MSc Thesis

On Dynamic Systems and Control

Christoph Pfister

Supervisor: Gheorghe Morosanu
Department of Mathematics and its Applications
Central European University
Budapest, Hungary

May 4, 2012

Contents

1 Introduction

The concept of a dynamical system in general is just anything that evolves with time. We can consider technical systems, such as vehicles, electrical circuits or power plants but also for example how a skyscraper or a bridge behaves while exposed to strong winds or an earthquake. Dynamical system do not only apply to engineering problems but are found in economics, biology and even social sciences.
Usually a system interacts with its environment via inputs and outputs. To study a dynamical system, we can observe what outputs we measure for given inputs. We can go further and try to steer the system to a desired state by applying certain inputs as a reaction to the measured outputs. The problem of forcing the dynamical system to the desired state is called control, and as for dynamical systems, the concept of control can be found in many areas.
Mathematical control theory deals with the analysis of the dynamical system and the design of an appropriate control method. Of course, we are not just satisfied with finding a control that steers our system to the defined target state, we are further interested to do this in an optimal way. So optimality is one of the main subjects to study in this area.
Basically, one can divide control theory in a classical and a modern theory. The classical approach analyzes the inputs and outputs of a system to determine its transfer function. This is usually done with help of the Laplace transform. From the knowledge of the transfer function, a feedback control can be designed that forces the system to follow a reference input. The thesis is not covering this but focuses on the modern control theory. The modern approach utilizes the time-domain state-space representation. The state-space representation is a mathematical model that relates the system state with the inputs and outputs of the system by first order differential equations. This area of mathematics is a widely studied subject, there are many books and papers covering various aspects and applications. Therefore this thesis can only cover some selected subjects to a certain extend.

1.1 History

Many natural systems maintain itself by some sort of control mechanism. So does for example the human body regulate the body temperature or it is able to keep the balance in different positions. Some of these natural systems were studied and described mathematically. So for example the interaction of different species with each other, some being predators and some being preys. These systems are described by the famous Lotka-Volterra Dynamics.
One can date back the origin of mathematical control to the description of the mechanical governor to control the speed of a steam engine by James Clerk Maxwell in 1868. He described the functionality of the device in a mathematical way, which led to further research in this subject. Many mathematicians contributed to this research. So for example Adolf Hurwitz and Edward J. Routh, who obtained the characterization

of stability for linear systems, Harry Nyquist and Hendrik W. Bode, who introduced feedback amplifiers for electrical circuits to assure their stability, or Norbert Wiener, who developed a theory of estimation for random processes.
The achievements mentioned were mainly done for the classical theory, hence they were restricted to time invariant systems with scalar inputs and outputs. During the 1950s, the modern control theory was introduced. One of the main contributors was Rudolf E. Kalman with his work on filtering, algebraic analysis of linear systems and linear quadratic control. In the area of optimal control theory, Richard E. Bellman and Lev S. Pontryagin contributed methods to obtain optimal control laws. Their work formed the basis for a large research effort in the 1960s which continuous to this day.

1.2 An Introductory Example

We shall state here one of the standard examples [21] on how to describe a system and define a simple control rule. We consider a problem from robotics, where we have a single link rotational joint using a motor placed at the pivot. Mathematically, this can be described as a pendulum to which one can apply a torque as an external force, see Figure 1. For simplicity, we take the following assumptions: friction is negligible and the mass is concentrated at the end of the rod.

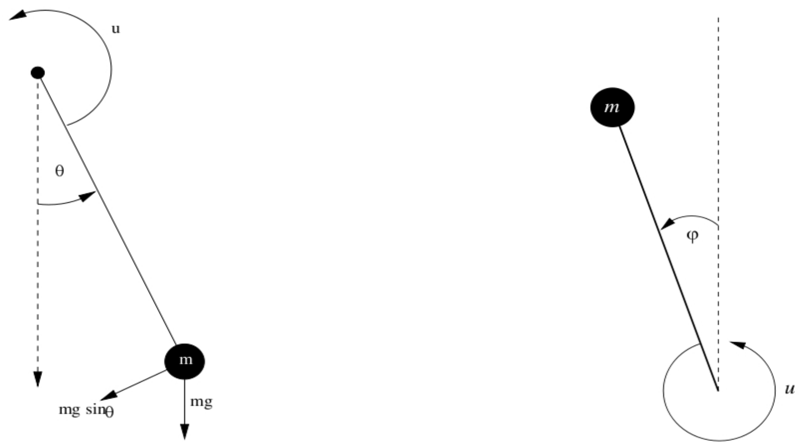

Figure 1: Pendulum and inverted pendulum [3, p. 4]

Let θ describe the counterclockwise rotation angle with respect to the vertical. Then we obtain the following second order nonlinear differential equation

$$m\ddot{\theta}(t) + m\frac{g}{L}\sin(\theta(t)) = u(t), \tag{1.1}$$

where m describes the mass, g the constant acceleration due to gravity, L the length of the rod and $u(t)$ the value of external torque at time t (counterclockwise being positive). For further simplicity assume we chose the constants so that $m = g = L = 1$.

Now our goal his to keep the pendulum in the upright position (inverted pendulum, see Figure 1, right) where we have $\theta = \pi$, $\dot{\theta} = 0$. This position is an unstable equilibrium point, therefore we need to apply torque to keep the pendulum in position as soon as some deviation from this equilibrium point is detected.

If we consider only small deviations, we can linearize the system around the equilibrium point by

$$\sin(\theta) = -(\theta - \pi) + o(\theta - \pi)^2, \tag{1.2}$$

where we drop the higher order term $o(\theta - \pi)^2$. By doing this, we can introduce a new variable $\varphi = \theta - \pi$ and instead of working with the nonlinear equation (1.1), we can work with the linear differential equation

$$\ddot{\varphi}(t) - \varphi(t) = u(t). \tag{1.3}$$

The objective is then to bring φ and $\dot{\varphi}$ to zero for any small nonzero initial values of $\varphi(0)$ and $\dot{\varphi}(0)$, preferably as fast as possible. We shall do this by simply applying proportional feedback, meaning that if the pendulum is to the left of the vertical, i.e. $\varphi = \theta - \pi > 0$, then we wish to move to the right and therefore apply negative torque. If instead the pendulum is to the right of the vertical, we apply positive torque. For a first try, the control law looks like

$$u(t) = -\alpha\varphi(t), \tag{1.4}$$

for α a positive real number. But one can easily show that this control law will lead to an oscillatory solution, the pendulum will oscillate around the equilibrium point. Therefore we modify the control law by adding a term that penalizes velocities. The new control law with the damping term is then

$$u(t) = -\alpha\varphi(t) - \beta\dot{\varphi}(t), \tag{1.5}$$

were we have β also a positive real number.
The control law described in (1.5) just brings us to the problem of *observability*. Realizing such a control would involve the measurement of both, angular position and velocity. If only the position is available, then the velocity must be estimated. This will be discussed in the chapters on *observability* and *stabilizability*. For now we assume that we can measure both and so we can construct the closed loop system from (1.3) and (1.5) as follows

$$\ddot{\varphi}(t) + \beta\dot{\varphi}(t) + (\alpha - 1)\varphi(t) = 0. \tag{1.6}$$

It can now be shown that (1.6) indeed is a stable system and thus the solutions of (1.3) converge to zero. We shall omit this and instead have a look at the state-space representation of this problem.
The state-space formalism means that instead of studying higher order differential equations, such as (1.1) or (1.3), we replace it by a system of first order differential equations. For (1.3) we introduce the first order equation

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1.7}$$

where

$$x(t) = \begin{bmatrix} \varphi(t) \\ \dot{\varphi}(t) \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{1.8}$$

Equation (1.7) is an example of a *linear, continuous time, time invariant, finite dimensional control system.*
In the matrix formalism, the feedback is given by $u(t) = Kx(t)$ where K is the vector

$$K = [-\alpha\ -\beta]\,. \tag{1.9}$$

The system matrix of the closed-loop system is then

$$A + BK = \begin{bmatrix} 0 & 1 \\ 1-\alpha & -\beta \end{bmatrix}, \tag{1.10}$$

from which we get the characteristic polynomial

$$\det(\lambda I - A - BK) = \det \begin{bmatrix} \lambda & -1 \\ \alpha - 1 & \lambda + \beta \end{bmatrix} = \lambda^2 + \beta\lambda + \alpha - 1. \tag{1.11}$$

From the characteristic polynomial we get the eigenvalues

$$\lambda_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4(\alpha - 1)}}{2}, \tag{1.12}$$

which have negative real part and hence the feedback system is asymptotically stable.
If we take $\beta^2 \geq 4(\alpha - 1)$ we have real eigenvalues and hence no oscillation results.
We can conclude that with a suitable choice of the values α and β, it is possible to attain the desired behavior, at least for the linear system.
This shows that state-space representation is a good and robust approach. Of course, if the system has higher dimensions and is maybe time variant, more sophisticated methods have to be used to analyze the system and define its control law.

1.3 Structure of the Thesis

Chapter 2 is an introduction to the theory on dynamic systems, where we focus on linear systems. We start by showing the existence and uniqueness of a solution to a homogeneous time-variant system. From this we can derive the *fundamental matrix* and the *transition matrix* with its basic properties. This properties are then used to obtain the solution to the linear controlled system.
In chapter 3 the system is analyzed on its stability. We define the main types of stability based on how the system behaves at an equilibrium point. The Theorem of *Lyapunov* is given, which states that if we can find a *Lyapunov function* that satisfies certain properties for the given system, stability of the system (possibly nonlinear) is given. Further we apply this theorem to a linear system and obtain the *Matrix Lyapunov Equation.*
In chapter 4 we introduce *observability* for a linear controlled system. Observability is

important if we have to deal with systems, where the state of a system is not directly available and we have to work with the system output instead. The definition of the *observability matrix* is given and we state the theorem that the system is observable if and only if the observability matrix is positive definite. Further, the *Kalman Observability Rank Condition* is stated, which applies to time-invariant systems.
Chapter 5 deals with *controllability*. Controllability of a system means that the system can be forced to attain a certain target state. For this we introduce the *controllability matrix* and state that if the controllability matrix is positive definite, then the system is controllable. For linear systems, we further state the *Kalman Controllability Rank Condition*. Finally it is shown that the two properties controllability and observability are in fact dual problems.
In chapter 6 we use the properties of controllability and observability to stabilize a system by *state feedback* if the states are accessible, or *output feedback* if we can only work with the output of the given system. It is shown that when the states are not directly available, we can construct an *observer*, which, together with the original system, allows us to realize the feedback control.
Finally in chapter 7 we try to find an optimal control by using *linear quadratic control*. This means that the performance of a controlled system is described by a cost functional, which we try to minimize. To do this, we follow Richard E. Bellman's approach using *dynamic programming* to obtain the *Hamilton-Jacobi-Bellman Equation*. If we apply this equation to a linear system, we obtain the *Differential Matrix Riccati Equation*. The solution to the Differential Matrix Riccati Equation then provides us an optimal state feedback control law. Further, the *steady-state LQR problem* is given, which handles the special case when the time horizon over which the optimization is performed is infinite and the system is time invariant. The result of this problem is the *Algebraic Riccati Equation*, which, as we will show, provides a feedback control law that makes the closed-loop system asymptotically stable.

2 Dynamical Systems

2.1 Nonlinear System

In this thesis, we will mainly focus on linear systems. But of course, most of the physical systems which can be encountered in real life are nonlinear. Nonlinear systems require intensive study to cover all their aspects. Regarding control, one therefore often prefers to linearize the system around some operating point. Nevertheless, we start by describing nonlinear systems, as they are the general form.
A nonlinear system is given by the equations:

$$\begin{aligned} \dot{x} &= f(x, u), \\ \dot{x} &= f(t, x, u), \end{aligned} \tag{2.1}$$

where $x \in \mathbf{R}^n$ is called the *state vector* and denotes the status of the system at a given time t. The *control vector* u takes values from a set $U \in \mathbf{R}^m$, which is the set of control

parameters. The function $f = col(f_1, f_2, ..., f_n)$ is an n-dimensional vector depending on x, u, and, in the time varying case, on t.
For a short discussion on solutions of nonlinear systems we consider the Cauchy problem

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)), \\ x(0) &= x_0. \end{aligned} \tag{2.2}$$

Unlike linear systems, existence and uniqueness of solutions is not guaranteed. In addition, a solution may only exist in an interval, called the maximal interval of existence. There are many results investigating (2.2) regarding this problem. We state here one without proof. For the proof of below theorem and more general results, refer to [1, 22] and many results are given in [18].

Theorem 2.1 : Assume for an arbitrary $x \in \mathbf{R}^n$ the function f is measurable and locally integrable in t and for the functions $L \in L_1^+(I)$ (integrable on interval $I \ni 0$) the Lipschitz condition holds:

$$\|f(t, x) - f(t, y)\| \leq L(t)\|x - y\|. \tag{2.3}$$

Then the system (2.2) has a unique solution x in the space of continuous functions, $x \in C(I, \mathbf{R}^n)$ for each $x_0 \in \mathbf{R}^n$.

Given that the a unique solution is found, by its continuity we can define the *solution operator* $\{T_{t,s},\, t \geq s \geq 0\}$ as in [1, p. 87] that maps the state ξ given at the time s to the current state, $x(t)$, at time t. This family of solution operators satisfies the evolution or semigroup properties:

$$\begin{aligned} &(T1): \ T_{s,s} = I, \ \forall s \geq 0, \\ &(T2): \ \lim_{t \downarrow s} T_{t,s}(\xi) = \xi, \ \forall s \geq 0, \ \forall \xi \in \mathbf{R}^n, \\ &(T3): \ T_{t,\tau}(T_{\tau,s}(\xi)) = T_{t,s}(\xi), \ \forall t \geq \tau \geq s \geq 0, \forall \xi \in \mathbf{R}^n. \end{aligned} \tag{2.4}$$

2.2 Linear Systems

A linear dynamic system of order n is described by a set of n first order ordinary differential equations:

$$\begin{aligned} \dot{x}_1(t) &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + ... + a_{1n}(t)x_n(t) + b_1(t), \\ \dot{x}_2(t) &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + ... + a_{2n}(t)x_n(t) + b_2(t), \\ &\vdots \\ \dot{x}_n(t) &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + ... + a_{nn}(t)x_n(t) + b_n(t). \end{aligned} \tag{2.5}$$

We can express (2.5) in matrix notation

$$\dot{x} = A(t)x(t) + b(t). \tag{2.6}$$

In order to guarantee existence and uniqueness of a solution, the coefficients $a_{ij}(t)$ are usually assumed to be continuous in t. A stronger result for $a_{ij}(t)$ locally integrable is shown in the next section.

Remark: Throughout this thesis, we use the *induced* or *operator norm* for the norm of a matrix: If the the vector norms on $\mathbf{R}^n$ and $\mathbf{R}^m$ are given, the operator norm for the matrix $A \in \mathbf{R}^{m\times n}$ is defined by

$$\|A\| = \sup\{\|Ax\| : x \in \mathbf{R}^n, \|x\| = 1\}. \tag{2.7}$$

The operator norm has the folowing important submultiplicative properties for the matrices A, B and vector x

$$\begin{aligned} \|Ax\| &\leq \|A\|\|x\|, \\ \|AB\| &\leq \|A\|\|B\|. \end{aligned} \tag{2.8}$$

In what follows, we shall first show existence and uniqueness of the solution. Then we will describe how such a solution looks like and finally we can define the *transition operator*, which solves the linear system.

2.2.1 Existence and Uniqueness of Solution

To start, we consider the homogeneous system

$$\begin{aligned} &\dot{x}(t) = A(t)x(t),\ t \geq s, \\ &x(s) = \xi. \end{aligned} \tag{2.9}$$

In (2.9) the vector $x(t) \in \mathbf{R}^n$ denotes the *state* of the system and the matrix $A(t) \in \mathbf{R}^{n\times n}$ represents the *system matrix* that characterizes the behavior of the system.

Theorem 2.2 : Consider the above system (2.9). Assume that the elements $a_{ij}(t)$ of matrix $A(t)$ are locally integrable in the sense that $\int_I |a_{ij}(t)|dt < \infty$, for all $i, j = 1, 2, \cdots, n$, and for every finite closed interval I, say $I = [s, T]$. Then our system has a unique and absolutely continuous solution which further is continuously dependent on the initial state ξ.

Proof : Consider the vector space of continuous functions $C(I, \mathbf{R}^n)$. Assign the supremum norm to a function $x(t)$ in this space:

$$\|x\| = \sup_{t\in I} \|x(t)\|. \tag{2.10}$$

Equipped with the supremum norm, the space $C(I, \mathbf{R}^n)$ forms a Banach space.
We will prove the theorem for the right intervall $I = [s, T]$, the proof for $[0, s]$ is similar. In fact, we can assume $s = 0$.

The Picard iteration process [5] provides a way to approximate the integral curve $x(t)$ as follows:

$$\begin{aligned} x_1(t) &= \xi,\ t \in I, \\ x_2(t) &= \xi + \int_0^t A(\tau)x_1(\tau)d\tau,\ t \in I, \\ &\vdots \\ x_{m+1}(t) &= \xi + \int_0^t A(\tau)x_m(\tau)d\tau,\ t \in I, \\ &\vdots \end{aligned} \tag{2.11}$$

We will show that the sequence $\{x_m\}$ converges to a unique $x(t) \in C(I, \mathbf{R}^n)$ in the supremum norm.
Define the function

$$h(t) \equiv \int_0^t \|A(\tau)\| d\tau. \tag{2.12}$$

By our assumption, the elements in the matrix $A(\tau)$ are locally integrable, hence the function $h(t)$ is non-decreasing, once differentiable and bounded for $t \in I$.
Now we investigate the sequence:

$$\begin{aligned} \|x_2(t) - x_1(t)\| &\leq \left(\int_0^t \|A(\tau)\| d\tau \right) \|\xi\| = h(t)\|\xi\|, \\ \|x_3(t) - x_2(t)\| &= \left\| \int_0^t A(\tau)x_2(\tau)\, d\tau - \int_0^t A(\tau)\xi\, d\tau \right\| \\ &= \left\| \int_0^t A(\tau) \left(\xi + \int_0^\tau A(\theta)\xi d\theta \right) - A(\tau)\xi\, d\tau \right\| \\ &\leq \int_0^t \underbrace{\|A(\tau)\|}_{h'(\tau)} \underbrace{\int_0^\tau \|A(\theta)\| d\theta}_{h(\tau)}\, d\tau \|\xi\| \end{aligned} \tag{2.13}$$

by the product rule we get

$$\begin{aligned} &= \int_0^t \frac{1}{2} \frac{d}{d\tau} \left(\int_0^\tau \|A(\theta)\| d\theta \right)^2 d\tau \|\xi\| \\ &= \frac{1}{2} \int_0^t \frac{d}{d\tau} h^2(\tau) d\tau \|\xi\| \\ &= \frac{1}{2} h^2(t) \|\xi\|. \end{aligned}$$

We can continue this process, the general step will then look like:

$$\|x_{m+1}(t) - x_m(t)\| \leq \frac{h^m(t)}{m!} \|\xi\|. \tag{2.14}$$

For any positive p we have

$$\begin{aligned}
\|x_{m+p}(t) - x_m(t)\| =& \|x_{m+p}(t) - x_{m+p-1}(t) + x_{m+p-1}(t) - x_{m+p-2}(t) + \\
& + x_{m+p-2}(t) - \cdots - x_{m+1}(t) + x_{m+1}(t) - x_m(t)\| \\
\leq & \|x_{m+p}(t) - x_{m+p-1}(t)\| + \|x_{m+p-1}(t) - x_{m+p-2}(t)\| + \\
& + \cdots + \|x_{m+1}(t) - x_m(t)\| \\
=& \sum_{k=m}^{m+p-1} \|x_{k+1}(t) - x_k(t)\| \qquad (2.15) \\
\leq & \sum_{k=m}^{m+p-1} \left(\frac{h^k(t)}{k!}\right) \|\xi\| \\
\leq & e^{(h(t))} \frac{h^m(t)}{m!} \|\xi\|.
\end{aligned}$$

We can now show that $\{x_m\}$ is a Cauchy sequence in the space $C(I, \mathbf{R}^n)$

$$\|x_{m+p} - x_m\| = \sup_{t \in I} \{\|x_{m+p}(t) - x_m(t)\|\} \leq e^{h(T)} \frac{h^m(T)}{m!} \|\xi\|. \qquad (2.16)$$

$h(T)$ is finite,

$$\lim_{m \to \infty} \|x_{m+p} - x_m\| = 0, \qquad (2.17)$$

uniformly with respect to p. Hence the sequence $\{x_m\}$ converges uniformly on the interval I to a continuous function $x(t)$. From the Picard iteration (2.11) by letting $m \to \infty$ we obtain

$$x(t) = \xi + \int_0^t A(\theta) x(\theta) d\theta, \ t \in I. \qquad (2.18)$$

Now for continuity with respect to initial data, take $\xi, \eta \in \mathbf{R}^n$ and denote by $x(t)$ the corresponding solution to the initial state ξ and $y(t)$ the solution to η.
By (2.18) we obtain

$$x(t) - y(t) = \xi - \eta + \int_0^t A(\theta)(x(\theta) - y(\theta)) d\theta, \ t \in I. \qquad (2.19)$$

Then by applying first the triangle inequality and then the Gronwall inequality (See for example [1, p. 44]) we get

$$\begin{aligned}
\|x(t) - y(t)\| &\leq \|\xi - \eta\| + \int_0^t \|A(\theta)\| \|x(\theta) - y(\theta)\| d\theta, \\
\|x(t) - y(t)\| &\leq \|\xi - \eta\| e^{h(t)} \leq e^{h(T)} \|\xi - \eta\| = c\|\xi - \eta\|,
\end{aligned} \qquad (2.20)$$

where $c = e^{h(T)} < \infty$. So the solution is Lipschitz with respect to the initial data and therefore depends continuously on it. □

2.2.2 Fundamental Matrix and Transition Matrix

The formula to solve a general linear system involves a matrix known as the *fundamental matrix.*

Definition 2.1: (The Fundamental Matrix) Let $\gamma_i : [0, \infty) \to \mathbf{R}^n$ with $i = \overline{1, n}$ be the unique solution of the initial value problem

$$\begin{aligned} \dot{x} &= A(t)x, \\ x(0) &= e_i. \end{aligned} \tag{2.21}$$

Here e_i, $i = \overline{1, n}$ denote the unit vectors in $\mathbf{R}^n$. The fundamental matrix is then the matrix $X(t)$ whose columns are the solutions of (2.21), $X = [\gamma_1(t), \gamma_2(t), \cdots, \gamma_n(t)]$.
It is easy to show that the vectors $\gamma_i(t)$, $i = \overline{1, n}$, are linearly independent and that every solution of the linear homogeneous system can be written as a unique linear combination of these vectors. In other words, the set of all solutions to the linear homogeneous problem is an n-dimensional vector space.

Theorem 2.3: If $X(t)$ is the fundamental matrix for $A(t)$, then $X(t)$ satisfies

$$\begin{aligned} \dot{X}(t) &= A(t)X(t), \; t > 0 \\ X(0) &= I \; (identity\, matrix). \end{aligned} \tag{2.22}$$

Proof: By differentiating $X(t)$ we get

$$\begin{aligned} \dot{X}(t) &= [\dot{\gamma}_1(t), \dot{\gamma}_2(t), \cdots, \dot{\gamma}_n(t)] \\ &= [A(t)\gamma_1(t), A(t)\gamma_2(t), \cdots, A(t)\gamma_n(t)] \\ &= A(t)[\gamma_1(t), \gamma_2(t), \cdots, \gamma_n(t)] \\ &= A(t)X(t). \end{aligned} \tag{2.23}$$

$X(0) = I$ is clear as $\gamma_i(0) = e_i$, $i = \overline{1, n}$ $\Box$

An important property of the fundamental matrix is that it is invertible for all $t \geq 0$. This can be shown by the use of Liouville's formula, which relates the determinant of X and the trace of A. We will omit this and just verify its non-singularity by means of the uniqueness of solutions and the *transition matrix* as in [1, p. 48].

Corollary 2.4: The transition matrix corresponding to the linear homogeneous system (2.9), denoted by $\Phi(t, s)$, $t \geq s$, is given by

$$\Phi(t, s) = X(t)X^{-1}(s), \; t \geq s. \tag{2.24}$$

And the solution of the linear homogeneous system (2.9) is given by

$$x(t) = \Phi(t, s)\xi, \; t \geq s. \tag{2.25}$$

Proof: Assume $X(t)$ is invertible, then we have

$$\dot{x} = A(t)X(t)X^{-1}(s)\xi = A(t)\Phi(t,s)\xi = A(t)x(t),\ t > 0, \tag{2.26}$$

and

$$\lim_{t \downarrow s} x(t) = \xi. \tag{2.27}$$

What remains to be shown is that $X(t)$ is non-singular. For this, consider the initial value problem

$$\begin{aligned} \dot{Y}(t) &= -Y(t)A(t),\ t \geq 0, \\ Y(0) &= I. \end{aligned} \tag{2.28}$$

Again, we consider the entries of A to be locally integrable, so by theorem 2.2 the equation has a unique and absolutely continuous solution $Y(t)$. By computing the derivative

$$\begin{aligned} \frac{d}{dt}(Y(t)X(t)) &= \dot{Y}(t)X(t) + Y(t)\dot{X}(t) \\ &= -Y(t)A(t)X(t) + Y(t)A(t)X(t) \\ &= 0, \end{aligned} \tag{2.29}$$

we see that the product of $Y(t)X(t)$ must be constant and from $Y(0)X(0) = I$, it follows that $Y(t)X(t) = I$ for all $t \geq 0$. Hence we have

$$X^{-1}(t) = Y(t), \tag{2.30}$$

which shows that the fundamental matrix $X(t)$ is non-singular and that its inverse is also a unique solution of a matrix differential equation and further $t \to X^{-1}(t)$ is absolutely continuous. □

Now we can state the basic properties of the transition matrix:

$$\begin{aligned} &(P1) : \Phi(t,t) = I, \\ &(P2) : \Phi(t,\theta)\Phi(\theta,s) = \Phi(t,s),\ s \leq \theta \leq t \text{ (evolution property)}, \\ &(P3) : \partial/\partial t\, \Phi(t,s) = A(t)\Phi(t,s), \\ &(P4) : \partial/\partial s\, \Phi(t,s) = -\Phi(t,s)A(s), \\ &(P5) : (\Phi(t,s))^{-1} = \Phi(s,t). \end{aligned} \tag{2.31}$$

Property $P5$ can be derived from $P2$, by letting $t = s$ we get
$\Phi(s,\theta)\Phi(\theta,s) = \Phi(s,s) = I \Rightarrow (\Phi(\theta,s))^{-1} = \Phi(s,\theta)$.

2.2.3 Fundamental Matrix for Time Invariant Systems

The fundamental matrix $X(t)$ has a particularly nice form for the time invariant system

$$\begin{aligned} \dot{x} &= Ax, \\ x(0) &= x_0 \in \mathbf{R}^n. \end{aligned} \tag{2.32}$$

Here the fundamental matrix is of the form $X(t) = e^{tA}$, where the matrix exponential function has the meaning

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}, \tag{2.33}$$

and so the solution of (2.32) is given by

$$x(t) = e^{tA} x_0,\ t \in \mathbf{R}_0. \tag{2.34}$$

2.2.4 Solution of the Linear Controlled System

The linear controlled system is of the form

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t),\ t > s, \\ x(s) &= \xi. \end{aligned} \tag{2.35}$$

Additionally to the state vector and the system matrix, we introduce the *control* $u(t) \in \mathbf{R}^d$ representing the control policy and the *control matrix* $B(t) \in \mathbf{R}^{n \times d}$.

Theorem 2.4: Assume the elements of the matrix $A(t)$ are locally integrable and those of $B(t)$ are essentially bounded, $u(t)$ is an L^1 function. Then a continuous solution of (2.35): $\varphi(t) \in \mathbf{R}^n$, $t \geq 0$, is given by

$$\varphi(t) = \Phi(t,s)\xi + \int_s^t \Phi(t,\theta)B(\theta)u(\theta)d\theta,\ t \geq s. \tag{2.36}$$

Proof: To show that φ satisfies the initial value problem (2.35), we differentiate (2.36) and by using the properties $P1$ and $P3$ of the transition matrix defined in (2.31), we obtain

$$\begin{aligned} \dot{\varphi}(t) &= A(t)\Phi(t,s)\xi + \int_s^t A(t)\Phi(t,\theta)B(\theta)u(\theta)d\theta + \Phi(t,t)B(t)u(t) \\ &= A(t)\varphi(t) + B(t)u(t). \end{aligned} \tag{2.37}$$

By letting $t \to s$ and applying $P1$ we get $\lim_{t \downarrow s} \varphi(t) = \Phi(t,t)\xi = \xi$. □

Further it is to be said that the solution is continuously dependent on all the data such as the initial state, the control and the system matrices. More on this is explained in [1, p. 51-54].

3 Stability Theory

3.1 Definition of Stability

Consider the time invariant, homogeneous, possibly nonlinear system $\dot{x}(t) = f(x(t))$, where $x(t) \in \mathbf{R}^n$ and $f : \mathbf{R}^n \to \mathbf{R}^n$ is continuous.

By investigating the stability of this system we either analyze the behavior of the system at an equilibrium point or, for systems having periodic solutions, we look at the stability of the periodic motion. Here we shall focus only on the stability of equilibrium points, for more details regarding periodic motion, refer to [19].
Mathematically, the stability of the system at such an equilibrium point can be reduced to the question of stability at the origin or equivalently, stability of the zero state. We shall therefore work with the following system

$$\dot{x} = f(x(t)),\ x(0) = x_0,\ with\, f(0) = 0. \tag{3.1}$$

Here the intensity of the perturbation is given by $\|x_0\|$.

We define the transition or solution operator T_t, $t \geq 0$ as in (2.4).

We state here three important definitions for stability with respect to stability of the zero state or null solution as in [1]:

Definition 3.1: The null solution of system (3.1) is called *stable* or *Lyapunov stable* if for every $R > 0$, there exists an $r = r(R) \leq R$ so that $T_t(B(0,r)) \subset B(0,R)$, $\forall t \geq 0$, where $B(0,r)$ and $B(0,R)$ denote the open balls centered at the origin with radius r and R respectively.
From definition 3.1 we immediately get the definition for instability, namely if for any given $R > 0$ we cannot find an $r > 0$ such that $T_t(B(0,r)) \subset B(0,R)$, $\forall t \geq 0$.

Definition 3.2: The null solution of system (3.1) is *asymptotically stable* if it is stable and for any $x_0 \in \mathbf{R}^n$, $\lim_{t\to\infty} T_t(x_0) = 0$.

Definition 3.3: The null solution of system (3.1) is *exponentially stable* if there exists an $M > 0$, $\alpha > 0$, such that $\|T_t(x_0)\| \leq Me^{-\alpha t}$, $t \geq 0$.

Generally spoken, if a system, whose null solution is stable in Lyapunov sense, gets kicked out from its equilibrium point, it stays nearby and executes motion around it. If the null solution of the system is asymptotically stable, it eventually returns to it and in case of exponential stability, the system goes back to the equilibrium point exponentially fast.

3.2 Lyapunov Stability Theory

There are basically two ways to analyze the stability of system (3.1), namely the linearization method and the Lyapunov method. To apply the linearization method, one computes the Taylor expansion around an equilibrium to obtain the Jacobian at this point and then determines the stability by computing its eigenvalues. The linearization method is not covered in this thesis, for further reading we refer to [19]. Instead we will focus on Lyapunov theory as it will be of great importance in the future chapters.

3.2.1 Lyapunov Functions and Stability Criterion

Definition 3.4: Consider system (3.1). Let Ω be an open set in $\mathbf{R}^n$ and $0 \in \Omega$. A function $V : \Omega \subseteq \mathbf{R}^n \to \mathbf{R}$ is said to be a *Lyapunov function* if it satisfies the following properties:

$$\begin{aligned} &(L1) : V \in C^1(\Omega) \\ &(L2) : V \text{ is positive definite: } V(x) > 0 \, \forall x \in \Omega \text{ and } V(0) = 0 \\ &(L3) : \dot{V}(x(t)) = (DV, f)(x(t)) \equiv (V_x(x(t)), f(x(t)) \leq 0, \\ &\qquad \text{along any solution trajectory } x(t) \equiv T_t(x_0) \equiv x(t, x_0),\ x, x_0 \in \Omega. \end{aligned} \tag{3.2}$$

Definition 3.4 shows that $\dot{V}(x(t))$ is a function of the state. Evaluated at a certain state $\overline{x}$, $\dot{V}(\overline{x}(t))$ gives the rate of increase of V along the trajectory of the system passing through $\overline{x}$. The great benefit of Lyapunov functions is that the computation of such rate does not require the preliminary computation of the system trajectory.
Unfortunately, there is no algorithm to determine a Lyapunov function. But in case we can find one, the stability of the null solution is guaranteed.

Theorem 3.1: (Lyapunov Theorem) Consider the dynamical system (3.1) and assume there exist a function V satisfying the properties $L1 - L3$ from (3.2), then the system is stable in Lyapunov sense with respect to the null solution. Moreover if strict inequality holds for $L3$, namely $\dot{V}(x(t)) < 0$, the the system (3.1) is asymptotically stable with respect to the null solution.

Proof: The proof follows [15, 18]. First suppose $L3$ holds, so $\dot{V}(x(t))$ is negative semi-definite. Given $\epsilon > 0$ consider the closed ball $\overline{B}(0, \epsilon)$. Its boundary $S(0, \epsilon)$ is closed and bounded, hence by Heine-Borel's Theorem it is compact. As V is a continuous function, by Weierstrass's Theorem V admits a minimum m on $S(0, \epsilon)$. This minimum is positive by $L2$:

$$\min_{x:\|x\|=\epsilon} V(x(t)) = m > 0. \tag{3.3}$$

Since V is continuous, in particular at the origin, there exists a $\delta > 0$, $0 < \delta \leq \epsilon$ such that $x_0 \in B(0, \delta) \Rightarrow V(x) - V(0) = V(x) < m$. This δ is the one required by the definition of stability, meaning that if a trajectory starts from within $B(0, \delta)$ it shall not leave $B(0, \epsilon)$. So choose an $x_0 \in B(0, \delta)$ as the initial condition for (3.1) and by contradiction suppose that the system trajectory $\varphi(t, x_0)$ leaves the ball $B(0, \epsilon)$. So there exist a time T on which the trajectory would intersect the boundary of $\overline{B}(0, \epsilon)$, this means $V(\varphi(T, x_0)) \geq m$. But $\dot{V}(x(t))$ is negative semi-definite and hence V is non-increasing along this trajectory, meaning $V(\varphi(T, x_0)) \leq V(x_0)$. Together with (3.3), this leads to the contradiction $m \leq V(\varphi(T, x_0)) \leq V(x_0) < m$.

To prove asymptotic stability, assume $\dot{V}(x(t)) < 0$, $x \in \Omega$, $x \neq 0$. By the previous reasoning, this implies stability, meaning $x(0) \in B(0, \delta)$ then $x(t) \in B(0, \epsilon)$, $\forall t \geq 0$. Now $V(x(t))$ is decreasing and bounded from below by zero. By contradiction suppose

$x(t)$ does not converge to zero. This implies that $V(x(t))$ is bounded from below, so there exists an $L > 0$, such that $V(x(t)) \geq L > 0$. Hence by continuity of $V(\cdot)$, there exists a δ', such that $V(x) < L$ for $x \in B(0,\delta')$, which further implies that $x(t) \notin B(0,\delta'), \forall t \geq 0$. Next we define the set $K = \overline{B}(0,\epsilon)\backslash B(0,\delta')$. Since K is compact and $\dot{V}(\cdot)$ is continuous and negative definite, we can define $L_1 \equiv \min_{x\in K} -\dot{V}(x(t))$. $L3$ implies $-\dot{V}(x(t)) \geq L_1$, $\forall x \in K$, or equivalently

$$V(x(t)) - V(x(0)) = \int_0^t \dot{V}(x(s))ds \leq -L_1 t, \tag{3.4}$$

and so for all $x(0) \in B(0,\delta)$

$$V(x(t)) \leq V(x(0)) - L_1 t. \tag{3.5}$$

By letting $t > \frac{V(x(0))-L}{L_1}$, it follows that $V(x(t)) < L$, which is a contradiction. Hence $x(t) \to 0$ as $t \to \infty$, which establishes asymptotic stability. □

3.2.2 Lyapunov's Theorem for Linear Systems

If we apply Lyapunov's theorem to time invariant linear systems, we obtain the *Matrix Lyapunov Equation* and the following theorem:

Theorem 3.2: [15, 19] Let $A \in \mathbf{R}^{n\times n}$ be the matrix of the system $\dot{x} = Ax$. Then the following statements are equivalent:

1. The matrix A is stable, meaning that all its eigenvalues have negative real part.
2. For all matrices $Q \in M_s^+$ (symmetric positive definite) there exists a unique solution $P \in M_s^+$ to the following matrix Lyapunov equation:

$$A'P + PA = -Q \tag{3.6}$$

Proof: $(1 \Rightarrow 2)$ Suppose the matrix A is stable. Let

$$P \equiv \int_0^\infty e^{A't}Qe^{At}dt. \tag{3.7}$$

If we consider the Jordan form of A, we see that the integral exists and is finite. Now we can verify that P is a solution to (3.6):

$$\begin{aligned} A'P + PA &= \int_0^\infty (A'e^{A't}Qe^{At} + e^{A't}Qe^{At}A)dt \\ &= \int_0^\infty \frac{d}{dt}\left(e^{A't}Qe^{At}\right)dt \\ &= \left[e^{A't}Qe^{At}\right]_0^\infty \\ &= -Q. \end{aligned} \tag{3.8}$$

Here we used the fact that $\lim_{t\to\infty} e^{A't}Qe^{At} = 0$.
To show that the solution is unique, let P_1 and P_2 be any two solutions satisfying

$$\begin{aligned} A'P_1 + P_1A + Q = 0 \\ A'P_2 + P_2A + Q = 0. \end{aligned} \tag{3.9}$$

Subtract the second from the first,

$$A'(P_1 - P_2) + (P_1 - P_2)A = 0, \tag{3.10}$$

and then

$$\begin{aligned} 0 &= e^{A't}(A'(P_1 - P_2) + (P_1 - P_2)A)e^{At} \\ &= e^{A't}A'(P_1 - P_2)e^{At} + e^{A't}(P_1 - P_2)Ae^{At} \\ &= \frac{d}{dt}\left(e^{A't}(P_1 - P_2)e^{At}\right). \end{aligned} \tag{3.11}$$

This shows that $e^{A't}(P_1 - P_2)e^{At}$ is constant and therefore has the same value for all t as for $t = 0$,

$$e^{A't}(P_1 - P_2)e^{At} = P_1 - P_2. \tag{3.12}$$

By letting $t \to \infty$ on the left hand side of (3.12), we obtain $P_1 - P_2 = 0$, hence the solution is unique.

($2 \Rightarrow 1$) Consider the linear time invariant system as given in the theorem and let $V(x) = x'Px$. Then

$$\begin{aligned} \dot{V}(x) &= \dot{x}'Px + x'P\dot{x} = (Ax)'Px + x'PAx \\ &= x'(A'P + PA)x. \end{aligned} \tag{3.13}$$

By our assumption, there exists a unique positive definite matrix P for any positive definite matrix Q satisfying (3.6). This means $\dot{V} = -x'Qx < 0$ for $x \neq 0$. So asymptotic stability follows from Lyapunov's Theorem (3.1). □

3.2.3 Linearization via Lyapunov Method

Consider system (3.1) with an equilibrium point at the origin. If f is continuously differentiable, we can linearize the system around the origin and analyze the corresponding linear time invariant system $\dot{x} = Ax(t)$, where $A = \frac{\partial f}{\partial x}(0) \in \mathbf{R}^{n\times n}$.

Theorem 3.3: [15] Let (3.1) have an equilibrium point at the origin and $f \in C^1(\mathbf{R}^n)$. If $A = \frac{\partial f}{\partial x}(0)$ is stable, then (3.1) is asymptotically stable with respect to the origin.

Proof: Suppose A is stable. We compute the Taylor expansion of f around the equilibrium point:

$$f(x) = f(0) + Ax + o(x)\|x\| = Ax + o(x)\|x\|, \tag{3.14}$$

where $\lim_{\|x\|\to 0} o(x) = 0$. As we assumed A to be a stability matrix, we have a solution $P \in M_s^+$ for the equation

$$A'P + PA + I = 0. \tag{3.15}$$

Now let $V(x) = x'Px$. Here $V(x)$ is a positive quadratic form and hence we can apply Lyapunov's Theorem by using V as a Lyapunov function.

$$\begin{aligned}\dot{V}(x) &= (Ax + o(x)\|x\|)'Px + x'P(Ax + o(x)\|x\|) \\ &= x'(A'P + PA)x + 2x'Po(x)\|x\| \\ &= -x'x + 2x'Po(x)\|x\| \\ &= \|x\|^2 \left(-1 + \frac{2x'Po(x)}{\|x\|}\right).\end{aligned} \tag{3.16}$$

Now by applying Cauchy Schwartz, we obtain

$$\begin{aligned}|2x'Po(x)| &= |(x, 2Po(x))| \\ &\le \|x\|\|2Po(x)\| \\ &\le 2\|x\|\|P\|\|o(x)\|.\end{aligned} \tag{3.17}$$

(3.17) shows that when $x \to 0$, also $\frac{2x'Po(x)}{\|x\|}$ tends to zero. So there exists an $\epsilon > 0$, such that $\dot{V}(x) < 0$ for all $x \in B(0,\epsilon)\backslash\{0\}$ and $\dot{V}$ is negative definite. By Lyapunov's Theorem, it follows that for the system (3.1) the origin is asymptotically stable. □

4 Observability

In many systems we may encounter in practice, the internal state $x(t) \in \mathbf{R}^n$ is not directly accessible. Therefore we introduce the following model of a system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),\ t \in I_{s,T} \equiv [s, T], \tag{4.1}$$

$$y(t) = H(t)x(t),\ t \in I_{s,T}. \tag{4.2}$$

The newly introduced vector $y(t)$ is the *output trajectory* and takes values from $\mathbf{R}^m$. Usually, the dimension of the output is less than the dimension of the system state, i.e. $m < n$. To have compatible dimensions for the matrices, given $A \in \mathbf{R}^{n\times n}$, the matrix H must be in $\mathbf{R}^{m\times n}$.

4.1 Observability Matrix

Now the question arises, if from the given output trajectory, the behavior of the entire system can be determined. To investigate this question, we first define the *observability* of a system.

Definition 4.1: The system defined by the equations (4.1) and (4.2) is observable over the time period $I_{s,T}$, if from the input data $u(t)$, $t \in I_{s,T}$ and the output trajectory $y(t)$, $t \in I_{s,T}$, the initial state, and hence the entire state trajectory, is uniquely identifiable.

As mentioned in the definition, mathematically the problem of observability of the system is equivalent to the problem of finding the initial state x_0. Therefore, we can approach the problem directly via the initial state.
We have seen that the solution for (4.1), with the initial state $x(s) = \xi$, is given by (2.36). By inserting (4.2) into this solution, we get

$$y(t,\xi) = H(t)\Phi(t,s)\xi + \int_s^t H(t)\Phi(t,\theta)B(\theta)u(\theta)d\theta,\ t \in I_{s,T}. \tag{4.3}$$

By definition 4.1 we have $y(t)$ and $u(t)$ given. So considering an initial state $\xi \in \mathbf{R}^n$ we can define

$$\tilde{y}(t,\xi) = y(t,\xi) - \int_s^t H(t)\Phi(t,\theta)B(\theta)u(\theta)d\theta = H(t)\Phi(t,s)\xi. \tag{4.4}$$

Now the matrix $H(t)\Phi(t,s)$ is of dimension $m \times n$ and therefore has no inverse. This means that we can not find ξ directly. To further investigate the problem, we introduce the *observability matrix*:

$$Q_T(s) \equiv \int_s^T \Phi'(t,s)H'(t)H(t)\Phi(t,s)dt. \tag{4.5}$$

The observability matrix $Q_T(t)$, considered as a function of the starting time $t \in I_{s,T}$, is symmetric positive semidefinite and is the solution of the matrix differential equation

$$\begin{aligned} \dot{Q}_T(t) &= -A'(t)Q_T(t) - Q_T(t)A(t) - H'(t)H(t),\ t \in [s,T), \\ Q_T(T) &= 0. \end{aligned} \tag{4.6}$$

That $Q_T(t)$ satisfies (4.6) can be seen by taking the derivative with respect to t in $Q_T(t) = \int_t^T \Phi'(\theta,t)H'(\theta)H(\theta)\Phi(\theta,t)d\theta$ using the Leibniz rule and applying the properties of the transition operator. The terminal condition is given by the integral definition of the observability matrix.
If the observability matrix is given, we have the following theorem to determine if a given system is observable.

Theorem 4.1: The system given by the equations (4.1) and (4.2) is observable over the time period $I_{s,T}$ $\Leftrightarrow$ the observability matrix $Q_T(s)$ is positive definite.

Proof: (Observability $\Rightarrow Q_T(s) > 0$) We assume the system is observable, meaning that two distinct initial states $\xi \neq \eta$ produce two distinct outputs $\tilde{y}(t,\xi) \neq y(t,\eta)$. We get

$$\begin{aligned}
&\int_s^T \|\tilde{y}(t,\xi) - y(t,\eta)\|^2 dt > 0,\ \forall \xi \neq \eta, \\
\Rightarrow &\int_s^T \|H(t)\Phi(t,s)(\xi - \eta)\|^2 dt > 0,\ \forall \xi \neq \eta, \\
\Rightarrow &\int_s^T (\Phi'(t,s)H'(t)H(t)\Phi(t,s)(\xi - \eta), (\xi - \eta))dt > 0\, \forall \xi \neq \eta, \\
\Rightarrow &(Q_T(s)(\xi - \eta), (\xi - \eta)) > 0,\ \forall \xi \neq \eta.
\end{aligned} \tag{4.7}$$

By (4.7) the observability matrix is positive definite whenever the system is observable.

($Q_T(s) > 0 \Rightarrow$ observability) By contradiction, suppose $Q_T(s) > 0$ but the system is not observable. In this case there exist two distinct initial states producing identical outputs, i.e. $\tilde{y}(t,\xi) = y(t,\eta), t \in I_{s,T}$. By (4.7) this results in

$$(Q_T(s)(\xi - \eta), (\xi - \eta)) = 0. \tag{4.8}$$

But $Q_T(s)$ is positive definite, which implies $\xi = \eta$. □

Since $A(t)$ and $H(t)$ are time varying, the system may be observable over one period of time but not so on another. Hence $Q_T(s)$ is a function of the time intervals.

4.2 Observability Rank Condition for Time Invariant Systems

According to previous result, to determine if the observability matrix is positive definite involves integration of matrix valued functions. This is not always easy. For a time invariant system

$$\begin{aligned}
\dot{x} &= Ax + Bu, \\
y &= Hx,
\end{aligned} \tag{4.9}$$

we have the following simpler alternative:

Theorem 4.2(Kalman Observability Rank Condition) The necessary and sufficient condition for observability of the system (4.9) is that the matrix $[H', A'H', (A')^2H', \cdots, (A')^{n-1}H']$ has full rank:

$$Observability \Leftrightarrow Rank([H', A'H', (A')^2H', \cdots, (A')^{n-1}H']) = n. \tag{4.10}$$

Proof: The proof follows [1, p. 157] and uses the fact from theorem 4.1 that observability of a system is equivalent to $Q_T > 0$. So it is sufficient to show

$$Q_T > 0 \Leftrightarrow Rank([H', A'H', (A')^2H', \cdots, (A')^{n-1}H']) = n. \tag{4.11}$$

($Q_T > 0 \Rightarrow$ Rank condition holds) Assume $Q_T > 0$, we have

$$(Q_T\xi,\xi) = \int_0^T \|He^{tA}\xi\|^2 dt > 0\,\forall \xi \neq 0. \tag{4.12}$$

From (4.12) the null space or kernel of He^{tA} must be $\{0\}$. We have

$$\{He^{tA}\xi = 0\,\forall t \in I_T\} \Leftrightarrow \xi = 0. \tag{4.13}$$

This is further equivalent to the statement

$$\{(He^{tA}\xi, z) = 0, \forall t \in I_T \,and\, \forall z \in \mathbf{R}^m\} \Leftrightarrow \xi = 0, \tag{4.14}$$

or to fit our notation for the rank condition,

$$\{(\xi, e^{tA'}H'z) = 0, \forall t \in I_T \,and\, \forall z \in \mathbf{R}^m\} \Leftrightarrow \xi = 0. \tag{4.15}$$

Now we define this to be a function

$$f(t) \equiv (\xi, e^{tA'}H'z),\ t \in I_T, \tag{4.16}$$

and compute its Taylor expansion around zero

$$f(t) = \sum_{k=0}^{\infty} f^{(k)}(0)\frac{t^k}{k!}. \tag{4.17}$$

This function can be identically zero if and only if all its derivatives at $t = 0$ vanish. This together with (4.15) leads to

$$\{(\xi, (A')^k H'z) = 0, \forall k \in \mathbf{N}_0 \,and\, \forall z \in \mathbf{R}^m\} \Leftrightarrow \xi = 0. \tag{4.18}$$

As k goes to infinity, this is an infinite sequence. By the Cayley-Hamilton theorem (see [23, p. 70]), this sequence can be reduced to an equivalent finite sequence as any square matrix $A \in \mathbf{R}^n$ satisfies its characteristic equation

$$P(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_{n-1}A^{n-1} + A^n = 0. \tag{4.19}$$

As a consequence of this A^n is a linear combination of $\{A^j,\ j = 1, \cdots, n-1\}$ and hence $A^k,\ k \geq n$ as well. This means that any integer power of the matrix A equal to n and beyond can be written as a function of powers of A up to $k = n-1$. Therefore, we can write (4.18) as

$$\{(\xi, (A')^k H'z) = 0, \forall k \in \{0, 1, 2, \cdots, n-1\} \,and\, \forall z \in \mathbf{R}^m\} \Leftrightarrow \xi = 0. \tag{4.20}$$

This means that the union $\bigcup_{k=0}^{n-1}(A')^k H'z$, $\forall z \in \mathbf{R}^m$ is all of $\mathbf{R}^n$. So

$$Range([H', A'H', (A')^2 H', \cdots, (A')^{n-1}H']) = \mathbf{R}^n, \tag{4.21}$$

and hence the matrix for the rank condition must span the whole space $\mathbf{R}^n$ and therefore

$$Rank([H', A'H', (A')^2H', \cdots, (A')^{n-1}H']) = n. \tag{4.22}$$

(Rank condition holds $\Rightarrow$ $Q_T > 0$) By contradiction suppose Q_T is not positive definite and hence there exists a vector $\xi \neq 0 \in \mathbf{R}^n$ such that $(Q_T\xi, \xi) = 0$. By theorem 4.1 this implies $He^{tA}\xi \equiv 0$, $\forall t \in I_T$. By the same reasoning as in the proof of the first part, we have

$$(He^{tA}\xi, \eta) = 0 \, \forall t \in I_T, \; and \, \forall \eta \in \mathbf{R}^m. \tag{4.23}$$

Again, to fit our notation and by following the same steps as before, we get the following expression

$$\{(\xi, (A')^k H'\eta) = 0 \, \forall k \in \{0, 1, 2, \cdots, n-1\}, \forall \eta \in \mathbf{R}^m\} \tag{4.24}$$

But by our assumption, the rank condition (4.22) holds and so it follows that $\xi = 0$, which is a contradiction to our assumption. Hence $Q_T > 0$. □

5 Controllability

Given a linear system, time variant or invariant, together with its initial state, we often want to drive the system to a desired target state. Here the problem consists of finding a suitable control strategy out of a set of admissible inputs, so that the system will reach said target state within a certain time interval. It might also be very likely that the system is unstable, then the question arises if the system can be stabilized. We will see in the next chapter on *stabilizability* that if the system is controllable, we can construct a feedback control that will stabilize the system. This can be done for both cases, when the internal state is available or only the output. However, if only the output is available, we need to introduce an observer as discussed in the next chapter.

5.1 Controllability Matrix

We consider the linear time variant system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \; t \in \mathbf{R}_0 \equiv (0, \infty), \\ x(s) &= x_0. \end{aligned} \tag{5.1}$$

Again we take the state vector $x \in \mathbf{R}^n$, then $A \in \mathbf{R}^{n \times n}$. Further let $u(t) \in \mathbf{R}^d$ and so $B \in \mathbf{R}^{n \times d}$. We define the admissible control strategies $\mathcal{U}_{ad}$ to be the space of locally square integrable functions with values in $\mathbf{R}^d$, i.e. $u \in \mathcal{U}_{ad} \equiv L_2^{loc}(\mathbf{R}_0, \mathbf{R}^d)$. Regarding physical systems, this definition for the admissible controls makes sense as the energy of the control is usually limited. For many systems not only the energy, but also the magnitudes of $u(t)$ are limited, therefore the control set is a closed and bounded, hence compact, subset of $\mathbf{R}^d$.
We now give the definition of controllability as in [1, p. 182]

Definition 5.1: The System (5.1) together with the admissible controls $\mathcal{U}_{ad}$ is said to be *controllable* at time T with respect to the pairs $\{x_0, x_1\}$, both in $\mathbf{R}^n$, if there exists a control strategy $u \in \mathcal{U}_{ad}$ over the time interval $I_{s,T} \equiv [s, T]$, so that the state attained at time T coincides with the given target state x_1. The system is said to be *globally controllable* over the period $I_{s,T}$, if controllability holds for any $x_0 \in \mathbf{R}^n$ to any $x_1 \in \mathbf{R}^n$.

Above definition describes controllability as the capability of the system to reach out to target states with the help of the admissible controls. Therefore it makes sense to describe these target states as attainable sets.

Definition 5.2: For a time interval $I_{s,T}$ the *attainable set* at time T is given by

$$\mathcal{A}_s(T) \equiv \left\{ z \in \mathbf{R}^n \;:\; z = \int_s^T \Phi(T,\theta)B(\theta)u(\theta)d\theta,\; u \in \mathcal{U}_{ad} \right\}. \tag{5.2}$$

As the initial state x_0 and the final state x_1 are arbitrary, we have $z(t) = x_1 - \Phi(t,s)x_0$ and therefore the definition of the attainable set in (5.2) does not include the initial state.

Similar to previous chapter on observability, to further investigate the controllability, we introduce the *controllability matrix*

$$C_s(t) \equiv \int_s^t \Phi(t,\theta)B(\theta)B'(\theta)\Phi'(t,\theta)d\theta,\; s \le t. \tag{5.3}$$

And again, as seen for observability, by differentiating equation (5.3), we get the following matrix differential equation from which the controllability matrix $C_s(t)$ can be obtained

$$\begin{aligned} &\dot{C}_s(t) = A(t)C_s(t) + C_s(t)A'(t) + B(t)B'(t),\; t \in (s,T], \\ &C_s(s) = 0. \end{aligned} \tag{5.4}$$

Now we state the theorem concerning global controllability.
Theorem 5.1: The system (5.1) is globally controllable over the time horizon $I_{s,T} \equiv [s,T] \;\Leftrightarrow\; C_s(T) > 0$.

Proof: The proof follows [1, p. 194].
($C_s(T) > 0 \;\Rightarrow$ global controllability) To show this, we actually construct a control that does the job. Given the initial state $x_0 \in \mathbf{R}^n$ at time s and the target state $x_1 \in \mathbf{R}^n$ at time T, define

$$z \equiv x_1 - \Phi(T,s)x_0. \tag{5.5}$$

We choose our control to be

$$u^*(t) \equiv B'(t)\Phi'(T,t)C_s^{-1}(T)z. \tag{5.6}$$

As by assumption $C_s(T) > 0$ and therefore nonsingular, u^* is well defined and as the elements of B' are essentially bounded we have $u^* \in L_\infty(I_{s,T}, \mathbf{R}^d) \subset L_2(I_{s,T}, \mathbf{R}^d)$. Now we substitute (5.6) in the expression

$$x(T) = \Phi(T,s)x_0 + \int_s^T \Phi(T,\tau)B(\tau)u(\tau)d\tau. \tag{5.7}$$

So we get

$$\begin{aligned} x(T) &= \Phi(T,s)x_0 + \int_s^T \Phi(T,\tau)B(\tau)B'(\tau)\Phi'(T,\tau)C_s^{-1}(T)z d\tau \\ &= \Phi(T,s)x_0 + \left[\int_s^T \Phi(T,\tau)B(\tau)B'(\tau)\Phi'(T,\tau)d\tau\right] C_s^{-1}(T)z \\ &= \Phi(T,s)x_0 + C_s(T)C_s^{-1}(T)z \\ &= \Phi(T,s)x_0 + [x_1 - \Phi(T,s)x_0] \\ &= x_1. \end{aligned} \tag{5.8}$$

So the control defined in (5.6) takes the system to the target state.

(Global controllability $\Rightarrow$ $C_s(T) > 0$) The vector z defined in (5.5) ranges the whole space $\mathbf{R}^n$. Hence for global controllability over the time interval $I_{s,T}$, also the attainable set (5.2) at time T must be the whole space, i.e. $\mathcal{A}_s(T) = \mathbf{R}^n$.
By our assumption global controllability holds, therefore

$$\{(a,\xi) = 0,\ \forall a \in \mathcal{A}_s(T)\} \ \Rightarrow \xi = 0. \tag{5.9}$$

Using the expression for the attainable set we have

$$\left\{\left(\int_s^T \Phi(T,\tau)B(\tau)u(\tau)d\tau, \xi\right) = 0, \forall u \in \mathcal{U}_{ad}\right\} \ \Rightarrow \xi = 0, \tag{5.10}$$

and by using the properties of the adjoint, we can write

$$\left\{\int_s^T (u(\tau), B'(\tau)\Phi'(T,\tau)\xi)d\tau = 0, \forall u \in \mathcal{U}_{ad}\right\} \Rightarrow \xi = 0. \tag{5.11}$$

This is only possible if

$$B'(t)\Phi'(T,t)\xi = 0,\ a.e.\, t \in I_{s,T}. \tag{5.12}$$

By (5.12) the expression given in (5.11) is equivalent to writing

$$\left\{\int_s^T \|B'(\tau)\Phi'(T,\tau)\xi)\|^2 d\tau = 0, \forall u \in \mathcal{U}_{ad}\right\} \Rightarrow \xi = 0. \tag{5.13}$$

From this it follows that by the definition of the controllability matrix (5.3) we have

$$\{(C_s(T)\xi, \xi) = 0\} \ \Rightarrow \ \xi = 0, \tag{5.14}$$

which shows that $C_s(T) > 0$. $\square$

5.2 Controllability Rank Condition for Time Invariant Systems

For a time invariant system,

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Hx, \end{aligned} \tag{5.15}$$

the global controllability can be determined using a rank condition like for the observability problem.

Theorem 5.2: (Kalman Controllability Rank Condition) The time invariant system (5.15) is globally controllable $\Leftrightarrow$ the following rank condition holds:

$$Rank(A \mid B) \equiv Rank([B, AB, A^2B, \cdots, A^{n-1}B]) = n. \tag{5.16}$$

Proof: Theorem 5.1 shows us that a system is globally controllable if and only if the controllability matrix is positive definite. Therefore, for any $T \in \mathbf{R}_0$, it is sufficient to prove that

$$C_T > 0 \;\Leftrightarrow Rank(A \mid B) = n. \tag{5.17}$$

($C_T > 0 \Rightarrow Rank(A \mid B) = n$) From the proof of theorem 5.1 and by the definition of the controllability matrix it is clear that for time invariant systems, positivity of C_T implies

$$Ker\{B'e^{(T-t)A'}, t \in I_T\} = \{0\}. \tag{5.18}$$

This is equivalent to the statement

$$\{(B'e^{(T-t)A'}\xi, v) = 0, \forall t \in I_T, \forall v \in \mathbf{R}^d\} \Leftrightarrow \xi = 0, \tag{5.19}$$

which is again equivalent to

$$\{(\xi, e^{(T-t)A}Bv) = 0, \forall t \in I_T, \forall v \in \mathbf{R}^d\} \Leftrightarrow \xi = 0. \tag{5.20}$$

Now we follow the same steps as we did in the proof of theorem 4.2 on the observability rank condition. We define (5.20) to be the function $f(t) = (\xi, e^{(T-t)A}Bv)$ and compute its Taylor expansion, this time around T. So we will obtain

$$\{(\xi, A^kBv) = 0, \forall k \in N_0 \; and \; \forall v \in \mathbf{R}^d\} \Leftrightarrow \xi = 0. \tag{5.21}$$

By applying the Cayley Hamilton Theorem, we can see that in (5.21) we only need the powers of A up to $k = n-1$.
Therefore we have that the union $\bigcup_{k=0}^{n-1} A^kBv, \forall v \in \mathbf{R}^d$ is all of $\mathbf{R}^n$. So the matrix for the rank condition must span the whole space $\mathbf{R}^n$ and hence

$$Rank([B, AB, A^2B, \cdots, A^{n-1}B]) = n. \tag{5.22}$$

($Rank(A \mid B) = n \Rightarrow C_T > 0$) As we assume the rank condition holds, this implies

$$Ker\{B'e^{(T-t)A'}, t \in I_T\} = \{0\} \tag{5.23}$$

holds. Hence

$$\int_0^T \|B'e^{(T-t)A'}\xi\|^2 dt = 0 \tag{5.24}$$

implies that $\xi = 0$, which gives $C_T > 0$. $\square$

5.3 Duality of Controllability and Observability

Observability and Controllability appear to be very similar properties of a linear system. Indeed, these two concepts are dual problems. To formalize this statement, note that given the system as described in (4.1) and (4.2), one can define the corresponding dual system by

$$\begin{aligned} \dot{\bar{x}} &= -A'(t)\bar{x}(t) + H'(t)\bar{u}(t),\ t \in I_{s,T} \\ \bar{y} &= B'(t)\bar{x}(t),\ t \in I_{s,T}. \end{aligned} \tag{5.25}$$

The duality Theorem states the relation between the two systems.

Theorem 5.3: (Duality Theorem) Let $\Phi(t,s)$ be the transition matrix of the system described by (4.1) and (4.2) and $\Psi(t,s)$ be the transition matrix of system (5.25). We have

1. $\Psi(t,s) = \Phi'(s,t)$.
2. The system (4.1) and (4.2) is controllable (observable) on $I_{s,T}$ $\Leftrightarrow$ the system (5.25) is observable (controllable) on $I_{s,T}$.

Proof:

1. By (2.31) (P3) we have

$$\dot{\Psi}(t,s) = -A'(t)\Psi(t,s). \tag{5.26}$$

Multiplying by $\Phi'(t,s)$ and using the property $\dot{\Phi}(t,s) = A(t)\Phi(t,s)$ we obtain

$$\begin{aligned} \Phi'(t,s)\dot{\Psi}(t,s) + \Phi'(t,s)A'(t)\Psi(t,s) &= 0, \\ \Phi'(t,s)\dot{\Psi}(t,s) + \dot{\Phi}'(t,s)\Psi(t,s) &= 0, \end{aligned} \tag{5.27}$$

i.e.

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\Phi'(t,s)\Psi(t,s)\right] = 0, \\ &\Rightarrow \Phi'(t,s)\Psi(t,s) = const. \end{aligned} \tag{5.28}$$

By (2.31) (P1) $\Phi'(s,s) = \Psi(s,s) = I$ at $t = s$ we have

$$\begin{aligned} &\Phi'(t,s)\Psi(t,s) = I, \\ &\Rightarrow \Psi(t,s) = \left[\Phi'(t,s)\right]^{-1} = \Phi'(s,t). \end{aligned} \tag{5.29}$$

In the last step we used property $P5$ of (2.31). This shows that the transition matrices of the two systems are duals of each other.

2. By comparing the definition of the controllability matrix

$$C_s(T) \equiv \int_s^T \Phi(T,\tau)B(\tau)B'(\tau)\Phi'(T,\tau)d\tau, \tag{5.30}$$

and the observability matrix

$$Q_T(s) \equiv \int_s^T \Phi'(\tau, s)H'(\tau)H(\tau)\Phi(\tau, s)d\tau, \tag{5.31}$$

we can see that the $C_s(T)$ is identical to $Q_T(s)$ associated with the pair $(-A'(t), B'(t))$ and conversely, $Q_T(s)$ is identical to $C_s(T)$ associated with the pair $(-A'(t), H'(t))$. $\square$

For time invariant linear systems, duality can be derived from the rank conditions. **Corollary 5.4:** If for a linear time invariant system the pair (A, B) is controllable, then the pair (A', B') is observable. Similarly if the pair (A, H) is observable, then the pair (A', H') is controllable.

Proof: We only prove the first statement, as the proof for the second statement is identical by applying the corresponding rank condition. Given the pair (A', B') is observable, we have $Rank(A' \mid B') = n$. Since

$$\begin{aligned} Rank(A' \mid B') &= Rank([(B')', (A')'(B')', \cdots, ((A')')^{n-1}(B')]) \\ &= Rank(A \mid B) = n, \end{aligned} \tag{5.32}$$

it follows that (A, B) is controllable. $\square$

6 Stabilizability

In the previous chapters we have seen that we can determine if a system is observable or controllable. In this chapter we will see that given these two properties, we can go further and design controllers that make the system behave in some desired way. In the proof of theorem 5.1 we already constructed a control policy that steers the system to a desired state. This is called an *open loop control*. However, this may not be sufficient in practice due to deviations in the initial states or the matrices involved. Especially for unstable systems, deviations can get arbitrarily large and open loop control may fail. The more robust way is to measure the state (or the output trajectory in case the state is not accessible) and as soon as the system starts to deviate from the desired trajectory, the input gets adapted accordingly. This method is called *feedback control*.
In this chapter we will see that if a system is controllable, then it is stabilizable by a state feedback control. If the states of the system are not accessible, but it is stabilizable and observable, we can construct a state estimator or observer, which will then be used to stabilize the original system.
During this chapter we will restrict ourselves to the linear time invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Hx. \end{aligned} \tag{6.1}$$

6.1 Control by State Feedback

To show that controllability of the system (6.1) implies stabilizability, we will first give a direct proof from [1, p. 208] using the Lyapunov theory for time invariant linear systems. After that a short description of the method of *Eigenvalue placement* or *Pole placement* is given.

Theorem 6.1: (State Feedback) If the system given in (6.1) is controllable, then it is stabilizable by a linear state feedback, where the feedback operator K and the corresponding feedback control are given by

$$K = -B'\tilde{C}_T^{-1}, \tag{6.2}$$

$$u = Kx, \tag{6.3}$$

where

$$\tilde{C}_T \equiv \int_0^T e^{-sA}BB'e^{-sA'}ds. \tag{6.4}$$

Proof: The proof shows that $(A + BK)$ is stable, i.e. all its eigenvalues have negative real parts. As we assume our system to be controllable, $\tilde{C}_T$ is invertible for $T > 0$ and hence the matrix K is well defined.
Consider the vector

$$\int_0^T \frac{d}{ds}\left(e^{-sA}BB'e^{-sA'}\right) ds, \tag{6.5}$$

and differentiate it once under the integral sign and once by using integration by parts. We obtain

$$\begin{aligned}\int_0^T -Ae^{-sA}BB'e^{-sA'} - e^{-sA}BB'e^{-sA'}A'ds &= e^{-sA}BB'e^{-sA'}\Big|_0^T \\ &= e^{-TA}BB'e^{-TA'} - BB',\end{aligned} \tag{6.6}$$

and so

$$A\tilde{C}_T + \tilde{C}_TA' = BB' - e^{-TA}BB'e^{-TA'}. \tag{6.7}$$

Introducing the feedback in the left hand side of above equation (6.7) and using (6.2) we obtain

$$\begin{aligned}(A + BK)\tilde{C}_T + \tilde{C}_T(A + BK)' &= (A\tilde{C}_T + \tilde{C}_TA') + BK\tilde{C}_T + \tilde{C}_TK'B' \\ &= (A\tilde{C}_T + \tilde{C}_TA') - 2BB' \\ &= -(BB' + e^{-TA}BB'e^{-TA'}).\end{aligned} \tag{6.8}$$

Defining

$$\Gamma \equiv (BB' + e^{-TA}BB'e^{-TA'}) \tag{6.9}$$

and $\tilde{A} \equiv (A+BK)$, we can rearrange equation (6.8) to a Lyapunov equation as described in chapter 3.2.2

$$\tilde{A}\tilde{C}_T + \tilde{C}_T\tilde{A}' = -\Gamma. \tag{6.10}$$

By our assumption of global controllability the second term of Γ is positive definite and so is the first. Hence we have $\Gamma \in M_s^+$, a symmetric positive definite square matrix. Now by referring to theorem 3.2, the matrix $\tilde{A}' = (A + BK)'$ is stable and so its transpose is stable as well. So by using K as our feedback operator, we obtain a stabilizing feedback control. □

6.1.1 Eigenvalue Placement by State Feedback

As mentioned, we will give here a short description on the method of Eigenvalue placement. For further reading please refer to [9] or [13].
In general, the eigenvalue placement method follows the principle that if the system is controllable, we can assign the eigenvalues of $(A + BK)$ by choice of the feedback operator K. This basically means that it is possible for the closed loop system to have any desired characteristic polynomial.

Theorem 6.2: (Eigenvalue placement) Let (A, B) be a controllable pair of matrices, i.e. $Rank(A \mid B) = n$. Then, given any n-th order monic real polynomial $p(\lambda) = \lambda^n + r_{n-1}\lambda^{n-1} + \ldots + r_1\lambda + r_0$, there is a real matrix K such that $A + BK$ has $p(\lambda)$ as its characteristic polynomial.

Here we shall only give an outline of the proof for a single input system, meaning $u(t) \in \mathbf{R}$ and hence $B \in \mathbf{R}^n$ is a vector b.
Before we start with the proof, we introduce the so called *controllable canonical form* for single input systems. The controllable canonical form is a special form of system matrices for which controllability always holds. It is given by:

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{6.11}$$

Its controllability matrix $(A \mid b)$ has rank n and the characteristic polynomial of A is given by

$$det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0. \tag{6.12}$$

Proof: Consider the single input system $\dot{x} = Ax + bu$ with the control law of the form $u = k'x$. The closed loop system is then given by $\dot{x} = (A + bk')x$. Our task is to find the vector $k' \in \mathbf{R}^n$.
By assumption the pair (A, b) is controllable. Given these properties one can show that there exists a coordinate transformation T (nonsingular) such that $(T^{-1}AT, T^{-1}b)$ is in controllable canonical form [13, p. 134]. We denote the transformed matrices as (A_c, b_c). Further the eigenvalues of $A + bk'$ are invariant under this transformation, i.e. $det(\lambda I - A) = det(\lambda - T^{-1}AT)$. So the coordinate transformation does not change the

system itself, only its representation and therefore fundamental properties such as stability, observability and controllability remain.

Let $z(t) = T^{-1}x(t)$, then

$$\dot{z} = T^{-1}ATz + T^{-1}bu. \tag{6.13}$$

Now suppose that we have solved the eigenvalue assignment problem for (6.13), so for a given monic polynomial $p(\lambda)$, there exists a vector k'_1 such that

$$det(\lambda I - T^{-1}AT - T^{-1}bk'_1) = p(\lambda). \tag{6.14}$$

Then since

$$det(\lambda I - T^{-1}AT - T^{-1}bk'_1) = det(\lambda I - A - bk'_1T^{-1}) \tag{6.15}$$

we find that the eigenvalue assignment problem can be solved by taking $k' = k'_1T^{-1}$.

We can consider our system (A_c, b_c) to be in the controllable canonical form exactly as in (6.11) and select k'_1 to be $k'_1 = [k_0, k_1, \ldots, k_n]$. Then we get

$$A_c + b_ck'_1 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ k_0 - a_0 & k_1 - a_1 & k_2 - a_2 & \ldots & k_{n-1} - a_{n-1} \end{bmatrix} \tag{6.16}$$

and hence for the eigenvalues we get

$$det(\lambda I - A_c - b_ck'_1) = \lambda^n + (a_{n-1} - k_{n-1})\lambda^{n-1} + \ldots + (a_1 - k_1)\lambda + (a_0 - k_0). \tag{6.17}$$

This can be made into any real monic polynomial of degree n, $p(\lambda) = \lambda^n + r_{n-1}\lambda^{n-1} + \ldots + r_1\lambda + r_0$ by simply choosing

$$k_1 = \begin{bmatrix} a_0 - r_0 \\ a_1 - r_1 \\ \vdots \\ a_{n-1} - r_{n-1} \end{bmatrix} \tag{6.18}$$

Hence we have found a way to determine k'. □

The eigenvalue placement problem for multi-variable systems is more involved, but its solution can be constructed from the single-valued case as described in [13, p. 143].

6.2 Control by Output Feedback

In reality one often has only access to the output of the system and the system state is hidden. The problem then becomes to design a feedback function that uses the measurements of the output to decide what input to apply to the system.

From the study of the state feedback, we have seen that we can construct a feedback operator to stabilize the system. Unfortunately we cannot construct such an operator directly for output feedback. One idea is now to calculate an estimate of the state using the accessible input and output of the system. So instead of using the true value of the state, we apply its estimate to a state feedback controller which gives us then the input to the system, see Figure 2 for an illustrative diagram.
One way to approach this problem is again by the eigenvalue placement method as illustrated in [9] or [13]. Here we will follow the theorems as in [1].

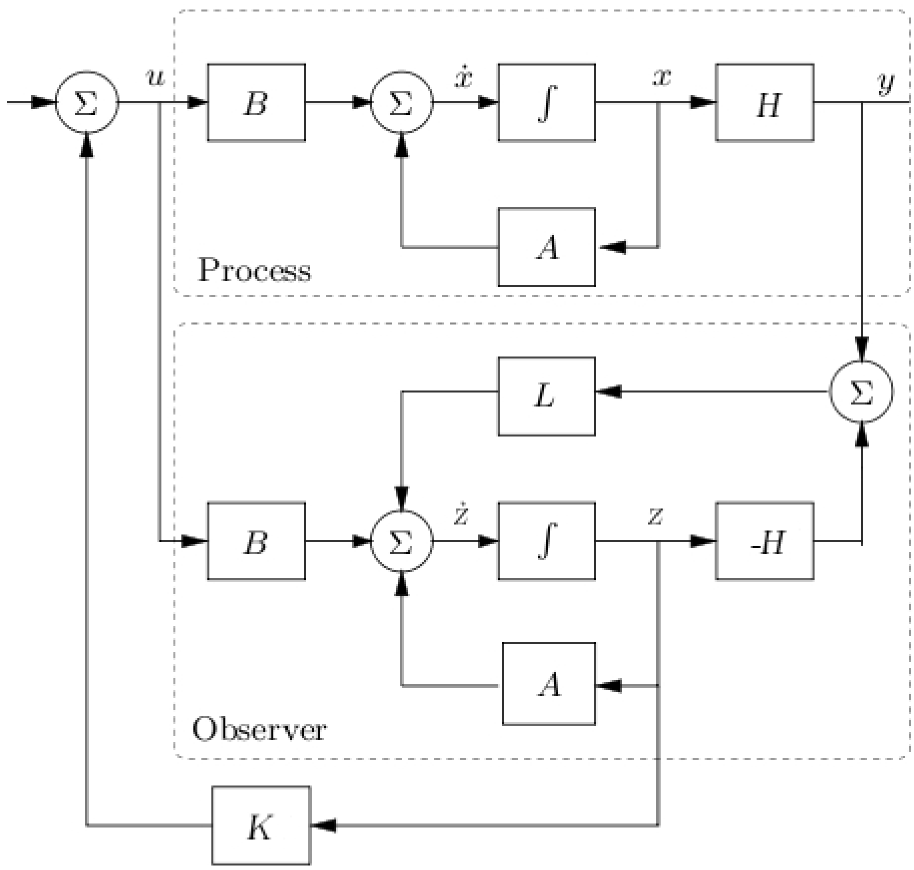

Figure 2: Block Diagram of Feedback Control with Observer

Consider the system (6.1). For what we call the *estimator problem* we need to produce an estimate $\tilde{x}$ of x from the observation of y, which has at least the asymptotic tracking property. This means that we introduce an estimator of the form

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + Bu + Ly, \tag{6.19}$$

where the matrices $\tilde{A}$ and L are chosen the way that

$$\lim_{t\to\infty} \|\tilde{x}(t) - x(t)\| = 0. \tag{6.20}$$

The solution to the estimator problem is stated in the following Theorem.
Theorem 6.3: If the system is observable, i.e. $Rank(A \mid H) = n$, the estimator problem has a solution. The estimator is given by

$$\dot{\tilde{x}} = (A - LH)\tilde{x} + Bu + Ly,\ L = \mathcal{C}_T^{-1}H', \tag{6.21}$$

where

$$\mathcal{C}_T \equiv \int_0^T e^{-tA'}H'He^{-tA}dt \tag{6.22}$$

is the observability matrix.

Proof: Define the error vector $e \equiv x - \tilde{x}$. By inserting the corresponding terms this leads to the differential equation

$$\dot{e} = (A - LH)e + (A - LH - \tilde{A})\tilde{x}. \tag{6.23}$$

Let $\tilde{A} = (A - LH)$, then above equation reduces to $\dot{e} = (A - LH)e$ and so we already obtain the desired form for the estimator as given in (6.21). If the system matrix $(A - LH)$ is asymptotically stable then the error vector tends to zero.
As we assume (A, H) to be observable, it follows by Corollary 5.4 that (A', H') is controllable. Hence the system $\dot{\xi} = A'\xi + H'v$ is globally controllable with the controllability matrix given by (6.22), which is positive definite. It follows from Theorem 6.1 that the control for this system is given by

$$v \equiv -H\mathcal{C}_T^{-1}\xi. \tag{6.24}$$

This control law stabilizes the system

$$\dot{\xi} = A'\xi + H'(-H\mathcal{C}_T^{-1})\xi = (A' - H'H\mathcal{C}_T^{-1})\xi. \tag{6.25}$$

From this it follows that both, the matrix $(A' - H'H\mathcal{C}_T^{-1})$ and its adjoint $(A - \mathcal{C}_T^{-1}H'H)$ are stability matrices. Now by choosing $L \equiv \mathcal{C}_T^{-1}H'$, the differential equation for the error vector can be written as

$$\dot{e} = (A - \mathcal{C}_T^{-1}H'H)e, \tag{6.26}$$

which is asymptotically stable with respect to the origin and so the error vector tends to zero which shows that (6.20) holds. Hence the estimator can be constructed as described in the Theorem. □

For the problem on how to stabilize the system by output feedback, consider the system

$$\dot{x} = Ax + Bu,\ y = Hx,\ u = Ky, \tag{6.27}$$

where the matrices $\{A, B, H, K\}$ have compatible dimension. From this we obtain the output feedback system of the form

$$\dot{x} = (A + BKH)x. \tag{6.28}$$

As mentioned already, it is usually not possible to find a matrix K that stabilizes the system directly, but if the pair (A, B) is controllable and the pair (A, H) is observable, we can construct a stabilizing compensator as stated in the next Theorem. For this we start by constructing a parallel system (observer) similar to the original system but with accessible states, which carry the same information as the true state x. This system will be of the form

$$\begin{aligned} \dot{x} &= Ax + Bu = Ax + BKz, \\ \dot{z} &= \tilde{A}z + Ly = \tilde{A}z + LHx, \end{aligned} \tag{6.29}$$

The matrices $\{\tilde{A}, L\}$ have to be chosen the way that the $2n$-dimensional compensated system

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A & BK \\ LH & \tilde{A} \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} \tag{6.30}$$

is asymptotically stable.

Theorem 6.4: (Output Feedback) If the original system (6.27) is controllable and observable, then there exists a stabilizing compensator (observer) given by

$$\begin{aligned} \dot{x} &= Ax + BKz, \\ \dot{z} &= (A + BK - LH)z + LHx, \end{aligned} \tag{6.31}$$

where K is given by the expression (6.2) and L is given by (6.21) from the previous theorem.

Proof: Consider the $2n$-dimensional system (6.30). We introduce the nonsingular matrix of dimension $2n \times 2n$

$$T \equiv \begin{bmatrix} I_n & 0 \\ I_n & -I_n \end{bmatrix}. \tag{6.32}$$

T is its own inverse as $T^2 = I_{2n}$. We continue by using T for the coordinate transformation

$$\begin{bmatrix} x \\ z \end{bmatrix} = T\begin{bmatrix} \xi \\ \eta \end{bmatrix}. \tag{6.33}$$

Under this transformation the system (6.30) becomes

$$\frac{d}{dt}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} A + BK & -BK \\ (A + BK - LH) - \tilde{A} & \tilde{A} - BK \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}. \tag{6.34}$$

By choosing $\tilde{A} = A + BK - LH$, the system matrix of (6.34) reduces to

$$\mathcal{A} = \begin{bmatrix} A + BK & -BK \\ 0 & A - LH \end{bmatrix}. \tag{6.35}$$

Since the matrix $\mathcal{A}$ is block diagonal, we find that the characteristic polynomial of the closed loop system is

$$det(\lambda I - \mathcal{A}) = det(\lambda I - A - BK)det(\lambda I - A + LH). \tag{6.36}$$

It follows that the set of eigenvalues of the closed loop system is the union of those of $A + BK$ and those of $A - LH$. So, if both $A + BK$ and $A - LH$ are asymptotically stable, the closed loop system is asymptotically stable as well. Now if the pair (A, B) is controllable, then, as seen in theorem 6.1, we can chose a matrix $K \in M_{d\times n}$ so that $A + BK$ is stable. Similarly, if the pair (A, H) is observable, then by duality (A', H') is controllable, i.e. the system $\dot{\xi} = A'\xi + H'v$ is globally controllable. Hence there exists a matrix $\tilde{K} \in M_{m\times n}$ such that $A' + H'\tilde{K}$ is stable and so is its transpose $A + \tilde{K}'H$. Thus one can chose the matrix $L = -\tilde{K}'$, which makes the compensated system stable and consequently the original system is stable too. □

7 Linear Quadratic Control

So far we have seen that if the system satisfies certain properties, it is possible to construct an open loop or feedback control to stabilize the system or force it to a certain state. Now we shall go further and try to obtain an optimal control. This means that given a system, we will express the performance of the controlled system in terms of a cost functional. The problem will then be to find an optimal controller that minimizes this cost functional. We will approach this problem by the method of *dynamic programming*, which was elaborated by Richard E. Bellman in the sixties. Bellman's central idea was to obtain the optimal control through a value function which satisfies the Hamilton-Jacobi equation. At around the same time in the Soviet Union, Lev S. Pontryagin formulated the *maximum principle* based on variational or multiplier methods (see [1, 10, 22]). These approaches are rather different and have their advantages and disadvantages; while dynamic programming provides a cosed loop control, the maximum principle is the more general approach.

The starting point here is the general optimal control problem. For the dynamic programming approach, we use *Bellman's Optimality Principle* to obtain the *Hamilton-Jacobi-Bellman* (HJB) equation. Applying this equation to a linear system and a quadratic cost functional, we obtain a solution to the finite-time *Linear Quadratic Regulator Problem (LQR)* in form of the state feedback solution by solving the *Matrix Riccati Differential Equation.*

Linear Quadratic Control is a largely studied subject. Its popularity is not only because it provides a full theory for time invariant and time variant linear systems, but also because it is a powerful design tool. Software algorithms to compute the corresponding matrices and solutions to the LQR problem can be applied to a given system.

In this chapter we will follow the lines in [8] and [6].

7.1 An Optimal Control Problem

For the general optimal control problem consider a nonlinear time varying system given by the equation

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)),\ t \in I_{s,T} \equiv [s, T], \\ x(s) &= x_0, \end{aligned} \tag{7.1}$$

where s, T, x_0 fixed are given. The problem is to determine the control function $u(t), t \in I_{s,T}$ that minimizes the cost functional given by

$$I(s, x_0, u(t)) = \int_s^T \phi(\tau, x(\tau), u(\tau)) d\tau + m(T, x(T)). \tag{7.2}$$

As usual, $x(t) \in \mathbf{R}^n$ represents the state vector, $u(t) \in \mathbf{R}^d$ is a square integrable function so that (7.1) has a unique solution. The scalar functions ϕ and m are continuous in their respective arguments and generally non-negative. The term $m(\cdot)$ represents a terminal penalty term and $\phi(\cdot)$ is the running cost function.

7.1.1 The Optimality Principle

Following Bellman, we introduce the value function defined as

$$V(t, x(t)) \equiv \inf_{u[t,T]} \int_t^T \phi(\tau, x, u) d\tau + m(T, x(T)), \tag{7.3}$$

for arbitrary $t \in I_{s,T}$ and arbitrary starting state $x(t)$. Note that in (7.3) we did not write x and u dependent on t as the problem of minimizing (7.2) can be found for any $x(t), t$.

We now introduce the *optimality principle*. The basic idea for the optimality principle is that the system can be characterized at time t by its state $x(t)$. This means that the current state completely summarizes all the effects of the input $u(\tau)$ prior to time t.

Theorem 7.1: (Optimality Principle) If $u^*(\tau)$ is optimal over the interval $[t, T]$, starting at state $x(t)$, then $u^*(\tau)$ is necessarily optimal over the subinterval $[t + \Delta t, T]$ for any Δt such that $T - t \geq \Delta t > 0$.

Proof: By contradiction assume that there exists a u^{**} that leads to a smaller value of

$$\int_{t+\Delta t}^T \phi(\tau, x, u) d\tau + m(T, x(T)), \tag{7.4}$$

than u^* over the subinterval $[t + \Delta t, T]$. So we define a new control $u(\tau)$ given by

$$u(\tau) = \begin{cases} u^*(\tau) & \text{for } t \leq \tau \leq t + \Delta t, \\ u^{**}(\tau) & \text{for } t + \Delta t \leq \tau \leq T. \end{cases} \tag{7.5}$$

Then over the interval $[t, T]$ we have

$$\begin{aligned}&\int_t^{t+\Delta t} \phi(\tau, x^*, u^*)d\tau + \int_{t+\Delta t}^T \phi(\tau, x^{**}, u^{**})d\tau + m(T, x^{**}(T)) \\ &< \int_t^{t+\Delta t} \phi(\tau, x^*, u^*)d\tau + \int_{t+\Delta t}^T \phi(\tau, x^*, u^*)d\tau + m(T, x^*(T)),\end{aligned} \tag{7.6}$$

which is a contradiction as we assumed u^* to be optimal over the interval $[t, T]$. □

Basically the optimality principle shows that the optimal control $u[t, T]$ can be found by concatenating the optimal $u[t, t_1]$ to the optimal $u[t_1, T]$ for $t_1 \in [t, T]$ as follows

$$\begin{aligned}V(t, x(t)) &= \inf_{u[t,T]} \left[\int_t^{t_1} \phi(\tau, x, u)d\tau + \int_{t_1}^T \phi(\tau, x, u)d\tau + m(T, x(T)) \right] \\ &= \inf_{u[t,t_1)\cup u[t_1,T]} \left[\int_t^{t_1} \phi(\tau, x, u)d\tau + \int_{t_1}^T \phi(\tau, x, u)d\tau + m(T, x(T)) \right] \\ &= \inf_{u[t,t_1)} \left[\int_t^{t_1} \phi(\tau, x, u)d\tau + \inf_{u[t_1,T]} \int_{t_1}^T \phi(\tau, x, u)d\tau + m(T, x(T)) \right].\end{aligned} \tag{7.7}$$

And so we can write

$$V(t, x(t)) = \inf_{u[t,t_1)} \left[\int_t^{t_1} \phi(\tau, x, u)d\tau + V(t_1, x(t_1)) \right]. \tag{7.8}$$

7.1.2 The Hamilton-Jacobi-Bellman Equation

We are now going to derive the Hamilton-Jacobi-Bellman Equation (HJB) for the general optimal control problem. The HJB is a differential form of (7.8) which we obtain by letting $t_1 \to t$. We assume that the equation is continuously differentiable in x and t as we will compute its Taylor expansion at one point in the proof. Write $t_1 = [t + \Delta t]$ and so we get for (7.8)

$$V(t, x(t)) = \inf_{u[t,t+\Delta t]} \left[\int_t^{t+\Delta t} \phi(\tau, x, u)d\tau + V(t + \Delta t, x(t + \Delta t)) \right]. \tag{7.9}$$

Now we apply the Mean Value Theorem to the integral in (7.9). As mentioned, we assume that we can compute the Taylor expansion of $V(t + \Delta t, x(t + \Delta t))$ (the partial derivatives exist and are bounded) around $V(t, x(t))$. So we get for some $\alpha \in [0, 1]$

$$
\begin{aligned}
V(t, x(t)) &= \inf_{u[t,t+\Delta t]} \Big[\phi(t + \alpha\Delta t, x(t + \alpha\Delta t), u(t + \alpha\Delta t))\Delta t + \\
&\quad + V(t, x(t)) + \frac{\partial V'}{\partial x}\dot{x}\Delta t + \frac{\partial V}{\partial t}\Delta t + o(\Delta t)^2\Big] \\
&= V(t, x(t)) + \frac{\partial V}{\partial t}\Delta t + \\
&\quad + \inf_{u[t,t+\Delta t]} \Big[\phi(t + \alpha\Delta t, x(t + \alpha\Delta t), u(t + \alpha\Delta t))\Delta t + \\
&\quad + \frac{\partial V'}{\partial x} f(t, x, u)\Delta t + o(\Delta t)^2\Big],
\end{aligned} \tag{7.10}
$$

where $o(\Delta t)^2$ represents the second and higher order terms of Δt. Now from the above equation follows

$$
\begin{aligned}
-\frac{\partial V}{\partial t} = \inf_{u[t,t+\Delta t]} &\Big[\phi(t + \alpha\Delta t, x(t + \alpha\Delta t), u(t + \alpha\Delta t)) + \\
&+ \frac{\partial V'}{\partial x} f(t, x, u) + o(\Delta t)\Big].
\end{aligned} \tag{7.11}
$$

Now let $\Delta t \to 0$, so we obtain

$$
-\frac{\partial V}{\partial t} = \inf_{u(t)} \left[\phi(t, x, u) + \frac{\partial V'}{\partial x} f(t, x, u)\right], \tag{7.12}
$$

which is the desired HJB Equation for optimality together with the boundary condition

$$
V(T, x(T)) = m(T, x(T)). \tag{7.13}
$$

(7.12) represents a partial differential equation for $V(t, x)$. Minimizing (7.12) gives $u(t)$ as a function of $x(t)$ and $\frac{\partial V}{\partial x}$. If a solution $V(t, x)$ to (7.12) and (7.13) can be found, this results in $u(t)$ as a function of $x(t)$, that is, a state feedback control. In the general case, it is very difficult or even not possible to obtain $u(t)$ analytically. Later we will see that for a linear system with a quadratic cost function an analytic solution can be found (LQR problem).

Now we define

$$
\begin{aligned}
J\left(t, x, u, \frac{\partial V}{\partial x}\right) &\equiv \phi(t, x, u) + \frac{\partial V'}{\partial x} f(t, x, u), \\
J^* &\equiv \inf_{u(t)} J\left(t, x, u, \frac{\partial V}{\partial x}\right),
\end{aligned} \tag{7.14}
$$

so we can rewrite (7.12) together with its boundary condition as

$$
\begin{aligned}
-\frac{\partial V}{\partial t} &= J^*, \\
V(T, x(T)) &= m(T, x(T)).
\end{aligned} \tag{7.15}
$$

Further we let

$$u^*(t) = Arg\left[\inf_{u(t)} J\left(t, x, u, \frac{\partial V}{\partial x}\right)\right],\ t \in I_{s,T}. \tag{7.16}$$

The above derivation of the HJB equation is only a necessary condition. For the sufficient conditions we have the following theorem.

Theorem 7.2: (Sufficient Conditions for Optimality) Assume that the function $V(t, x)$ satisfying (7.12) on $([s, T] \times \mathbf{R}^n)$ together with its boundary condition (7.13) exist. Further, a given control $u^*[s, T]$ satisfying (7.16) exist. Then $u^*[s, T]$ is the optimal control minimizing the cost function (7.2) and the minimum value of the cost function is $V(s, x_0)$.

Proof: First note that

$$\int_s^T \frac{dV(\tau, x(\tau))}{d\tau} d\tau = V(T, x(T)) - V(s, x_0), \tag{7.17}$$

and

$$\frac{dV(\tau, x(\tau))}{d\tau} = \frac{\partial V(\tau, x(\tau))}{\partial \tau} + \frac{\partial V'(\tau, x(\tau))}{\partial x} f(\tau, x(\tau), u(\tau)). \tag{7.18}$$

So we can rewrite (7.2) by adding (7.17) the following way

$$\begin{aligned} &I(s, x_0, u(t)) = \\ =&\int_s^T \left(\phi(\tau, x(\tau), u(\tau)) + \frac{\partial V'(\tau, x(\tau)}{\partial x} f(\tau, x(\tau), u(\tau)) + \frac{\partial V(\tau, x(\tau)}{\partial \tau}\right) d\tau + \\ &+ m(T, x(T)) - V(T, x(T)) + V(s, x_0) \\ =&V(s, x_0) + \int_s^T \left(J\left(\tau, x, u, \frac{\partial V}{\partial x}\right) - J^*\left(\tau, x, u^*, \frac{\partial V}{\partial x}\right)\right) d\tau. \end{aligned} \tag{7.19}$$

We obtained the last result by the usage of the boundary condition and the definition of J^* (7.15). By our assumption for u^* we have for all u

$$J^*\left(t, x, u^*, \frac{\partial V}{\partial x}\right) \le J\left(t, x, u, \frac{\partial V}{\partial x}\right). \tag{7.20}$$

From this equation we see that the integral of (7.19) has a minimum value of zero, when

$$u[s, T] = u^*[s, T], \tag{7.21}$$

and hence the minimum cost is $V(s, x_0)$, i.e.

$$I(s, x_0, u[s, T]) \ge I(s, x_0, u^*[s, T]) = V(s, x_0) \tag{7.22}$$

for all $u[s, T]$. □

7.2 Finite Time LQR Problem

We now use the previous results to obtain a solution to the LQR problem. Since we try to find the optimal control $u^*(t)$ over the finite interval $I_{s,T}$, we refer to it as the finite time LQR problem.
Consider the linear time variant system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t),\ t \in I_{s,T}, \\ x(s) &= x_0,\ x_0 \in \mathbf{R}^n, \end{aligned} \tag{7.23}$$

with the usual dimensions for the matrices. Then we can formulate (7.2) for this case by letting the running cost functional $\phi(t, x(t), u(t)) = (Q(t)x(t), x(t)) + (R(t)u(t), u(t))$ and the terminal cost $m(T, x(T)) = (Mx(T), x(T))$. $Q(t), R(t), M$ are symmetric matrices of appropriate size. Further the matrices $Q(t)$ and M are positive semidefinite, $R(t)$ is positive definite. The cost functional is then given by

$$I(s, x_0, u(t)) = \int_s^T \left[(Q(\tau)x(\tau), x(\tau)) + (R(\tau)u(\tau), u(\tau))\right] d\tau + (Mx(T), x(T)). \tag{7.24}$$

7.2.1 Derivation of the Matrix Riccati Equation

In this chapter we will show that the solution to the finite time LQR problem can be found by solving a matrix differential equation, the famous *Differential Matrix Riccati Equation.*

Theorem 7.3 Consider the linear system given in (7.23) with the quadratic cost functional (7.24). The optimal state feedback control law is given by

$$u^*(t) \equiv \underbrace{-R^{-1}(t)B'(t)P(t)}_{K(t)} x(t), \tag{7.25}$$

where $P(t)$ is the solution to the Differential Matrix Riccati Equation

$$-\dot{P}(t) = Q(t) + P(t)A(t) + A'(t)P(t) - P(t)B(t)R^{-1}(t)B'(t)P(t),\ t \in I_{s,T} \tag{7.26}$$

with boundary condition

$$P(T) = M. \tag{7.27}$$

Proof: The equations (7.24) and (7.15) result in

$$\begin{gathered} \frac{\partial V(t, x(t))}{\partial t} = \\ \inf_{u(t)} \left[(Q(t)x(t), x(t)) + (R(t)u(t), u(t)) + \frac{\partial V'}{\partial x}(A(t)x(t) + B(t)u(t))\right], \\ V(T, x(T)) = (Mx(T), x(T)). \end{gathered} \tag{7.28}$$

The minimization of (7.28) is done by setting its gradient with respect to $u(t)$ to the zero vector

$$2R(t)u^*(t) + B'(t)\frac{\partial V}{\partial x} = 0, \tag{7.29}$$

and as $R(t)$ is positive definite we get for $u^*(t)$

$$u^*(t) = -\frac{1}{2}R^{-1}(t)B'(t)\frac{\partial V}{\partial x}. \tag{7.30}$$

Now substitute $u(t)$ in (7.28) with the expression in (7.30) to get the partial differential equation which $V(t, x(t))$ should satisfy

$$\begin{gathered} -\frac{\partial V}{\partial t} = (Q(t)x(t), x(t)) + \frac{\partial V'}{\partial x}A(t)x(t) - \frac{1}{4}\frac{\partial V'}{\partial x}B(t)R^{-1}(t)B'(t)\frac{\partial V}{\partial x}, \\ V(T, x(T)) = (Mx(T), x(T)). \end{gathered} \tag{7.31}$$

As we defined the matrices in the cost function to be in quadratic form, it is reasonable to assume that the function V is also quadratic in x. So we take as a candidate function for V

$$V(t, x(t)) = (P(t)x(t), x(t)), \tag{7.32}$$

where $P(t)$ is symmetric. The gradient of V with respect to x is then $2P(t)x(t)$ Then (7.31) becomes

$$\begin{aligned} -(\dot{P}(t)x(t), x(t)) =& (Q(t)x(t), x(t)) + 2x(t)P(t)A(t)x(t) - \\ & - x'(t)P(t)B(t)R^{-1}(t)B'(t)P(t)x(t), \\ (P(T)x(T), x(T)) =& (Mx(T), x(T)). \end{aligned} \tag{7.33}$$

We can write $2P(t)A(t)$ as the sum of a symmetric and an antisymmetric term

$$2P(t)A(t) = \underbrace{P(t)A(t) + A'(t)P(t)}_{\text{symmetric}} + \underbrace{P(t)A(t) - A'(t)P(t)}_{\text{antisymmetric}}, \tag{7.34}$$

and since for an antisymmetric matrix $S(t)$ it holds that

$$(S(t)x(t), x(t)) = -(S(t)x(t), x(t)), \tag{7.35}$$

we can write

$$2x'(t)P(t)A(t)x(t) = x'(t)(A'(t)P(t) + P(t)A(t))x(t). \tag{7.36}$$

Then we can rewrite the matrix differential equation in (7.33) as follows

$$\begin{aligned} -x'(t)\dot{P}(t)x(t) =& x'(t)\big[Q(t) + P(t)A(t) + A'(t)P(t) - \\ & - P(t)B(t)R^{-1}(t)B'(t)P(t)\big]x(t), \end{aligned} \tag{7.37}$$

which leads us to (7.26) for $t \in I_{s,T}$ with the boundary condition (7.27). If a solution $P(t)$ can be found, from (7.30) we obtain the optimal control law as stated in (7.25), which is a feedback control law. To obtain the control, $P(t)$ must be precomputed from the boundary condition and satisfy (7.26). □

7.2.2 Solution to the Matrix Riccati Equation

Here we show that the nonlinear Riccati Equation can be solved by transforming it to an equivalent linear system.

Theorem 7.4 Suppose that the matrices A, Q have locally integrable entries, the elements of B are locally square integrable and the elements of R and R^{-1} are essentially bounded measurable functions. Then for every $M \in M_{n\times n}$ the equation (7.26) and (7.27) have a unique absolutely continuous solution $P(t) \in (I_{s,T}, M_{n\times n})$.

Proof: There are several ways to prove this result. We will show a direct method as given in [1, p. 311] or [8, p. 16] by transforming the nonlinear equation into an equivalent linear system , which is then solved by its corresponding transition operator. Another way to solve this can be found in [14].
Consider the linear system

$$\dot{Z} = \mathcal{A}(t)Z,\ Z(T) = \begin{bmatrix} I \\ M \end{bmatrix}. \tag{7.38}$$

The matrix $\mathcal{A}(t) \in M_{2n\times 2n}$ plays an important role in linear quadratic problems and is called the *Hamiltonian matrix.* It is given by

$$\mathcal{A}(t) \equiv \begin{bmatrix} A(t) & -B(t)R^{-1}(t)B'(t) \\ -Q(t) & -A'(t) \end{bmatrix}. \tag{7.39}$$

(7.38) with the Hamiltonian matrix is a linear differential equation in $M_{2n\times n}$. By our assumptions for the matrices, the elements of $\mathcal{A}$ are locally integrable. Hence there exist a unique transition operator $\Psi(t, s)$ which gives a unique solution to (7.38)

$$Z(t) = \Psi^{-1}(T, t) \begin{bmatrix} I \\ M \end{bmatrix},\ t \in [s, T]. \tag{7.40}$$

Let $X(t)$ and $Y(t)$ be $n \times n$ matrices so that

$$Z(t) \equiv \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}. \tag{7.41}$$

Now we show that the solution of the (7.26) and (7.27) is given by $P(t) = Y(t)X^{-1}(t)$. By taking the derivative with respect to time and using the identity $\frac{d}{dt}(X^{-1}(t)) = -X^{-1}(t)\dot{X}(t)X^{-1}(t)$, we get

$$\begin{aligned} \dot{P}(t) &= \dot{Y}(t)X^{-1}(t) + Y(t)\dot{X}^{-1}(t) \\ &= (-Q(t)X(t) - A'(t)Y(t))X^{-1}(t) - Y(t)X^{-1}(t)\dot{X}(t)X^{-1}(t) \\ &= -Q(t) - A'(t)P(t) - P(t)(A(t)X(t) - B(t)R^{-1}(t)B'(t)Y(t))X^{-1}(t) \\ &= -Q(t) - A'(t)P(t) - P(t)A(t) + P(t)B(t)R^{-1}(t)B'(t)P(t). \end{aligned} \tag{7.42}$$

This is equation (7.26) and as $Y(T) = M$ and $X(T) = I$, also $P(T) = M$ holds. □

When the matrices $\{A, B, Q, R\}$ are time-varying, the matrix (7.39) and hence the linear equation (7.38) is also time-varying and an analytical solution my not be available. However, we can express the solution for $X(t)$ and $Y(t)$ in terms of the transition matrix $\Phi(t, T)$. For this, we partition the $2n \times 2n$ transition matrix into four equally sized blocks

$$\Phi(t, T) = \begin{bmatrix} \Phi_{11}(t, T) & \Phi_{12}(t, T) \\ \Phi_{21}(t, T) & \Phi_{22}(t, T) \end{bmatrix}. \tag{7.43}$$

Together with the boundary condition in (7.38) we get

$$\begin{aligned} X(t) &= \Phi_{11}(t, T)X(T) + \Phi_{12}(t, T)Y(T) \\ &= \Phi_{11}(t, T) + \Phi_{12}(t, T)M, \\ Y(t) &= \Phi_{21}(t, T)X(T) + \Phi_{22}(t, T)Y(T) \\ &= \Phi_{21}(t, T) + \Phi_{22}(t, T)M. \end{aligned} \tag{7.44}$$

So finally the solution for $P(t)$ becomes

$$P(t) = [\Phi_{21}(t, T) + \Phi_{22}(t, T)M]\,[\Phi_{11}(t, T) + \Phi_{12}(t, T)M]^{-1}. \tag{7.45}$$

When the matrices are time invariant, an explicit solution is available in terms of matrix exponential functions. An outline on how to obtain these solutions is given in [8, p. 17].

Remark: In the above theorem and proof, we did not require the matrices $\{Q, R, M\}$ to be symmetric positive (semi)definite. However we used this assumption in the derivation of the Matrix Riccati Equation and will see next that this is also used to obtain stability for the feedback system.

7.3 Steady State LQR Problem

In practical applications, the time variant problem might be difficult as one has to know the time varying matrices in the feedback control over the period of of optimization. For time invariant systems the problem becomes simpler as the matrices $\{A, B, Q, R\}$ are time invariant. Further, the optimization interval is infinite, i.e, T approaches infinity. Then the problem is to find an optimal control that minimizes the cost functional

$$I(x_0, u[0, \infty)) = \int_0^\infty [(Qx, x) + (Ru, u)]\, dt, \tag{7.46}$$

subject to the dynamical constraint

$$\dot{x} = Ax + Bu,\ x(0) = x_0 \in \mathbf{R}^n. \tag{7.47}$$

As before, the matrix Q is symmetric and positive semidefinite and the matrix R is symmetric positive definite.

First, we start with the general infinite horizon optimal control problem before we go over to the linear quadratic case.

7.3.1 General Conditions for Optimality

Consider the system dynamics described by

$$\dot{x}(t) = f(x(t), u(t)),\ x(0) = x_0 \in \mathbf{R}^n, \tag{7.48}$$

for which the following cost function is formulated

$$I(x_0, u(t)) = \int_0^\infty \psi(x(\tau), u(\tau))d\tau. \tag{7.49}$$

Again, to minimize the cost function I we look for feedback controls of the form $u(t) = \mu(x(t))$. So instead of (7.49), we write

$$I(x_0, \mu) = \int_0^\infty \psi(x(\tau), \mu(x(\tau)))d\tau. \tag{7.50}$$

The closed loop system is also required to be asymptotically stable, meaning

$$\lim_{t\to\infty} x(t) = 0,\ \forall x_0 \in \mathbf{R}^n. \tag{7.51}$$

So the set of admissible controls $\mathcal{U}_{ad}$ contains the $\mu((x(t)))$, for which (7.48) has a unique solution $x(t)$ which satisfies (7.51).

The optimal control problem is to find a control law, $\mu^*(x(t)) \in \mathcal{U}_{ad}$ that minimizes (7.50)

$$\mu^* = Arg\left[\inf_{\mu\in\mathcal{U}_{ad}} \int_0^\infty \psi(x(\tau), \mu(x(\tau)))d\tau\right]. \tag{7.52}$$

For the value function of this problem, we define

$$V(x_0) \equiv \inf_{\mu\in\mathcal{U}_{ad}} I(x_0, \mu). \tag{7.53}$$

This suggests to search for the general value function $V(x(t))$ with the given problem being the evaluation of V at $x = x_0$. The following result provides this approach.

Theorem 7.5: If there exists a control law $u = \mu^*(x(t)) \in \mathcal{U}_{ad}$ and a $V(x(t)) \in C^1$ such that $0 \leq V(x) \leq (Tx, x)$, $\forall x \in \mathbf{R}^n$, for some symmetric positive definite matrix T and

(a)

$$\frac{\partial V'}{\partial x} f(x(t), \mu^*(x(t))) + \psi(x(t), \mu^*(x(t))) = 0,\ \forall x \in \mathbf{R}^n \tag{7.54}$$

(b)

$$\frac{\partial V'}{\partial x} f(x(t), u(t)) + \psi(x(t), u(t)) \geq 0, \; \forall x \in \mathbf{R}^n, \; \forall u \in \mathcal{U}_{ad} \tag{7.55}$$

that is

$$J\left(x, u, \frac{\partial V}{\partial x}\right) \geq J\left(x, \mu^*(x), \frac{\partial V}{\partial x}\right) = 0.$$

Then $\mu^*(x(t))$ is the optimal control minimizing (7.49).

Proof: Let $u(t) = \mu^*(x(t))$, then

$$\begin{aligned} \frac{dV(x(t))}{dt} &= \frac{\partial V'}{\partial x} f(x(t), \mu^*(x(t))) + \underbrace{\frac{\partial V}{\partial t}}_{=0} \\ &= -\psi(x(t), \mu^*(x(t))), \end{aligned} \tag{7.56}$$

as in (7.54).
No we integrate from 0 to τ

$$V(x(\tau)) - V(x_0) = -\int_0^\tau \psi(x(t), \mu^*(x(t)))dt, \tag{7.57}$$

and since by the assumption $V(x(\tau)) \leq (Tx(\tau), x(\tau))$ and $x(\tau) \to 0$, it follows,

$$\lim_{\tau \to \infty} V(x(\tau)) = 0. \tag{7.58}$$

From this we have

$$V(x_0) = I(x_0, \mu^*(x(t))). \tag{7.59}$$

Now let $u(t) = \mu(x(t))$ be an arbitrary admissible control law and let $x(t)$ denote the solution to (7.48). By integrating (7.55)

$$V(x(\tau)) - V(x_0) = \int_0^\tau \frac{\partial V'}{\partial x} f(x(t), \mu(x(t))dt \quad \geq -\int_0^\tau \psi(x(t), \mu(x(t)))dt, \tag{7.60}$$

or

$$V(x_0) \leq V(x(\tau)) + \int_0^\tau \psi(x(t), \mu(x(t)))dt. \tag{7.61}$$

By letting $\tau \to \infty$ and using (7.58) and (7.59), we obtain

$$I(x_0, \mu^*(x(t))) \leq I(x_0, \mu(x(t))), \tag{7.62}$$

and further

$$V(x_0) = \inf_{\mu \in \mathcal{U}_{ad}} I(x_0, \mu(x(t))). \tag{7.63}$$

□

We now continue to the infinite horizon or steady-state LQR problem.

7.3.2 Algebraic Riccati Equation

For the steady-state LQR problem we again consider the cost functional given in (7.46) and the system (7.47) and try to find the functions $V(x(t))$ and $\mu^*(x(t))$ satisfying (7.54) and (7.55). So we have for optimality

$$\frac{\partial V'}{\partial x}(Ax + B\mu^*) + (Qx, x) + (R\mu^*, \mu^*) = 0, \tag{7.64}$$

and for arbitrary admissible $\mu(x(t))$

$$\frac{\partial V'}{\partial x}(Ax + B\mu) + (Qx, x) + (R\mu, \mu) \geq 0. \tag{7.65}$$

Now if we take

$$V(x) = (\bar{P}x, x) \tag{7.66}$$

and

$$\mu^*(x(t)) = -R^{-1}B'\bar{P}x, \tag{7.67}$$

then the conditions (7.64) and (7.65) hold if the following matrix equation holds

$$A'\bar{P} + \bar{P}A + Q - \bar{P}BR^{-1}B'\bar{P} = 0. \tag{7.68}$$

The above equation is the famous *Algebraic Matrix Ricatti Equation* (ARE).
Now what remains to be shown is that this equation has a unique solution under certain conditions and that, given such a solution can be found, the feedback system is stable.

Theorem 7.6: (Existence and Stability of the Steady-State LQR Solution) Given the LQR problem with $M = 0, R > 0, Q = D'D$, where the pair (A, D) is observable and the pair (A, B) is controllable, then the solution of the steady-state LQR problem exists, in particular, there exists a unique positive definite solution $\bar{P}$ to the (ARE) (7.68). Further, the optimal closed-loop system $\dot{x} = (A - BK)$, with $K = R^{-1}B'\bar{P}$, is asymptotically stable.

Proof: We will first show that $\bar{P}$ is positive definite by the observability and controllability assumptions, then we use the Lyapunov Theorem to show that the closed loop system is stable.
As we have $V(t, x) = (\bar{P}x, x)$ and by our assumptions for the matrices, $M = 0, R > 0, Q = D'D$ we can see that $\bar{P} \geq 0$. For $Q = D'D$ we usually have the matrix D the way that $rank(Q) =$ number of rows in D.
Now we show that $P(t)$ is bounded above for all T and is monotonically increasing with increasing T, hence it converges. To show that $P(t)$ is monotonically increasing, consider the inequality

$$\int_t^{T_1} \phi(\tau, x^*, \mu^*)d\tau \leq \int_t^{T_2} \phi(\tau, x^*, \mu^*)d\tau = V(T_2),\ \forall T_2 \geq T_1,\ \forall x \in \mathbf{R}^n \tag{7.69}$$

where μ^* denotes the control that is optimal over the interval $[t, T_2]$, with the corresponding state x^*. The inequality (7.69) follows directly from the additive property of the integral and by $\phi(\tau, x^*, \mu^*) = (Qx, x) + (Ru, u)$ non-negative. Then if we denote μ^{**} to be the control that is optimal over the interval $[t, T_1]$, we have the other inequality

$$V(T_1) = \int_t^{T_1} \phi(\tau, x^{**}, \mu^{**}) d\tau \leq \int_t^{T_1} \phi(\tau, x^*, \mu^*) d\tau, \; \forall x \in \mathbf{R}^n. \tag{7.70}$$

So we have $V(T_1) \leq V(T_2), T2 \geq T_1$ and as $V(t, x)$ is in quadratic form of $P(t)$, $P(t)$ is monotonically increasing with respect to T.
To show that $P(t)$ is bounded above, we use the controllability assumption. By controllability, we have seen that there exists a constant matrix K, such that the closed-loop system $\dot{x} = (A - BK)x$ with the feedback control $u(t) = -Kx(t)$ is asymptotically stable, meaning $\lim_{t\to\infty} x(t) = 0$, $\forall x \in \mathbf{R}^n$. But this control might not be optimal. However, the value of the cost function $\tilde{V}$ corresponding to the given control law is bounded as $x(t) \to 0$ exponentially and hence $u(t) = -Kx(t)$ also converges to zero exponentially. $\tilde{V}$ is also in quadratic form, we denote this by $(\tilde{P}x, x)$. So for any $V(t, x) = (P(t)x, x)$, which is optimal over (t, ∞) we must have $P(t) \leq \tilde{P}$, which establishes our upper bound.

Now we show that if (A, D) is observable, it follows that $\bar{P} > 0$. By contradiction, assume that $\bar{P}$ is only positive semidefinite. Then there exists an initial state $x_0 (\neq 0) \in \mathbf{R}^n$ for which $(\bar{P}x_0, x_0) = 0$. This means

$$(\bar{P}x_0, x_0) = \int_0^{\infty} \left[(D'Dx, x) + (Ru, u) \right] dt = 0. \tag{7.71}$$

Since $R > 0$, this implies that $u(t) = 0$, *a.e.*. So the first term in the integral must also be zero and this in turn implies that $Dx(t, x_0) = 0 \; \forall t \geq 0$. But we have (A, D) observable, which implies $x_0 = 0$, so we obtained a contradiction.

We now show that the optimal closed-loop system is asymptotically stable. By Lyapunov, this means that the derivative of any solution of the closed loop system

$$\dot{x}(t) = (A - BR^{-1}B'\bar{P})x(t), \; x(0) = x_0 \in \mathbf{R}^n \tag{7.72}$$

is negative definite. Let $x_0 \in \mathbf{R}^n$ be arbitrary and $x(t) = x(t, x_0)$ be the corresponding solution of the closed loop system. By differentiating V along this trajectory, we obtain

$$\begin{aligned} \dot{V}(t, x) &= (\dot{\bar{P}}x, x) + (\bar{P}\dot{x}, x) + (\bar{P}x, \dot{x}) \\ &= ((\bar{P}A - \bar{P}BR^{-1}B'\bar{P})x, x) + (\bar{P}x, (A - BR^{-1}B'\bar{P})x). \end{aligned} \tag{7.73}$$

Further we have

$$\dot{V}(t, x) = -\left[(D'Dx, x) + (\bar{P}BR^{-1}B'\bar{P}x, x) \right]. \tag{7.74}$$

From this we can rewrite the (ARE) (7.68) as

$$A_c'\bar{P} + \bar{P}A_c = -D'D - \bar{P}BR^{-1}B'\bar{P}, \tag{7.75}$$

where A_c is the closed loop system matrix given by

$$A_c = A - BR^{-1}B'\bar{P}. \tag{7.76}$$

From the fact that $\bar{P}$ is known to be positive definite and by assumption the pair (A, D) is observable, it follows by the Lyapunov Theorem for linear systems that A_c must be a stability matrix. This can be shown directly by writing the second term in (7.74) as $x'(R^{-1}B'\bar{P})'R(R^{-1}B'\bar{P})x$. So for $\dot{V}(t,x) \equiv 0$ we must have $R^{-1}B'\bar{P}x(t) \equiv 0$ since $R > 0$. But this implies that $Dx(t, x_0) = 0\, \forall t \geq 0$. Again from observability it follows that $x_0 = 0$ and hence $x(t) \equiv 0$.

What remains to be shown is the uniqueness of the solution to the (ARE). By contradiction suppose there are two solutions $\bar{P}$ and $\tilde{P}$. Then we define $E \equiv \bar{P} - \tilde{P}$, and by subtracting the corresponding equations from each other, we obtain a matrix equation for E

$$A_1E + EA_2 = 0, \tag{7.77}$$

where

$$A_1 \equiv (A - BR^{-1}B'\tilde{P})' \text{ and } A_2 \equiv (A - BR^{-1}B'\bar{P}). \tag{7.78}$$

The matrix equations of the form $A_1E + EA_2 = L$ have the following solution (see for example the proof of Theorem 3.2)

$$E = \int_0^\infty e^{tA_1} L e^{tA2} dt. \tag{7.79}$$

We have seen that both, A_1 and A_2 are stability matrices, so the integral is well defined. Here we have $L = 0$, from which $E = 0$ and hence uniqueness follows. □

Actually, the existence and stability result for the (ARE) can be obtained for the weaker conditions of stabilizability and detectability. For definition of detectability, refer to [1, p. 167]. One can show that stabilizability is enough for $P(t)$ bounded and detectability is all that is needed to guarantee that A_c is stable. With the condition of observability replaced by detectability, however, the matrix $\bar{P}$ may only be positive semidefinite. In particular, one can show that stabilizability and detectability are all that is required from the system dynamics for the Hamiltonian matrix as defined in (7.39) to have no purely imaginary eigenvalues. This is elaborated in detail in [6, p. 651].

7.4 Example

Consider the inverted pendulum as in the introduction (1.1). Here will create the feedback control law with help of the ARE.
Again we assume that the friction is negligible and the mass is centered at the end of the rod with $m = 1$. For the linearized system about the equilibrium point we have

$$\ddot{\varphi}(t) - \omega^2\varphi(t) = u(t), \text{ with } \omega^2 = g/L. \tag{7.80}$$

The state vector is $x(t) = (\varphi(t), \dot{\varphi}(t))' = (x_1(t), x_2(t))'$. The linear system can then be written as

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega^2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \tag{7.81}$$

For the cost functional we define the matrix $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $R = \frac{1}{c^2}$ for a real number c. So we obtain for the cost functional

$$I = \int_0^\infty \varphi(t)^2 + \frac{1}{c^2} u(t)^2 dt. \tag{7.82}$$

Now let $\bar{P} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$ be a symmetric matrix. So finally we get for the ARE (7.68)

$$\begin{bmatrix} \omega^2 p_2 & \omega^2 p_3 \\ p_1 & p_2 \end{bmatrix} + \begin{bmatrix} p_2\omega^2 & p_1 \\ p_3\omega^2 & p_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} c^2 p_2^2 & c^2 p_2 p_3 \\ c^2 p_2 p_3 & c^2 p_3^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{7.83}$$

which leads to the following equations

$$\begin{aligned} &2\omega^2 p_2 + 1 - c^2 p_2^2 = 0 \;\Rightarrow\; p_2 = \frac{\omega^2 \pm \sqrt{\omega^4 + c^2}}{c^2}, \\ &p_1 + p_3\omega^2 - c^2 p_2 p_3 = 0 \;\; \text{(appeares twice)}, \\ &2p_2 - c^2 p_3^2 = 0 \;\Rightarrow\; p_3 = \pm\frac{1}{c}\sqrt{2p_2}. \end{aligned} \tag{7.84}$$

As p_3 is a diagonal term, it needs to be real and positive, hence p_2 needs to be positive, so we have

$$\begin{aligned} p_2 &= \frac{\omega^2 + \sqrt{\omega^4 + c^2}}{c^2}, \\ p_3 &= \frac{1}{c}\sqrt{2p_2}, \\ p_1 &= c^2 p_2 p_3 - p_3\omega^2. \end{aligned} \tag{7.85}$$

For the feeback operator K we get

$$K = R^{-1} B' \bar{P} = \begin{bmatrix} c^2 p_2 \\ c^2 p_3 \end{bmatrix}', \tag{7.86}$$

and for the control

$$u(t) = Kx(t) = c^2 (p_2 \varphi(t) + p_3 \dot{\varphi}(t)). \tag{7.87}$$

To analyze our obtained solution, first consider the uncontrolled system. It is easy to see that the original system is unstable by computing the eigenvalues of the system matrix A

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -1 \\ -\omega^2 & \lambda \end{bmatrix} \right) = \lambda^2 - \omega^2 = 0 \;\Rightarrow\; \lambda_{1,2} = \pm\omega. \tag{7.88}$$

(7.88) shows that we have one eigenvalue with positive real part, hence the system is unstable.
For the closed loop system we have the system matrix

$$A_{CL} = A - BK = \begin{bmatrix} 0 & 1 \\ \omega^2 - c^2 p_2 & -c^2 p_3 \end{bmatrix}. \tag{7.89}$$

First, define $\Omega^2 = \sqrt{\omega^4 + c^2}$. Then, after some computation, we obtain the characterisitc polynomial for the closed loop system

$$\det(\lambda_{CL} I - A_{CL}) = \lambda_{CL}^2 + \sqrt{2(\omega^2 + \Omega^2)}\lambda_{CL} + \Omega^2, \tag{7.90}$$

and from this the eigenvalues

$$\lambda_{CL,1,2} = -\frac{1}{\sqrt{2}}\sqrt{(\omega^2 + \Omega^2)} \pm i\frac{1}{\sqrt{2}}\sqrt{(\Omega^2 - \omega^2)}. \tag{7.91}$$

By our definition of $\Omega^2 = \sqrt{\omega^4 + c^2} > \omega^2$, we have both eigenvalues with negative real part and hence the closed loop system is asymptotically stable.

Above example showed that the computation of the corresponding matrices can already be very technical for small dimensions. For higher dimensional systems, usually computer software is used to compute the solution. Various algorithms and numerical methods about linear quadratic optimization are described in [20].

8 Conclusion

Given a linear dynamical system, the state-space approach to analyze the system and construct a control law, both open loop and closed loop, is a comprehensive theory and offers many powerful methods. Once we obtained the system properties such as stability, observability and controllability, our aim is to find a control law that first stabilizes a possibly unstable system and second steers the system to a desired target state.
We have shown that linear quadratic optimization can be used to construct an optimal feedback control law in combination with a cost functional. These design methods are applicable to both, time-varying and time invariant systems and are therefore not only an interesting concept from a mathematical point of view, but also very useful in practice. The presented topics in this thesis are the foundation to the latest research and methods in control theory such as the linear quadratic Gaussian control problem (also referred to as H^2 problem) or the H^∞ control for robust performance. Concerning these methods, the interested reader may refer to [8, 2].

References

[1] Ahmed, N. U., *Dynamic Systems and Control with Applications.* World Scientific Publishing Co. Pte. Ltd., Singapore, 2006, ISBN: 981-270-053-6.

[2] Basar, T., Bernhard, P., *H^∞-Optimal Control and Related Minimax Problems. A Dynamic Game Approach. Second Edition.* Birkhaeuser, Boston, 2008, ISBN: 978-0-8176-4756-8.

[3] Benner, P., *Control Theory.* Fakultaet fuer Mathematik, Technische Universitaet Chemnitz. Chemnitz, 2006.

[4] Bernstein, D., S., *Matrix Mathematics. Theory, Facts and Formulas. Second Edition.* Princeton University Press, Princeton, 2009, ISBN: 978-0-691-13287-7.

[5] Betounes, D., *Differential Equations. Theory and Applications, Second Edition.* Springer, New York, 2010, ISBN: 978-1-4419-1162-9.

[6] Bhattacharyya, S., P., Datta, A., Keel, L., H., *Linear Control Theory. Structure, Robustness and Optimization.* Taylor and Francis Group, LLC, Boca Raton, 2009, ISBN: 978-0-8493-4063-5.

[7] Chen, C. T., *Linear System Theory and Design, Third Edition.* Oxford University Press, New York, 1999, ISBN: 0-19-511777-8.

[8] Dorato, P., Abdallah, C. T., Cerone, V., *Linear Quadratic Control. An Introduction.* Krieger Publishing Company, Malabar Florida, 2000, ISBN: 1-57524-156-0.

[9] Fairman, F. W., *Linear Control Theory. The State Space Approach.* John Wiley and Sons, Ltd., Chichester, 1998, ISBN: 0-471-97489-7.

[10] Kirk, D., E., *Optimal Control Theory. An Introduction.* Dover Publications, Inc., Mineola, 2003, ISBN: 0-486-43484-2.

[11] Levin, J., J., *On the Matrix Riccati Equation.* Proceedings of the American Mathematical Society, Vol. 10, No. 4 (Aug., 1959), pp. 519-524.

[12] Luenberger, D. G., *Dynamic Systems. Theory, Models, and Applications.* John Wiley and Sons, Inc., New York, 1979, ISBN: 0-471-02594-1.

[13] Lygeros, J., Ramponi, F., *Lecture Notes on Linear System Theory.* ETH Zurich, Zurich, 2010.

[14] Nazarzadeh, J., Razzaghi, M., Nikravesh, K., Y., *Solution of the Matrix Riccati Equation for the Linear Quadratic Control Problems.* Math. Comput. Modelling vol. 27, No. 7 (1998), pp. 51-55.

[15] Ramponi, F., *Notes on Lyapunov's Theorem.*
`control.ee.ethz.ch/~ifalst/docs/lyapunov.pdf`

[16] Reid, W., T., *A Matrix Differential Equation of Riccati Type.* American Journal of Mathematics, Vol.68, No. 2 (Apr., 1946), pp 237-246.

[17] Rugh, W., J,, *Linear System Theory, Second Edition.* Prentice Hall, Inc., New Jersey, 1996, ISBN: 0-13-441205-2.

[18] Haddad, W. M., Chellaboina, V., *Nonlinear Dynamical Systems and Control. A Lyapunov-Based Approach.* Princeton University Press, Princeton, 2007, ISBN: 978-0-6911-3329-4.

[19] Hsu, Sze-Bi, *Ordinary Differential Equations with Applications.* World Scientific Publishing Co. Pte. Ltd., Singapore, 2005, ISBN: 981-256-319-9.

[20] Sima, V., *Algorithms for Linear-Quadratic Optimization.* Marcel Dekker, Inc. New York, 1996, ISBN: 0-8247-9612-8.

[21] Sontag, E., D., *Mathematical Control Theory. Deterministic Finite Dimensional Systems. Second Edition.* Springer, New York, 1998, ISBN: 0-387-98489-5.

[22] Zabczyk, J., *Mathematical Control Theory. An Introduction.* Birkhaeuser, Boston, 2008, ISBN: 978-0-8176-4732-2.

[23] Zhang, F., *Matrix Theory. Basic Results and Techniques.* Springer, New York, 1999, ISBN: 0-387-98696-0.