A multiplicity result for an elliptic anisotropic differential inclusion involving variable exponents

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June 21, 2012

Abstract In this paper we are concerned with the study of a class of quasilinear elliptic differential inclusions involving the anisotropic $\overrightarrow{p}(\cdot)$-Laplace operator, on a bounded open subset of $\mathbb{R}^n$ which has a smooth boundary. The abstract framework required to study this kind of differential inclusions lies at the interface of three important branches in analysis: *nonsmooth analysis*, *the variable exponent Lebesgue-Sobolev spaces theory* and *the anisotropic Sobolev spaces theory*. Using the concept of *nonsmooth critical point* we are able to prove that our problem admits at least two non-trivial weak solutions.

Keywords Clarke's generalized gradient · Differential inclusion · Nonhomogeneous differential operator · Anisotropic Sobolev spaces · Nonsmooth critical point · Multiple weak solutions

Mathematics Subject Classification (2010) 58K05 · 47J30 · 34A60 · 49J52 · 35D30

1 Introduction

In this paper we study the weak solvability of a differential inclusion involving a nonhomogeneous anisotropic differential operator of the following type

$$\begin{cases} -\sum\limits_{i=1}^{n} \partial_i \left(|\partial_i u|^{p_i(x)-2} \partial_i u \right) \in \lambda \partial_C \alpha(x,u) + \partial_C \beta(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a bounded open set with smooth boundary, $\lambda > 0$ is a real parameter, $\alpha, \beta : \Omega \times \mathbb{R} \to \mathbb{R}$ are two locally Lipschitz functions with respect to the second variable and, for each $i \in \{1, \ldots, n\}$, $p_i : \bar{\Omega} \to \mathbb{R}$ is a continuous function such that $2 \leq p_i(x) < n$ for all $x \in \bar{\Omega}$. The notation $\partial_i u$ stands for the partial derivative of u with respect to the x_i component, that is $\partial u / \partial x_i$, while $\partial_C \alpha(x,t)$ denotes the Clarke generalized gradient of the function $t \mapsto \alpha(x,t)$. The definition and main properties of the Clarke generalized gradient will be given in the next section.

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We point out the fact that, if $\alpha(x,t)=\frac{1}{q(x)}|t|^{q(x)}$ and $\beta\equiv const.$, then problem (1) reduces to the following nonhomogeneous anisotropic eigenvalue problem

$$\begin{cases} -\sum\limits_{i=1}^{n}\partial_i\left(|\partial_i u|^{p_i(x)-2}\partial_i u\right)=\lambda|u|^{q(x)-2}u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

which was studied by Mihăilescu, Pucci and Rădulescu [22,23]. In these papers the authors show that the "competition" between the growth rates of the functions p_i and q deeply influence the existence or nonexistence of the weak solutions. To our best knowledge these are the first papers dealing with the *anisotropic variable exponent* $\overrightarrow{p}(\cdot)$*-Laplace operator*, i.e.

$$\Delta_{\overrightarrow{p}(\cdot)}u=\sum_{i=1}^{n}\partial_i\left(|\partial_i u|^{p_i(x)-2}\partial_i u\right),$$

where $\overrightarrow{p}:\bar{\Omega}\to\mathbb{R}^n$ is the vectorial function $\overrightarrow{p}(\cdot)=(p_1(\cdot),\dots,p_n(\cdot))$. Also in these papers it was introduced for the first time the anisotropic exponent Sobolev space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ that allowed an accurate study of problems of the type (2). We point out that the aforementioned space can be viewed as a natural generalization of the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ (when $p_1(\cdot)=\dots=p_n(\cdot)=p(\cdot)$) as well as a natural generalization of the classical anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$ (when p_i are constant functions, $i\in\{1,\dots,n\}$).

On the other hand, let us consider the case when $\alpha\equiv const.$ and β is the primitive of some Carathéodory function $f:\Omega\times\mathbb{R}\to\mathbb{R}$

$$\beta(x,t)=\int_0^t f(x,s)\,ds.$$

Then the function $t\mapsto\beta(x,t)$ is differentiable and thus $\partial_C\beta(x,t)=\{f(x,t)\}$ and problem (1) reduces to the following nonhomogeneous anisotropic problem

$$\begin{cases} -\sum\limits_{i=1}^{n}\partial_i\left(|\partial_i u|^{p_i(x)-2}\partial_i u\right)=f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

which was studied recently by Boureanu, Pucci and Rădulescu [3], by using the symmetric mountain-pass theorem of Ambrosetti and Rabinowitz.

The abstract framework required to study differential inclusions of the type (1) lies at the interface of three important directions in analysis:

- *the nonsmooth analysis*: the need for such theory comes naturally whenever we deal with functions which are not differentiable everywhere, but are convex or locally Lipschitz (see, e.g., Andrei, Costea and Matei [2], Chang [5], Clake [7], Costea and Varga [9,10], Kristály, Rădulescu and Varga [18], Motreanu and Panagiotopoulos [27], Motreanu and Rădulescu [28], Naniewicz and Panagiotopoulos [31], Panagiotopoulos [32]).
- *the variable exponent Lebesgue-Sobolev spaces theory*: problems involving the isotropic $p(x)$-Laplace operator have captured special attention in the last decades since they can model various phenomena which arise in elastic mechanics (see, e.g., Zhikov [39]), image restoration (see, e.g., Chen, Levine and Rao [6]) or electrorheological fluids (see, e.g., Acerbi and Mingione [1], Diening [11], Diening, Harjulehto, Hästö and Ružička [12], Halsey [16], Ružička [35], Costea and Mihăilescu [8], Mihăilescu and Rădulescu [20,21]).

- *the anisotropic Sobolev spaces theory*: the need for such theory comes naturally whenever we deal with materials possessing inhomogeneities that have different behavior on different space directions (see, e.g., Edmunds and Edmunds [14], Nikol'skii [30], Rákosník [33,34], Troisi [38]).

Although the $\overrightarrow{p}(\cdot)$-Laplace operator was introduced recently (in 2007 by Mihăilescu, Rădulescu and Pucci), problems involving this operator, or similar operators, have captured special attention in the last years (see Boureanu, Pucci and Rădulescu [3], Fan [13], Mihăilescu and Moroşanu [26], Mihăilescu, Moroşanu and Rădulescu [24,25], Stancu-Dumitru [36]). However, in all the works we are aware of, the *energy functional* attached to the problem is smooth, while differential inclusions like problem (1), for which the attached energy functional is only locally Lipschitz and not differentiable, have not yet been studied. Thus, this paper represents the first contribution in this direction.

2 Mathematical background

For the convenience of the reader we present in this section some notations and preliminary results from nonsmooth analysis as well as some basic properties of the variable exponent Lebesgue-Sobolev spaces that will be used throughout the paper. For a given Banach space $(X, \|\cdot\|_X)$ we denote by X^* its dual space and by $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X. The inner product and the euclidian norm in $\mathbb{R}^m$ $(m \geq 1)$ will be denoted by " $\cdot$ " and $|\cdot|$, respectively.

We recall that a functional $\varphi : X \to \mathbb{R}$ is said to be *locally Lipschitz* if, for every $u \in X$ there exists a neighborhood V of u and a positive constant L, which depends on the neighborhood V, such that

$$|\varphi(w) - \varphi(v)| \leq L\|w - v\|_X, \quad \text{for all } v, w \in V.$$

Definition 1 Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional. The *Clarke generalized derivative* of φ at $u \in X$ with respect to the direction $v \in X$, denoted $\varphi^0(u; v)$, is defined by

$$\varphi^0(u; v) = \limsup_{\substack{\tilde{u} \to u \\ \mu \downarrow 0}} \frac{\varphi(\tilde{u} + \mu v) - \varphi(\tilde{u})}{\mu},$$

while the *Clarke generalized gradient* of φ at a point $u \in X$, denoted $\partial_C \varphi(u)$, is the subset of X^* defined by

$$\partial_C \varphi(u) = \{\zeta \in X^* : \ \varphi^0(u; v) \geq \langle \zeta, v \rangle, \ \text{ for all } v \in X\}.$$

We point out the fact that for each $u \in X$ the Clarke generalized gradient $\partial_C \varphi(u)$ is a nonempty subset of X^*, which is a direct consequence of the Hahn-Banach theorem (see, e.g., Brezis [4]).

The following lemma points out some important properties of generalized derivatives that will be used in the sequel.

Lemma 1 *Let $\varphi, \psi : X \to \mathbb{R}$ be two locally Lipschitz functionals. Then*

- *for each fixed $u \in X$, the function $v \mapsto \varphi^0(u; v)$ is finite, subadditive and satisfies*

$$|\varphi^0(u; v)| \leq L\|v\|_X,$$

  where $L > 0$ is the Lipschitz constant near the point u;
- *the application $(u, v) \mapsto \varphi^0(u; v)$ is upper semicontinuous;*
- *$\varphi^0(u; -v) = (-\varphi)^0(u; v)$ for all $u, v \in X$;*
- *$\varphi^0(u; \mu v) = \mu\varphi^0(u; v)$ for all $u, v \in X$ and all $\mu > 0$;*
- *$(\varphi + \psi)^0(u; v) \leq \varphi^0(u; v) + \psi^0(u; v)$ for all $u, v \in X$.*

The proof can be found in Clarke [7].

Definition 2 A point $u \in X$ is said to be a *nonsmooth critical point* of the locally Lipschitz functional $\varphi : X \to \mathbb{R}$, if $0 \in \partial_C \varphi(u)$.

According to the previous definition and the definition of the Clarke generalized gradient a point $u \in X$ is a nonsmooth critical point φ if and only if $\varphi^0(u; v) \geq 0$, for all $v \in X$. We point out the fact that every local extremum point of the locally Lipschitz function φ is a nonsmooth critical point of φ in the sense of Definition 2.

Definition 3 A locally Lipschitz functional $\varphi : X \to \mathbb{R}$ is said to satisfy the *nonsmooth Palais-Smale condition at level* $c \in \mathbb{R}$ ($(PS)_c$-condition in short) if any sequence $\{u_k\} \subset X$ which satisfies

- $\varphi(u_k) \to c$, as $k \to \infty$;
- there exists $\{\epsilon_k\} \subset \mathbb{R}$, $\epsilon_k \downarrow 0$, such that $\varphi^0(u_k; v - u_k) + \epsilon_k \|v - u_k\|_X \geq 0$, for all $v \in X$ and all $k \geq 0$;

admits a convergent subsequence.

If this is true for every $c \in \mathbb{R}$, we say that φ satisfies the *nonsmooth Palais-Smale condition* ((PS)-condition in short).

We recall next the zero-altitude version of the nonsmooth Mountain Pass Theorem due to Motreanu and Varga [29] which will play an important role in proving our main result.

Theorem 1 *Let $I : X \to \mathbb{R}$ be a locally Lipschitz functional which satisfies the nonsmooth (PS)-condition and let $u_1, u_2 \in X$ and $r \in (0, \|u_2 - u_1\|_X)$ be such that*

$$\inf_{u \in \partial \bar{B}(u_1; r)} I(u) \geq \max\{I(u_1), I(u_2)\}.$$

Then I has a nonsmooth critical point $\tilde{u} \in X \setminus \{u_1, u_2\}$ such that $I(\tilde{u}) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$ and $I(\tilde{u}) \geq \max\{I(u_1), I(u_2)\}$.

In the previous theorem we have denoted by Γ the family of all continuous paths connecting the points u_1 and u_2, that is

$$\Gamma = \{\gamma \in C([0,1]; X) : \ \gamma(0) = u_1, \ \gamma(1) = u_2\}.$$

Let us recall next some definitions and basic properties of the variable exponents Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W_0^{1,p(\cdot)}(\Omega)$ and $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ in the case when Ω is a bounded open subset of $\mathbb{R}^n$, with smooth boundary. We consider the set $C_+(\bar{\Omega}) = \left\{\varphi \in C(\bar{\Omega}) : \ \min_{x \in \bar{\Omega}} \varphi(x) > 1\right\}$ and for each $\varphi \in C_+(\bar{\Omega})$ we denote

$$\varphi^- = \inf_{x \in \Omega} \varphi(x) \quad \text{and} \quad \varphi^+ = \sup_{x \in \Omega} \varphi(x).$$

Moreover, let

$$\varphi^*(x) = \begin{cases} \frac{n\varphi(x)}{n - \varphi(x)} & \text{if } \varphi(x) < n, \\ +\infty & \text{otherwise.} \end{cases}$$

For a function $p \in C_+(\bar{\Omega})$ the *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{u : \Omega \to \mathbb{R} : \ u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty\right\},$$

and can be endowed with the norm (called *Luxemburg norm*) defined by

$$\|u\|_{p(\cdot)} = \inf \left\{ \zeta > 0 : \int_\Omega \left| \frac{u(x)}{\zeta} \right|^{p(x)} dx \leq 1 \right\}.$$

It can be proved that $\left(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}\right)$ is a reflexive and separable Banach space (see, e.g., Kováčik and Rákosník [17]). If we denote by $p'(x) = \frac{p(x)}{p(x)-1}$ the pointwise conjugate exponent of $p(x)$, then for all $u \in L^{p(\cdot)}(\Omega)$ and all $v \in L^{p'(\cdot)}(\Omega)$ the following Hölder-type inequality holds

$$\left| \int_\Omega u(x)v(x)\, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}.$$

We also remember the definition of the $p(\cdot)$-*modular* of the space $L^{p(\cdot)}(\Omega)$, which is the application $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx.$$

This application is extremely useful in manipulating the variable exponent Lebesgue-Sobolev spaces as it satisfies the following relations

$$\|u\|_{p(\cdot)} > 1 (< 1;\ = 1) \text{ if and only if } \rho_{p(\cdot)}(u) > 1 (< 1;\ = 1), \tag{4}$$

$$\|u\|_{p(\cdot)} > 1 \text{ implies } \|u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+}, \tag{5}$$

$$\|u\|_{p(\cdot)} < 1 \text{ implies } \|u\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-}. \tag{6}$$

Clearly, if $p(x) = p_0$ for all $x \in \bar{\Omega}$, then the Luxemburg norm reduces to norm of the classical Lebesgue space $L^{p_0}(\Omega)$, that is

$$\|u\|_{p_0} = \left(\int_\Omega |u(x)|^{p_0}\, dx \right)^{1/p_0}.$$

For a $p \in C_+(\bar{\Omega})$ the isotropic variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ can be defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)} :\ \partial_i u \in L^{p(\cdot)}(\Omega) \text{ for all } i \in \{1, \ldots, n\} \right\},$$

and endowed with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)},$$

becomes a separable and reflexive Banach space. Moreover, if p is log-Hölder continuous (there exists $M > 0$ such that $|p(x) - p(y)| \leq \frac{-M}{\log(|x-y|)}$, for all $x, y \in \Omega$ satisfying $|x - y| < 1/2$), then the space $C_0^\infty(\Omega)$ is dense in $W^{1,p(\cdot)}(\Omega)$ and we can define the Sobolev space with zero boundary values $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(\cdot)}$. Note that if $q \in C_+(\bar{\Omega})$ is a function such that $q(x) < p^*(x)$ for all $x \in \bar{\Omega}$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.

We recall now the definition of the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, where $\overrightarrow{p} : \bar{\Omega} \to \mathbb{R}^n$ is of the form

$$\overrightarrow{p}(x) = (p_1(x), \ldots, p_n(x)), \qquad \text{for all } x \in \bar{\Omega},$$

and for each $i \in \{1, \dots, n\}$, $p_i : \bar{\Omega} \to \mathbb{R}$ is a log-Hölder continuous function. The space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{p}(\cdot)} = \sum_{i=1}^n \|\partial_i u\|_{p_i(\cdot)},$$

and this space is a reflexive Banach space with respect to the above norm (see, e.g., Mihăilescu, Pucci and Rădulescu [23]).

For an easy manipulation of the space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ we introduce $p_M, p_m : \bar{\Omega} \to \mathbb{R}$ and $P^* \in \mathbb{R}$ as follows

$$p_M(x) = \max_{1 \le i \le n} p_i(x), \quad p_m(x) = \min_{1 \le i \le n} p_i(x), \quad P^* = n \left(\sum_{i=1}^n \frac{1}{p_i^-} - 1 \right)^{-1}.$$

We close this section recalling an important result due to Mihăilescu, Pucci and Rădulescu [23] concerning the embedding of $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$.

Theorem 2 *Assume $\Omega \subset \mathbb{R}^n$ $(n \ge 3)$ is an open bounded set having smooth boundary and, for each $i \in \{1, \dots, n\}$, $p_i : \bar{\Omega} \to \mathbb{R}$ is a log-Hölder continuous function such that the following relation holds true*

$$\sum_{i=1}^n \frac{1}{p_i^-} > 1.$$

Then, for any $q \in C_+(\bar{\Omega})$ satisfying $1 < q(x) < \max\{p_m^+, P^*\}$ for all $x \in \bar{\Omega}$, the embedding $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.

3 The main results

In this section we prove a multiplicity result concerning the weak solutions of problem (1). Before defining the concept of weak solution we denote by X the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and by $\|\cdot\|$ the norm defined on this space, that is $\|\cdot\|_{\overrightarrow{p}(\cdot)}$.

Definition 4 A function $u \in X$ is called a *weak solution* for problem (1) if, for almost every $x \in \Omega$, there exist $\xi(x) \in \partial_C \alpha(x, u(x))$ and $\zeta(x) \in \partial_C \beta(x, u(x))$ such that

$$\int_\Omega \sum_{i=1}^n |\partial_i u|^{p_i(x)-2} \partial_i u \partial_i v \, dx = \lambda \int_\Omega \xi v \, dx + \int_\Omega \zeta v \, dx, \qquad \text{for all } v \in X.$$

In order to obtain our main result we shall assume fulfilled the following hypotheses.

$\mathcal{H}(\overrightarrow{p})$: For each $i \in \{1, \dots, n\}$ the function $p_i \in C_+(\bar{\Omega})$ is log-Hölder continuous, $2 \le p_i(x) < n$ for all $x \in \bar{\Omega}$ and $p_M^+ < P^*$;

$\mathcal{H}(\alpha)$: The function $\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies
- (i) $\alpha(x, 0) = 0$ for almost every $x \in \Omega$;
- (ii) $x \mapsto \alpha(x, t)$ is measurable for all $t \in \mathbb{R}$;
- (iii) $t \mapsto \alpha(x, t)$ is locally Lipschitz for almost every $x \in \Omega$;

(iv) there exist $c_\alpha > 0$ and $q \in C_+(\bar{\Omega})$ such that $1 < q^- \leq q^+ < p_m^-$ and

$$|\xi(x)| \leq c_\alpha |t|^{q(x)-1}$$

for almost every $x \in \Omega$, all $t \in \mathbb{R}$ and all $\xi(x) \in \partial_C \alpha(x,t)$;

(v) there exist $\mu \in (0,1)$, $\alpha_0 > 0$ and $t_0 > 0$ such that

$$\alpha(x,t) \leq 0, \qquad \text{for all } |t| < \mu \text{ and almost every } x \in \Omega,$$

and

$$\alpha(x,t_0) \geq \alpha_0 > 0, \qquad \text{for almost every } x \in \Omega.$$

$\mathcal{H}(\beta)$: The function $\beta : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

(i) $\beta(x,0) = 0$ for almost every $x \in \Omega$;

(ii) $x \mapsto \beta(x,t)$ is measurable for all $t \in \mathbb{R}$;

(iii) there exist $r \in C_+(\bar{\Omega})$ with the property that $1 < r(x) < P^*$ and $K \in L^{r'(\cdot)}(\Omega)$ such that

$$|\beta(x,t_1) - \beta(x,t_2)| \leq K(x)|t_1 - t_2|$$

for almost every $x \in \Omega$ and all $t_1, t_2 \in \mathbb{R}$;

(iv) $\beta(x,t) \leq 0$, for almost every $x \in \Omega$ and all $t \in \mathbb{R}$.

The main result of this paper is given by the following theorem.

Theorem 3 *Assume that $\mathcal{H}(\overrightarrow{p})$, $\mathcal{H}(\alpha)$ and $\mathcal{H}(\beta)$ hold. Then there exists $\lambda^* > 0$ such that for any $\lambda \in (\lambda^*, +\infty)$ problem (1) admits at least two non-zero weak solutions.*

Before proving the main result we need the following extension of the Aubin-Clarke Theorem (see Clarke [7], Theorem 2.7.5) concerning the subdifferentiation of integral functionals defined on variable exponent Lebesgue spaces.

Let $s \in C_+(\bar{\Omega})$ and $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that $x \mapsto \varphi(x,t)$ is measurable for all $t \in \mathbb{R}$ and, in addition, suppose φ satisfies one of the following conditions

(a) there exist $m \in L^{s'(\cdot)}(\Omega)$ such that

$$|\varphi(x,t_1) - \varphi(x,t_2)| \leq m(x)|t_1 - t_2|, \qquad \text{for almost every } x \in \Omega \text{ and all } t_1, t_2 \in \mathbb{R}$$

or,

(b) the application $t \mapsto \varphi(x,t)$ is locally Lipschitz for almost every $x \in \Omega$ and there exists $c_\varphi > 0$ such that

$$|\xi| \leq c_\varphi |t|^{s(x)-1}, \qquad \text{for almost every } x \in \Omega, \text{ all } t \in \mathbb{R} \text{ and all } \xi \in \partial_C \varphi(x,t).$$

We introduce next the functional $\phi : L^{s(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\phi(w) = \int_\Omega \varphi(x, w(x))\, dx, \qquad \text{for all } w \in L^{s(\cdot)}(\Omega). \tag{7}$$

Lemma 2 *Assume $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a functional such that $x \mapsto \varphi(x,t)$ is measurable for all $t \in \mathbb{R}$ and either* (a) *or* (b) *holds. Then, the functional $\phi : L^{s(\cdot)}(\Omega) \to \mathbb{R}$ defined by (7) is locally Lipschitz and satisfies*

$$\phi^0(w;z) \leq \int_\Omega \varphi^0(x, w(x); z(x))\, dx, \qquad \textit{for all } w, z \in L^{s(\cdot)}(\Omega). \tag{8}$$

Moreover, if $\varphi(x,\cdot)$ is regular at $w(x)$ for almost every $x \in \Omega$, then ϕ is regular at w and equality takes place in (8).

We recall that a locally Lipschitz functional $h : X \to \mathbb{R}$ is said to be *regular* at $u \in X$ if, for all $v \in X$ the usual one-sided directional derivative $h'(u; v)$ exists and $h'(u; v) = h^0(u; v)$.

Proof We start by proving that ϕ is a locally Lipschitz functional. Let us fix $w \in L^{s(\cdot)}(\Omega)$, $r > 0$ and choose $w_1, w_2 \in \bar{B}_{L^{s(\cdot)}(\Omega)}(w; r)$.

CASE 1. Condition (a) holds.

Then, using the Hölder-type inequality we have

$$\begin{aligned} |\phi(w_1) - \phi(w_2)| &= \left| \int_\Omega \varphi(x, w_1(x)) - \varphi(x, w_2(x)) \, dx \right| \\ &\le \int_\Omega |\varphi(x, w_1(x)) - \varphi(x, w_2(x))| \, dx \\ &\le \int_\Omega m(x) |w_1(x) - w_2(x)| \, dx \\ &\le 2\|m\|_{s'(\cdot)} \|w_1 - w_2\|_{s(\cdot)}, \end{aligned}$$

i.e., ϕ is Lipschitz continuous on $\bar{B}_{L^{s(\cdot)}(\Omega)}(w; r)$ with the Lipschitz constant $L_\phi = 2\|m\|_{s'(\cdot)}$.

CASE 2. Condition (b) holds.

According to Lebourg's mean value theorem (see [19]) there exist $t_0 \in (0, 1)$, $\bar{w} = t_0 w_1 + (1 - t_0) w_2$ and $\bar{\xi}(x) \in \partial_C \varphi(x, \bar{w}(x))$ such that

$$\varphi(x, w_1(x)) - \varphi(x, w_2(x)) = \bar{\xi}(x) \left(w_1(x) - w_2(x) \right).$$

Thus we obtain the following estimates

$$\begin{aligned} |\phi(w_1) - \phi(w_2)| &= \left| \int_\Omega \varphi(x, w_1(x)) - \varphi(x, w_2(x)) \, dx \right| \\ &\le \int_\Omega |\varphi(x, w_1(x)) - \varphi(x, w_2(x))| \, dx \\ &= \int_\Omega |\bar{\xi}(x)| \, |w_1(x) - w_2(x)| \, dx \\ &\le \int_\Omega c_\varphi |\bar{w}(x)|^{s(x)-1} |w_1(x) - w_2(x)| \, dx \\ &\le 2c_\varphi \left\| \bar{\psi} \right\|_{s'(\cdot)} \|w_1 - w_2\|_{s(\cdot)}, \end{aligned}$$

where $\bar{\psi}(x) = |\bar{w}(x)|^{s(x)-1}$. Now, we only need to prove that $\left\|\bar{\psi}\right\|_{s'(\cdot)}$ is bounded from above by a constant $L_\phi > 0$ which depends only on the ball $\bar{B}_{L^{s(\cdot)}(\Omega)}(w; r)$. Obviously we need to discuss only the case when $\left\|\bar{\psi}\right\|_{s'(\cdot)} > 1$. Using the relations (4)-(6) we get

$$\left\|\bar{\psi}\right\|_{s'(\cdot)} \le \left\|\bar{\psi}\right\|_{s'(\cdot)}^{s'^-} \le \rho_{s'(\cdot)}\left(\bar{\psi}\right) = \int_\Omega \left(|\bar{w}(x)|^{s(x)-1} \right)^{s'(x)} dx = \int_\Omega |\bar{w}(x)|^{s(x)} \, dx = \rho_{s(\cdot)}(\bar{w}).$$

On the other hand,

$$\rho_{s(\cdot)}(\bar{w}) \le \begin{cases} \|\bar{w}\|_{s(\cdot)}^{s^-}, & \text{if } \|\bar{w}\|_{s(\cdot)} < 1, \\ \|\bar{w}\|_{s(\cdot)}^{s^+}, & \text{if } \|\bar{w}\|_{s(\cdot)} > 1, \end{cases}$$

and choosing $L_\phi = \max\left\{2c_\varphi;\ 2c_\varphi \|\bar{w}\|_{s(\cdot)}^{s^-};\ 2c_\varphi \|\bar{w}\|_{s(\cdot)}^{s^+}\right\}$ we deduce that

$$|\phi(w_1) - \phi(w_2)| \le L_\phi \|w_1 - w_2\|_{s(\cdot)},$$

which shows that the functional ϕ is Lipschitz continuous on $\bar{B}_{L^{s(\cdot)}(\Omega)}(w; r)$.

Let us check now that

$$\phi^0(w;z) \le \int_\Omega \varphi^0(x,w(x);z(x))\,dx, \qquad \text{for all } w,z \in L^{s(\cdot)}(\Omega).$$

We denote by $h_{\mu,\delta}(w(x),z(x))$ the difference quotient

$$h_{\mu,\delta}(w(x),z(x)) = \frac{\varphi(x,w(x)+\delta+\mu z(x)) - \varphi(x,w(x)+\delta)}{\mu}.$$

Clearly, $x \mapsto h_{\mu,\delta}(w(x),z(x))$ is measurable. Moreover, if condition (a) holds then

$$|h_{\mu,\delta}(w(x),z(x))| \le m(x)|z(x)|, \text{ for almost every } x \in \Omega.$$

On the other hand, if condition (b) holds we can apply Lebourg's mean value theorem to deduce that

$$h_{\mu,\delta}(w(x),z(x)) = \zeta_x z(x),$$

for some $\zeta_x \in \partial_C \varphi(x,\tilde{w}(x))$ with $\tilde{w}(x) \in \{\nu\,(w(x)+\delta) + (1-\nu)\,(w(x)+\delta+\mu z(x)) :\ \nu \in (0,1)\}$.
Now, let $b : \Omega \to \mathbb{R}$ be defined by

$$b(x) = \begin{cases} m(x), & \text{if (a) holds,} \\ c_\varphi |\tilde{w}(x)|^{s(x)-1}, & \text{if (b) holds.} \end{cases}$$

It is easy to see that $b \in L^{s'(\cdot)}(\Omega)$ and

$$|h_{\mu,\delta}(w(x),z(x))| \le |b(x)|\,|z(x)|, \qquad \text{for almost every } x \in \Omega.$$

Thus, we can apply Fatou's lemma to get the following estimate

$$\limsup_{\substack{\delta\to 0\\ \mu\downarrow 0}} \int_\Omega h_{\mu,\delta}(w(x),z(x))\,dx \le \int_\Omega \limsup_{\substack{\delta\to 0\\ \mu\downarrow 0}} h_{\mu,\delta}(w(x),z(x))\,dx,$$

which shows that

$$\phi^0(w;z) \le \int_\Omega \varphi^0(x,w(x);z(x))\,dx, \qquad \text{for all } w,z \in L^{s(\cdot)}(\Omega).$$

Finally, let us prove that ϕ is regular at w if $\varphi(x,\cdot)$ is regular at $w(x)$ for almost every $x \in \Omega$. Using Fatou's lemma we have

$$\begin{aligned}
\phi^0(w,z) &\ge \liminf_{\mu\downarrow 0} \frac{\phi(w+\mu z)-\phi(w)}{\mu} \\
&\ge \int_\Omega \liminf_{\mu\downarrow 0} \frac{\varphi(z,w(x)+\mu z(x)) - \varphi(x,w(x))}{\mu}\,dx \\
&\ge \int_\Omega \lim_{\mu\downarrow 0} \frac{\varphi(z,w(x)+\mu z(x)) - \varphi(x,w(x))}{\mu}\,dx \\
&= \int_\Omega \varphi'(x,w(x);z(x))\,dx \\
&= \int_\Omega \varphi^0(x,w(x);z(x))\,dx \\
&\ge \phi^0(w;z).
\end{aligned}$$

Thus, everywhere above we have equality, $\phi'(w;z)$ exists for all $z \in L^{s(\cdot)}(\Omega)$ and

$$\phi'(w;z) = \int_\Omega \varphi'(x,w(z);z(x))\,dx = \int_\Omega \varphi^0(x,w(z);z(x))\,dx = \phi^0(w,z).$$

□

Proof of Theorem 3. Let us introduce the functionals $J : X \to \mathbb{R}$, $\Lambda : L^{q(\cdot)}(\Omega) \to \mathbb{R}$ and $\Theta : L^{r(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$J(u) := \int_\Omega \sum_{i=1}^n \frac{|\partial_i u|^{p_i(x)}}{p_i(x)}\, dx, \qquad \Lambda(w) := \int_\Omega \alpha(x, w(x))\, dx, \qquad \Theta(z) := \int_\Omega \beta(x, z(x))\, dx.$$

The idea is to prove that the functional $\mathcal{E}_\lambda : X \to \mathbb{R}$ defined by $\mathcal{E}_\lambda(u) = J(u) - \Theta(u) - \lambda\Lambda(u)$ admits at least two non-zero nonsmooth critical points and every nonsmooth critical point of $\mathcal{E}_\lambda$ is a weak solution for problem (1). With this view in mind we divide the proof into several steps as follows.

Step 1. The functional $\mathcal{E}_\lambda$ is locally Lipschitz.

By a standard argument it can be proved that $J \in C^1(X;\mathbb{R})$ and its Fréchet derivative is given by

$$\langle J'(u), v\rangle = \int_\Omega \sum_{i=1}^n |\partial_i u|^{p_i(x)-2}\, \partial_i u\, \partial_i v\, dx.$$

According to Theorem 2 the functionals $\Lambda : L^{q(\cdot)}(\Omega) \to \mathbf{R}$ and $\Theta : L^{r(\cdot)}(\Omega) \to \mathbf{R}$ are locally Lipschitz.

In order to check that the functional $\mathcal{E}_\lambda$ is also locally Lipschitz let us fix $u \in X$, $\lambda > 0$, $r > 0$ and $u_1, u_2 \in \bar{B}_X(u; r)$. Since $J \in C^1(X;\mathbb{R})$ we have

$$|J(u_1) - J(u_2)| = |\langle J'(\bar{u}), u_1 - u_2\rangle| \le \|J'(\bar{u})\|_{X^*}\, \|u_1 - u_2\|,$$

where $\bar{u} = t_1 u_1 + (1 - t_1) u_2$ for some $t_1 \in (0, 1)$. But, X is a reflexive Banach space therefore the ball $\bar{B}_X(u; r)$ is weakly compact and thus we get the existence of a positive constant M such that $\|J'(\bar{u})\|_{X^*} \le M$. We also point out that our hypotheses ensure that we can apply Theorem 2 to conclude that the embeddings $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ and $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ are both compact, hence there exist two positive constants c_q and c_r such that

$$\|u\|_{q(\cdot)} \le c_q \|u\| \quad \text{and} \quad \|u\|_{r(\cdot)} \le c_r \|u\|, \quad \text{for all } u \in X.$$

Finally we obtain the following estimates

$$\begin{aligned} |\mathcal{E}_\lambda(u_1) - \mathcal{E}_\lambda| &\le |J(u_1) - J(u_2)| + |\Theta(u_1) - \Theta(u_2)| + \lambda|\Lambda(u_1) - \Lambda(u_2)| \\ &\le M\|u_1 - u_2\| + L_\Theta \|u_1 - u_2\|_{r(\cdot)} + \lambda L_\Lambda \|u_1 - u_2\|_{q(\cdot)} \\ &\le (M + c_r L_\Theta + \lambda c_q L_\Lambda)\, \|u_1 - u_2\|, \end{aligned}$$

which show that the functional $\mathcal{E}_\lambda$ is locally Lipschitz.

Step 2. Any nonsmooth critical point of the functional $\mathcal{E}_\lambda$ is a weak solution of problem (1).

Let $u \in X$ be a nonsmooth critical point of $\mathcal{E}_\lambda$ and let $v \in X$ be arbitrary fixed. Then

$$\begin{aligned} 0 &\le \mathcal{E}_\lambda^0(u; v) \\ &= (J - \Theta - \lambda\Lambda)^0\, (u; v) \\ &\le J^0(u; v) + (-\Theta)^0(u; v) + (-\lambda\Lambda)^0(u; v) \\ &= \langle J'(u), v\rangle + \Theta^0(u; -v) + \Lambda^0(u; -\lambda v). \end{aligned}$$

Using Lemma 2 we derive

$$\Lambda^0(w_1; w_2) \le \int_\Omega \alpha^0(x, w_1(x); w_2(x))\, dx, \quad \text{for all } w_1, w_2 \in L^{q(\cdot)}(\Omega)$$

and

$$\Theta^0(z_1; z_2) \le \int_\Omega \beta^0(x, z_1(x); z_2(x))\, dx, \qquad \text{for all } z_1, z_2 \in L^{r(\cdot)}(\Omega).$$

Choosing $w_1 = u$, $w_2 = -\lambda v$, $z_1 = u$, $z_2 = -v$ we conclude that

$$0 \le \langle J'(u), v\rangle + \int_\Omega \alpha^0(x, u(x); -\lambda v(x))\, dx + \int_\Omega \beta^0(x, u(x); -v(x))\, dx.$$

We apply now Proposition 2.1.2 from Clarke [7] and for almost every $x \in \Omega$ we find $\tilde{\xi}(x) \in \partial_C \alpha(x, u(x))$ and $\tilde{\zeta}(x) \in \partial_C \beta(x, u(x))$ such that for all $t \in \mathbb{R}$ the following relations hold true

$$\alpha^0(x, u(x); t) = \tilde{\xi}(x)t = \max\{\boldsymbol{\xi} t :\ \boldsymbol{\xi} \in \partial_C \alpha(x, u(x))\}$$

and

$$\beta^0(x, u(x); t) = \tilde{\zeta}(x)t = \max\{\boldsymbol{\zeta} t :\ \boldsymbol{\zeta} \in \partial_C \beta(x, u(x))\}.$$

Thus,

$$0 \le \langle J'(u); v\rangle - \int_\Omega \tilde{\zeta} v\, dx - \lambda \int_\Omega \tilde{\xi} v\, dx.$$

Taking $\bar{v} = -v$ we obtain

$$0 \ge \langle J'(u); v\rangle - \int_\Omega \tilde{\zeta} v\, dx - \lambda \int_\Omega \tilde{\xi} v\, dx.$$

Combining the above two relations with the fact that

$$\langle J'(u); v\rangle = \int_\Omega \sum_{i=1}^n |\partial_i u|^{p_i(x)-2}\, \partial_i u \partial_i v\, dx$$

and the fact that $v \in X$ was chosen arbitrarily we infer that u is a weak solution for problem (1) in the sense of Definition 4.

Step 3. The functional $\mathcal{E}_\lambda$ is sequentially weakly lower semicontinuous on X.
Let $\{u_k\} \subset X$ be a sequence which converges weakly to some $u \in X$. Using the same arguments as in the proof of Lemma 3.4 of [20] we can prove that the functional J is sequentially weakly lower semicontinuous and thus

$$\liminf_{k\to\infty} J(u_k) \ge J(u).$$

Taking into account that X is compactly embedded in $L^{q(\cdot)}(\Omega)$ we conclude that $u_k \to u$ in $L^{q(\cdot)}(\Omega)$ and by Fatou's lemma, passing eventually to a subsequence, we have

$$\begin{aligned}\liminf_{k\to\infty} [-\Lambda(u_k)] &= -\limsup_{k\to\infty} \Lambda(u_k) = -\limsup_{k\to\infty} \int_\Omega \alpha(x, u_k(x))\, dx \ge -\int_\Omega \limsup_{k\to\infty} \alpha(x, u_k(x))\, dx\\ &= -\int_\Omega \alpha(x, u(x))\, dx = -\Lambda(u).\end{aligned}$$

Using the same arguments as above, we can prove that

$$\liminf_{k\to\infty} [-\Theta(u_k)] \ge -\Theta(u).$$

Gathering the above information we arrive at the following estimates

$$\begin{aligned}\liminf_{k\to\infty}\mathcal{E}_\lambda(u_k) &= \liminf_{k\to\infty}\,[J(u_k)-\Theta(u_k)-\lambda\Lambda(u_k)]\\ &\geq \liminf_{k\to\infty} J(u_k)+\liminf_{k\to\infty}\,[-\Theta(u_k)]+\liminf_{k\to\infty}\,[-\lambda\Lambda(u_k)]\\ &\geq J(u)-\Theta(u)-\lambda\Lambda(u)\\ &= \mathcal{E}_\lambda(u).\end{aligned}$$

Therefore the functional $\mathcal{E}_\lambda$ is sequentially weakly lower semicontinuous.

STEP 4. The functional $\mathcal{E}_\lambda$ is coercive.

Let us fix $u\in X$ such that $\|u\|>1$. Applying Lebourg's mean value theorem we deduce that there exist $\nu\in(0,1)$ and $\bar{\xi}(x)\in\partial_C\alpha(x,\nu u(x))$ such that

$$\alpha(x,u(x))=\alpha(x,u(x))-\alpha(x,0)=\bar{\xi}(x)u(x).$$

A simple computation yields

$$\Lambda(u)=\int_\Omega\alpha(x,u(x))\,dx\leq\int_\Omega|\bar{\xi}(x)|\,|u(x)|\,dx\leq c_\alpha\int_\Omega|\nu u(x)|^{q(x)-1}\,|u(x)|\,dx\leq c_\alpha\int_\Omega|u(x)|^{q(x)}\,dx.$$

Using hypothesis $\mathcal{H}(\beta)$ we can derive

$$\begin{aligned}\Theta(u)=\int_\Omega\beta(x,u(x))\,dx &\leq \int_\Omega|\beta(x,u(x))-\beta(x,0)|\,dx\\ &\leq\int_\Omega K(x)|u(x)|\,dx\leq 2\|K\|_{r'(\cdot)}\|u\|_{r(\cdot)}\\ &\leq 2c_r\|K\|_{r'(\cdot)}\|u\|.\end{aligned}$$

On the other hand,

$$J(u)=\int_\Omega\sum_{i=1}^n\frac{|\partial_i u|^{p_i(x)}}{p_i(x)}\,dx\geq\frac{1}{p_M^+}\sum_{i=1}^n\int_\Omega|\partial_i u|^{p_i(x)}\,dx.$$

Let us define $I_1=\big\{i\in\{1,\ldots,n\}:\ \|\partial_i u\|_{p_i(\cdot)}\leq 1\big\}$ and $I_2=\big\{i\in\{1,\ldots,n\}:\ \|\partial_i u\|_{p_i(\cdot)}>1\big\}$ and assume that $|I_1|=n_0\leq n$. Then according to relations (4)-(6) we have

$$\begin{aligned}J(u)&\geq\frac{1}{p_M^+}\left(\sum_{i\in I_1}\|\partial_i u\|_{p_i(\cdot)}^{p_M^+}+\sum_{i\in I_2}\|\partial_i u\|_{p_i(\cdot)}^{p_m^-}\right)\\ &\geq\frac{1}{p_M^+}\left[\sum_{i=1}^n\|\partial_i u\|_{p_i(\cdot)}^{p_m^-}-\sum_{i\in I_1}\left(\|\partial_i u\|_{p_i(\cdot)}^{p_m^-}-\|\partial_i u\|_{p_i(\cdot)}^{p_M^+}\right)\right]\\ &\geq\frac{1}{p_M^+}\left(\sum_{i=1}^n\|\partial_i u\|_{p_i(\cdot)}^{p_m^-}-n_0\right).\end{aligned}$$

Let us consider now the function $h:\mathbb{R}_+\to\mathbb{R}_+$ defined by $h(t)=t^{p_m^-}$. Obviously, h is convex and taking $t_i=\|\partial_i u\|_{p_i(\cdot)}$, $i\in\{1,\ldots,n\}$ we have

$$\frac{1}{n^{p_m^-}}\|u\|^{p_m^-}=\left(\frac{1}{n}\sum_{i=1}^n\|\partial_i u\|_{p_i(\cdot)}\right)^{p_m^-}=h\left(\frac{1}{n}\sum_{i=1}^n t_i\right)\leq\frac{1}{n}\sum_{i=1}^n h(t_i)=\frac{1}{n}\sum_{i=1}^n\|\partial_i u\|_{p_i(\cdot)}^{p_m^-}.$$

From the above information we deduce that

$$J(u) \geq \frac{1}{p_M^+}\left(\frac{1}{n^{p_m^- -1}}\|u\|^{p_m^-} - n_0\right).$$

We consider two cases: $\|u\|_{q(\cdot)} \geq 1$ and $\|u\|_{q(\cdot)} < 1$. We will discuss only the first case, the proof for the second case being similar. We have

$$\begin{aligned}
\mathcal{E}_\lambda(u) &= J(u) - \Theta(u) - \lambda \Lambda(u) \\
&\geq \frac{1}{p_M^+ n^{p_m^- -1}}\|u\|^{p_m^-} - \frac{n_0}{p_M^+} - 2c_r\|K\|_{r'(\cdot)}\|u\| - \lambda c_\alpha \rho_{q(\cdot)}(u) \\
&\geq \frac{1}{p_M^+ n^{p_m^- -1}}\|u\|^{p_m^-} - \frac{n_0}{p_M^+} - 2c_r\|K\|_{r'(\cdot)}\|u\| - \lambda c_\alpha \|u\|_{q(\cdot)}^{q^+} \\
&\geq \frac{1}{p_M^+ n^{p_m^- -1}}\|u\|^{p_m^-} - \frac{n_0}{p_M^+} - 2c_r\|K\|_{r'(\cdot)}\|u\| - \lambda c_\alpha c_q^{q^+}\|u\|^{q^+}.
\end{aligned}$$

Since by our hypotheses $1 < q^- \leq q^+ < p_m^-$ we infer that $\mathcal{E}_\lambda(u) \to \infty$ as $\|u\| \to \infty$, so the functional $\mathcal{E}_\lambda$ is coercive.

Step 5. There exists $\lambda^* > 0$ such that for any $\lambda > \lambda^*$ we can determine $u_0 \in X$ for which $\mathcal{E}_\lambda(u_0) < 0$. Let $x_0 \in \operatorname{int} \Omega$ be such that the distance from x_0 to the boundary of Ω is maximal and let R_0 be this distance ($R_0 = \max\limits_{x\in\Omega} \min\limits_{y\in\partial\Omega} |x-y|$). Clearly, for $0 < R < R_0/2$, we have $\bar{B}(x_0; 2R) \subseteq \Omega$. It can be easily seen that there exists $u_0 \in C_0^\infty(B(x_0; 2R))$ such that

$$\begin{cases} u_0(x) = t_0 & \text{for } x \in \bar{B}(x_0; R) \\ 0 \leq u_0(x) \leq t_0 & \text{for } x \in B(x_0; 2R) \setminus B(x_0; R). \end{cases}$$

Since $u_0 \in C_0^\infty(B(x_0; 2R))$, for each $i \in \{1, \ldots, n\}$ there exists $m_i > 0$ such that $|\partial_i u_0(x)| \leq m_i$ in $B(x_0; 2R)$. Then for $m := \max\{1, m_1 \ldots, m_n\}$ we have

$$\begin{aligned}
\mathcal{E}_\lambda(u_0) &= \int_\Omega \sum_{i=1}^n \frac{|\partial_i u_0(x)|^{p_i(x)}}{p_i(x)}\, dx - \int_\Omega \beta(x, u_0(x))\, dx - \lambda \int_\Omega \alpha(x, u_0(x))\, dx \\
&\leq \int_{B(x_0;2R)} \sum_{i=1}^n \frac{m^{p_M^+}}{p_m^-}\, dx - \int_{B(x_0;2R)} \beta(x, u_0(x))\, dx - \lambda \int_{B(x_0;2R)} \alpha(x, u_0(x))\, dx.
\end{aligned}$$

Obviously,

$$-\int_{B(x_0;2R)} \beta(x, u_0(x))\, dx = \int_{B(x_0;2R)} \beta(x, 0) - \beta(x, u_0(x))\, dx \leq \int_{B(x_0;2R)} K(x) u_0(x)\, dx \leq \beta_1,$$

for a suitable constant $\beta_1 > 0$.
On the other hand, splitting $B(x_0; 2R)$ into the sets

$$D_1 = \{x \in B(x_0; 2R) : \ \alpha(x, u_0(x)) \leq 0\}$$

and

$$D_2 = \{x \in B(x_0; 2R) : \ \alpha(x, u_0(x)) > 0\},$$

we observe that $B(x_0;R) \subset D_2$. Applying Lebourg's mean value theorem and taking into account hypothesis $\mathcal{H}(\alpha)$ we have

$$\begin{aligned}
\int_{B(x_0;2R)} \alpha(x,u_0(x))\,dx &= \int_{D_1} \alpha(x,u_0(x))\,dx + \int_{D_2} \alpha(x,u_0(x))\,dx \\
&\geq \int_{D_1} \left(\alpha(x,u_0(x)) - \alpha(x,0)\right)\,dx + \int_{B(x_0;R)} \alpha(x,u_0(x))\,dx \\
&\geq -\int_{D_1} |\bar{\xi}(x)|\,|u_0(x)|\,dx + \alpha_0 \frac{\omega_n R^n}{n} \\
&\geq -c_\alpha \int_{D_1} \nu^{q(x)-1} |u_0(x)|^{q(x)}\,dx + \alpha_0 \frac{\omega_n R^n}{n} \\
&\geq -\alpha_1 + \alpha_0 \frac{\omega_n R^n}{n},
\end{aligned}$$

where $\nu \in (0,1)$, $\bar{\xi}(x) \in \partial_C \alpha(x, \nu u_0(x))$, ω_n is the area of the unit sphere in $\mathbb{R}^n$ and $\alpha_1 > 0$ is a suitable constant.

Thus

$$\mathcal{E}_\lambda(u_0) \leq \frac{m^{p_M^+} 2^n \omega_n R^n}{p_m^-} + \beta_1 + \alpha_1 - \lambda \alpha_0 \frac{\omega_n R^n}{n} < 0,$$

for any $\lambda > \frac{n 2^n m^{p_M^+} \omega_n R^n + n(\beta_1 + \alpha_1) p_m^-}{\alpha_0 \omega_n R^n}$.

STEP 6. The functional $\mathcal{E}_\lambda$ satisfies the nonsmooth (PS)-condition.

In order to prove the above statement let us fix $c \in \mathbb{R}$ and consider $\{u_k\} \subset X$ a sequence which satisfies the following conditions

- $\mathcal{E}_\lambda(u_k) \to c$ as $k \to \infty$;
- there exists $\{\epsilon_k\} \subset \mathbb{R}$ such that $\epsilon_k \downarrow 0$ as $k \to \infty$ and

$$\mathcal{E}_\lambda^0(u_k; v - u_k) + \epsilon_k \|v - u_k\| \geq 0, \text{ for all } v \in X.$$

Obviously the above sequence is bounded due to the coercivity of the functional $\mathcal{E}_\lambda$, therefore it admits a subsequence (still denoted $\{u_k\}$ for simplicity) which converges weakly to some $u \in X$. Taking $v = u$ and applying Lemma 1 we obtain

$$\begin{aligned}
0 &\leq \epsilon_k \|u - u_k\| + \mathcal{E}_\lambda^0(u_k; u - u_k) \\
&= \epsilon_k \|u - u_k\| + (J - \Theta - \lambda \Lambda)^0(u_k; u - u_k) \\
&\leq \epsilon_k \|u - u_k\| + J^0(u_k; u - u_k) + (-\Theta)^0(u_k; u - u_k) + (-\lambda \Lambda)^0(u_k; u - u_k) \\
&= \epsilon_k \|u - u_k\| + \langle J'(u_k), u - u_k \rangle + \Theta^0(u_k; u_k - u) + \Lambda^0(u_k; \lambda(u_k - u)).
\end{aligned}$$

Since X is compactly embedded in $L^{q(\cdot)}(\Omega)$ and $L^{r(\cdot)}(\Omega)$ respectively, $u_k \to u$ as $k \to \infty$ in $L^{q(\cdot)}(\Omega)$ and $u_k \to u$ as $k \to \infty$ in $L^{r(\cdot)}(\Omega)$. On the other hand, $\epsilon_k \downarrow 0$ as $k \to \infty$ and $\{u_k\}$ is bounded hence

$$\limsup_{k \to \infty} \epsilon_k \|u - u_k\| = 0.$$

Using the above information and taking the superior limit as $k \to \infty$ we get

$$\begin{aligned}
\limsup_{k \to \infty} \langle J'(u_k), u_k - u \rangle &\leq \limsup_{k \to \infty} \left[\epsilon_k \|u - u_k\| + \Theta^0(u_k; u_k - u) + \Lambda^0(u_k; \lambda(u_k - u)) \right] \\
&\leq \limsup_{k \to \infty} \epsilon_k \|u - u_k\| + \limsup_{k \to \infty} \Theta^0(u_k; u_k - u) + \limsup_{k \to \infty} \Lambda^0(u_k; \lambda(u_k - u)) \\
&\leq \Theta^0(u; 0) + \Lambda^0(u; 0) = 0.
\end{aligned}$$

Finally we can apply the same arguments as in the proof of Theorem 3.1 in [15] to conclude that $J' : X \to X^*$ has the $(\mathcal{S})_+$ property, i.e. if $u_k \rightharpoonup u$ in X and $\limsup_{k\to\infty}\langle J'(u_k), u_k - u\rangle \leq 0$, then $u_k \to u$. We have proved thus that the functional $\mathcal{E}_\lambda$ satisfies the nonsmooth (PS)-condition.

Step 7. The functional $\mathcal{E}_\lambda$ admits two nonsmooth critical points $u_1, u_2 \in X \setminus \{0\}$, provided that $\lambda \in (\lambda^*, +\infty)$.

Step 3 and Step 4 enable us to apply the direct method in the calculus of variations (see, e.g., Struwe [37], Theorem 1.2) to obtain the existence of an element $u_1 \in X$ such that $\mathcal{E}_\lambda(u_1) = \min_{u\in X} \mathcal{E}_\lambda(u)$. Obviously u_1 is a nonsmooth critical point of $\mathcal{E}_\lambda$ as it is a global minimizer, while Step 5 ensures that $\mathcal{E}_\lambda(u_1) < 0$, which means that $u_1 \neq 0$. Furthermore, if there exists $\rho \in (0, \|u_1\|)$ such that $\inf_{\partial B(0;\rho)} \mathcal{E}_\lambda \geq 0 = \max\{\mathcal{E}_\lambda(0), \mathcal{E}_\lambda(u_1)\}$, then we can apply Theorem 1 to obtain another nonsmooth critical point $u_2 \in X \setminus \{0, u_1\}$.

Let us consider $s \in C_+(\bar{\Omega})$ such that $p_M^+ < s^- \leq s^+ < P^*$ and choose $\rho > 0$ such that $\rho < \min\{1, 1/c_s, \|u_1\|\}$, where $c_s > 0$ is the constant given by the compact inclusion $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$, i.e.,

$$\|u\|_{s(\cdot)} \leq c_s\|u\|, \qquad \text{for all } u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

Then for each $u \in \partial B(0;\rho)$ we have $\|\partial_i u\|_{p_i(\cdot)} < 1$ $(1 \leq i \leq n)$ and $\|u\|_{s(\cdot)} < 1$. Thus for $u \in X$ such that $\|u\| = \rho$ we have

$$\begin{aligned}
\mathcal{E}_\lambda(u) &= J(u) - \Theta(u) - \lambda\Lambda(u) \\
&\geq \frac{1}{p_M^+}\sum_{i=1}^n \int_\Omega |\partial_i u|^{p_i(x)}\, dx - \int_\Omega \beta(x,u(x))\, dx - \lambda\int_\Omega \alpha(x,u(x))\, dx \\
&\geq \frac{1}{p_M^+}\sum_{i=1}^n \|\partial_i u\|_{p_i(\cdot)}^{p_M^+} - \lambda\int_\Omega \alpha(x,u(x))\, dx.
\end{aligned}$$

Using the convexity of the function $h : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $h(t) = t^{p_M^+}$ we deduce that

$$\sum_{i=1}^n \|\partial_i u\|_{p_i(\cdot)}^{p_M^+} \geq \frac{1}{n^{p_M^+ - 1}}\|u\|^{p_M^+}.$$

Defining $\Omega_1 = \{x \in \Omega :\ |u(x)| < \mu\}$ and $\Omega_2 = \{x \in \Omega :\ |u(x)| \geq \mu\}$ and having in mind hypothesis $\mathcal{H}(\alpha) - (v)$ and Lebourg's mean value theorem we get

$$\begin{aligned}
\int_\Omega \alpha(x,u(x))\, dx &= \int_{\Omega_1} \alpha(x,u(x))\, dx + \int_{\Omega_2} \alpha(x,u(x))\, dx \\
&\leq \int_{\Omega_2} \alpha(x,u(x))\, dx \leq \int_{\Omega_2} |\bar{\xi}(x)|\, |u(x)|\, dx \\
&\leq c\int_{\Omega_2} |u(x)|^{q(x)}\, dx = c\int_{\Omega_2} |u(x)|^{s(x)}\frac{|u(x)|^{q(x)}}{|u(x)|^{s(x)}}\, dx \\
&\leq c\mu^{q^+ - s^-}\int_{\Omega_2} |u(x)|^{s(x)}\, dx \leq c\mu^{q^+ - s^-}\int_\Omega |u(x)|^{s(x)}\, dx \\
&\leq c\mu^{q^+ - s^-}\|u\|_{s(\cdot)}^{s^-} \leq c\mu^{q^+ - s^-} c_s^{s^-}\|u\|^{s^-},
\end{aligned}$$

for a suitable constant $c > 0$.

Thus, for $u \in \partial B(0;\rho)$

$$\mathcal{E}_\lambda(u) \geq \frac{1}{p_M^+ n^{p_M^+-1}}\rho^{p_M^+} - \lambda c\mu^{q^+-s^-} c_s^{s^-}\rho^{s^-} = \rho^{p_M^+}\left(\frac{1}{p_M^+ n^{p_M^+-1}} - \lambda c\mu^{q^+-s^-} c_s^{s^-}\rho^{s^- - p_M^+}\right).$$

Finally, we observe that the function $h:[0,1]\to\mathbb{R}$ defined by

$$h(t) = \frac{1}{p_M^+ n^{p_M^+-1}} - \lambda c\mu^{q^+-s^-} c_s^{s^-} t^{s^- - p_M^+}$$

is continuous on $[0,1]$ and $h(0) = \frac{1}{p_M^+ n^{p_M^+-1}} > 0$, hence $h>0$ in a small neighborhood at the right of the origin. Choosing $\rho>0$ such that ρ belongs to this neighborhood and $\rho < \min\{1, 1/c_s, \|u_1\|\}$ we deduce that $\mathcal{E}_\lambda(u) > 0$ for all $u \in \partial B(0;\rho)$ and this completes the proof.

□

We close this paper with several examples of functions $\alpha,\beta : \Omega\times\mathbb{R}\to\mathbb{R}$ for which conditions $\mathcal{H}(\alpha)$ and $\mathcal{H}(\beta)$ are fulfilled.

Example 1 Let $\{\varepsilon_k\}$ be a sequence of positive real numbers such that $\varepsilon_k \downarrow 0$ as $k\to+\infty$. Let $q \in C_+(\bar{\Omega})$ be such that $1 < q^- \leq q^+ < p_m^-$ and let $\alpha_k, \beta_k : \Omega\times\mathbb{R}\to\mathbb{R}$ to be defined by

$$\alpha_k(x,t) = \begin{cases} \frac{1}{q(x)}|t+\varepsilon_k|^{q(x)}, & \text{for } t \in (-\infty, -\varepsilon_k], \\ 0, & \text{for } t \in (-\varepsilon_k, \varepsilon_k), \\ \frac{1}{q(x)}|t-\varepsilon_k|^{q(x)}, & \text{for } t \in [\varepsilon_k, +\infty), \end{cases}$$

and

$$\beta_k(x,t) = 0, \text{ for all } (x,t) \in \Omega\times\mathbb{R}.$$

Then, $\partial_C\beta(x,t) = 0$ for all $(x,t)\in\Omega\times\mathbb{R}$, while

$$\partial_C\alpha_k(x,t) = \begin{cases} |t+\varepsilon_k|^{q(x)-2}(t+\varepsilon_k), & \text{for } t \in (-\infty, -\varepsilon_k), \\ 0, & \text{for } t \in [-\varepsilon_k, \varepsilon_k], \\ |t-\varepsilon_k|^{q(x)-2}(t-\varepsilon_k), & \text{for } t \in (\varepsilon_k, +\infty). \end{cases}$$

Thus, for any $\xi(x) \in \partial_C\alpha(x,t)$, we have

$$|\xi(x)| \leq \begin{cases} |t|^{q(x)-1} \leq \frac{1}{\varepsilon_k^{q^+-1}}|t|^{q(x)-1}, & \text{for } |t| \in [1+\varepsilon_k, +\infty), \\ 1 \leq \left(\frac{|t|}{\varepsilon_k}\right)^{q(x)-1} \leq \frac{1}{\varepsilon_k^{q^+-1}}|t|^{q(x)-1}, & \text{for } |t| \in (\varepsilon_k, 1+\varepsilon_k), \\ 0 \leq \frac{1}{\varepsilon_k^{q^+-1}}|t|^{q(x)-1}, & \text{for } |t| \in [0, \varepsilon_k). \end{cases}$$

We point out the fact that when $k\to+\infty$ then problem (1) with α_k, β_k defined above reduces to problem (2), hence this example shows that slightly perturbing problem (2) around the origin we can obtain two nontrivial weak solutions instead of only one weak solution as Theorem 3 [23] states.

Example 2 Let $\mu \in (0,1)$, $q_1, q_2 \in C_+(\bar{\Omega})$ be such that $1 < q_1^- \leq q_1^+ < q_2^- \leq q_2^+ < p_m^-$ and let $a \in L^\infty(\Omega)$ be such that $a(x) < 0$ for almost every $x\in\Omega$. We consider the functions $\alpha,\beta:\Omega\times\mathbb{R}\to\mathbb{R}$ to be defined by

$$\alpha(x,t) = \begin{cases} 0, & \text{for } t \in (-\infty, \mu), \\ \max\left\{(t-\mu)^{q_1(x)}, (t-\mu)^{q_2(x)}\right\}, & \text{for } t \in [\mu, +\infty), \end{cases}$$

and

$$\beta(x,t) = a(x)|t|.$$

Then

$$\partial_C \alpha(x,t) = \begin{cases} 0, & \text{for } t \in (\infty, \mu], \\ q_1(x)(t-\mu)^{q_1(x)-1}, & \text{for } t \in (\mu, 1+\mu), \\ [q_1(x), q_2(x)], & \text{for } t = 1+\mu, \\ q_2(x)(t-\mu)^{q_2(x)-1}, & \text{for } t \in (1+\mu, +\infty), \end{cases}$$

and

$$\partial_C \beta(x,t) = \begin{cases} -a(x), & \text{for } t \in (-\infty, 0), \\ [a(x), -a(x)], & \text{for } t = 0, \\ a(x), & \text{for } t \in (0, +\infty). \end{cases}$$

Thus for every $\xi(x) \in \partial_C \alpha(x,t)$ we have

$$|\xi(x)| \leq \begin{cases} 0 \leq \frac{q_2^+}{\mu^{q_2^+-1}} |t|^{q_2(x)-1}, & \text{for } t \in (-\infty, \mu], \\ q_1^+ \leq q_2^+ \left(\frac{|t|}{\mu}\right)^{q_2(x)-1} \leq \frac{q_2^+}{\mu^{q_2^+-1}} |t|^{q_2(x)-1}, & \text{for } t \in (\mu, 1+\mu), \\ q_2^+ \leq \frac{q_2^+}{\mu^{q_2^+-1}} |t|^{q_2(x)-1}, & \text{for } t = 1+\mu, \\ q_2^+ |t-\mu|^{q_2(x)-1} \leq \frac{q_2^+}{\mu^{q_2^+-1}} |t|^{q_2(x)-1}, & \text{for } t \in [1+\mu, +\infty). \end{cases}$$

Example 3 Let $f, g \in L^\infty_{\text{loc}}(\Omega \times \mathbb{R})$ and consider $\alpha, \beta : \Omega \times \mathbb{R}$ be defined by

$$\alpha(x,t) = \int_0^t f(x,s)\, ds \quad \text{and} \quad \beta(x,t) = \int_0^t g(x,s)\, ds.$$

Obviously, $t \mapsto \alpha(x,t)$ and $t \mapsto \beta(x,t)$ are locally Lipschitz and according to Proposition 1.7 in Motreanu and Panagiotopoulos [27], we have

$$\partial_C \alpha(x,t) = \left[\, \underline{f}(x,t), \overline{f}(x,t)\, \right] \quad \text{and} \quad \partial_C \beta(x,t) = \left[\, \underline{g}(x,t), \overline{g}(x,t)\, \right],$$

where for a function $h \in L^\infty_{\text{loc}}(\Omega \times \mathbb{R})$ we denote by

$$\underline{h}(x,t) = \lim_{\delta \downarrow 0} \inf_{|s-t|<\delta} h(x,t) \quad \text{and} \quad \overline{h}(x,t) = \lim_{\delta \downarrow 0} \sup_{|s-t|<\delta} h(x,t).$$

Clearly, there are many ways in which we can choose f and g such that conditions $\mathcal{H}(\alpha)$ and $\mathcal{H}(\beta)$ are fulfilled.

References

1. E. Acerbi and G. Mingione, Gradient estimates for the $p(x)$-Laplacean system, *J. Reine Angew. Math.* **584** (2005), 117-148.
2. I. Andrei, N. Costea and A. Matei, Antiplane shear deformation of piezoelectric bodies in contact with a conductive support, *J. Global Optim.* (2011), DOI: 10.1007/s10898-011-9815-x.
3. M.M. Boureanu, P. Pucci and V. Rădulescu, Multiplicity of solutions for a class of anisotropic elliptic equations with variable exponent, *Complex Var. Elliptic Equ.* **56** (2011), 755-767.
4. H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York/Dordrecht/London, 2011.
5. K.-C. Chang, Variational methods for non-differentiable functionals and their applications to partial differential equations, *J. Math. Anal. Appl.* **80** (1981), 102-129.

6. Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image processing, *SIAM J. Appl. Math.* **66** (2006), No. 4, 1383-1406.
7. F.H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley & Sons, New York, 1983.
8. N. Costea and M. Mihăilescu, On an eigenvalue problem involving variable exponent growth conditions, *Nonlinear Anal. T.M.A.* **71** (2009), 4271-4278.
9. N. Costea and Cs. Varga, Multiple critical points for non-differentiable parametrized functionals and applications to differential inclusions, *J. Global Optim.* (2011), DOI: 10.1007/s10898-011-9801-3.
10. N. Costea and Cs. Varga, Systems of nonlinear hemivariational inequalities and applications, *Topol. Methods Nonlinear Anal.* (2012), in press.
11. L. Diening, *Theoretical and Numerical Results for Electrorheological Fluids*, Ph.D. Thesis, University of Freiburg, Germany, 2002.
12. L. Diening, P. Harjulehto, P. Hästö and M. Ružička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Lecture Notes in Mathematics vol. 2017, Heidelberg/Dordrecht/London/New York, 2011.
13. X.L. Fan, Anisotropic variable exponent Sobolev spaces and $\overrightarrow{p}(x)$-Laplacian equations, *Complex Var. Elliptic Equ.* **56** (2011), 623-642.
14. D.E. Edmunds and R.M. Edmunds, Embeddings of anisotropic Sobolev spaces, *Arch. Ration. Mech. Anal.* **94** (3) (1986), 245-252.
15. X.L. Fan and Q.H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Anal. TMA* **52** (2003), 1843-1853.
16. T.C. Halsey, Electrorheological fluids, *Science* **258** (1992), 761-766.
17. O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.* **41** (1991), 592-618.
18. A. Kristály, V. Rădulescu and Cs. Varga, *Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics and its Applications, No. 136, Cambridge University Press, Cambridge, 2010.
19. G. Lebourg, Valeur moyenne pour gradient généralisé, *C. R. Acad. Sci. Paris* **281** (1975), 795-797.
20. M. Mihăilescu and V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. Roy. Soc. London Ser. A* **462** (2006), 2625-2641.
21. M. Mihăilescu, V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proc. Amer. Math. Soc.* **135** (2007), 2929-2937.
22. M. Mihăilescu, P. Pucci and V. Rădulescu, Nonhomogeneous boundary value problems in anisotropic Sobolev spaces, *C. R. Acad. Sci. Paris, Ser. I* **345** (2007), 561-566.
23. M. Mihăilescu, P. Pucci and V. Rădulescu, Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponents, *J. Math. Anal. Appl.* **340** (2008), 687-698.
24. M. Mihăilescu, G. Moroşanu and V. Rădulescu, Eigenvalue problems in anisotropic Orlicz-Sobolev spaces, *C. R. Acad. Sci. Paris, Ser., I* **347** (2009), 521-526.
25. M. Mihăilescu, G. Moroşanu and V. Rădulescu, Eigenvalue problems for anisotropic elliptic equations: An Orlicz-Sobolev space setting, *Nonlinear Anal. T.M.A.* **73** (2010), 3239-3253.
26. M. Mihăilescu and G. Moroşanu, Existence and multiplicity of solutions for an anisotropic elliptic problem involving variable exponent growth conditions, *Appl. Anal.* **89** (2) (2010), 257-271.
27. D. Motreanu and P.D. Panagiotopoulos, *Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities and Applications*, Kluwer Academic Publishers, Nonconvex Optimization and its Applications, vol. 29, Boston/Dordrecht/London, 1999.
28. D. Motreanu and V. Rădulescu, *Variational and Non-variational Methods in Nonlinear Analysis and Boundary Value Problems*, Kluwer Academic Publishers, Boston/Dordrecht/London, 2003.
29. D. Motreanu and Cs. Varga, Some critical point results for locally Lipschitz functionals, *Comm. Appl. Nonlinear Anal.* **4** (1997), 17-33.
30. S.M. Nikol'skii, On imbedding, continuation and approximation theorems for differentiable functions of several variables, *Russ. Math. Surv.* **16** (1961), 55-104.
31. Z. Naniewicz and P.D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Marcel Dekker, New York, 1995.
32. P.D. Panagiotopoulos, Nonconvex energy functions. Hemivariational inequalities and substationarity principles, *Acta Mech.* **42** (1983), 160-183.
33. J. Rákosník, Some remarks to anisotropic Sobolev spaces: I, *Beiträge zur Anal.* **13** (1979), 55-68.
34. J. Rákosník, Some remarks to anisotropic Sobolev spaces: II, *Beiträge zur Anal.* **15** (1981), 127-140.
35. M. Ruzicka, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2002.

36. D. Stancu-Dumitru, Multiplicity of solutions for anisotropic quasilinear elliptic equations with variable exponents, *Bull. Belg. Math. Soc. Simon Stevin* **17** (2010), 875-889.
37. M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, Heidelberg, 1996.
38. M. Troisi, Teoremi di inclusione per spazi di Sobolev non isotropi, *Ric. Mat.* **18** (1969), 3-24.
39. V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, *Math. USSR Izv.* **29** (1987), 33-66.