The Geometry and Algebra of Spherical Spaces

Johannes Wachs

Department of Mathematics and its Applications

Central European University

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Supervisor:
Professor Róbert Szőke
ELTE

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Contents

1 Introduction

Groups and smooth manifolds are fundamental objects in mathematics. Both are well studied and have highly developed theories. The case in which a smooth manifold also has an appropriately compatible group structure is especially distinguished. These spaces are called Lie groups. They arise in many areas of mathematics and physics, and have many applications. The algebra, geometry and their synthesis contribute to the richness of the theory.

This thesis is about a special class of quotients of Lie groups known as spherical spaces. They are worth studying both as generalizations of symmetric spaces and in their own right because of their special representation-theoretic properties. The spherical space $X = SU(p+q)/SU(p) \times SU(q)$ for $p > q \geq 1$ will be the examined from several perspectives. To whet the appetite of the reader, we present the following facts about X:

1. In the case $q = 1$, $X = SU(p+1)/SU(p) \cong S^{2p+1}$, the $2p+1$-dimensional sphere,
2. X is a principal S^1 bundle over the Grassmannian, $G_{p+q}(q) = SU(p+q)/S(U(p) \times U(q))$,
3. X is a spherical space which is not a symmetric space when $q > 1$.

These properties are related. Consider the homogeneous fiber bundle $S^1 \to X \to Gr_{p+q}(q)$ and set $q = 1$. We obtain the bundle $S^1 \to S^{2p+1} \to \mathbb{C}P^p$. This is the famous complex Hopf fibration [Hu]. The third fact suggests that X is a good candidate to help us understand the difference between symmetric spaces and spherical spaces while the second hints that we should compare X directly with a famous symmetric space, the Grassmannian.

As seen above, X is involved in the generalization of several important mathematical concepts. On the other hand, it is not such a vast generalization that intuition and techniques coming from the original concepts become useless. In fact, the relationship X has with the Grassmannian will be helpful in computing the curvature and topological invariants of X.

We are also interested in Lie algebraic calculations related to X, in the context of generalizing the simpler case of symmetric spaces. These calculations come up in the study of geometric quantization. The process of geometric quantization associates to a physical classical system a set of quantum states. Mathematically, a Hilbert space encoding the quantum states is associated to a Riemannian manifold's cotangent bundle (which carries a symplectic structure) encoding the classical system. In this complicated process one must make choices and there is not a canonical Hilbert space. Instead we obtain a family of Hilbert spaces associated to M parametrized by the set of choices. For a given Riemannian manifold M, it is a topic of current research to study the corresponding family of Hilbert spaces, see for example the paper of Lempert and Szőke [LSz10]. For example we would like to know when the Hilbert spaces in the family associated to M are canonically isomorphic. In the case where M is a compact symmetric space terms from the Lie algebra are used in calculations to check if this property holds. Even if the Hilbert spaces are not isomorphic, the Lie algebra calculations help us understand how far the space is from this situation. When we generalize to spherical spaces, additional Lie algebra calculations must be made, as the Lie algebras arising in this situation do not have all of the nice relations as the ones in the symmetric case.

The goal of the thesis is to introduce spherical spaces and to obtain information about the space X. In the second section, background information about Lie groups, homogeneous spaces, and symmetric spaces will be given. In the third section spherical spaces will be defined and the classification of spherical spaces arising as quotients of compact simple Lie groups by Krämer will be sketched by showing that X is spherical. In the fourth section Lie algebra calculations relating to X will be motivated and carried out. In the fifth section some topological invariants of X will be calculated. The thesis ends with a conclusion and an appendix surveying the literature on spherical spaces in various areas of mathematics.

2 Background

This section collects various facts and theorems about Lie groups and related spaces. The theme throughout is the interplay between the algebraic and geometric aspects of the theory. We introduce homogeneous spaces as interesting quotients of Lie groups with special properties. We then present symmetric spaces, first describing their origins in Riemannian geometry, and then discussing their characterization as quotients of Lie groups following Cartan. Examples, especially the ones relevant to the subsequent sections, are emphasized. The main reference is [He2]. For an introduction to Lie groups see [St]. For more about Riemannian geometry, see [dC].

Definition 2.1. A Lie group is a set G with both smooth manifold and group structures, where group multiplication and inversion are smooth maps.

Example 2.2. The real line is a Lie group with addition as the operation. We denote it by $\mathbb{R}_+$.

Example 2.3. The unit circle in the complex plane is a one dimensional smooth manifold. The elements of the circle form a group under multiplication. We can identify elements of this Lie group with the set of 1×1 unitary matrices, which we call $U(1)$.

Example 2.4. The group of n by n unitary matrices is a Lie group, denoted $U(n)$. Its intersection with $SL_n(\mathbb{C})$, the group of invertible n by n matrices with determinant 1, is also a Lie group called the special unitary group or $SU(n)$ for short.

First we will demonstrate that it is worthwhile to restrict our attention to the Lie groups that are either connected or finite. We will not consider finite Lie groups in this thesis, as every finite group is a zero dimensional Lie group, and the theory of finite groups is complicated enough.

Theorem 2.5. Let G be a Lie group. Denote by G^o the identity component of G, that is the connected component of G containing the group's identity element. Then G^o is a Lie group and a normal subgroup of G. The quotient G/G^o is discrete.

Proof. First we show that G^o is a subgroup of G. We know that the identity is its own inverse. Since the continuous image of a connected set is connected, we see that G^o is closed under taking inverses. Let e be the identity element of G. Similarly, $e * e = e$ shows that G^o is closed under group multiplication. So $G^o \leq G$. To check that G^o is normal in G we must show that $ghg^{-1} \in G^o$ for any $h \in G^o$ and $g \in G$. Conjugation by g is a continuous map so it will send G^o to a connected component. Since conjugation maps e to e, this componenet is G^o. So $G^o \triangleleft G$. □

Moreover, it follows from the theory of covering spaces that much of the study of connected Lie groups can be carried out entirely in the realm of simply connected Lie groups by way of the universal cover and lifting.

Now we introduce the concept of the Lie algebra associated to a Lie group. The idea is to study a locally defined 'linearized' object, the Lie algebra, and to obtain as much data as possible about the global object, the Lie group. One interpretation of the Lie algebra $\mathfrak{g}$ associated to a Lie group G is as the tangent space of G at the identity element e. The Lie algebra contains a great deal of information about the Lie group. For example, if G and H are connected Lie groups, then any Lie group morphism from G to H is uniquely determined by the induced linear map on the Lie algebras. We now give the algebraic definition of an abstract Lie algebra.

Definition 2.6. A Lie algebra is a vector space $\mathfrak{g}$ over a field F (though in this thesis we only examine Lie algebras over C) with a bilinear map

$$[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

called the Lie bracket which is bilinear,

$$[ax + by, z] = a[x, z] + b[y, z],\ [z, ax + by] = a[z, x] + b[z, y]$$

for scalars a and b from the field and x, y, and z from $\mathfrak{g}$;
alternating,

$$[x, x] = 0$$

for all x in $\mathfrak{g}$;
and satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all x, y, z in $\mathfrak{g}$.

Example 2.7. The Lie algebra associated to $SU(n)$ is the space of $n \times n$ traceless skew-hermitian complex matrices, denoted $\mathfrak{s}u(n)$. These are the matrices equal to the negative of their conjugate transpose and with 0 trace.

For a matrix Lie group G, the exponential map $\exp(A) = 1 + A + \dfrac{A^2}{2!} + \ldots$ maps elements of $\mathfrak{g}$ to G. An appropriate generalization maps the Lie algebra of a general Lie group to the Lie group. It is a diffeomorphism from a neighborhood of 0 in the vector space $\mathfrak{g}$ and a neighborhood of the identity in G. We cannot resist quoting

Chevalley, who so succinctly explains why it is worthwhile to view elements of G as exponentials of elements of $\mathfrak{g}$:

"The property of a matrix being orthogonal or unitary is defined by a system of nonlinear relationships between its coefficients; the exponential mapping gives a parametric representation of the set of unitary (or orthogonal) matrices by matrices whose coefficients satisfy *linear* relations."[St]

The spaces we will be studying arise as quotients of Lie groups. First the notion of a Lie subgroup must be made clear. H is a Lie subgroup of G if it is both a subgroup and an imbedded submanifold. Some care must be taken as the quotients of groups and the quotients of topological spaces can be complicated individually. It is true that Lie subgroups are always closed and that the quotient of a Lie group G by a Lie subgroup H has the structure of a smooth manifold. In addition, when H is normal, the quotient is again a Lie group. Considering Lie algebras, we note that the Lie algebra of a Lie subgroup is a subalgebra of the Lie algebra of the original Lie group. We summarize some of these and other results below.

Theorem 2.8. Let G be a Lie group.

1. Let H be a Lie subgroup of G. Then H is closed in G.
2. Any closed subgroup of a Lie group is a Lie subgroup.
3. If G is connected and U is a neighborhood of e, then U generates G.
4. If G has dimension n and $H \subset G$ is a Lie subgroup of dimension K then the space of cosets G/H is a manifold of dimension $n - k$ and the canonical map $G \to G/H$ is a fiber bundle with fiber H.

Again we digress for a moment to talk about the importance of Lie algebras. At first glance it is not clear how well Lie algebras help us distinguish Lie groups. In the search for simple Lie groups, one naturally encounters simple Lie algebras. In fact, simple Lie algebras are easier to find than the simple Lie groups. It seems like a good idea to start then with a simple Lie algebra and to see if we can find a simple Lie group. Unfortunately the Lie algebra $\mathfrak{g}$ cannot see the finite subgroups of G because they have 0-dimensional tangent spaces: they are invisible. It turns out that $\mathfrak{g}$ can see all normal subgroups of G except for those in the center of G. The centers are relatively easy to calculate, and so this problem is not as bad as it first appears, and again we see evidence for the great utility of the Lie algebra.

Now that we have defined Lie subgroups, we can discuss one of the primary sources of examples in practice. One of the most fruitful ideas in group theory is the idea of an action. We now define the analogue for Lie groups, and suggest how to obtain subgroups and quotient spaces from an action.

Definition 2.9. Let G be a Lie group and M a manifold. Denote by $\mathrm{Diff}(M)$ the group of diffeomorphisms of M. G acts on M if we assign to each g in G a diffeomorphism $\rho(g)$ in $\mathrm{Diff}(M)$ satisfying $\rho(e) = 1$, $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$, and the additional condition that the map

$$G \times M \to M \ (g, m) \mapsto \rho(g)(m)$$

is smooth.

Example 2.10. The group $SU(n)$ acts on the sphere $S^{2n-1} \subset \mathbb{C}^n$. The action is transitive. To see this, it is enough to show that $SU(n)$ maps the vector $e_1 = (1, 0, \ldots, 0)$ to any other vector of length one in $\mathbb{C}^n$. Given $x \in S^{2n-1}$, there is a $g \in SU(n)$ with $ge_1 = x$, which we now construct. Begin with a matrix with x as its leftmost column. Find additional vectors to form a basis for $\mathbb{C}^n$ with x. Then apply the Gram-Schmidt process to find an orthonormal basis $x, v_2, v_3, ..v_n$ for $\mathbb{C}^n$. In the unitary group we require $g^*g = I$, which means that the columns of g form an orthonormal basis. So the matrix g consisting of the column vectors $x, v_2, v_3, ..v_n$ maps e_1 to x. To insure that g has determinant 1, replace v_n by $(detg)^{-1}v_n$. This preserves orthonormality and insures that g is in $SU(n)$.

As in the case with finite groups we can define the notion of an orbit. This will lead us to the definition of a homogeneous space. These are quotients of Lie groups coming from group actions.

Definition 2.11. Let G be a Lie group and M a manifold. Let G act on M. The *orbit* of a point $m \in M$ is defined to be the set $\mathcal{O}_m = Gm = \{gm | g \in G\}$.

Definition 2.12. Fix a point $m \in M$. Define the set of group elements which fix m, denoted $I_G(m) = \{g \in G | gm = m\}$, as the *isotropy subgroup* of m.

The inclusion of the word subgroup suggests the following lemma:

Lemma 2.13. Let $H = I_G(m)$. Then H is a Lie subgroup in G. Furthermore, the map $g \mapsto gm$ is an injective immersion $G/H \hookrightarrow M$. The image is precisely the orbit $\mathcal{O}_m$.

A corollary of the lemma is that if the orbit $\mathcal{O}_m$ is closed, then the map in the lemma is a diffeomorphism $G/H \cong \mathcal{O}_m$. The case where there is only a single orbit in a group action is especially interesting. We call such an action transitive.

Definition 2.14. A *homogeneous space* is a manifold with a transitive action of G.

Applying the corollary, homogeneous spaces are diffeomorphic to coset spaces G/H. In this way, they are also fiber bundles.

Example 2.15. The action of $SU(n)$ on the sphere S^{2n+1} is transitive, as seen in the previous example. The isotropy group of the last basis vector $e_n = (0, \ldots, 0, 1)$ is $\left\{ \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} : A \in SU(n-1) \right\}$. The isotropy group is isomorphic to $SU(n-1)$. So by the results above, $SU(n)/SU(n-1) \cong S^{2n-1}$.

The next step is to define symmetric spaces. Symmetric spaces have their origins in Riemannian geometry. A Riemannian locally symmetric space is a Riemannian manifold for which the curvature tensor is invariant under all parallel translations. E. Cartan set out to classify such spaces and suceeded in 1926. He carried out the classification in two different ways. One of them involves reducing the problem to the classification of simple Lie algebras over $\mathbb{R}$. The first step of Cartan's program was to express the problem in terms of Lie groups. The idea is the following: the invariance of the curvature tensor under parallel translations is equivalent to geodesic symmetry with respect to each point being a local isometry. A Riemannian globally symmetric space is a space for which the geodesic symmetry extends to a global isometry. We refer to these spaces as Riemannian symmetric spaces. Riemannian symmetric spaces have a transitive group of isometries. Invoking the theory of homogeneous spaces outlined above, Riemannian symmetric spaces are precisely coset spaces G/H where G is a connected Lie group, and H is the fixed point set of an involutive automorphism of G.

Before giving a precise definition and examples, let us take a moment to motivate the study of symmetric spaces. In some sense the notion of curvature of a Riemannian manifold is a measure of its complexity. By this measure, spaces of constant curvature are the least complicated. Considering simply connected ones there are only three examples: Euclidean spaces, spheres and hyperbolic spaces. One way to relax the condition of constant curvature is to instead assume that the covariant derivative of the curvature is 0. There are many more examples and these are precisely the symmetric spaces. As mentioned above they have many different interpretations, including one in terms of Lie groups. This interpretation is especially fruitful for computations. They are classified into a list that is long enough to contain many interesting examples, but short enough not to be overwhelming.

Definition 2.16. Let M be a Riemannian manifold. Let G be the group of isometries of M. M is a *symmetric space* if for any $m \in M$ there is $g_x \in G$ such that $g_x x = x$ and $dg_x = -I$.

We can immediately note several consequences from the definition. For one, M is homogeneous. To see this note that any two points a and b can be connected by a geodesic. The isometry g_m associated to the midpoint m of the geodesic connecting a and b maps a to b. Hence G acts transitively. In the other direction, we have that if M is a homogeneous space, it is a symmetric space if and only if there is a point x and an isometry g_x satisfying the conditions above. Finally we note that the isotropy subgroups of G are compact. Here are some examples:

1. The Euclidean space $\mathbb{R}^n$ is a symmetric space. The isometry associated to each point x is $g_x(x + v) = x - v$.

2. The sphere S^n is a symmetric space with the isometry associated to a point x reflection at the line $\mathbb{R}x$. Specifically, $g_x(y) = -y + 2\langle y, x\rangle x$.

3. Define $Gr_k(n)$ as the set of all k-dimensional linear subspaces of $\mathbb{C}^n$. This space is called the *Grassmannian.* The group $SU(n)$ acts transitively on it, with the isotropy of $\mathbb{C}^k$ equal to $S(U(k) \times U(n-k))$. The subgroup $S(U(k) \times U(n-k))$ denotes block $n \times n$ matrices with $k \times k$ unitary matrices in the top left position, $(n-k) \times (n-k)$ unitary matrices in bottom right, zeros elsewhere and the final condition that the determinant of the entire matrix is one. The symmetry g_x at the point x on the Grassmannian is the reflection fixing x. It is the linear transformation with eigenvalue 1 on x and -1 elsewhere. In the case when $k = 1$, we obtain complex projective space.

4. Any compact Lie group is a symmetric space.

Cartan classified symmetric spaces in two different ways [Ca1][Ca2]. His first method was by studying groups arising from differential-geometric properties of the symmetric spaces called holonomy groups. The second method was to characterize symmetric groups as the quotients of Lie groups and to study their Lie algebras. We now introduce the Lie-theoretic definition of symmetric spaces, and show that it is equivalent to the Riemannian-geometric definition.

Theorem 2.17. Let G be a connected Lie group with automorphism of order 2, $\sigma : G \to G$ and a left invariant metric which is also right invariant in the closed subgroup

$$K = \text{Fix}(\sigma) = \{g \in G | \sigma g = g\}.$$

Then any closed subgroup H satisfying

$$K^o \subset H \subset K$$

defines a symmetric space $G/H = S$. Every symmetric space is of this form.

Proof. We begin by showing the last statement, namely that any symmetric space arises in this way. Let S be a symmetric space. From our first definition, we know that symmetric spaces are homogeneous spaces with a symmetry at each point. We know that the group of isometries of S, call it G', acts transitively and contains a geodesic symmetry at some point $p \in S$, which we call g_p. Conjugation by this symmetry defines an automorphism σ of G',

$$\sigma(g) = g_p g g_p^{-1} = g_p g g_p.$$

Now observe that g_p^2 is the identity element of G' since it is an isometry with the same value and derivative at p as the identity element. So σ defines an involution. We can pass to the identity component of G' which we will refer to as G, and since the identity component is preserved by any automorphism, σ is also an involution on G. We proceed by relating g_p to the isotropy group at p. Specifically, g_p acts as $-I$ on the tangent space of S at p, which we denote T_pS. Therefore it commutes with the action of the isotropy group at p, which we call H. Hence H is contained in the fixed point set of σ in G. Conversely, if g is in the fixed point set of σ, it commutes with g_p. So it leaves the subset of elements of S fixed by g_p invariant. However, p is isolated from the other fixed points of σ in S as no nonzero vectors in T_pS are fixed by the dx_p, as they are mapped to their negative. So g fixes p exactly when g can be connected to the identity of the fixed point set of σ. In other words, if g is in the identity component of elements fixed by σ. We have obtained the containments

$$\text{Fix}(\sigma)^o \subset H \subset \text{Fix}(\sigma).$$

Now we must find some left invariant metric on G which is also right invariant on the fixed point set of σ. Begin by noting that the map

$$\pi : G \to S \;\; g \mapsto gp$$

is a submersion with fibers $\pi^{-1}(gp) = \{gh; h \in H\} = gH$. Therefore S is diffeomorphic to G/H, the coset space. Taking the derivative of π at e, we obtain a map of the Lie algebra $\mathfrak{g}$ of G to the tangent space of S at p. By the diffeomorphism above, $d\pi_e Ad(h)X = ad(h)d\pi_e X$ for any $h \in H$ and vector field X in the $\mathfrak{g}$, where Ad denotes the adjoint representation and ad its derivative. Hence taking the derivative of π at e is an H-equivariant linear map which is onto. The kernel is precisely the tangent space at the identity of H, which we denote by $\mathfrak{h}$. $\mathfrak{h}$ is precisely the eigenspace corresponding to the +1 eigenvalue of $d\sigma$. The complement is the -1 eigenspace, which we will call $\mathfrak{p}$. The isomorphism $d\pi|_{\mathfrak{p}} : \mathfrak{p} \to T_pS$ allows us to transfer the inner product on the tangent space of S at p to $\mathfrak{p}$. We can extend it to all of $\mathfrak{g}$ by taking an $Ad(H)$-invariant metric on $\mathfrak{h}$ and setting $\mathfrak{h}$ orthogonal to $\mathfrak{p}$. The metric is left invariant on G, right invariant with respect to H and so we are done.

Now we assume that we have G, H and σ as in the first part of the theorem and show that we have a symmetric space. First we note that the quotient G/H is homogeneous as the metric on G induces a metric on the coset space $S = G/H$. To show it is a symmetric space in the sense of our original definition, we find some g_p satisfying the symmetric space conditions at $p = eH \in S$. Because $H \subset Fix(\sigma)$, we know that H is invariant under σ and so the map $\sigma : G \to G$ induces a diffeomorphism $\widehat{\sigma} : G/K \to G/K$. As in the proof of the converse, we obtain a decomposition of the Lie algebra $\mathfrak{g}$ of G in $\mathfrak{k} + \mathfrak{p}$ as the +1/-1 eigenspaces of the derivative of σ. As a result, we have $\widehat{\sigma}(p) = p$ and $d\widehat{\sigma} = -I$ because of the following:

$$T_p(G/H) = \mathfrak{g}/\mathfrak{h} = \{X + \mathfrak{h} | X \in \mathfrak{g}\} = \{X + \mathfrak{h} | X \in \mathfrak{p}\}, \text{ and}$$

$$dg_p(X + \mathfrak{h}) = d\sigma(X) + \mathfrak{h} = -X + \mathfrak{h} \text{ for all } X \in \mathfrak{p}.$$

Finally we must show that g_p is actually an isometry. In this context we need to show that $dg_{g'p} : T_{g'H}S \to T_{\sigma(g'H)}S$ preserves the inner product for $g' \in G$. In the case when $g' = e$, this is clear as the derivative of g_p is $-I$. We extend this to arbitrary g' by the following observations:

$$g_x(g'g''H) = \sigma g' \sigma g'' H = \sigma g' g_p(g''H).$$

Hence

$$g_x \circ L_{g'} = L_{\sigma g'} \circ g_x, \text{ for any } g' \in G,$$

where $L_{g'}(g''H) = g'g''H$. Take the derivative at $p = eH$ of this equality and obtain

$$dg_{g'p} \circ (dL_{g'})_p = (dL_{\sigma g'})_p \circ dg_x.$$

Since all of the maps are bijective, and the latter three preserve inner products, so must the first. Hence g_x is an isometry and the quotient $S = G/H$ is a symmetric space in the sense of our first definition. □

The Lie algebras one encounters when dealing with symmetric spaces are very important. Cartan's aforementioned classification using our new interpretation of symmetric spaces reduces the problem of classifying symmetric spaces to that of classifying simple real Lie algebras. Cartan had already solved that problem in 1914. The classification is carried out in [He2] and is beyond the scope of this thesis. One fact about Lie algebras which is used in the classification will be of great use to us in the next chapters.

Definition 2.18. Let σ be an involution on a Lie algebra $\mathfrak{g}$. The *Cartan decomposition* of $\mathfrak{g}$ is the splitting of $\mathfrak{g}$ into two eigenspaces corresponding to the eigenvalues ± 1: $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The Cartan decomposition has the following three properties:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k},\ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \text{ and } [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

Additionally, given a Lie algebra decomposition with these properties, one can recover an involution on the Lie algebra with eigenvalues ± 1 on the components.

We have already used this idea in the proof of the theorem relating our two definitions of symmetric spaces. Indeed, from the definition of symmetric space involving an involution, we see that if $S = G/K$ is a symmetric space, then the Lie algebra of G, $\mathfrak{g}$, decomposes as the sum of $\mathfrak{k} + \mathfrak{p}$. This is a Cartan decomposition and the bracket relations hold. The first is automatic for homogeneous spaces. The second condition is the distinguishing feature of reductive homogeneous spaces, of which symmetric spaces are just one class of examples. The final condition, that $\mathfrak{p}$ brackets into $\mathfrak{k}$ is a key feature of symmetric spaces. Finally, we note that the fact that we can recover an involutive automorphism from a Cartan decomposition is a small taste of the classification program of Cartan.

One further consequence of the properties of the Cartan decomposition for symmetric spaces is the simplification of the formula for the curvature tensor usually defined in the case of homogeneous spaces. Here we will restrict ourselves to the compact case, but similar, though more complicated formulae exist for the general case. Besse's [Bes] text contains this material, while [Ber] has some examples.

Definition 2.19. Let G be a compact connected Lie group. Let G/K be a homogeneous space. Consider the Cartan decomposition of $\mathfrak{g}$ into $\mathfrak{k} + \mathfrak{p}$. Then for X and Y in $\mathfrak{p}$, the sectional curvature $K(X, Y)$ is equal to $\frac{1}{4} \|[X, Y]_{\mathfrak{k}}\|^2 + \|[X, Y]_{\mathfrak{p}}\|^2$. Here $[X, Y]$ refers to the Lie algebra bracket.

Since we are taking elements of $\mathfrak{p}$, we know, from the bracket relations, that in the symmetric case the formula simplifies. Since $\mathfrak{p}$ brackets entirely into $\mathfrak{k}$, the second term vanishes.

In the next section we will study a generalization of symmetric spaces called spherical spaces. At first glance the connection will not be apparent, as the spaces will be defined in the language of representation theory. One of our first tasks will be to show that symmetric spaces are contained in this class. We summarize the hierarchy of spaces from the thesis here:

$$\begin{gathered}\text{Symmetric Spaces} \subset \text{Spherical Spaces} \subset \text{Homogeneous Spaces} \subset \\ \subset \text{Riemannian Manifolds}\end{gathered}$$

All of the containments are proper. We are interested in the left-most on. Namely, we will look for examples which are spherical, yet not symmetric, and then analyze them.

3 Spherical Groups

In this section we introduce spherical groups. We first note where they come from and why they are worth studying. We mentioned some examples and then proceed to describe the process by which Krämer [Kr79] classified those arising as subgroups of compact simple Lie groups, with the details of a specific example worked out.

Let G be a compact Lie group and $H \subset G$ be a closed subgroup. A finite dimensional complex unitary representation is a homomorphism $\rho : G \to U_n(\mathbb{C})$. We will focus entirely on finite dimensional represenions so from here on, all representations are finite dimensional. Finite dimensional unitary representations of compact Lie groups are special because of the following properties:

- They are completely reducible, and so they are the sum of irreducible representations.
- Every finite dimensional representation of G is equivalent to a unitary one.

The first connection we make between representation theory and the material from the previous chapter is that representations can be viewed as group actions in the following sense. Let $x \in C^n$. Then for $g \in G$, the image under a unitary representation $\rho : G \to U_n(\mathbb{C})$ of g is a unitary matrix which acts on ^{n} in the natural way. With this point of view, we can import one of the useful ideas from the last chapter, namely isotropy subgroups.

Definition 3.1. Let $x \in \mathbb{C}^n$. The Isotropy subgroup of G, denoted G_x, is the set of $g \in G$ such that for a given representation ρ, $\rho(g)(x) = x$.

In [Kr76], Krämer poses the question, that given a subgroup H of G, is there an irreducible representation $\rho : G \to U_n(\mathbb{C})$ such that $\rho(H)$ is an isotropy subgroup of the operation of $\rho(G)$ on $\mathbb{C}^n$? Krämer notes that for H maximal, and for G connected and H finite the answer is yes. He then proves that for simple G and any H, the answer is yes "modulo connectedness". More precisely, when G is connected and simple, and H is a closed connected subgroup, then there exist complex irreducible representations $\rho : G \to U_n(\mathbb{C})$ with isotropy group K such that K^o, the identity component of K, is H.

In the course of proving this result, Krämer notes a remarkable property of closed subgroups of connected compact Lie groups. Given a closed subgroup $H \subset G$, either

- For any irreducible unitary representation ρ of G, the fixed point set of H under ρ is at most one dimensional, or
- For any natural number N, there exists an irreducible unitary representation ρ of G such that the dimension of the fixed point set of H under ρ is greater than N.

In other words, closed subgroups of G have either very small fixed point sets under any representation of G, or they have arbitrarily large ones. Krämer calls those subgroups H with the former property *spherical*, and the pair (G, H) with H spherical in G a *spherical pair*. The quotient space G/H is called a spherical space. The name is inspired by the groups $O(n) \subset O(n+1)$ with $O(n+1)/O(n) \cong S^n$.

In [Kr79], Krämer suggests three reasons for studying these spaces. First, in light of the alternative and the importance of unitary representations, spherical subgroups are very distinguished. Second, they help us understand the orbit structure of the representations of G. Finally, they are a generalization of the well studied Riemannian symmetric spaces, which afford us with many examples of spherical spaces. We are also curious about examples which are not symmetric spaces, which provide intuition about the difference between these classes of spaces.

- As noted above the sphere S^n is the quotient of $O(n+1)/O(n)$ and so is, as one might expect, a spherical space. It is also a symmetric space.
- The complex Grassmannian $G_k(n)$ is the set of all k-dimensional linear subspaces of $\mathbb{C}^n$. This is also a symmetric space. The group $SU(n)$ acts transitively on the set, and the isotropy group of $\mathbb{C}^k \subset \mathbb{C}^n$ is $S(U(k) \times U(n-k))$.
- The group $SU(p+q)$ also contains the subgroup $SU(p) \times SU(q)$. If $p \neq q$, then the quotient, which we denote by X, is spherical space, but it is not symmetric.

Let us digress into a brief discussion of the fact that symmetric spaces are spherical. In his paper, Kr̈amer refers to Helgason's text [He2] for the result. Helgason's approach to spherical spaces is analytic.

Definition 3.2. Let G be a compact connected Lie group and H a subgroup. The set of complex-valued continuous functions on G which are constant on the double conjugacy classes HgH for $g \in G$ form a vector space, which we denote $C^\dagger(G)$.

Using the theory of the Haar integral one can make $C^\dagger$ into an algebra with convolution as the operation. It can be shown that H is spherical in G exactly when this algebra is commutative. Helgason, in [He1], proceeds to show that any symmetric space has this property. We will discuss this and other characterizations of spherical spaces only in the appendix. For the rest of the main body of the thesis, we can stick to the first definition.

Krämer's paper [Kr79] classifies the spherical subgroups of compact connected simple Lie groups. He uses the classification of compact simple Lie groups and two technical lemmas to create a list of candidates, and then analyzes them case by case. Here we recall the lemmas and show that X is a candidate, and then outline the proof that it is indeed spherical (and not spherical in the case where $p = q$). The goal is to give an idea of Kramer's classification method.

Lemma 3.3. Let G be simple and $H \subset G$ be a connected spherical subgroup. The H-module $Ad(G)_{|H}$ is the direct sum of n real components. Then $n \leq 4$ if G is a classical group with the exception of $SO(8)$, for which n is at most 5. In the case of the exceptional groups, n is at most 3.

Lemma 3.4. Suppose G is semisimple and $H \subset G$ is a connected subgroup. If H is spherical in G, then

$$2 \cdot dim\ H + Rank\ G \geq dim\ G,$$

where the rank of a Lie group is the dimension of its Cartan subgroup.

One can interpret this lemma as saying that H has to be a relatively large subgroup. One can perhaps say that Krämer's paper is showing by inspection that simple groups G don't have too many large subgroups. This should also provide some heuristic evidence for why the classification of spherical spaces which are quotients of non-simple G might be difficult, at least without different ideas.

First we show that X satisfies these two lemmas. Without loss of generality, let $p \geq q$. From the tables in [Kr75] we see that the module $Ad(G)_{|H}$ is the direct summand of 3 real components. So X satisfies lemma 3.1.

We check lemma 3.2 with dim $H = p^2 - 1 + q^2 - 1$, Rank $G = p + q - 1$ and dim $G = (p+q)^2 - 1$

$$\begin{aligned} 2 \cdot dim\ H + Rank\ G - dim\ G &= 2 \cdot (p^2 + q^2 - 2) + p + q - 1 - ((p+q)^2 - 1) \\ &= \mathrm{p}^2 + q^2 - 4 + p + q - 2pq = (p+q)(p-q) - 4 + p + q \geq 0 \end{aligned}$$

Krämer makes heavy use of several lemmas relating spherical spaces to specific symmetric spaces. We begin with the definition of separability of subgroups of a group, and then proceed with a lemma.

Definition 3.5. Let F and H be subgroups of a compact connected Lie group with $F \subset H \neq G$. F and H are separable if there exists a simple G-module V with the following property: there exist nontrivial fixed points of V under the action of H, and nontrivial fixed points of V under F which are not fixed under H.

Lemma 3.6. Let G be a compact connected Lie group with F and H subgroups. Furthermore, suppose that $F \subset H$ and H is connected. Then the following are equivalent.

1. F and H are separable.
2. The dimension of a principle orbit of the natural operation of H on G/H is greater than the dimension of a principle orbit of the restriction of this operation to F.

A further lemma allows the application of the previous lemma to our situation.

Lemma 3.7. Let G be a compact connected Lie group and let $H \subset G$ be a Riemannian symmetric pair. Denote by t the rank of G/H as a symmetric space. Then the dimension of a principle orbit of the natural operation of H on G/H equal to $dim\ G/H - t$.

One final lemma provides a criterion allowing us to use separability and symmetric spaces to show that a subgroup of G is spherical if it is separable from a symmetric subgroup of G.

Lemma 3.8. Let F, H, G be compact connected Lie groups with $F \subset H \subset G$ and F the semisimple part of H. Suppose $H \subset G$ is spherical. Then $F \subset G$ is spherical if and only F and H are not separable in G.

These lemmas are useful in our situation, and indeed in the situation of nearly all of the other candidates, as our subgroup $F = SU(p) \times SU(q)$ is contained in another subgroup of $SU(p+q)$, namely $H = S(U(p) \times U(q))$, and the quotient $SU(p+q)/S(U(p) \times U(q))$ is a symmetric space. By Lemma 3.5, it suffices to check that F and H are separable.

Theorem 3.9. The space $X = SU(p+q)/SU(p) \times SU(q)$, for $p \neq q$ is spherical.

Proof. We denote $SU(p+q)$ by G. First note that $\text{Rank}G/H = q$ and $\text{dim}G/H = 2pq$. It follows from Lemma 3.4 that $\dim O(H) = 2pq - q$, where $O(H)$ denotes the principle orbit of the natural operation of H on G/H.

To study the operation of $F = SU(p) \times SU(q)$ on the quotient G/F, we must introduce a new concept.

Definition 3.10. The *realification* of a complex vector space of dimension n, $\mathbb{C}^n$, is the real vector space that is the same as $\mathbb{C}^n$ as a group, with multiplication by real scalars defined as usual, but multiplication by complex scalars not defined.

Now consider the usual representation of $SU(p)$ on $\mathbb{C}^p$. For $SU(q)$ we consider the contragradient (or dual) of its usual representation on $\mathbb{C}^q$. Now we can describe the operation of F as the realification of the complex tensor product of the above two representations.

Now denote by $U(1)^q$ the q-fold product of the group $U(1)$ in the multiplicative group $(\mathbb{C}^*)^q$. By $S(U(1)^q)$ we mean the subgroup of $U(1)^q$ in which the product of the entries is 1. Specifically those elements $t = (t_1, t_2, \ldots, t_q) \in U(1)^q$ with $\prod_{i=1}^q t_1 = 1$.

It is a fact from [HH] that the isotropy group of F is isomorphic to $SU(p-q) \times S(U(1)^q)$. So the isotropy group of F has dimension $(p-q)^2 + (q-2)$ for $p > q$ and dimension $q - 1$ when $p = q$. Hence the dimension of $O(F)$ is $2pq - q$ which is also the dimension of $O(H)$ for $p > q$. For the case $p = q$, $\text{dim}O(F) = 2pq - q - 1 = \dim O(H) - 1$.

From here we apply the lemmas. By Lemmas 3.3 and 3.4, we see that the dimension of the principle orbit of F relative to H determines separability, and hence if F is spherical. In the case that $p = q$, we see that $\text{dim}O(F)$ is less than $\text{dim}O(H)$. By Lemma 3.3, F and H are separable, hence by Lemma 3.5 F is not a spherical subgroup. In the case $p \neq q$, the principle orbits have the same dimension, hence, by Lemma 3.3, F and H are separable and so, by Lemma 3.5, F is a spherical subgroup in G. $\square$

From now on, we will assume that p is greater than q. Most of the other spherical subgroups of the compact simple Lie groups in Krämer's paper are verified in a similar way.

In [Ng], the spherical pairs in Krämer's list are split into six families. The space X is one of three S^1-bundles over hermitian symmetric spaces. In the case of X, the hermitian symmetric space is the complex Grassmannian. The other spherical pairs are $SU(n) \subset SO(2n)$ for $n > 3$ and odd, and $D_5 \subset E_6$ where D_5 is $Spin(10)$. As we have seen from this section, X's relationship to the Grassmannian helps us understand it. In the following two sections we will continue the process. First we will carry out some calculations in the decomposition of the Lie algebra corresponding to X. Later we will relate the homotopy groups and cohomology groups of the Grassmannian to those of X.

4 Lie Algebra Calculations

In this section we continue our study of the spherical space $X = SU(p+q)/SU(p) \times SU(q)$. We are going to perform several calculations in the Lie algebra decomposition of $\mathfrak{su}(p+q)$ arising from this quotient. To fix the notation, the decomposition of $\mathfrak{su}(p+q)$ corresponding to this quotient will be denoted $(\mathfrak{su}(p) \oplus \mathfrak{su}(q)) \oplus \mathfrak{m}$. We have two distinct motivations. The first comes from the curvature formula for homogeneous spaces defined in chapter 2. We recall the formula here:

Definition 4.1. Let G be a compact connected Lie group. Let G/K be a homogeneous space. Consider the Cartan decomposition of $\mathfrak{g}$ into $\mathfrak{k} + \mathfrak{p}$. Then for X and Y in $\mathfrak{p}$, the sectional curvature $K(X,Y)$ is equal to $\frac{1}{4} \|[X,Y]_{\mathfrak{k}}\|^2 + \|[X,Y]_{\mathfrak{p}}\|^2$. Here $[X,Y]$ refers to the Lie algebra bracket.

In chapter two we noticed that because of Cartan decomposition of symmetric spaces, this formula simplifies, with the second term vanishing. For X, and indeed all non-symmetric spherical spaces, this term does not vanish. For this reason we would like to calculate $[Y,Z]_{\mathfrak{m}}$, the commutator of two elements of $\mathfrak{m}$ composed with projection onto $\mathfrak{m}$. The hope is that the operator is not too complicated and that the sectional curvature calculation can be carried out.

We are also interested in writing down the operator which arises when $Y \in \mathfrak{m}$ above is fixed. The motivation for this calculation is to facilitate functional calculus in relation to the space X. By functional calculus we mean being able to apply functions like the exponential function or trigonometric functions to elements of $\mathfrak{m}$. If, for example, the operator winds up being diagonalizable, it is very simple to take powers of it, as one might like to do when calculating a Taylor series expansion. This sort of information is valuable in calculations arising in the theory of Hilbert families mentioned in the introduction of the paper. Specifically, the recent paper of Szőke and Lempert [LSz10] comes to several conclusions about the Hilbert families associated to symmetric spaces by analogous calculations. In this section we will calculate these two operators.

We have seen that the complex Grassmanian can be realized as the quotient $SU(p+q)/S(U(p) \times U(q))$. As the Grassmanian is a symmetric space, we obtain the Cartan decomposition of the Lie algebra:

$$\mathfrak{su}(p+q) = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{k}$ is the Lie algebra of $S(U(p) \times U(q))$ and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{su}(p+q)$, and $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$. We note here that $\mathfrak{p}$ can be interpreted as the tangent space of the Grassmannian. We should also define the inner product to which we have implicitly referred when we used the word orthogonal complement:

Definition 4.2. Let I and J be arbitrary elements of $\mathfrak{su}(p+q)$. The map $h(I,J) = -\frac{1}{2}\mathrm{Tr}(IJ)$ defines an inner product on $\mathfrak{su}(p+q)$.

From the paper of Krämer we have another interesting quotient of $SU(p+q)$, namely the space X, which we have related to the Grassmannian:

$$X = SU(p+q)/SU(p) \times SU(q).$$

We again decompose the Lie algebra and obtain

$$\mathfrak{s}u(p+q) = (\mathfrak{s}u(p) \oplus \mathfrak{s}u(q)) \oplus \mathfrak{m}.$$

We have noted that this space is a spherical space which is not symmetric. On the Lie algebra level we see the contrast by noting that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, while $[\mathfrak{m}, \mathfrak{m}] \not\subset (\mathfrak{s}u(p) \oplus \mathfrak{s}u(q))$. This is a significant departure from the realm of symmetric spaces, as this specific bracket property makes calculations in the Lie algebra of the symmetric space easier. For example, for Y and Z in $\mathfrak{m}$, $[Y, Z]_{\mathfrak{m}}$ would be 0 in a symmetric space. In some sense, the complexity of this operator gives us an idea of how far this spherical space is from being symmetric in a concrete way. We now proceed to calculate this operator, first fixing notation.

Definition 4.3. Let Y and Z be in $\mathfrak{m}$. Denote by $\eta_{\mathfrak{m}}$ the following map

$$\eta_{\mathfrak{m}} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$$

$$Y \times Z \to [Y, Z]$$

We denote projection of elements from $\mathfrak{s}u(p+q)$ onto $\mathfrak{m}$ by $P_{\mathfrak{m}}$.

To write down an element of $\mathfrak{s}u(p+q)$ in terms of the spherical space decomposition we note that elements in $(\mathfrak{s}u(p) \times \mathfrak{s}u(q))$ are of the form

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$$

where A is in $\mathfrak{s}u(p)$ and B is in $\mathfrak{s}u(q)$.

Elements in $\mathfrak{m}$ are more complicated. A generic element looks like this:

$$Y = \left[\begin{array}{c|c} \frac{c\cdot(I_p)}{p} & M \\ \hline -M^* & \frac{-c\cdot(I_q)}{q} \end{array}\right].$$

Here c denotes a purely imaginary complex number, I_p and I_q denote the identity p by p and q by q matrices, respectively, and M^* is the conjugate transpose of the matrix M. To justify this, keep in mind that we are working in $\mathfrak{s}u(p+q)$. Anything not in $\mathfrak{s}u(p) \oplus \mathfrak{s}u(q)$ must be in $\mathfrak{m}$. Hence the top right block can be anything. Since elements of this Lie algebra must equal their own conjugate transpose, picking the top right determines uniquely the bottom left. As for the elements in the diagonal blocks, we must include those matrices which have non-zero traces in their top left and bottom right blocks, taken separately, but trace zero when combined. These matrices are not in $\mathfrak{s}u(p) \oplus \mathfrak{s}u(q)$, as both blocks are assumed to have trace 0. Therefore we include them in $\mathfrak{m}$.

We now fix Z as another generic element of $\mathfrak{m}$ and proceed to the first calculation.

$$Z = \left[\begin{array}{c|c} \frac{d\cdot(I_p)}{p} & N \\ \hline -N^* & \frac{-d\cdot(I_q)}{q} \end{array}\right]$$

Theorem 4.4. Define $u := \frac{1}{p} + \frac{1}{q}$. Then

$$P_{\mathfrak{m}} \circ \eta_{\mathfrak{m}}(Y,Z) = \left[\begin{array}{c|c} \frac{\operatorname{tr}(NM^*-MN^*)}{p} I_p & u(cN-dM) \\ \hline u(cN^*-dM^*) & \frac{-\operatorname{tr}(NM^*-MN^*)}{q} I_q \end{array}\right].$$

Proof. First we will take the bracket, which in our case is just the commutator, and then later project to $\mathfrak{m}$.

$$[Y,Z] = YZ - ZY = \left[\begin{array}{c|c} NM^*-MN^* & u(cN-dM) \\ \hline u(cN^*-dM^*) & N^*M-M^*N \end{array}\right]$$

Refering to the decomposition of $\mathfrak{su}(p+q)$ into $\mathfrak{k} \oplus \mathfrak{p}$, we decompose this matrix into

$$\left[\begin{array}{c|c} NM^*-MN^* & 0 \\ \hline 0 & N^*M-M^*N \end{array}\right] + \left[\begin{array}{c|c} 0 & u(cN-dM) \\ \hline u(cN^*-dM^*) & 0 \end{array}\right],$$

where the first matrix is in $\mathfrak{k}$ and the second is contained in $\mathfrak{p}$, hence also in $\mathfrak{m}$. To complete the calculation we need to project the first matrix to $\mathfrak{m}$. We need to discard any nondiagonal entries, and we must make sure that the trace of the upper left block is the negative of the trace of the bottom left block. This is the same reasoning we applied when we first described elements of $\mathfrak{m}$, namely that we must include diagonal matrices with trace 0, whose top left and bottom right blocks have non-zero trace. We arrive at the following

$$P_{\mathfrak{m}}(\left[\begin{array}{c|c} NM^*-MN^* & 0 \\ \hline 0 & N^*M-M^*N \end{array}\right]) = \left[\begin{array}{c|c} \frac{\operatorname{tr}(NM^*-MN^*)}{p} I_p & 0 \\ \hline 0 & \frac{-\operatorname{tr}(NM^*-MN^*)}{q} I_q \end{array}\right].$$

Combining this projection with the $\mathfrak{p}$-part of the commutator, we arrive at the formula

$$P_{\mathfrak{m}} \circ \eta_{\mathfrak{m}}(Y,Z) = \left[\begin{array}{c|c} \frac{\operatorname{tr}(NM^*-MN^*)}{p} I_p & u(cN-dM) \\ \hline u(cN^*-dM^*) & \frac{-\operatorname{tr}(NM^*-MN^*)}{q} I_q \end{array}\right].$$

$\square$

Now we shift our attention to the operator when Y is fixed, i.e. the adjoint representation of Y. In our notation we would like to calculate $P_{\mathfrak{m}} \circ ad_Y(\mathfrak{m})$ which we abbreviate to T_Y. When we fix Y we obtain a decomposition of $\mathfrak{m}$

$$\mathfrak{m} = e_1 \oplus e_2 \oplus e_3 \oplus (e_2 \oplus e_3)^{\perp_{\mathfrak{p}}}$$

where

$$e_1 = \left[\begin{array}{c|c} \frac{i\cdot(I_p)}{p} & 0 \\ \hline 0 & \frac{-i\cdot(I_q)}{q} \end{array}\right],$$

$$e_2 = \left[\begin{array}{c|c} 0 & M \\ \hline -M^* & 0 \end{array}\right],$$

and

$$e_3 = Je_2 = \left[\begin{array}{c|c} 0 & iM \\ \hline -(iM)^* & 0 \end{array}\right],$$

where $J = \mathfrak{p} \to \mathfrak{p}$ is the complex structure on $\mathfrak{p} = T_*(Gr_n(k))$:

$$J : \left[\begin{array}{c|c} 0 & N \\ \hline -N^* & 0 \end{array}\right] \mapsto \left[\begin{array}{c|c} 0 & iN \\ \hline -(iN)^* & 0 \end{array}\right].$$

Theorem 4.5. With respect to this decomposition of $\mathfrak{m}$,

$$\mathrm{T}_Y(\mathfrak{m}) = \left[\begin{array}{ccc|c} 0 & 0 & 2 \cdot \mathrm{tr}(MM^*) & \\ 0 & 0 & ua & \mathbf{0} \\ -u & ua & 0 & \\ \hline & \mathbf{0} & & uaJ \end{array}\right].$$

Proof. We calculate the images of e_1, e_2, e_3, and v, where v is an arbitrary member of $(e_2 \oplus e_3)^{\perp_{\mathfrak{p}}}$. We define a as the real constant $c = ai$ for as c is purely imaginary.

$$T_Y(e_1) = P_{\mathfrak{m}}([Y, e_1]) = -ui \left[\begin{array}{c|c} 0 & M \\ \hline M^* & 0 \end{array}\right] = -u \left[\begin{array}{c|c} 0 & iM \\ \hline -(iM)^* & 0 \end{array}\right] = -uJe_2 = -ue_3.$$

$$T_Y(e_2) = P_{\mathfrak{m}}[Y, e_2] = cu \left[\begin{array}{c|c} 0 & M \\ \hline M^* & 0 \end{array}\right] = ua \left[\begin{array}{c|c} 0 & iM \\ \hline -(iM)^* & 0 \end{array}\right] = uaJe_2 = uae_3.$$

$$T_Y(e_3) = P_{\mathfrak{m}}[Y, e_3] = P_{\mathfrak{m}}(\left[\begin{array}{c|c} iMM^* - iMM^* & cuiM \\ \hline cu(iM)^* & (iM)^*M - iMM^* \end{array}\right]) =$$

$= P_{\mathfrak{m}}(\left[\begin{array}{c|c} iMM^* - iMM^* & 0 \\ \hline 0 & (iM)^*M - iM^*M \end{array}\right] + \left[\begin{array}{c|c} 0 & cuiM \\ \hline cu(iM)^* & 0 \end{array}\right]) = 2 \cdot \mathrm{tr}(MM^*)e_1 +$
$uae_2.$

$$T_Y(v) = P_{\mathfrak{m}}[Y, v] = P_{\mathfrak{m}}(\left[\left[\begin{array}{c|c} \frac{c \cdot (I_p)}{p} & M \\ \hline -M^* & \frac{-c \cdot (I_q)}{q} \end{array}\right], \left[\begin{array}{c|c} 0 & N \\ \hline -N^* & 0 \end{array}\right]\right]) =$$

$$= P_{\mathfrak{m}}(\left[\begin{array}{c|c} NM^* - MN^* & cuN \\ \hline cuN^* & NM^* - M^*N \end{array}\right]) = \left[\begin{array}{c|c} NM^* - MN^* & 0 \\ \hline 0 & NM^* - M^*N \end{array}\right] + \left[\begin{array}{c|c} 0 & cuN \\ \hline cuN^* & 0 \end{array}\right]$$

When projecting this onto $\mathfrak{m}$, the first component disappears, as v is orthogonal to e_1 and ensuring that the trace of the projection is 0 leads to taking the inner product. The second component becomes

$$ua \left[\begin{array}{c|c} 0 & iN \\ \hline -(iN)^* & 0 \end{array}\right] = uaJv.$$

To summarize, we review the results and write down the matrix of T_Y.

- $T_Y(e_1) = -uJe_2 = -ue_3$
- $T_Y(e_2) = uaJe_2 = uae_3$
- $T_Y(e_3) = 2 \cdot \mathrm{tr}(MM^*)e_1 + uae_2$
- $T_Y(v) = uaJv$

$$T_Y(\mathfrak{m}) = \left[\begin{array}{ccc|c} 0 & 0 & 2 \cdot \mathrm{tr}(MM^*) & \\ 0 & 0 & ua & \mathbf{0} \\ -u & ua & 0 & \\ \hline & \mathbf{0} & & uaJ \end{array}\right].$$

□

The eigenvalues of the top left 3 by 3 matrix are 0 and $\pm\sqrt{(ua)^2 - 2u||M||}$, where $||M|| = \mathrm{tr}(MM^*)$.

In this section we made two primary calculations in the Lie algebra decomposition of X. The first was motivated by the formula for curvature of a homogeneous space. The second calculation gave us an explicit reprepresentation of the operator for the adjoint representation of an element of $\mathfrak{m}$ projected back to $\mathfrak{m}$. This kind of matrix is useful in performing calculations arising in the field of geometric quantization. Throughout we made use of and reference to the Cartan decomposition of the Grassmannian. Both of the calculations carried out here would have yielded 0 if we had considered the Grassmannian instead of X. In the next section, we will consider topological invariants, again comparing X with the Grassmannian.

5 Topological Calculations

It is a theorem that if M is a hermitian symmetric space of compact type, then M is simply connected [He2]. There are many results linking the curvature of a manifold to its topological invariants. Classically, one of the first such results was Gauss' calculation of the linking number of two curves by a double integral. Another example is the Gauss-Bonnet formula, which relates the Gaussian curvature and the Euler characteristic of a space. Besides their relationships to curvature, topological invariants have many other uses and are worth calculating in their own right. In this section we will calculate some cohomology and homotopy groups of the space X. We make extensive use of the fact that X is a principal S^1 bundle over $Gr_k(n)$. This allows us to relate the cohomology groups of these two spaces using the Gysin sequence. For background on the topological material, see [BT] or [Br]. Specific homotopy groups of Lie groups come from [Hu].

A chain of closed subgroups $H \subset K \subset G$ yields a homogeneous fiber bundle

$$K/H \longrightarrow G/H \longrightarrow G/K$$

Furthermore, if H is normal in K, it is a principal K/H bundle [Bredon/DeVito]. In our case, $H = SU(p) \times SU(q)$, $K = S(U(p) \times U(q))$ and $G = SU(p+q)$. K/H is ismorphic as a Lie group to $U(1)$, the circle group, by sending an element of H to the determinant of the $U(p)$ part. So X is a principal S^1 bundle over $Gr_k(n)$.

First we calculate some homotopy groups. We consider the two fibrations

$$(1) \qquad S(U(P) \times U(q)) \longrightarrow SU(p+q) \longrightarrow Gr_k(n)$$

$$(2) \qquad SU(P) \times SU(q) \longrightarrow SU(p+q) \longrightarrow X$$

These fibrations give rise to a long exact sequence of homotopy groups

$$\begin{aligned}
(1) \ldots &\longrightarrow \pi_n(S(U(p) \times U(q))) \longrightarrow \pi_n(SU(p+q)) \longrightarrow \pi_n(Gr_k(n)) \\
&\longrightarrow \pi_{n-1}(S(U(p) \times U(q))) \longrightarrow \quad \cdots \quad \longrightarrow \pi_2(Gr_k(n)) \\
&\longrightarrow \pi_1(S(U(p) \times U(q))) \longrightarrow \pi_1(SU(p+q)) \longrightarrow \pi_1(Gr_k(n)) \\
&\longrightarrow \quad \cdots
\end{aligned}$$

$$\begin{aligned}
(2) \ldots &\longrightarrow \pi_n(SU(p) \times SU(q)) \longrightarrow \pi_n(SU(p+q)) \longrightarrow \pi_n(X) \\
&\longrightarrow \pi_{n-1}(SU(p) \times SU(q)) \longrightarrow \quad \cdots \quad \longrightarrow \pi_2(X) \\
&\longrightarrow \pi_1(SU(p) \times SU(q)) \longrightarrow \pi_1(SU(p+q)) \longrightarrow \pi_1(X) \\
&\longrightarrow \quad \cdots
\end{aligned}$$

Some of these groups are well known. We summarize those that we use in the following table.

Table 1: Homotopy Groups of Spaces in the Fibrations

Space	π_1	π_2	π_3
$SU(p+q)$	0	$\mathbb{Z}$	0
$SU(p) \times SU(q)$	0	$\mathbb{Z} \times \mathbb{Z}$	0
$S(U(p) \times U(q))$	$\mathbb{Z}$	?	?

Plugging what we know into the first long exact sequence, we see that the Grassmannian is simply connected and that $\pi_2(Gr_k(n)) = \mathbb{Z}$. By the Hurwicz theorem, $H^2(Gr_k(n)) = \mathbb{Z}$. In the second long exact sequence we have

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_3(SU(p)\times SU(q)) \arrow[r] & \pi_3(SU(p+q)) \arrow[r] & \pi_3(X) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
 & \pi_2(SU(p)\times SU(q)) \arrow[r] & \pi_2(SU(p+q)) \arrow[r] & \pi_2(X) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
 & \pi_1(SU(p)\times SU(q)) \arrow[r] & \pi_1(SU(p+q)) \arrow[r] & \pi_1(X) \arrow[r] & .
\end{tikzcd}$$

plugging in we get

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}\times\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \pi_3(X) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
 & 0 \arrow[r] & 0 \arrow[r] & \pi_2(X) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
 & 0 \arrow[r] & 0 \arrow[r] & \pi_1(X) \arrow[r] & {}
\end{tikzcd}$$

and we see $\pi_3(X) = \mathbb{Z}$, $\pi_2(X) = 0$, and $\pi_1(X) = 0$. By the Hurwicz theorem, $H^2(X)$ is 0.

Now we must introduce a notion from algebraic topology which will allow us to relate the Grassmannian and X. The idea begins with the fact that X is a principal S^1 bundle over the Grassmannian. There is a great deal of interaction between the topology of the bundles of a space and the underlying space itself. For instance, one can define the orientability of a smooth manifold as the orientability of its tangent bundle, as a bundle. One particular topological invariant of a vector bundle is known as the Euler class.

Definition 5.1. Let $E \to B$ be a real oriented vector bundle of rank r, with B the base space. Let $s : B \to E$ be a section such that $s(B)$ and the zero section intersect transversely. The intersection, which we call I, is a dimension $dim(B) - r$ submanifold of E, which, by inclusion into the zero section, can be thought of a submanifold of E. The *Euler class* is the Poincaré dual in B of I in $H^r(B)$.

It is not apparent from the definition what the use of the Euler class is. It is often mentioned that it measures how twisted the vector bundle is. As it is traditional to view cohomology classes as obstructions, we can think of the Euler class's existence as an obstruction to being able to implement the analogues of polar coordinates on trivializations of the bundle.

In the case where $E = \mathbb{R}^2 \times B$ is a trivial vector bundle with projection onto B, then we define the form ϕ as the pullback of $\frac{1}{2\pi}d\theta$ under the projection map defined everywhere but the origin. The Euler class χ is defined by

$$d\phi = -\pi^*\chi.$$

Here χ is zero since ϕ is closed. This is because we can choose polar coordinates everywhere. In more complicated situations, the Euler class detects the failure to patch together angular coordinates on triple intersections. For more detail on this approach to the Euler class see [BT].

Given the name Euler class and being represented by the symbol χ, the reader might wonder if the Euler class is connected to the Euler characteristic. It is a fact that in the case of the tangent bundle of a smooth manifold the Euler class of the bundle is precisely the Euler characteristic of the manifold. Hence we view the Euler class as the generalization of the Euler characteristic.

The Euler class of X as a principal S^1-bundle over the Grassmannian is a generator of $H^2(Gr_k(n))$. Knowing the Euler class allows us to use the Gysin sequence.

Definition 5.2. Let $\pi : E \to M$ be a fiber bundle with fiber S^k. Then there exists a long exact sequence of cohomology groups called the Gysin sequence:

$$\ldots \longrightarrow H^n(E) \xrightarrow{a} H^{n-k}(M) \xrightarrow{\wedge e} H^{n+1}(M) \xrightarrow{\pi_*} H^{n+1}(E) \ldots$$

Where a denotes integration along the fiber, $\wedge e$ denotes multiplication by the Euler class, and π_* is the pullback.

In our case, $k = 1$, and we know the generator of the Euler class. We obtain the following long exact sequence, which will enable us to calculate $H^*(X)$ in terms of the Grassmannian's cohomology ring, a complicated, but well studied object.

$$\begin{tikzcd}
0 \arrow[r] & H^1(X) \arrow[r] & H^0(Gr_k(n)) \arrow[r] & H^2(Gr_k(n)) \arrow[dll, rounded corners] \\
& H^2(X) \arrow[r] & H^1(Gr_k(n)) \arrow[r] & H^3(Gr_k(n)) \arrow[dll, rounded corners] \\
& H^3(X) \arrow[r] & H^2(Gr_k(n)) \arrow[r] & H^4(Gr_k(n)) \arrow[r] & \ldots
\end{tikzcd}$$

We plug in the previously calculated cohomology groups, and note that the odd cohomology groups of the Grassmannian are 0.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[dll, rounded corners] \\
& 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, rounded corners] \\
& H^3(X) \arrow[r] & \mathbb{Z} \arrow[r] & H^4(Gr_k(n)) \arrow[dll, rounded corners] \\
& H^4(X) \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H^5(X) \arrow[r] & \ldots
\end{tikzcd}$$

The cohomology ring of the Grassmannian is a difficult object to understand and is beyond the scope of this work. We refer the read to [BT] for further details, taking only the above assumption that the odd cohomology groups of the Grassmannian vanish, and, for illustrative purposes the fact that $H^4(Gr_k(n)) = \mathbb{Z}^2$. Plugging in we see that $H^3(X) = \mathbb{Z}$ and $H^4(X) = \mathbb{Z}^2$. In general, given the ability to calculate the cohomology groups of $Gr_k(n)$, we can calculate those of X. The cohomology groups of the Grassmannian are combinatorially complicated, but calculatable, especially with the assistance of computer packages like SAGE.

In this section we calculated some topological invariants of X. As X is a homogeneous space, we can use the long exact sequence of homotopy groups arising from the fibration to calculate several homotopy groups of X. Notably, we see that X is simply connected. Then, using the Hurwicz theorem, we obtained the first couple cohomology groups of X. We carried out the same calculations with the Grassmannian, and then, as in the previous sections, related X to the Grassmannian. Specifically, we noted that X is a principal S^1 bundle over the Grassmannian. We noted that the Euler class of X as a principal S^1 bundle over the Grassmannian is the generator for the second cohomology group of $Gr_n(k)$. Hence, using the Gysin sequence, we were able to calculate the cohomology of X completely in terms of the cohomology of the Grassmannian.

6 Conclusion

This thesis was an attempt to introduce an interesting class of spaces with many connections to other objects in mathematics. Given this motivation, it should be clear that any effort to understand these spherical spaces must involve relating them to other objects. In the thesis, this involved both general theory, for instance examining the containments

$$\text{symmetric spaces} \subset \text{spherical spaces} \subset \text{homogeneous spaces},$$

and working with the specific examples X and the Grassmannian.

With the benefit of hindsight, let us now review the reasons why it is nice to study these spaces. First recall that spherical spaces are a special case of homogeneous spaces. Homogeneous spaces occur as manifolds on which a given group has a special action. Specifically the group action must be transitive. Given the fruitfulness of the theory of group actions throughout mathematics, it should not be surprising that there are many spaces which arise this way, and that a great deal can be said about them. On the other hand, more can be said about specific subclasses. One such naturally occuring class of spaces are the symmetric spaces. They are very well studied, and the irreducible and semisimple symmetric spaces are classified. We saw previously that spherical spaces fit exactly between symmetric spaces and homogeneous spaces. Hence spherical spaces stimulate mathematical curiousity both as generalizations and as special cases.

We saw several specific examples of this idea in the thesis. For example, the formula for the curvature of a homogeneous space is somewhat complicated. In the symmetric space it simplifies, and we carried out the 'extra' calculations needed for a the curvature formula of the specific spherical space X. In the section on topological invariants, we made use of both the long exact sequence of homotopy groups associated to any fibration coming from a homogeneous space and the close relationship that X had to a specific symmetric space to calculate many invariants of X. Indeed, this sandwich effect has proven useful several times. It would like be much more difficult to say as much about X using only its representation theoretic characterization as a spherical space.

Let us now mention again a specific motivation for studying spherical spaces as generalizations of symmetric spaces, namely geometric quantization, with a few more details taken from [LSz10]. We described the process of geometric quantization as associating to a Riemannian manifold a family of Hilbert spaces 'above' the manifold. Let us make this more precise now: we associate to the cotangent bundle X of a Riemannian manifold M a Hermitian line bundle $L \to X$, and a Hilbert space H of its sections. The cotangent bundle has a canonical symplectic structure. There is not always a canonical way to find L; choices are made involving the complex structure and so one must deal with a family of line bundles and Hilbert spaces, parametrized by the choices. In the case when the set of choices form a complex manifold S, Hitchen in [Hi] and Axelrod, Della Pietra, and Witten [ADW] decided to view the Hilbert spaces as fibers of a holomorphic Hilbert bundle $H \to S$. Then, using connection and parallel transport they attempted to identify the individual Hilbert spaces as fibers. The curvature of the connection is important in this instance, as it measures how transport between fibers varies between paths. When this curvature is a scalar operator the Hilbert spaces are isomorphic. This is not always the case. In this sense, view the curvature as an obstruction to uniformity among the Hilbert spaces. In the case of compact symmetric spaces, special techniques can be used to better understand this situation. For example, in [LSz12] Lempert and Szőke come to some results about compact, simply connected symmetric spaces of rank-1. It is hoped that these methods can be extended, with perhaps slight modifications to a new level of generality. We are referring here to compact spherical spaces. Understanding operators like T_Y calculated in Section 4 helps us understand a more general case and form hypotheses about similar cases.

7 Appendix Spherical Spaces Throughout Mathematics

There are several important characterizations of spherical spaces which arise in different fields of mathematics. In this thesis we considered spherical spaces G/H in the following sense: the set of invariant vectors of H under any irreducible representation of G is a vector space of dimension 1 or 0. We are now going to mention alternative characterizations, where they come from, and why they are useful. We will also examine a related situation in the following sense. Throughout the thesis G was a compacted connected Lie group and H was a closed subgroup; the analogous algebraic situation is when G is a reductive connected algebraic group and H is an algebraic subgroup. We will survey new results and new questions related to spherical spaces. Finally we will survey recent results and papers taking any of these approaches, and mention some open questions.

7.1 A Survey of the Characterizations

Theorem 7.1. Let G be a compact Lie group, let H be a closed subgroup. The following are equivalent:

1. G/H is spherical
2. For any irreducible unitary representation ρ of G, the fixed point set of H under is at most one dimensional
3. The multiplicity of any continuous irreducible representation of G in $L^2(G/H)$ is at most one dimensional
4. The algebra of complex-valued continuous functions on G which are constant on the double coset space HgH for $g \in G$ with convolution, denoted $C^\dagger(G)$, is commutative.
5. The Poisson algebra of G-invariant functions on the cotangent bundle of G/H is commutative

In [Ti] all of these equivalences and more are listed and their origins are cited. What is clear is that spherical spaces come up in many different areas of math and that many different tools can be used to study them.

7.2 As Algebraic Groups: The Theory of Spherical Varieties

In 1986 Brion, Luna, and Vust [BLV] wrote a paper considering the algebraic analogue of spherical spaces.

Definition 7.2. An *algebraic group* is a group that is an algebraic variety with group multiplication and inversion given by regular functions.

We say that an algebraic group G over an algebraically closed field is *reductive* if the unipotent radical (i.e. the group of unipotent elements of the radical of G) is trivial. Any semisimple algebraic group is reductive. For more background information on algebraic groups, see [Bo] and [Sp]. Now let G be a connected reductive algebraic group over an algebraically closed field of characteristic 0, and H a closed algebraic subgroup. Brion, Luna, and Vust mention the following characterizations of spherical varieties.

Theorem 7.3. The following are equivalent:

1. G/H is a spherical variety.

2. The group H has an open orbit in G/B, where B is a Borel subgroup of G.

3. H acts on G/B with finitely many orbits.

4. There exists an element θ of the automorphism group of G that fixes H with the property that for any $g \in G$, $\theta(g) \in Hg^{-1}H$.

When appropriately translated into the language of algebraic groups, the different notions of spherical spaces discussed in the previous section apply to spherical varieties as well. Examples of spherical varieties include the following

- When G is a torus and H is trivial.

- When H is the set of fixed points of an involution of G, called symmetric varieties.

- When H contains a maximal unipotent subgroup of G, called horospherical varieties, of which flag varieties are an example.

The theory of spherical varieties unifies these specific ideas. Luna and Vust [LV] classified all embeddings of spherical varieties as a special case of several results on the embeddings of arbitrary homogeneous varieties. Since then the program of so-called Luna-Vust theory has been to classify and study spherical properties using algebraic geometry and combinatorics arising from generalizations of the theories of flag varieties and toric varieties. There are several informal lecture notes available on the web aimed at different backgrounds and with different goals in mind.

We now discuss briefly the relationship between algebraic groups and Lie groups. Specifically, we will compare compact real Lie groups with reductive algebraic groups over $\mathbb{C}$.

Theorem 7.4. The following hold

1. To any compact group G there exists a canonically associated reductive linear algebraic group G' having the representative functions T_G as the coordinate ring $\mathcal{O}(G')$.

2. G' has the "same" representations as G, meaning you can obtain any finite dimensional representation of G by restricting from G'.

3. G is a maximimal compact subgroup of G', and G is Zariski-dense in G'.

4. If V is faithful as a G-module, it is faithful as a G'-module.

5. For any G-invariant Hermitian scalar product on V, G' is self-adjoint.

A good example is the relationship between $SU(2)$ the Lie group and $SL_2(\mathbb{C})$ the linear algebraic group. Now some more general statements:

Theorem 7.5.

1. If $G \subset GL_n(\mathbb{C})$ is a self-adjoint Lie group, with finitely many connected components and a complex Lie algebra, then G is a reductive algebraic group.

2. Conversely, given a reductive algebraic group G and a representation of G on V, there exists a Hermitian scalar product on V such that G is self-adjoint.

3. If V is faithful as a G-module, then the unitary elements in G form a maximal compact subgroup K and $G = K \cdot e^{i\mathfrak{K}}$ (this is sometimes called polar decomposition).

4. All maximal compact subgroups of G are conjugate in G, and every compact Lie group arises that way.

We summarize these theorems and the different passages between Lie groups and algebraic groups in the following diagram:

$$\begin{tikzcd}
\begin{array}{c}\text{Compact Real}\\ \text{Lie Groups}\end{array} \arrow[r] & \begin{array}{c}\text{Compact Real}\\ \text{Lie Algebra}\end{array} \arrow[r, "1"] & \begin{array}{c}\text{Finite Dimensional}\\ \text{Complex Lie Algebra}\end{array} \arrow[d, "2"] \arrow[dl, "3"] \\
\begin{array}{c}\text{Reductive Complex}\\ \text{Linear Algebraic}\\ \text{Group*}\end{array} \arrow[r] & \begin{array}{c}\text{Simply Connected}\\ \text{Cover}\end{array} \arrow[r] \arrow[ul, "4"] & \begin{array}{c}\text{Complex Linear}\\ \text{Algebraic Group}\end{array}
\end{tikzcd}$$

The * on reductive complex linear algebraic group denotes that we only consider those with simple Lie algebras. The numbered arrows describe the following:

1. Complexification of the real Lie algebra,

2. Taking the adjoint group,

3. The one to one correspondence of representations,

4. Passing to maximal compact subgroups.

Hopefully it is now apparent that it is no coincidence that the theory of compact Lie groups and reductive linear algebraic groups are so similar. Indeed, recent work on spherical spaces and spherical varieties often jumps back and forth or takes results from one area and translates it, without much difficulty, to the other.

7.3 New Results, New Questions

As mentioned in the thesis, Krämer classified spherical spaces G/H for simple G. More recently Mikityuk [Mi] and Brion [Bri] classified the spherical spaces of non-simple semisimple algebraic groups. Mikityuk approaches the problem from a symplectic geometry perspective, while Brion uses the algebraic geometry of algebraic groups.

Another landmark paper focusing on the symplectic geometry view of spherical spaces is [GS], in which the authors prove the equivalence of the representation theoretic definition of spherical spaces and the Poisson bracket commutativity condition. They also give an alternative proof of Krämer's lemma 3.4, which is the most productive lemma in the Krämer's classification. They claim that it is less complicated, but that probably depends on the reader's background.

There has been a trend in the recent literature to compare the class of spherical spaces with extensions of the class of the symmetric spaces. One such example is in [Dz], where Dzyadyk calls a spherical nonsymmetric space G/H *almost symmetric* if the quotient $\dfrac{G/(Z(G)\cap H)}{H/(Z(G)\cap H)}$ is symmetric. Dzyadyk classifies irreducible almost symmetric spaces and proves several nice properties about them which distinguishes them from both both symmetric spaces and spherical spaces. This puts another layer right in the middle of our sandwich.

Another generalization of symmetric spaces turns out to be more directly compatible with spherical spaces.

Definition 7.6. A *weakly symmetric space* is a complete Riemannian manifold with the property that any two points can be exchanged by an isometry.

This is indeed a generalization of symmetric spaces as points on a symmetric space can be exchanged by an isometry of order 2. In [Ng], it is shown, using Krämer's classification, that the spherical spaces G/H with compact connected simple G are exactly the weakly symmetric spaces arising as quotients of G. More is true: Let G be a connected complex reductive algebraic group and H a reductive algebraic subgroup. Let G' be a connected real form of G such that $H' := H \cup G'$ is a compact real form of H. Then G'/H' is a weakly symmetric space if and only if G/H is a spherical space. In that case there is a Weyl involution ν of G that preserves and restricts to Weyl involutions on G', H and H' such that $\nu|_{G'}$ is a weak symmetry for G'/H'. This is the primary theorem of [AV]. Both of these results are discussed and proven in [Wo]. The book also collects the classification results of [Mi] and [Bri].

Now we introduce two more subclasses of homogeneous spaces.

Definition 7.7. Let G/H be a homogeneous space. G/H is *commutative* if the algebra of all invariant differential operators on G/H is commutative.

Definition 7.8. Let G/H be a homogeneous space. For a manifold X denote by $P(T^*X)$ the algebra of functions on the contangent space of X which are polynomials on the fibers with respect to multiplication and some Poisson bracket. G/H is *weakly commutative* if $P(T^*(G/H))$ is commutative when restricted to G invariant functions.

In Vinberg's survey article [Vi] these classes are related. For general homogeneous space G/H we have the following inclusions:

$$
\begin{tikzcd}
\begin{array}{c}\text{Symmetric}\\ \text{Spaces}\end{array} \arrow[r, hook] & \begin{array}{c}\text{Weakly}\\ \text{Symmetric}\\ \text{Spaces}\end{array} \arrow[r, hook] & \begin{array}{c}\text{Commutative}\\ \text{Spaces}\end{array} \arrow[r, hook] & \begin{array}{c}\text{Weakly}\\ \text{Commutative}\\ \text{Spaces}\end{array}
\end{tikzcd}
$$

For homogeneous spaces of reductive groups, the last three classes coincide. Furthermore, in this case these three classes are the same as the class of spherical spaces. To conclude:

$$
\begin{tikzcd}
\begin{array}{c}\text{Weakly}\\ \text{Commutative}\\ \text{Spaces}\end{array} \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & \begin{array}{c}\text{Commutative}\\ \text{Spaces}\end{array} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \begin{array}{c}\text{Weakly}\\ \text{Symmetric}\\ \text{Spaces}\end{array} \arrow[dl, leftrightarrow] \\
 & \text{Spherical Spaces} &
\end{tikzcd}
$$

References

[ADW] S. Axelrod, S. Della Pietra, E. Witten, *Geometric quantization of Chern-Simons gauge theory*, J. Diff. Geo. **33** (1991), 787–902.

[AV] D. N. Akhiezer, E. B. Vinberg, *Weakly symmetric spaces and spherical varieties*, Transf. Groups, **4** (1994), 3–24.

[Ber] M. Berger, *A Panoramic View of Riemannian Geometry*, Springer-Verlag, Berlin (2006).

[Bes] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin (1987).

[BLV] M. Brion, D. Luna, Th. Vust, *Espaces homogènes sphériques*, Inv. Math. **84** (1986), 617–632.

[Bo] A. Borel, *Linear Algebraic Groups*, GTM, vol. 126, Springer-Verlag, Berlin (1991)

[Br] G. Bredon, *Topology and Geometry*, Springer, New York (1993).

[Bri] M. Brion, *Classification des espaces homogènes sphériques*, Compositio Math. **63** (1987), 189–208.

[BT] R. Bott, L. Tu, *Differential Forms in Algebraic Topology*, Springer, New York (1982).

[Ca1] E. Cartan, *Sur une classe remarquable d'espaces de Riemann, I*, Bull. de la Soc. Math. de France **54** (1926), 214–216.

[Ca2] E. Cartan, *Sur une classe remarquable d'espaces de Riemann, II*, Bull. de la Soc. Math. de France **55** (1927), 114–2.

[dC] M. do Carmo, *Riemannian Geometry*, Birkhäuser, Boston (1992).

[Dz] Y. V. Dzyadyk, *A new class of compact spherical spaces*, Ukrainian Math. J., **50**, no. 11 (1998), 1780–1784.

[GS] V. Guillemin, S. Sternberg, *Multiplicity-free spaces*, J. Differential Geom. **19** (1984), 31–56.

[He1] S. Helgason, *Differential geometry and symmetric spaces.* Academic Press, New York-London (1962).

[He2] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Amer. Math. Soc., Providence, 2001.

[HH] W.C. Hsiang and W.Y. Hsiang, *Differentiable actions of compact connected classical groups II*, Ann. of Math., **92** (1970), 189-223.

[Hi] N. Hitchin, *Flat connections and geometric quantization*, Comm. Math. Phys., **131** (1990), 357–380.

[Hu] D. Husemöller, *Fibre Bundles*, Springer, New York, (1994).

[Kr75] M. Krämer, *Eine Klassifikation bestimmter Untergruppen kompakter zusammenhängender Liegruppen*, Communic. in Algebra, **3** (1975), 691–737.

[Kr76] M. Krämer, *Über Untergruppen kompakter Liegruppen als Isotropiegruppen bei linearen Aktionen*, Math. Z., **147** (1976), 207-224.

[Kr79] M. Krämer, *Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen*, Comp. Math. **38** n. 2 (1979), 129–153.

[LSz10] L. Lempert, R. Szőke, *Uniqueness in geometric quantization*, arXiv: 1004.4863 (2010).

[LSz12] L. Lempert, R. Szőke, *Curvature of fields of quantum Hilbert spaces*, arXiv:1204.0963 (2012).

[LV] D. Luna, Th. Vust, *Plongements d'espaces homogènes.* Comment. Math. Helv. **58** (1983), 186–245.

[Mi] I. V. Miktiyuk, *On the integrability of invariant Hamiltonian systems with homogeneous configuration spaces*, Mat. Sb. **129** (1986), 514–534. English translation: Math. USSR-Sb. **57** (1987), 527–546.

[Ng] H. Nguyen, *Compact weakly symmetric spaces and spherical pairs*, Proc. Am. Math. Soc., **128** n. 11 (2000), 3425-3433.

[Sp] T. A. Springer, *Linear Algebraic Groups*, Progress in Mathematics, vol. 9, Brikhäuser-Verlag, Berlin (1998).

[St] J. Stillwell, *Naive Lie Theory*, Springer, New York (2008).

[Ti] D.A. Timashev, *Homogeneous Spaces and Equivariant Embeddings*, Encyclopaedia of Mathematical Sciences 138, Springer-Verlag, Berlin (2011).

[Vi] E. B. Vinberg, *Commutative homogeneous spaces and co-isotropic symplectic actions*, Russ. Math. Surv., **56** n. 1 (2001).

[Wo] J. A. Wolf, *Harmonic Analysis on Commutative Spaces*, Mathematical Surveys and Monographs, Vol. 142, AMS, Providence (2007).