | \lambda I - A \text{ is invertible}, \lambda \in \mathbb{C}\}$. The resolvent of A is called the family $R(\lambda, A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ of bounded linear operators.

Note that for $\lambda \in \mathbb{C}$ such that $R(\lambda, A)x = \int_0^{\infty} e^{-\lambda s} T(s)x ds$ exists $\forall x \in X$, we call $R(\lambda, A)$ the integral representation of the resolvent. The integral can be interpreted here as Riemann integral, i.e.,

$$R(\lambda, A)x = \lim_{t \rightarrow \infty} \int_0^t e^{-\lambda s} T(s)x ds, \quad \forall x \in X$$

which can be further written as

$$(2.3) \quad R(\lambda, A) = \int_0^{\infty} e^{-\lambda s} T(s) ds$$

We prove (2.3) later on.

Lemma 2.16. *Let A be a closed densely defined operator in $D(A)$, such that the resolvent set $\rho(A)$ of A contains \mathbb{R}^+ and for every $\lambda > 0$ we have $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$, then for $\lambda \rightarrow \infty$ we have*

- i) $\lambda R(\lambda, A)x \rightarrow x$ for all $x \in X$,
- ii) $\lambda A R(\lambda, A)x = \lambda R(\lambda, A)Ax \rightarrow Ax$ for all $x \in D(A)$.

Proof. i) By the definition of the resolvent of A at the point λ , $R(\lambda, A) := (\lambda, A)^{-1}$, we get that

$$\lambda R(\lambda, A) - AR(\lambda, A) = I$$

and

$$\lambda R(\lambda, A)x = R(\lambda, A)Ax + x.$$

Then for $x \in D(A)$ and $\lambda \rightarrow \infty$

$$\|R(\lambda, A)Ax\| \leq \frac{1}{\lambda}\|Ax\| \rightarrow 0.$$

Since $D(A)$ dense in X and $\|\lambda R(\lambda, A)\| \leq 1$, we have for all $x \in X$

$$\lambda R(\lambda, A)x \rightarrow x.$$

ii) This is a consequence of the first statement, namely

$$\lim_{\lambda \rightarrow \infty} \lambda R(\lambda, A)Ax = Ax.$$

■

For every $\lambda > 0$ we define the Yosida Approximation of A by

$$\lambda AR(\lambda, A) = A_\lambda = \lambda^2 R(\lambda, A) - \lambda I$$

So the lemma indicates which bounded operators A_λ we have to take in order to approximate the unbounded operator A . The following Hille-Yosida theorem will characterize the generators of strongly continuous one-parameter semigroups of linear operators on Banach spaces. It is a special case on contraction semigroups of the general case Feller-Miyadera-Phillips theorem.

Theorem 2.17. (*Hille-Yosida*) *Let A be a linear unbounded operator, that is $A : D(A) \rightarrow X$, then the following are equivalent*

i) $(A, D(A))$ generates a C_0 -semigroup of contractions.

ii) $(A, D(A))$ is closed, densely defined and for every $\lambda > 0$ we have $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}.$$

Proof. i) \Rightarrow ii) Theorem 2.13 proves that A is a closed, densely defined operator in $D(A)$.

Given that $\{e^{-\lambda t}T(t), t \in \mathbb{R}\}$ is a semigroup of contractions

$$\|R(\lambda, A)x\| \leq \int_0^\infty e^{-\lambda t} \|T(t)\| \|x\| dt \leq \int_0^\infty e^{-\lambda t} \|x\| dt \leq \frac{1}{\lambda} \|x\|$$

where $x \in X$ and $\lambda > 0$. So we defined a bounded linear operator $R(\lambda, A)$. For $\lambda > 0$, we have the semigroup of contractions $\{e^{-\lambda t}T(t), t \in \mathbb{R}\}$. Let's compute its generator

$$A_1 x = \lim_{t \rightarrow 0^+} \frac{e^{-\lambda t}T(t)x - x}{t} = \lim_{t \rightarrow 0^+} \frac{-\lambda e^{-\lambda t}T(t)x + e^{-\lambda t}Ax}{1} = -\lambda x + Ax = (-\lambda I + A)x$$

where A_1 is the generator and namely in this case we have $A_1 = -\lambda I + A$ in the domain $D(A)$. By Theorem 2.12 b) we know that for $x \in X$,

$$(-\lambda I + A) \int_0^t e^{-\lambda s} T(s)x ds = e^{-\lambda t} T(t)x - x, \quad x \in X.$$

Namely,

$$x = (\lambda I - A) \int_0^t e^{-\lambda s} T(s)x ds + e^{-\lambda t} T(t)x, \quad x \in X.$$

Let $t \rightarrow \infty$. For A being closed, $\int_0^\infty e^{-\lambda s} T(s)x ds \in D(A)$. Using the Dominated Convergence Theorem,

$$x = (\lambda I - A) \int_0^\infty e^{-\lambda s} T(s)x ds, \quad x \in D(A).$$

This means that

$$x = (\lambda I - A)R(\lambda, A)x, \quad \text{for } x \in D(A).$$

By the other side for $x \in D(A)$ we have

$$\begin{aligned} AR(\lambda, A)x &= A \int_0^\infty e^{-\lambda t} T(t)x dt = \int_0^\infty e^{-\lambda t} AT(t)x dt \\ &= \int_0^\infty e^{-\lambda t} T(t)Ax dt = R(\lambda, A)Ax. \end{aligned}$$

So for all $\lambda > 0$ we have that $R(\lambda, A)$ is the inverse of $\lambda I - A$, meaning that $\lambda \in \rho(A)$. So the conditions *ii*) are necessary.

ii) \Rightarrow *i*) Let the uniformly continuous semigroups be given by

$$T_\lambda(t) := e^{tA_\lambda}, \quad t \geq 0.$$

where A_λ is the Yosida approximation of A and is bounded. e^{tA_λ} is a semigroup of contractions since

$$\|e^{tA_\lambda}\| = e^{-t\lambda} \|e^{t\lambda^2 R(\lambda, A)}\| \leq e^{-t\lambda} e^{t\lambda^2 \|R(\lambda, A)\|} \leq 1.$$

For every $x \in X, \lambda, \eta > 0$ we know that A_λ and A_η commute with each other so

$$\begin{aligned} \|T_\lambda(t)x - T_\eta(t)x\| &= \left\| \int_0^t \frac{d}{ds} (T_\lambda(t-s)T_\eta(s)x) ds \right\| \\ &\leq \int_0^t t \|T_\eta(t-s)T_\lambda(s)(A_\lambda - A_\eta)\| ds \leq t \|A_\lambda x - A_\eta x\|. \end{aligned}$$

By Lemma 2.16 we have that for $x \in D(A)$, $e^{tA_\lambda}x$ converges to $T(t)x$ as $\lambda \rightarrow \infty$, because for each $x \in D(A)$, where $D(A)$ dense in X , $T_\lambda(t)x$, for $\lambda \in \mathbb{N}$ converges uniformly on bounded intervals.

So we will have

$$\lim_{\lambda \rightarrow \infty} e^{tA_\lambda}x = T(t)x.$$

This implies that the limit family $T(t), t \geq 0$ satisfies the semigroup property, that is $T(0) = I$ and $\|T(t)\| \leq 1$, hence is a semigroup of contractions. On the other side $t \mapsto T(t)x, t \geq 0$ is uniform limit of continuous functions $t \mapsto e^{tA_\lambda}x$, so it is strongly continuous.

Let $(B, D(B))$ be the generator of $\{T(t), t \geq 0\}$ and $x \in D(A)$. On bounded intervals the differentiated functions $e^{tA_\lambda}A_\lambda x$ converge uniformly to $T(t)Ax$. So the functions $t \mapsto T_\lambda(t)x$ are differentiable and at point $t = 0$ we have $T_\lambda(t)A_\lambda x = T(t)Ax$. This implies $x \in D(B)$ and $Ax = Bx$, so $D(A) \subset D(B)$.

For $\lambda > 0$, by definition of the resolvent we have that $\lambda I - A : D(A) \rightarrow X$ is a bijection and $\lambda \in \rho(A)$. Since B is the infinitesimal generator of $T(t)$, where $T(t)$ is a contraction semigroup, it follows that $\lambda \in \rho(B)$. So $\lambda I - B : D(B) \rightarrow X$ is also a bijection. On the other side on $D(A)$ we have

$$(\lambda I - A)D(A) = (\lambda I - B)D(A) = X$$

so $\lambda - B$ coincides to $\lambda - A$, implying that $D(B) = D(A)$ and therefore $A = B$. ■

2.2.2 The Feller-Miyadera-Phillips Theorem

Theorem 2.18. *Let $(A, D(A))$ be a linear operator on a Banach space X and let $\omega \in \mathbb{R}, M \geq 1$ be constants. Then the following properties are equivalent*

i) $(A, D(A))$ generates a C_0 -semigroup $\{T(t), t \geq 0\}$ satisfying

$$\|T(t)\| \leq Me^{\omega t}, \quad t \geq 0.$$

ii) $(A, D(A))$ is a closed, densely defined and for every $\lambda \in \mathbb{C}$ where $\operatorname{Re}(\lambda) > \omega$ we have $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^n}, \quad \forall n \in \mathbb{N}.$$

Proof. *ii*) \Rightarrow *i*) Let $A : D(A) \subseteq X \rightarrow X$ be linear operator satisfying $(0, +\infty) \subseteq \sigma(A)$ and $\|\lambda^n R(\lambda, A)^n\| \leq M$ for each $n \in \mathbb{N}$ and $\lambda > 0$.

For $\mu > 0$ define the norm $|\cdot|_\mu : X \rightarrow R_+$ by

$$|x|_\mu = \sup_{n \in \mathbb{N}} \|\mu^n R(\mu, A)^n x\|.$$

We can see that these norms have the following properties

$$(a) \quad \|x\| \leq |x|_\mu \leq M\|x\|$$

$$(b) \quad |\mu R(\mu, A)|_\mu \leq |x|_\mu$$

By the resolvent equation we have

$$R(\lambda, A)x = R(\mu, A)x + (\mu - \lambda)R(\mu, A)R(\lambda, A)x = R(\mu, A)(x + (\mu - \lambda)R(\lambda, A)x).$$

therefore

$$|R(\lambda, A)x|_\mu \leq \frac{1}{\mu}|x|_\mu + \frac{\mu - \lambda}{\mu}|R(\lambda, A)x|_\mu.$$

that is

$$(c) \quad |\lambda R(\lambda, A)x|_\mu \leq |x|_\mu \text{ for all } \lambda \in (0, \mu].$$

From (a) and (c) and each $n \in \mathbb{N}$ and $\lambda \in (0, \mu]$, we have that

$$(d) \quad \|\lambda^n R(\lambda, A)^n x\| \leq |\lambda^n R(\lambda, A)^n x|_\mu \leq |x|_\mu.$$

For $n \in \mathbb{N}$ and for each $\lambda \in (0, \infty]$ the $\sup_{n \in \mathbb{N}} \|\lambda^n R(\lambda, A)^n x\|$ gives

$$(e) \quad |x|_\lambda \leq |x|_\mu.$$

We can define a norm $|\cdot|$ on X such that satisfies the properties

$$(f) \quad \|x\| \leq |x| \leq M\|x\|$$

$$(g) \quad |\lambda R(\lambda, A)x| \leq |x| \text{ for all } x \in X \text{ and } \lambda > 0.$$

Notice that the property (g) satisfies the Hille-Yosida condition on C_0 -semigroups on contractions, so A generates a strongly continuous semigroup of contractions $\{S(t), t \geq 0\}$.

$$\|S(t)x\| \leq |S(t)x| \leq |x| \leq M\|x\|, \quad \forall t \geq 0, x \in X.$$

so here $\{S(t), t \geq 0\}$ is of type $(M, 0)$. Considering the general case we will have that the semigroup will be of type (M, ω) with generator A .

$i) \Rightarrow ii)$ From (2.3), for $Re\lambda > \omega$ and $x \in X$ we have,

$$\frac{d}{d\lambda}R(\lambda, A)x = \frac{d}{d\lambda} \int_0^\infty e^{-\lambda s}T(s)x ds = - \int_0^\infty s e^{-\lambda s}T(s)x ds$$

Thus by induction

$$\begin{aligned} R(\lambda, A)^n x &= \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{d^{n-1}\lambda} R(\lambda, A)x = \frac{(-1)^{n-1}}{(n-1)!} \int_0^\infty \frac{d^{n-1}}{d^{n-1}\lambda} e^{-\lambda s}T(s)x ds \\ &= \frac{1}{(n-1)!} \int_0^\infty s^{n-1} e^{-\lambda s}T(s)x ds. \end{aligned}$$

for $Re\lambda > \omega$ and all $x \in X$. From Theorem 2.10 we have that $\|T(t)\| \leq M e^{\omega t}$, so

$$\begin{aligned} \|R(\lambda, A)^n x\| &= \frac{1}{(n-1)!} \left\| \int_0^\infty s^{n-1} e^{-\lambda s}T(s)x ds \right\| \leq \frac{1}{(n-1)!} \int_0^\infty s^{n-1} e^{-Re\lambda s} M e^{\omega s} ds \cdot \|x\| \\ &\leq \frac{M}{(n-1)!} \frac{(n-1)!}{(Re\lambda - \omega)^n} \|x\| = \frac{M}{(Re\lambda - \omega)^n} \cdot \|x\| \end{aligned}$$

for all $x \in X$.

Assume that $\omega = 0$. We will try to find a norm equivalent to the one defined above, namely define $||| \cdot ||| : X \rightarrow \mathbb{R}_+$ by

$$|||x||| := \sup_{t \geq 0} \|S(t)x\|$$

where $\{S(t), t \geq 0\}$ is a C_0 -semigroup. We want to show that actually this is a semigroup of contractions. The norm defined above is equivalent to $\|\cdot\|$ because it satisfies

$$(f') \quad \|x\| \leq |||x||| \leq M\|x\|, \text{ for all } x \in X.$$

On the other side we have that

$$|||S(t)x||| = \sup_{s \geq 0} \|S(s)S(t)x\| \leq \sup_{t \geq 0} \|S(t)x\| = |||x|||,$$

and the property

$$(g') \quad |||\lambda R(\lambda, A)x||| \leq |||x|||.$$

So the semigroup $\{S(t), t \geq 0\}$ satisfies the condition of the Hille-Yosida Theorem 2.17 for the norm $||| \cdot |||$ by generating a C_0 -semigroup of contractions. From the Generation Theorem on semigroups of contractions we have that A is closed, densely defined and for every $\lambda \in \mathbb{C}$ where $Re\lambda > \omega$ we have $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(Re\lambda - \omega)^n} \text{ for all } n \in \mathbb{N}.$$

■

Notes

Many books on semigroups give the definitions of strongly continuous semigroups and their properties. On this thesis we referred to ([Bátkai [1], Chap. 1, pg. 3-11], [Goldstein [7], pg. 151], [Nagel [4], Chap. 1 and 2, pg. 1-19, 34-46] and [Pazy [13], Chap. 1, pg. 1-8]).

The generation theorem proof is taken from ([Engel [5], Chap. 2, pg., 73-76], [Nagel [4], Chap. 2, pg. 63-74], [Hille [9], Chap. 12, pg. 237., Thm. 12.2.1], [Pazy [13], Chap. 1, pg. 8-13], [Vrabie [10], Chap. 3, pg. 51-56], [Yosida [18], pg. 15-21]). The Generation Theorem for the general case can be found on (Engel [[5], Chap. 2, pg., 77-79],[Vrabie [10], Chap. 3, pg. 56-58]).

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