

PROXIMAL POINT ALGORITHMS

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Submitted to
Central European University
Department of Mathematics and its Applications

In partial fulfillment of the requirements for the degree of
Master of Science

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Budapest, Hungary

2013

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Introduction

Many nonlinear operator equations (inclusions) are of the form $0 \in A(x)$, where A is a (possibly set-valued) monotone operator in a Hilbert space H . There are different iterative methods that are used to approximate the solutions of this equation. One of the most popular methods is the so-called proximal point algorithm (PPA), first introduced by B. Martinet (1970) and later developed extensively by R.T. Rockafellar and many other researchers. The researchers have investigated the convergence of this iterative process and in some cases gave the rate of convergence of this method. Many applications have been investigated as well. In particular, this algorithm is used to approximate the solutions of some variational inequalities or minimizers of convex cost functionals.

Let us first describe the PPA, as formulated by Rockafellar. Let $A : D(A) \subset H \rightarrow H$ be a (possibly set-valued) maximal monotone operator.

Starting from an arbitrary $x_0 \in H$, the PPA generates recursively a sequence of points $\{x_n\}$ as follows

$$x_{n+1} = (I + \beta_n A)^{-1}(x_n) + e_n, \text{ for all } n \geq 0,$$

where $\{\beta_n\} \subset (0, \infty)$ and $\{e_n\}$ is the error sequence.

Rockafellar proved in 1976 that for (β_n) bounded below away from zero and $\sum_{n=0}^{\infty} \|e_n\| < \infty$, the sequence $\{x_n\}$ generated by the above PPA converges weakly in H to a zero of A , whenever the set of zeros is nonempty.

In this thesis, we first provide a description of the particular case introduced by Martinet. Then, it is shown how Rockafellar derived the general PPA from Martinet's algorithm. Unfortunately, the sequence $\{x_n\}$ converges only weakly, as shown by O. Güler (1991). Many mathematicians (M. V. Solodov and B. F. Svaiter [26], S. Kamimura and W. Takahashi [15], H. K. Xu [28] and others) have tried to modify the PPA in such a way that the new iterative methods generate strongly convergent sequences. On the other hand, the above summability condition on errors is too strong from a numerical point of view. This summability condition can be relaxed and there are already several results in this direction (for examples, see [16]). Furthermore, there are extensions of the PPA to the case of two monotone operators A

and B . These extensions are generalizations of the old method of alternating projections introduced by J. von Neumann in early thirtieths.

The structure of the thesis is organized as follows. In Chapter 1, we recall some fundamental concepts, notations and results, which will be used in the following chapters. Chapter 2 presents a short introduction to the proximal point algorithms. Then, in Chapter 3, we include some important results regarding the boundedness and convergence of the sequences generated by the PPA. In Chapter 4, we discuss some generalizations of the regularization method. In particular, the modified two parameter method as well as the modified four parameter method will be discussed. Finally, Chapter 5 is concerned with the method of alternating resolvents, i.e., the proximal point algorithm involving two monotone operators.

Chapter 1

Preliminaries

In this chapter, we recall some definitions, notations, and results which will be useful tools in proving the results in the following chapters.

1.1 Some concepts and results in nonlinear analysis

Let H be a real Hilbert space with scalar product (\cdot, \cdot) and the corresponding Hilbertian norm $\|x\| = \sqrt{(x, x)}$.

Definition 1.1 *An operator $A : D(A) \subset H \rightarrow H$ is said to be monotone (strongly monotone) if $(x_1 - x_2, y_1 - y_2) \geq 0$ ($(x_1 - x_2, y_1 - y_2) \geq \alpha \|x_1 - x_2\|^2$, for some $\alpha > 0$, respectively), for all $[x_1, y_1], [x_2, y_2] \in A$, i. e., $[x_1, y_1], [x_2, y_2] \in \{(x, y) \in H \times H : x \in D(A), y \in Ax\}$.*

We also say that A is a monotone subset of $H \times H$.

An operator A is said to be *maximal monotone* if in addition to being a monotone operator, $A \subset H \times H$ is not properly included in any other monotone subset of $H \times H$.

Obviously, if A is maximal monotone, so is A^{-1} .

If A is a maximal monotone operator, we define the resolvent and Yosida approximation of A as follows:

Resolvent: $J_t = (I + tA)^{-1}$, $t > 0$,

Yosida approximation: $A_t = t^{-1}(I - J_t)$, $t > 0$.

We also recall that a map $T : H \rightarrow H$ is called *nonexpansive* if for every $x, y \in H$, we have

$$\|Tx - Ty\| \leq \|x - y\|.$$

T is said to be *firmly nonexpansive* if for every $x, y \in H$, we have

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2.$$

It is clear that firmly nonexpansive mappings are also nonexpansive. To characterize a firmly nonexpansive mapping, we use the following lemma.

Lemma 1.2 ([12]). *T is firmly nonexpansive if and only if $2T - I$ is nonexpansive.*

It is worth noting that for a maximal monotone operator A , the resolvent of A , J_t , $t > 0$, is well defined on the whole space H , and is single-valued. One can also see that the projection operator and the resolvent of A are firmly nonexpansive for every $t > 0$. Moreover, the Yosida approximation of A is maximal monotone for every $t > 0$.

We shall now give an important definition.

Definition 1.3 *Let $\varphi : H \rightarrow (-\infty, \infty]$ be a proper, convex, and lower semicontinuous function. The subdifferential of φ is the operator $\partial\varphi : H \rightarrow H$ defined by*

$$\partial\varphi(x) = \{\omega \in H : \varphi(x) - \varphi(v) \leq (\omega, x - v), \text{ for all } v \in H\}.$$

We will also use the following lemma, its proof can be found in many books (see, e.g., [23], page 5).

Lemma 1.4 ([24]). *Let C be a nonempty subset of H . Assume that the sequence (x_n) satisfies the conditions*

- (a) $\lim_{n \rightarrow \infty} \|x_n - q\| = \rho(q)$ exists, for all $q \in C$,
- (b) any weak cluster point of (x_n) belongs to C .

Then, there exists a point $p \in C$ such that (x_n) converges weakly to p .

The following inequality is important in proving some theorems in the following chapters.

Lemma 1.5 (Subdifferential Inequality). *For all $x, y \in H$, we have*

$$\|x + y\|^2 \leq \|y\|^2 + 2(x, x + y).$$

We also need the following inequality:

Lemma 1.6 ([27]). *For any $x \in H$, and $\mu \geq \beta > 0$,*

$$\|x - J_\beta x\| \leq 2\|x - J_\mu x\|,$$

where J_t , $t > 0$, denotes the resolvent of a maximal monotone operator $A : D(A) \subset H \rightarrow H$.

1.2 Some results on sequences and series of real numbers

We start with the following lemma:

Lemma 1.7 Suppose (a_n) and (b_n) are positive sequences such that $\sum_{n=0}^{\infty} b_n = \infty$ and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \alpha \in \mathbb{R}. \text{ Then } \lim_{m \rightarrow \infty} \frac{\sum_{n=1}^m a_n}{\sum_{n=1}^m b_n} = \alpha.$$

Lemma 1.8 Suppose (a_n) , (b_n) , and (y_n) are positive sequences satisfying the following inequality:

$$b_n y_n \leq y_{n-1} - y_n + a_n.$$

Then we have:

- (i) If $(\frac{a_n}{b_n})$ is bounded, then the sequence (y_n) is bounded;
- (ii) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$, then there exists $\lim_{n \rightarrow \infty} y_n$;
- (iii) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=0}^{\infty} b_n = \infty$, then $\lim_{n \rightarrow \infty} y_n = 0$.

Lemma 1.9 (O. A. Boikanyo and G. Moroşanu). Let (s_n) be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - \alpha_n)(1 - \lambda_n)s_n + \alpha_n b_n + \lambda_n c_n + d_n, \quad n \geq 0,$$

where (α_n) , (λ_n) , (b_n) , (c_n) , and (d_n) satisfy the conditions:

- (i) $\alpha_n, \lambda_n \in [0, 1]$, with $\lim_{n \rightarrow \infty} \alpha_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} b_n \leq 0$;
- (iii) $\limsup_{n \rightarrow \infty} c_n \leq 0$;
- (iv) $d_n \geq 0$ for all $n \geq 0$ with $\sum_{n=0}^{\infty} d_n < \infty$.

Then $\lim_{n \rightarrow \infty} s_n = 0$.

For $\lambda_n = 0, \forall n \geq 0$, we obtain

Lemma 1.10 ([28]). Let (s_n) be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n b_n + c_n, \quad n \geq 0,$$

where (α_n) , (b_n) , and (c_n) satisfy the conditions:

- (i) $\alpha_n \in [0, 1]$, with $\lim_{n \rightarrow \infty} \alpha_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $c_n \geq 0$ for all $n \geq 0$, $\sum_{n=0}^{\infty} c_n < \infty$;

(iii) $\limsup_{n \rightarrow \infty} b_n \leq 0$.

Then $\lim_{n \rightarrow \infty} s_n = 0$.

Lemma 1.11 ([18]). *Let (s_n) be a sequence of real numbers that does not decrease that infinity, in the sense that there exists a subsequence (s_{n_i}) of (s_n) such that $s_{n_i} \leq s_{n_{i+1}}$ for all $i \geq 0$. Define an integer sequence $(\tau(n))_{n \geq n_0}$ as*

$$\tau(n) = \max\{n_0 \leq k \leq n : s_k < s_{k+1}\}.$$

Then $\lim_{n \rightarrow \infty} \tau_n = 0$ and for all $n \geq n_0$, $\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}$.

Chapter 2

Description of the Proximal Point Algorithm

Let H be a real Hilbert space with scalar product (\cdot, \cdot) and the Hilbertian norm $\|x\| := \sqrt{(x, x)}$. Let $T : H \rightarrow H$ be a single-valued, monotone and hemicontinuous operator (i.e., for all $x, y \in X$, $\lim_{\lambda \rightarrow 0} T(x + \lambda y) = Tx$ with respect to the weak topology of H). Let $C \subset H$ be a nonempty, convex, closed set.

Let us assume that there exists at least a solution of the variational inequality

$$x \in C \quad \text{and} \quad (Tx, v - x) \geq 0 \quad \text{for all } v \in C. \quad (2.1)$$

In 1970, B. Martinet [20] formulated the following algorithm:

Given $x_n \in C$ one determines $x_{n+1} \in C$ such that

$$(Tx_{n+1}, v - x_{n+1}) + (x_{n+1} - x_n, v - x_{n+1}) \geq 0 \quad \forall v \in C \quad (2.2)$$

for a given starting point $x_0 \in H$.

It is easy to see that

$$x_{n+1} = (I + A)^{-1}x_n, \quad n = 0, 1, 2, \dots \quad (2.3)$$

with $A = T + N_C$, where $N_C(z)$ is the normal cone to C at z :

$$N_C(z) = \{w \in H : (z - v, w) \geq 0 \quad \forall v \in C\}.$$

Obviously, if $z \in \text{Int}C$, then $N_C(z) = \{0\}$.

It is well known that N_C is a maximal monotone operator and therefore $A = T + N_C$ is so (see [23], Theorem 1.4 and 1.6).

On the other hand, (2.1) can be rewritten as

$$0 \in Ax. \quad (2.4)$$

According to Martinet [19], [20], the sequence x_n generated by (2.2) (or (2.3)) is weakly convergent to a solution of (2.1) (or (2.4)) for any starting point $x_0 \in H$.

Obviously, the result above works for cT instead of T with $c > 0$ arbitrary, so A in (2.3) can be replaced by cA since

$$cT + N_C = cT + cN_C = cA.$$

Hence, Martinet's result extends to the algorithm given by

$$x_{n+1} = (I + cA)^{-1}x_n, \quad n = 0, 1, 2, \dots \quad (2.5)$$

where $A = T + N_C$, $c > 0$.

R. T. Rockafellar [25] further extended this algorithm. More precisely, he considered the following iterative scheme

$$x_{n+1} = (I + c_n A)^{-1}x_n + e_n, \quad n = 0, 1, 2, \dots \quad (\text{PPA})$$

for a general maximal monotone operator $A : D(A) \subset H \rightarrow H$, where (c_n) is a sequence of positive numbers, $(e_n) \subset H$ is the sequence of computational errors, and $x_0 \in H$ is a given starting point.

Following the terminology of J. J. Moreau [21], $P = (I + cA)^{-1}$ is called the proximal mapping associated with cA , so Rockafellar called the algorithm above the proximal point algorithm (denoted here (PPA)). In fact, (PPA) is a first order difference equation. Since $c_n > 0$ ($n = 0, 1, 2, \dots$), for each $x_0 \in H$ there exists a unique sequence (x_n) satisfying this difference equation.

In what follows, we elaborate on the concepts of proximal point and proximal mapping following J. J. Moreau [21].

Let $f : H \rightarrow (-\infty, \infty]$ be a proper, convex, lower semicontinuous function. It is well known (see, e.g., [23], Theorem 1.12, p. 36) that its subdifferential, ∂f , is a maximal monotone operator. Consider the function

$$\varphi(u) = \frac{1}{2}\|u - z\|^2 + f(u), \quad u \in H,$$

for a given $z \in H$. Obviously φ is proper, strictly convex, and lower semicontinuous. Let x be its (unique) minimizer, i.e.,

$$0 \in \partial\varphi(x). \quad (2.6)$$

It is easy to see that $D(\partial\varphi) = D(\partial f)$ and

$$\partial\varphi(u) = u - z + \partial f(u), \quad \forall u \in D(\partial f).$$

Thus, (2.6) can be written as

$$z - x \in \partial f(x) \Leftrightarrow x = (I + \partial f)^{-1}z.$$

Point x is said to be the proximal point of z with respect to function f , and is denoted $x = \text{prox}_f z$. The mapping $P = (I + \partial f)^{-1}$, i.e.,

$$z \mapsto Pz = x \quad \forall z \in H,$$

is called the proximal mapping associated with f (or ∂f). Rockafellar extended these definitions to a general maximal monotone operator A . Thus x_{n+1} in (PPA) represents the proximal point of x_n with respect to $c_n A$, calculated with some error, denoted e_n .

If A is the subdifferential of a proper, convex, lower semicontinuous function f , then

$$\begin{aligned} x_{n+1} &= \arg \min_{u \in H} \frac{1}{2} \|u - x_n\|^2 + c_n f(u) \\ &= \arg \min_{u \in H} \Phi_n(u), \quad n = 0, 1, 2, \dots \end{aligned}$$

where

$$\Phi_n(u) = \frac{1}{2c_n} \|u - x_n\|^2 + f(u).$$

The main result provided by Rockafellar [25] is the following

Theorem 2.1 ([25]). *Assume that $A : D(A) \subset H \rightarrow H$ is maximal monotone, with $0 \in R(A)$; $(c_n) \subset (0, \infty)$, and $\liminf_{n \rightarrow \infty} c_n > 0$; $\sum_{n=0}^{\infty} \|e_n\| < \infty$. Then for every $x_0 \in H$, the sequence (x_n) generated by (PPA) converges weakly to a zero of A (i.e., a solution of (2.4)).*

The proof of Theorem 2.1 is omitted here, since there is a better result due to H. Brézis and P. L. Lions [9] that will be discussed later.

The study on (PPA) has shown that it has applications in nonlinear analysis and convex optimization. In particular, (PPA) is an important iterative method that can be used to approximate minimizers of convex functions. In the above case, x_{n+1} is exactly the minimizer of the function Φ_n , and, the sequence (x_n) converges to a minimizer of f .

The question whether the convergence of (x_n) given by (PPA) is strong was left as an open one by Rockafellar. Unfortunately, the answer is negative, as proved by O. Güler [13]. On the other hand, the summability condition on errors ($\sum_{n=0}^{\infty} \|e_n\| < \infty$) is too strong from a numerical point of view. This summability condition can be relaxed. In the next chapter, some important results regarding the PPA will be presented.

Chapter 3

Main Results on the Proximal Point Algorithm

The aim of this chapter is to systematically present the most significant results concerning the proximal point algorithm (PPA). The main results regarding the boundedness and convergence of the sequence generated by the PPA will be given in the following sections. In addition, the rate of convergence to the minimum value of a convex, proper, and lower semicontinuous (LSC) function will be discussed.

As usual, throughout this chapter, H is a real Hilbert space with scalar product (\cdot, \cdot) and the corresponding Hilbertian norm $\|x\| = \sqrt{(x, x)}$. We denote here the weak convergence by \rightharpoonup and the strong convergence by \rightarrow . In the sequel, $A : D(A) \subset H \rightarrow H$ is a maximal monotone operator. Recall that the PPA generates a sequence (x_n) as follows

$$x_{n+1} = (I + c_n A)^{-1} x_n + e_n, \quad n = 0, 1, 2, \dots \quad (\text{PPA})$$

where $(c_n) \subset (0, \infty)$, $(e_n) \subset H$ is the sequence of computational errors, and $x_0 \in H$ is a given starting point.

If we set $z_n = x_n - e_{n-1}$ and $f_n = -e_{n-1}$ for all $n = 1, 2, \dots$, (PPA) becomes

$$z_{n+1} = (I + c_n A)^{-1} (z_n - f_n), \quad n = 0, 1, 2, \dots \quad (\text{PPA}')$$

where $(c_n) \subset (0, \infty)$, $(f_n) \subset H$ is the sequence of computational errors, and $z_0 = x_0 \in H$ is a given starting point.

For the purpose of stating the results on the boundedness and convergence of the PPA, in this chapter, we shall consider both forms (PPA) and (PPA').

3.1 Boundedness

It is worth noting that in Theorem 2.1, Chapter 2, the sufficient condition provided for the maximal monotone operator A was that $A^{-1}0 \neq \emptyset$. This assumption is also given in Theorems 3.6 and 3.9 in this Chapter. Rockaffelar [25], and later, B. Djafari Rouhani and H. Khatbzadeh [11] showed that under the summability on the error sequence and suitable conditions on the parameter sequence, the set $A^{-1}0$ is nonempty if and only if the sequence generated by the PPA is bounded.

In the following, we state some results concerning the boundedness of the sequence (x_n) generated by the PPA, including the special case when A is the subdifferential of a proper, convex, and LSC function. In particular, we will see in this section that the boundedness of (z_n) is possible without assuming summability of the error sequence.

Morosanu [23] showed that the boundedness of (x_n) can be obtained under reasonable hypotheses on (c_n) and (e_n) .

Theorem 3.1 ([23]). *Assume that $A : D(A) \subset H \rightarrow H$ is maximal monotone and coercive, i.e., A satisfies*

$$\lim_{\substack{\|\xi\| \rightarrow \infty \\ [\xi, \mu] \in A}} \frac{(\mu, \xi)}{\|\xi\|} = \infty. \quad (3.1)$$

Assume further that

$$c_n \geq \epsilon > 0 \text{ and } \|e_n\| < C, n = 0, 1, \dots \quad (3.2)$$

Then the (x_n) given by (PPA) is bounded.

In 2012, H. Khathzadeh [16] extended Theorem 3.1 above. He showed the boundedness of (z_n) (equivalently, the boundedness of (x_n)) under weaker conditions on the errors and parameters.

Theorem 3.2 ([16]). *Let $A : D(A) \subset H \rightarrow H$ be a coercive maximal monotone operator. If the sequence $\left(\frac{\|f_n\|}{c_n}\right)$ is bounded, then for every $x_0 \in H$, the sequence (z_n) generated by (PPA') is bounded.*

Proof. There exists $C > 0$ such that for each $n \geq 0$, $\frac{\|f_n\|}{c_n} < C$. Since A is coercive, there is a $K > 0$, and $z' \in H$ such that for all $[z, \omega] \in A$, with $\|z\| > K$, $\frac{(\omega, z - z')}{\|z - z'\|} > C$.

Suppose that there exists n such that $\|z_{n+1} - z'\| > K$. By (PPA'), we have

$$z_n - z_{n+1} - f_n \in c_n A z_{n+1}. \quad (3.3)$$

which implies that

$$c_n C + \|z_{n+1} - z'\| \leq \|z_n - z'\| + \|f_n\|. \quad (3.4)$$

It follows that

$$\|z_{n+1} - z'\| \leq \|z_n - z'\| + c_n \left(\frac{\|f_n\|}{c_n} - C \right) < \|z_n - z'\|$$

for each $n \geq 0$, such that $\|z_{n+1} - z'\| > K$.

Thus, for all $n \geq 0$,

$$\|z_{n+1} - z'\| \leq \max\{\|x_0 - z'\|, K\}. \blacksquare$$

The boundedness of (x_n) generated by (PPA) leads to the interesting fact that provided $A^{-1}0 \neq \emptyset$, under the suitable conditions on the errors, (PPA) is equivalent to (PPA').

Theorem 3.3 ([23]). *Let (x_n) be the sequence generated by (PPA), where $A = \partial\varphi$, $\varphi : H \rightarrow (-\infty, \infty]$ is proper, convex, and LSC; $F := A^{-1}0 \neq \emptyset$; $(\|e_n\|) \in l^2$, $c_n \geq \epsilon > 0$, $n = 0, 1, \dots$*

If in addition (x_n) is bounded, then

$$\lim_{n \rightarrow \infty} \varphi(x_n - e_{n-1}) = \inf_{z \in H} \varphi(z).$$

Proof. Setting $z_n = x_n - e_{n-1}$, (PPA) becomes

$$\begin{cases} z_n + e_{n-1} \in z_{n+1} + c_n A z_{n+1}, & n = 1, 2, \dots; \\ z_0 = x_0. \end{cases} \quad (3.5)$$

By multiplying (3.5) by $z_{n+1} - z_n$ and using simple technical computations, one can get

$$\frac{1}{2c_n} \|z_{n+1} - z_n\|^2 + \varphi(z_{n+1}) - \varphi(z_n) \leq \frac{1}{2\epsilon} \|e_{n-1}\|^2, \quad n = 1, 2, \dots \quad (3.6)$$

which implies that

$$\varphi(z_{n+1}) - \frac{1}{2\epsilon} \sum_{i=0}^n \|e_i\|^2 \leq \varphi(z_n) - \frac{1}{2\epsilon} \sum_{i=0}^{n-1} \|e_i\|^2, \quad n = 1, 2, \dots \quad (3.7)$$

Using the fact that $\inf_{z \in H} \varphi(z) = \varphi(p) > -\infty$, $\forall p \in F$ and $(\|e_n\|) \in l^2$, by (3.7), one can easily see that

$$\lim_{n \rightarrow \infty} \varphi(z_{n+1}) = l \in \mathbb{R}. \quad (3.8)$$

From (3.6) and (3.8), it follows that

$$\lim_{n \rightarrow \infty} c_n^{-\frac{1}{2}} \|z_{n+1} - z_n\| = 0.$$

Taking the limit in

$$0 \leq \varphi(z_n) - \varphi(p) \leq (z_n, z_n - p), \quad \forall p \in F$$

with noting that (z_n) is bounded (since (x_n) is so) one can get

$$\lim_{n \rightarrow \infty} \varphi(z_n) = \lim_{z \in H} \varphi(z). \quad \blacksquare$$

We shall see now that the boundedness of the sequences generated by either (PPA) or (PPA') guarantees $F := A^{-1}0 \neq \emptyset$, under reasonable conditions on errors and parameters. For the sequence generated by (PPA), we consider a refinement of the following result due to Rockafellar [25].

Theorem 3.4 ([25]). *Let $A : D(A) \subset H \rightarrow H$ be a maximal monotone operator, with $0 \in R(A)$. Assume that*

$$\sum_{n=0}^{\infty} c_n = \infty; \tag{3.9}$$

and

$$\sum_{n=0}^{\infty} \|e_n\| < \infty. \tag{3.10}$$

Then the sequence (x_n) generated by (PPA) is bounded if and only if $A^{-1}0 =: F$ is nonempty.

H. Khatibzadeh [16] showed that under the non-summability condition on the error sequence, if the sequence generated by (PPA') is bounded, then the set of all zeros of A is nonempty.

Theorem 3.5 ([16]). *Let (z_n) be a bounded sequence generated by (PPA'). If $\lim_{n \rightarrow \infty} \frac{\|f_n\|}{c_n} = 0$ and $\sum_{n=0}^{\infty} c_n = \infty$ then $A^{-1}0 =: F$ is nonempty.*

Remark 3.1.1. In the case when $A = \partial\varphi$ where $\varphi : H \rightarrow (-\infty, \infty]$ is proper, convex, and LSC; if $\lim_{\|z\| \rightarrow \infty} \varphi(z) = \infty$, and $(\|e_n\|) \in l^2$, $c_n \geq \epsilon > 0$, $n = 0, 1, \dots$, then the sequence (x_n) generated by (PPA) is bounded for any given starting point $x_0 \in H$. Indeed, this conclusion can be shown by the same technical steps as in the proof of Theorem 3.2, using the definition of the subdifferential.

Remark 3.1.2. It follows from Theorem 3.1 and Theorem 3.3 that the conclusion of Theorem 3.3 still holds if we replace the boundedness of (x_n) by the following assumption

$$\lim_{\|u\| \rightarrow \infty} \varphi(u) = \infty.$$

This assumption is the sufficient condition in order to get the boundedness of (x_n) , which is essential in Theorem 3.3, according to Theorem 3.1. Furthermore, if F in Theorem 3.3 is singleton, say $F = \{p\}$, then we can see that $x_n \rightarrow p$ for any given starting point $x_0 \in H$.

3.2 Convergence

3.2.1 Weak convergence

In this section, we shall first present the generalization of Rockfellar's result due to H. Brézis and P. L. Lions. The most general results on weak convergence of PPA will also be stated.

Theorem 3.6 ([9]). *Let $A : D(A) \subset H \rightarrow H$ be the subdifferential of a proper, convex, and LSC function $\varphi : H \rightarrow (-\infty, \infty]$, with $0 \in R(A)$. Assume that*

$$\sum_{n=0}^{\infty} c_n = \infty; \quad (3.11)$$

and

$$\sum_{n=0}^{\infty} \|e_n\| < \infty. \quad (3.12)$$

Then the sequence (x_n) generated by (PPA) converges weakly to some point $p \in A^{-1}0 =: F$.

This conclusion still holds if A is assumed to be a general maximal monotone operator, but replacing (3.11) by the stronger assumption

$$\sum_{n=0}^{\infty} c_n^2 = \infty. \quad (3.13)$$

Proof. It is sufficient to prove the theorem in the exact case (i.e., $e_n = 0$, $n = 0, 1, \dots$). Indeed, we assume that the theorem is proved in the exact case. For a given k , let the sequence $(\xi_n(k))$ be defined by

$$\xi_0(k) = x_k, \quad \xi_1(k) = (I + c_k A)^{-1} x_k, \quad \dots, \quad \xi_{n+1}(k) = (I + c_{n+k} A)^{-1} (\xi_n(k)), \quad \dots,$$

where (x_n) is the sequence generated by (PPA). Moreover, one can see that

$$\|\xi_n(k) - \xi_{n+1}(k-1)\| \leq \|e_{k-1}\| \quad (3.14)$$

which implies

$$\|\xi(k) - \xi(k-1)\| \leq \|e_{k-1}\|.$$

Thus, $(\xi_n(k))$ is a Cauchy sequence and, hence, converges to a point a . Again, from (3.14), we have

$$\|x_k - \xi_{n+1}(k-n-1)\| \leq \sum_{i=k-n-1}^{k-1} \|e_i\| \text{ for } k > n.$$

It follows that

$$\|x_{k+n} - \xi_{n+1}(k-1)\| \leq \sum_{i=k-1}^{k+n-1} \|e_i\|.$$

On the other hand,

$$x_{k+n} - a = [x_{k+n} - \xi_{n+1}(k-1)] + [\xi_{n+1}(k-1) - \xi(k-1)] + [\xi(k-1) - a].$$

Therefore, for k is large enough,

$$\sum_{i=k-1}^{k+n-1} \|e_i\| + \|\xi(k) - a\| \leq \epsilon.$$

Then

$$\limsup_{n \rightarrow \infty} |(x_{n+k} - a, h)| \leq \epsilon \|h\|, \quad \forall h \in H,$$

or $x_n \rightharpoonup a$.

In the following, (x_n) denotes the sequence generated by (PPA) with $e_n = 0$, $n = 0, 1, \dots$

We shall consider first the proof of the second part. We have

$$x_n \in x_{n+1} + c_n A x_{n+1}, \quad n = 0, 1, \dots \quad (3.15)$$

Setting $y_{n+1} = \frac{1}{c_n}(x_{n+1} - x_n)$, $n = 0, 1, \dots$, from (3.15) and the maximal monotonicity of A , we get

$$(y_{n+1} - y_n, y_n) \leq 0, \quad n = 0, 1, \dots$$

which implies that the sequence $(\|y_n\|)$ is nonincreasing.

On the other hand, it follows from (3.15) that for every $q \in F$,

$$\|x_{n+1} - q\|^2 + c_n^2 \|y_{n+1}\|^2 \leq \|x_n - q\|^2, \quad n = 0, 1, \dots \quad (3.16)$$

Thus,

$$\|y_{n+1}\| \leq \left(\sum_{i=0}^n c_i^2 \right)^{-\frac{1}{2}} \text{dist}(x_0, F). \quad (3.17)$$

From (3.13) and (3.17), $y_n \rightarrow 0$.

It follows from (3.16) that for any $q \in F$, the sequence $(\|x_n - q\|)$ is nonincreasing. Applying Opial's Lemma, there exists a $p \in F$ such that $x_n \rightharpoonup p$.

Now, assume that A is the subdifferential of a proper, convex, and LSC function $\varphi : H \rightarrow (-\infty, \infty]$ with $F \neq \emptyset$. Here, F is the set of all minimizers of φ . Note that $(\varphi(x_n))$ is a nonincreasing sequence. Indeed,

$$\begin{aligned} \varphi(x_{n+1}) - \varphi(x_n) &\leq (A x_{n+1}, x_{n+1} - x_n) \\ &= \frac{1}{c_n} (x_n - x_{n+1}, x_{n+1} - x_n) \\ &\leq 0, \quad n = 1, 2, \dots \end{aligned}$$

On the other hand, for any $q \in F$, we have

$$\begin{aligned}
c_k[\varphi(x_{k+1}) - \inf \varphi] &= c_k[\varphi(x_{k+1}) - \varphi(q)] \\
&\leq c_k(Ax_{k+1}, x_{k+1} - q) \\
&= (x_k - x_{k+1}, x_{k+1} - q) \\
&\leq \frac{1}{2}\|x_k - q\|^2 - \frac{1}{2}\|x_{k+1} - q\|^2,
\end{aligned}$$

which implies

$$\sum_{k=0}^{n-1} c_k[\varphi(x_{k+1}) - \inf \varphi] \leq \frac{1}{2}\|x_0 - q\|^2.$$

Since $(\varphi(x_n))$ is nonincreasing, it follows that

$$\varphi(x_n) - \inf \varphi \leq \frac{\|x_0 - q\|^2}{2 \sum_{k=0}^{n-1} c_k}.$$

Thus, $\varphi(x_n) \rightarrow \inf \varphi$, which shows that every weak cluster point of (x_n) belongs to F . The conclusion follows by Opial's Lemma. \blacksquare

Remark 3.2.1. It is clear that the second part of Theorem 3.6 is a significant generalization of Theorem 2.1, Chapter 2. Indeed, instead of the assumption on parameters as in (3.11), the weak convergence of (x_n) can be obtained even with the condition (3.13) only.

An interesting fact is that in the results on weak convergence of (x_n) generated by (PPA) mentioned above, the assumption $F := A^{-1}0 \neq \emptyset$ was provided. However, it was shown in Theorems 3.4 and 3.5 that the boundedness of (x_n) ((z_n) respectively) implies the existence of a zero of A , under suitable assumptions on error and parameter sequences.

We are now concerned with the PPA in the form (PPA'). In the sequel, we denote by y_n the element $\frac{z_{n-1} - z_n - f_{n-1}}{c_{n-1}} \in H$, $n \geq 1$.

Theorem 3.7 ([16]). *Let (z_n) be a bounded sequence generated by (PPA'), and $\sum_{n=0}^{\infty} c_n^2 = \infty$. If $\sum_{n=0}^{\infty} \frac{\|f_n\|^2}{c_n^2} < \infty$, and $\lim_{n \rightarrow \infty} \frac{\|f_n\|}{c_n^2} = 0$, then every weak cluster point of (z_n) belongs to $A^{-1}0$.*

Proof. Since A is monotone,

$$\left(y_n - y_{n+1}, y_{n+1} + \frac{f_n}{c_n} \right) \geq 0$$

which implies that

$$\|y_{n+1}\|^2 \leq \|y_n\|^2 + \frac{\|f_n\|^2}{c_n^2}. \quad (3.18)$$

Using the assumption $\sum_{n=1}^{\infty} \frac{\|f_n\|^2}{c_n^2} < \infty$ and applying the result of H. Khatibzadeh [16] (see Theorem 4.1) we get that $A^{-1}0 \neq \emptyset$. Assume $p \in A^{-1}0$. The monotonicity of A implies that

$$(z_n - z_{n+1} - f_n, z_{n+1} - p) \geq 0.$$

It follows that

$$\|z_{n+1} - z_n\|^2 \leq \|z_n - p\|^2 - \|z_{n+1} - p\|^2 + M\|f_n\|,$$

where $M := 2 \sup_{n \geq 0} \|z_n - p\|$.

Summing up from $i = 0$ to n , we have

$$\sum_{i=0}^n c_i^2 \left(\|y_{i+1}\|^2 + \frac{\|f_i\|^2}{c_i^2} - 2\|y_{i+1}\| \frac{\|f_i\|}{c_i} \right) \leq \|x_0 - p\|^2 - \|z_{n+1} - p\|^2 + M \sum_{i=0}^n \|f_i\|.$$

Thus,

$$\sum_{i=0}^n c_i^2 \|y_{i+1}\|^2 \leq 2L \sum_{i=0}^n c_i \|f_i\| + \|x_0 - p\|^2 - \|z_{n+1} - p\|^2 + M \sum_{i=0}^n \|f_i\|,$$

where $L := \sup_{n \geq 0} \|y_n\|$.

From (3.18), it is easy to see that for any $i < n + 1$,

$$\|y_{n+1}\|^2 \leq \|y_{i+1}\|^2 + \sum_{j=i}^n \frac{\|f_j\|^2}{c_j^2}.$$

Thus,

$$\|y_{n+1}\|^2 \sum_{i=0}^n c_i^2 \leq \sum_{i=0}^n c_i^2 \sum_{j=i}^n \frac{\|f_j\|^2}{c_j^2} + 2L \sum_{i=0}^n c_i \|f_i\| + \|x_0 - p\|^2 - \|z_{n+1} - p\|^2 + M \sum_{i=0}^n \|f_i\|.$$

Applying Lemma 1.7, $\lim_{n \rightarrow \infty} y_n = 0$. If $z_{n_k} \rightharpoonup q$, then $q \in A^{-1}0$ since A is demiclosed. ■

Theorem 3.8 ([16]). *Let (z_n) be a bounded sequence generated by (PPA'), and $A = \partial\varphi$, where $\varphi : H \rightarrow (-\infty, \infty]$ is a proper, convex, and LSC function. If $\sum_{n=0}^{\infty} c_n = \infty$, $\sum_{n=0}^{\infty} \frac{\|f_n\|^2}{c_n} < \infty$, and $\lim_{n \rightarrow \infty} \frac{\|f_n\|}{c_n} = 0$, then every weak cluster point of (z_n) belongs to $A^{-1}0$, and $\lim_{n \rightarrow \infty} \varphi(z_n) = \inf_{z \in H} \varphi(z)$.*

The techniques for proving Theorem 3.8 are similar to those in proof of Theorem 3.7. One can prove Theorem 3.8 by applying Lemma 1.7 starting from the subdifferential inequality to get that $A^{-1}0$ is nonempty. The proof will be completed by using the hypothesis to obtain that $\varphi(z_n)$ converges to $\varphi(p)$ with $p \in A^{-1}0$.

Remark 3.2.2. Theorems 3.7 and 3.8 cannot guarantee the weak convergence of (z_n) to a point $p \in A^{-1}0$. However, if in addition to the hypotheses of Theorem 3.7, A is strictly monotone, and if φ in Theorem 3.8 is strictly convex, in other words, $A^{-1}0$ is singleton, then Theorems 3.7 and 3.8 show that $z_n \rightharpoonup p$, where p is the unique element of $A^{-1}0$.

Remark 3.2.3. It is worth mentioning that the boundedness of (z_n) is essential in both Theorems 3.7 and 3.8. Besides, they cannot imply the weak convergence of (z_n) to some point $p \in A^{-1}0$ unless A and φ satisfy the conditions as mentioned above such that $A^{-1}0$ is singleton. However, they improve the errors in the proximal point algorithm. We can see in these theorems that under a suitable assumption on the parameter sequence, the sequence of errors is allowed to be unbounded.

3.2.2 Strong convergence

Worth noticing is that an efficient algorithm should generate a strongly convergent sequence. We will consider the strong convergence of the PPA in the form (PPA'). H. Khatibzadeh [16] showed that under appropriate assumptions on errors and parameters without summability assumption on the error sequence, (PPA') generates a strongly convergent sequence.

Theorem 3.9 ([16]). *Let (z_n) be the sequence generated by (PPA'), and A be a maximal monotone and strongly monotone operator. If $\sum_{n=0}^{\infty} c_n = \infty$, and $\lim_{n \rightarrow \infty} \frac{\|f_n\|}{c_n} = 0$ then (z_n) converges strongly to the unique element p of $A^{-1}0$.*

Proof. Applying Theorem 3.2, the sequence (z_n) is bounded. The boundedness of (z_n) implies that $A^{-1}0$ is nonempty. In other words, $A^{-1}0$ has a unique element p , by the strong monotonicity of A .

Since A is strongly monotone, for some $\alpha > 0$

$$(z_n - z_{n+1} - f_n, z_{n+1} - p) \geq \alpha c_n \|z_{n+1} - p\|^2$$

which implies

$$2\alpha c_n \|z_{n+1} - p\|^2 \leq \|z_n - p\|^2 - \|z_{n+1} - p\|^2 + 2M \|f_n\|^2,$$

where $M := \sup_{n \geq 0} \|z_n - p\|$.

The proof is completed by applying Lemma 1.8. ■

In the case when A is the subdifferential of a proper, convex, LSC, and even function, H. Brézis and P. L. Lions showed that under the condition that $\sum_{n=1}^{\infty} \|e_n\| < \infty$ only, the sequence (x_n) generated by (PPA) is strongly convergent. It is not sure that the limit of the sequence (x_n) is a zero of A unless the parameters satisfy $\sum_{n=1}^{\infty} c_n = \infty$. More precisely, we have,

Theorem 3.10 ([9]). *Let $A = \partial\varphi$ where $\varphi : H \rightarrow (-\infty, \infty]$ is proper, convex, LSC, and even. If*

$$\sum_{n=0}^{\infty} \|e_n\| < \infty$$

then for every $(c_n) \subset (0, \infty)$ and for every $x_0 \in H$, (x_n) generated by (PPA) is strongly convergent. If in addition

$$\sum_{n=0}^{\infty} c_n = \infty$$

then $x_n \rightarrow p \in A^{-1}0$.

Remark 3.2.4. We can also see that, the sequence (x_n) generated by (PPA) is strongly convergent if in addition to the hypotheses of Theorem 3.6, one of the following assumptions is satisfied:

1. Admitting that for some $c > 0$ (or, equivalently, for every $c > 0$) $(I + cA)^{-1}$ is a compact operator;
2. A is strongly monotone.

3.3 Rate of convergence

A rate of convergence to the minimum value of a proper, convex, and lower semicontinuous function φ was first announced by Güler [13] with an important assumption that the sequence (x_n) generated by (PPA) with $A = \partial\varphi$ converges strongly to a minimizer p of φ . Recently, H. Khatibzadeh [16] generalized the result of Güler without assuming $x_n \rightarrow p$, but assuming only the boundedness of (x_n) .

Theorem 3.11 ([16]). *Let $A = \partial\varphi$ where $\varphi : H \rightarrow (-\infty, \infty]$ is proper, convex, and LSC. Suppose that the sequence (x_n) generated by (PPA) with $e_n = 0$, $n = 0, 1, \dots$ is bounded. If $\sum_{n=0}^{\infty} c_n = \infty$ then*

$$\varphi(x_n) - \varphi(p) = o\left(\left(\sum_{i=0}^{n-1} c_i\right)^{-1}\right),$$

where p is a minimum point of φ .

Proof. By the definition of subdifferential, we have

$$\begin{aligned}c_n(\varphi(x_{n+1}) - (p)) &\leq (x_n - x_{n+1}, x_{n+1} - p) \\ &\leq \frac{1}{2}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2)\end{aligned}$$

which implies that

$$\sum_{i=0}^n c_i(\varphi(x_{i+1}) - \varphi(p)) < \frac{1}{2}\|x_0 - p\|^2.$$

Since the sequence $(\varphi(x_{n+1}) - \varphi(p))$ is nonincreasing the conclusion follows. ■

Chapter 4

Modified Proximal Point Algorithms

Based on the previous information that the proximal point algorithm fails to generate a strongly convergent sequence in general (to a solution of the equation (inclusion) $0 \in Ax$ as mentioned above), many researchers have been studying in many directions in order to get an efficient algorithm. Modifying Rockafellar's algorithm has been such a way that the strong convergence can be ensured. One of the most famous methods is the so-called Tikhonov method or the regularization method. The sequence (x_n) is generated by this method as follows:

$$x_n = J_{\beta_n}(0), \quad n = 0, 1, \dots \quad (4.1)$$

where $(\beta_n) \subset (0, \infty)$, and $\lim_{n \rightarrow \infty} \beta_n = \infty$, and $J_\beta = (I + \beta A)^{-1}$ is the resolvent operator of A .

In their 1996 paper, Lehdili and Moudafi [17] introduced an algorithm which was obtained by combining the regularization method of Tikhonov and the PPA. This algorithm generates a sequence (x_n) as follows:

$$x_{n+1} = J_{\beta_n}(\alpha_n x_n + e_n), \quad n = 0, 1, \dots \quad (4.2)$$

where $(\beta_n) \subset (0, \infty)$, $(\alpha_n) \subset (0, 1)$, and $\lim_{n \rightarrow \infty} \alpha_n = 1$, $(e_n) \subset H$ is the sequence of computational errors, and $x_0 \in H$ is a given starting point.

Xu [27] extended the algorithm given by (4.2) to

$$x_{n+1} = J_{\beta_n}(\alpha_n u + (1 - \alpha_n)x_n + e_n), \quad n = 0, 1, \dots \quad (4.3)$$

for given vectors $x_0, u \in H$, $(\beta_n) \subset (0, \infty)$, $(\alpha_n) \subset (0, 1)$, and $(e_n) \subset H$ is the computational error sequence.

It was shown that under the summability condition on the errors and reasonable conditions on parameters, the sequences (x_n) generated by either (4.2) or (4.3) strongly converges to a zero of A . After that, much research has been devoted to generalize Xu's algorithm in order to get a more efficient algorithm.

In this chapter, we will discuss generalizations of the regularization method. We shall start first with the convergence of the modified two parameter method under the suitable conditions on the two parameters. Finally, a four parameter regularization method will be discussed.

4.1 The two parameter regularization method

Recently, O. A. Boikanyo and G. Moroşanu [4] proposed a generalization of Xu's algorithm. This new algorithm generates a sequence (x_n) as follows

$$x_{n+1} = J_{\beta_n}(\alpha_n u + (1 - \alpha_n)(x_n + e_n)), \quad n = 0, 1, \dots \quad (4.4)$$

where $(\beta_n) \subset (0, \infty)$, $(\alpha_n) \subset (0, 1)$, and $(e_n) \subset H$ is the sequence of computational errors, for given vectors $x_0, u \in H$.

Unsurprisingly, provided $\lim_{n \rightarrow \infty} \alpha_n = 0$, for n large enough, the argument of the resolvent operator in (4.4) becomes arbitrarily close to that in the PPA. Based on the fact that for large β , J_β approximates the zero of A which is the projection of u onto $F := A^{-1}0$, the following theorem shows us a different approach. Instead of $\lim_{n \rightarrow \infty} \alpha_n = 0$, O. A. Boikanyo and G. Moroşanu have proposed the assumption that $\lim_{n \rightarrow \infty} \alpha_n = 1$, which implies that the argument of J_{β_n} tends to u as n tends to infinity.

Theorem 4.1 ([4]). *Assume that $A : D(A) \subset H \rightarrow H$ is a maximal monotone operator, with $F := A^{-1}0 \neq \emptyset$. For any fixed $x_0, u \in H$, let the sequence (x_n) be generated by (4.4) where $(\beta_n) \subset (0, \infty)$, and $(\alpha_n) \subset (0, 1)$, for all $n \geq 0$. If (e_n) is bounded, $\lim_{n \rightarrow \infty} \alpha_n = 1$, and $\lim_{n \rightarrow \infty} \beta_n = \infty$, then (x_n) converges strongly to $P_F u$, the metric projection of u onto F .*

Proof. The key idea to prove this theorem is to show that the sequence (x_n) is bounded. This is the conclusion of the following Lemma, given by O. A. Boikanyo and G. Moroşanu [4].

Lemma 4.2 *Assume that $A : D(A) \subset H \rightarrow H$ is a maximal monotone operator, with $F := A^{-1}0 \neq \emptyset$. For any fixed $x_0, u \in H$, let the sequence (x_n) be generated by (4.4) where $(\beta_n) \subset (0, \infty)$, and $(\alpha_n) \subset (0, 1)$, for all $n \geq 0$. If $\frac{(1-\alpha_n)}{\alpha_n} \|e_n\| \leq C$ for some $C > 0$, then (x_n) is bounded.*

We shall now prove the Lemmar 4.2 in order to prove the theorem. Setting $z_n = \alpha_n u + (1 - \alpha_n)(x_n + e_n)$, $n = 0, 1, \dots$. Since J_{β_n} is nonexpansive, for each $p \in F$, we have

$$\|x_{n+1} - p\| = \|J_{\beta_n} z_n - J_{\beta_n} p\| \leq \|z_n - p\|.$$

Thus, it is sufficient to show that (z_n) is bounded.

By the definition,

$$z_{n+1} = \alpha_{n+1} u + (1 - \alpha_{n+1})e_{n+1} + (1 - \alpha_{n+1})J_{\beta_n} z_n, \quad n = 0, 1, \dots \quad (4.5)$$

There exists $M > 0$ such that for some $p \in F$,

$$\|u - p\| + \frac{1 - \alpha_n}{\alpha_n} \|e_n\| \leq M \text{ and } \|z_0 - p\| \leq 2M, \text{ for all } n \geq 0.$$

From (4.5), we obtain

$$\|z_{n+1} - p\|^2 \leq (1 - \alpha_{n+1})^2 \|z_n - p\|^2 + 2M\alpha_{n+1} \|z_{n+1} - p\|. \quad (4.6)$$

Assuming that for some n , $\|z_n - p\| \leq 2M$, we get from (4.6)

$$(\|z_{n+1} - p\| - M\alpha_{n+1})^2 \leq (2M - M\alpha_{n+1})^2,$$

which shows that $\|z_{n+1} - p\| \leq 2M$.

Therefore, (z_n) is bounded.

Now we have that (x_n) is bounded. Moreover, one can see that

$$\begin{aligned} \|x_{n+1} - P_F u\| &\leq \|x_{n+1} - J_{\beta_n} u\| + \|J_{\beta_n} u - P_F u\| \\ &\leq (1 - \alpha_n) \|x_n - u + e_n\| + \|J_{\beta_n} u - P_F u\|. \end{aligned} \quad (4.7)$$

Using the result proved independently by R. E. Buck, Jr. [10] and G. Moroşanu [22], we have $\lim_{n \rightarrow \infty} \|J_{\beta_n} u - P_F u\| = 0$.

The proof is completed by passing to the limit in (4.7) as n tends to infinity. ■

We can see from the proof of Theorem 4.1 that $x_{n+1} = J_{\beta_n}(z_n)$, $n = 0, 1, \dots$. However, for (e_n) bounded and $\lim_{n \rightarrow \infty} \alpha_n = 1$, algorithms (4.4) and (4.5) are not equivalent. Indeed, under the conditions provided in Lemma 4.2, the sequence (z_n) is bounded, and so is (x_n) . Consequently, for any $u \notin F$, (z_n) converges strongly to $u \neq P_F u$, while $P_F u$ is the strong limit of (x_n) .

4.2 The four parameter regularization method

In this section, we shall discuss an algorithm for estimating the convergence rate of a sequence that approximates minimum values of certain functionals. It is the four

parameter proximal point algorithm introduced by O. A. Boikanyo and G. Moroşanu [3]. This generalized regularization method generates a sequence (x_n) as follows:

$$x_{n+1} = J_{\beta_n}(\alpha_n u + \lambda_n x_n + \gamma_n T x_n + e_n), \quad n = 0, 1, \dots \quad (4.8)$$

where $T : H \rightarrow H$ is a nonexpansive map, $(\beta_n) \subset (0, \infty)$, $\alpha_n, \lambda_n, \gamma_n \in [0, 1]$, with $\alpha_n + \lambda_n + \gamma_n = 1$, for all $n \geq 0$.

It is worth noting that for $\lambda_n = 0, n = 0, 1, \dots$ and T is the identity operator, the algorithm (4.8) is exactly the Xu's regularization method ([27]) as given by (4.4).

They showed in [3] that under some reasonable conditions on the parameters and errors, the sequence (x_n) strongly converges to the projection of u on $F := A^{-1}0$, provided that $\emptyset \neq F \subset \text{Fix}T$, where $\text{Fix}T = \{x \in H : x = Tx\}$ is the set of all fixed points of T . Later, as in the following theorem, it was shown by the same authors that under weaker assumptions on $\beta_n, \alpha_n, \lambda_n, \gamma_n$ and e_n , (4.8) can generate a strongly convergent sequence (x_n) whose limit is a zero of A nearest to u .

Theorem 4.3 ([2]). *Let $A : D(A) \subset H \rightarrow H$ be a maximal monotone operator, and $T : H \rightarrow H$ be a nonexpansive map with $\emptyset \neq F := A^{-1}0 \subset \text{Fix}T$, where $\text{Fix}T$ is the fixed point set of T . For arbitrary but fixed $x_0, u \in H$, let (x_n) be the sequence generated by (4.8), where $(\beta_n) \subset (0, \infty)$, and $\alpha_n, \lambda_n, \gamma_n \in [0, 1]$, with $\alpha_n + \lambda_n + \gamma_n = 1$, for all $n \geq 0$. Assume that $\lim_{n \rightarrow \infty} \alpha_n = 0$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\beta_n \geq \beta$ for some $\beta > 0$.*

If either $\sum_{n=0}^{\infty} \|e_n\| \leq \infty$ or $\lim_{n \rightarrow \infty} \frac{\|e_n\|}{\alpha_n} = 0$, then (x_n) converges strongly to $P_F u$, the metric projection of u onto F .

Proof. Denote by (z_n) the sequence generated by (4.8) in the exact iterative process, i.e., $e_n = 0, n = 0, 1, \dots$

$$z_{n+1} = J_{\beta_n}(\alpha_n u + \lambda_n z_n + \gamma_n T z_n), \quad n = 0, 1, \dots \quad (4.9)$$

We shall first show that (x_n) is bounded and so is (z_n) . Indeed, assume that $\sum_{n=0}^{\infty} \|e_n\| \leq \infty$. For some $p \in F$, we have

$$\|x_{n+1} - p\| \leq \alpha_n \|u - p\| + (1 - \alpha_n) \|x_n - p\| + \|e_n\|,$$

which implies that

$$\|x_{n+1} - p\| \leq \left[1 - \prod_{k=0}^n (1 - \alpha_k) \right] \|u - p\| + \prod_{k=0}^n (1 - \alpha_k) \|x_0 - p\| + \sum_{k=0}^n \|e_k\|.$$

This shows that (x_n) is bounded.

Now, assume that $\lim_{n \rightarrow \infty} \frac{\|e_n\|}{\alpha_n} = 0$. Then there is $M > 0$ such that for all $n \geq 0$,

$$\|u - p\| + \frac{\|e_n\|}{\alpha_n} \leq M.$$

From (4.8) we have

$$\|x_{n+1} - p\| \leq \alpha_n M + (1 - \alpha_n) \|x_n - p\|.$$

By induction,

$$\|x_{n+1} - p\| \leq \left[1 - \prod_{k=0}^n (1 - \alpha_k) \right] M + \prod_{k=0}^n (1 - \alpha_k) \|x_0 - p\|,$$

which implies the boundedness of (x_n) .

Thus, (z_n) is bounded.

Since T is nonexpansive, we get

$$\|x_{n+1} - z_{n+1}\| \leq (1 - \alpha_n) \|x_n - z_n\| + \|e_n\|.$$

From Lemma 1.10, $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$. Therefore, it is sufficient to prove that (z_n) strongly converges to $P_F u$.

Multiplying

$$\alpha_n(u - p) + \lambda_n(z_n - p) + \gamma_n(Tz_n - p) \in z_{n+1} - p + \beta_n A z_{n+1}$$

by $z_{n+1} - p$, and using the monotonicity of A , we have

$$\begin{aligned} 2\|z_{n+1} - p\|^2 &\leq (1 - \alpha_n)(\|z_{n+1} - p\|^2 + \|z_n - p\|^2) + 2\alpha_n(u - p, z_{n+1} - p) \\ &\quad - \lambda_n\|z_{n+1} - z_n\|^2 - \gamma_n\|Tz_n - z_{n+1}\|^2. \end{aligned}$$

Thus

$$\begin{aligned} (1 + \alpha_n)\|z_{n+1} - p\|^2 &\leq (1 - \alpha_n)\|z_n - p\|^2 + 2\alpha_n(u - p, z_{n+1} - p) \\ &\quad - \lambda_n\|z_{n+1} - z_n\|^2 - \gamma_n\|Tz_n - z_{n+1}\|^2. \end{aligned} \quad (4.10)$$

Set $s_n = \|z_n - P_F u\|^2$, $n = 0, 1, \dots$. Since (z_n) is bounded, it follows from (4.10) that

$$s_{n+1} - s_n + \lambda_n\|z_{n+1} - z_n\|^2 + \gamma_n\|Tz_n - z_{n+1}\|^2 \leq \alpha_n M, \quad (4.11)$$

for some $M > 0$.

On the other hand, from (4.9), we get:

$$\|z_{n+1} - J_\beta z_{n+1}\| \leq 2(\alpha_n \|u - z_{n+1}\| + \lambda_n \|z_n - z_{n+1}\| + \gamma_n \|Tz_n - z_{n+1}\|). \quad (4.12)$$

We will consider the two possible cases of (s_n) in order to complete the proof.

Case 1. (s_n) is eventually decreasing, i.e., there is $N \geq 0$ such that (s_n) is decreasing for all $n \geq N$. In this case, (s_n) is convergent. From (4.11) and (4.12), we have $\lim_{n \rightarrow \infty} \|z_n - J_\beta z_n\| = 0$.

Thus, every weak cluster point of (z_n) belongs to F . One can extract a subsequence (z_{n_k}) of (z_n) such that (z_{n_k}) converges weakly to some $y \in F$, and

$$\limsup_{n \rightarrow \infty} (u - P_F u, z_n - P_F u) = \lim_{k \rightarrow \infty} (u - P_F u, z_{n_k} - P_F u) = (u - P_F u, y - P_F u) \leq 0.$$

Then, from (4.10),

$$\|z_{n+1} - P_F u\|^2 \leq (1 - \alpha_n) \|z_n - P_F u\|^2 + 2\alpha_n (u - P_F u, z_{n+1} - P_F u),$$

which implies that (z_n) strongly converges to $P_F u$.

Case 2. (s_n) is not eventually decreasing, i.e., there exists a subsequence (s_{n_i}) of (s_n) such that $s_{n_i} < s_{n_{i+1}}$, for all $i \geq 0$.

Define an integer sequence $(\tau(n))$ as in Lemma 1.11 so that for all $n \geq n_0$, $s_{\tau(n)} \leq s_{\tau(n)+1}$.

From (4.11) and (4.12),

$$\lim_{n \rightarrow \infty} \|z_{\tau(n)+1} - J_\beta z_{\tau(n)+1}\| = 0,$$

which implies that

$$\limsup_{n \rightarrow \infty} (u - P_F u, z_{\tau(n)+1} - P_F u) \leq 0.$$

Therefore, from (4.10), for $n \geq n_0$,

$$s_{\tau(n)+1} \leq (u - P_F u, z_{\tau(n)+1} - P_F u).$$

Passing to the limit in the above inequality, $\lim_{n \rightarrow \infty} s_n = 0$, or, in other words, $x_n \rightarrow P_F u$ as $n \rightarrow \infty$. ■

Chapter 5

Extensions to Proximal Point Algorithms Involving Two Monotone Operators

Let K_1 and K_2 be nonempty, closed, and convex subsets of a real Hilbert space H , and $K_1 \cap K_2 \neq \emptyset$. Consider the following convex feasibility problem:

$$\text{find an } x \in H \text{ such that } x \in K_1 \cap K_2. \quad (5.1)$$

One of the methods for solving this problem in the particular case when K_1 and K_2 are subspaces of H was introduced by von Neumann in 1933. It is an iterative procedure of alternating projections defined as

$$H \ni x_0 \mapsto x_1 = P_{K_1}x_0 \mapsto x_2 = P_{K_2}x_1 \mapsto x_3 = P_{K_1}x_2 \mapsto x_4 = P_{K_2}x_3 \mapsto \dots \quad (5.2)$$

As we shall see below, it was showed by von Neumann that this iterative process strongly converges to the projection of x_0 on $K_1 \cap K_2$, which is a solution of problem (5.1).

The method of alternating projections in the general case has been an interesting open topic. After years since von Neumann announced his strong convergence result, in 1965, Bregman [8] showed that in general, when K_1 and K_2 are two arbitrary nonempty, closed, and convex subsets of H , with $K_1 \cap K_2 \neq \emptyset$, the sequence (x_n) generated by (5.2) is weakly convergent to a point in $K_1 \cap K_2$. Nearly four decades later, H. Hundal [14], showed in his 2004 paper, that the strong convergence of this iterative method fails to hold in general.

Let $A : D(A) \subset H \rightarrow H$ and $B : D(B) \subset H \rightarrow H$ be two maximal monotone operators. We can restate the problem (5.1) as follows

$$\text{find an } x \in D(A) \cap D(B) \text{ such that } x \in A^{-1}0 \cap B^{-1}0. \quad (5.3)$$

We denote here the resolvent of a maximal monotone operator A by $J_\lambda^A := (I + \lambda A)^{-1}$, $\lambda > 0$. Based on the fact that the projection operator is the resolvent of a normal cone, an iterative method for solving the problem (5.3) was introduced. It is called the *method of alternating resolvents* which is defined as follows: for $\beta_n, \mu_n \in (0, \infty)$, and two maximal operators A and B ,

$$\begin{aligned} x_{2n+1} &= J_{\beta_n}^A(x_{2n} + e_n) \quad \text{for } n = 0, 1, \dots \\ x_{2n} &= J_{\mu_n}^B(x_{2n-1} + e'_n) \quad \text{for } n = 1, 2, \dots \end{aligned} \quad (5.4)$$

where (e_n) and (e'_n) are sequences of computational errors.

In 2005, Bauschke [6] gave a weak convergence result for the exact algorithm (5.4) when $e_n = e'_n = 0$ and $\beta_n = \mu_n = \lambda > 0$, $n = 0, 1, \dots$. He showed that the sequence (x_n) generated by (5.4) in the exact case converges weakly to one fixed point of the composition mapping $J_\lambda^A J_\lambda^B$, provided the fixed set of $J_\lambda^A J_\lambda^B$ is nonempty, which is a solution of the problem (5.3). This conclusion still holds in the general case (5.4). However, this process does not converge strongly in general.

In order to obtain strong convergence of the method of alternating resolvents, some authors have generalized and modified the mentioned method. In this chapter, we consider the iterative process given by O. A. Boikanyo [1] (2012). Precisely, for any two maximal monotone operators A and B , the sequence (x_n) is generated as follows: for given $x_0, u \in H$, $\alpha_n, \delta_n, \gamma_n, \lambda_n \in [0, 1]$ with $\alpha_n + \delta_n + \gamma_n = 1$ and $\beta_n, \mu_n \in (0, \infty)$,

$$\begin{aligned} x_{2n+1} &= \alpha_n u + \delta_n x_{2n} + \gamma_n J_{\beta_n}^A x_{2n} + e_n \quad \text{for } n = 0, 1, \dots \\ x_{2n} &= J_{\mu_n}^B(\lambda_n u + (1 - \lambda_n)x_{2n-1} + e'_n) \quad \text{for } n = 1, 2, \dots \end{aligned} \quad (5.5)$$

Now we will go back to the strong result given by von Neumann on two subspaces with nonempty intersection.

Theorem 5.1 ([7]). *Let K_1 and K_2 be closed linear subspaces of a real Hilbert space H , and $K_1 \cap K_2 \neq \emptyset$. Then the sequence (x_n) defined by (5.2) strongly converges to $P_{K_1 \cap K_2} x_0$.*

Proof. The proof follows Bauschke [7].

For all $n \geq 0$, we have

$$\|x_n\|^2 = \|x_{n+1}\|^2 + \|x_n - x_{n+1}\|^2. \quad (5.6)$$

In particular, $(\|x_n\|)$ is convergent since it is decreasing and nonnegative.

We shall show that for all n, k, l , with $1 \leq k = l - n$,

$$\|x_k - x_l\|^2 \leq \|x_k\|^2 - \|x_l\|^2. \quad (5.7)$$

Indeed, by induction on n , (5.7) holds for $n = 0$ and $n = 1$ (by (5.6)). Assume that (5.7) holds for some $n \geq 1$, take k, l such that $1 \leq k = l - (n + 1)$. If n is even, or $n + 1 = l - k$ is odd, then $x_{k+1} = P_{K_1}x_k$, $x_l = P_{K_1}x_{l-1} \in K_1$, whereas $x_k - x_{k+1} = (I - P_{K_1})x_k \in K_1^\perp$. Hence,

$$(x_k - x_{k+1}, x_{k+1} - x_l) = 0. \quad (5.8)$$

Similarly, if l is even, one can get (5.8) by replacing K_1 by K_2 .

Thus, from (5.6) and (5.8), we have (5.7).

In the case when n is odd, or $n + 1 = l - k$ is even, one can see that

$$(x_k - x_l, x_l - x_{l-1}) = 0.$$

Again, by the property of $(\|x_n\|)$ and using (5.6), we can get (5.7).

Therefore, (x_n) is Cauchy, and hence convergent.

Let $x = \lim_{n \rightarrow \infty} x_n$. Since $(x_{2n+1}) \subset K_1$ and $(x_{2n}) \subset K_2$, $x \in K_1 \cap K_2$, and

$$x = \lim_{n \rightarrow \infty} P_{K_1 \cap K_2} x_n.$$

Fix n and $t \in \mathbb{R}$, and set $z = (1 - t)P_{K_1 \cap K_2}x_n + tP_{K_1 \cap K_2}x_{n+1}$. Then $z \in \text{Fix}(P_{K_1}) \cap \text{Fix}(P_{K_2})$, and $P_{K_1}z = P_{K_2}z = z$. Also, $x_{n+1} \in \{P_{K_1}x_n, P_{K_2}x_n\}$. Since, P_{K_1} and P_{K_2} are nonexpansive,

$$\|x_{n+1} - z\| \leq \|x_n - z\|,$$

which implies

$$(1 - 2t)\|P_{K_1 \cap K_2}x_{n+1} - P_{K_1 \cap K_2}x_n\|^2 + \|P_{(K_1 \cap K_2)^\perp}x_{n+1}\|^2 \leq \|P_{(K_1 \cap K_2)^\perp}x_n\|^2.$$

Therefore, $P_{K_1 \cap K_2}x_n = P_{K_1 \cap K_2}x_{n+1}$, that is $x = \lim_{n \rightarrow \infty} x_n = P_{K_1 \cap K_2}x_0$. ■

In the following, we shall consider the more general case when K_1 and K_2 are just closed and convex subsets of H . In this case, the strong convergence of the sequence (x_n) generated by (5.5) shows us that (5.5) is an efficient algorithm. Under reasonable assumptions on the parameters and errors, the inexact iterative process strongly converges if the exact one does.

Theorem 5.2 ([1]). *Let $A : D(A) \subset H \rightarrow H$ and $B : D(B) \subset H \rightarrow H$ be two maximal monotone operators with $A^{-1}0 \cap B^{-1}0 := F \neq \emptyset$. For any given $x_0, u \in H$, let the sequence (x_n) be generated by (5.5), where $\alpha_n, \delta_n, \gamma_n, \lambda_n \in [0, 1]$ with $\alpha_n + \delta_n + \gamma_n = 1$ and $\beta_n, \mu_n \in (0, \infty)$. Assume that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\gamma_n \geq \gamma$*

for some $\gamma > 0$ and $\lim_{n \rightarrow \infty} \lambda_n = 0$, either $\sum_{n=0}^{\infty} \alpha_n = \infty$ or $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\beta_n \geq \beta$ and $\mu_n \geq \mu$ for some $\beta, \mu > 0$. Then (x_n) converges strongly to the projection of u on F , provided that any of the following conditions is satisfied:

- (a) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $\sum_{n=1}^{\infty} \|e'_n\| < \infty$;

- (b) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $\lim_{n \rightarrow \infty} \frac{\|e'_n\|}{\alpha_n} = 0$;
- (c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $\lim_{n \rightarrow \infty} \frac{\|e'_n\|}{\lambda_n} = 0$;
- (d) $\lim_{n \rightarrow \infty} \frac{\|e_n\|}{\alpha_n} = 0$ and $\sum_{n=1}^{\infty} \|e'_n\| < \infty$;
- (e) $\lim_{n \rightarrow \infty} \frac{\|e_n\|}{\lambda_n} = 0$ and $\sum_{n=1}^{\infty} \|e'_n\| < \infty$;
- (f) $\lim_{n \rightarrow \infty} \frac{\|e_n\|}{\alpha_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\|e'_n\|}{\alpha_n} = 0$;
- (g) $\lim_{n \rightarrow \infty} \frac{\|e_n\|}{\alpha_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\|e'_n\|}{\lambda_n} = 0$;
- (h) $\lim_{n \rightarrow \infty} \frac{\|e_n\|}{\lambda_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\|e'_n\|}{\alpha_n} = 0$;
- (i) $\lim_{n \rightarrow \infty} \frac{\|e_n\|}{\lambda_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\|e'_n\|}{\lambda_n} = 0$;
- (j) $\lim_{n \rightarrow \infty} \frac{\|e_n\|}{\alpha_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\|e'_n\|}{\alpha_{n-1}} = 0$;
- (k) $\lim_{n \rightarrow \infty} \frac{\|e_{n-1}\|}{\lambda_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\|e'_n\|}{\alpha_{n-1}} = 0$;
- (l) $\lim_{n \rightarrow \infty} \frac{\|e_{n-1}\|}{\lambda_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\|e'_n\|}{\lambda_n} = 0$;
- (m) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $\lim_{n \rightarrow \infty} \frac{\|e'_n\|}{\alpha_{n-1}} = 0$;
- (n) $\lim_{n \rightarrow \infty} \frac{\|e_{n-1}\|}{\lambda_n} = 0$ and $\sum_{n=1}^{\infty} \|e'_n\| < +\infty$.

Proof. Let the sequence (z_n) be generated by (5.5) in the exact case, i.e.,

$$\begin{aligned} z_{2n+1} &= \alpha_n u + \delta_n z_{2n} + \gamma_n J_{\beta_n}^A z_{2n} \text{ for } n = 0, 1, \dots \\ z_{2n} &= J_{\mu_n}^B (\lambda_n u + (1 - \lambda_n) z_{2n-1}) \text{ for } n = 1, 2, \dots \end{aligned} \quad (5.9)$$

We will start with showing that the sequence (z_n) strongly converges to the projection of u onto F . First, we shall prove that (z_n) is bounded. Indeed, with $p \in F$, since $J_{\beta_n}^A$ and $J_{\mu_n}^B$ are nonexpansive, one can see that

$$\|z_{2n} - p\| \leq \lambda_n \|u - p\| + (1 - \lambda_n) \|z_{2n-1} - p\|,$$

which implies that

$$\|z_{2n+1} - p\| \leq [1 - (1 - \alpha_n)(1 - \lambda_n)] \|u - p\| + (1 - \alpha_n)(1 - \lambda_n) \|z_{2n-1} - p\|.$$

By induction,

$$\|z_{2n+1} - p\| \leq \left[1 - \prod_{k=1}^n (1 - \alpha_k)(1 - \lambda_k) \right] \|u - p\| + \|z_1 - p\| \prod_{k=1}^n (1 - \alpha_k)(1 - \lambda_k). \quad (5.10)$$

Thus, (z_n) is bounded. Now, set $s_n := \|z_{2n-1} - P_F u\|^2$. We shall show that (s_n) converges strongly to zero. Since $J_{\beta_n}^A$ is firmly nonexpansive, we have

$$\|J_{\beta_n}^A z_{2n} - p\|^2 \leq \|z_{2n} - p\|^2 - \|z_{2n} - J_{\beta_n}^A z_{2n}\|^2,$$

which implies

$$(z_{2n} - p, J_{\beta_n}^A z_{2n} - p) \leq \|z_{2n} - p\|^2 - \|z_{2n} - J_{\beta_n}^A z_{2n}\|^2, \quad (5.11)$$

and

$$\begin{aligned} \|\delta_n(z_{2n} - p) + \gamma_n(J_{\beta_n}^A z_{2n} - p)\|^2 &\leq (1 - \alpha_n)^2 \|z_{2n} - p\|^2 \\ &\quad - \gamma_n(\gamma_n + 2\delta_n) \|z_{2n} - J_{\beta_n}^A z_{2n}\|^2. \end{aligned} \quad (5.12)$$

By the properties of norms, we have

$$\|z_{2n+1} - p\|^2 \leq (1 - \alpha_n) \|z_{2n} - p\|^2 - \epsilon \|z_{2n} - J_{\beta_n}^A z_{2n}\|^2 + 2\alpha_n(u - p, z_{2n+1} - p), \quad (5.13)$$

where $\epsilon > 0$ is such that $\epsilon \leq \gamma_n(\gamma_n + 2\delta_n)$. On the other hand, we can obtain from (5.9) that

$$\lambda_n(u - p) + (1 - \lambda_n)(z_{2n-1} - p) \in z_{2n} - p + \mu_n B z_{2n}. \quad (5.14)$$

It follows from (5.14) that

$$\|z_{2n} - p\|^2 \leq (1 - \lambda_n) [\|z_{2n-1} - p\|^2 - \|z_{2n} - z_{2n-1}\|^2] + 2\lambda_n(u - p, z_{2n} - p). \quad (5.15)$$

From (5.13), one can get that

$$\begin{aligned} \|z_{2n+1} - p\|^2 &\leq (1 - \alpha_n)(1 - \lambda_n) \|z_{2n-1} - p\|^2 - \epsilon \|z_{2n} - J_{\beta_n}^A z_{2n}\|^2 \\ &\quad + 2\alpha_n(u - p, z_{2n+1} - p) - (1 - \alpha_n)(1 - \lambda_n) \|z_{2n} - z_{2n-1}\|^2 \\ &\quad + 2\lambda_n(1 - \alpha_n)(u - p, z_{2n} - p). \end{aligned} \quad (5.16)$$

Therefore, for some $M > 0$, we have

$$s_{n+1} - s_n + \|z_{2n} - z_{2n-1}\|^2 + \epsilon \|z_{2n} - J_{\beta_n}^A z_{2n}\|^2 \leq (\alpha_n + \lambda_n)M. \quad (5.17)$$

We consider the following two cases on the sequence (s_n) :

Case 1. (s_n) is eventually decreasing, i.e., there is $N \geq 0$ such that $(s_n)_{n \geq N}$ is decreasing. In this case, (s_n) is convergent. From (5.17), we have

$$\lim_{n \rightarrow \infty} \|z_{2n} - z_{2n-1}\| = \lim_{n \rightarrow \infty} \|z_{2n} - J_{\beta_n}^A z_{2n}\| = 0. \quad (5.18)$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, from (5.18), we have

$$\lim_{n \rightarrow \infty} \|z_{2n+1} - z_{2n}\| = \lim_{n \rightarrow \infty} (\alpha_n \|u - J_{\beta_n}^A z_{2n}\| + \|J_{\beta_n}^A z_{2n} - z_{2n}\|) = 0 \quad (5.19)$$

and

$$\lim_{n \rightarrow \infty} \|z_{2n} - J_{\beta}^A z_{2n}\| = \lim_{n \rightarrow \infty} 2\|z_{2n} - J_{\beta_n}^A z_{2n}\| = 0. \quad (5.20)$$

Since A_{β}^{-1} , the inverse of the Yosida approximation of A , is demiclosed, all the weak cluster points of (z_{2n}) belong to $A^{-1}0$.

On the other hand, since J_{μ}^B is nonexpansive, we have

$$\|z_{2n} - J_{\mu}^B z_{2n}\| \leq 2\|z_{2n} - J_{\mu_n}^B z_{2n}\| \leq 2(\lambda_n \|u - z_{2n-1}\| + \|z_{2n-1} - z_{2n}\|). \quad (5.21)$$

Since B_{μ}^{-1} , the inverse of the Yosida approximation of B , is demiclosed, all the weak cluster points of (z_{2n}) belong to $B^{-1}0$. Thus, they belong to F . That is, there exists a subsequence (ω_k) of (z_{2n}) weakly converging to some $z \in F$ such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} (u - P_F u, z_{2n} - P_F u) &= \limsup_{k \rightarrow \infty} (u - P_F u, \omega_k - P_F u) \\ &= (u - P_F u, z - P_F u) \leq 0. \end{aligned} \quad (5.22)$$

From (5.19), we also get

$$\limsup_{n \rightarrow \infty} (u - P_F u, z_{2n+1} - P_F u) \leq 0. \quad (5.23)$$

From (5.16), we have

$$\begin{aligned} \|z_{2n+1} - P_F u\|^2 &\leq (1 - \alpha_n)(1 - \lambda_n)\|z_{2n-1} - P_F u\|^2 \\ &\quad + 2\alpha_n(u - P_F u, z_{2n+1} - P_F u) \\ &\quad + 2\lambda_n(1 - \alpha_n)(u - P_F u, z_{2n} - P_F u). \end{aligned} \quad (5.24)$$

Applying Lemma 1.9, we obtain $\lim_{n \rightarrow \infty} \|z_{2n+1} - P_F u\| = 0$. Passing to the limit in (5.15), we get $\lim_{n \rightarrow \infty} \|z_{2n} - P_F u\| = 0$ as well. That is, $\lim_{n \rightarrow \infty} \|z_n - P_F u\| = 0$.

Case 2. (s_n) is not eventually decreasing, i.e., there exists a subsequence (s_{n_i}) of (s_n) such that $s_{n_i} < s_{n_i+1}$, for all $i \geq 0$.

Define an integer sequence $(\tau(n))_{n \geq n_0}$ as in Lemma 1.11 so that for all $n \geq n_0$, $s_{\tau(n)} \leq s_{\tau(n)+1}$.

It follows from (5.17) that

$$\lim_{n \rightarrow \infty} \|z_{2\tau(n)} - z_{2\tau(n)-1}\| = \lim_{n \rightarrow \infty} 2\|z_{2\tau(n)} - J_{\beta_{\tau(n)}}^A z_{2\tau(n)}\| = 0.$$

From (5.5), we also get

$$\lim_{n \rightarrow \infty} \|z_{2\tau(n)+1} - z_{2\tau(n)}\| = \lim_{n \rightarrow \infty} (\alpha_{\tau(n)} \|u - J_{\beta_{\tau(n)}}^A z_{2\tau(n)}\| + \gamma_{\tau(n)} \|J_{\beta_{\tau(n)}}^A z_{2\tau(n)} - z_{2\tau(n)}\|) = 0.$$

Arguing as in Case 1, one can see that all the weak cluster point of $(z_{2\tau(n)})$ belong to F . Consequently,

$$\limsup_{n \rightarrow \infty} (u - P_F u, z_{2\tau(n)} - P_F u) \leq 0.$$

It follows from (5.24) that, for some $K > 0$,

$$\begin{aligned} \|z_{2n+1} - P_F u\|^2 &\leq (1 - \alpha_n)(1 - \lambda_n) \|z_{2n-1} - P_F u\|^2 + \alpha_n K \|z_{2n+1} - z_{2n}\| \\ &\quad + 2[\lambda_n(1 - \alpha_n) + \alpha_n](u - P_F u, z_{2n} - P_F u). \end{aligned}$$

Therefore, for all $n \geq n_0$, we have

$$\begin{aligned} s_{\tau(n)+1} &\leq (1 - \alpha_{\tau(n)})(1 - \lambda_{\tau(n)})s_{\tau(n)} + \alpha_{\tau(n)}K \|z_{2\tau(n)+1} - z_{2\tau(n)}\| \\ &\quad + 2[\lambda_{\tau(n)}(1 - \alpha_{\tau(n)}) + \alpha_{\tau(n)}](u - P_F u, z_{2\tau(n)} - P_F u) \\ &\leq 2(u - P_F u, z_{2\tau(n)} - P_F u) + K \|z_{2\tau(n)+1} - z_{2\tau(n)}\|. \end{aligned} \quad (5.25)$$

Passing to the limit in (5.25), we get $\lim_{n \rightarrow \infty} s_{\tau(n)+1} = 0$ which implies that $\lim_{n \rightarrow \infty} s_n = 0$. Thus, $\lim_{n \rightarrow \infty} z_{2n+1} = P_F u$. Moreover, from (5.15) for some $L > 0$, we have

$$\|z_{2n} - P_F u\|^2 \leq (1 - \lambda_n) \|z_{2n-1} - P_F u\|^2 + 2\lambda_n L,$$

which implies that $\lim_{n \rightarrow \infty} z_{2n} = P_F u$. Therefore, $\lim_{n \rightarrow \infty} z_n = P_F u$.

Finally, we shall show that $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$.

Since $J_{\beta_n}^A$ and $J_{\mu_n}^B$ are nonexpansive, we have

$$\|x_{2n} - z_{2n}\| \leq (1 - \lambda_n) \|x_{2n-1} - z_{2n-1}\| + \|e'_n\|; \quad (5.26)$$

and

$$\begin{aligned} \|x_{2n+1} - z_{2n+1}\| &\leq (1 - \alpha_n) \|x_{2n} - z_{2n}\| + \|e_n\| \\ &\leq (1 - \alpha_n)(1 - \lambda_n) \|x_{2n-1} - z_{2n-1}\| + \|e'_n\| + \|e_n\|. \end{aligned} \quad (5.27)$$

Therefore, if (e_n) and (e'_n) satisfy any of the assumptions (a) to (i), by Lemma 1.9, we obtain $\lim_{n \rightarrow \infty} \|x_{2n+1} - z_{2n+1}\| = 0$ and $\lim_{n \rightarrow \infty} \|x_{2n} - z_{2n}\| = 0$.

Otherwise, if the parameters and the errors satisfy any of the other assumptions, from (5.26) and (5.27), we get

$$\|x_{2n} - z_{2n}\| \leq (1 - \alpha_{n-1})(1 - \lambda_n) \|x_{2n-2} - z_{2n-2}\| + \|e_{n-1}\| + \|e'_n\|.$$

Thus, $\lim_{n \rightarrow \infty} \|x_{2n} - z_{2n}\| = 0$ and $\lim_{n \rightarrow \infty} \|x_{2n+1} - z_{2n+1}\| = 0$. ■

Remark. A result similar to Theorem 5.2 above holds for the following iterative process:

$$\begin{aligned} x_{2n+1} &= J_{\beta_n}^A(\alpha_n u + (1 - \alpha_n)x_{2n} + e_n) \text{ for } n = 0, 1, \dots \\ x_{2n} &= J_{\mu_n}^B(\lambda_n u + (1 - \lambda_n)x_{2n-1} + e'_n) \text{ for } n = 1, 2, \dots \end{aligned}$$

The proof of this remark can be found in [5].

Obviously, for $A = B$, we reobtain the algorithm (4.4) (or algorithm (4.8) with $T = I$) discussed in Chapter 4.

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