Intersection of longest paths

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1 Introduction

Three tourist companies in Budapest advertise "We show you the most possible sites without showing you any site twice!" They strategically start their tours from places where they can travel the maximum distance allowed by the streets of Budapest, and they never retrace their steps. At the end of the day three tourists, Alice, Bob, and Charlie, who each took a different tour find themselves in a coffee shop. They share their experiences from the day, and they realize that they each saw some pretty different sites. They begin to wonder, "Is there a site which we all saw?"

Alice, Bob, and Charlie take the time to compare their maps eventually realizing the answer is "yes" they were all at Deák square at some point in time. Had their routes been positioned in such a way that there were no common visited site, then they would have solved an old question of graph theory. To this day, we still do not know if Alice, Bob, and Charlie *necessarily* saw a common site.

The question can be rephrased mathematically: Do three longest paths in a connected graph share a common vertex? We know that any two longest paths must intersect. We also have a graph (due to Schmitz, [15], 1975) for which seven longest paths do *not* intersect. Yet it is unknown what happens when we consider between three and six longest paths. Zamfirescu [27] has been asking these questions and related ones since the 1980s.

Unfortunately the questions about longest paths are beyond the capabilities of computing power. We have no polynomial-time algorithms determining the length of a longest path unless $P = NP$. To see this, we observe that an algorithm outputting the length of a longest path (or for the decision version "yes" if its length is $\geq k$ and "no" otherwise) could also solve the Hamiltonian path problem, a well-known NP-complete task. This shows that the problem is NP-hard, and it is in fact NP-complete.

The aim of this paper is two-fold. We will 1) survey the question of intersecting *all* longest paths and cycles and 2) discuss in detail the progress on the problem of intersecting *three* longest paths, providing some new results in the process.

In Section 2 we provide a historical overview of longest path- and cycle-intersection problems. In particular, when is it that we have examples of k-connected graphs such that given any j vertices we can find a longest path missing (i.e. not intersecting) those vertices. We seek graphs of smallest possible order. At the end of the section we show how recent developments (such as finding smaller, planar hypohamiltonian graphs) can produce examples of smaller order.

In Section 3 we compile a list of open questions on the subject of longest path intersections. In Section 4 we discuss research trends in attacking these open problems.

In Section 5 we turn our attention to the problem of whether or not three longest paths share a common point. We conjecture the affirmative and analyze the structure of a minimal counterexample to the conjecture. Section 5 is broken down further into two subsections. In the first subsection we present basic structural requirements of a minimal counterexample, and in the second subsection we present some more structural requirements which give rise to a generalization of a theorem of Axenovich in [2].

As the topic of longest paths is basic in nature we do not provide a separate "definitions" section. We assume the reader is familiar with the basics of graph theory, and we define more specific concepts as they appear. Knowledge of words such as k-regular graph, k-connected, girth, Hamiltonian, chordal, or block, for example, should be enough to understand everything in the paper. The reader could check any elementary graph theory book, e.g. [4], for unknown words or concepts.

2 Intersecting all longest paths or cycles

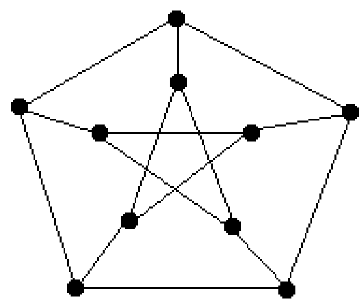


Figure 1: The Petersen graph

The graph in Figure 1 is the well-known Petersen graph. It is 3-regular (or *cubic*), 3-connected, has girth 5, and is very symmetric (the view from any vertex is identical). In addition to these nice properties, more important to this discussion is the fact that the Petersen graph is *hypohamiltonian*, that is the graph itself is not Hamiltonian but if we delete any vertex it becomes Hamiltonian. This is easily verified by the symmetry of the graph. It follows that the intersection of the longest cycles of the Petersen graph is empty. Letting $\mathbf{C}(G)$ and $\mathbf{P}(G)$ denote the set of longest cycles and the set of longest paths of a graph, respectively, then for the Petersen graph we have $\bigcap \mathbf{C}(G) = \emptyset$.

At a colloquim in 1966, Tibor Gallai [6] posed the analagous question for longest paths in a graph. (Here and henceforth when we say "graph" we mean a "finite, connected graph").

Question 1 (Gallai 1966)**.** Do all longest paths share a common point?

Does $\bigcap \mathbf{P}(G) \neq \emptyset$ hold for all graphs G? A few years later, Walther [22] produced a counterexample on 25 vertices. The simplest counterexample was found by both Walther [23] and Zamfirescu [26] independently and has 12 vertices, see Figure 2.

Let G be the graph in Figure 2. To see that $\bigcap \mathbf{P}(G) = \emptyset$, observe that by identifying the 3 degree 1 vertices we obtain the Petersen graph. Taking a longest cycle in the Petersen graph which uses the identified vertex we obtain a path of length 10 in G. Any path in G must miss at least one of the degree 1 vertices, and if it misses exactly one it cannot use every other vertex in the graph, as the Petersen graph is not Hamiltonian. This proves that no path of length 11 exists. It is easy to find a path of length 10 missing any vertex in G simply by taking the corresponding longest cycle in the Petersen graph (here we use hypohamiltonicity).

Zamfirescu later refined Gallai's question. Can we find graphs such that for any j vertices there exists a longest path missing those j vertices? Can we find k-connected graphs with the same property? Can we even find planar graphs? Zamfirescu asks identical questions with "path" replaced by "cycle".

If there is a k-connected graph with longest paths (cycles) missing any j vertices then we denote P_k^j (C_k^j) to be the minimum number of vertices of such a graph (following

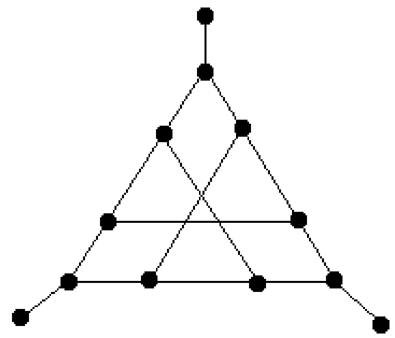


Figure 2: A graph with $\bigcap \mathbf{P}(G) = \emptyset$ due to Walther and Zamfirescu

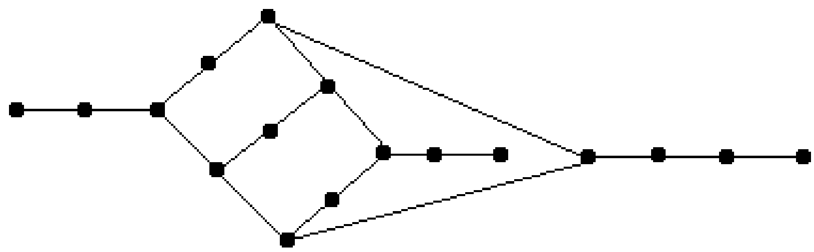


Figure 3: A planar graph with $\bigcap \mathbf{P}(G) = \emptyset$ due to Schmitz

notation started by Zamfirescu). If no such graph exists then we set $P_k^j = \infty$. We define $\overline{P}_k^j$ ($\overline{C}_k^j$) similarly with the restriction that the graph is planar. Since the Petersen graph is 3-connected, the examples above show that $C_3^1 \leq 10$ (and hence $C_2^1, C_1^1 \leq 10$) and $P_1^1 \leq 12$. Sometimes we write a graph "solves C_k^j" to mean that the graph is k connected and its longest cycles can miss any j vertices.

A number of people, namely Grünbaum, Hatzel, Schmitz, Thomassen, Walther, C. Zamfirescu and T. Zamfirescu in for example [8], [9], [15], [18], [24], [25], [26], have worked on proving the finiteness and providing better upperbounds for the P_k^j's and C_k^j's. Zamfirescu [27] provides an overview of who has done what and a nice table is presented in Voss's book ([21], page 79) showing the best upper bounds as of 1991. Tables 1 and 2 show slightly more updated versions of the tables originally presented in Voss's book.

The general P_k^j's and C_k^j's remain unchanged since the 1970s. The planar $\overline{P}_3^j$'s and $\overline{C}_3^j$'s, however, have seen some recent improvements. Before discussing these recent developments, let's look at some of the other entries.

A celebrated theorem of Tutte [20] states that any 4-connected planar graph is Hamiltonian, immediately implying $\overline{P}_4^j, \overline{C}_4^j = \infty$.

We have already mentioned that Figure 2 shows $P_1^1 \leq 12$, however it is not planar. Schmitz [15] presents a 17 vertex planar graph (see Figure 3) whose longest paths have empty intersection, i.e. $\overline{P}_1^1 \leq 17$, improving on Zamfirescu's previous bound of 19 (in [26]). Skupień [17] shows $P_2^1 \leq 26$, also improving a bound of Zamfirescu's. In the same paper, Skupień generalizes the construction of Schmitz's graph by contsructing for all $k \geq 7$ a graph with k longest paths having empty intersection while any $k-1$ longest

P_k^j

k \ j	1	2	3
1	≤ 12	≤ 93	?
2	≤ 26	≤ 270	?
3	≤ 36	≤ 270	?
4	?	?	?

C_k^j

k \ j	1	2	3
1	$C_1^j = 3j + 3$		
2	$= 10$	≤ 75	?
3	$= 10$	≤ 75	?
4	?	?	?

Table 1

$\overline{P}_k^j$

k \ j	1	2	3
1	≤ 17	≤ 308	?
2	≤ 32	≤ 914	?
3	≤ 188	$\leq 16,926$	?
4	∞	∞	∞

$\overline{C}_k^j$

k \ j	1	2	3
1	$\overline{C}_1^j = 3j + 3$		
2	≤ 15	≤ 135	?
3	≤ 48	$\leq 4,277$	?
4	∞	∞	∞

Table 2

paths still share a common point. Skupień's contsruction yields Schmitz's planar graph for $k = 7$.

The C_1^j and $\overline{C}_1^j$ cases have been completely solved. Zamfirescu [26] observed $C_1^j \leq 3j + 3$ from the graph in Figure 4. It consists of $j + 1$ triangles, so for any j points chosen, there is still an available triangle (i.e. longest cycle) missing those points. Zamfirescu conjectured $C_1^j = 3j+3$ for which Thomassen [18] gives an affirmative answer by showing that the longest cycles of a graph on n vertices can be covered by at most $\lfloor \frac{n}{3} \rfloor$ vertices.

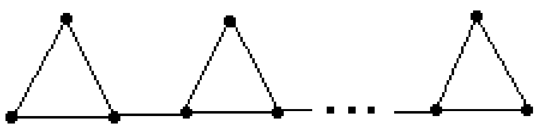


Figure 4: A graph which solves C_1^j

2-connectivity is a more natural requirement to impose when working with cycles (as studying cycles of a 1-connected graph amounts to studying its 2-connected blocks separately). Walther [22] was the first to prove the finiteness of C_2^2 with a 220 vertex graph. Grünbaum [8] gave a better bound on C_2^2 and C_3^2 by giving a 90 vertex, 3-connected graph whose longest cycles can miss any 2 vertices, see Figure 5. Zamfirescu [26] contracts some superfluous edges of Grünbaum's graph to show $C_3^2, C_2^2 \leq 75$.

2.1 A method for upperbounding C_3^2 and $\overline{C}_3^2$

Grünbaum's idea in [8] demonstrates a method for constructing these graphs. Take a hypohamiltonian graph F and "open it up" by splitting a degree 3 vertex into 3 open edges (like how Figure 2 was obtained from the Petersen graph). Denote this open graph F^*. Insert this opened up graph into a "nice" (to be explained) 3-regular hypohamiltonian graph H at each vertex and attach the edges. We call edges originally belonging to H *attachment edges*. Figure 5 depicts G when both F and H are the Petersen graph, which is precisely the graph used by Grünbaum in [8].

We claim that the resulting graph G solves C_3^2 for any hypohamiltonian F (with at least one degree 3 vertex) and any "nice" cubic hypohamiltonian H. G will be 3-connected because hypohamiltonian graphs are 3-connected (remove 2 vertices and there is still a Hamiltonian path). It remains to show that any 2 vertices can be missed by some longest cycle. If the 2 vertices in question belong to the same copy of F^* we simply choose a longest cycle which misses this copy of F^* (all longest cycles in G must miss 1 copy of F^* since H is hypohamiltonian). Suppose now that the 2 vertices x and y belong to different copies of F^*, denoted F_x^* and F_y^*. Observe that when a longest cycle C passes through F_x^* without touching x then it travels a longest path in F_x^*. This may force C to use or not use certain attachment edges adjacent to F_x^* (e.g. if there is a unique longest path in F^* missing x). In any case at most one attachment edge is forbidden by x. The only potential problem is that an attachment edge forbidden by x is simultaneously forced by y. However, if H is "nice" - that is, *if any 2 edges can be missed by a longest cycle* - then we can easily find a C which will avoid the (at most) two edges forbidden by x and y.

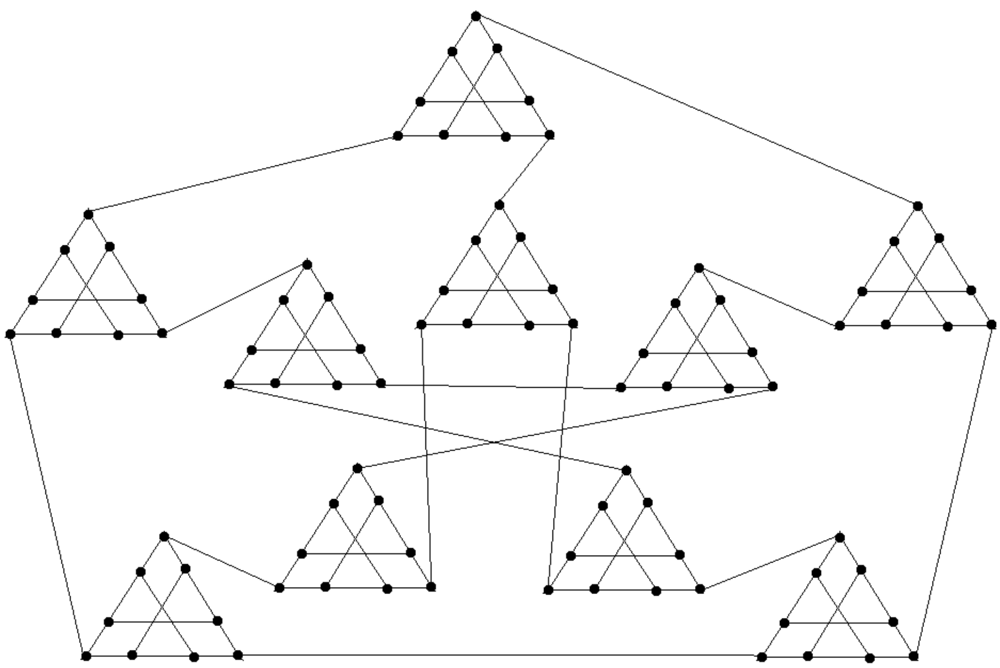


Figure 5: Grünbaum's graph solving C_3^2

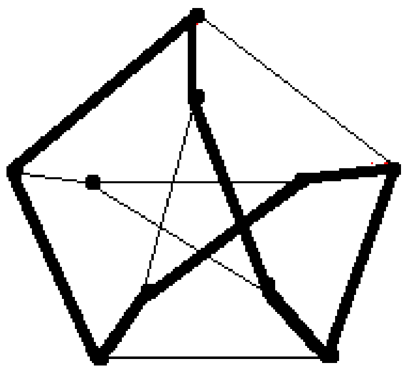


Figure 6: A longest cycle in the Petersen graph

Based on the above argument, to verify Grünbaum's graph solves C_3^2 all we need to verify is that the Petersen graph is "nice". (We note that Grünbaum in [8] skims over this technicality by checking only the case that the two forbidden edges are adjacent. Fortunately, however, it is not difficult to check either case for the Petersen graph.) If the two forbidden edges are adjacent, hypohamiltonicity immediately implies we can find a cycle missing those two edges. The incredible amount of symmetry of the Petersen graph actually shows that any pair of nonadjacent edges is essentially the same. Thus if the forbidden edges are nonadjacent it is enough to check it for one given pair of edges. Figure 6 gives an example of a longest cycle missing two nonadjacent edges, so the Petersen graph is indeed nice.

To obtain an even better upperbound for C_3^2 we can contract all attachment edges (Zamfirescu, [26]). All longest cycles use the same number of attachment edges ($|V(H)| - 1$) so lengths will be uniformly decreased. To check that any two vertices x and y can still be missed by a longest cycle, one only needs to address the problem that a longest cycle before contraction may not be a cycle after contraction, as is the case when two vertices u and v being used by a longest cycle C are both adjacent to an unused attachment edge. If $u \notin F_x^*$ then we can modify the path taken through F_u^* so as to miss u which we can do by the hypohamiltonicity of F. So we may assume both $u \in F_x^*$ and $v \in F_y^*$. Then the attachment edges forbidden by x and y must be adjacent (say to F_x^*) since edge uv is not used, in which case the longest cycle can be taken to entirely miss F_x^*. Modifying the cycle in this way as needed, we can miss any two vertices. Zamfirescu does this to improve Grünbaum's bound to $C_3^2 \leq 75$ by deleting the 15 attachment edges of the Petersen graph.

Of course the same arguments can be used to construct planar graphs solving $\overline{C}_3^2$. We just need a planar hypohamiltonian graph F and a nice, cubic planar hypohamiltonian graph H. Note that F need not be cubic, but that it needs to have at least one vertex of degree 3 which can be opened up and inserted into the cubic graph H at each vertex.

Zamfirescu [26] gave the first answer to $\overline{C}_3^2$ with a graph on 14,818 vertices. Hatzel [9] improves this result to 6,758 using the argument mentioned above where F is a 57 vertex planar hypohamiltonian graph (found by Hatzel himself) and H is a 124 vertex cubic planar hypohamiltonian graph (first presented by Grünbaum in [8]). Indeed, if we

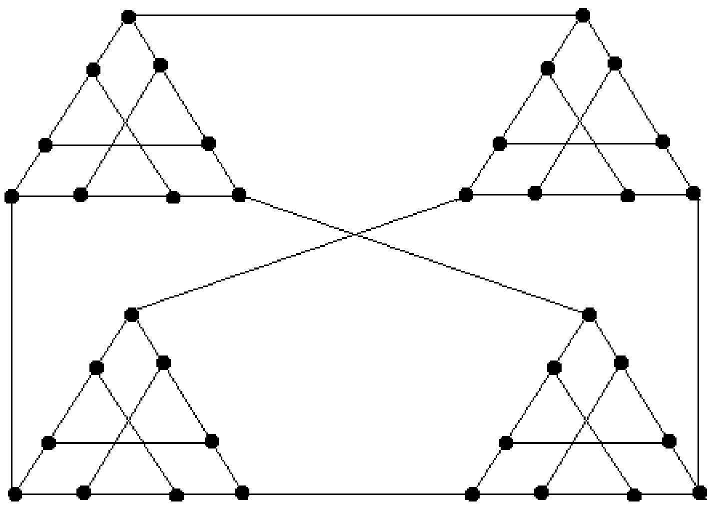


Figure 7: Petersen graph inserted into K_4 shows $P_3^1 \leq 36$

open up a 57 vertex graph at a degree 3 vertex, insert into a 124 vertex graph, and contract the $124 \times 3/2 = 186$ attachment edges we will obtain $56 \times 124 - 186 = 6,758$ vertices.

As we mentioned earlier $\overline{C}_3^1$ and $\overline{C}_3^2$ have seen some recent improvements. In 2007, Zamfirescu and Zamfirescu [28] presented a 48 vertex planar hypohamiltonian graph, immediately improving $\overline{C}_3^1$ to 48. Inserting their graph at each vertex of the 94 vertex cubic planar hypohamiltonian graph (first presented by Thomassen in [19]), they improve $\overline{C}_3^2$ to 4277 (this uses the fact that Thomassen's graph is "nice"). In fact, the table of $\overline{C}_k^j$'s and $\overline{P}_k^j$'s is not entirely accurate. Just recently Araya and Wiener (in [1], to appear) have found a 42 vertex planar hypohamiltonian graph.

Using the above arguments, either finding a planar hypohamiltonian graph smaller than Araya and Wiener's 42 vertex graph or finding a cubic planar hypohamiltonian graph smaller than Thomassen's 94 vertex graph would immediately result in better upper bounds for the $\overline{C}_3^j$'s. As we shall see in the next section it also results in better upperbounds for the $\overline{P}_3^j$'s.

2.2 A method for upperbounding P_3^j and $\overline{P}_3^j$

Zamfirescu [26] shows $P_3^1 \leq 36$ by replacing each vertex of K_4 with an opened Petersen graph, see Figure 7. It is easy to see that any vertex can be missed by a longest path as the hypohamiltonicity of the Petersen graph aids us once again.

Grünbaum [8] is the first to prove P_3^2 is finite, showing $P_3^2 \leq 324$. He replaces every vertex of K_4 by the opened Petersen graph and then replaces every vertex again by the opened Petersen graph. Zamfirescu [26] proceeds similarly but contracts the attachment edges after each step, to show $P_3^2 \leq 270$. The idea behind these proofs is identical to that of the previous section.

As smaller planar hypohamiltonian graphs are being found, upon inserting these graphs (opened at a degree 3 vertex) into K_4 we obtain a better bound on $\overline{P}_3^1$. To obtain a bound on $\overline{P}_3^2$ we replace every vertex of K_4 by Thomassen's 94 vertex graph from [19] opened at a vertex. Then in the resulting graph replace every vertex by the opened Araya-Wiener 42 vertex planar hypohamiltonian graph. So we are inserting a graph solving C_3^2 into K_4 at each vertex. Similar arguments combined with Thomassen's graph being "nice" shows that any 2 vertices can be avoided by a longest path.

3 Open problems

Perhaps the most remarkable thing about Tables 1 and 2 are the question marks. We restate the most elementary unresolved questions here for emphasis. Some of these questions appear already in [27] and [21]. The same questions can be posed where "paths" is replaced by "cycles" everywhere, in which case we require that the graph be 2-connected.

Question 2. Do there exists graphs whose longest paths miss any 3 prescribed vertices?

Question 3. More generally, does there exist an integer k such that every graph has a set of k vertices covering all the longest paths? Or if the graph is planar, can we find such a k?

Question 4. Do there exist 4-connected graphs whose intersection of longest paths or cycles is empty?

Question 5. More generally, does there exist an integer k such that if the graph is k-connected, then the intersection of its longest paths is nonempty?

As was mentioned before, Tutte's theorem [20] shows that $k = 4$ answers the above question for planar graphs.

We can also ask what happens when we intersect not *all* of the longest paths (or cycles), but rather k of them. It is easy to show that any two longest paths share a common point, arguing by contradiction and using the connectivity of the graph to construct a longer path. But if we consider 3 longest paths, we immediately encounter an open problem.

Question 6. Do 3 longest paths always share a common vertex?

Schmitz's graph in Figure 3 has 7 longest paths with empty intersection. A 2-connected example of Jendrol and Skupień [11] has 7 longest cycles with empty intersection. But 6 paths (or cycles) still presents a challenge.

Question 7. Do 6 longest paths always share a common vertex?

Question 8. More generally, what is the maximal k such that k longest paths necessarily share a common vertex?

Schmitz, Jendrol, and Skupień's results imply $2 \leq k \leq 6$.

Several attempts have been made to study Question 6 for specific types of graphs. Balister et al. [3] determine that all longest paths in interval graphs, a special family of chordal graphs, share a common point. However, the following question remains open.

Question 9. Do any 3 longest paths in a chordal graph share a common point?

We can also ask questions about the number of points in the intersection of longest paths or cycles. We know that any 2 cycles meet in a 2-connected graph. It can be shown that they meet in atleast 2 points. Similarly, for a 3-connected graph any 2 cycles meet in 3 points. More generally, we have the following question, conjectured with an affirmative answer by Smith in 1979.

Question 10. Do any 2 cycles in a k-connected graph share k common points ($k \geq 2$)?

Grötschel [7] shows this is true for $k \leq 6$. Chen et al. [5] show that there are $\Omega(k^{3/5})$ common points.

4 Directions of recent research

Instead of searching for graphs whose intersection of longest paths is empty, we can ask the question “when is it that all longest paths *necessarily* share a common point?” In 1990, Klavžar and Petkovšek [12] gave a characterization of graphs G with $\cap\mathbf{P}(G) \neq \emptyset$. Their characterization is useful in that it turns a global question into a local one. To find a vertex common to all paths in $\mathbf{P}(G)$ it suffices to find a vertex common to those longest paths passing through an arbitrary block B of G. Let $\mathbf{P}_B(G)$ be the subset of paths in $\mathbf{P}(G)$ which have at least 1 edge in the block B of G.

Theorem 11 (Klavžar, Petkovšek, [12]). *$\cap\mathbf{P}(G) \neq \emptyset$ if and only if $\mathbf{P}_B(G) \neq \emptyset$ for every block B of G.*

Sketch of proof. In one direction there is nothing to prove since $\mathbf{P}_B(G) \subset \mathbf{P}(G)$. Now suppose $\cap\mathbf{P}_B(G) \neq \emptyset$ for all B in G.

Case 1: For every pair of paths, there is a block in which they both have an edge. In this case we look at the tree $T(G)$ associated with G where the vertices are blocks and cutpoints and the edges represent intersection. (Here and in the proof of Klavžar and Petkovšek “block” means either a maximal 2-connected subgraph of G or a bridge, and the cutpoints are simply the points where the blocks meet). Denote by $f(P)$ the image of a longest path P under this reduction. Then the condition implies all pairs of paths in $\{f(P) | P \in \mathbf{P}(G)\}$ intersect. A well-known Helly-type property of trees states that a pairwise intersecting family of subtrees is completely-intersecting, i.e. there exists a vertex $v \in \bigcap_{P\in\mathbf{P}(G)} f(P)$. If v is a cutpoint we are done. Otherwise every longest path belongs to $\mathbf{P}_B(G)$ for some B, and $\cap\mathbf{P}(G) = \cap\mathbf{P}_B(G) \neq \emptyset$, as desired.

Case 2: There exists paths P and Q which do not have an edge in a common block. In this case P and Q do not share an edge nor do they intersect in more than one point (as this would create a cycle whose edges belong to P and Q, giving them a common block). We claim that the lone vertex $x \in P \cap Q$ is common to all longest paths. If there is some longest path R such that $x \notin R$ then we can use R to construct a block in which both P and Q have edges. To do so, consider the subgraph $P \cup Q \cup R$. Using the fact that each path intersects pairwise we pick the closest point in this subgraph of $R \cap Q$ to x and the closest point of $R \cap P$ to x. These two points combined with x form a cycle in which both P and Q share edges, a contradiction. □

Remark 12. The Helly-type property of subtrees mentioned in the proof of Theorem 11 get its name from the mathematician Eduard Helly (1884 - 1943). Helly’s more general theorem states that a finite family of $\geq n+1$ convex sets in $\mathbb{R}^n$ shares a point in common provided that any $n+1$ of them shares a common point. Statements of the type “ if for all $\mathcal{F} \subset \mathcal{G}, |\mathcal{F}| = k$, $\bigcap_{F\in\mathcal{F}} F \neq \emptyset$ then $\bigcap \mathcal{G} \neq \emptyset$” are called Helly-type statements. The Helly-type statement that a pairwise intersecting family of subtrees of a tree is completely intersecting is most easily proved by induction on the number of vertices of the tree.

Klavžar and Petkovšsek’s characterization implies that if the blocks of G are Hamilton-connected (for all x, y there is a hamiltonian path joining x and y), almost Hamilton-connected (a Hamilton-connected analog for bipartite graphs), or a cycle then $\cap\mathbf{P}(G) \neq \emptyset$.

Among the collection of open problems presented in the previous section, Question 8 seems to be the most popular one to attack, due to its extremely elementary nature. All we know is $2 \leq k \leq 6$. We can chip away at this inequality from above by finding a graph with 6 longest paths not sharing a common point. Or we may chip away at this inequality from below by proving any 3 longest paths share a common point. Of course it makes no sense to try to prove 6 longest paths share a common point (since 3 should be easier if it is true) nor does it make much sense to look for a graph with 3 longest paths not sharing a common point (since using 6 paths allows for more freedom to miss a point).

Skupień [17] works on the inequality from above, but in a different sort of way. It may happen that in a graph G, v longest paths have empty intersection and so do $v - 1$. For example, take Zamfirescu and Walther's graph and append an edge to one of the corner vertices of the triangle. In doing so we have created many new longest paths, however we need not intersect all of them to achieve the empty set. Skupień examines for what integers v can we find graphs where any $v-1$ longest paths share a common point but some v of them do not, i.e. what v's can serve as threshold values. He constructs examples for all $v \geq 7$.

How to construct a graph with 6 longest paths not sharing a point remains a challenging question. Building up a graph from other graphs whose longest path (or cycle) behavior is well-known is a good approach. Skupień's examples were constructed by taking Hamiltonian graphs, subdividing some edges, and then splitting a vertex of degree 3 into 3 degree 1 vertices. The "splitting" operation cannot be emphasized enough as a useful tool, as this technique allows us to obtain a graph solving P_1^1 from a graph solving C_2^1 as in the way we obtained Zamfirescu and Walther's graph from the Petersen graph.

To prove 3 longest paths necessarily intersect, it seems natural to start with certain classes of graphs, as well as small graphs and very restrictive graphs. To this end Balister, Győri, Lehel, and Schelp [3] examine longest paths in chordal graphs. They are able to prove that all longest paths intersect in interval graphs (a subfamily of chordal graphs), but the question of even 3 paths is still elusive for chordal graphs in general (see Question 9). This is somewhat surprising as chordal graphs are relatively easily traversable (triangles are everywhere). Klavžar and Petkovšek had also shown in [12] that all longest paths intersect in a different subfamily of chordal graphs called *split graphs*. A split graph is a graph which is chordal and whose complement is also chordal. A useful characterization of a split graph is that its vertices can be partitioned into an independent set and a clique.

The approach of Balister et al. and Klavzar et al. shows a natural extension of 3 path results. If we can show that any 3 longest paths intersect can we say the same about *all* longest paths. Lehel conjectures that all longest paths in a chordal graph intersect. If we define the *path transversal number* of a graph to be the minimum number of vertices needed to cover all longest paths, then Lehel's conjecture can be stated as "chordal graphs have a path transversal number of 1".

Another class of graphs for which the intersection of 3 longest paths has been examined are *outerplanar graphs*. An outerplanar graph is a planar graph with a distinguished face touching every vertex, i.e. the graph can be realized by placing the vertices on a circle and then drawing chords and arcs. Axenovich [2] succeeds in showing that any 3 longest paths share a common point in an outerplanar graph. We will examine the method used by Axenovich more in the next section as well as sketch a useful means of generalizing it.

We should mention that intersections of longest paths and cycles have been considered in a variety of other contexts as well, such as digraphs. A conjecture of Laborde, Payan, and Xuong [13] generalizing a theorem of Gallai and Roy can be stated as “every digraph has an independent set intersecting every longest path.” See for example [10].

5 Examining 3 longest paths

Henceforth we will focus our attention on Question 6 and the problem of proving that any 3 longest paths share a common vertex. Based on the difficulty of finding a counterexample (even for 6 paths) and for the number of classes which it has been shown to be true, I conjecture "Yes!" with optimism.

Conjecture 13. *For any 3 longest paths P_1, P_2, P_3 in a connected graph G, we have $P_1 \cap P_2 \cap P_3 \neq \emptyset$.*

As mentioned in the previous section, some people have proven Conjecture 13 holds for specific graphs and in most cases even the stronger result $\mathbf{P}(G) \neq \emptyset$ holds. In the case of outerplanar graphs, however, we have only the result for 3 paths.

Theorem 14 (Axenovich [2])**.** *In a connected outerplanar graph G, any three longest paths share a common vertex.*

The methods which Axenovich uses to prove the result for outerplanar graphs is what we will examine further. We provide tools for generalization based on Axenovich's idea of examining a minimal counterexample to Conjecture 13. While this will lead to a new proof of Theorem 14 and a slight generalization, the real advantage of examining a general minimal counterexample is that we are not restricted to one specific class of graphs and so results can be extended more easily (just as these results are an extension of general lemmas presented in the first half of Axenovich's paper). Ideally we prove more and more structure in this minimal counterexample until we reach a contradiction.

5.1 Basic structure of a minimal counterexample G

Henceforth let G be a minimal counterexample to Conjecture 13 with respect to edges. Then the first observation (important enough to deserve its own line) is

$$G = P_1 \cup P_2 \cup P_3$$

where P_1, P_2, P_3 are longest paths of G with $P_1 \cap P_2 \cap P_3 = \emptyset$. Otherwise, we can delete edges not in $P_1 \cup P_2 \cup P_3$, contradicting minimality. Another easy observation is that $\deg x \leq 4$ for all vertices $x \in V(G)$.

We have the following very useful lemma which renders Klavžar's characterization (see Theorem 11) somewhat useless for our purposes.

Lemma 15 (Axenovich [2])**.** *G has exactly one nontrivial block.*

Proof. If G contains no nontrivial blocks then G is a tree and the result follows immediately. It is not difficult to show that all longest paths in a tree must intersect in the "middle" vertex. See for example [14], p. 31.

If G contains ≥ 2 nontrivial blocks, then we pick a cutpoint u on a bridge between 2 blocks, B_1 and B_2 (this "bridge" might just be a single point). As $u \notin P_1 \cap P_2 \cap P_3$ we have that one of the paths, say P_3, does not touch one of the blocks, say B_1. Then we can write $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \{u\}$ and $P_3 \subset G_2$, $B_1 \subset G_1$.

Since P_1 and P_2 must intersect P_3, we have $u \in P_1 \cap P_2$. Next observe that $|P_1[G_1]| = |P_2[G_1]|$, otherwise we would have a longer path by traveling e.g. $P_1[G_1] \cup P_2[G_2]$. Since

$B_1 \subset G_1$ the edge set of $P_1[G_1] \setminus P_2[G_1]$ is nonempty. Delete $P_1[G_1] \setminus P_2[G_1]$ and define P_1' to travel $P_2[G_1] \cup P_1[G_2]$. Then P_1', P_2, P_3 are three longest paths in $P_1' \cup P_2 \cup P_3$ sharing no common point, but this is a contradiction since $P_1' \cup P_2 \cup P_3$ has fewer edges than G. □

We can already begin to see the advantage of working with a minimal counterexample. Immediately we can assume that G has just one block, a serious restriction on the structure of G.

There is a subtle point about Lemma 15 which needs to be mentioned. Suppose our goal is to prove that 3 longest paths share a common point for a *specific* class of graphs, for the sake of example let's just say "green" graphs. We might want to argue by contradiction, taking a minimal counterexample among all green graphs. To use the above lemma, however, we must be sure that green graphs are *closed under the operation of edge/subgraph deletion*. Otherwise the graph $P_1' \cup P_2 \cup P_3$ in the proof of Lemma 15 may not be a green graph and thereby *not* contradict the minimality of $P_1 \cup P_2 \cup P_3$. As outerplanar graphs are clearly closed under subgraph deletion, the minimal counterexample from the space of outerplanar counterexamples indeed has only 1 nontrivial block. If you remove an edge from a chordal graph, however, the resulting graph may not be chordal. A family of graphs which is closed under subgraph deletion is sometimes called a *monotone* family.

This same sort of reasoning applies to different types of reduction procedures, not just subgraph deletion. When arguing by contradiction and taking a minimal counterexample, we hope to reach a contradiction by finding a smaller (think "more efficient") counterexample. We may delete, contract, subdivide, add, glue, replace, etc. to obtain this smaller graph. These arguments are only valid if the resulting graph is still from the same class of graphs we were considering in the first place. Fortunately, as we are working with G, a minimal counterexample from the space of all counterexamples, we are free to manipulate G in any way we choose and we do not need to worry about this distinction.

Before presenting more proofs we make a quick note regarding notation.

Remark 16 (**Regarding notation**)**.** Maintaining absolute precision when referring to vertices, edges, and lengths of paths can become tedious and unnecessary. When expressing membership we often write P instead of $V(P)$ or $E(P)$. The length of P, denoted $l(P)$, will always refer to $|E(P)|$, but in inequalities we may simply write $P + Q$ to represent $|E(P)| + |E(Q)|$. Concatenation PQ is the best way to represent travelling path P and then path Q, but we may also write $P \cup Q$.

Our first result concerning G shows that G has girth ≥ 4. The proof demonstrates how we may manipulate G to form a "more edge-efficient" graph (in this case starting from a triangle).

Lemma 17. *G is triangle-free.*

Proof. Suppose $x, y, z \in V(G)$ form a triangle. First we claim that we can assume $x, y, z \in P_1 \cap P_2$ and so $x, y, z \notin P_3$. If $xy \in P_1$ then $z \in P_1$ as well, otherwise P_1 can travel a longer route by travelling xz and zy instead of xy. As triangle Δxyz has edges from atleast two of P_1, P_2, P_3, we can assume P_1 and P_2 both have edges in Δxyz. Then $x, y, z \in P_1 \cap P_2$ as claimed.

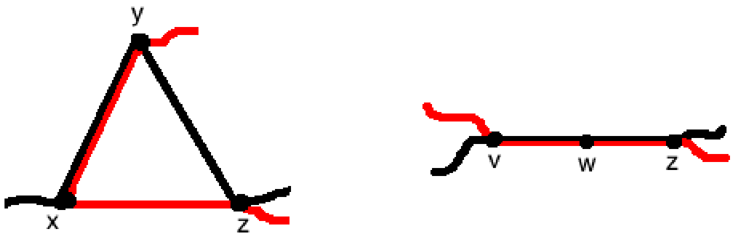


Figure 8: Case 1

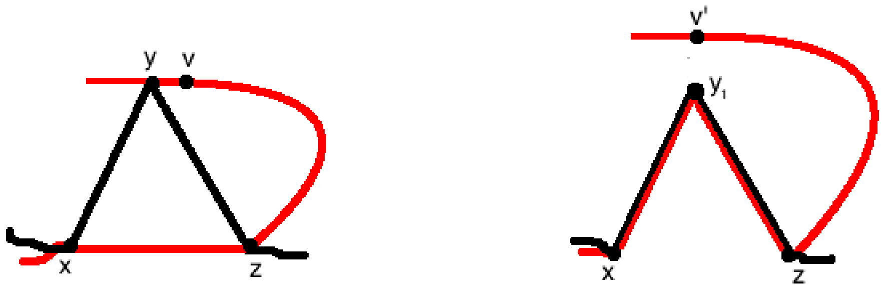


Figure 9: Case 2

Furthermore we can assume P_1 uses two edges of Δxyz, say xy and yz. Then $xz \in P_2$ and we have two cases: Case 1) P_2 uses another edge, say xy, of Δxyz or Case 2) P_2 does not use another edge of Δxyz but just touches the triangle at y.

Case 1. Contract vertices x and y to a vertex v, then subdivide vz by adding one vertex w between them, see Figure 8. We claim that the images of P_1, P_2, P_3 under this operation, which we denote P_1', P_2', P_3', are 3 longest paths in the graph $P_1' \cup P_2' \cup P_3'$ and $P_1' \cap P_2' \cap P_3' = \emptyset$, thereby contradicting the minimality of G. The fact that their intersection is empty is immediate. We only need to prove that no path longer than $l(P_1') = l(P_1)$ exists in $P_1' \cup P_2' \cup P_3'$.

Suppose R is a path with $l(R) > l(P_1)$. R uses at least one of vw, wz otherwise R is in a contracted subgraph of G. If R uses exactly one of vw, wz, say vw, then we can find a path of equal length in G which travels the preimages of vertices in $R \setminus w$ (i.e. we gain back edge xy), and similarly if R uses just wz. So we can assume R uses the entire segment vz. There are at most 4 remaining cases depending on which 'outgoing' edges R uses adjacent to v and to z. But in any case, we can find a length 2 path in Δxyz between the necessary vertices so as to travel the same edges R uses 'outside' of the triangle. Hence R cannot possibly be longer.

Case 2. Detach P_1 and P_2 at y into $y_1 \in P_1$ and $y_2 \in P_2$, for v a neighbor of y_2 (y_2 has at most 2 neighbors as $\deg y \leq 4$) contract edge vy_2 to the point v', then delete xz. See Figure 9. In the resulting graph, let P_1' and P_3' travel the images of paths P_1 and P_3 respectively (which are essentially unchanged), and let P_2' travel the images of the edges of $P_2 \setminus xz$ as well as xy_1z (i.e. P_2 gains one edge from the triangle and loses one outside).

We claim that the resulting graph with P_1', P_2', P_3' contradicts the minimality of G. As it clearly has fewer edges and $P_1' \cap P_2' \cap P_3' = \emptyset$, we need only show that no longer path was created. Suppose R is a path with $l(R) > l(P'_1)$. R uses atleast one of xy_1, y_1z otherwise it is in a contracted subgraph of G. If R uses xy_1z then it must also use v' (otherwise we can easily find a suitable path of equal length in G), but we can find a path of equal length in G which travels xz and the preimages of the vertices in $R \setminus xy_1z$. If R uses xy_1 and not y_1z, then by similar reasoning it must use both vertices v' and z. To find a path of equal length in G, we simply take the preimage of $R \setminus xy_1$. Hence, no path longer than $l(P'_1)$ exists, as desired. □

Corollary 18. *There are no chordal graphs in the space of minimal counterexamples.*

Lemma 17 does not imply that if H is chordal then any 3 of its longest paths share a common point (see Question 9). But it says that in some sense it is not so important to consider chordal graphs (or any graph containing a triangle), as these are not minimal counterexamples.

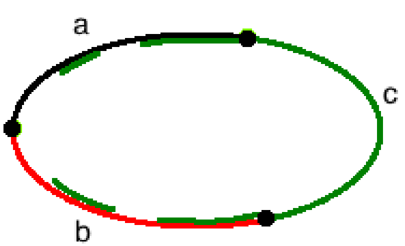
It should be clear by now that the main questions we want to explore are the following: What does G look like? What structures/subgraphs are forbidden in G? What structures/subgraphs *must* be in G? To be more specific, these questions can be asked with varying degrees of generality.

Question 19. What structure must G have based on the fact that G is

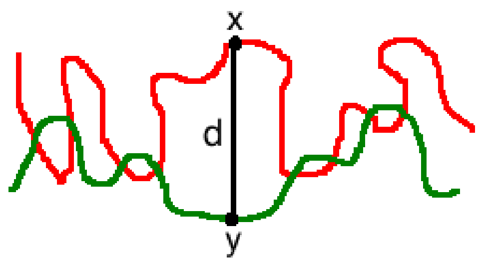
1. the union of three longest paths?
2. the union of three longest paths with empty intersection?
3. the union of three longest paths with empty intersection and minimal edges?

Thus far we have shown that G is a triangle-free graph with precisely one nontrivial block. Each of these results used the minimality of G. We can, however, show that certain structures are forbidden in G without using the condition that G is minimal. Axenovich [2] shows that the graphs in Figure 10 are forbidden as subgraphs of any graph which is the union of three longest paths. We stick to the initial labeling of Axenovich by calling these graphs configuration Q_1 and Q_2 respectively.

Lemma 20 (Axenovich, [2]). *G does not contain configuration Q_1 (see Figure 10a). That is, G does not contain a cycle which is the union of internally disjoint, positive (edge) length segments a, b, c of P_1, P_2, P_3 respectively, such that 1) the interior vertices of $a \cup c$ are disjoint from P_2 and 2) the interior vertices of $b \cup c$ are disjoint from P_1.*



(a) forbidden configuration Q_1



(b) forbidden configuration Q_2

Figure 10

Proof. The direction P_1 travels around the cycle implies $c + b \leq a$. The direction P_2 travels yields $a + c \leq b$. Then $c = 0$, a contradiction. □

Lemma 21 (Axenovich, [2]). *G does not contain configuration Q_2. That is, G does not contain a segment d of positive (edge) length with the following properties:*

(a) d has endpoints $x \in P_2$, $y \in P_3$;
(b) the internal vertices of d belong to P_1 only;
(c) P_2 and P_3 can be oriented so that $P_2 = P_2'P_2''$, $P_2' \cap P_2'' = \{x\}$, $P_3 = P_3'P_3''$, $P_3' \cap P_3'' = \{y\}$, and $(P_2' \cup P_3') \cap (P_2'' \cup P_3'') = \{x, y\}$.

Proof. Based on the fact that P_2' does not use d to switch over to P_3'' we obtain the inequality $P_3'' + d \leq P_2''$. Similarly $P_2'' + d \leq P_3''$. Then $d = 0$, a contradiction. □

The proof strategy involved here is a common and important one. Wherever we have a vertex of degree 3, e.g. when P_1 and P_2 split, we have an inequality lurking in the background. The fact that P_1, a longest path, bypassed the opportunity to travel in some direction, must mean that the longest road down that direction is at most as long as the remainder of P_1.

Armed with this simple observation it is possible to show that if G is a small graph, e.g. n vertices and $n, n+1$, or $n+2$ edges, then $P_1 \cap P_2 \cap P_3 \neq \emptyset$. However, for $n+3$ edges the number of cases to consider becomes quite large.

Using the forbidden configurations of Figure 10, we can now impose another structural requirement on G. Again we do not need the minimality of G, but we do need that P_1, P_2, P_3 have empty intersection.

Lemma 22 (Axenovich, [2]). *$P_i \cup P_j$ contains at least 2 cycles for* $1 \leq i < j \leq 3$.

Proof. Suppose $P_2 \cup P_3$ has at most one cycle. Because the paths must intersect pairwise we can take a segment d of P_1 whose internal vertices belong to P_1 only and whose endpoints are $x \in P_1 \cap P_2$ and $y \in P_2 \cap P_3$. If $P_2 \cup P_3$ is acyclic then we have forbidden configuration Q_2. Suppose C is the lone cycle of $P_2 \cup P_3$. If $x, y \in C$ or if $x, y \notin C$ then we have forbidden configuration Q_2. If $x \in C$, $y \notin C$, then after some inspection we have a forbidden configuration Q_1 (where P_2 plays the role of P_3 in Figure 10a). □

5.2 More structure on G

In the previous section we showed the following basic facts concerning the structure of G.

Lemma 15. G has one nontrivial block B.

Lemma 17. G does not contain a triangle.

Lemma 20/21. G does not contain configurations Q_1 or Q_2.

Lemma 22. The union of any 2 of P_1, P_2, P_3 contains at least 2 cycles.

We would like to know more about the fundamental structure of G. Of particular interest would be results concerning connectivity as well as more forbidden substructures. For example, can G contain a square? Or a pentagon? What does the block B look like? How connected is B?

In this section our goal is to add the following main lemmas to our knowledge of G.

Lemma 23. *G does not contain configuration Q_3 (see Figure 11).*

Lemma 24. *Every cycle of G contains ≥ 3 branching points.*

A *branching point* in G is a vertex x such that $\deg_B(x) \geq 3$. A vertex x with $\deg_G(x) \geq 3$ but $\deg_B(x) = 2$ is *not* a branching point in our notation, but rather a *terminal-type point*. The name "terminal-type" point comes from the observation that some path exits B at that point. In fact, we can say a little more. A terminal-type point is the starting point of precisely one path outside of the block B, which we call a *terminal branch*.

Lemma 25. *$G \setminus B$ consists of ≤ 6 (vertex) disjoint paths and one endpoint of each path is a terminal-type point.*

Sketch of proof. Use minimality in a similar manner as the proof of Lemma 15. □

As a corollary of Lemma 24, we can obtain the following generalization of Axenovich's result on outerplanar graphs (Theorem 14).

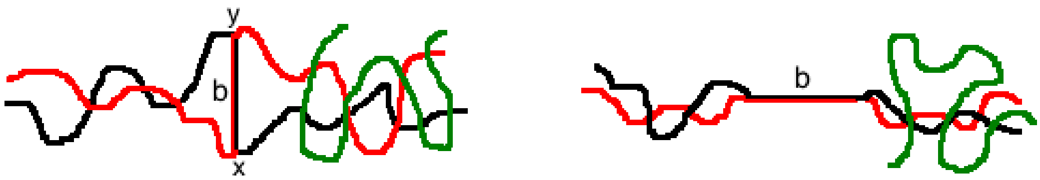
Corollary 26. *If H is topologically outerplanar, then any three longest paths of H share a common point.*

Proof. We can take a minimal counterexample G from the space of topologically outerplanar counterexamples. As these graphs form a monotone family, all the results of the previous section hold and so does Lemma 24. Observe that all branching points of G lie on the outer circle of the outerplanar graph which G is a subdivision of.

Take a cycle C_1 of G. As C_1 has ≥ 3 branching points we can find a cycle C_2 which shares a (subdivided) chord with C_1. Similarly we can find a cycle $C_3 \neq C_1$ such that C_3 shares a (subdivided) chord with C_2. We construct a sequence of cycles in this way such that $C_k \neq C_{k-1}$. Moreover this sequence is nonrepeating for if $C_k = C_j$ for $j \leq k-2$ then there is a branching point in the interior of the circle, a contradiction. As the graph is finite, this sequence must terminate. Yet the sequence can only terminate when a cycle has ≤ 2 branching points, a contradiction. □

Now we can really see the advantage of analyzing a minimal counterexample G. The results are general and do not restrict us to one specific family of graphs. On the other hand, the results can be immediately applied to monotone families of graphs. Ideally we can place an increasing number of restrictions on the structure of G so as to rule out wider classes of monotone families of graphs until eventually we can rule out the most important monotone family of graphs: *all* graphs. Of course a more realistic and natural next step after outerplanar graphs would be to consider planar graphs.

We have seen now the utility of Lemma 24, but how do we prove the lemma itself? Unfortunately it is a tedious case-by-case analysis involving repeated use of all the lemmas used up to now as well as some easy observations regarding the minimality of G. The separation of cases reflects all the different ways 3 non-intersecting longest paths can be



(a) forbidden configuration Q_3 (b) forbidden configuration Q_3, trivial case

Figure 11: P_1 = black, P_2 = red, P_3 = green

situated to form a cycle C with only 2 branching points. In each case, we stare at C and ask ourselves "why is this not an efficient way to use edges?", then we replace the cycle with a more edge-efficient configuration. We redefine P_1, P_2, P_3 for the new graph, prove that they are indeed longest paths, and then we have contradicted the minimality of G, as desired.

In a moment we will sketch the proof, but more important than the proof itself are the collection of lemmas and observations we use throughout. For this reason, I have highlighted Lemma 23 as it is particularly useful in simplifying cases, and in my opinion, it can likely be generalized to a more powerful lemma. The lemma (stated above) outlaws the configurations in Figure 11 . To be more precise, we have a reduction technique where if configuration Q_3 appears in G then we can reduce the graph to a more edge-efficient one.

Configuration Q_3 is defined by a maximal segment $b \in P_1 \cap P_2$ with the following properties:

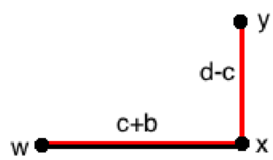
(a) We can write $P_1 = P_1'P_1''$ and $P_2 = P_2'P_2''$ where b is the final segment of P_i' $i = 1, 2$, such that $(P_1' \cup P_2')$ and $(P_1'' \cup P_2'')$ intersect only at the endpoint(s) of b;

(b) $P_3 \cap (P_1' \cup P_2') = \emptyset$;

(c) $E(P_1' \setminus P_2')$ and $E(P_2' \setminus P_1')$ are each nonempty.

Proof of Lemma 23. If G contains configuration Q_3 then it must look like one of the graphs from Figure 11. In the trivial case removing a vertex from b disconnects G so we can proceed in a manner similar as in the proof of Axenovich's Lemma 15 and in the proof of Lemma 25. Observe that P_1' and P_2' are of equal lengths, delete $P_1' \setminus P_2'$, redefine P_1' to be P_2', and we have contradicted the minimality of G.

Suppose G looks like Figure 11a. Let $c = |E(P_1' \setminus b)|$ and $d = |E(P_2' \setminus b)|$. If $d = c$ we can replace $P_1' \cup P_2'$ by a segment of length $d + b$, redefine P_1' and P_2' to travel this segment, and we have a smaller graph. Otherwise suppose $d > c$. Based on the directions P_1 and P_2 travel we have the inequality $d \leq b + c$. Replace $P_1' \cup P_2'$ by the following graph



where x and y are identified with the endpoints of b as depicted in Figure 11a. Redefining $P_1' = wx$ and $P_2' = wxy$, we clearly have not changed the lengths of P_1 and P_2. The resulting graph has fewer edges than G with $P_1 \cap P_2 \cap P_3 = \emptyset$. It remains to show that no path longer than P_1 exists. Suppose R is a longer path than P_1. R uses wx otherwise it is in a contracted subgraph of G (recall $d - c \leq b$). R must also use y (o.w. $l(R) \leq l(P_1)$). If R uses all of xy then it cannot be longer than P_2. So R uses wx, exits at x, returns to y, and travels all but the final edge of xy (recall $\deg y = 3$ in G forces R to 'reenter' yx). Then $l(R) = d + b - 1 + s$ where s is the length of the segment outside of $P_1' \cup P_2'$. But in G there is a path traveling $d + b + s - 1$ where we use the edges of P_2' and the edges of R 'outside' stopping one edge shy of x. Hence, no longer path exists. □

The above proof that $Q_3 \not\subset G$ resembles the proof that G does not contain a triangle. In each case we suppose $I \subset G$ and then show that we can replace I by a more efficient subgraph I'. It is convenient to call this subgraph I the 'inside' and $O = G \setminus I$ the 'outside'. Then we can say something like 'P_1 enters (or exits) at x' to mean P_1 comes inside from the outside (or vice versa) at the vertex $x \in O \cap I$.

We are almost ready to begin proving Lemma 24 which states that each cycle in G has at least 3 branching points. Before jumping into the proof we make a few observations which we leave as exercises and which are often needed in the proof.

Observation 27. Let x and y be 'essentially adjacent' branching points in B, i.e. there is a segment xy whose internal vertices are not branching points. Then P_1, P_2, P_3 cannot all have edges on this segment. (Hint: There will be a forbidden configuration Q_2.)

Observation 28. Each half of P_1 intersects both P_2 and P_3.

Observation 29. Let $u \in P_1 \cap P_2$, and let v and w be neighbors of u such that $v \in P_1 \setminus (P_2 \cap P_3)$ and $w \in P_2 \setminus (P_1 \cap P_3)$. Then G is not minimal. (Hint: Identify v and w.)

Observation 30. Let v be a terminal type point of P_1 only (i.e. P_1 exits B at v while P_2 and P_3 do not). All neighbors of v are either in P_1 or $P_2 \cap P_3$. If v is an endpoint of P_1 then all neighbors of v are in P_1. (Hint: Identify the neighbor of v on the terminal branch with another available neighbor of v.)

Sketch of proof of Lemma 24. Suppose $C \subset G$ has ≤ 2 branching points. If C has only 1 branching point then the removal of this vertex disconnects the block, a contradiction. We may assume that C has precisely 2 branching points x and y.

Define the inside I to be the union of C and all of the terminal branches adjacent to C. And the outside $O = G \setminus I$. In the subdivision of cases consider x and y as vertices of I rather than O. Let an 'arc' of C refer to one of the two paths in C connecting x and y.

Instead of giving a detailed proof, we show how the cases can be grouped, and then we show a more edge-efficient graph to use as a replacement graph. It is left to the ambitious reader to verify all the details. Proving one or two of the tougher cases in detail will give a sense of how to prove the other cases. The first thing we do is gather usable inequalities. Then we experiment with potential replacement graphs until we have found the one which does not produce a longer path.

Tree of cases

Case 1 All endpoints of P_1, P_2, P_3 are outside.

Case 2 P_1 has both endpoints inside.

- **Case 2.1** $P_1 \subseteq I$.
- **Case 2.2** $P_1 \nsubseteq I$.
  - **Case 2.2.1** Each of P_1, P_2, P_3 has a vertex in I.
  - **Case 2.2.2** P_3 is vertex disjoint from I.
    - **Case 2.2.2.1** P_1's terminal type points are on the same arc.
    - **Case 2.2.2.2** P_1's terminal type points are on different arcs.

Case 3 P_1 has one endpoint inside and one outside.

- **Case 3.1** P_3 is vertex disjoint from I.
  - **Case 3.1.1** P_2 exits at x.
  - **Case 3.1.2** P_2 does not exit at x.
- **Case 3.2** Each of P_1, P_2, P_3 has a vertex in I.
  - **Case 3.2.1** P_3 is edge disjoint from C.
  - **Case 3.2.2** Each of P_1, P_2, P_3 has at least one edge in C.

Case 1: All endpoints of P_1, P_2, P_3 are outside.

We can assume P_1 and P_2 both enter at x and exit at y. If P_2 travels the same arc as P_1 then P_3 travels the other arc and we have $x \in P_1 \cap P_2 \cap P_3$. So we can assume P_2 travels the opposite arc from P_1, and P_3 is vertex disjoint from I. Observe that each arc must be the same length, otherwise we can find a longer path traveling the longer arc inside and using the other path outside. Delete the arc which P_2 travels and redefine P_2 to travel the same as P_1 inside. The new graph has fewer edges, no longer path, and P_1, P_2, P_3 have no common point, a contradiction.

Case 2.1: $P_1 \subset I$.

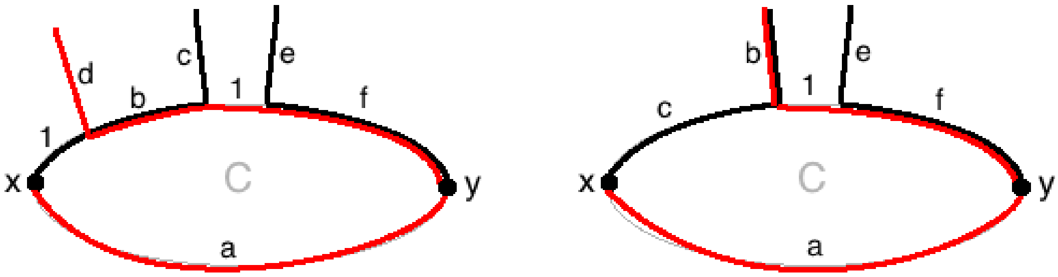
Show that $P_1 \cup P_2$ has at most 1 cycle, contradicting one of Axenovich's lemmas.

Case 2.2.1: P_1 has both endpoints inside, $P_1 \not\subset I$, and each of P_1, P_2, P_3 has a vertex in I.

Separate into two pictures based on which arc P_1's terminal type points are on. Show that one of the pictures is impossible because it puts us back in Case 2.1. Show that the other picture produces configuration Q_2.

Case 2.2.2.1: $P_1 \not\subset I, P_3 \cap I = \emptyset, P_1$'s terminal type points lie on the same arc

Show that I looks like one of the graphs from Figure 12.



(a) (b)

Figure 12

Consider Figure 12a first. Use minimality to show that we can assume $|c - e| \leq 1$. If $d = 0$, P_2 can travel the opposite direction around C. So $d > 0$, and by Lemma 23 we have $c = 0$. Notice that $E(P_2 \cap I) \leq E(P_1 \cap I) + 1$ (o.w. P_2 can travel longer outside), which gives us $d = a = e = 1$. Replace I by the following graph with P_1 and P_2 redefined according to the colors in the picture (black = P_1, red = P_2), and show that the resulting graph has no longer path.

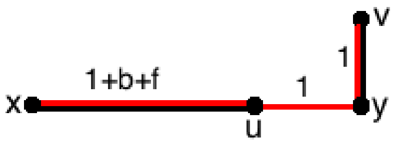


Figure 13: Use to replace Figure 12a.

For Figure 12b, simply replace I by the following graph, proving that no longer path exists.

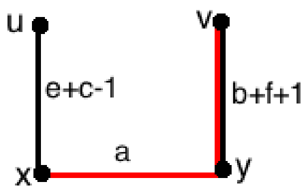


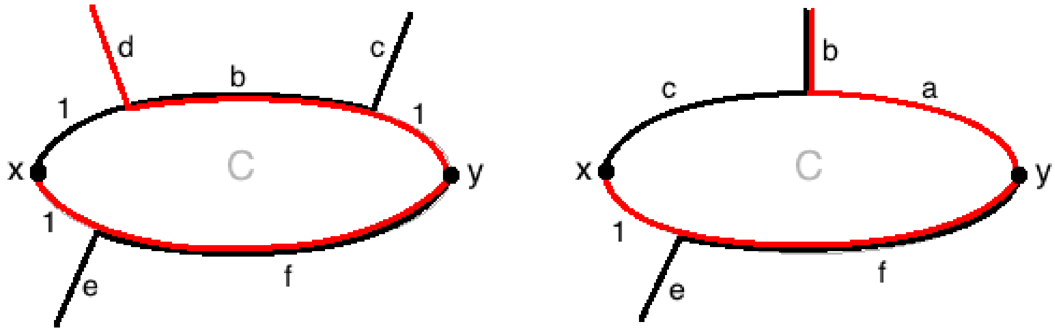
Figure 14: Use to replace Figure 12b.

Case 2.2.2.2: $P_1 \not\subset I, P_3 \cap I = \emptyset, P_1$'s terminal type points lie on different arcs.

We can assume $P_2 \cap I$ is connected. Then we have one of the graphs from Figure 15.

For Figure 15a, we can assume $d > 0$ (o.w. P_2 can travel the other direction around C) which implies $c = 0$ by Lemma 23. In fact we have $d > e$. But the inequality $E(P_2 \cap I) \leq E(P_1 \cap I) + 1$ implies $d \leq e$.

For Figure 15b, prove that we have $a + 1 \leq c + e$ and $c + e \leq b + 1$ by comparing paths. Replace I by the graph in Figure 16, and use these inequalities to show that no longer path exists.



(a) (b)

Figure 15

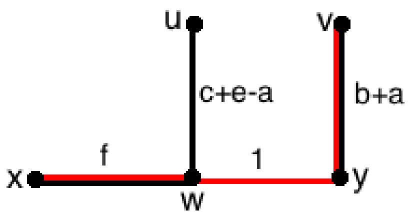


Figure 16: use as a replacement for Figure 15b

Case 3.1.1: P_1 has one endpoint in and one out, $P_3 \cap I = \emptyset$, P_2 exits at x

Replace the graph of Figure 17 by Figure 18.

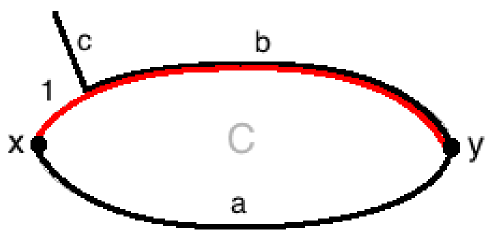


Figure 17

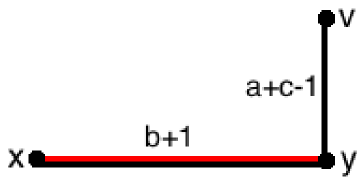


Figure 18: Use to replace Figure 17.

Case 3.1.2: P_1 **has one endpoint in and one out,** $P_3 \cap I = \emptyset, P_2$ **does not exit at** x**.**

Observe that the only case not producing a forbidden configuration Q_3 is the graph in Figure 19. In a similar manner as the proof of Lemma 23, replace Figure 19 by Figure 20.

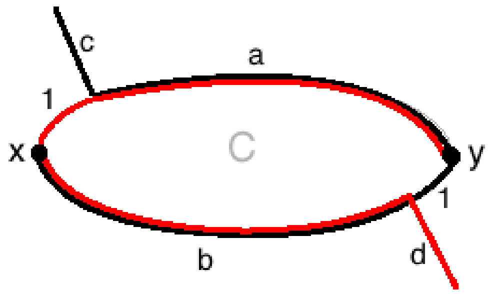


Figure 19

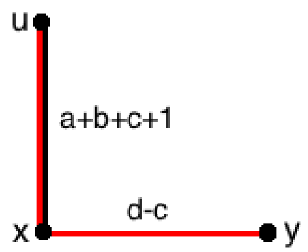


Figure 20: Use to replace Figure 19.

Case 3.2.1: P_1 **has one endpoint in and one out,** P_3 **just 'touches'** I

Show that P_3 can touch at only one of x or y. Delete and redefine P_1 and P_2 as needed inside, using the minimality of G.

Case 3.2.2: Each of P_1, P_2, P_3 **has at least one edge in** C

In any case we can readily find a forbidden configuration Q_2. This concludes the proof. □

Another way of stating Lemma 24 is that if we delete $G \setminus B$ and then contract vertices of degree 2, the resulting graph will have no multiple edges. More results on this 'reduced graph' would be of interest. What can we say about its connectivity? Can we strengthen Lemma 24 to say that any cycle in G contains ≥ 4 branching points? How traversable is this reduced graph? What conditions on this graph would be enough to prove the conjecture in full? Or for planar graphs? Unanswered, interesting questions abound.

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