Multiple critical points for non-differentiable parametrized functionals and applications to differential inclusions

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Abstract

In this paper we deal with a class of non-differentiable functionals defined on a real reflexive Banach space X and depending on a real parameter of the form $\mathcal{E}_\lambda(u) = L(u) - (J_1 \circ T)(u) - \lambda(J_2 \circ S)(u)$, where $L : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous C^1 functional, $J_1 : Y \to \mathbb{R}$, $J_2 : Z \to \mathbb{R}$ (Y, Z Banach spaces) are two locally Lipschitz functionals, $T : X \to Y$, $S : X \to Z$ are linear and compact operators and $\lambda > 0$ is a real parameter. We prove that this kind of functionals posses at least three nonsmooth critical points for each $\lambda > 0$ and there exists $\lambda^* > 0$ such that the functional $\mathcal{E}_{\lambda^*}$ possesses at least four nonsmooth critical points. As an application, we study a nonhomogeneous differential inclusion involving the $p(x)$-Laplace operator whose weak solutions are exactly the nonsmooth critical points of some "*energy functional*" which satisfies the conditions required in our main result.

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1 Introduction and preliminary results

Motivated mostly by various real life phenomena arising in Mathematical Physics or Mechanics many authors studied the existence of critical points of various functions. The critical point theory for locally Lipschitz functionals began with the work of K-C. Chang [4], who used the properties of generalized gradients given by F.H. Clarke in [5] to propose a generalization of the the well known Palais-Smale

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compactness condition and obtained several mini-max principles concerning the existence of nonsmooth critical points. Since then, many results have been obtained in this direction, see e.g. the the book of D. Motreanu and P.D. Panagiotopoulos [24] and more recently A. Kristály, V. Rădulescu and Cs. Varga [16], or the papers of S. Adly, G. Buttazzo and M. Théra [1], D. Motreanu [20], D. Motreanu and Cs. Varga [21], D. Motreanu and V. Rădulescu [22], V. Rădulescu [26], A. Kristály, W. Marzantowicz and Cs. Varga [15], and the references therein.

The aim of this paper is to extend a very recent four critical points of B. Ricceri [28] to locally Lipschitz functionals, providing also an application in the form of a differential inclusion with a nonlinear boundary condition of Steklov type.

For the convenience of the reader we present next some notations and preliminary results from nonsmooth analysis that will be used throughout the paper. For a given Banach space $(X, \|\cdot\|_X)$ we denote by X^* its dual space and by $\langle\cdot,\cdot\rangle$ the duality pairing between X^* and X. The inner product and the euclidian norm in $\mathbb{R}^N$ $(N \geq 1)$ will be denoted by " $\cdot$ " and $|\cdot|$, respectively.

We recall that a functional $h : X \to \mathbb{R}$ is called *locally Lipschitz* if for every $u \in X$ there exists a neighborhood U of u and a constant $L = L(U) > 0$ such that

$$|h(w) - h(v)| \leq L\|w - v\|_X, \quad \text{for all } v, w \in U.$$

Definition 1.1. *Let $h : X \to \mathbb{R}$ be a locally Lipschitz functional. The generalized derivative of h at $u \in X$ in the direction $v \in X$, denoted $h^0(u;v)$, is defined by*

$$h^0(u;v) = \limsup_{\substack{w\to u\\ t\downarrow 0}} \frac{h(w+tv) - h(w)}{\lambda}.$$

The generalized gradient of h at a point $u \in X$, denoted $\partial h(u)$, is the subset of X^* defined by

$$\partial h(u) = \{\zeta \in X^* : \ h^0(u;v) \geq \langle \zeta, v\rangle, \ \ \textit{for all } v \in X\}.$$

We point out the fact that for each $u \in X$ we have $\partial h(u) \neq \emptyset$. In order to see that it suffices to apply the Hahn-Banach theorem (see e.g. Brezis [3]).

The next lemma points out important properties of generalized derivatives.

Lemma 1.1. *Let $g, h : X \to \mathbb{R}$ be locally Lipschitz of rank L near the point $u \in X$. Then*

- *the function $v \mapsto h^0(u;v)$ is finite, subadditive and satisfies*

$$|h^0(u;v)| \leq L\|v\|_X;$$

- *the application $(u,v) \mapsto h^0(u;v)$ is upper semicontinuous;*
- *$(g+h)^0(u;v) \leq g^0(u;v) + h^0(u;v)$ for all $u,v \in X$;*
- *$(-h)^0(u;v) = h^0(u;-v)$ and $h^0(u;\lambda v) = \lambda h^0(u;v)$ for all $u,v \in X$ and all $\lambda > 0$.*

The proof can be found in Clarke [5].

Definition 1.2. *A point $u \in X$ is nonsmooth a critical point of the locally Lipschitz functional $h : X \to \mathbb{R}$, if $0 \in \partial h(u)$, that is*

$$h^0(u; v) \geq 0, \quad \text{for all } v \in X.$$

The number $c = h(u)$ is called critical value of the functional h corresponding to the critical point u.

We point out the fact that every local extremum of the locally Lipschitz h is a nonsmooth critical point of h in the sense of Definition 1.2.

Definition 1.3. *A locally Lipschitz functional $h : X \to \mathbb{R}$ is said to satisfy the nonsmooth Palais-Smale condition at level $c \in \mathbb{R}$ (for brevity we shall use the notation $(PS)_c$-condition) if any sequence $\{u_n\} \subset X$ which satisfies*

- *$h(u_n) \to c$;*
- *there exists $\{\varepsilon_n\} \subset \mathbb{R}$, $\varepsilon_n \downarrow 0$ such that $h^0(u_n; v - u_n) + \varepsilon_n \|v - u_n\|_X \geq 0$, for all $v \in X$ and all $n \in \mathbb{N}$*

admits a convergent subsequence.

If this is true for every $c \in \mathbb{R}$, we say that h satisfies the nonsmooth (PS)-condition.

We recall next the zero-altitude version of the nonsmooth Mountain Pass Theorem due to D. Motreanu and Cs. Varga [21] which will play an important role in proving our main result.

Theorem 1.1. *Let $E : X \to \mathbb{R}$ be a locally Lipschitz functional which satisfies the nonsmooth (PS)-condition and let $u_1, u_2 \in X$ and $R \in (0, \|u_2 - u_1\|_X)$ be such that*

$$\inf_{u \in \partial \bar{B}(u_1; R)} E(u) \geq \max\{E(u_1), E(u_2)\}.$$

Then $c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E(\gamma(t)) \geq \max\{E(u_1), E(u_2)\}$ is a critical value for E and there exists $\tilde{u} \in X \setminus \{u_1, u_2\}$ such that $E(\tilde{u}) = c$.

Here and hereafter, Γ is the family of all continuous paths connecting the points u_1 and u_2, that is

$$\Gamma = \{\gamma \in C([0,1]; X) : \ \gamma(0) = u_1, \ \gamma(1) = u_2\}.$$

We recall now that for a functional $h : X \to \mathbb{R}$, the sets of the type $h^{-1}((-\infty, c])$ with $c \in \mathbb{R}$ are called *sub-level sets*. The functional h is said to be *quasi-concave* if the set $h^{-1}([c, +\infty))$ is convex for all $c \in \mathbb{R}$.
The second tool that will play a key role in the proof of our main result is given by the following theorem due to B. Ricceri [27]. Note that no smoothness assumption is required on the functional.

Theorem 1.2. *Let X be a topological space, $I \subseteq \mathbb{R}$ an open interval and $f : X \times I \to \mathbb{R}$ a functional satisfying the following conditions:*

- $\lambda \mapsto f(u,\lambda)$ *is quasi-concave and continuous for all* $u \in X$*;*

- $u \mapsto f(u,\lambda)$ *has closed and compact sub-level sets for all* $\lambda \in I$*;*

- $\sup\limits_{\lambda\in I} \inf\limits_{u\in X} f(u,\lambda) < \inf\limits_{u\in X} \sup\limits_{\lambda\in I} f(u,\lambda)$.

Then there exists $\lambda^* \in I$ *such that the functional* $u \mapsto f(u,\lambda^*)$ *admits at least two global minima.*

2 Main result: a nonsmooth Ricceri-type multiplicity result

Let X be a real reflexive Banach space and Y, Z two Banach spaces such that there exist $T : X \to Y$ and $S : X \to Z$ linear and compact. Let $L : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous C^1 functional such that $L' : X \to X^*$ has the $(\mathcal{S})_+$ property, i.e. if $u_n \rightharpoonup u$ in X and $\limsup\limits_{n\to\infty}\langle L'(u_n), u_n - u\rangle \le 0$, then $u_n \to u$. Assume in addition that $J_1 : Y \to \mathbb{R}$, $J_2 : Z \to \mathbb{R}$ are two locally Lipschitz functionals.

We are interested in studying the existence of nonsmooth critical points for functionals $\mathcal{E}_\lambda : X \to \mathbb{R}$ of the following type

$$\mathcal{E}_\lambda(u) := L(u) - (J_1 \circ T)(u) - \lambda(J_2 \circ S)(u), \tag{2.1}$$

where $\lambda > 0$ is a real parameter.

We point out the fact that it makes sense to talk about nonsmooth critical points for the functional defined in (2.1) as $\mathcal{E}_\lambda$ is locally Lipschitz. In order to see this let us fix $u \in X$, $\lambda > 0$ and $r > 0$ and choose $v, w \in \bar{B}(u;r)$. Since $L \in C^1(X;\mathbb{R})$ we have

$$|L(w) - L(v)| = |\langle L'(z), w - u\rangle| \le \|L'(z)\|_{X^*}\|w - v\|_X,$$

where $z = tw + (1-t)v$ for some $t \in (0,1)$. But, $\bar{B}(u;r)$ is weakly compact thus there exists $M > 0$ such that $\|L'(z)\|_{X^*} \le M$ on $\bar{B}(u;r)$. Using the fact that J_1, J_2 are locally Lipschitz functionals we get

$$\begin{aligned}
|\mathcal{E}_\lambda(w) - \mathcal{E}_\lambda(v)| &\le |L(w) - L(v)| + |(J_1\circ T)(w) - (J_1\circ T)(v)| + \lambda|(J_2\circ S)(w) - (J_2\circ S)(v)| \\
&\le M\|w - v\|_X + m_1\|Tw - Tv\|_Y + \lambda m_2\|Sw - Sv\|_Z \\
&\le \left[M + m_1\|T\|_{\mathcal{L}(X,Y)} + \lambda m_2\|S\|_{\mathcal{L}(X,Z)}\right]\|w - v\|_X,
\end{aligned}$$

which shows that $\mathcal{E}_\lambda$ is locally Lipschitz.

We also point out the fact that the functional $\mathcal{E}_\lambda$ is sequentially weakly lower semicontinuous since we assumed L to be sequentially weakly lower semicontinuous and T, S to be compact operators.

In order to prove our main result we shall assume the following conditions are fulfilled:

$(\mathcal{H}_1)$ there exists $u_0 \in X$ such that u_0 is a strict local minimum for L and $L(u_0) = (J_1\circ T)(u_0) = (J_2\circ S)(u_0) = 0$;

$(\mathcal{H}_2)$ for each $\lambda > 0$ the functional $\mathcal{E}_\lambda$ is coercive and we can determine $u_\lambda^0 \in X$ such that $\mathcal{E}_\lambda(u_\lambda^0) < 0$;

$(\mathcal{H}_3)$ there exists $R_0 > 0$ such that

$$(J_1 \circ T)(u) \leq L(u) \quad \text{and} \quad (J_2 \circ S)(u) \leq 0, \quad \text{for all } u \in \bar{B}(u_0; R_0) \setminus \{u_0\};$$

$(\mathcal{H}_4)$ there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda>0} \inf_{u\in X} \{\lambda\,[L(u) - (J_1 \circ T)(u) + \rho] - (J_2 \circ S)(u)\} < \inf_{u\in X} \sup_{\lambda>0} \{\lambda\,[L(u) - (J_1 \circ T)(u) + \rho] - (J_2 \circ S)(u)\}.$$

Our main result is given by the following theorem and it extends the result obtained recently by B. Ricceri (see [28], Theorem 1) to the case of non-differentiable locally Lipschitz functionals.

Theorem 2.1. *Assume that conditions $(\mathcal{H}_1) - (\mathcal{H}_3)$ are fulfilled. Then for each $\lambda > 0$ the functional $\mathcal{E}_\lambda$ defined in (2.1) has at least three nonsmooth critical points. If in addition $(\mathcal{H}_4)$ holds, then there exists $\lambda^* > 0$ such that $\mathcal{E}_{\lambda^*}$ has at least four nonsmooth critical points.*

Proof. The proof of Theorem 2.1 will be carried out in four steps an relies essentially on the zero altitude mountain pass theorem (see Theorem 1.1) combined with a technique of finding global minima for parametrized functions developed by B. Ricceri in [27] (see Theorem 1.2). Let us first fix $\lambda > 0$ and assume that $(\mathcal{H}_1) - (\mathcal{H}_3)$ are fulfilled.

STEP 1. u_0 is a nonsmooth critical point for $\mathcal{E}_\lambda$.

Since $u_0 \in X$ is a strict local minimum for L there exists $R_1 > 0$ such that

$$L(u) > 0, \quad \text{for all } u \in \bar{B}(u_0; R_1) \setminus \{u_0\}. \tag{2.2}$$

From $(\mathcal{H}_3)$ we deduce that

$$\frac{(J_1 \circ T)(u) + \lambda(J_2 \circ S)(u)}{L(u)} \leq 1, \quad \text{for all } u \in \bar{B}(u_0; R_0) \setminus \{u_0\}. \tag{2.3}$$

Taking $R_2 = \min\{R_0, R_1\}$ from (2.2) and (2.3) we have

$$\mathcal{E}_\lambda(u) = L(u) - (J_1 \circ T)(u) - \lambda(J_2 \circ S)(u) \geq 0 = \mathcal{E}_\lambda(u_0), \quad \text{for all } u \in \bar{B}(u_0; R_2) \setminus \{u_0\}. \tag{2.4}$$

We have proved thus that $u_0 \in X$ is a local minimum for $\mathcal{E}_\lambda$, therefore it is a nonsmooth critical point for this functional.

STEP 2. The functional $\mathcal{E}_\lambda$ admits a global minimum point $u_1 \in X \setminus \{u_0\}$.

Indeed, such a point exists since the functional $\mathcal{E}_\lambda$ is sequentially weakly lower semicontinuous and coercive, therefore it admits a global minimizer denoted u_1. Moreover, from $(\mathcal{H}_2)$ we deduce that $\mathcal{E}_\lambda(u_1) < 0$, hence $u_1 \neq u_0$.

STEP 3. There exists $u_2 \in X \setminus \{u_0, u_1\}$ such that u_2 is a nonsmooth critical point for $\mathcal{E}_\lambda$.

We shall prove first that the functional $\mathcal{E}_\lambda$ satisfies the nonsmooth (PS)-condition.

Let $c \in \mathbb{R}$ be fixed and $\{u_n\} \subset X$ be a sequence such that

- $\mathcal{E}_\lambda(u_n) \to c$;
- there exists $\{\varepsilon_n\} \subset \mathbb{R}$, $\varepsilon_n \downarrow 0$ such that $\mathcal{E}_\lambda^0(u_n; v - u_n) + \varepsilon_n \|v - u_n\|_X \geq 0$, for all $v \in X$.

Obviously $\{u_n\}$ is bounded due to the fact that $\mathcal{E}_\lambda$ is coercive. Then there exists $u \in X$ such that, up to a subsequence, $u_n \rightharpoonup u$ in X. Taking into account that T, S are compact operators we infer that $Tu_n \to Tu$ in Y and $Su_n \to Su$ in Z. For $v = u$ we have

$$\begin{aligned} 0 &\leq \varepsilon_n \|u - u_n\|_X + \mathcal{E}_\lambda^0(u_n; u - u_n) \\ &= \varepsilon_n \|u - u_n\|_X + (L - J_1 \circ T - \lambda J_2 \circ S)^0(u_n; u - u_n) \\ &\leq \varepsilon_n \|u - u_n\|_X + L^0(u_n; u - u_n) + J_1^0(Tu_n; Tu_n - Tu) + (\lambda J_2)^0(Su_n; Su_n - Su). \end{aligned}$$

But, $L \in C^1(X; \mathbb{R})$ and thus

$$L^0(u_n; u - u_n) = \langle L'(u_n), u - u_n \rangle.$$

On the other hand $\varepsilon_n \downarrow 0$ and $\{u_n\}$ is bounded hence $\limsup_{n\to\infty} \varepsilon_n \|u - u_n\|_X = 0$. Taking into account Lemma 1.1 we deduce that

$$\limsup_{n\to\infty} J_1^0(Tu_n; Tu_n - Tu) \leq J_1^0(Tu; 0) = 0$$

and

$$\limsup_{n\to\infty} (\lambda J_2)^0(Su_n; Su_n - Su) \leq (\lambda J_2)^0(Su; 0) = 0.$$

Gathering the above information we have

$$\begin{aligned} \limsup_{n\to\infty} \langle L'(u_n), u_n - u \rangle &\leq \limsup_{n\to\infty} \varepsilon_n \|u - u_n\|_X + \limsup_{n\to\infty} J_1^0(Tu_n; Tu_n - Tu) \\ &\quad + \limsup_{n\to\infty} (\lambda J_2)^0(Su_n; Su_n - Su) \\ &\leq 0. \end{aligned}$$

But, $L' : X \to X^*$ has the $(\mathcal{S})_+$ property, and this allows us to conclude that $\{u_n\}$ has a convergent subsequence, therefore $\mathcal{E}_\lambda$ satisfies the nonsmooth (PS)-condition.

According to Step 2 there exists $u_1 \in X$ such that $\mathcal{E}_\lambda(u_1) < 0$. On the other hand, $\mathcal{E}_\lambda(u_0) = 0$ and we can choose $0 < r < \min\{R_2, \|u_1 - u_0\|_X\}$ such that

$$\mathcal{E}_\lambda(u) \geq \max\{\mathcal{E}_\lambda(u_0), \mathcal{E}_\lambda(u_1)\} = 0, \qquad \text{for all } u \in \partial \bar{B}(u_0; r).$$

Applying Theorem 1.1 we obtain that there exists a nonsmooth critical point $u_2 \in X \setminus \{u_0, u_1\}$ for $\mathcal{E}_\lambda$ and $\mathcal{E}_\lambda(u_1) \geq 0$. This completes the proof of the first part of the theorem, i.e. the functional $\mathcal{E}_\lambda$ has at least three distinct critical points.

Step 4. If in addition $(\mathcal{H}_4)$ holds, then there exists $\lambda^* > 0$ such that $\mathcal{E}_{\lambda^*}$ has two global minima.

Let us consider the functional $f : X \times (0,\infty) \to \mathbb{R}$ defined by

$$f(u,\mu) = \mu\,[L(u) - (J_1 \circ T)(u) + \rho] - (J_2 \circ S)(u) = \mu\mathcal{E}_{1/\mu}(u) + \mu\rho,$$

where $\rho \in \mathbb{R}$ is the number from $(\mathcal{H}_4)$.

We observe that for each $u \in X$ the functional $\mu \mapsto f(u,\mu)$ is affine, therefore it is quasi-concave. We also note that for each $\mu > 0$ the mapping $u \mapsto f(u,\mu)$ is sequentially weakly lower semicontinuous. Therefore for each $\mu > 0$, the sub-level sets of $u \mapsto f(u,\mu)$ are sequentially weakly closed.

Let us consider now the set $S^\mu(c) = \{u \in X : \ f(u,\mu) \leq c\}$ for some $c \in \mathbb{R}$ and a sequence $\{u_n\} \subset S^\mu(c)$. Obviously $\{u_n\}$ is bounded due to the fact that the functional $u \mapsto f(u,\mu)$ is coercive, which is clear since $f(u,\mu) = \mu\mathcal{E}_{1/\mu}(u) + \mu\rho$, $\mathcal{E}_{1/\mu}$ is coercive and $\mu > 0$. According to the Eberlein-Smulyan Theorem $\{u_n\}$ admits a subsequence, still denoted $\{u_n\}$, which converges weakly to some $u \in X$. Keeping in mind that $u_n \in S^\mu(c)$ for $n > 0$ we deduce that

$$\mathcal{E}_{1/\mu}(u_n) \leq \frac{c - \mu\rho}{\mu}, \qquad \text{for all } n > 0.$$

Combining the above relation with the fact that $\mathcal{E}_{1/\mu}$ is sequentially weakly lower semicontinuous we get

$$\mathcal{E}_{1/\mu}(u) \leq \liminf_{n\to\infty} \mathcal{E}_{1/\mu}(u_n) \leq \frac{c - \mu\rho}{\mu},$$

which shows that $f(u,\mu) \leq c$, therefore the set $S^\mu(c)$ is a sequentially weakly compact subset of X. We have proved thus that, for each $\mu > 0$, the sub-level sets of $u \mapsto f(u,\mu)$ are sequentially weakly compact. Taking into account Remark 1 in [27] which states that we can replace "closed and compact" by "sequentially closed and sequentially compact" in Theorem 1.2 and using condition $(\mathcal{H}_4)$ we can apply Theorem 1.2 for the weak topology of X and conclude that there exists $\mu^* > 0$ for which $f(u,\mu^*) = \mu^*\mathcal{E}_{1/\mu^*}(u) + \mu^*\rho$ has two global minima. It is easy to check that any global minimum point of $f(u,\mu^*)$ is also a global minimum point for $\mathcal{E}_{1/\mu^*}$, and thus we get the existence of a point $u_3 \in X \setminus \{u_1\}$ such that

$$\mathcal{E}_{1/\mu^*}(u_1) = \mathcal{E}_{1/\mu^*}(u_3) \leq \mathcal{E}_{1/\mu^*}(u^0_{1/\mu^*}) < 0 = \mathcal{E}_{1/\mu^*}(u_0) \leq \mathcal{E}_{1/\mu^*}(u_2),$$

which shows that $u_3 \in X \setminus \{u_0, u_1, u_2\}$. Taking $\lambda^* = 1/\mu^*$ the proof of Theorem 2.1 is now complete.

□

Remark 2.1. *A similar result to Theorem 2.1 has recently been obtained jointly by F. Faraci and A. Iannizzotto [12] (in a as yet unpublished paper) who also give a very interesting application to the existence of three non-zero periodic solutions for an ordinary differential inclusion.*

3 A differential inclusion involving the $p(x)$-Laplace operator and Steklov-type boundary conditions

In this section we are concerned with the study of a differential inclusion of the type

$$(P_\lambda): \begin{cases} -\text{div}\,(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u \in \partial\phi(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_{p(x)}} \in \lambda \partial\psi(x,u) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $\lambda > 0$ is a real parameter, $p : \overline{\Omega} \to \mathbb{R}$ is a continuous function such that $\inf_{x\in\overline{\Omega}} p(x) > N$, $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\psi : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are locally Lipschitz functionals with respect to the second variable and $\frac{\partial u}{\partial n_{p(x)}} = |\nabla u|^{p(x)-2}\nabla u \cdot n$, n being the unit outward normal on $\partial\Omega$.

In the case when $p(x) \equiv p$, $\phi(x,t) \equiv 0$ and $\psi(x,t) = \frac{1}{q}|t|^q$ the problem (P_λ) becomes

$$(\mathcal{P}): \begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = \lambda |u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

and it was studied by J. Fernández Bonder and J.D. Rossi [13] in the case $1 < q < p^* = \frac{p(N-1)}{N-p}$ by using variational arguments combined with the Sobolev trace inequality. In [13] it is also proved that if $p = q$ then problem $(\mathcal{P})$ admits a sequence of eigenvalues $\{\lambda_n\}$, such that $\lambda_n \to \infty$ as $n \to \infty$. Furthermore, S. Martinez and J.D. Rossi [18] proved that the first eigenvalue λ_1 of problem $(\mathcal{P})$ (that is, $\lambda_1 \leq \lambda$ for any other eigenvalue) when $p = q$ is isolated and simple. In the linear case, that is $p = q = 2$, problem $(\mathcal{P})$ is known in the literature as the *Steklov* problem (see e.g. I. Babuška and J. Osborn [2]).

Let us present next some basic notions and results from the theory of Lebesgue-Sobolev spaces with variable exponent. For more details one can consult the book by J. Musielak [25] and the papers by D.E. Edmunds et al. [6, 7, 8], O. Kováčik and J. Rákosník [14], X.L. Fan et al. [9, 10], M. Mihăilescu and V. Rădulescu [19].

Set

$$C_+(\overline{\Omega}) = \{\varphi \in C(\overline{\Omega}) : \ \varphi(x) > 1, \ \text{for all } x \in \overline{\Omega}\},$$

and for $\varphi \in C_+(\overline{\Omega})$ we denote

$$\varphi^- = \inf_{x\in\overline{\Omega}} \varphi(x) \quad \text{and} \quad \varphi^+ = \sup_{x\in\overline{\Omega}} \varphi(x).$$

For a function $p \in C_+(\overline{\Omega})$ we define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \left\{u : \ u \text{ is a real valued-function and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}$$

which can be endowed with the so-called *Luxemburg norm* given by the formula

$$|u|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\zeta > 0 : \ \int_\Omega \left|\frac{u(x)}{\zeta}\right|^{p(x)} dx \leq 1\right\}.$$

We recall that $\left(L^{p(\cdot)}(\Omega), |\cdot|_{L^{p(\cdot)}(\Omega)}\right)$ is a separable and reflexive Banach space. If $0 < \text{meas}(\Omega) < \infty$ and p, q are variable exponents such that $p(x) \leq q(x)$ in Ω, then the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous. We also remember that the following Hölder type inequality holds

$$\int_\Omega |u(x)v(x)|\, dx \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{L^{p(\cdot)}(\Omega)} |v|_{L^{p'(\cdot)}(\Omega)}, \tag{3.1}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and all $v \in L^{p'(\cdot)}(\Omega)$, where by $p'(x)$ we have denoted the conjugated exponent of $p(x)$, that is $p'(x) = \frac{p(x)}{p(x)-1}$.

An important role in manipulating the variable exponent Lebesgue spaces is played by the so-called *modular* of the $L^{p(\cdot)}(\Omega)$ space, that is $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) :\to \mathbb{R}$

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx$$

which has the following properties:

$$|u|_{L^{p(\cdot)}(\Omega)} > 1 \Longrightarrow |u|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq |u|^{p^+}_{L^{p(\cdot)}(\Omega)}, \tag{3.2}$$

$$|u|_{L^{p(\cdot)}(\Omega)} < 1 \Longrightarrow |u|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq |u|^{p^-}_{L^{p(\cdot)}(\Omega)}. \tag{3.3}$$

The variable exponent Sobolev space is defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) :\ |\nabla u| \in L^{p(\cdot)}(\Omega)\right\}$$

and can be endowed with the norm

$$\|u\| = |u|_{L^{p(\cdot)}(\Omega)} + |\nabla u|_{L^{p(\cdot)}(\Omega)}.$$

We recall that $\left(W^{1,p(\cdot)}(\Omega), \|\cdot\|\right)$ is a separable and reflexive Banach space. If we set

$$I(u) = \int_\Omega \left(|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)}\right)\, dx$$

then for $u \in W^{1,p(\cdot)}(\Omega)$ the following relations hold true

$$\|u\| > 1 \Longrightarrow \|u\|^{p^-} \leq I(u) \leq \|u\|^{p^+}, \tag{3.4}$$

$$\|u\| < 1 \Longrightarrow \|u\|^{p^+} \leq I(u) \leq \|u\|^{p^-}. \tag{3.5}$$

Denote

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N, \end{cases}$$

and

$$p_*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N. \end{cases}$$

If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact, while if $r \in C_+(\partial\Omega)$ and $r(x) < p_*(x)$ for all $x \in \partial\Omega$, then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ is also compact.

Remark 3.1. *If $N < p^- \leq p(x)$ for any $x \in \overline{\Omega}$ it is clear from above that for any $q \in C_+(\overline{\Omega})$ and any $r \in C_+(\partial\Omega)$ the space $W^{1,p(\cdot)}(\Omega)$ is compactly embedded in $L^{q(\cdot)}(\Omega)$ and $L^{r(\cdot)}(\partial\Omega)$, respectively. On the other hand, according to Theorem 2.2 from [11] the space $W^{1,p(\cdot)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$, and, since $N < p^-$ it follows that $W^{1,p(\cdot)}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Therefore, there exists a positive constant $c_\infty > 0$ such that*

$$\|u\|_\infty \leq c_\infty \|u\|, \qquad \text{for all } u \in W^{1,p(\cdot)}(\Omega), \tag{3.6}$$

where by $\|\cdot\|_\infty$ we have denoted the usual norm on $C(\overline{\Omega})$, that is $\|u\|_\infty = \sup_{x\in\overline{\Omega}} |u(x)|$.

Definition 3.1. *We say that $u \in W^{1,p(\cdot)}(\Omega)$ is a solution of problem (P_λ) if there exist $\xi(x) \in \partial\phi(x, u(x))$ and $\zeta(x) \in \partial\psi(x, u(x))$ for almost every $x \in \overline{\Omega}$ such that for all $v \in W^{1,p(\cdot)}(\Omega)$ we have*

$$\int_\Omega \Big(-\text{div }(|\nabla u(x)|^{p(x)-2}\nabla u(x)) + |u(x)|^{p(x)-2}u(x)\Big)\, v(x)\, dx = \int_\Omega \xi(x)v(x)\, dx$$

and

$$\int_{\partial\Omega} \frac{\partial u}{\partial n_{p(\cdot)}} v(x)\, d\sigma = \lambda \int_{\partial\Omega} \zeta(x)v(x)\, d\sigma.$$

Here, and hereafter we shall assume the the following hypotheses hold:

(H^1) $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a functional such that

- (i) $\phi(x, 0) = 0$ for almost every $x \in \Omega$;
- (ii) the function $x \mapsto \phi(x, t)$ is measurable for every $t \in \mathbb{R}$;
- (iii) the function $t \mapsto \phi(x, t)$ is locally Lipschitz for almost every $x \in \Omega$;
- (iv) there exist $c_\phi > 0$ and $q \in C(\overline{\Omega})$ with $1 < q(x) \leq q^+ < p^-$ such that

$$|\xi(x)| \leq c_\phi |t|^{q(x)-1}$$

  for almost every $x \in \Omega$, every $t \in \mathbb{R}$ and every $\xi(x) \in \partial\phi(x, t)$.
- (v) there exists $\delta_1 > 0$ such that $\phi(x, t) \leq 0$ when $0 < |t| \leq \delta_1$, for almost every $x \in \Omega$.

(H^2) $\psi : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a functional such that

- (i) $\psi(x, 0) = 0$ for almost every $x \in \partial\Omega$;
- (ii) the function $x \mapsto \psi(x, t)$ is measurable for every $t \in \mathbb{R}$;
- (iii) the function $t \mapsto \psi(x, t)$ is locally Lipschitz for almost every $x \in \partial\Omega$;

(iv) there exist $c_\psi > 0$ and $r \in C(\partial\Omega)$ with $1 < r(x) \le r^+ < p^-$ such that

$$|\zeta(x)| \le c_\psi |t|^{r(x)-1}$$

for almost every $x \in \partial\Omega$, every $t \in \mathbb{R}$ and every $\zeta(x) \in \partial\psi(x,t)$;

(v) there exists $\delta_2 > 0$ such that $\psi(x,t) \le 0$ when $0 < |t| \le \delta_2$, for almost every $x \in \partial\Omega$.

(H^3) There exist $\eta > \max\{\delta_1, \delta_2\}$ such that $\eta^{p(x)} \le p(x)\phi(x,\eta)$ for almost every $x \in \Omega$ and $\psi(x,\eta) > 0$ for almost every $x \in \partial\Omega$.

(H^4) There exists $m \in L^1(\Omega)$ such that $\phi(x,t) \le m(x)$ for all $t \in \mathbb{R}$ and almost every $x \in \Omega$.

(H^5) There exists $\mu > \max\left\{c_\infty(p^+\|m\|_{L^1(\Omega)})^{1/p^-}; c_\infty(p^+\|m\|_{L^1(\Omega)})^{1/p^+}\right\}$ such that

$$\sup_{|t|\le\mu} \psi(x,t) \le \psi(x,\eta) < \sup_{t\in\mathbb{R}} \psi(x,t).$$

The main result of this section is given by the following theorem.

Theorem 3.1. *Assume that (H^1)-(H^3) hold true. Then for each $\lambda > 0$ problem (P_λ) admits at least two non-zero solutions. If in addition (H^4) and (H^5) hold, then there exists $\lambda^* > 0$ such that problem (P_{λ^*}) admits at least three non-zero solutions.*

Proof. Let us denote $X = W^{1,p(\cdot)}(\Omega)$, $Y = L^{q(\cdot)}(\Omega)$, $Z = L^{r(\cdot)}(\partial\Omega)$, and consider $T : X \to Y$ to be the embedding operator and $S : X \to Z$, $S = i \circ \gamma$ where $\gamma : W^{1,p(\cdot)}(\Omega) \to L^{p(\cdot)}(\partial\Omega)$ is the trace operator ($\gamma(u) = u|_{\partial\Omega}$ for all $u \in W^{1,p(\cdot)}(\Omega)$) and $i : L^{p(\cdot)}(\partial\Omega) \to L^{r(\cdot)}(\partial\Omega)$ is the embedding operator. It is clear that T, S are compact operators and for the sake of simplicity, everywhere below, we will omit to write Tu and Su to denote the above operators, writing u instead of Tu or Su. The compactness of these operators involves the existence of two positive constants $c_r, c_q > 0$ such that

$$|u|_{L^{q(\cdot)}(\Omega)} \le c_q\|u\|, \quad \text{for all } u \in X,$$

and

$$|u|_{L^{r(\cdot)}(\partial\Omega)} \le c_r\|u\|, \quad \text{for all } u \in X.$$

We introduce next $L : X \to \mathbb{R}$, $J_1 : Y \to \mathbb{R}$ and $J_2 : Z \to \mathbb{R}$ as follows

$$L(u) = \int_\Omega \frac{1}{p(x)} \left[|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)}\right] dx, \quad \text{for } u \in X,$$

$$J_1(y) = \int_\Omega \phi(x, y(x))\, dx, \quad \text{for } y \in Y,$$

and

$$J_2(z) = \int_{\partial\Omega} \psi(x, z(x))\, d\sigma, \quad \text{for } z \in Z.$$

We point out the fact that L is sequentially weakly lower semicontinuous and $L' : X \to X^*$,

$$\langle L'(u), v\rangle = \int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot\nabla v(x) + |u(x)|^{p(x)-2}u(x)v(x)\,dx$$

has the $(\mathcal{S})_+$ property according to X.L. Fan and Q.H. Zhang [10], see Theorem 3.1.

The idea is to prove that the functional $\mathcal{E}_\lambda : X \to \mathbb{R}$ defined by $\mathcal{E}_\lambda(u) = L(u) - J_1(u) - \lambda J_2(u)$ satisfies the conditions of Theorem 2.1 and each nonsmooth critical point of this functional is a solution of problem (P_λ) in the sense of Definition 3.1. With this end in view we divide the proof in several steps.

Step 1. The functionals J_1 and J_2 defined above are locally Lipschitz.

Let $y \in Y$, $R > 0$ and $y_1, y_2 \in \bar{B}_Y(y;R)$ be fixed. According to Lebourg's mean value theorem (see e.g. [17], or [23] Theorem 1.1) there exists $\bar{y} = t_0 y_1 + (1-t_0)y_2$ and $\xi^*(x) \in \partial\phi(x,\bar{y}(x))$, for some $t_0 \in (0,1)$, such that

$$\phi(x,y_1(x)) - \phi(x,y_2(x)) = \xi^*(x)(y_1(x) - y_2(x)).$$

Thus,

$$\begin{aligned}
|J_1(y_1) - J_1(y_2)| &= \left|\int_\Omega \phi(x,y_1(x)) - \phi(x,y_2(x))\,dx\right| \\
&\le \int_\Omega |\phi(x,y_1(x)) - \phi(x,y_2(x))|\,dx \\
&= \int_\Omega |\xi^*(x)|\,|y_1(x) - y_2(x)|\,dx \\
&\le \int_\Omega c_\phi |\bar{y}(x)|^{q(x)-1}|y_1(x) - y_2(x)|\,dx \\
&\le c_\phi\left(\frac{1}{q^-} + \frac{1}{q'^-}\right)\left||\bar{y}|^{q(x)-1}\right|_{L^{q'(\cdot)}(\Omega)}|y_1 - y_2|_{L^{q(\cdot)}(\Omega)}
\end{aligned}$$

We only need to prove now that $\left||\bar{y}|^{q(x)-1}\right|_{L^{q'(\cdot)}(\Omega)}$ is bounded. The conclusion is obvious when $\left||\bar{y}|^{q(x)-1}\right|_{L^{q'(\cdot)}(\Omega)} \le 1$, therefore we shall treat only the case $\left||\bar{y}|^{q(x)-1}\right|_{L^{q'(\cdot)}(\Omega)} > 1$. Keeping in mind (3.2)-(3.3) we have

$$\begin{aligned}
\left||\bar{y}|^{q(x)-1}\right|_{L^{q'(\cdot)}(\Omega)} &\le \left||\bar{y}|^{q(x)-1}\right|^{q^-}_{L^{q'(\cdot)}(\Omega)} \\
&\le \rho_{q'(\cdot)}\left(|\bar{y}|^{q(x)-1}\right) \\
&= \int_\Omega \left(|\bar{y}|^{q(x)-1}\right)^{q'(x)}\,dx \\
&= \int_\Omega |\bar{y}|^{q(x)}\,dx \\
&= \rho_{q(\cdot)}(\bar{y}).
\end{aligned}$$

On the other hand we have

$$\rho_{q(\cdot)}(\bar{y}) \leq \begin{cases} |\bar{y}|^{q^-}_{L^{q(\cdot)}(\Omega)}, & \text{if } |\bar{y}|_{L^{q(\cdot)}(\Omega)} < 1, \\ |\bar{y}|^{q^+}_{L^{q(\cdot)}(\Omega)}, & \text{if } |\bar{y}|_{L^{q(\cdot)}(\partial\Omega)} > 1. \end{cases}$$

In a similar way we prove that J_2 is a locally Lipschitz functional.

Step 2. $u_0 = 0$ satisfies hypothesis $(\mathcal{H}_1)$.

Indeed, $L(0) = J_1(0) = J_2(0) = 0$ and for each $R > 0$ we have

$$L(u) > 0, \qquad \text{for all } u \in \bar{B}_X(0; R) \setminus \{0\},$$

which shows that $u_0 = 0$ is a strict minimum point for L.

Step 3. The functional $\mathcal{E}_\lambda$ is coercive.

Let $u \in X$ be fixed. According to Lebourg's mean value theorem there exist $s_0, s_1 \in (0, 1)$ and $\xi^*(x) \in \partial\phi(x, s_0u(x))$, $\zeta^*(x) \in \partial\psi(x, s_1u(x))$ such that

$$\phi(x, u(x)) - \phi(x, 0) = \xi^*(x)u(x) \quad \text{and} \quad \psi(x, u(x)) - \psi(x, 0) = \zeta^*(x)u(x).$$

Thus,

$$\begin{aligned} J_1(u) &= \int_\Omega \phi(x, u(x)) - \phi(x, 0)\, dx \\ &\leq \int_\Omega |\xi^*(x)||u(x)|\, dx \\ &\leq c_\phi \int_\Omega |s_0u(x)|^{q(x)-1}|u(x)|\, dx \\ &\leq c_\phi \int_\Omega |u(x)|^{q(x)}\, dx, \end{aligned}$$

and

$$\begin{aligned} J_2(u) &= \int_{\partial\Omega} \psi(x, u(x)) - \psi(x, 0)\, d\sigma \\ &\leq \int_{\partial\Omega} |\zeta^*(x)||u(x)|\, d\sigma \\ &\leq c_\psi \int_{\partial\Omega} |s_1u(x)|^{r(x)-1}|u(x)|\, d\sigma \\ &\leq c_\psi \int_{\partial\Omega} |u(x)|^{r(x)}\, d\sigma. \end{aligned}$$

Hence for $u \in X$ with $\|u\| > 1$, $|u|_{L^{q(\cdot)}(\Omega)} > 1$ and $|u|_{L^{r(\cdot)}(\partial\Omega)} > 1$ we have

$$\begin{aligned}
\mathcal{E}_\lambda(u) &= L(u) - J_1(u) - \lambda J_2(u) \\
&= \int_\Omega \frac{1}{p(x)} \left[|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)} \right] dx - \int_\Omega \phi(x, u(x))\, dx - \lambda \int_{\partial\Omega} \psi(x, u(x))\, d\sigma \\
&\geq \frac{1}{p^+} \|u\|^{p^-} - c_\phi |u|^{q^+}_{L^{q(\cdot)}(\Omega)} - \lambda c_\psi |u|^{r^+}_{L^{r(\cdot)}(\partial\Omega)} \\
&\geq \frac{1}{p^+} \|u\|^{p^-} - c_\phi c_q^{q^+} \|u\|^{q^+} - \lambda c_\psi c_r^{r^+} \|u\|^{r^+}.
\end{aligned}$$

We conclude that $\mathcal{E}_\lambda(u) \to \infty$ as $\|u\| \to \infty$ since $r^+ < p^-$ and $q^+ < p^-$.

STEP 4. There exists $\bar{u}_0 \in X$ such that $\mathcal{E}_\lambda(\bar{u}_0) < 0$.

Choosing $\bar{u}_0(x) = \eta$ for all $x \in \overline{\Omega}$ and taking into account (H^3) we conclude that

$$\begin{aligned}
\mathcal{E}_\lambda(\bar{u}_0) &= L(\bar{u}_0) - J_1(\bar{u}_0) - \lambda J_2(\bar{u}_0) \\
&= \int_\Omega \frac{1}{p(x)} \eta^{p(x)}\, dx - \int_\Omega \phi(x, \eta)\, dx - \lambda \int_{\partial\Omega} \psi(x, \eta)\, d\sigma < 0.
\end{aligned}$$

STEP 5. There exists $R_0 > 0$ such that $J_1(u) \leq L(u)$ and $\frac{J_2(u)}{L(u)} \leq 0$ for all $u \in B(0; R_0) \setminus \{0\}$.

Let us define $R_0 < \min\left\{ \frac{\delta_1}{c_\infty}; \frac{\delta_2}{c_\infty} \right\}$ where c_∞ is given in (3.6) and δ_1, δ_2 are given in (H^1) and (H^2), respectively. For an arbitrarily fixed $u \in B(0; R_0)$, taking into account the way we defined the operators T and S, we have

$$|u(x)| \leq \|u\|_\infty \leq c_\infty \|u\| \leq c_\infty R_0 < \delta_1, \qquad \text{for all } x \in \Omega$$

and

$$|u(x)| \leq \|u\|_\infty \leq c_\infty \|u\| \leq c_\infty R_0 < \delta_2, \qquad \text{for all } x \in \partial\Omega.$$

Hypotheses (H^1) and (H^2) ensure that $\phi(x, u(x)) \leq 0$ and $\psi(x, u(x)) \leq 0$ for all $u \in B(0; R_0)$, therefore $J_1(u) \leq 0 < L(u)$ and $J_2(u) \leq 0$ for all $u \in B(0; R_0) \setminus \{0\}$.

STEP 6. There exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda > 0} \inf_{u \in X} \lambda \left[L(u) - J_1(u) + \rho \right] - J_2(u) < \inf_{u \in X} \sup_{\lambda > 0} \lambda \left[L(u) - J_1(u) + \rho \right] - J_2(u).$$

Using the same arguments as B. Ricceri [27] (see the proof of Theorem 2) we conclude that it suffices to find $\rho \in \mathbb{R}$ and $\bar{u}_1, \bar{u}_2 \in X$ such that

$$L(\bar{u}_1) - J_1(\bar{u}_1) < \rho < L(\bar{u}_2) - J_1(\bar{u}_2) \tag{3.7}$$

and

$$\frac{\sup_{u \in A} J_2(u) - J_2(\bar{u}_1)}{\rho - L(\bar{u}_1) + J_1(\bar{u}_1)} < \frac{\sup_{u \in A} J_2(u) - J_2(\bar{u}_2)}{\rho - L(\bar{u}_2) + J_1(\bar{u}_2)}, \tag{3.8}$$

where $A = (L - J_1)^{-1}((-\infty, \rho])$.

Let us define $\bar{u}_1 \equiv \eta$ and choose $\bar{u}_2$ such that

$$\psi(x, \bar{u}_2(x)) > \sup_{|t| \leq \mu} \psi(x, t).$$

We point out the fact that a $\bar{u}_2$ satisfying the above relation exists due to (H^5). Next we define

$$\rho = \min \left\{ \frac{1}{p^+} \left(\frac{\mu}{c_\infty} \right)^{p^+} - \|m\|_{L^1(\Omega)}; \frac{1}{p^+} \left(\frac{\mu}{c_\infty} \right)^{p^-} - \|m\|_{L^1(\Omega)} \right\}$$

and observe that $\rho > 0$.

We shall prove next that for any $u \in A$ we have $\|u\|_\infty \leq \mu$. In order to do this, let us fix $u \in A$. Keeping in mind (H^4) and the way we defined ρ we distinguish the following cases:

- $\|u\| \leq 1$.

  In this case we have $\|u\|^{p^+} \leq I(u)$ and we obtain the following estimates:

$$\begin{aligned}
\frac{1}{p^+} \|u\|^{p^+} &\leq \frac{1}{p^+} I(u) \\
&\leq \int_\Omega \frac{1}{p(x)} \left[|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)} \right] dx \\
&\leq \rho + \int_\Omega \phi(x, u(x)) \, dx \\
&\leq \rho + \int_\Omega m(x) \, dx \\
&\leq \frac{1}{p^+} \left(\frac{\mu}{c_\infty} \right)^{p^+} - \|m\|_{L^1(\Omega)} + \|m\|_{L^1(\Omega)} \\
&= \frac{1}{p^+} \left(\frac{\mu}{c_\infty} \right)^{p^+}.
\end{aligned}$$

  We conclude from above that $\|u\| \leq \frac{\mu}{c_\infty}$ therefore we must have $\|u\|_\infty \leq \mu$.

- $\|u\| > 1$.

  In this case we have $\|u\|^{p^-} \leq I(u)$ and we obtain the following estimates:

$$\begin{aligned}
\frac{1}{p^+} \|u\|^{p^-} &\leq \frac{1}{p^+} I(u) \\
&\leq \int_\Omega \frac{1}{p(x)} \left[|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)} \right] dx \\
&\leq \rho + \int_\Omega \phi(x, u(x)) \, dx \\
&\leq \rho + \int_\Omega m(x) \, dx \\
&\leq \frac{1}{p^+} \left(\frac{\mu}{c_\infty} \right)^{p^-}.
\end{aligned}$$

The above computations enable us to conclude that $\|u\| \leq \frac{\mu}{c_\infty}$ therefore we must have $\|u\|_\infty \leq \mu$.

We only have to check that (3.7) and (3.8) hold for $\bar{u}_1$ and $\bar{u}_2$ chosen as above. From above we conclude that $\bar{u}_2 \notin A$ and thus

$$\sup_{u\in A} J_2(u) \leq \sup_{\|u\|_\infty \leq \mu} J_2(u) \leq J_2(\bar{u}_1), \quad \sup_{u\in A} J_2(u) \leq \sup_{\|u\|_\infty \leq \mu} J_2(u) \leq J_2(\bar{u}_2),$$

and

$$L(\bar{u}_1) - J_1(\bar{u}_1) \leq 0 < \rho < L(\bar{u}_2) - J_1(\bar{u}_2).$$

Step 7. Any nonsmooth critical point of the functional $\mathcal{E}_\lambda$ is a solution of problem (P_λ).

It is easy to check that $u \in W^{1,p(\cdot)}(\Omega)$ is a solution of problem (P_λ), if and only if there exist $\xi(x) \in \partial\phi(x, u(x))$ and $\zeta(x) \in \partial\psi(x, u(x))$ such that for all $v \in W^{1,p(\cdot)}(\Omega)$

$$\begin{aligned} 0 \;\;&=\;\; \int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot\nabla v(x) + |u(x)|^{p(x)-2}u(x)v(x)\;dx + \int_\Omega \xi(x)(-v(x))\;dx \\ &\quad + \int_{\partial\Omega} \zeta(x)(-\lambda v(x))\;d\sigma. \end{aligned}$$

Applying a similar argument as in the proof of the Aubin-Clarke Theorem (see e.g. Clarke [5] Theorem 2.7.5 or Motreanu-Rădulescu [23] Theorem 1.3) we can conclude that

$$J_1^0(y_1; y_2) \leq \int_\Omega \phi^0(x, y_1(x); y_2(x))\;dx, \qquad \text{for all } y_1, y_2 \in L^{q(\cdot)}(\Omega),$$

and

$$J_2^0(w_1; w_2) \leq \int_{\partial\Omega} \psi^0(x, w_1(x); w_2(x))\;dx, \text{ for all } z_1, z_2 \in L^{r(\cdot)}(\partial\Omega).$$

Let $u \in X$ be a nonsmooth critical point of $\mathcal{E}_\lambda$ and $v \in X$ be fixed. Taking into account Lemma 1.1 we obtain

$$\begin{aligned} 0 \;&\leq\; \mathcal{E}_\lambda^0(u; v) \\ &=\; (L - J_1 - \lambda J_2)^0(u; v) \\ &\leq\; L^0(u; v) + (-J_1)^0(u; v) + \lambda(-J_2)^0(u; v) \\ &\leq\; \langle L'(u), v\rangle + J_1^0(u; -v) + J_2(u; -\lambda v) \\ &\leq\; \int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot\nabla v(x) + |u(x)|^{p(x)-2}u(x)v(x)\;dx + \int_\Omega \phi^0(x, u(x); -v(x))\;dx \\ &\quad + \int_{\partial\Omega} \psi^0(x, u(x); -\lambda v(x))\;d\sigma. \end{aligned}$$

On the other hand, Proposition 2.1.2 in [5] (p. 27) ensures that for almost every $x \in \Omega$ there exists $\xi(x) \in \partial\phi(x, u(x))$ such that, for all $t \in \mathbb{R}$, we have

$$\phi^0(x, u(x); t) = \xi(x)t = \max\{zt : z \in \partial\phi(x, u(x))\}.$$

In a similar way we deduce that for almost every $x \in \partial\Omega$ there exist $\zeta(x) \in \partial\psi(x, u(x))$ such that

$$\psi^0(x, u(x); t) = \zeta(x)t = \max\{\tilde{z} : \tilde{z} \in \partial\psi(x, u(x))\}.$$

Combining the above relations we conclude that any critical point u of $\mathcal{E}_\lambda$ satisfies

$$\begin{aligned} 0 \quad \leq \quad & \int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x) \cdot \nabla v(x) + |u(x)|^{p(x)-2} u(x) v(x)\, dx + \int_\Omega \xi(x)(-v(x))\, dx \\ & + \int_{\partial\Omega} \zeta(x)(-\lambda v(x))\, d\sigma. \end{aligned}$$

Replacing v with $-v$ in the above relation we get

$$\begin{aligned} 0 \quad = \quad & \int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x) \cdot \nabla v(x) + |u(x)|^{p(x)-2} u(x) v(x)\, dx + \int_\Omega \xi(x)(-v(x))\, dx \\ & + \int_{\partial\Omega} \zeta(x)(-\lambda v(x))\, d\sigma, \end{aligned}$$

which shows that u is a solution of (P_λ).

□

Example. Let us provide next an example of two functions $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\psi : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ which satisfy the conditions required in Theorem 3.1. Let Ω be an open bounded subset of $\mathbb{R}^N$ with smooth boundary and assume meas $(\Omega) \geq 1$. Let $p, q \in C_+(\overline{\Omega})$ be such that $p^- > N$ and $q^+ < p^-$ and $r \in C(\partial\Omega)$ such that $1 < r(x) < r^+ < p^-$. We consider $\mu > \max\left\{1; c_\infty \left[2p^+ \text{meas}(\Omega)\right]^{1/p^-}\right\}$, $0 < \delta < \min\left\{\frac{1}{3}, \left(\frac{N}{2p^-}\right)^{1/(p^- - q^+)}\right\}$. We consider now $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\psi : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ to be two nonsmooth locally Lipschitz functionals defined by

$$\phi(x, t) = \begin{cases} 0, & t \leq \delta \\ (t - \delta)^{q(x)}, & \delta \leq t < \delta + \pi \\ \sin(t - \delta), & \delta + 1 \leq t, \end{cases}$$

and

$$\psi(x, t) = \begin{cases} |t + \mu|^{r(x)}, & t \leq -\mu \\ 0, & -\mu < t \leq \delta \\ (t - \delta)(3\delta - t), & \delta \leq t < 3\delta \\ 0, & 3\delta \leq t, \end{cases}$$

and prove that hypotheses (H^1)-(H^5) are satisfied.

First we observe that

$$\partial\phi(x,t) = \begin{cases} 0, & t \leq \delta \\ q(x)(t-\delta)^{q(x)-1}, & \delta < t < \delta + \pi \\ \left\{\nu q(x)\pi^{q(x)-1} - (1-\nu) : \nu \in [0,1]\right\}, & t = \delta + \pi \\ \cos(t-\delta), & t > \delta + \pi \end{cases}$$

and

$$\partial\psi(x,t) = \begin{cases} -r(x)(-t-\mu)^{r(x)-1}, & t < -\mu \\ 0, & -\mu \leq t < \delta \\ [0, 2\delta], & t = \delta \\ -2t + 4\delta, & \delta < t < 3\delta \\ [-2\delta, 0], & t = 3\delta \\ 0, & t > 3\delta. \end{cases}$$

Thus, for any $\xi(x) \in \partial\phi(x,t)$ and any $\zeta(x) \in \psi(x,t)$, we have

$$|\xi(x)| \leq \begin{cases} 0 < |t|^{q(x)-1}, & t \leq \delta \\ q^+|t-\delta|^{q(x)-1} < q^+ < q^+\left(\frac{|t|}{\delta}\right)^{q(x)-1} < \frac{q^+}{\delta^{q^+-1}}|t|^{q(x)-1}, & \delta < t < \delta + 1 \\ q^+|t-\delta|^{q(x)-1} < q^+|t|^{q(x)-1}, & \delta + 1 < t < \delta + \pi \\ 1 < |t|^{q(x)-1}, & t \geq \delta + \pi \end{cases}$$

and

$$|\zeta(x)| \leq \begin{cases} r^+|t+\mu|^{r(x)-1} < r^+|t|^{r(x)-1}, & t < -1-\mu \\ r^+|t+\mu|^{r(x)-1} < r^+ < r^+\left(\frac{|t|}{\mu}\right)^{r(x)-1} < \frac{r^+}{\mu^{r^--1}}|t|^{r(x)-1}, & -1-\mu < t < -\mu \\ 0 < |t|^{r(x)-1}, & -\mu \leq t < \delta \\ 2\delta \leq 2\delta\left(\frac{|t|}{\delta}\right)^{r(x)-1} < \frac{2\delta}{\delta^{r^--1}}|t|^{r(x)-1}, & \delta \leq t \leq 3\delta \\ 0 < |t|^{r(x)-1}, & t > 3\delta. \end{cases}$$

It is clear from above that (H^1) and (H^2) hold. In order to see that (H^3)-(H^5) are satisfied we point out that the functional ϕ is bounded and choose $\eta = 2\delta < 1$. We have

$$\eta^{p(x)} = (2\delta)^{p(x)} \leq (2\delta)^{p^-} \leq N\delta^{q^+} \leq N\delta^{q(x)} \leq p(x)\delta^{q(x)} = p(x)\phi(x,\eta).$$

On the other hand we observe that $\psi(x,t)$ attains its maximum at $t = 2\delta$ on $[\delta, 3\delta]$, $\psi(x, 2\delta) = \delta^2 > 0$ and $\psi(x,t) = 0$ on $[-\mu, \delta] \cup [3\delta, \mu]$, which shows that $\sup_{|t|\leq\mu}\psi(x,t) = \psi(x,\eta)$, while $\sup_{t\in\mathbb{R}}\psi(x,t) = \infty$.

As a final remark we point out the fact that the nonsmooth Ricceri-type multiplicity results proved in Section 2 can be successfully applied to other problems. An example can be the following ordinary differential inclusion with periodic boundary conditions

$$(\mathbf{ODI}): \begin{cases} -u'' + u \in \lambda\alpha(t)\partial F(u) + \beta(t)\partial G(u) & \text{in } [0,1] \\ u(0) = u(1) \\ u'(0) = u'(1) \end{cases}$$

where $\lambda > 0$ is a real parameter, $F, G : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz and $\alpha, \beta : [0,1] \to [0,\infty)$ are nonconstant functions. It can be proved that, under suitable assumptions, for each $\lambda > 0$ problem (**ODI**) has at least two nonzero solutions and there exists $\lambda^* > 0$ for which problem (**ODI**) has at least three nonzero solutions. Due to the similarity with the problem studied by F. Faraci and A. Iannizzotto [12] we shall not provide a proof and the interested reader can consult the aforementioned paper.

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