On atomicity of free algebras in certain cylindric-like varieties

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Abstract: In this paper we show that the one-generated free three dimensional polyadic and substitutional algebras $\mathsf{Fr_1PA_3}$ and $\mathsf{Fr_1SCA_3}$ are not atomic. What is more, their corresponding logics have the Gödel's incompleteness property. This provides a partial solution to a longstanding open problem of Németi and Maddux going back to Alfred Tarski via the book [Tarski–Givant].

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1 Introduction

Let $\mathcal{L} = \langle \mathsf{Fm}, \vdash, \mathsf{M}, \models \rangle$ be a logic in the sense of [1] extending propositional calculus where Fm is the set of formulas, M is the class of models, and $\vdash$ and $\models$ are the syntactic and semantic consequence relations. For such a logic, one can define syntactic and semantic properties depending on which one of the symbols $\vdash$ or $\models$ is used in the definition. E.g. completeness of a theory T can be understood both as semantic or syntactic completeness (which are not the same if the consequence relation $\vdash$ is not sound or complete). From now on, for simplicity, we will define these two kinds of notions at the same time by using the symbol $\Vdash$ for both $\vdash$ and $\models$. This will cause no confusion and where it is important we emphasize which one we have in mind.

A set T of formulas is said to be *recursively incomplete* if any recursive and consistent extension $T \subseteq K \subseteq \mathsf{Fm}$ of T is incomplete, that is, for some $\phi \in \mathsf{Fm}$ we have $K \nVdash \phi$ and $K \nVdash \neg\phi$. We say that T is *weakly incomplete* if the same holds as before but with K finite. Using these definitions Gödel's incompleteness property can be formulated as follows. $\mathcal{L}$ has Gödel's incompleteness property if there exists a consistent formula that cannot be extended to a consistent, complete, recursive theory, i.e. it is recursively incomplete. If we replace the word recursive by finite, then we get the *weak incompleteness property.*

One measure of "strength" of a logic is whether Gödel's incompleteness property holds for it. Another measure is the number of variables it uses. The fewer variables it has, the "weaker" our logic is. It was proved by Németi [7], improving a result of Tarski [Tarski–Givant], that the logical counterpart of CA_3 i.e. first order logic using three variables has Gödel's incompleteness property (both syntactic and semantic). In fact, there is a consistent, recursively incomplete formula whose language contains one binary relation symbol only. A consequence of this is that the one-generated free three dimensional cylindric algebra is not atomic. For further detail we refer to Theorem 1 (a) and (b) of [7]. For completeness we note that the same hold for the classes PEA_3 and RPEA_3 and for their corresponding logics: Németi's proofs work in these cases as well, see [6], [7]. It is also known that first order logic with two variables (the corresponding logic of CA_2) is decidable thus it does not enjoy Gödel's incompleteness property (not even the weak incompleteness property; a result of Henkin, see [5] 2.5.7 (ii), 4.2.7-9).

In the quest of searching for the "weakest strong" logic there remained the following cases lying between CA_2 and CA_3.

(1) First order logic with three variables without equality with permutations and substitutions of variables (PA_3);

(2) First order logic with three variables without equality with substitutions (SCA_3);

(3) First order logic with three variables without equality (Df_3).

In this paper we give answers to cases **(1)** and **(2)**. In more detail we show that the mentioned logics in **(1)** and **(2)** have both the syntactic and semantic Gödel's incompleteness properties. For completeness, we note that in the case of **(1)** we use permutations only (i.e. our proof works in the subcase of substitution-free reduct as well). Its algebraic consequences are that the one-generated free three dimensional substitution algebra and the one-generated free polyadic algebra are not atomic. We note that this result have been announced by Németi [8] but remained unpublished. Here we give a simpler proof.

The structure of the rest of the paper is as follows. In section 2 we present the main ideas lying behind our proofs in a quite general framework. Then we turn to the special cases: in section 3 we summarize our notation and describe the logical and algebraic framework we use. Then in section 4 we prove our main results about the above mentioned equality-free logics. Finally we give the algebraic consequences in section 5.

2 A general scenario

In this section we present the main ideas of our approach in a general situation. From now on, we will deal only with such logics that have an algebraic counterpart which is a BAO variety. In the following sections we are going to show the existence of functions which are described in the next definition.

Definition 1 *For $i \in 2$ let $\mathcal{L}_i = \langle \mathsf{Fm}_i, \mathsf{M}_i, \vdash_i, \models_i \rangle$ be two logics. Then the pair $\langle \mathcal{L}_0, \mathcal{L}_1 \rangle$ is said to be a* good pair of logics *if the following hold:*

(i) $\mathsf{Fm}_1 \subseteq \mathsf{Fm}_0$ *and*

(ii) $\vdash_0$ *extends* $\vdash_1$*, i.e.* $\vdash_1 \subseteq \vdash_0$.

In this situation a function $\mathsf{tr} : \mathsf{Fm}_0 \to \mathsf{Fm}_1$ *is called a* formula translator function *if the following properties hold.*

(1) tr *is a homomorphism w.r.t. the common connectives of the appropriate formula algebras;*

(2) $\mathsf{tr}_{|\mathsf{Fm}_1} = \mathsf{id}_{\mathsf{Fm}_1}$*;*

(3) tr *is recursive;*

If Σ *is a set of formulas then* $\mathsf{tr}(\Sigma)$ *denotes the set* $\{\mathsf{tr}(\sigma) : \sigma \in \Sigma\}$.

A function $\mathsf{t} : \mathsf{M}_1 \to \mathsf{M}_0$ *is said to be a* semantic adjoint of tr *if it satisfies the following.*

(1) $\mathsf{M}_1 \ni \mathcal{A} \vDash_1 \mathsf{tr}(\phi)$ *iff* $\mathsf{t}(\mathcal{A}) \vDash_0 \phi$ *for all* $\phi \in \mathsf{Fm}_0$*;*

(2) t *is surjective.*

Proposition 2 *Let* $\langle \mathcal{L}_0, \mathcal{L}_1 \rangle$ *be a good pair of logics, and let* tr *be a corresponding translator function. Suppose there exists a recursive* $T \subseteq \mathsf{Fm}_0$ *such that*

(1) T *is recursively syntactically incomplete in* $\mathcal{L}_0$*;*

(2) $T \vdash_0 \varphi \leftrightarrow \mathsf{tr}(\varphi)$ *for all* $\varphi \in \mathsf{Fm}_0$.

Then $\mathsf{tr}(T)$ *is recursively syntactically incomplete in* $\mathcal{L}_1$.

Proof: Let $T \subseteq \mathsf{Fm}_0$ be a recursively incomplete theory in $\mathcal{L}_0$ and set $K = \mathsf{tr}(T)$. By way of contradiction suppose K is not recursively incomplete, that is, there is some recursive set of formulas $K' \subseteq \mathsf{Fm}_1$ such that $K \cup K'$ is complete (in $\mathcal{L}_1$). Note that $K' = \mathsf{tr}(K')$ and by assumption $T \cup K \cup K'$ is incomplete, thus for some $\varphi \in \mathsf{Fm}_0$ we have $T \cup K \cup K' \nvdash_0 \varphi$ and $T \cup K \cup K' \nvdash_0 \neg\varphi$. By completeness of $K \cup K'$ we get, say, $K \cup K' \vdash_1 \mathsf{tr}(\varphi)$. But this deduction is also a deduction of $\mathcal{L}_0$ hence $K \cup K' \vdash_0 \mathsf{tr}(\varphi)$. Then it follows that $T \cup K \cup K' \vdash_0 \mathsf{tr}(\varphi)$ and by assumption (2) we get $T \cup K \cup K' \vdash_0 \varphi$ which is a contradiction. ∎

Proposition 3 *Let* $\langle \mathcal{L}_0, \mathcal{L}_1 \rangle$ *be a good pair of logics with the corresponding translator function* tr *with semantic adjoint* t. *Suppose there exists a recursive* $T \subseteq \mathsf{Fm}_0$ *which is recursively semantically incomplete in* $\mathcal{L}_0$. *Then* $\mathsf{tr}(T)$ *is recursively semantically incomplete in* $\mathcal{L}_1$.

Proof: By way of contradiction suppose there exists a set $K \subseteq \mathsf{Fm}_1$ such that $\mathsf{tr}(T) \cup K$ is complete. Observe that $K = \mathsf{tr}(K)$. Let $\phi \in \mathsf{Fm}_0$ be a formula which is independent from $T \cup K$, that is, for some models $\mathcal{B}, \mathcal{B}' \in M_0$ we have $\mathcal{B} \vDash_0 T \cup K \cup \{\phi\}$ and $\mathcal{B}' \vDash_0 T \cup K \cup \{\neg\phi\}$. Since t is surjective there exist $\mathcal{A}, \mathcal{A}' \in \mathsf{M}_1$ such that $\mathcal{B} = \mathsf{t}(\mathcal{A})$ and $\mathcal{B}' = \mathsf{t}(\mathcal{A}')$. Then by property (1) of t we get $\mathcal{A} \vDash \mathsf{tr}(T) \cup K \wedge \mathsf{tr}(\phi)$ and $\mathcal{A}' \vDash \mathsf{tr}(T) \cup K \wedge \neg\mathsf{tr}(\phi)$, thus $\mathsf{tr}(\phi)$ is independent from $\mathsf{tr}(T) \cup K$ in $\mathcal{L}_1$, which is a contradiction. ∎

3 Variants of first order logic with n variables.

This part overviews some background around the connection between certain logics and their corresponding classes of algebras. For more detail we suggest consulting [1].

Let $n \in \omega$ and let $V_n = \{v_i : i < n\}$ be our set of variables. We use one n-ary relation symbol R. The set of atomic formulas of the logic $\mathcal{L}_n$ is $\{R(v_0, v_1, \ldots, v_{n-1})\}$, and the set of connectives is $\{\neg,\ \wedge,\ v_i = v_j,\ \exists v_i : i, j < n\}$. This defines the set Fm_n of formulas. The class of models and consequence relations $\vDash_n$ and $\vdash_n$ are defined as usual.

Proof theory and model theory of $\mathcal{L}_n$ correspond to the classes CA_n of cylindric algebras and RCA_n of representable cylindric algebras, respectively. In more detail any formula $\varphi \in \mathsf{Fm}_n$ can be identified with a term in the algebraic language of CA_n such that R is considered as a variable, assuming that we identify the operations of CA_n with connectives of $\mathcal{L}_n$. Hence $\varphi = 1$ is an equation in the language of CA_n. We recall the following theorem.

Theorem (see [1], p.225) Let $\varphi \in \mathsf{Fm}_n$, n any ordinal. Then (i) and (ii) below hold.
(i) $\vdash_n \varphi$ iff $\mathsf{CA}_n \vDash \varphi = 1$;
(ii) $\vDash_n \varphi$ iff $\mathsf{RCA}_n \vDash \varphi = 1$.

There are at least three ways of making this logic weaker by dropping equalities. One is to drop equalities but keep substitutions as connectives. Other is to forget about equalities but keep transpositions, and the third one is to simply remove equalities. By these processes we get different logics with different corresponding classes of algebras. The following table tries to summarize the situation.

Connectives	Logic	Algebra classes
$\{\neg,\ \wedge,\ \exists v_i,\ v_i = v_j : i, j < n\}$	$\mathcal{L}_n$	CA_n, RCA_n
$\{\neg,\ \wedge,\ \exists v_i,\ [v_i/v_j] : i, j < n\}$	$\mathcal{L}_n^s$	SCA_n, RSCA_n
$\{\neg,\ \wedge,\ \exists v_i,\ [v_i, v_j],\ [v_i/v_j] : i, j < n\}$	$\mathcal{L}_n^p$	PA_n, RPA_n
$\{\neg,\ \wedge,\ \exists v_i : i < n\}$	$\mathcal{L}_n^f$	Df_n, RDf_n

The classes of Diagonal Free Cylindric Algebras (Df_n), Polyadic Algebras (PA_n) and Substitution Algebras (SCA_n) have been intensively studied since the 1960's; we refer to [4] and [5] as standard references. For more recent related investigations see e.g. [2], [3], [10], [11] and [9].

Results of section 4 are true both for $\mathcal{L}_3^s$ and $\mathcal{L}_3^p$ so for simplicity, by a slight abuse of notation, in the rest of the paper by $\mathcal{L}_3^{\neq}$ we understand one of $\mathcal{L}_3^s$ or $\mathcal{L}_3^p$. The reason for this is the following. Instead of using a ternary relation symbol R we use a binary relation symbol $\in$ which can be defined both using substitutions or permutations as follows.

Definitions in $\mathcal{L}_3^p$.

Logical framework			Algebraic framework		
$v_0 \in v_1$	$=$	$\exists v_2 R(v_0, v_1, v_2)$	$\in_{01}$	$=$	$c_2 R$
$v_0 \in v_2$	$=$	$\exists v_1 R(v_0, v_1, v_2)$	$\in_{02}$	$=$	$c_1 R$
$v_1 \in v_2$	$=$	$\exists v_0 R(v_0, v_1, v_2)$	$\in_{12}$	$=$	$c_0 R$
$v_1 \in v_0$	$=$	$\exists v_2 [v_0, v_1] R(v_0, v_1, v_2)$	$\in_{10}$	$=$	$c_2 p_{01} R$
$v_2 \in v_0$	$=$	$\exists v_1 [v_0, v_2] R(v_0, v_1, v_2)$	$\in_{20}$	$=$	$c_1 p_{02} R$
$v_2 \in v_1$	$=$	$\exists v_0 [v_1, v_2] R(v_0, v_1, v_2)$	$\in_{21}$	$=$	$c_0 p_{12} R$

Definitions in $\mathcal{L}_3^s$.

Logical framework			Algebraic framework		
$v_0 \in v_1$	$=$	$\exists v_2 R(v_0, v_1, v_2)$	$\in_{01}$	$=$	$c_2 R$
$v_0 \in v_2$	$=$	$\exists v_1 R(v_0, v_1, v_2)$	$\in_{02}$	$=$	$c_1 R$
$v_1 \in v_2$	$=$	$\exists v_0 R(v_0, v_1, v_2)$	$\in_{12}$	$=$	$c_0 R$
$v_1 \in v_0$	$=$	$\exists v_2 [v_0/v_2][v_2/v_1][v_1/v_0] R(v_0, v_1, v_2)$	$\in_{10}$	$=$	$c_2 s_2^0 s_1^2 s_0^1 R$
$v_2 \in v_0$	$=$	$\exists v_1 [v_0/v_1][v_1/v_2][v_2/v_0] R(v_0, v_1, v_2)$	$\in_{20}$	$=$	$c_1 s_1^0 s_2^1 s_0^2 R$
$v_2 \in v_1$	$=$	$\exists v_0 [v_1/v_0][v_0/v_2][v_2/v_1] R(v_0, v_1, v_2)$	$\in_{21}$	$=$	$c_0 s_0^1 s_2^0 s_1^2 R$

So after deciding which logic we use $\mathcal{L}_3^s$ or $\mathcal{L}_3^p$, it is legal to write formulas like $v_i \in v_j$. We will do this without any further warning.

4 Logical counterpart

Let $\mathsf{Ax}_{\mathsf{eq}}$ and $\mathsf{Ax}_{\mathsf{cong}}$ be the following sets of formulas:

$$\mathsf{Ax}_{\mathsf{eq}} = \big\{\forall xy(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)) : \ \{x, y, z\} = \{v_0, v_1, v_2\}\big\},$$
$$\mathsf{Ax}_{\mathsf{cong}} = \big\{\forall xy(\forall z(z \in x \leftrightarrow z \in y) \longrightarrow \forall z(x \in z \leftrightarrow y \in z)) : \ \{x, y, z\} = \{v_0, v_1, v_2\}\big\}.$$

For completeness, we note that the corresponding algebraic terms of $\mathsf{Ax}_{\mathsf{eq}}$ are the following.

$$\mathsf{Ax}_{\mathsf{eq}} = \big\{d_{ij} = -c_k\big[(\in_{ki} - \in_{kj}) + (\in_{kj} - \in_{ki}))\big] : \{i, j, k\} = 3\big\}.$$

We define a function which translates a formula $\varphi \in \mathsf{Fm}_3$ into a formula $\mathsf{tr}(\varphi) \in \mathsf{Fm}_3^{\neq}$. Suppose a formula $\varphi \in \mathsf{Fm}_3$ is given. Then for all $\{x, y, z\} = \{v_0, v_1, v_2\}$ let us replace all the occurrences of $x = y$ by $\forall z(z \in x \leftrightarrow z \in y)$ simultaneously. Denote this new formula with $\mathsf{tr}(\varphi)$. If Σ is a set of formulas then $\mathsf{tr}(\Sigma) = \{\mathsf{tr}(\varphi) : \varphi \in \Sigma\}$. Note that tr is a homomorphism with respect to the common connectives of the corresponding formula algebras, that is, $\mathsf{tr}(\neg\varphi) = \neg\mathsf{tr}(\varphi)$, $\mathsf{tr}(\phi_1 \wedge \phi_2) = \mathsf{tr}(\phi_1) \wedge \mathsf{tr}(\phi_2)$ and $\mathsf{tr}(\exists v\varphi) = \exists v\mathsf{tr}(\varphi)$.

4.1 $\mathcal{L}_3^{\neq}$ has the syntactic Gödel's incompleteness property

We arrived at proving that $\mathcal{L}_3^{\neq}$ has the syntactic Gödel's incompleteness property. For brevity we will denote $\vdash_3$ by $\vdash$. In this subsection we use the syntactic versions of our notions, e.g. completeness means syntactic completeness, etc.

Lemma 4 *If $T \vdash \alpha \leftrightarrow \beta$ then by replacing all the occurrences of α in a closed formula ϕ with β and denoting this new formula by ϕ' we get $T \vdash \phi \leftrightarrow \phi'$.*

PROOF: The proof goes by induction on the complexity of the formula ϕ. By symmetry we will concentrate on only one of the directions, say on $T \vdash \phi \to \phi'$. The other direction is completely similar.
Case 1. Suppose we know

$$T \vdash \phi_0 \leftrightarrow \phi_0' \text{ and}$$
$$T \vdash \phi_1 \leftrightarrow \phi_1'.$$

We would like to get $T \vdash \phi_0 \wedge \phi_1 \to \phi_0' \wedge \phi_1'$. Converting the formula we get

$$\phi_0 \wedge \phi_1 \leftrightarrow \phi_0' \wedge \phi_1' \equiv (\phi_0 \wedge \phi_1 \to \phi_0') \wedge (\phi_0 \wedge \phi_1 \to \phi_1').$$

Since the inference system $\vdash$ extends propositional calculus, we have if $T \vdash \psi_0$ and $T \vdash \psi_1$ then $T \vdash \psi_0 \wedge \psi_1$ and also if $T \vdash \phi_0 \leftrightarrow \phi_i'$ then $T \vdash (\phi_0 \wedge \phi_1) \to \phi_i'$. From these two observations one can conclude the desired deduction.
Case 2. Suppose $T \vdash \phi \leftrightarrow \phi'$. We would like to obtain $T \vdash \neg\phi \to \neg\phi'$. Note that $T \vdash \neg\varphi$ if and only if $T \cup \{\varphi\}$ is inconsistent (again by some known properties of propositional calculus). So $T \vdash \neg\phi \to \neg\phi'$ if and only if $T \cup \{\neg\phi\} \vdash \neg\phi'$ if and only if $T \cup \{\neg\phi, \phi'\}$ is inconsistent. But $T \vdash \phi' \leftrightarrow \phi$ thus $T \cup \{\neg\phi, \phi'\}$ is equiconsistent with $T \cup \{\neg\phi, \phi\}$ which is, of course, inconsistent.
Case 3. Finally suppose again $T \vdash \phi \leftrightarrow \phi'$. Then using generalization $T \vdash \forall x(\phi \leftrightarrow \phi')$ and by a logical axiom we get $T \vdash \forall x\phi \leftrightarrow \forall x\phi'$ which finishes the induction and the proof. ∎

Lemma 5 $\mathsf{Ax_{eq}} \vdash \varphi \leftrightarrow \mathsf{tr}(\varphi)$, *for all* $\varphi \in \mathsf{Fm}_3$.

PROOF: For the formula $x = y$ the statement is clear from the definition of $\mathsf{Ax_{eq}}$ and for $x \in y$ it is also clear because $\mathsf{tr}(x \in y) = x \in y$. The rest follows from Lemma 4. ∎

Theorem 6 *Suppose $T \subseteq \mathsf{Fm}_3$ is a recursive set of formulas such that*
(i) T is recursively incomplete in $\mathcal{L}_3$;
(ii) $\mathsf{Ax_{eq}} \subseteq T$.
Then $\mathsf{tr}(T)$ is recursively incomplete in $\mathcal{L}_3^{\neq}$.

PROOF: Its easy to see that $\langle \mathcal{L}_3, \mathcal{L}_3^{\neq} \rangle$ is a good pair of logics, and tr is a translator function for this pair. By (ii) and lemma 5 the conditions of proposition 2 hold, hence the result follows from its consequence. ∎

Theorem 7 *$\mathcal{L}_3^{\neq}$ has the syntactic Gödel's incompleteness property.*

Proof: We recall that in [7] it was shown that there exists a finite and consistent set $T \subseteq \mathsf{Fm}_3$ of formulas which is recursively incomplete in $\mathcal{L}_3$. The key step of the proof was showing that classical first order arithmetic (formulated in the language of $\in$) can be built up using three variables. For further detail we refer to both [7] and [6]. This T is semantically consistent because it has a model. In fact the standard model of arithmetic is such. But in the standard model ZF's extensionality holds hence $T \cup \{\mathsf{Ax_{eq}}\}$ is also semantically (and therefore syntactically) consistent. Thus by theorem 6 the proof is complete. ∎

4.2 $\mathcal{L}_3^{\neq}$ has the semantic Gödel's incompleteness property

In what follows, we show that the semantic incompleteness property also holds for $\mathcal{L}_3^{\neq}$. Throughout this subsection we use the semantic versions of our notions, e.g. completeness means semantic completeness, etc. In the next lemma $\mathsf{Fm}_\omega^{\neq}$ denotes the set of those first order formulas (the classical ones) which don't contain equality. We call a homomorphism *strong* if in addition to being homomorphism it preserves relation symbols in both directions.

Lemma 8 *Let $h : \mathcal{A} \to \mathcal{B}$ be a surjective strong homomorphism. Then it preserves all the formulas not containing equality. In more detail for any formula $\varphi \in \mathsf{Fm}_\omega^{\neq}$ and any valuation k we have*

$$\mathcal{A} \vDash \varphi[k] \longleftrightarrow \mathcal{B} \vDash \varphi[h \circ k].$$

Proof: We prove it by induction on the complexity of the formula. For atomic formulas it is true by definition of a strong homomorphism. Suppose h preserves ϕ_i $(i \in 2)$ that is $\mathcal{A} \vDash \phi_i[\bar{a}] \longleftrightarrow \mathcal{B} \vDash \phi_i[h(\bar{a})]$.

If $\psi = \phi_0 \wedge \phi_1$ then

$$\mathcal{A} \vDash \phi_0 \wedge \phi_1[\bar{a}] \Leftrightarrow \left\{ \begin{array}{ccc} \mathcal{A} \vDash \phi_0[\bar{a}] & \Leftrightarrow & \mathcal{B} \vDash \phi_0[h(\bar{a})] \\ \mathcal{A} \vDash \phi_1[\bar{a}] & \Leftrightarrow & \mathcal{B} \vDash \phi_1[h(\bar{a})] \end{array} \right\} \Leftrightarrow \mathcal{B} \vDash \phi_0 \wedge \phi_1[h(\bar{a})].$$

If $\psi = \exists v \phi_0$ then $\mathcal{A} \vDash \exists v \phi_0$ if and only if there exists $a \in A$ such that $\mathcal{A} \vDash \phi_0[v/a]$. By induction this is equivalent to $\mathcal{B} \vDash \varphi[x/h(a)]$, thus $\mathcal{B} \vDash \exists v \phi_0$. The converse implication follows similarly from surjectivity of h.

If $\psi = \neg \phi_0$, and $\mathcal{A} \vDash \neg \phi_0$ then suppose $\mathcal{B} \vDash \phi_0[\bar{c}]$. Since h is surjective there exist $\bar{a}$ with $h(\bar{a}) = \bar{c}$. Thus $\mathcal{B} \vDash \phi[h(\bar{a})]$ and then by induction $\mathcal{A} \vDash \phi_0[\bar{a}]$, but then $\mathcal{A} \nvDash \neg \phi_0[\bar{a}]$, which is a contradiction. ∎

As a next step we present a model construction. Let $\mathcal{A} \in \mathsf{Mod}^{\neq}$ and further let $\approx \subseteq A \times A$ such that $x \approx y \longleftrightarrow \forall z (z \in x \leftrightarrow z \in y)$. It is easy to see that $\approx$ is an equivalence relation on A.

We claim that if $\mathcal{A} \vDash \mathsf{Ax_{cong}}$ then $\approx$ is also a strong congruence relation, that is, it preserves the relation $\in$ in both directions. We note that $\approx$ is sometimes called the Leibniz congruence of $\mathcal{A}$. It follows that one can define $\mathcal{A}/_{\approx}$ as usual. To this end, suppose $a \approx a'$ and $b \approx b'$ and $a \in b$. Then we have to prove $a' \in b'$. If $a \in b$ and $a \approx a'$ then by $\mathsf{Ax_{cong}}$ we have $a' \in b$. But $b \approx b'$ thus $a' \in b'$ as desired. Observe that equality in $\mathcal{A}/_{\approx}$ is the same as $\approx$ in $\mathcal{A}/_{\approx}$, consequently $(\mathcal{A}/_{\approx})/_{\approx} = \mathcal{A}/_{\approx}$. It is important to note that if $\mathcal{B} \vDash \varphi$ then $\mathcal{B} \vDash \mathsf{tr}(\varphi)$, since equality implies $\approx$.

The following lemma is the semantic version of Lemma 5.

Lemma 9 *For all formula $\phi \in \mathsf{Fm}_3$ and for all structure $\mathcal{A} \in \mathsf{Mod}^{\neq}(\mathsf{Ax_{cong}})$ we have*

$$\mathcal{A} \vDash \mathsf{tr}(\phi) \Longleftrightarrow \mathcal{A}/_{\approx} \vDash \phi.$$

PROOF: To prove the statement consider two cases.

In the first case suppose $\mathcal{A} \vDash \mathsf{tr}(\phi)$. The natural mapping $a \mapsto a/_{\approx}$ is clearly a surjective strong homomorphism, thus by Lemma 8, we get $\mathcal{A}/_{\approx} \vDash \mathsf{tr}(\phi)$. But by our remark, in $\mathcal{A}/_{\approx}$ equality is the same as $\approx$, so we get $\mathcal{A}/_{\approx} \vDash \phi$.

For the second case suppose $\mathcal{A}/_{\approx} \vDash \phi$. Then $\mathcal{A}/_{\approx} \vDash \mathsf{tr}(\phi)$ since equality trivially implies $\approx$. Now if $\mathcal{A} \vDash \neg\mathsf{tr}(\phi)[k]$ would be true for some valuation k, then again by Lemma 8 we would get $\mathcal{A}/_{\approx} \vDash \neg\mathsf{tr}(\phi)[k/_{\approx}]$, which is a contradiction. ∎

Lemma 10 *For every model $\mathcal{B} \in \mathsf{Mod}^{=}(\mathsf{Ax_{eq}})$ there exists a model $\mathcal{A} \in \mathsf{Mod}^{\neq}$ such that*

$$\mathcal{B} \cong \mathcal{A}/_{\approx}.$$

PROOF: If $\mathcal{B} \vDash \mathsf{Ax_{eq}}$ then $\mathcal{B}/_{\approx} \cong \mathcal{B}$, so $\mathcal{A} = \mathcal{B}$ will be good. Note that we do not need $\mathsf{Ax_{cong}}$ to be true in $\mathcal{B}$ to define $\mathcal{B}/_{\approx}$, because since $\mathcal{B}/_{\approx} \cong \mathcal{B}$ it remains a model (thus well defined). ∎

Theorem 11 *Suppose $T \subseteq \mathsf{Fm}_3$ is a recursive set of formulas such that*

(i) T is recursively incomplete in $\mathcal{L}_3$;

(ii) $\mathsf{Ax_{eq}}, \mathsf{Ax_{cong}} \subseteq T$.

Then $\mathsf{tr}(T)$ is recursively incomplete in $\mathcal{L}_3^{\neq}$.

PROOF: Let the function $t : \mathsf{Mod}^{\neq} \to \mathsf{Mod}$ be defined as $\mathcal{A} \mapsto \mathcal{A}/_{\approx}$. Then by lemma 10 and lemma 9 this function is a semantic adjoint of tr. Hence the result follows from proposition 3. ∎

Theorem 12 *$\mathcal{L}_3^{\neq}$ has the semantic Gödel's incompleteness property.*

PROOF: The proof is the same as in 7. Since ω is a model of arithmetic (formulated in the language of $\in$) and $T \cup \{\mathsf{Ax_{eq}}, \mathsf{Ax_{cong}}\}$, the proof can be completed using theorem 11. ∎

5 Algebraic counterpart

In this section we investigate the algebraic consequences of theorems 7 and 12. As usual $\mathsf{Fr}_k\mathfrak{A}$ denotes the k-generated free algebra in the variety $\mathfrak{A}$.

Theorem 13 *The following algebras are not atomic:*

(i) $\mathsf{Fr}_1\mathsf{SCA}_3$ *and* $\mathsf{Fr}_1\mathsf{RSCA}_3$*;*

(ii) $\mathsf{Fr}_1\mathsf{PA}_3$ *and* $\mathsf{Fr}_1\mathsf{RPA}_3$*.*

PROOF: According to section 3, results of theorems 7 and 12 apply to both $\mathcal{L}_3^s$ and $\mathcal{L}_3^p$. Therefore proofs of (i) and (ii) are essentially the same. Thus it is enough to prove (i).

Consider the one-generated free three dimensional substitution algebra $\mathsf{Fr}_1\mathsf{SCA}_3$. Every element of this algebra can be considered as a term in the algebraic language of SCA's which can be (and will be) identified with a formula of $\mathcal{L}_3^s$ (because we used only one ternary relation symbol). Let ϕ be the formula showing syntactic Gödel's incompleteness property for $\mathcal{L}_3^s$.

Every atom of $\mathsf{Fr}_1\mathsf{SCA}_3$ defines a syntactically complete theory which contains those elements which are elements of the ultrafilter containing the atom. Since ϕ has no finite complete extensions there can be no atoms below it.

The case of $\mathsf{Fr}_1\mathsf{RSCA}_3$ is similar but using semantic incompleteness property. Note that in this part of the proof we used the weaker weakly incompleteness instead of recursive incompleteness. In fact, as it is easy to see, the weak incompleteness property is equivalent to the non-atomicity of the appropriate algebras (see also Proposition 1.8 of [7]). ∎

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