Weak solvability of quasilinear elliptic inclusions with mixed boundary conditions

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Abstract

The main goal of this paper is to establish an existence result for differential inclusions governed by a quasilinear elliptic operator and two multivalued functions given by Clarke's generalized gradient of some locally Lipschitz functionals. We divide the boundary $\partial\Omega$ of our domain into two open measurable parts Γ_1 and Γ_2 and we impose a nonhomogeneous Neumann boundary condition on Γ_1, while on Γ_2 we impose a Dirichlet boundary condition. This kind of problems have been treated in the past by various authors. However, in all the work we are aware of, either a Neumann, or a Dirichlet boundary condition was imposed on the entire boundary. Another main point of interest of this paper is that our problem depends on a real parameter $\lambda > 0$ and we are able to prove the existence of solutions for any $\lambda \in (0, +\infty)$.

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1 Introduction and preliminary results

Inequality problems and differential inclusions have within a very short period of time undergone a remarkable development both in pure and applied mathematics as well as in other sciences such as mechanics, engineering and economics. This development has facilitated the solution of many open questions in mechanics and engineering sciences and also allowed the mathematical study of new classes of interesting problems.

Hemivariational inequalities were introduced by P.D. Panagiotopoulos at the beginning of 1980's as variational formulation for several classes of mechanical problems with nonsmooth and nonconvex energy superpotentials (see [23, 24]). In the case of convex superpotentials, hemivariational inequalities

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reduce to variational inequalities which were studied earlier by many authors (see e.g. the work of G. Fichera [12] or P. Hartman and G. Stampacchia [13]). We point out the fact that unlike variational inequalities, the hemivariational inequalities are not equivalent to minimum problems but, give rise to substationarity problems. For a comprehensive treatment of the theory of hemivariational inequalities and differential inclusions involving Clarke's generalized gradient the reader can consult the monographs [20, 21, 22] and more recently, [16]. For more information and connections regarding hemivariational inequalities and differential inclusions see the recent papers [2, 3, 8, 9, 10, 11, 19] and the references therein.

Hemivariational inequalities and differential inclusions are closely related, in the sense that almost all problems formulated in terms of hemivariational inequalities can be reformulated equivalently as multivalued differential inclusions involving the concept of Clarke's generalized gradient of a locally Lipschitz function.

For the convenience of the reader we present next some notations and preliminary results from nonsmooth analysis that will be used throughout the paper. For a given Banach space $(X, \|\cdot\|_X)$ we denote by X^* its dual space and by $\langle\cdot,\cdot\rangle$ the duality pairing between X^* and X. The inner product and the euclidian norm in $\mathbb{R}^N$ $(N \geq 1)$ will be denoted by " $\cdot$ " and $|\cdot|$, respectively.

We recall that a functional $h : X \to \mathbb{R}$ is called *locally Lipschitz* if for every $u \in X$ there exists a neighborhood U of u and a constant $L = L(U) > 0$ such that

$$|h(w) - h(v)| \leq L\|w - v\|_X, \qquad \text{for all } v, w \in U.$$

Definition 1.1. *Let $h : X \to \mathbb{R}$ be a locally Lipschitz functional. The generalized derivative of h at $u \in X$ in the direction $v \in X$, denoted $h^0(u; v)$, is defined by*

$$h^0(u; v) = \limsup_{\substack{w \to u \\ t \downarrow 0}} \frac{h(w + tv) - h(w)}{\lambda}.$$

Lemma 1.1. *Let $h : X \to \mathbb{R}$ be locally Lipschitz of rank L near the point $u \in X$. Then*

(a) The function $v \mapsto h^0(u; v)$ is finite, positively homogeneous, subadditive and satisfies

$$|h^0(u; v)| \leq L\|v\|_X;$$

(b) $h^0(u; v)$ is upper semicontinuous as a function of (u, v).

The proof can be found in Clarke [7], Proposition 2.1.1.

Definition 1.2. *The generalized gradient of a locally Lipschitz functional $h : X \to \mathbb{R}$ at a point $u \in X$, denoted $\partial h(u)$, is the subset of X^* defined by*

$$\partial h(u) = \{\zeta \in X^* : \ h^0(u; v) \geq \langle \zeta, v\rangle, \ \ \textit{for all } v \in X\}.$$

We point out the fact that for each $u \in X$ we have $\partial h(u) \neq \emptyset$. In order to see that it suffices to apply the Hahn-Banach theorem (see e.g. Brezis [1], p. 1).

The following theorem for set valued mappings is due to Lin (see [17], Theorem 1) and will be one of the key arguments in the sequel.

Theorem 1.1. *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Let $\mathcal{P} \subseteq K \times K$ be a subset such that*

(i) for each $\eta \in K$ the set $\Lambda(\eta) = \{\zeta \in K : \ (\eta, \zeta) \in \mathcal{P}\}$ is closed in K;

(ii) for each $\zeta \in K$ the set $\Theta(\zeta) = \{\eta \in K : \ (\eta, \zeta) \notin \mathcal{P}\}$ is either convex or empty;

(iii) $(\eta, \eta) \in \mathcal{P}$ for each $\eta \in K$;

(iv) K has a nonempty compact convex subset K_0 such that the set

$$B = \{\zeta \in K : \ (\eta, \zeta) \in \mathcal{P} \text{ for all } \eta \in K_0\}$$

is compact.

Then there exists a point $\zeta_0 \in B$ such that $K \times \{\zeta_0\} \subset \mathcal{P}$.

2 Formulation of the problem

Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \geq 3$) and $1 < p < +\infty$. We denote $\partial\Omega = \Gamma$ the boundary of Ω and we assume that $\Gamma_1, \ \Gamma_2$ are two open measurable parts that form a partition of Γ (i.e. $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$) such that $\text{meas}(\Gamma_2) > 0$.

We are interested in studying nonsmooth quasilinear elliptic inclusions with mixed boundary conditions of the following type:

$$(P_\lambda): \quad \begin{cases} \operatorname{div}(a(x, \nabla u)) \in \lambda \partial j_1(x, u) - g(x), & \text{in } \Omega \\ -a(x, \nabla u) \cdot n \in \mu(x, u) \partial j_2(x, u), & \text{on } \Gamma_1, \\ u = 0, & \text{on } \Gamma_2, \end{cases}$$

where $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is of the form $a(x, \xi) = (a_1(x, \xi), \ldots, a_N(x, \xi))$, with $a_i : \Omega \times \mathbb{R}^N \to \mathbb{R}$ for $i \in \{1, \ldots, N\}$, $\lambda > 0$ is a real parameter, $\partial j_i(x, u)$ stands for the Clarke generalized gradient of the mapping $u \mapsto j_i(x, u)$ $(i = 1, 2)$ and n is the unit outward normal to $\partial\Omega$.

Examples.

1. Set $a(x, \xi) = |\xi|^{p-2}\xi$. Then $a(x, \xi)$ is the continuous derivative with respect to the second variable of the mapping $\mathcal{A}(x, \xi) = \frac{1}{p}|\xi|^p$, i.e. $a(x, \xi) = \nabla_\xi \mathcal{A}(x, \xi)$. Then we get the $p-$Laplace operator

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

2. Set $a(x,\xi) = (1+|\xi|^2)^{(p-2)/2}\xi$. Then $a(x,\xi)$ is the continuous derivative with respect to the second variable of the mapping $\mathcal{A}(x,\xi) = \frac{1}{p}\left[(1+|\xi|^2)^{p/2} - 1\right]$, i.e. $a(x,\xi) = \nabla_\xi \mathcal{A}(x,\xi)$. Then we get the the mean curvature operator
$$\operatorname{div}\left((1+|\nabla u|^2)^{(p-2)/2}\nabla u\right).$$

We point out the fact that our operator is not necessarily a potential operator, but we have chosen these examples due to the fact that boundary value problems involving the above mentioned operators were studied intensively in the last decades since quasilinear operators can model a variety of physical phenomena (e.g. the p-Laplacian it is used in non-Newtonian fluids, reaction-diffusion problems as well as in flow through porous media).

Let us introduce the functional space

$$V = \{v \in W^{1,p}(\Omega) : \ \gamma v = 0 \text{ on } \Gamma_2\}$$

where $\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ is the Sobolev trace operator. For simplicity, everywhere below, we will omit to write γv to indicate the Sobolev trace on the boundary, writing v instead of γv. Since $\operatorname{meas}(\Gamma_2) > 0$, it is well known that V is a closed subspace of $W^{1,p}(\Omega)$ and can be endowed with the norm
$$\|v\|_V = \|\nabla v\|_{L^p(\Omega)},$$
which is equivalent to the usual norm on $W^{1,p}(\Omega)$ due to the Poincaré-Friedrichs inequality (see e.g. Proposition 2.94 in [4]).

Definition 2.1. *We say that $u \in V$ is a weak solution of the problem (P_λ) if there exist $\zeta_1 \in L^{p'}(\Omega)$ satisfying $\zeta_1(x) \in \partial j_1(x, u(x))$ and $\zeta_2 \in L^{p'}(\partial\Omega)$ satisfying $\zeta_2(x) \in \partial j_2(x, u(x))$ for almost every $x \in \overline{\Omega}$ such that*

$$\int_\Omega \operatorname{div} a(x,\nabla u)\cdot(v-u)\,dx = \lambda\int_\Omega \zeta_1(v-u)\,dx - \int_\Omega g(x)(v-u)\,dx$$

and

$$-\int_{\Gamma_1} a(x,\nabla u)\cdot n(v-u)\,d\sigma = \int_{\Gamma_1} \zeta_2\mu(x,u)(v-u)\,d\sigma,$$

for all $v \in V$.

Let us turn now our attention towards the terms given by Clarke's generalized gradient. At our best knowledge differential inclusions of the type (P_λ) were only studied in the past either with Neumann condition, or with Dirichlet condition on the entire boundary. This cases can be obtained when $\Gamma_1 = \Gamma$, or $\Gamma_2 = \Gamma$. We present next several particular cases of our problem that have been treated in the last years by various authors.

1. $\boldsymbol{\Gamma_1 = \Gamma}$ (Neumann problem).

- If j_1 and j_2 are primitives of some Carathéodory functions $f:\Omega\times\mathbb{R}\to\mathbb{R}$ and $h:\partial\Omega\times\mathbb{R}\to\mathbb{R}$

$$j_1(x,t)=\int_0^t f(x,s)\,ds \quad \text{and} \quad j_2(x,t)=\int_0^t h(x,s)\,ds$$

then the functions $t\mapsto j_1(x,t)$ and $t\mapsto j_2(x,t)$ are differentiable. Thus $\partial j_1(x,t)=\{f(x,t)\}$, $\partial j_2(x,t)=\{g(x,t)\}$ and (P_λ) reduces to the following eigenvalue problem

$$\begin{cases} \operatorname{div}(a(x,\nabla u))=\lambda f(x,u)-g(x) & \text{in } \Omega \\ -a(x,\nabla u)\cdot n=\mu(x,u)h(x,u) & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

A particular case of problem (2.1) was studied by Y.X. Huang [14] (there the author studies the case when $a(x,\xi)=|\xi|^{p-2}\xi$, $f(x,u)=m(x)|u|^{p-2}u$, $g\equiv 0$ and $h\equiv 0$).

- In the case when the functionals f and h from the previous example are only locally bounded, i.e. $f\in L^\infty_{\text{loc}}(\Omega\times\mathbb{R})$ and $h\in L^\infty_{\text{loc}}(\partial\Omega\times\mathbb{R})$ then $t\mapsto j_1(x,t)$ and $t\mapsto j_2(x,t)$ are locally Lipschitz functionals and, according to Proposition 1.7 in [20] we have

$$\partial j_1(x,t)=\left[\underline{f}(x,t),\overline{f}(x,t)\right] \quad \text{and} \quad \partial j_2(x,t)=\left[\underline{h}(x,t),\overline{h}(x,t)\right],$$

where

$$\underline{f}(x,t)=\lim_{\delta\downarrow 0}\ \text{ess}\inf_{|s-t|<\delta} f(x,s) \qquad \overline{f}(x,t)=\lim_{\delta\downarrow 0}\ \text{ess}\sup_{|s-t|<\delta} f(x,s)$$

and

$$\underline{h}(x,t)=\lim_{\delta\downarrow 0}\ \text{ess}\inf_{|s-t|<\delta} h(x,s) \qquad \overline{h}(x,t)=\lim_{\delta\downarrow 0}\ \text{ess}\sup_{|s-t|<\delta} h(x,s).$$

In this case problem (P_λ) reduces to

$$\begin{cases} \operatorname{div}(a(x,\nabla u))\in\lambda\left[\underline{f}(x,u),\overline{f}(x,u)\right]-g(x) & \text{in } \Omega \\ -a(x,\nabla u)\cdot n\in\mu(x,u)\left[\underline{h}(x,u),\overline{h}(x,u)\right] & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

A particular case of problem (2.2) was studied by F. Papalini [25] in the case of the p-Laplacian. The approach is variational and it is based on the nonsmooth critical point theory for locally Lipschitz functionals developed by K.-C. Chang in [6].

- In the case when $g\equiv 0$ and $\mu(x,t)\equiv\mu>0$ problem (P_λ) becomes

$$\begin{cases} \operatorname{div}(a(x,\nabla u))\in\lambda\partial j_1(x,u), & \text{in } \Omega \\ -a(x,\nabla u)\cdot n\in\mu\partial j_2(x,u), & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

A problem similar to problem (2.3) was studied A. Kristály, W. Marzantowicz and Cs. Varga in [15] where the authors use a nonsmooth three critical points theorem to prove that there exists a compact interval $[a,b]$ such that for every $\lambda\in[a,b]$ there exists $\mu_0\in(0,\lambda+1)$ such the for each $\mu\in[0,\mu_0]$, the studied problem admits at least three distinct solutions.

2. $\mathbf{\Gamma_2} = \mathbf{\Gamma}$ (Dirichlet problem)

In this case our problem can be rewritten equivalently as follows:

$$u \in W_0^{1,p}(\Omega) : \quad \mathcal{A}u + \lambda \partial j_1(\cdot, u) \ni g \quad \text{in } W^{-1,p'}(\Omega), \tag{2.4}$$

where $\mathcal{A}u(x) = -\text{div } a(x, \nabla u(x))$.

Problem (2.4) was treated in the case $\lambda = 1$ and $g \equiv 0$ by S. Carl and D. Motreanu [5] who used the method of sub and supersolutions to obtain general comparison results. We also remember the work of Z. Liu and G. Liu [18] and ji-an Wang [26] who studied eigenvalue problems for elliptic hemivariational inequalities that can be rewritten equivalently as differential inclusions similar to (2.4). In [18] and [26] the authors used the surjectivity of multivalued pseudomonotone operators to prove the existence of solutions.

As we have seen above, in most papers dealing with differential inclusions of the type (P_λ) nonsmooth critical point theory, or the pseudomonotonicity of a certain multivalued operator played an essential role in obtaining the existence of solutions. However, in all the works we are aware of, additional assumptions on the structure of the elliptic operator and/or the generalized Clarke's gradient are needed to obtain the existence of the solution (e.g. the elliptic operator is of potential type, or the locally Lipschitz functional is required to be regular, or to satisfy some conditions of Landesman-Lazer type, or the Clarke's generalized gradient is supposed to satisfy more restrictive growth conditions). In this paper our approach is topological and the novelty is that we are able to obtain the existence of at least one weak solution for any $\lambda \in (0, +\infty)$ without assuming any of the above restrictions.

3 Hypotheses and the main result

In this section we present the conditions that need to be imposed in order to prove the existence of weak solutions and we prove our main result.

Here and hereafter, we shall assume fulfilled the following conditions:
$(\mathcal{H}_1)$ $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is an operator of the form $a(x, \xi) = (a_1(x, \xi), \dots, a_N(x, \xi))$ which satisfies

(i) for each $i \in \{1, \dots, N\}$ $a_i : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function and there exists $c_0 > 0$ and $\alpha \in L^{p'}(\Omega)$ such that

$$|a_i(x, \xi)| \leq \alpha(x) + c_0 |\xi|^{p-1},$$

for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^N$;

(ii) there exist $c_1 > 0$ and $\beta \in L^1(\Omega)$ such that

$$a(x, \xi) \cdot \xi \geq c_1 |\xi|^p - \beta(x)$$

for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^N$;

(iii) for almost every $x \in \Omega$ and all $\xi_1, \xi_2 \in \mathbb{R}^N$

$$[a(x,\xi_1) - a(x,\xi_2)] \cdot (\xi_1 - \xi_2) \geq 0.$$

$(\mathcal{H}_2)$ $j_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function which satisfies:

(i) for all $t \in \mathbb{R}$ the function $x \mapsto j_1(x,t)$ is measurable;

(ii) for almost every $x \in \Omega$ the function $t \mapsto j_1(x,t)$ is locally Lipschitz;

(iii) there exists $c_2 > 0$ such that for almost every $x \in \Omega$ and all $t \in \mathbb{R}$

$$|\partial j_1(x,t)| \leq c_2(1 + |t|^{p-1});$$

(iv) there exists $\gamma_1 \in L^p(\Omega)$ such that for almost every $x \in \Omega$ and all $t \in \mathbb{R}$

$$|j_1^0(x,t;-t)| \leq \gamma_1(x)|t|^{p-1}.$$

$(\mathcal{H}_3)$ $j_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to the first variable and there exists $\gamma_2 \in L^{p'}(\partial\Omega)$ such that

$$|j_2(x,t_1) - j_2(x,t_2)| \leq \gamma_2(x)|t_1 - t_2|$$

for almost every $x \in \partial\Omega$ and all $t_1, t_2 \in \mathbb{R}$.

$(\mathcal{H}_4)$ $\mu : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exists $\mu^* > 0$ such that

$$0 \leq \mu(x,t) \leq \mu^*$$

for almost every $x \in \partial\Omega$ and all $t \in \mathbb{R}$.

$(\mathcal{H}_5)$ $g \in L^{p'}(\Omega)$.

Let us introduce now the operator $A : V \to V^*$ defined by

$$\langle Au, v\rangle = \int_\Omega a(x, \nabla u) \cdot \nabla v \, dx \tag{3.1}$$

and denote by ϕ the element of V^* given by the Riesz-Fréchet representation theorem as follows,

$$\langle \phi, v\rangle = \int_\Omega g(x) v \, dx.$$

Taking into account the above notations and the definition of the Clarke generalized gradient the weak solvability of problem (P_λ) reduces to finding solutions for the following *hemivariational inequality* $(HI)_\lambda$ Find $u \in V$ such that

$$\langle Au, v-u\rangle + \lambda \int_\Omega j_1^0(x,u;v-u)\, dx + \int_{\Gamma_1} \mu(x,u) j_2^0(x,u;v-u)\, d\sigma \geq \langle \phi, v-u\rangle$$

for all $v \in V$.

We point out the fact that we do not deal with a classical hemivariational inequality due to the presence of the term $\int_{\Gamma_1} \mu(x,u) j_2^0(x,u;v-u)\, d\sigma$ in the left-hand side of the inequality and consequently several difficulties occur in determining the existence of solutions since the classical methods fail to be applied directly. Our main result is given by the following theorem.

Theorem 3.1. *Suppose that conditions $(\mathcal{H}_1) - (\mathcal{H}_5)$ are fulfilled. Then for each $\lambda \in (0, +\infty)$ problem (P_λ) admits at least one weak solution.*

Proof. The proof of Theorem 3.1 will be carried out in several steps and relies essentially on topological arguments.

First we point out the fact that under $(\mathcal{H}_1)$ the operator $A : V \to V^*$ defined in (3.1) is well defined and satisfies the following properties:

- there exists $c_3 > 0$ such that $\langle Au, u\rangle \geq c_1\|u\|_V^p - c_3$, for all $u \in V$;
- $\langle Av - Au, v - u\rangle \geq 0$ for all $u, v \in V$.
- $\langle Au_n, v\rangle \to \langle Au, v\rangle$ for all $v \in V$, whenever $u_n \to u$ in V.

Let us fix $\lambda > 0$. We shall prove next that there exists at least one $u \in V$ which solves $(HI)_\lambda$. In order to do this let us fix $R > 0$ and define $K = \bar{B}_V(0, R)$ and

$$\mathcal{P} = \left\{(v, u) \in K \times K : \ \langle Au, v - u\rangle + \lambda \int_\Omega j_1^0(x, u; v - u)\, dx + \int_{\Gamma_1} \mu(x, u) j_2^0(x, u; v - u)\, d\sigma \geq \langle \phi, v - u\rangle\right\}$$

Step 1. For each $v \in K$ the set $\Lambda(v) = \{u \in K : \ (v, u) \in \mathcal{P}\}$ is weakly closed.

Let $v \in K$ be fixed and $\{u_n\}_n \subset \Lambda(v)$ be a sequence which converges weakly to some $u \in V$. Using the fact that the embeddings $V \subset L^p(\Omega)$ and $V \subset L^p(\Gamma_1)$ are compact we conclude that

$$u_n \to u, \qquad \text{in } L^p(\Omega)$$

and

$$u_n \to u, \qquad \text{in } L^p(\Gamma_1).$$

Combining the fact that $u_n \in \Lambda(v)$ for all $n \in \mathbb{N}$ with the monotonicity of A we deduce that

$$\langle Aw, w - u_n\rangle + \lambda \int_\Omega j_1^0(x, u_n; w - u_n)\, dx + \int_{\Gamma_1} \mu(x, u_n) j_2^0(x, u_n; w - u_n)\, d\sigma \geq \langle \phi, w - u_n\rangle$$

for all $w \in V$ and passing to $\limsup$ as $n \to \infty$ in the above relation and using Lemma 1.1 we get the following estimates

$$\begin{aligned}
\langle \phi, w - u\rangle &= \limsup_{n\to\infty} \langle \phi, w - u_n\rangle \\
&\leq \limsup_{n\to\infty} \left[\langle Aw, w - u_n\rangle + \lambda \int_\Omega j_1^0(x, u_n; w - u_n)\, dx + \int_{\Gamma_1} \mu(x, u_n) j_2^0(x, u_n; w - u_n)\, d\sigma\right] \\
&\leq \langle Aw, w - u\rangle + \lambda \int_\Omega \limsup_{n\to\infty} j_1^0(x, u_n; w - u_n)\, dx + \int_{\Gamma_1} \mu(x, u) \limsup_{n\to\infty} j_2^0(x, u_n; w - u_n)\, d\sigma \\
&\quad + \int_{\Gamma_1} \limsup_{n\to\infty} |\mu(x, u_n) - \mu(x, u)||j_2^0(x, u_n; w - u_n)|\, d\sigma \\
&\leq \langle Aw, w - u\rangle + \lambda \int_\Omega j_1^0(x, u; w - u)\, dx + \int_{\Gamma_1} \mu(x, u) j_2^0(x, u; w - u)\, d\sigma
\end{aligned}$$

Taking $w = u + t(v-u)$ for $t \in (0,1)$ and keeping in mind Lemma 1.1 we obtain

$$t\left[\langle Aw, v-u\rangle + \lambda \int_\Omega j_1^0(x,u;v-u)\,dx + \int_{\Gamma_1} \mu(x,u) j_2^0(x,u;v-u)\,d\sigma\right] \geq t\langle \phi, v-u\rangle.$$

Finally we divide by $t>0$, then let $t\to 0$ to obtain that $u\in\Lambda(v)$, which shows that $\Lambda(v)$ is a weakly closed subset of K.

STEP 2. For each $u\in K$ the set $\Theta(u) = \{v \in K :\ (v,u)\notin \mathcal{P}\}$ is either empty or convex.

Let us fix $u\in K$ and assume that $\Theta(u)$ is nonempty. For $v_1, v_2 \in \Theta(u)$ and $t\in(0,1)$ let us define $w = v_1 + t(v_2 - v_1)$. A simple computation leads to

$$\begin{aligned}
&\langle Au, w-u\rangle + \lambda \int_\Omega j_1^0(x,u;w-u)\,dx + \int_{\Gamma_1} \mu(x,u) j_2^0(x,u;w-u)\,d\sigma\\
&= (1-t)\left[\langle Au, v_1-u\rangle + \lambda \int_\Omega j_1^0(x,u;v_1-u)\,dx + \int_{\Gamma_1} \mu(x,u) j_2^0(x,u;v_1-u)\,d\sigma\right]\\
&\quad +t\left[\langle Au, v_2-u\rangle + \lambda \int_\Omega j_1^0(x,u;v_2-u)\,dx + \int_{\Gamma_1} \mu(x,u) j_2^0(x,u;v_2-u)\,d\sigma\right]\\
&< (1-t)\langle \phi, v_1-u\rangle + t\langle\phi, v_2-u\rangle = \langle \phi, w-u\rangle,
\end{aligned}$$

which shows that $w\in\Lambda(u)$, therefore $\Theta(u)$ is a convex subset of K.

STEP 3. The set $B = \{u\in K :\ (v,u)\in\mathcal{P} \text{ for all } v\in K\}$ is weakly compact.

We observe that $B = \bigcap_{v\in K} \Lambda(v)$ is weakly closed as it is an intersection of weakly closed sets. On the other hand, B is a subset of K which is weakly compact therefore B is weakly compact.

STEP 4. For each positive integer n the restriction of $(HI)_\lambda$ to $\bar{B}_V(0,n)$ admits at least one solution.

From the above steps, for each positive integer n we can apply Theorem 1.1 with $K_0 = K = \bar{B}_V(0,n)$ (obviously $(u,u)\in\mathcal{P}$ for all $u\in\bar{B}_V(0,n)$) and obtain the existence of an element $u_n \in \bar{B}_V(0,n)$ such that $\bar{B}_V(0,n)\times\{u_n\}\subseteq\mathcal{P}$, which can be rewritten equivalently as

$$\langle Au_n, v-u_n\rangle + \lambda \int_\Omega j_1^0(x,u_n, v-u_n)\,dx + \int_{\Gamma_1} j_2^0(x,u_n;v-u_n)\,d\sigma \geq \langle\phi, v-u_n\rangle, \tag{3.2}$$

for all $v\in\bar{B}_V(0,n)$.

STEP 5. There exists $n^*>0$ such that $u_{n^*}\in \operatorname{int}\bar{B}_V(0,n^*)$.

Arguing by contradiction let us assume that $\|u_n\|_V = n$ for all $n>0$. Taking $v=0$ in (3.2) we obtain

$$\begin{aligned}
\langle Au_n, u_n\rangle &\leq \langle\phi, u_n\rangle + \lambda\int_\Omega j_1^0(x,u_n;-u_n)\,dx + \int_{\Gamma_1}\mu(x,u_n) j_2^0(x,u_n;-u_n)\,d\sigma\\
&\leq \|\phi\|_{V^*}\|u_n\|_V + \lambda\int_\Omega \gamma_1(x)|u_n|^{p-1}\,dx + \mu^*\int_{\Gamma_1}\gamma_2(x)|u_n|\,d\sigma\\
&\leq \|\phi\|_{V^*}\|u_n\|_V + \lambda\|\gamma_1\|_{L^p(\Omega)}\|u_n\|^{p-1}_{L^p(\Omega)} + \mu^*\|\gamma_2\|_{L^{p'}(\Gamma_1)}\|u_n\|_{L^p(\Gamma_1)}\\
&\leq \tilde{c}_1\|u_n\|_V + \tilde{c}_2\|u_n\|_V^{p-1},
\end{aligned}$$

for a suitable constants $\tilde{c}_1, \tilde{c}_2 > 0$. On the other hand, we know that

$$\langle Au_n, u_n\rangle \geq c_1\|u_n\|_V^p - c_3.$$

Combining the above estimates and keeping in mind that $1 < p$ and $\|u_n\|_V = n$ for all $n > 0$ we arrive at

$$c_1 n^p - c_3 \leq \tilde{c}_1 n^{p-1} + \tilde{c}_2 n.$$

Dividing by n^{p-1} and letting $n \to \infty$ we get a contradiction as the left-hand term of the inequality diverges while the right-hand term remains bounded.

Step 6. u_{n^*} solves $(HI)_\lambda$.

Let $v \in V$ be fixed. From Step 5. we know that $u_{n^*} < n^*$ which allows us to choose $t \in (0,1)$ such that $w = u_{n^*} + t(v - u_{n^*}) \in \bar{B}_V(0, n^*)$. Plugging w in (3.2) we have

$$\begin{aligned} t\langle \phi, v - u_{n^*}\rangle &= \langle \phi, w - u_{n^*}\rangle \\ &\leq \langle Au_{n^*}, w - u_{n^*}\rangle + \lambda\int_\Omega j_1^0(x, u_{n^*}; w - u_{n^*})\,dx + \int_{\Gamma_1}\mu(x, u_n^*) j_2^0(x, u_{n^*}; w - u_{n^*})\,d\sigma \\ &= t\left[\langle Au_{n^*}, v - u_{n^*}\rangle + \lambda\int_\Omega j_1^0(x, u_{n^*}; v - u_{n^*})\,dx + \int_{\Gamma_1}\mu(x, u_n^*) j_2^0(x, u_{n^*}; v - u_{n^*})\,d\sigma\right]. \end{aligned}$$

Dividing the above relation by $t > 0$ we conclude that u_{n^*} is indeed a solution for $(HI)_\lambda$.

□

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