

ELLIPTIC-LIKE REGULARIZATION OF SEMILINEAR EVOLUTION EQUATIONS

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ABSTRACT. Consider in a real Hilbert space the Cauchy problem $(P_0) : u'(t) + Au(t) + Bu(t) = f(t), 0 \leq t \leq T; u(0) = u_0$, where $-A$ is the generator of a C_0 -semigroup of linear contractions and B is a smooth nonlinear operator. We associate with (P_0) the following problem: $(P_1^\varepsilon) : -\varepsilon u''(t) + u'(t) + Au(t) + Bu(t) = f(t), 0 \leq t \leq T; u(0) = u_0, u(T) = u_1$, where $\varepsilon > 0$ is a small parameter. Existence, uniqueness and higher regularity for both (P_0) and (P_1^ε) are investigated and an asymptotic expansion for the solution of problem (P_1^ε) is established, showing the presence of a boundary layer near $t = T$.

1. INTRODUCTION

Let H be a real Hilbert space with scalar product (\cdot, \cdot) and the induced norm $\|\cdot\|$. Denote by (P_0) the following Cauchy problem

$$\begin{cases} u'(t) + Au(t) + Bu(t) = f(t), & 0 \leq t \leq T, & (E) \\ u(0) = u_0, & & (IC) \end{cases} \quad (P_0)$$

where $T > 0$ is a given time instant, $f : [0, T] \rightarrow H$, $A : D(A) \subset H \rightarrow H$ is a linear operator, such that

(Hyp 1) $-A$ generates a C_0 -semigroup of contractions, say $\{S(t) : H \rightarrow H; t \geq 0\}$,

(Hyp 2) $B : D(B) = H \rightarrow H$ is a differentiable nonlinear operator satisfying $\sup_{x \in H} \|B'(x)\|_{L(H)} = C < \infty$;

$u_0 \in H$ is a given initial state. So equation (E) is a semilinear one.

Consider the second order equation

$$-\varepsilon u''(t) + u'(t) + Au(t) + Bu(t) = f(t), \quad 0 \leq t \leq T, \quad (E_\varepsilon)$$

where $\varepsilon > 0$ is a small parameter. In this paper we intend to investigate whether the solution of (P_0) could be approximated by a solution of (E_ε) which is expected to be more regular. Thus, it is natural to associate with (E_ε) the same condition (IC) . Since the problem $(E_\varepsilon), (IC)$ is incomplete, we need to associate with (E_ε) an additional condition. Let it be

$$u(T) = u_1. \quad (C)$$

So we have the problem

$$(E_\varepsilon), (IC), (C). \quad (P_1^\varepsilon)$$

A typical example.

Let $H = L^2(\Omega)$ with the usual scalar product and norm, where Ω is a bounded open subset of \mathbb{R}^3 with a sufficiently smooth boundary $\partial\Omega$,

$$A = -\Delta \quad \text{with} \quad D(A) = H_0^1(\Omega) \cap H^2(\Omega),$$

$$Bu = b \circ u,$$

where $b : \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function, with $0 \leq b'(r) \leq C < \infty \forall r \in \mathbb{R}$. In this case, (E) is the nonlinear heat equation

$$u_t - \Delta_x u + b \circ u = f(x, t), \quad x \in \Omega, \quad 0 \leq t \leq T,$$

while (E_ε) is an elliptic type equation. Thus it is natural to call (E_ε) an elliptic-like regularization of (E).

It is worth pointing out A. Perjan's work on the behavior of solutions of the hyperbolic-like problem $\varepsilon u''(t) + u'(t) + Au(t) + Bu(t) = f(t)$, $0 \leq t \leq T$, $u(0) = u_0$, $u'(0) = u_1$ in a Hilbert space H as $\varepsilon \rightarrow 0$, where A is a linear, symmetric, strong positive operator and B is a nonlinear operator. See, *e.g.*, [6] and [7]. See also [3], Chapter 11, for a particular case, when $H = L^2(\Omega)$.

The rest of this paper is organized as follows: Sections 2 and 3 are concerned with the existence, uniqueness, and regularity of the solutions of problems (P_0) and (P_1^ε) . While the results of these sections are important in themselves, we need them in Section 4 in order to validate the asymptotic expansion for the solution of problem (P_1^ε) that will be first derived heuristically. It is worth mentioning that our analysis covers many applications, in particular the initial-boundary value problem for the semilinear heat equation that is discussed in our paper.

2. EXISTENCE AND REGULARITY FOR PROBLEM (P_0)

Throughout this paper we adopt the usual definitions of a strong and weak solution for the Cauchy problem associated with a nonlinear operator $Q : D(Q) \subset H \rightarrow H$,

$$u'(t) + Qu(t) = f(t), \quad 0 \leq t \leq T, \quad u(0) = u_0. \quad (P)$$

Definition 2.1. Let $f \in L^1(0, T; H)$. A function $u \in C([0, T]; H)$ is said to be a strong solution of problem (P) if: u is absolutely continuous on every compact subinterval of $(0, T)$; $u(0) = u_0$ and u satisfies the above equation of (P) for a.e. $t \in (0, T)$.

Definition 2.2. A function $u \in C([0, T]; H)$ is said to be a weak solution of problem (P) if there exist sequences $\{u_n\} \subset C([0, T]; H)$ and $\{f_n\} \subset L^1(0, T; H)$ such that each u_n is absolutely continuous on every compact subinterval of $(0, T)$ and $u_n'(t) + Qu_n(t) = f_n(t)$ a.e. $t \in (0, T)$, for each n ; $u_n \rightarrow u$ in $C([0, T]; H)$; $u(0) = u_0$; and $f_n \rightarrow f$ in $L^1(0, T; H)$.

Lemma 2.3. Let X be a Banach space, and $B : X \rightarrow X$ be differentiable satisfying $\sup_{x \in H} \|B'(x)\|_{L(X)} = C < \infty$, then

$$\|Bx - By\| \leq C\|x - y\| \quad \forall x, y \in X. \quad (2.1)$$

Proof. The proof is not new but we give it for the convenience of the reader. Let $f \in X^*$ be arbitrary. Consider $g : [0, 1] \rightarrow \mathbb{R}$ defined as

$$g(t) = (f \circ B)(y + t(x - y)).$$

By mean value theorem for real valued functions, there exists a $c \in (0, 1)$ such that

$$g(1) - g(0) = g'(c).$$

So,

$$\begin{aligned} f(Bx) - f(By) &= f[B'(y + c(x - y))(x - y)] \\ &\Rightarrow f(Bx - By) = f[B'(y + c(x - y))(x - y)]. \end{aligned}$$

If $Bx - By = 0$, then (2.1) is trivial. Assume that $Bx - By \neq 0$, then by the Hahn-Banach theorem, there exists $f \in X^*$ satisfying $\|f\| = 1$, and $f(Bx - By) = \|Bx - By\|$. From this (2.1) follows easily. \square

Remark 2.4. If B satisfies (Hyp 2), then by Lemma 2.3 B is a Lipschitz operator. If in addition B is assumed to be monotone, then B is maximal monotone and since A is maximal monotone (cf. (Hyp 1)), it follows that $A + B$ is maximal monotone as well.

We have the following existence results for problem (P_0) :

Theorem 2.5. *Assume (Hyp 1) and (Hyp 2). If $u_0 \in H$ and $f \in L^1(0, T; H)$, then (P_0) has a unique mild solution $u \in C([0, T]; H)$, i.e.,*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)[f(s) - Bu(s)] ds, \quad 0 \leq t \leq T. \quad (2.2)$$

If $u_0 \in D(A)$ and $f \in W^{1,1}(0, T; H)$ then $u \in C^1([0, T]; H)$ and it is a strong solution of (P_0) , satisfying (E) of (P_0) for all $t \in [0, T]$.

Proof. The first part follows by Theorem 2.0.28 in [3], p. 30. Now, if $u_0 \in D(A)$ and $f \in W^{1,1}(0, T; H)$, then according to [5], Theorem 2.1, p. 48 and Remark 2.1, p. 53, the mild solution u belongs to $W^{1,\infty}(0, T; H)$ and it is a strong solution. Moreover, since

$$\begin{aligned} u'(t) &= S(t)(f(0) - Au_0 - Bu_0) \\ &\quad + \int_0^t S(t-s)[f'(s) - (Bu)'(s)] ds, \quad 0 \leq t \leq T, \end{aligned} \quad (2.3)$$

it follows that $u' \in C([0, T]; H)$. This completes the proof. \square

Remark 2.6. If $u_0 \in H$ and $f \in L^1(0, T; H)$ then the mild solution u of problem (P_0) is also a weak solution. Indeed, let $(u_0^n, f_n) \in D(A) \times W^{1,1}(0, T; H)$ approximate (u_0, f) in $H \times L^1(0, T; H)$. Denote by u_n the strong solution of (P_0) with $u_0 := u_0^n$ and $f := f_n$. Then,

$$u_n(t) = S(t)u_0^n + \int_0^t S(t-s)[f_n(s) - Bu_n(s)] ds, \quad 0 \leq t \leq T. \quad (2.4)$$

Therefore, since $S(t)$ is a contraction for each $t \geq 0$ and B is Lipschitzian, we have

$$\|u_n(t) - u_m(t)\| \leq \|u_0^n - u_0^m\| + \|f_n - f_m\|_{L^1(0, T; H)} + C \int_0^t \|u_n(s) - u_m(s)\| ds,$$

for all $t \in [0, T]$.

It follows by Gronwall's Lemma that

$$\|u_n(t) - u_m(t)\| \leq \left(\|u_0^n - u_0^m\| + \|f_n - f_m\|_{L^1(0, T; H)} \right) e^{Ct}, \quad \forall t \in [0, T].$$

This shows that u_n converges in $C([0, T]; H)$ and its limit \tilde{u} is a weak solution of problem (P_0) . By passing to the limit in (2.4) we can see that \tilde{u} is also a mild solution of the same problem, so by the uniqueness of the mild solution $\tilde{u} = u$.

Theorem 2.7. *Assume that (Hyp 1) and (Hyp 2) hold and, in addition, that A is self-adjoint. If $u_0 \in H$ and $f \in L^2(0, T; H)$, then problem (P_0) has a unique strong solution u , such that $t^{1/2}u' \in L^2(0, T; H)$.*

Proof. Since A is self-adjoint, it follows that $A = \partial\varphi$, where

$$\varphi(x) = \begin{cases} \frac{1}{2}\|A^{1/2}x\|^2 & x \in D(A^{1/2}), \\ +\infty, & \text{otherwise,} \end{cases}$$

that is convex and lower semicontinuous (see, e.g., [5], p. 43). Thus the result follows by the well known Brezis' result (see, e.g., [3], Theorem 2.0.30, p. 31). \square

Theorem 2.8 (Higher Regularity). *Assume that (Hyp 1) and (Hyp 2) hold and, in addition, that A is self-adjoint. If $u_0 \in D(A)$ and $f \in W^{1,2}(0, T; H)$, then the solution u of problem (P_0) belongs to $C^1([0, T]; H)$ and u' is differentiable, i.e., with $t^{1/2}u'' \in L^2(0, T; H)$. If in addition $f(0) - Au_0 - Bu_0 \in D(A^{1/2})$, then $u \in W^{2,2}(0, T; H)$.*

Proof. If $u_0 \in D(A)$ and $f \in W^{1,2}(0, T; H)$ it follows by Theorem 2.5 that $u \in C^1([0, T]; H)$. Obviously u satisfies the equation

$$u'(t) = S(t)(f(0) - Au_0 - Bu_0) + \int_0^t S(t-s)[f'(s) - B'(u(s))u'(s)] ds \quad (2.5)$$

for all $t \in [0, T]$. Now, consider the equation (obtained from (E) by formal differentiation)

$$\begin{cases} v'(t) + Av(t) + B'(u(t))v(t) = f'(t), & 0 \leq t \leq T, \\ v(0) = f(0) - Au_0 - Bu_0. \end{cases} \quad (CP)$$

(CP) has a mild solution $v = v(t) \in C([0, T]; H)$,

$$v(t) = S(t)(f(0) - Au_0 - Bu_0) + \int_0^t S(t-s)[f'(s) - B'(u(s))v(s)] ds, \quad (2.6)$$

for all $t \in [0, T]$. From (2.5) and (2.6) we derive

$$\|v(t) - u'(t)\| \leq \int_0^t \|B'(u(s))\|_{L(H)} \cdot \|v(s) - u'(s)\| ds, \quad 0 \leq t \leq T$$

which implies $v(t) = u'(t)$ for all $0 \leq t \leq T$.

In fact, since A is self-adjoint, the above Cauchy problem (CP) has a strong solution v , with $\sqrt{t}v' \in L^2(0, T; H)$ (cf. [3], Theorem 2.0.30, p. 31). Therefore, $\sqrt{t}u'' \in L^2(0, T; H)$. Now, if in addition $f(0) - Au_0 - Bu_0 \in D(A^{1/2})$, then the solution $v = u'$ of problem (CP) belongs to $W^{1,2}(0, T; H)$ (cf. [3], Theorem 2.0.24, pp. 28-29). So the proof is complete. \square

If A is not self-adjoint, then a higher regularity result holds under more restrictive conditions, as shown in the next theorem.

Theorem 2.9. *Assume (Hyp 1) and (Hyp 2). If $f \in W^{2,\infty}(0, T; H)$, $u_0 \in D(A)$, $f(0) - Au_0 - Bu_0 \in D(A)$, and B is twice differentiable with B'' bounded on bounded sets, then problem (P_0) has a unique solution $u \in C^2([0, T]; H)$.*

Proof. Taking into account Theorem 2.5, it suffices to prove that the above (CP) has a solution $v \in C^1([0, T]; H)$. Since $v = u'$ this would conclude the proof. We will apply Kato's theorem (see [3], Theorem 2.0.31, p. 31). To this purpose, let us replace (CP) by an equivalent one which fits in the framework of Kato's theorem. Let $C_1 > C$. Multiply the equation (CP) by $e^{-C_1 t}$. Denoting $w(t) = e^{-C_1 t} v(t)$, we obtain the following Cauchy problem (which is equivalent to (CP))

$$\begin{cases} w'(t) + Aw(t) + (C_1 I + B'(u(t)))w(t) = e^{-C_1 t} f'(t), & 0 \leq t \leq T, \\ w(0) = f(0) - Au_0 - Bu_0. \end{cases} \quad (\widetilde{CP})$$

The operator $A + C_1 I + B'(u(t))$ is maximal monotone for all $t \in [0, T]$ and its domain is $D(A)$ for all $t \in [0, T]$. Thus, by Kato's existence result, (\widetilde{CP}) has a unique solution $w \in W^{1,\infty}(0, T; H)$. Therefore, $v = u' \in W^{1,\infty}(0, T; H)$. In fact, $v = u'$ is a mild solution of problem (CP) (see (2.6)) and satisfies

$$\begin{aligned} u''(t) = v'(t) = S(t)v'(0) + \int_0^t S(t-s)[f''(s) \\ - B''(u(s))u'(s)v(s) - B'(u(s))v'(s)]ds, \quad 0 \leq t \leq T. \end{aligned}$$

which shows that $u'' \in C([0, T]; H)$. This concludes the proof. \square

Remark 2.10. It is worth pointing out that by using the above ideas every level of regularity for the solution of (P_0) can be reached under appropriate conditions.

3. EXISTENCE AND REGULARITY FOR PROBLEM (P_1^ε)

In this section we assume $\varepsilon = 1$ without any loss of generality. So (P_1^ε) becomes

$$\begin{cases} -u'' + u' + Au + Bu = f(t), & 0 \leq t \leq T, \\ u(0) = u_0, \quad u(T) = u_1. \end{cases} \quad (P_1)$$

Theorem 3.1. *If (Hyp 1), (Hyp 2) hold, B is monotone, $f \in L^2(0, T; H)$, and $u_0, u_1 \in D(A)$, then problem (P_1) has a unique solution $u \in W^{2,2}(0, T; H)$.*

Proof. $Q = A + B$ is a maximal monotone operator, with $D(Q) = D(A)$ (see Remark 2.4 above). Therefore, one can apply [1] (see also [3], theorem 2.0.37, p. 34) to derive the existence of at least one solution $u \in W^{2,2}(0, T; H)$ for problem (P_1) . Indeed, all conditions of Theorem 2.0.37 in [3] are fulfilled, with $\beta_1 = \beta_2 =$ the subdifferential of the indicator function φ of the set $\{0\} \subset H$, i.e., $\varphi(0) = 0$ and $\varphi(x) = +\infty$ for all $x \in H \setminus \{0\}$.

In fact, u is unique. Indeed, if v is another solution of (P_1) , then

$$\begin{cases} -(u-v)'' + (u-v)' + A(u-v) + Bu - Bv = 0, & 0 \leq t \leq T, \\ (u-v)(0) = 0, \quad (u-v)(T) = 0. \end{cases}$$

If we multiply by $(u-v)$ the above equation and use the monotonicity of Q , we obtain

$$-\int_0^T ((u-v)'', u-v) dt + \int_0^T ((u-v)', u-v) dt \leq 0.$$

This implies

$$\int_0^T \|u' - v'\|^2 dt + \underbrace{\frac{1}{2} \|u - v\|^2}_{=0} \Big|_{t=0}^{t=T} \leq 0,$$

hence $u' - v' \equiv 0$, i.e., $u - v$ is a constant function. In fact $u \equiv v$ since $u(0) = v(0)$. \square

Theorem 3.2. *Assume (Hyp 1) and (Hyp 2) hold and B is monotone. If $f \in L^2(0, T; H)$ and $u_0, u_1 \in H$, then problem (P_1) has a unique solution $u \in C([0, T]; H) \cap W_{\text{loc}}^{2,2}(0, T; H)$, with $t^{1/2}(T-t)^{1/2}u'$, $t^{3/2}(T-t)^{3/2}u'' \in L^2(0, T; H)$.*

Proof. Note that $\overline{D(A)} = \overline{D(Q)} = H$, where $Q = A + B$. We will use a technique similar to that of [4]. We continue the proof with

Claim 1. If u, v are two solution of (P_1) with the properties specified in the statement of Theorem 3.2, then $t \mapsto e^{-t}\|u(t) - v(t)\|^2$ is a convex function on $[0, T]$, and

$$\|u - v\|_{C([0, T]; H)} \leq \max \{e^{T/2}\|u(0) - v(0)\|, \|u(T) - v(T)\|\}, \quad (3.1)$$

$$\int_0^T t(T-t)\|u' - v'\|^2 dt \leq 2T(e^T\|u(0) - v(0)\|^2 + \|u(T) - v(T)\|^2). \quad (3.2)$$

Proof of Claim 1. Let

$$g(t) = \frac{1}{2}\|u(t) - v(t)\|^2, \quad 0 \leq t \leq T.$$

Obviously, $g \in C([0, T])$ and

$$g'' = (u'' - v'', u - v) + \|u' - v'\|^2, \quad (3.3)$$

for a.e. $t \in (0, T)$. From (3.3) and (P_1) we get

$$\begin{aligned} g'' &= (u' - v', u - v) + (Qu - Qv, u - v) + \|u' - v'\|^2 \\ &\geq g' + \|u' - v'\|^2 \quad \text{a.e. } t \in (0, T). \end{aligned} \quad (3.4)$$

Therefore,

$$\begin{aligned} (e^{-t}g)'' &= e^{-t}(g'' - 2g' + g) \\ &\geq e^{-t}(\|u' - v'\|^2 - g' + g) \\ &= e^{-t}\left(\|u' - v'\|^2 - (u' - v', u - v) + \frac{1}{2}\|u - v\|^2\right) \\ &\geq \frac{1}{2}e^{-t}\|u' - v'\|^2 \end{aligned} \quad (3.5)$$

for a.e. $t \in (0, T)$, hence $t \mapsto e^{-t}g(t)$ is a convex function. This yields

$$e^{-t}g(t) \leq \max \{g(0), e^{-T}g(T)\} \quad \forall t \in [0, T],$$

i.e., (3.1) holds. In order to prove estimation (3.2), consider the function (as in [4]) $\beta_\delta(t) = \min\{t - \delta, T - \delta - t\}$ for a small $\delta > 0$. Multiply (3.5) by β_δ and then

integrate over $[\delta, T - \delta]$

$$\begin{aligned} \frac{1}{2} \int_{\delta}^{T-\delta} e^{-t} \beta_{\delta}(t) \|u' - v'\|^2 dt &\leq \int_{\delta}^{T-\delta} \beta_{\delta}(t) (e^{-t} g)'' dt = - \int_{\delta}^{T-\delta} \beta'_{\delta}(t) (e^{-t} g)' dt \\ &= e^{-\delta} g(\delta) + e^{-T+\delta} g(T - \delta) - 2e^{-T/2} g(T/2) \\ &\leq g(\delta) + e^{-T+\delta} g(T - \delta). \end{aligned}$$

Letting $\delta \rightarrow 0^+$ and applying the Fatou's lemma yields

$$\frac{1}{2} \int_0^T \beta(t) \|u' - v'\|^2 dt \leq e^T g(0) + g(T),$$

where $\beta(t) := \min\{t, T - t\}$. This inequality implies (3.2), so the proof of Claim 1 is complete.

Claim 2. Let $u \in W^{2,2}(0, T; H)$ be the solution of (P_1) with $u_0, u_1 \in D(A)$, and $f \in L^2(0, T; H)$. Let $u_{\lambda} \in W^{2,2}(0, T; H)$ be the solution of the problem

$$-u''_{\lambda} + u'_{\lambda} + A_{\lambda} u_{\lambda} = f - Bu, \quad 0 \leq t \leq T, \quad (3.6a)$$

$$u_{\lambda}(0) = u_0, \quad u_{\lambda}(T) = u_1, \quad (3.6b)$$

where $\lambda > 0$ and A_{λ} denotes the Yosida approximation of A . Then,

$$u_{\lambda} \rightarrow u, \quad u'_{\lambda} \rightarrow u' \text{ in } C([0, T]; H), \quad (3.7)$$

$$\text{and } u''_{\lambda} \rightarrow u'' \text{ weakly in } L^2(0, T; H) \text{ as } \lambda \rightarrow 0^+. \quad (3.8)$$

Proof of Claim 2. Define for $\lambda > 0$ and $t \in [0, T]$

$$\begin{cases} u^*(t) = \frac{T-t}{T} u_0 + \frac{t}{T} u_1, \\ v_{\lambda}(t) = u_{\lambda}(t) - u^*(t). \end{cases}$$

Obviously, v_{λ} satisfies the problem

$$\begin{cases} -v''_{\lambda} + v'_{\lambda} + A_{\lambda} v_{\lambda} = f - Bu - A_{\lambda} u^* + \frac{1}{T}(u_0 - u_1), & 0 \leq t \leq T, \\ v_{\lambda}(0) = 0 = v_{\lambda}(T). \end{cases} \quad (3.9)$$

If we multiply (3.9) by v_{λ} and integrate over $[0, T]$ we obtain

$$\begin{aligned} & - \int_0^T (v''_{\lambda}, v_{\lambda}) dt + \underbrace{\frac{1}{2} \int_0^T \frac{d}{dt} \|v_{\lambda}\|^2 dt}_{=0} + \underbrace{\int_0^T (A_{\lambda} v_{\lambda}, v_{\lambda}) dt}_{\geq 0} \\ &= \int_0^T \left(f - Bu + \frac{1}{T}(u_0 - u_1) - A_{\lambda} u^*, v_{\lambda} \right) dt. \end{aligned} \quad (3.10)$$

Since

$$\begin{aligned} \|A_{\lambda} u^*(t)\| &= \left\| \frac{T-t}{T} A_{\lambda} u_0 + \frac{t}{T} A_{\lambda} u_1 \right\| \\ &\leq \frac{T-t}{T} \|A u_0\| + \frac{t}{T} \|A u_1\| \leq \max\{\|A u_0\|, \|A u_1\|\} < \infty, \end{aligned} \quad (3.11)$$

for all $t \in [0, T]$, we derive from (3.10)

$$\int_0^T \|v'_{\lambda}\|^2 dt \leq K \left(\int_0^T \|v_{\lambda}\|^2 dt \right)^{1/2}, \quad (3.12)$$

where $K > 0$ is a constant. On the other hand,

$$v_\lambda(t) = \int_0^t v'_\lambda(s) ds, \quad 0 \leq t \leq T. \quad (3.13)$$

Combining (3.12) and (3.13), we get

$$\{v'_\lambda; \lambda > 0\} \text{ is bounded in } L^2(0, T; H), \quad (3.14)$$

$$\{v_\lambda; \lambda > 0\} \text{ is bounded in } C([0, T]; H). \quad (3.15)$$

Since

$$\begin{aligned} \frac{d}{dt} (v'_\lambda, A_\lambda v_\lambda) &= (v''_\lambda, A_\lambda v_\lambda) + \underbrace{(v'_\lambda, A_\lambda v'_\lambda)}_{\geq 0} \\ &\geq (v''_\lambda, A_\lambda v_\lambda) \quad \text{for a.e. } t \in (0, T), \end{aligned}$$

we can write, by using (3.9),

$$0 \geq \int_0^T (v''_\lambda, A_\lambda v_\lambda) dt = \int_0^T (v'_\lambda + A_\lambda v_\lambda - f_\lambda, A_\lambda v_\lambda) dt,$$

where

$$f_\lambda(t) := f(t) - Bu(t) - A_\lambda u^*(t) + \frac{1}{T}(u_0 - u_1).$$

Therefore, by (3.11) and (3.14), one gets

$$\begin{aligned} \|A_\lambda v_\lambda\|_{L^2}^2 &\leq \|v'_\lambda\|_{L^2} \|A_\lambda v_\lambda\|_{L^2} + \|f_\lambda\|_{L^2} \|A_\lambda v_\lambda\|_{L^2} \\ &\leq K_1 \|A_\lambda v_\lambda\|_{L^2}, \end{aligned}$$

so both

$$\{u''_\lambda; \lambda > 0\} \text{ and } \{A_\lambda u_\lambda; \lambda > 0\} \text{ are bounded in } L^2 := L^2(0, T; H). \quad (3.16)$$

For $\lambda, \mu > 0$ we have from (3.6)

$$\begin{aligned} - \int_0^T (u''_\lambda - u''_\mu, u_\lambda - u_\mu) dt &+ \int_0^T (u'_\lambda - u'_\mu, u_\lambda - u_\mu) dt \\ &+ \int_0^T (A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu) dt = 0, \end{aligned}$$

which implies that

$$\begin{aligned} \int_0^T \|u'_\lambda - u'_\mu\|^2 dt &= - \int_0^T \underbrace{(A_\lambda u_\lambda - A_\mu u_\mu, J_\lambda u_\lambda - J_\mu u_\mu)}_{\geq 0} dt \\ &\quad - \int_0^T (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu) dt \\ &\leq K_2(\lambda + \mu), \end{aligned}$$

where $J_\lambda = (I + \lambda A)^{-1}$. This shows that $\{u'_\lambda; \lambda > 0\}$ is a Cauchy sequence in L^2 , hence convergent in L^2 as $\lambda \rightarrow 0^+$. Since

$$\begin{aligned} \|u_\lambda(t) - u_\mu(t)\| &= \left\| \int_0^T (u'_\lambda(s) - u'_\mu(s)) ds \right\| \\ &\leq T^{1/2} \|u'_\lambda - u'_\mu\|_{L^2} \quad \forall t \in [0, T], \end{aligned}$$

$\{u_\lambda; \lambda > 0\}$ converges in $C([0, T]; H)$. Denote its limit by \hat{u} . Summarizing, we have $\hat{u} \in W^{2,2}(0, T; H)$ and

$$u_\lambda \rightarrow \hat{u} \quad \text{in } C([0, T]; H), \quad (3.17)$$

$$u'_\lambda \rightarrow \hat{u}' \quad \text{in } L^2(0, T; H), \quad (3.18)$$

$$u''_\lambda \rightarrow \hat{u}'' \quad \text{weakly in } L^2(0, T; H), \text{ as } \lambda \rightarrow 0^+. \quad (3.19)$$

In fact, since by (3.19) the sequence $\{u'_\lambda; \lambda > 0\}$ is equicontinuous,

$$u'_\lambda \rightarrow \hat{u}' \quad \text{in } C([0, T]; H) \text{ as } \lambda \rightarrow 0^+. \quad (3.20)$$

It is easily seen that

$$J_\lambda u_\lambda \rightarrow \hat{u} \quad \text{in } C([0, T]; H). \quad (3.21)$$

Indeed, for all $t \in [0, T]$,

$$\begin{aligned} \|J_\lambda u_\lambda(t) - \hat{u}(t)\| &\leq \|J_\lambda u_\lambda(t) - u_\lambda(t)\| + \|u_\lambda(t) - \hat{u}(t)\| \\ &= \lambda \|A_\lambda u_\lambda(t)\| + \|u_\lambda(t) - \hat{u}(t)\| \\ &\leq K_3 \lambda + \|u_\lambda(t) - \hat{u}(t)\| \quad (\text{by (3.16)}), \end{aligned}$$

which confirms our assertion.

Using the above pieces of information on u_λ , we can pass to the limit as $\lambda \rightarrow 0^+$ in (3.6a) regarded as an equation in L^2 to obtain

$$-\hat{u}'' + \hat{u}' + A\hat{u} = f - Bu, \quad 0 \leq t \leq T. \quad (3.22a)$$

We also have (see (3.6b) and (3.17))

$$\hat{u}(0) = u_0, \quad \hat{u}(T) = u_1. \quad (3.22b)$$

Now, from (P_1) , (3.22a) and (3.22b) we can easily see that $\hat{u} \equiv u$. This completes the proof of Claim 2.

Claim 3. Let $u \in W^{2,2}(0, T; H)$ be the solution of (P_1) with $u_0, u_1 \in D(A)$ and $f \in L^2(0, T; H)$. Then, there exist constants $C_1, C_2 > 0$ such that

$$\|u''\|_{L^{2*}} \leq C_1 (\|f\|_{L^2} + \|u'\|_{L^2_*} + \|u\|_{C([0, T]; H)}) + C_2, \quad (3.23)$$

where (as in [4]) $L^2_* := L^2(0, T; H; \beta(t)dt)$, $L^{2*} := L^2(0, T; H; \beta^3(t)dt)$.

Proof of Claim 3. Consider again problem (3.6). From the obvious inequality

$$\begin{aligned} \frac{d}{dt} (u'_\lambda, A_\lambda u_\lambda) &= (u''_\lambda, A_\lambda u_\lambda) + (u'_\lambda, A_\lambda u'_\lambda) \\ &\geq (u''_\lambda, A_\lambda u_\lambda), \end{aligned}$$

we derive by multiplication by β^3 and integration over $[0, T]$,

$$-3 \int_0^T \beta^2 \beta' (u'_\lambda, A_\lambda u_\lambda) dt \geq \int_0^T \beta^3 (u'_\lambda + A_\lambda u_\lambda - f + Bu, A_\lambda u_\lambda) dt.$$

It follows that

$$\begin{aligned} \|A_\lambda u_\lambda\|_{L^{2**}}^2 &\leq 3\|u'_\lambda\|_{L^2} \|A_\lambda u_\lambda\|_{L^{2**}} + \|u'_\lambda\|_{L^{2*}} \|A_\lambda u_\lambda\|_{L^{2*}} + \\ &\quad + \|f\|_{L^{2**}} \|A_\lambda u_\lambda\|_{L^{2**}} + \|Bu\|_{L^{2**}} \|A_\lambda u_\lambda\|_{L^{2**}} \\ &\leq K_4 \|A_\lambda u_\lambda\|_{L^{2**}} \left(\|f\|_{L^2} + \|u'_\lambda\|_{L^2} + \|u\|_{C([0,T];H)} \right) + K_5. \end{aligned}$$

and so

$$\|A_\lambda u_\lambda\|_{L^{2**}} \leq K_4 \left(\|f\|_{L^2} + \|u'_\lambda\|_{L^2} + \|u\|_{C([0,T];H)} \right) + K_6. \quad (3.24)$$

From (3.6a) we then derive

$$\|u''_\lambda\|_{L^{2**}} \leq C_1 \left(\|f\|_{L^2} + \|u'_\lambda\|_{L^2} + \|u\|_{C([0,T];H)} \right) + C_2. \quad (3.25)$$

By (3.7), (3.8) and (3.25) we obtain (3.23).

Proof of Theorem 3.2 (continuation). Let us approximate $u_0, u_1 \in H$ by $u_{0n}, u_{1n} \in D(A)$, i.e.,

$$\|u_{0n} - u_0\| \rightarrow 0, \quad \|u_{1n} - u_1\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By Theorem 3.1, problem (P_1) with $u(0) = u_{0n}$, $u(T) = u_{1n}$ has a unique solution $u_n \in W^{2,2}(0, T; H)$. Now, estimates (3.1), (3.2) and (3.23) come into play showing that there exists a function $u \in C([0, T]; H) \cap W_{\text{loc}}^{2,2}(0, T; H)$, with $u' \in L^2_*$, $u'' \in L^{2**}$, such that

$$u_n \rightarrow u \quad \text{in } C([0, T]; H), \quad (3.26)$$

$$u'_n \rightarrow u' \quad \text{in } L^2_*, \quad (3.27)$$

$$u''_n \rightarrow u'' \quad \text{weakly in } L^{2**}. \quad (3.28)$$

Regarding the equation

$$-u''_n + u'_n + Au_n + Bu_n = f$$

as one in the space $L^2(\delta, T - \delta; H)$ for positive small δ 's, we obtain by (3.26), (3.27) and (3.28) that u satisfies for a.e. $t \in (0, T)$ the equation

$$-u'' + u' + Au + Bu = f.$$

In addition, by (3.26),

$$u(0) = \lim u_n(0) = u_0, \quad u(T) = \lim u_n(T) = u_1.$$

The uniqueness of the solution follows by (3.1). \square

Theorem 3.3. *Assume (Hyp 1) and (Hyp 2) hold and B is monotone. If $u_0, u_1 \in D(A)$ and $f \in W^{1,2}(0, T; H)$, then problem (P_1) has a unique solution $u \in W^{2,2}(0, T; H) \cap W_{\text{loc}}^{3,2}(0, T; H)$, with $t^{3/2}(T - t)^{3/2}u''' \in L^2(0, T; H)$. If $u_0, u_1 \in H$ and $f \in W^{1,2}(0, T; H)$, then $u \in C([0, T]; H) \cap W_{\text{loc}}^{3,2}(0, T; H)$, with $t^{1/2}(T - t)^{1/2}u'$, $t^{3/2}(T - t)^{3/2}u''$, $t^{5/2}(T - t)^{5/2}u''' \in L^2(0, T; H)$.*

Proof. Assume first $u_0, u_1 \in D(A)$ and $f \in W^{1,2}(0, T; H)$. By Theorem 3.1 problem (P_1) has a unique solution $u \in W^{2,2}(0, T; H)$. Consider again problem (3.6). We know that u_λ approximates u in the sense of (3.7) and (3.8). Note that $u_\lambda \in W^{3,2}(0, T; H)$ and

$$-u'''_\lambda + u''_\lambda + A_\lambda u'_\lambda = (f - Bu)'' \quad \text{for a.e. } t \in (0, T). \quad (3.29)$$

Now, if we multiply by β^3 the inequality

$$\frac{d}{dt} \left(u''_\lambda, A_\lambda u'_\lambda \right) \geq \left(u'''_\lambda, A_\lambda u'_\lambda \right), \quad (3.30)$$

and then integrate over $[0, T]$, we get

$$-3 \int_0^T \beta^2 \beta' \left(u''_\lambda, A_\lambda u'_\lambda \right) dt \geq \int_0^T \beta^3 \left(u''_\lambda - f' + (Bu)' + A_\lambda u'_\lambda, A_\lambda u'_\lambda \right) dt.$$

As in the proof of Claim 3, we find

$$\|u'''_\lambda\|_{L^2_{**}} \leq \widetilde{C}_1 \left(\|f'\|_{L^2} + \|u''_\lambda\|_{L^2_*} + \|u'\|_{C([0, T]; H)} \right) + \widetilde{C}_2. \quad (3.31)$$

According to (3.7), (3.8) and (3.31) $u''' \in L^2_{**}$ and $\beta^{3/2} u''' \rightarrow \beta^{3/2} u'''$ weakly in L^2 , as $\lambda \rightarrow 0^+$.

Now assume $u_0, u_1 \in H$ and $f \in W^{1,2}(0, T; H)$. By Theorem 3.2 above, problem (P_1) has a unique solution $u \in C([0, T]; H) \cap W^{2,2}_{\text{loc}}(0, T; H)$, with $u' \in L^2_*$, $u'' \in L^2_{**}$. So all we have to prove is that $u \in W^{3,2}_{\text{loc}}(0, T; H)$ and $t^{5/2}(T-t)^{5/2} u''' \in L^2(0, T; H)$. As usual, we approximate u_0, u_1 by $u_{0n}, u_{1n} \in D(A)$, and denote by u_n the solution of the problem

$$\begin{cases} -u''_n + u'_n + Au_n + Bu_n = f, & 0 \leq t \leq T, \\ u_n(0) = u_{0n}, \quad u_n(T) = u_{1n}. \end{cases} \quad (3.32)$$

From the proof of Theorem 3.2, we know that (u_n) satisfies (3.26) – (3.28). Now, for an $n \in \mathbb{N}$ (arbitrary but fixed) and $\lambda > 0$, denote by $u_{n\lambda}$ the solution of the problem

$$\begin{cases} -u''_{n\lambda} + u'_{n\lambda} + A_\lambda u_{n\lambda} = f - Bu_n, & 0 \leq t \leq T, \\ u_{n\lambda}(0) = u_{0n}, \quad u_{n\lambda}(T) = u_{1n}. \end{cases} \quad (3.33)$$

We know from Claim 2 that

$$\begin{cases} u_{n\lambda} \rightarrow u_n, \quad u'_{n\lambda} \rightarrow u'_n \text{ in } C([0, T]; H) \text{ and} \\ u''_{n\lambda} \rightarrow u''_n \text{ weakly in } L^2(0, T; H) \text{ as } \lambda \rightarrow 0^+. \end{cases} \quad (3.34)$$

Obviously, $u_{n\lambda} \in W^{3,2}(0, T; H)$ and satisfies the equation

$$-u'''_{n\lambda} + u''_{n\lambda} + A_\lambda u'_{n\lambda} = (f - Bu_n)' \text{ for a.e. } t \in (0, T). \quad (3.35)$$

Let us multiply by β^5 the inequality

$$\frac{d}{dt} \left(u''_{n\lambda}, A_\lambda u'_{n\lambda} \right) \geq \left(u'''_{n\lambda}, A_\lambda u'_{n\lambda} \right),$$

and integrate over $[0, T]$. Thus we derive the estimate

$$\left(\int_0^T \beta^5 \|u'''_{n\lambda}\|^2 \right)^{1/2} \leq \widehat{C}_1 \left(\|u''_{n\lambda}\|_{L^2_{**}} + \|u'_{n\lambda}\|_{L^2_*} + \|f'\|_{L^2} \right) + \widehat{C}_2. \quad (3.36)$$

As in the proof of Claim 3, we obtain from (3.33) an estimate similar to (3.25)

$$\|u''_{n\lambda}\|_{L^2_{**}} \leq C_1 \left(\|f\|_{L^2} + \|u'_{n\lambda}\|_{L^2_*} + \|u_n\|_{C([0, T]; H)} \right) + C_2. \quad (3.37)$$

Using (3.37) in (3.36) we obtain

$$\begin{aligned} \left(\int_0^T \beta^5 \|u_{n\lambda}'''\|^2 \right)^{1/2} &\leq D_1 \left(\|f\|_{L^2} + \|f'\|_{L^2} + \|u'_{n\lambda}\|_{L^2_*} \right. \\ &\quad \left. + \|u'_{n\lambda}\|_{L^2_*} + \|u_n\|_{C([0,T];H)} \right) + D_2, \end{aligned} \quad (3.38)$$

where D_1, D_2 are positive constants. Since $u'_{n\lambda} \rightarrow u'_n$ in $C([0, T]; H)$ as $\lambda \rightarrow 0^+$ (see (3.34)), we deduce from (3.38) that $\{u_{n\lambda}'''; \lambda > 0\}$ is bounded in $L^2(0, T; H; \beta^5(t)dt)$, so $\beta^{5/2}u_{n\lambda}''' \rightarrow \beta^{5/2}u_n'''$ weakly in $L^2(0, T; H)$. Letting $\lambda \rightarrow 0^+$ in (3.38) we get

$$\begin{aligned} \|\beta^{5/2}u_n'''\|_{L^2} &\leq D_1 (\|f\|_{L^2} + \|f'\|_{L^2} + 2\|u'_n\|_{L^2_*} \\ &\quad + \|u_n\|_{C([0,T];H)}) + D_2. \end{aligned} \quad (3.39)$$

Since (u_n) satisfies (3.26) and (3.27), it follows from (3.39) that $\beta^{5/2}u_n''' \in L^2(0, T; H)$. Thus the proof of the Theorem 3.3 is complete. \square

Remark 3.4. By using the above method, we can obtain higher regularity for u under appropriate regularity assumptions on f and B .

4. ASYMPTOTIC EXPANSION OF ORDER ZERO FOR (P_1^ε)

From the above analysis, we see that the solutions of problem (P_1^ε) are more regular than those of (P_0) , as expected. For example, if $u_0, u_1 \in H$ and $f \in L^2(0, T; H)$, then the solution of (P_0) belongs to $C([0, T]; H)$, while the solution of (P_1^ε) belongs to $C([0, T]; H) \cap W_{\text{loc}}^{2,2}(0, T; H)$. Now if $u_0, u_1 \in D(A)$ and $f \in W^{1,2}(0, T; H)$, then the solution of (P_0) belongs to $C^1([0, T]; H)$, while that of (P_1^ε) belongs to $W^{2,2}(0, T; H) \cap W_{\text{loc}}^{3,2}(0, T; H)$ (cf. Theorems 2.5 and 3.3).

We also expect that the solution of (P_1^ε) approximates the solution of (P_0) as $\varepsilon \rightarrow 0^+$. We will show in what follows that this is indeed the case under suitable conditions on the data. However, a boundary layer occurs near $t = T$ and so the solution u_ε of (P_1^ε) must be corrected by adding a boundary layer function in order to obtain a good approximation for the solutions of (P_0) .

According to the previous results related to the particular cases of (P_1^ε) ([3], p. 211) the following expansion is expected to hold

$$u_\varepsilon(t) = u(t) + i(\tau) + r_\varepsilon(t), \quad 0 \leq t \leq T, \quad (4.1)$$

where $\tau := \frac{T-t}{\varepsilon}$ is the stretched (fast) variable, $u = u(t)$ is the solution of the reduced problem (P_0) , $i = i(\tau)$ is the boundary layer function, and $r_\varepsilon = r_\varepsilon(t)$ is the remainder (of order zero).

Assuming that all functions involved in (4.1) are smooth enough, we can identify these functions by heuristic arguments. So if we use (4.1) in (E_ε) , and identify the coefficients of $\varepsilon^{-1}, \varepsilon^0$, we get

$$\frac{d^2 i}{d\tau^2} + \frac{di}{d\tau} = 0, \quad \tau > 0 \quad \text{with} \quad i(0) = u_1 - u(T), \quad (4.2)$$

u satisfies (P_0) , and r_ε satisfies

$$\begin{cases} -\varepsilon(u + r_\varepsilon)'' + r'_\varepsilon + Ar_\varepsilon + B(u_\varepsilon) = B(u) - Ai, \\ r_\varepsilon(0) = -i(T/\varepsilon), \quad r_\varepsilon(T) = 0. \end{cases} \quad (R_\varepsilon)$$

From (4.2) we get (note that $i(\infty) = 0$)

$$i(\tau) = (u_1 - u(T)) e^{-\tau}. \quad (4.3)$$

Condition $i(\infty) = 0$ should be read as: i is negligible away from the boundary layer. For more details on the heuristic procedure to determine asymptotic expansions, see, e.g., [3]. In what follows we validate expansion (4.1).

Theorem 4.1. *Assume that (Hyp 1) and (Hyp 2) hold, B is a monotone operator, A is strongly positive, $u_0, u_1 \in D(A)$ and $f \in W^{1,1}(0, T; H)$. Then, for every $\varepsilon > 0$, the solution u_ε of problem (P_1^ε) admits the asymptotic expansion (4.1), where u is the solution of problem (P_0) , $i = i(\tau)$ is boundary layer function defined by (4.3), and the remainder $r_\varepsilon = r_\varepsilon(t)$ satisfies problem (R_ε) and the following estimate*

$$\|r_\varepsilon\|_{C([0, T]; H)} = \mathcal{O}(\varepsilon^{1/4}). \quad (4.4)$$

Proof. By Theorems 2.5 and 3.1, we have

$$r_\varepsilon = u_\varepsilon - u - i \in C^1([0, T]; H), \quad \text{and} \quad (4.5)$$

$$u + r_\varepsilon = u_\varepsilon - i \in W^{2,2}(0, T; H). \quad (4.6)$$

Note that $u(T) \in D(A)$, so $i(\tau) \in D(A)$ for all $\tau \geq 0$. It is easy to check that r_ε satisfies problem (R_ε) .

In order to homogenize the boundary condition for r_ε , we set

$$\bar{r}_\varepsilon(t) = r_\varepsilon(t) + \alpha_\varepsilon(t), \quad 0 \leq t \leq T, \quad (4.7)$$

where

$$\alpha_\varepsilon(t) = (1 - t/T)i(T/\varepsilon). \quad (4.8)$$

Obviously, \bar{r}_ε satisfies the problem

$$-\varepsilon(u + \bar{r}_\varepsilon)'' + \bar{r}_\varepsilon' + A\bar{r}_\varepsilon + Bu_\varepsilon = h_\varepsilon + Bu, \quad 0 \leq t \leq T, \quad (4.9a)$$

$$\bar{r}_\varepsilon(0) = 0, \quad \bar{r}_\varepsilon(T) = 0, \quad (4.9b)$$

where

$$h_\varepsilon(t) := -i(T/\varepsilon) + A\alpha_\varepsilon(t) - Ai(\tau). \quad (4.10)$$

Multiplying (4.9a) by \bar{r}_ε and integrating over $[0, T]$, we obtain

$$\begin{aligned} \varepsilon \int_0^T \left((u + \bar{r}_\varepsilon)', \bar{r}_\varepsilon' \right) dt + \frac{1}{2} \int_0^T \frac{d}{dt} \|\bar{r}_\varepsilon\|^2 dt + \int_0^T (A\bar{r}_\varepsilon, \bar{r}_\varepsilon) dt \\ + \int_0^T (Bu_\varepsilon, \bar{r}_\varepsilon) dt = \int_0^T (h_\varepsilon, \bar{r}_\varepsilon) dt + \int_0^T (Bu, \bar{r}_\varepsilon) dt. \end{aligned} \quad (4.11)$$

Since B is monotone, we derive from (4.11)

$$\begin{aligned} \varepsilon \int_0^T \|\bar{r}_\varepsilon'\|^2 dt + \int_0^T (A\bar{r}_\varepsilon, \bar{r}_\varepsilon) dt \leq \int_0^T (Bu - B(u - \alpha_\varepsilon + i), \bar{r}_\varepsilon) dt \\ + \varepsilon \|u'\|_{L^2} \|\bar{r}_\varepsilon'\|_{L^2} + \|h_\varepsilon\|_{L^2} \|\bar{r}_\varepsilon\|_{L^2}. \end{aligned} \quad (4.12)$$

Note that

$$\begin{aligned} \|i(T/\varepsilon)\| &= \mathcal{O}(\varepsilon^j) \quad \forall j \geq 1, \quad \|i\|_{L^2} = \mathcal{O}(\varepsilon^{1/2}), \\ \|Ai\|_{L^2} &= \|e^{-\tau} A(u_1 - u(T))\|_{L^2} \\ &= \|A(u_1 - u(T))\|_{L^2} \|e^{-\tau}\|_{L^2(0, T)} = \mathcal{O}(\varepsilon^{1/2}). \end{aligned}$$

We estimate

$$\|B(u + i - \alpha_\varepsilon) - Bu\|_{L^2} \leq C\|i - \alpha_\varepsilon\|_{L^2} = \mathcal{O}(\varepsilon^{1/2}). \quad (4.13)$$

From (4.12) and (4.13) we obtain

$$\varepsilon \|\bar{r}'_\varepsilon\|_{L^2}^2 + c \|\bar{r}_\varepsilon\|_{L^2}^2 = \mathcal{O}(\varepsilon^{1/2}) \|\bar{r}_\varepsilon\|_{L^2} + \|u'\|_{L^2} \|\bar{r}'_\varepsilon\|_{L^2} \varepsilon,$$

since A is strongly positive with a constant $c > 0$. This estimate implies (4.4). \square

Comments

1. The results presented in this paper cover many specific problems in PDE, in particular the example mentioned in Section 1. For this particular example, one can obtain, in addition to (4.4) (which reads in this case $\|r_\varepsilon\|_{C([0,T];L^2(\Omega))} = \mathcal{O}(\varepsilon^{1/4})$) the following estimate

$$\|u_\varepsilon - u\|_{L^2(0,T;H_0^1(\Omega))} = \mathcal{O}(\varepsilon^{1/2}). \quad (4.14)$$

Indeed, an inspection of the proof of Theorem 4.1, shows that

$$(A\bar{r}_\varepsilon, \bar{r}_\varepsilon)_{L^2(0,T;L^2(\Omega))} = \int_0^T \int_\Omega \nabla_x \bar{r}_\varepsilon \cdot \nabla_x \bar{r}_\varepsilon \, dx \, dt = \mathcal{O}(\varepsilon),$$

which implies (4.14), since

$$\begin{aligned} \|\bar{r}_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} &= \|u_1 - u(T)\|_{H_0^1(\Omega)} \left(\int_0^T e^{-2(\frac{T-t}{\varepsilon})} \, dt \right)^{1/2} \\ &= \mathcal{O}(\varepsilon^{1/2}). \end{aligned}$$

So in this case the boundary layer function i can be included in the remainder term r_ε , *i.e.*, i disappears from expansion (4.1). In other words, the boundary layer is not visible in $L^2(0, T; H_0^1(\Omega))$ and problem (P_1^ε) is regularly perturbed in this space (while it is singularly perturbed in $C([0, T]; L^2(\Omega))$).

2. In fact, Theorem 4.1 holds under weaker assumptions on B , more precisely it suffices to assume that

$$B : H \rightarrow H \text{ is monotone and Lipschitz on bounded sets.} \quad (4.15)$$

Obviously, if

(Hyp 2)' B is differentiable on H and $B' : H \rightarrow L(H)$ is bounded on bounded sets, then B is Lipschitz on bounded sets.

To argue, let us revisit the proof of Theorem 4.1. First, we used Theorem 3.1 and the second part of Theorem 2.5.

Theorem 3.1 is still true if we assume the weaker condition (4.15), since we need only the fact that $Q = A + B$ is maximal monotone.

The second part of Theorem 2.5 is also valid if B satisfies (4.15). Indeed, for $r > 0$, let us consider the operator $B_r = B \circ \phi_r$, where ϕ_r is the radial retraction,

$$\phi_r(x) = \begin{cases} x & \text{if } \|x\| \leq r, \\ \frac{r}{\|x\|}x & \text{if } \|x\| \geq r. \end{cases}$$

Since ϕ_r is Lipschitz (see, *e.g.*, [2], p. 55), it follows that B_r is Lipschitz on H (with a Lipschitz constant depending on r). If $u_0 \in D(A)$ and $f \in W^{1,1}(0, T; H)$, then

problem (P_0) with B_r instead of B has a strong solution $u_r \in C^1([0, T]; H)$ (cf. Theorem 2.5). If we multiply by $u_r(t)$ the equation

$$u_r'(t) + Au_r(t) + Bu_r(t) - B0 = f(t) - B0$$

and take into account the monotonicity of A and B (A need not be strongly positive) we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_r(t)\|^2 \leq (\|f(t)\| + \|B0\|) \|u_r(t)\|$$

which implies that

$$\|u_r\|_{C([0, T]; H)} \leq M,$$

where M is a constant depending on $\|f\|_{C([0, T]; H)}$ and $\|B0\|$ (and independent of r). Therefore, if we choose $r > M$, then u_r is a solution of (P_0) . The rest of the proof of Theorem 4.1 works well for B satisfying (4.15). We just point out that in (4.13) the arguments of B belong to a ball in H whose radius depends on $\|u\|_{C([0, T]; H)}$ but is independent of ε . So Theorem 4.1 actually covers a larger class of B 's.

Further relaxations are also possible in other theorems. For example, Theorem 2.8 is valid under the weaker condition (Hyp 2)'. This condition is also sufficient to guarantee the conclusions of Theorem 3.2, as one can see by a careful inspection of the proof of Theorem 3.2.

3. It is expected that higher order asymptotic expansions hold for problem (P_1^ε) under appropriate additional assumptions on u_0, u_1, f and B . We will investigate this conjecture in a forthcoming paper.

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