Systems of nonlinear hemivariational inequalities and applications

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Abstract

In this paper we prove several existence results for a general class of systems of nonlinear hemivariational inequalities by using a fixed point theorem of Lin [21]. Our analysis includes both the cases of bounded and unbounded closed convex subsets in real reflexive Banach spaces. In the last section we apply the abstract results obtained to extend some results concerning nonlinear hemivariational inequalities, to establish existence results of Nash generalized derivative points and to prove the existence of at least one weak solution for an electroelastic contact problem.

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1 Introduction

In the last decades the theory of hemivariational inequalities captured special attention as many papers were dedicated to the study of existence and multiplicity of solutions for this kind of inequalities (see e.g. [2, 3, 5, 6, 7, 9, 10, 12, 17, 23]). The notion of *hemivariational inequality* was introduced by P.D. Panagiotopoulos at the beginning of the 1980's (see e.g. [29, 30]) as a variational formulation for several classes of unilateral mechanical problems with nonsmooth and nonconvex energy functionals. If the involved functionals are convex, then hemivariational inequalities reduce to *variational inequalities* which were studied earlier by many authors (see e.g. Fichera [13] or Hartman and Stampacchia [15]). In almost three decades the theory of hemivariational inequalities has produced an abundance of important results both in pure and applied mathematics as well as in other domains such as mechanics and engineering sciences as it allowed mathematical formulations for new classes of interesting problems (see e.g. the monographs [14, 19, 24, 25, 26, 31]).

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The aim of this paper is to prove the existence of at least one solution for a general class of systems of nonlinear hemivariational inequalities on bounded or unbounded closed and convex subsets without using critical point theory. The proofs strongly rely on a fixed point theorem involving set-valued mappings due to Lin [21].

The rest of paper the paper is structured as follows. In Section 2 we introduce some notation and preliminaries. In Section 3 we formulate the problem that will be studied and the main results are proved. In Section 4 we present three applications of the abstract results obtained in the previous section.

2 Notation and preliminaries

For the convenience of the reader we present in this section some notations and preliminary results from nonsmooth analysis that will be used throughout the paper. For a given Banach space $(E, \|\cdot\|_E)$ we denote by E^* its dual space and by $\langle\cdot,\cdot\rangle_E$ the duality pairing between E^* and E. The inner product and the euclidian norm in $\mathbb{R}^m$ $(m \geq 1)$ will be denoted by " $\cdot$ " and $|\cdot|$, respectively.

We recall that a functional $\phi : E \to \mathbb{R}$ is called *locally Lipschitz* if for every $u \in E$ there exists a neighborhood U of u and a constant $L_u > 0$ such that

$$|\phi(w) - \phi(v)| \leq L_u\|w - v\|_E, \quad \text{for all } v, w \in U.$$

Definition 2.1. *Let $\phi : E \to \mathbb{R}$ be a locally Lipschitz functional. The generalized derivative of ϕ at $u \in E$ in the direction $v \in E$, denoted $\phi^0(u; v)$, is defined by*

$$\phi^0(u; v) = \limsup_{\substack{w \to u \\ \lambda \downarrow 0}} \frac{\phi(w + \lambda v) - \phi(w)}{\lambda}.$$

For a function $\varphi : E_1 \times \ldots \times E_k \times \ldots \times E_n \to \mathbb{R}$ which is locally Lipschitz in the k^{th} variable we denote by $\varphi^0_{,k}(u_1, \ldots, u_k, \ldots, u_n; v_k)$ *the partial generalized derivative* of $\varphi(u_1, \ldots, u_k, \ldots, u_n)$ at the point $u_k \in E_k$ in the direction $v_k \in E_k$, that is

$$\varphi^0_{,k}(u_1, \ldots, u_k, \ldots, u_n; v_k) = \limsup_{\substack{w_k \to u_k \\ \lambda \downarrow 0}} \frac{\varphi(u_1, \ldots, w_k + tv_k, \ldots, u_n) - \varphi(u_1, \ldots, w_k, \ldots, u_n)}{\lambda}.$$

Lemma 2.1. *Let $\phi : E \to \mathbb{R}$ be locally Lipschitz of rank L_u near the point $u \in E$. Then*

(a) the function $v \rightsquigarrow \phi^0(u; v)$ is finite, positively homogeneous, subadditive and satisfies

$$|\phi^0(u; v)| \leq L_u\|v\|_E;$$

(b) $\phi^0(u; v)$ is upper semicontinuous as a function of (u, v).

The proof can be found in Clarke [8], Proposition 2.1.1.

Definition 2.2. *The generalized gradient of a locally Lipschitz functional $\phi : E \to \mathbb{R}$ at a point $u \in E$, denoted $\partial\phi(u)$, is the subset of E^* defined by*

$$\partial\phi(u) = \{\zeta \in E^* : \ \phi^0(u; v) \geq \langle\zeta, v\rangle_E, \ \text{ for all } v \in E\}.$$

We point out the fact that for each $u \in E$ we have $\partial\phi(u) \neq \emptyset$. In order to see that it suffices to apply the Hahn-Banach theorem (see e.g. Brezis [4], Chapter I).

For a function $\varphi : E_1 \times \ldots \times E_k \times \ldots \times E_n \to \mathbb{R}$ which is locally Lipschitz in the k^{th} variable we denote by $\partial_k\varphi(u_1, \ldots, u_k, \ldots, u_n)$ the partial generalized gradient of the mapping $u_k \rightsquigarrow \varphi(u_1, \ldots, u_k, \ldots, u_n)$, that is

$$\partial_k\varphi(u_1, \ldots, u_k, \ldots, u_n) = \left\{\eta_k \in E_k^* : \ \varphi^0_{,k}(u_1, \ldots, u_k, \ldots, u_n; v_k) \geq \langle\eta_k, v_k\rangle_{E_k}, \ \text{ for all } v_k \in E_k\right\}.$$

The next lemma points out important properties of generalized gradients.

Lemma 2.2. *Let $\phi : E \to \mathbb{R}$ be locally Lipschitz of rank L_u near the point $u \in E$. Then*

(a) $\partial\phi(u)$ is a convex, weak compact subset of E^* and*

$$\|\zeta\|_{E^*} \leq L_u, \quad \text{for all } \zeta \in \partial\phi(u);$$

(b) For each $v \in E$, one has

$$\phi^0(u; v) = \max\{\langle\zeta, v\rangle_E : \ \zeta \in \partial\phi(u)\}.$$

The proof can be found in Clarke [8], Proposition 2.1.2.

Definition 2.3. *Let E be a Banach space and let $\phi : E \to \mathbb{R}$ be a locally Lipschitz functional. We say that ϕ is regular at $u \in E$, if for all $v \in E$ the usual one-sided directional derivative $\phi'(u; v)$ exists and $\phi'(u; v) = \phi^0(u; v)$.*

If this is true at every $u \in E$, we say that ϕ is regular.

It is a fact that in general neither of the sets $\partial\varphi(u_1, \ldots, u_n)$ and $\partial_1\varphi(u_1, \ldots, u_n) \times \ldots \times \partial_n\varphi(u_1, \ldots, u_n)$ need to be contained in the other (see e.g. Clarke [8] Section 2.5). For regular functions, however, a general relationship does hold between these sets.

Lemma 2.3. *Let $\varphi : E_1 \times \ldots \times E_n \to \mathbb{R}$ be a regular, locally Lipschitz functional. Then the following assertions hold true:*

(i) $\partial\varphi(u_1, \ldots, u_k, \ldots, u_n) \subseteq \partial_1\varphi(u_1, \ldots, u_k, \ldots, u_n) \times \ldots \times \partial_k\varphi(u_1, \ldots, u_k, \ldots, u_n) \times \ldots \times \partial_n\varphi(u_1, \ldots, u_k, \ldots, u_n)$;

(ii) $\varphi^0(u_1, \ldots, u_k, \ldots, u_n; v_1, \ldots, v_k, \ldots, v_n) \leq \sum_{k=1}^{n} \varphi^0_{,k}(u_1, \ldots, u_k, \ldots, u_n; v_k)$;

(iii) $\varphi^0(u_1, \ldots, u_k, \ldots, u_n; 0, \ldots, v_k, \ldots, 0) \leq \varphi^0_{,k}(u_1, \ldots, u_k, \ldots, u_n; v_k)$.

The following fixed point theorem for set valued mappings is due to Lin (see [21], Theorem 1) and will be one of the key arguments in the sequel.

Theorem 2.1. *Let K be a nonempty convex subset of a Hausdorff topological vector space E. Let $\mathcal{A} \subseteq K \times K$ be a subset such that*

- *for each $x \in K$ the set $\mathcal{N}(x) = \{y \in K : \ (x, y) \in \mathcal{A}\}$ is closed in K;*
- *for each $y \in K$ the set $\mathcal{M}(y) = \{x \in K : \ (x, y) \notin \mathcal{A}\}$ is either convex or empty;*
- *$(x, x) \in \mathcal{A}$ for each $x \in K$;*
- *K has a nonempty compact convex subset K_0 such that the set*

$$B = \{y \in K : \ (x, y) \in \mathcal{A} \text{ for all } x \in K_0\}$$

  is compact.

Then there exists a point $y_0 \in B$ such that $K \times \{y_0\} \subset \mathcal{A}$.

3 Formulation of the problem and the main results

Let n be a positive integer, let $X_1, \ldots, X_n$ be real reflexive Banach spaces and let $Y_1, \ldots, Y_n$ be real Banach spaces such that there exist linear and compact operators $T_k : X_k \to Y_k$, for $k \in \{1, \ldots, n\}$.

Our aim is to study the following *system of nonlinear hemivariational inequalities* :

(SNHI) Find $(u_1, \ldots, u_n) \in K_1 \times \ldots \times K_n$ such that for all $(v_1, \ldots, v_n) \in K_1 \times \ldots \times K_n$

$$\begin{cases} \psi_1(u_1, \ldots, u_n, v_1) + J^0_{,1}(\hat{u}_1, \ldots, \hat{u}_n; \hat{v}_1 - \hat{u}_1) \geq \langle F_1(u_1, \ldots, u_n), v_1 - u_1\rangle_{X_1} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \psi_n(u_1, \ldots, u_n, v_n) + J^0_{,n}(\hat{u}_1, \ldots, \hat{u}_n; \hat{v}_n - \hat{u}_n) \geq \langle F_n(u_1, \ldots, u_n), v_n - u_n\rangle_{X_n}, \end{cases}$$

where for each $k \in \{1, \ldots, n\}$

$K_k \subseteq X_k$ is a nonempty closed and convex subset;

$\psi_k : X_1 \times \ldots \times X_k \times \ldots \times X_n \times X_k \to \mathbb{R}$ is a nonlinear functional;

$J : Y_1 \times \ldots \times Y_n \to \mathbb{R}$ is a regular locally Lipschitz functional;

$F_k : X_1 \times \ldots \times X_k \times \ldots \times X_n \to X_k^*$ is a nonlinear operator;

$\hat{u}_k = T_k(u_k)$.

In order to establish the existence of at least one solution for problem **(SNHI)** we shall assume fulfilled the following hypotheses:

(H1) For each $k \in \{1, \ldots, n\}$, the functional $\psi_k : X_1 \times \ldots \times X_k \times \ldots \times X_n \times X_k \to \mathbb{R}$ satisfies

(i) $\psi_k(u_1, \ldots, u_k, \ldots, u_n, u_k) = 0$ for all $u_k \in X_k$;

(ii) For each $v_k \in X_k$ the mapping $(u_1, \ldots, u_n) \rightsquigarrow \psi_k(u_1, \ldots, u_n, v_k)$ is weakly upper semicontinuous;

(iii) For each $(u_1, \ldots, u_n) \in X_1 \times \ldots \times X_n$ the mapping $v_k \rightsquigarrow \psi_k(u_1, \ldots, u_n, v_k)$ is convex.

(H2) For each $k \in \{1, \ldots, n\}$, $F_k : X_1 \times \ldots \times X_k \times \ldots \times X_n \to X_k^*$ is a nonlinear operator such that

$$\liminf_{m \to \infty} \langle F_k \left(u_1^m, \ldots, u_n^m\right), v_k - u_k^m \rangle_{X_k} \geq \langle F_k \left(u_1, \ldots, u_n\right), v_k - u_k \rangle_{X_k}$$

whenever $(u_1^m, \ldots, u_n^m) \rightharpoonup (u_1, \ldots, u_n)$ as $m \to \infty$ and $v_k \in X_k$ is fixed.

The first main result of this paper refers to the case when the sets K_k are bounded, closed and convex and it is given by the following theorem.

Theorem 3.1. *For each $k \in \{1, \ldots, n\}$ let $K_k \subset X_k$ be a nonempty, bounded, closed and convex set and let us assume that conditions (H1)-(H2) hold true. Then, the system of nonlinear hemivariational inequalities* **(SNHI)** *admits at least one solution.*

The existence of solutions for our system will be a direct consequence of the fact that a *vector hemivariational inequality* admits solutions. Let us introduce the following notations:

- $X = X_1 \times \ldots \times X_n$, $K = K_1 \times \ldots \times K_n$ and $Y = Y_1 \times \ldots \times Y_n$;
- $u = (u_1, \ldots, u_n)$ and $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_n)$;
- $\Psi : X \times X \to \mathbb{R}, \quad \Psi(u, v) = \sum\limits_{k=1}^{n} \psi_k(u_1, \ldots, u_k, \ldots, u_n, v_k)$;
- $F : X \to X^*, \quad \langle Fu, v \rangle_X = \sum\limits_{k=1}^{n} \langle F_k(u_1, \ldots, u_n), v_k \rangle_{X_k}$.

and formulate the following vector hemivariational inequality

(VHI) Find $u \in K$ such that

$$\Psi(u, v) + J^0(\hat{u}; \hat{v} - \hat{u}) \geq \langle Fu, v - u \rangle_X, \qquad \text{for all } v \in K.$$

Remark 3.1. *If (H1)-*(i) *holds, then any solution $u^0 = (u_1^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_n$ of the vector hemivariational inequality* **(VHI)** *is also a solution of the system* **(SNHI)**.

Indeed, if for a $k \in \{1, \ldots, n\}$ we fix $v_k \in K_k$ and for $j \neq k$ we consider $v_j = u_j^0$, using Lemma 2.3 and the fact that u^0 solves **(VHI)** we obtain

$$\begin{aligned}
\left\langle F_k \left(u_1^0, \ldots, u_n^0\right), v_k - u_k^0 \right\rangle_{X_k} &= \sum_{j=1}^{n} \left\langle F_j \left(u_1^0, \ldots, u_n^0\right), v_j - u_j^0 \right\rangle_{X_j} \\
&= \left\langle Fu^0, v - u^0 \right\rangle_X \\
&\leq \Psi \left(u^0, v\right) + J^0 \left(\hat{u}^0; \hat{v} - \hat{u}^0\right) \\
&\leq \sum_{j=1}^{n} \psi_j \left(u_1^0, \ldots, u_j^0, \ldots, u_n^0, v_j\right) + \sum_{j=1}^{n} J_{,j}^0 \left(\hat{u}_1^0, \ldots, \hat{u}_n^0; \hat{v}_j - \hat{u}_j^0\right) \\
&= \psi_k \left(u_1^0, \ldots, u_k^0, \ldots, u_n^0, v_k\right) + J_{,k}^0 \left(\hat{u}_1^0, \ldots, \hat{u}_n^0; \hat{v}_k - \hat{u}_k^0\right).
\end{aligned}$$

As $k \in \{1, \ldots, n\}$ and $v_k \in K_k$ were arbitrarily fixed, we conclude that $(u_1^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_n$ is a solution of our system **(SNHI)**.

Proof of Theorem 3.1. According to Remark 3.1 it suffices to prove that problem **(VHI)** admits a solution. With this end in view we consider the set $\mathcal{A} \subset K \times K$ as follows

$$\mathcal{A} = \left\{(v, u) \in K \times K : \ \Psi(u, v) + J^0(\hat{u}; \hat{v} - \hat{u}) - \langle Fu, v - u\rangle_X \geq 0\right\}.$$

We shall prove next that the set $\mathcal{A}$ satisfies the conditions required in Theorem 2.1 for the weak topology of the space X.

STEP 1. For each $v \in K$ the set $\mathcal{N}(v) = \{u \in K : \ (v, u) \in \mathcal{A}\}$ is weakly closed in K.

In order to prove the above assertion, for a fixed $v \in K$ we consider the functional $\alpha : K \to \mathbb{R}$ defined by

$$\alpha(u) = \Psi(u, v) + J^0(\hat{u}; \hat{v} - \hat{u}) - \langle Fu, v - u\rangle_X$$

and we shall prove that it is weakly upper semicontinuous. Let us consider a sequence $\{u^m\} \subset K$ such that $u^m \rightharpoonup u$ as $m \to \infty$. Taking into account that T_k is compact for each $k \in \{1, \ldots, n\}$ we deduce that $\hat{u}^m \to \hat{u}$ as $m \to \infty$. Using (H1)-(ii) we obtain

$$\begin{aligned}
\limsup_{m\to\infty} \Psi(u^m, v) &= \limsup_{m\to\infty} \sum_{k=1}^{n} \psi_k(u_1^m, \ldots, u_n^m, v_k) \\
&\leq \sum_{k=1}^{n} \limsup_{m\to\infty} \psi_k(u_1^m, \ldots, u_n^m, v_k) \\
&\leq \sum_{k=1}^{n} \psi_k(u_1, \ldots, u_n, v_k) \\
&= \Psi(u, v).
\end{aligned}$$

On the other hand, using Lemma 2.1 we deduce that

$$\limsup_{m\to\infty} J^0(\hat{u}^m; \hat{v} - \hat{u}^m) \leq J^0(\hat{u}; \hat{v} - \hat{u})$$

Finally, using (H2) we have

$$\begin{aligned}
\limsup_{m\to\infty} \left[- \langle Fu^m, v - u^m\rangle_X\right] &= -\liminf_{m\to\infty} \langle Fu^m, v - u^m\rangle_X \\
&= -\liminf_{m\to\infty} \sum_{k=1}^{n} \langle F_k(u_1^m, \ldots, u_n^m), v_k - u_k^m\rangle_{X_k} \\
&\leq -\sum_{k=1}^{n} \langle F_k(u_1, \ldots, u_n), v_k - u_k\rangle_{X_k} \\
&= -\langle Fu, v - u\rangle_X
\end{aligned}$$

It is clear from the above relations that the functional α is weakly upper semicontinuous, therefore the set

$$[\alpha \geq \lambda] = \{u \in K : \alpha(u) \geq \lambda\}$$

is weakly closed for any $\lambda \in \mathbb{R}$. Taking $\lambda = 0$ we obtain that the set $\mathcal{N}(v)$ is weakly closed.

STEP 2. For each $u \in K$ the set $\mathcal{M}(u) = \{v \in K : \ (v, u) \notin \mathcal{A}\}$ is either convex or empty.

Let us fix $u \in K$ and assume that $\mathcal{M}(u)$ is nonempty. Let v^1, v^2 be two elements of $\mathcal{M}(u)$, $t \in (0,1)$ and $v^t = tv^1 + (1-t)v^2$. Using (H1) -(iii) we obtain

$$\begin{aligned}
\Psi(u, v^t) &= \sum_{k=1}^{n} \psi_k \left(u_1, \ldots, u_n, tv_k^1 + (1-t)v_k^2\right) \\
&\le t \sum_{k=1}^{n} \psi_k \left(u_1, \ldots, u_n, v_k^1\right) + (1-t) \sum_{k=1}^{n} \psi_k \left(u_1, \ldots, u_n, v_k^2\right) \\
&= t\Psi(u, v^1) + (1-t)\Psi(u, v^2),
\end{aligned}$$

which shows that the mapping $v \rightsquigarrow \Psi(u,v)$ is convex. On the other hand Lemma 2.1 ensures that the mapping $v \rightsquigarrow J^0(\hat{u}; \hat{v} - \hat{u})$ is convex. Using the fact that the mapping $v \rightsquigarrow \langle Fu, v - u\rangle_X$ is affine we are led to

$$\begin{aligned}
\Psi(u, v^t) + J^0(\hat{u}; \hat{v}^t - \hat{u}) - \langle Fu, v^t - u\rangle_X &\le t\left[\Psi(u, v^1) + J^0(\hat{u}; \hat{v}^1 - \hat{u}) - \langle Fu, v^1 - u\rangle_X\right] \\
&\quad +(1-t)\left[\Psi(u, v^2) + J^0(\hat{u}; \hat{v}^2 - \hat{u}) - \langle Fu, v^2 - u\rangle_X\right] \\
&< 0,
\end{aligned}$$

which means that $v^t \in \mathcal{M}(u)$, therefore $\mathcal{M}(u)$ is a convex set.

STEP 3. $(u,u) \in \mathcal{A}$ for each $u \in K$.

Let $u \in K$ be fixed. Using (H1)-(i) we obtain

$$\Psi(u,u) + J^0(\hat{u}; \hat{u} - \hat{u}) - \langle Fu, u - u\rangle_X = \sum_{k=1}^{n} \psi_k(u_1, \ldots, u_k, \ldots, u_n, u_k) = 0,$$

and this is equivalent to $(u,u) \in \mathcal{A}$.

STEP 4. The set $B = \{u \in K : (v,u) \in \mathcal{A} \text{ for all } v \in K\}$ is compact.

First we observe that K is a weakly compact subset of the reflexive space X as it is bounded, closed and convex. Then, we observe that the set B can be rewritten in the following way

$$B = \bigcap_{v \in K} \mathcal{N}(v).$$

This shows that B is also a weakly compact set as it is an intersection of weakly closed subsets of K.

We are now able to apply Lin's theorem and conclude that there exists $u^0 \in B \subseteq K$ such that $K \times \{u^0\} \subset \mathcal{A}$. This means that

$$\Psi(u^0, v) + J^0(\hat{u}^0; \hat{v} - \hat{u}^0) \ge \langle Fu^0, v - u^0\rangle_X, \quad \text{for all } v \in K,$$

therefore u^0 solves problem **(VHI)** and, accordingly to Remark 3.1, it is a solution of our system of nonlinear hemivariational inequalities **(SNHI)**, the proof of Theorem 3.1 being now complete.

□

We will show next that if we change the hypotheses on the nonlinear functionals ψ_k we are still able to prove the existence of at least one solution for our system. Let us consider that instead of (H1) we have the following set of hypotheses

(H3) For each $k \in \{1, \ldots, n\}$, the functional $\psi_k : X_1 \times \ldots \times X_k \times \ldots \times X_n \times X_k \to \mathbb{R}$ satisfies

(i) $\psi_k(u_1, \ldots, u_k, \ldots, u_n, u_k) = 0$ for all $u_k \in X_k$;

(ii) For each $k \in \{1, \ldots, n\}$ and any pair $(u_1, \ldots, u_k, \ldots, u_n)$, $(v_1, \ldots, v_k, \ldots, v_n) \in X_1 \times \ldots \times X_k \times \ldots \times X_n$ we have

$$\psi_k(u_1, \ldots, u_k, \ldots, u_n, v_k) + \psi_k(v_1, \ldots, v_k, \ldots, v_n, u_k) \geq 0;$$

(iii) For each $(u_1, \ldots, u_n) \in X_1 \times \ldots \times X_n$ the mapping $v_k \rightsquigarrow \psi_k(u_1, \ldots, u_n, v_k)$ is weakly lower semicontinuous;

(iv) For each $v_k \in X_k$ the mapping $(u_1, \ldots, u_n) \rightsquigarrow \psi_k(u_1, \ldots, u_n, v_k)$ is concave.

We are now in position to state our second main result of the paper, which concerns the case when the sets K_k are bounded, closed and convex for each $k \in \{1, \ldots, n\}$.

Theorem 3.2. *For each $k \in \{1, \ldots, n\}$ let $K_k \subset X_k$ be a nonempty, bounded, closed and convex set and let us assume that conditions (H2)-(H3) hold true. Then, the system of nonlinear hemivariational inequalities* **(SNHI)** *admits at least one solution.*

In order to prove Theorem 3.2 we will need the following lemma.

Lemma 3.1. *Assume that (H3) holds. Then*

(a) $\Psi(u, v) + \Psi(v, u) \geq 0$ for all $u, v \in X$;

(b) For each $v \in X$ the mapping $u \rightsquigarrow -\Psi(v, u)$ is weakly upper semicontinuous;

(c) For each $u \in X$ the mapping $v \rightsquigarrow -\Psi(v, u)$ is convex.

Proof.

(a) Taking into account (H3)-(ii) and the way the functional $\Psi : X \times X \to \mathbb{R}$ was defined we find

$$\Psi(u, v) + \Psi(v, u) = \sum_{k=1}^{n} \left[\psi_k(u_1, \ldots, u_k, \ldots, u_n, v_k) + \psi_k(v_1, \ldots, v_k, \ldots, v_n, u_k)\right] \geq 0.$$

(b) Let $v \in X$ be fixed and let $\{u^m\} \subset X$ be a sequence which converges weakly to some $u \in X$. Using (H3)-(iii) and the fact that $u^m \to u$ we obtain

$$\begin{aligned}
\limsup_{m\to\infty} \left[-\Psi(v, u^m)\right] &= -\liminf_{m\to\infty} \Psi(v, u^m) \\
&= -\liminf_{m\to\infty} \sum_{k=1}^{n} \psi_k(v_1, \ldots, v_n, u_k^m) \\
&\leq -\sum_{k=1}^{n} \liminf_{m\to\infty} \psi_k(v_1, \ldots, v_n, u_k^m) \\
&\leq -\sum_{k=1}^{n} \psi_k(v_1, \ldots, v_n, u_k) \\
&= -\Psi(v, u).
\end{aligned}$$

(c) Let $u, v^1, v^2 \in X$ and $t \in (0, 1)$. Keeping (H3)-(iv) in mind we deduce that

$$\begin{aligned}
\Psi\left(tv^1 + (1-t)v^2, u\right) &= \sum_{k=1}^{n} \psi_k\left(tv_1^1 + (1-t)v_1^2, \ldots, tv_n^1 + (1-t)v_n^2, u_k\right) \\
&\geq \sum_{k=1}^{n} t\psi_k\left(v_1^1, \ldots, v_n^1, u_k\right) + (1-t)\psi_k\left(v_1^2, \ldots, v_n^2, u_k\right) \\
&= t\Psi(v^1, u) + (1-t)\Psi(v^2, u).
\end{aligned}$$

We have prove that the mapping $v \rightsquigarrow \Psi(v, u)$ is concave, hence the application $v \rightsquigarrow -\Psi(v, u)$ must be convex.

□

Proof of Theorem 3.2. Let us consider the set $\mathcal{A} \subset K \times K$ defined by

$$\mathcal{A} = \{(v,u) \in K \times K : -\Psi(v,u) + J^0(\hat{u}; \hat{v} - \hat{u}) - \langle Fu, v-u \rangle_X \geq 0\}.$$

Lemma 3.1 ensures that we can follow the same steps as in the proof of Theorem 3.1 to conclude that the conditions required in Lin's theorem are fulfilled. Thus we get the existence of an element $u^0 \in K$ such that $K \times \{u^0\} \subset \mathcal{A}$ which is equivalent to

$$-\Psi(v,u^0) + J^0(\hat{u}^0; \hat{v} - \hat{u}^0) \geq \langle Fu^0, v - u^0 \rangle_X \quad \text{for all } v \in K. \tag{3.1}$$

On the other hand Lemma 3.1 tells us that

$$\Psi(u^0, v) + \Psi(v, u^0) \geq 0, \quad \text{for all } v \in K. \tag{3.2}$$

Combining relations (3.1) and (3.2) we deduce that u^0 solves problem **(VHI)**, therefore it is a solution of problem **(SNHI)**.

□

Let us consider now the case when at least one of the subsets K_k is unbounded and either conditions (H1)-(H2) or (H2)-(H3) hold. We shall denote next by $B_E(0;R)$ the closed ball of the space E centered in the origin and of radius R, that is

$$B_E(0;R) = \{v \in E : \ \|v\|_E \leq R\}.$$

Let $R > 0$ be such that the set $K_{k,R} = K_k \cap B_{X_k}(0;R)$ is nonempty for every $k \in \{1,\ldots,n\}$. Then, for each $k \in \{1,\ldots,n\}$ the set $K_{k,R}$ is nonempty, bounded, closed and convex and according to Theorem 3.1 or Theorem 3.2 the following problem

(SR) Find $(u_1,\ldots,u_n) \in K_{1,R} \times \ldots \times K_{n,R}$ such that for all $(v_1,\ldots,v_n) \in K_{1,R} \times \ldots \times K_{n,R}$

$$\begin{cases} \psi_1(u_1,\ldots,u_n,v_1) + J^0_{,1}(\hat{u}_1,\ldots,\hat{u}_n;\hat{v}_1 - \hat{u}_1) \geq \langle F_1(u_1,\ldots,u_n), v_1 - u_1 \rangle_{X_1} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \psi_n(u_1,\ldots,u_n,v_n) + J^0_{,n}(\hat{u}_1,\ldots,\hat{u}_n;\hat{v}_n - \hat{u}_n) \geq \langle F_n(u_1,\ldots,u_n), v_n - u_n \rangle_{X_n}, \end{cases}$$

admits at least one solution.

We have the following existence result concerning the case of at least one unbounded subset.

Theorem 3.3. *For each $k \in \{1,\ldots,n\}$ let $K_k \subset X_k$ be a nonempty, closed and convex set and assume that there exists at least one index $k_0 \in \{1,\ldots,n\}$ such that K_{k_0} is unbounded. Assume in addition that either (H1)-(H2) or (H2)-(H3) hold. Then, the system of nonlinear hemivariational inequalities* **(SNHI)** *admits at least one solution if and only if the following condition holds true:*

(H4) there exists $R > 0$ such that $K_{k,R}$ is nonempty for every $k \in \{1,\ldots,n\}$ and at least one solution $(u_1^0,\ldots,u_n^0)$ of problem **(SR)** *satisfies*

$$u_k^0 \in \operatorname{int} B_{X_k}(0;R), \quad \textit{for all } k \in \{1,\ldots,n\}.$$

Proof. The necessity is obvious.

In order to prove the sufficiency for each $k \in \{1,\ldots,n\}$ let us fix $v_k \in K_k$ and define the scalar

$$\lambda_k = \begin{cases} \frac{1}{2} & \text{if } u_k^0 = v_k \\ \min\left\{\frac{1}{2}; \frac{R - \|u_k^0\|_{X_k}}{\|v_k - u_k^0\|_{X_k}}\right\} & \text{otherwise.} \end{cases}$$

Condition (H4) ensures that $\lambda_k \in (0,1)$, therefore $w_{\lambda_k} = u_k^0 + \lambda_k(v_k - u_k^0)$ is an element of $K_{k,R}$ due to the convexity of the set K_k.

CASE 1. (H1)-(H2) hold.

Using the fact $(u_1^0,\dots,u_n^0)$ is a solution of **(SR)** for each $k\in\{1,\dots,n\}$ we have

$$\psi_k(u_1^0,\dots,u_n^0,w_{\lambda_k})+J^0_{,k}(\hat u_1^0,\dots,\hat u_n^0;\hat w_{\lambda_k}-\hat u_k^0)\geq\langle F_k(u_1^0,\dots,u_n^0),w_{\lambda_k}-u_k^0\rangle_{X_k} \tag{3.3}$$

In this case relation (3.3) leads to

$$\begin{aligned}\lambda_k\langle F_k(u_1^0,\dots,u_n^0),v_k-u_k^0\rangle_{X_k} &= \langle F_k(u_1^0,\dots,u_n^0),w_{\lambda_k}-u_k^0\rangle_{X_k}\\ &\leq \lambda_k\psi_k(u_1^0,\dots,u_n^0,v_k)+(1-\lambda_k)\psi_k(u_1^0,\dots,u_n^0,u_k^0)\\ &\quad+\lambda_k J^0_{,k}(\hat u_1^0,\dots,\hat u_n^0;\hat v_k-\hat u_k^0)\\ &= \lambda_k\left[\psi_k(u_1^0,\dots,u_n^0,v_k)+J^0_{,k}(\hat u_1^0,\dots,\hat u_n^0;\hat v_k-\hat u_k^0)\right]\end{aligned}$$

Dividing by λ_k the above inequality and taking into account that $v_k\in K_k$ was arbitrary fixed we conclude that $(u_1^0,\dots,u_n^0)$ is a solution of **(SNHI)**.

CASE 2. (H2)-(H3) hold.

Theorem 3.2 ensures that (see (3.1))

$$-\Psi(w,u^0)+J^0(\hat u^0;\hat w-u^0)\geq\langle Fu^0,w-u^0\rangle,\quad \text{for all } w\in K_R=K_{1,R}\times\ldots\times K_{n,R}.$$

Choosing $w_k=w_{\lambda_k}$ and $w_j=u_j^0$ for $j\neq k$ in the above relation we obtain

$$\begin{aligned}\lambda_k\left\langle F_k(u_1^0,\dots,u_n^0),v_k-u_k^0\right\rangle_{X_k} &= \left\langle F_k(u_1^0,\dots,u_n^0),w_{\lambda_k}-u_k^0\right\rangle_{X_k}\\ &= \sum_{j=1}^n\left\langle F_k(u_1^0,\dots,u_n^0),w_j-u_j^0\right\rangle_{X_k}\\ &= \langle Fu^0,w-u^0\rangle_X\\ &\leq -\Psi(w,u^0)+J^0(\hat u^0;\hat w-\hat u^0)\\ &= -\sum_{j=1}^n\psi_j(w_1,\dots,w_j,\dots,w_n,u_j^0)+\sum_{j=1}^n J^0_{,j}(\hat u_1^0,\dots,\hat u_n^0;\hat w_j-\hat u_j^0)\\ &= -\psi_k(u_1^0,\dots,w_{\lambda_k},\dots,u_n^0,u_k^0)+J^0_{,k}(\hat u_1^0,\dots,\hat u_n^0;\hat w_{\lambda_k}-\hat u_k^0)\\ &\leq -\lambda_k\psi_k(u_1^0,\dots,v_k,\dots,u_n^0,u_k^0)-(1-\lambda_k)\psi_k(u_1^0,\dots,u_k^0,\dots,u_n^0,u_k^0)\\ &\quad+\lambda_k J^0_{,k}(\hat u_1^0,\dots,\hat u_n^0;\hat v_k-\hat u_k^0)\\ &\leq \lambda_k\left[-\psi_k(u_1^0,\dots,v_k,\dots,u_n^0,u_k^0)+J^0_{,k}(\hat u_1^0,\dots,\hat u_n^0;\hat v_k-\hat u_k^0)\right]\end{aligned}$$

Dividing by λ_k we obtain that

$$-\psi_k(u_1^0,\dots,v_k,\dots,u_n^0,u_k^0)+J^0_{,k}(\hat u_1^0,\dots,\hat u_n^0;\hat v_k-\hat u_k^0)\geq\left\langle F_k(u_1^0,\dots,u_n^0),v_k-u_k^0\right\rangle_{X_k}.$$

Combining the above inequality and (H3)-(ii) we deduce the for each $k\in\{1,\dots,n\}$ the following inequality takes place

$$\psi_k(u_1^0,\dots,u_k^0,\dots,u_n^0,v_k)+J^0_{,k}(\hat u_1^0,\dots,\hat u_n^0;\hat v_k-\hat u_k^0)\geq\left\langle F_k(u_1^0,\dots,u_n^0),v_k-u_k^0\right\rangle_{X_k},$$

which means that $(u_1^0,\dots,u_n^0)$ is a solution of **(SNHI)**, since $v_k\in K_k$ was arbitrary fixed.

□

Corollary 3.1. *For each $k\in\{1,\dots,n\}$ let $K_k\subset X_k$ be a nonempty, closed and convex set and assume that there exists at least one index $k_0\in\{1,\dots,n\}$ such that K_{k_0} is unbounded. Assume in addition that either (H1)-(H2) or (H2)-(H3) hold. Then, a sufficient condition for* **(SNHI)** *to admit at least one solution is*

(H5) there exists $R_0 > 0$ such that K_{k,R_0} is nonempty for every $k \in \{1,\dots,n\}$ and for each $(u_1,\dots,u_n) \in K_1 \times \ldots \times K_n \setminus K_{1,R_0} \times \ldots \times K_{n,R_0}$ there exists $(v_1^0,\dots,v_n^0) \in K_{1,R_0} \times \ldots \times K_{n,R_0}$ such that

$$\psi_k(u_1,\dots,u_n,v_k^0) + J^0_{,k}(\hat{u}_1,\dots,\hat{u}_n;\hat{v}_k^0 - \hat{u}_k) < \langle F_k(u_1,\dots,u_n), v_k^0 - u_k\rangle_{X_k}, \tag{3.4}$$

for all $k \in \{1,\dots,n\}$.

Proof. Let us fix $R > R_0$. According to Theorem 3.1 or Theorem 3.2 problem **(SR)** admits at least one solution. Let $(u_1,\dots,u_n) \in K_{1,R} \times \ldots \times K_{n,R}$ be a solution of **(SR)**. We shall prove that $(u_1,\dots,u_n)$ also solves **(SNHI)**.

CASE 1. $u_k \in \mathrm{int}B_{X_k}(0,R)$ for all $k \in \{1,\dots,n\}$.

In this case we have nothing to prove as Theorem 3.3 ensures that $(u_1,\dots,u_n)$ is a solution of **(SNHI)**.

CASE 2. There exists at least one index $j_0 \in \{1,\dots,n\}$ such that $u_{j_0} \notin \mathrm{int}B_{X_{j_0}}(0,R)$.

In this case $\|u_{j_0}\|_{X_{j_0}} = R > R_0$, therefore $(u_1,\dots,u_n) \notin K_{1,R_0} \times \ldots \times K_{n,R_0}$ and according to (H5) there exist $(v_1^0,\dots,v_n^0) \in K_{1,R_0} \times \ldots \times K_{n,R_0}$ such that (3.4) holds.

For each $k \in \{1,\dots,n\}$ let us fix $v_k \in K_k$ and define the scalar

$$\lambda_k = \begin{cases} \frac{1}{2} & \text{if } v_k = v_k^0 \\ \min\left\{\frac{1}{2}, \frac{R-R_0}{\|v_k - v_k^0\|_{X_k}}\right\} & \text{otherwise.}\end{cases}$$

Obviously $\lambda_k \in (0,1)$ and $w_{\lambda_k} = v_k^0 + \lambda_k(v_k - v_k^0) \in K_{k,R}$. Furthermore, we observe that

$$w_{\lambda_k} - u_k = v_k^0 - u_k + \lambda_k v_k - \lambda_k v_k^0 + \lambda_k u_k - \lambda_k u_k = \lambda_k(v_k - u_k) + (1-\lambda_k)(v_k^0 - u_k).$$

Case 2.1 (H1)-(H2) hold.

Using the fact that $(u_1,\dots,u_n)$ solves **(SR)** we obtain the following estimates

$$\begin{aligned}\langle F_k(u_1,\dots,u_n), w_{\lambda_k} - u_k\rangle &= \lambda_k\langle F_k(u_1,\dots,u_n), v_k - u_k\rangle_{X_k} + (1-\lambda_k)\langle F_k(u_1,\dots,u_n), v_k^0 - u_k\rangle_{X_k} \\ &\le \psi_k(u_1,\dots,u_n,w_{\lambda_k}) + J^0_{,k}(\hat{u}_1,\dots,\hat{u}_n;\hat{w}_{\lambda_k} - \hat{u}_k) \\ &\le \lambda_k\left[\psi_k(u_1,\dots,u_n,v_k) + J^0_{,k}(\hat{u}_1,\dots,\hat{u}_n;\hat{v}_k - \hat{u}_k)\right] \\ &\quad + (1-\lambda_k)\left[\psi_k(u_1,\dots,u_n,v_k^0) + J^0_{,k}(\hat{u}_1,\dots,\hat{u}_n;\hat{v}_k^0 - \hat{u}_k)\right]\end{aligned}$$

Combining the above relation and (3.4) we obtain that

$$F_k(u_1,\dots,u_n), v_k - u_k\rangle_{X_k} \le \psi_k(u_1,\dots,u_n,v_k) + J^0_{,k}(\hat{u}_1,\dots,\hat{u}_n;v_k - u_k) \ \text{ for all } k \in \{1,\dots,n\},$$

which means that $(u_1,\dots,u_n)$ is a solution of **(SNHI)**.

Case 2.2. (H2)-(H3) hold.

The fact that $(u_1,\dots,u_n)$ solves **(SR)** and relation (3.1) allow us to conclude that

$$-\Psi(w,u) + J^0(\hat{u},\hat{w} - \hat{u}) \ge \langle Fu, w-u\rangle_X, \quad \text{for all } w \in K_R = K_{1,R} \times \ldots \times K_{n,R}.$$

Choosing $w_k = w_{\lambda_k}$ and $w_j = u_j$ for $j \neq k$ in the above relation and using (H3)-(iv) we obtain

$$\begin{aligned}
\langle F_k(u_1,\ldots,u_n), w_{\lambda_k} - u_k\rangle_{X_k} &= \lambda_k \langle F_k(u_1,\ldots,u_n), v_k - u_k\rangle_{X_k} + (1-\lambda_k)\left\langle F_k(u_1,\ldots,u_n), v_k^0 - u_k\right\rangle_{X_k} \\
&= \sum_{j=1}^{n} \langle F_k(u_1,\ldots,u_n), w_j - u_j\rangle_{X_k} \\
&= \langle Fu, w-u\rangle \\
&\leq -\Psi(w,u) + J^0(\hat{u};\hat{w}-\hat{u}) \\
&= -\sum_{j=1}^{n} \psi_j(w_1,\ldots,w_j,\ldots,w_n,u_j) + \sum_{j=1}^{n} J^0_{,j}(\hat{u}_1,\ldots,\hat{u}_n;\hat{w}_j-\hat{u}_j) \\
&= -\psi_k(u_1,\ldots,w_{\lambda_k},\ldots,u_n,u_k)) + J^0_{,k}(\hat{u}_1,\ldots,\hat{u}_n;\hat{w}_{\lambda_k}-\hat{u}_k) \\
&\leq -\lambda_k\psi_k(u_1,\ldots,v_k,\ldots,u_n,u_k) - (1-\lambda_k)\psi_k(u_1,\ldots,v_k^0,\ldots,u_n,u_k) \\
&\quad +\lambda_k J^0_{,k}(\hat{u}_1,\ldots,\hat{u}_n;\hat{v}_k-\hat{u}_k) + (1-\lambda_k)J^0(\hat{u}_1,\ldots,\hat{u}_n;\hat{v}_k^0-\hat{u}_k)
\end{aligned}$$

Using (H3)-(ii) and (3.4) we deduce that

$$F_k(u_1,\ldots,u_n), v_k - u_k\rangle_{X_k} \leq \psi_k(u_1,\ldots,u_n,v_k) + J^0_{,k}(\hat{u}_1,\ldots,\hat{u}_n;v_k-u_k) \quad \text{for all } k\in\{1,\ldots,n\},$$

which means that $(u_1,\ldots,u_n)$ is a solution of **(SNHI)**.

□

In order to simplify some computations let us assume next that $0 \in K_k$ for each $k \in \{1,\ldots,n\}$. In this case $K_{k,R} \neq \emptyset$ for every $k \in \{1,\ldots,n\}$ and every $R > 0$.

Corollary 3.2. *For each $k \in \{1,\ldots,n\}$ let $K_k \subset X_k$ be a nonempty, closed and convex set and assume that there exists at least one index $k_0 \in \{1,\ldots,n\}$ such that K_{k_0} is unbounded and either (H1)-(H2) or (H2)-(H3) hold. Assume in addition that for each $k \in \{1,\ldots,n\}$ the following conditions hold*

(H6) There exists a function $c : \mathbb{R}_+ \to \mathbb{R}_+$ with the property that $\lim_{t\to\infty} c(t) = +\infty$ such that

$$-\sum_{k=1}^{n} \psi_k(u_1,\ldots,u_k,\ldots,u_n,0) \geq c(\|u\|_X)\|u\|_X,$$

for all $(u_1,\ldots,u_n) \in X_1\times\ldots\times X_n$, where $u = (u_1,\ldots,u_k,\ldots,u_n)$ and $\|u\|_X = \left(\sum_{k=1}^{n}\|u_k\|^2_{X_k}\right)^{1/2}$;

(H7) There exists $M_k > 0$ such that

$$J^0_{,k}(w_1,\ldots,w_k,\ldots,w_n;-w_k) \leq M_k\|w_k\|_{Y_k}, \quad \text{for all } (w_1,\ldots,w_n) \in Y_1\times\ldots\times Y_n;$$

(H8) There exists $m_k > 0$ such that

$$\|F_k(u_1,\ldots,u_k,\ldots,u_n)\|_{X_k^*} \leq m_k, \quad \text{for all } (u_1,\ldots,u_n) \in X_1\times\ldots\times X_n.$$

Then the system **(SNHI)** *admits at least one solution.*

Proof. For each $R > 0$ Theorem 3.1 (or Theorem 3.2) enables us to conclude that there exists a solution $(u_{1R},\ldots,u_{nR}) \in K_{1,R}\times\ldots\times K_{n,R}$ of problem **(SR)**. We shall prove that there exists $R_0 > 0$ such that

$$u_{kR_0} \in \operatorname{int} B_{X_k}(0;R_0), \quad \text{for all } k\in\{1,\ldots,n\},$$

which, according to Theorem 3.3, means that $(u_{1R_0},\ldots,u_{nR_0})$ is a solution of the system **(SNHI)**.

Arguing by contradiction let us assume that for each $R > 0$ there exists at least one index $j_0 \in \{1, \ldots, n\}$ such that $u_{j_0 R} \notin \operatorname{int} B_{X_{j_0}}(0, R)$, therefore $\|u_{j_0 R}\|_{X_{j_0}} = R$. Using the fact that $(u_{1R}, \ldots, u_{nR})$ solves **(SR)** we conclude that for each $k \in \{1, \ldots, n\}$ the following inequality holds

$$\psi_k(u_{1R}, \ldots, u_{nR}, v_k) + J^0_{,k}(\hat{u}_{1R}, \ldots, \hat{u}_{nR}; \hat{v}_k - \hat{u}_{kR}) \geq \langle F_k(u_{1R}, \ldots, u_{nR}), v_k - u_{kR} \rangle_{X_k}, \tag{3.5}$$

for all $v_k \in K_{k,R}$.

Taking $v_k = 0$ in (3.5), summing and using (H6)-(H8) we have

$$\begin{aligned} c(\|u\|_X)\|u\|_X &\leq -\sum_{k=1}^{n} \psi_k(u_{1R}, \ldots, u_{j_0 R}, \ldots, u_{nR}, 0) \\ &\leq \sum_{k=1}^{n} \left[\langle F_k(u_{1R}, \ldots, u_{nR}), u_{kR} \rangle_{X_k} + J^0_{,k}(\hat{u}_{1R}, \ldots, \hat{u}_k, \ldots, \hat{u}_{nR}; -\hat{u}_k) \right] \\ &\leq \sum_{k=1}^{n} \left(\|F_k(u_{1R}, \ldots, u_{nR})\|_{X_k^*} \|u_k\|_{X_k} + M_k \|\hat{u}_{kR}\|_{Y_k} \right) \\ &\leq \sum_{k=1}^{n} \left[(m_k + M_k \|T_k\|) \|u_{kR}\|_{X_k} \right] \\ &\leq C\|u\|_X. \end{aligned}$$

Dividing by $\|u\|_X$ and letting $R \to +\infty$ we obtain a contradiction since the left-hand term of the inequality is unbounded while the the right-hand term remains bounded. $\square$

4 Applications

4.1 Nonlinear hemivariational inequalities

Let us consider X, Y to be real reflexive Banach spaces such that there exists a linear and compact operator $T : X \to Y$. If K is a nonempty closed subset of X and $n = 1$, then the system **(SNHI)** reduces to the following nonlinear hemivariational inequality

$(\mathcal{P}_\psi)$ Find $u \in K$ such that

$$\psi(u, v) + J^0(\hat{u}; \hat{v} - \hat{u}) \geq \langle Fu, v - u \rangle_X, \quad \text{for all } v \in K.$$

1. Let us consider $\Omega \subseteq \mathbb{R}^k$ $(k \geq 1)$ to be open, bounded with smooth boundary and assume that X is compactly embedded in $L^q(\Omega; \mathbb{R}^k)$ for some $q \in (1, +\infty)$. Let us assume in addition that the following conditions hold

   $\mathcal{H}_1(j)$ $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ is a functional which satisfies

   (i) $x \rightsquigarrow j(x, y)$ is measurable, for every $y \in \mathbb{R}^k$;

   (ii) either there exists $\alpha \in L^{q/(q-1)}(\Omega; \mathbb{R}^k)$ such that

   $$|j(x, v_1) - j(x, v_2)| \leq \alpha(x)|v_1 - v_2|, \quad \text{for almost every } x \in \Omega \text{ and every } v_1, v_2 \in \mathbb{R}^k;$$

   or, $y \rightsquigarrow j(x, y)$ is locally Lipschitz for almost every $x \in \Omega$ and there exists $c > 0$ such that

   $$|\partial_2 j(x, y)| \leq c(1 + |y|^{q-1}), \text{ for almost every } x \in \Omega \text{ and every } y \in \mathbb{R}^k;$$

   (iii) $y \rightsquigarrow j(x, y)$ is regular for almost every $x \in \Omega$.

Choosing $Y = L^q(\Omega;\mathbb{R}^k)$, $T = i$ (here i is the embedding operator between X and $L^q(\Omega;\mathbb{R}^k)$), $Fu = f$ ($f \in X^*$) for all $u \in X$ and $J : L^q(\Omega;\mathbb{R}^k) \to \mathbb{R}$ defined by

$$J(w) = \int_\Omega j(x, w(x))\, dx,$$

our inequality $(\mathcal{P}_\psi)$ becomes

$(\mathcal{P}^1_\psi)$ Find $u \in K$ such that

$$\psi(u,v) + \int_\Omega j^0_{,2}(x,\hat{u}(x);\hat{v}(x) - \hat{u}(x))\, dx \geq \langle f, v-u\rangle_X \qquad \text{for all } v \in K$$

this kind of hemivariational inequalities being studied by Costea and Rădulescu in [11]. Comparing our results with the ones obtained by Costea and Rădulescu we observe that Corollary 3.1 extends Theorem 3 from [11] while Theorem 3.2 extends Theorem 4 from [11].

Remark 4.1. *It can be proved that problem $(\mathcal{P}^1_\psi)$ admits solutions even in the case when $\Omega \subseteq \mathbb{R}^k$ is unbounded. In this case we need to replace $\mathcal{H}_1(j)$ with an appropriate condition which ensures the existence of the integral term $\int_\Omega j^0_{,2}(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))\, dx$ (see e.g. the work of Kristály and Varga [20] or Lisei et al. [22]).*

2. Let us consider (T,μ) to be a measure space of finite and positive measure and assume that X is compactly embedded in $L^q(T)$ for some $q \in (1,+\infty)$. Assume in addition that the following conditions hold true

   $\mathcal{H}_2(j)$ $j : T \times \mathbb{R} \to \mathbb{R}$ is a functional which satisfies

   (i) $x \rightsquigarrow j(x,y)$ is measurable, for every $y \in \mathbb{R}$;

   (ii) either there exists $\beta \in L^{q/(q-1)}(T;\mathbb{R}^k)$ such that

   $$|j(x,v_1) - j(x,v_2)| \leq \beta(x)|v_1 - v_2|, \qquad \text{for almost every } x \in T \text{ and every } v_1, v_2 \in \mathbb{R};$$

   or, $y \rightsquigarrow j(x,y)$ is locally Lipschitz for almost every $x \in T$ and there exists $c > 0$ such that

   $$|\partial_2 j(x,y)| \leq c(1 + |y|^{q-1}), \text{ for almost every } x \in T \text{ and every } y \in \mathbb{R};$$

   (iii) $y \rightsquigarrow j(x,y)$ is regular for almost every $x \in T$.

   $\mathcal{H}(f)$ $f : T \times \mathbb{R} \to \mathbb{R}$ is a functional such that

   (i) $x \rightsquigarrow f(x,y)$ is measurable for every $y \in \mathbb{R}$;

   (ii) $y \rightsquigarrow f(x,y)$ is continuous for almost every $x \in T$;

   (iii) there exists $\gamma_1 \in L^{q/(q-1)}(T)$ and $\gamma_2 \in L^\infty(T)$ such that

   $$|f(x,y)| \leq \gamma_1(x) + \gamma_2(x)|y|^{q-1}, \qquad \text{for almost every } x \in T \text{ and every } y \in \mathbb{R}.$$

   Choosing $Y = L^q(T)$, $T = i$ (where $i : X \to L^q(T)$ is the embedding operator), $F : X \to X^*$ defined by

   $$\langle Fu, v\rangle_X = \int_T f(x,u(x))\, v(x)\, d\mu$$

   and $J : L^q(T) \to \mathbb{R}$ defined by

   $$J(w) = \int_T j(x,w(x))\, d\mu$$

   our inequality $(\mathcal{P}_\psi)$ becomes

   $(\mathcal{P}^2_\psi)$ Find $u \in K$ such that

   $$\psi(u,v) + \int_T j^0_{,2}(x,u(x);v(x)-u(x))\, d\mu \geq \int_T f(x,u(x))(v(x)-u(x))\, d\mu,$$

   The above inequality is similar to the one studied by Andrei and Costea in [1] in the case $h(x,y) = 1$ for all $x \in T$ and $y \in \mathbb{R}$. Comparing the results we observe that Theorem 3.1 extends Theorem 2.1 from [1], Theorem 3.2 extends Theorem 2.2 from [1] while Corollary 3.2 extends Theorem 2.3 from [1].

4.2 Existence of Nash generalized derivative points

Let $E_1, \ldots, E_n$ be Banach spaces and for each $k \in \{1, \ldots, n\}$ let K_k be a nonempty subset of E_k. We also assume that $g_k : K_1 \times \ldots \times K_k \times \ldots \times K_n \to \mathbb{R}$ are given functionals. We recall below the notion of Nash equilibrium point (see [27, 28]).

Definition 4.1. *An element $(u_1, \ldots, u_k, \ldots, u_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ is a Nash equilibrium point for the functionals $g_1, \ldots, g_k, \ldots, g_n$, if for every $k \in \{1, \ldots, n\}$ and every $(v_1, \ldots, v_k, \ldots, v_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ we have*

$$g_k(u_1, \ldots, v_k, \ldots, u_n) \geq g_k(u_1, \ldots, u_k, \ldots, u_n).$$

Let $D_k \subset E_k$ be an open set such that $K_k \subset D_k$ for all $k \in \{1, \ldots, n\}$. For each $k \in \{1, \ldots, n\}$ we consider the functional $g_k : K_1 \times \ldots \times D_k \times \ldots \times K_n \to \mathbb{R}$ such that $u_k \rightsquigarrow g_k(u_1, \ldots, u_k, \ldots, u_n)$ is locally Lipschitz. The following notion was introduced by Kristály in [18].

Definition 4.2. *An element $(u_1, \ldots, u_k, \ldots, u_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ is a Nash generalized derivative point for the functionals $g_1, \ldots, g_k, \ldots, g_n$ if for every $k \in \{1, \ldots, n\}$ and every $(v_1, \ldots, v_k, \ldots, v_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ we have*

$$g^0_{k,k}(u_1, \ldots, u_k, \ldots, u_n; v_k - u_k) \geq 0.$$

We point out the fact that the above definition coincides with the notion of *Nash stationary point* introduced by Kassay, Kolumbán and Páles in [16] if every functional g_k is differentiable with respect to the k^{th} variable. Moreover, every Nash equilibrium point a Nash generalized derivative point.

1. For each $k \in \{1, \ldots, n\}$ let $D_k \subseteq X_k$ be an open and consider the functional $g_k : K_1 \times \ldots \times D_k \times \ldots \times K_n \to \mathbb{R}$ such that g_k is locally Lipschitz with respect to the k^{th} variable and for each $v_k \in X_k$ the mapping $(u_1, \ldots, u_k, \ldots, u_n) \rightsquigarrow g^0_{k,k}(u_1, \ldots, u_k, \ldots, u_n; v_k)$ is weakly upper semicontinuous. Let us choose next $\psi_k(u_1, \ldots, u_k, \ldots, u_n, v_k) = g^0_{k,k}(u_1, \ldots, u_k, \ldots, u_n; v_k - u_k)$, $J \equiv 0$, $F_k \equiv 0$.

   (i) If for each $k \in \{1, \ldots, n\}$ the set $K_k \subset X_k$ is nonempty, bounded, closed and convex, then Theorem 3.1 implies that there exists at least one point $(u^0_1, \ldots, u^0_k, \ldots, u^0_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ such that for all $(v_1, \ldots, v_k, \ldots, v_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ we have

   $$g^0_{k,k}(u^0_1, \ldots, u^0_k, \ldots, u^0_n; v_k - u^0_k) \geq 0, \quad \text{for all } k \in \{1, \ldots, n\},$$

   that is, $(u^0_1, \ldots, u^0_k, \ldots, u^0_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ is a Nash generalized derivative point for the functionals $g_1, \ldots, g_k, \ldots g_n$.

   (ii) Let us assume now that the sets K_k are nonempty, closed and convex and at least one of them is unbounded. Assume in addition that there exists $R_0 > 0$ such that K_{k,R_0} is nonempty for every $k \in \{1, \ldots, n\}$ and for each $(u_1, \ldots, u_k, \ldots, u_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n \setminus K_{1,R_0} \times \ldots \times K_{k,R_0} \times \ldots \times K_{n,R_0}$ there exists $(v^0_1, \ldots, v^0_k, \ldots, v^0_n) \in K_{1,R_0} \times \ldots \times K_{k,R_0} \times \ldots \times K_{n,R_0}$ such that

   $$g^0_{k,k}(u_1, \ldots, u_k, \ldots, u_n; v^0_k - \hat{u}_k) < 0, \quad \text{for all } k \in \{1, \ldots, n\}.$$

   Then, according to Corollary 3.1, there exists at least one point $(u^0_1, \ldots, u^0_k, \ldots, u^0_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ such that for all $(v_1, \ldots, v_k, \ldots, v_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ we have

   $$g^0_{k,k}(u^0_1, \ldots, u^0_k, \ldots, u^0_n; v_k - u^0_k) \geq 0, \quad \text{for all } k \in \{1, \ldots, n\},$$

   which means that $(u^0_1, \ldots, u^0_k, \ldots, u^0_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ is a Nash generalized derivative point for the functionals $g_1, \ldots, g_k, \ldots, g_n$.

   (iii) Let us assume now that the sets K_k are nonempty, closed and convex and at least one of them is unbounded. Assume in addition that there exists a function $c : \mathbb{R}_+ \to \mathbb{R}_+$ with the property that $\lim_{t \to \infty} c(t) = +\infty$ such that

   $$-\sum_{k=1}^{n} g^0_{k,k}(u_1, \ldots, u_k, \ldots, u_n; -u_k) \geq c(\|u\|_X)\|u\|_X, \quad \text{for all } (u_1, \ldots, u_n) \in K_1 \times \ldots \times K_n;$$

Then, according to Corollary 3.2, there exists at least one point $(u_1^0, \ldots, u_k^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ such that for all $(v_1, \ldots, v_k, \ldots, v_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ we have

$$g_{k,k}^0(u_1^0, \ldots, u_k^0, \ldots, u_n^0; v_k - u_k^0) \geq 0, \quad \text{for all } k \in \{1, \ldots, n\},$$

which means that $(u_1^0, \ldots, u_k^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ is a Nash generalized derivative point for the functionals $g_1, \ldots, g_k, \ldots g_n$.

2. Let us consider that for each $k \in \{1, \ldots, n\}$ we have $\psi_k \equiv 0$, $J \equiv 0$ and $F_k : X_1 \times \ldots \times X_k \times \ldots \times X_n \to X_k^*$ a nonlinear operator such that (H2) holds.

   (i) For each $k \in \{1, \ldots, n\}$ we assume that the set $K_k \subset X_k$ is nonempty, bounded, closed and convex. Then Theorem 3.1 implies that there exists at least one point $(u_1^0, \ldots, u_k^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ such that for all $(v_1, \ldots, v_k, \ldots, v_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ we have

   $$-\langle F_k(u_1^0, \ldots, u_k^0, \ldots, u_n^0), v_k - u_k^0 \rangle_{X_k} \geq 0, \quad \text{for all } k \in \{1, \ldots, n\},$$

   which means that $(u_1^0, \ldots, u_k^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ is a Nash stationary point for the functionals $g_1, \ldots, g_k, \ldots, g_n$, where $g_k : K_1 \times \ldots \times X_k \times \ldots \times K_n \to \mathbb{R}$ is differentiable with respect to the k^{th} variable and $g'_{k,k} = -\tilde{F}_k$ (here $\tilde{F}_k$ is the restriction of F_k to $K_1 \times \ldots \times X_k \times \ldots \times K_n$).

   (ii) Let us assume now that the sets K_k are nonempty, closed and convex and at least one of them is unbounded. Assume in addition that there exists $R_0 > 0$ such that K_{k,R_0} is nonempty for every $k \in \{1, \ldots, n\}$ and for each $(u_1, \ldots, u_k, \ldots, u_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n \setminus K_{1,R_0} \times \ldots \times K_{k,R_0} \times \ldots \times K_{n,R_0}$ there exists $(v_1^0, \ldots, v_k^0, \ldots, v_n^0) \in K_{1,R_0} \times \ldots \times K_{k,R_0} \times \ldots \times K_{n,R_0}$ such that

   $$\langle F_k(u_1, \ldots, u_k, \ldots, u_n), v_k^0 - u_k \rangle_{X_k} > 0, \quad \text{for all } k \in \{1, \ldots, n\}.$$

   Then, according to Corollary 3.1, there exists at least one point $(u_1^0, \ldots, u_k^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ such that for all $(v_1, \ldots, v_k, \ldots, v_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ we have

   $$-\langle F_k(u_1^0, \ldots, u_k^0, \ldots, u_n^0), v_k - u_k^0 \rangle_{X_k} \geq 0, \quad \text{for all } k \in \{1, \ldots, n\},$$

   which means that $(u_1^0, \ldots, u_k^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ is a Nash stationary point for the functionals $g_1, \ldots, g_n$, where $g_k : K_1 \times \ldots \times X_k \times \ldots \times K_n \to \mathbb{R}$ is differentiable with respect to the k^{th} variable and $g'_{k,k} = -\tilde{F}_k$.

4.3 Weak solvability of frictional problems for piezoelectric bodies in contact with a conductive foundation

This subsection focuses on the weak solvability of a mechanical model describing the contact between a piezoelectric body and a conductive foundation. The piezoelectric effect is characterized by the coupling between the mechanical and the electrical properties of the materials. This coupling leads to the appearance of electric potential when mechanical stress is present and, conversely, mechanical stress is generated when electric potential is applied.

Before describing the problem let us first present some notations and preliminary material which will be used throughout this subsection.

Let m be a positive integer and denote by $\mathcal{S}_m$ the linear space of second order symmetric tensors on $\mathbb{R}^m$ ($\mathcal{S}_m = \mathbb{R}_s^{m \times m}$). We recall that the inner product and the corresponding norm on $\mathcal{S}_m$ are given by

$$\tau : \sigma = \tau_{ij}\sigma_{ij}, \quad \|\tau\|_{\mathcal{S}_m} = \sqrt{\tau : \tau}, \quad \text{for all } \tau, \sigma \in \mathcal{S}_m.$$

Here, and hereafter the summation over repeated indices is used, all indices running from 1 to m.

Let $\Omega \subset \mathbb{R}^m$ be an open bounded subset with a Lipschitz boundary Γ and let ν denote the outward unit normal vector to Γ. We introduce the spaces

$$H = L^2(\Omega; \mathbb{R}^m), \qquad \mathcal{H} = \{\tau = (\tau_{ij}) : \ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\} = L^2(\Omega; \mathcal{S}_m),$$
$$H_1 = \{u \in H : \ \varepsilon(u) \in \mathcal{H}\} = H^1(\Omega; \mathbb{R}^m), \qquad \mathcal{H}_1 = \{\tau \in \mathcal{H} : \ \text{Div}\, \tau \in H\},$$

where $\varepsilon : H_1 \to \mathcal{H}$ and Div $: \mathcal{H}_1 \to H$ denote the deformation and the divergence operators, defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)),\ \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad \mathrm{Div}\,\tau = \left(\frac{\partial \tau_{ij}}{\partial x_j}\right),$$

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are Hilbert spaces endowed with the following inner products

$$(u,v)_H = \int_\Omega u_i v_i\, dx, \qquad (\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma : \tau\, dx,$$
$$(u,v)_{H_1} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \qquad (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (\mathrm{Div}\,\sigma, \mathrm{Div}\,\tau)_H.$$

The associated norms in H, $\mathcal{H}$, H_1, $\mathcal{H}_1$ will be denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively.

Given $v \in H_1$ we denote by v its trace γv on Γ, where $\gamma : H^1(\Omega;\mathbb{R}^m) \to H^{1/2}(\Gamma;\mathbb{R}^m) \subset L^2(\Gamma;\mathbb{R}^m)$ is the Sobolev trace operator. Given $v \in H^{1/2}(\Gamma;\mathbb{R}^m)$ we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ, that is $v_\nu = v \cdot \nu$ and $v_\tau = v - v_\nu \nu$. Similarly, for a regular tensor field $\sigma : \Omega \to \mathcal{S}_m$, we define its normal and tangential components to be the normal and the tangential components of the Cauchy vector $\sigma\nu$, that is $\sigma_\nu = (\sigma\nu)\cdot\nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$. Recall that the following Green formula holds:

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (\mathrm{Div}\,\sigma, v)_H = \int_\Gamma \sigma\nu \cdot v\, d\Gamma, \qquad \text{for all } v \in H_1. \tag{4.1}$$

We shall describe next the model for which we shall derive a variational formulation. Let us consider body $\mathcal{B}$ made of a piezoelectric material which initially occupies an open bounded subset $\Omega \subset \mathbb{R}^m$ $(m = 2,3)$ with smooth a boundary $\partial\Omega = \Gamma$. The body is subjected to volume forces of density f_0 and has volume electric charges of density q_0, while on the boundary we impose mechanical and electrical constraints. In order to describe these constraints we consider two partitions of Γ: the first partition is given by three mutually disjoint open parts Γ_1, Γ_2 and Γ_3 such that meas$(\Gamma_1 > 0)$ and the second partition consists of three disjoint open parts Γ_a, Γ_b and Γ_c such that meas$(\Gamma_a) > 0$, $\Gamma_c = \Gamma_3$ and $\overline{\Gamma}_a \cup \overline{\Gamma}_b = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$. The body is clamped on Γ_1 and a surface traction of density f_2 acts on Γ_2. Moreover, the electric potential vanishes on Γ_a and a surface electric charge of density q_b is applied on Γ_b. On $\Gamma_3 = \Gamma_c$ the body comes in frictional contact with a conductive obstacle, called foundation which has the electric potential φ_F.

Denoting by $u : \Omega \to \mathbb{R}^m$ the displacement field, by $\varepsilon(u) = (\varepsilon_{ij}(u))$ the strain tensor, by $\sigma : \Omega \to \mathcal{S}_m$ the stress tensor, by $D : \Omega \to \mathbb{R}^m$, $D = (D_i)$ the electric displacement field and by $\varphi : \Omega \to \mathbb{R}$ the electric potential we can now write the strong formulation of the problem which describes the above process:

$(\mathcal{P}_M)$ Find a displacement field $u : \Omega \to \mathbb{R}^m$ and an electric potential $\varphi : \Omega \to \mathbb{R}$ such that

$$\mathrm{Div}\,\sigma + f_0 = 0 \ \text{ in } \Omega, \tag{4.2}$$
$$\mathrm{div}\,D = q_0 \ \text{ in } \Omega, \tag{4.3}$$
$$\sigma = \mathcal{E}\varepsilon(u) + \mathcal{P}^\intercal \nabla\varphi \ \text{ in } \Omega, \tag{4.4}$$
$$D = \mathcal{P}\varepsilon(u) - \mathcal{B}\nabla\varphi \ \text{ in } \Omega, \tag{4.5}$$
$$u = 0 \ \text{ on } \Gamma_1, \tag{4.6}$$
$$\varphi = 0 \ \text{ on } \Gamma_a, \tag{4.7}$$
$$\sigma\nu = f_2 \ \text{ on } \Gamma_2, \tag{4.8}$$
$$D \cdot \nu = q_b \ \text{ on } \Gamma_b, \tag{4.9}$$
$$-\sigma_\nu = S \ \text{ on } \Gamma_3, \tag{4.10}$$
$$-\sigma_\tau \in \partial_2 j(x, u_\tau) \ \text{ on } \Gamma_3, \tag{4.11}$$
$$D \cdot \nu \in \partial_2 \phi(x, \varphi - \varphi_F) \ \text{ on } \Gamma_3. \tag{4.12}$$

We point out the fact that once the displacement field u and the electric potential φ are determined, the stress tensor σ and the electric displacement field D can be obtained via relations (4.4) and (4.5), respectively.

Let us now provide explanation of the equations and the conditions (4.2)-(4.12) in which, for simplicity, we have omitted the dependence of the functions on the spatial variable x.

First, equations (4.2)-(4.3) are the governing equations consisting of the equilibrium conditions, while equations (4.4)-(4.5) represent the electro-elastic constitutive law.

We assume that $\mathcal{E} : \Omega \times \mathcal{S}_m \to \mathcal{S}_m$ is a nonlinear elasticity operator, $\mathcal{P} : \Omega \times \mathcal{S}_m \to \mathbb{R}^m$ and $\mathcal{P}^\intercal : \Omega \times \mathbb{R}^m \to \mathcal{S}_m$ are the piezoelectric operator (third order tensor field) and its transpose, respectively and $\mathcal{B} : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ denotes the electric permittivity operator (second order tensor field) which is considered to be linear. The tensors $\mathcal{P}$ and $\mathcal{P}^\intercal$ satisfy the equality

$$\mathcal{P}\tau \cdot \zeta = \tau : \mathcal{P}^\intercal \zeta, \qquad \text{for all } \tau \in \mathcal{S}_m \text{ and all } \zeta \in \mathbb{R}^m$$

and the components of the tensor $\mathcal{P}^\intercal$ are given by $p^\intercal_{ijk} = p_{kij}$.

When $\tau \rightsquigarrow \mathcal{E}(x,\tau)$ is linear, $\mathcal{E}(x,\tau) = \mathcal{C}(x)\tau$ with the elasticity coefficients $\mathcal{C} = (c_{ijkl})$ which may be functions indicating the position in a nonhomogeneous material. The decoupled state can be obtained by taking $p_{ijk} = 0$, in this case we have purely elastic and purely electric deformations.

Conditions (4.6) and (4.7) model the fact that the displacement field and the electrical potential vanish on Γ_1 and Γ_a, respectively, while conditions (4.8) and (4.9) represent the traction and the electric boundary conditions showing that the forces and the electric charges are prescribed on Γ_2 and Γ_b, respectively.

Conditions (4.10)-(4.12) describe the contact, the frictional and the electrical conductivity conditions on the contact surface Γ_3, respectively. Here, S is the normal load imposed on Γ_3, the functions $j : \Gamma_3 \times \mathbb{R}^m \to \mathbb{R}^m$ and $\phi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ are prescribed and φ_F is the electric potential of the foundation.

The strong formulation of problem $(\mathcal{P}_M)$ consists in finding $u : \Omega \to \mathbb{R}^m$ and $\varphi : \Omega \to \mathbb{R}$ such that (4.2)-(4.12) hold. However, it is well known that, in general, the strong formulation of a contact problem does not admit any solution. Therefore, we reformulate problem $(\mathcal{P}_M)$ in a weaker sense, i.e. we shall derive its variational formulation. With this end in view, we introduce the functional spaces for the displacement field and the electrical potential

$$V = \left\{ v \in H^1(\Omega;\mathbb{R}^m) : \ v = 0 \text{ on } \Gamma_1 \right\}, \qquad W = \left\{ \varphi \in H^1(\Omega) : \ \varphi = 0 \text{ on } \Gamma_a \right\}$$

which are closed subspaces of H_1 and $H^1(\Omega)$. We endow V and W with the following inner products and the corresponding norms

$$\begin{aligned} (u,v)_V &= (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}\,, \qquad && \|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}} \\ (\varphi,\chi)_W &= (\nabla\varphi, \nabla\chi)_H\,, \qquad && \|\chi\|_W = \|\nabla\chi\|_H \end{aligned}$$

and conclude that $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ are Hilbert spaces.

Assuming sufficient regularity of the functions involved in the problem, using the Green formula (4.1), the relations (4.2)-(4.12), the definition of the Clarke generalized gradient and the equality

$$\int_{\Gamma_3} (\sigma\nu)\cdot v \, d\Gamma = \int_{\Gamma_3} \sigma_\nu v_\nu \, d\Gamma + \int_{\Gamma_3} \sigma_\tau \cdot v_\tau \, d\Gamma$$

we obtain the following variational formulation of problem $(\mathcal{P}_M)$ in terms of the displacement field and the electric potential:

$(\mathcal{P}_V)$ Find $(u,\varphi) \in V \times W$ such that for all $(v,\chi) \in V \times W$

$$(\mathcal{E}\varepsilon(u), \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} + (\mathcal{P}^\intercal \nabla\varphi, \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} + \int_{\Gamma_3} j^0_{,2}(x, u_\tau; v_\tau - u_\tau)\, d\Gamma \ \geq \ (f, v-u)_V \tag{4.13}$$

$$(\mathcal{B}\nabla\varphi, \nabla\chi - \nabla\varphi)_H - (\mathcal{P}\varepsilon(u), \nabla\chi - \nabla\varphi)_H + \int_{\Gamma_3} \phi^0_{,2}(x, \varphi - \varphi_F; \chi - \varphi)\, d\Gamma \ \geq \ (q, \chi - \varphi)_W\,, \tag{4.14}$$

where $f \in V$ and $q \in W$ are the elements given by the Riesz's representation theorem as follows

$$(f, v-u)_V = \int_\Omega f_0 \cdot v \, dx + \int_{\Gamma_2} f_2 \cdot v \, d\Gamma - \int_{\Gamma_3} S v_\nu \, d\Gamma,$$

$$(q,\chi)_W = \int_\Omega q_0 \chi \, dx - \int_{\Gamma_b} q_2 \chi \, d\Gamma.$$

In the study of problem $(\mathcal{P}_V)$ we shall assume fulfilled the following hypotheses:

($\mathcal{H}_\mathcal{E}$) The elasticity operator $\mathcal{E} : \Omega \times \mathcal{S}_m \to \mathcal{S}_m$ such that

(i) $x \rightsquigarrow \mathcal{E}(x, \tau)$ is measurable for all $\tau \in \mathcal{S}_m$;

(ii) $\tau \rightsquigarrow \mathcal{E}(x, \tau)$ is continuous for almost every $x \in \Omega$;

(iii) there exist $c_1 > 0$ and $\alpha \in L^2(\Omega)$ such that $\|\mathcal{E}(x, \tau)\|_{\mathcal{S}_m} \leq c(\alpha(x) + \|\tau\|_{\mathcal{S}_m})$ for all $\tau \in \mathcal{S}_m$ and almost every $x \in \Omega$;

(iv) $\tau \rightsquigarrow \mathcal{E}(x, \tau) : (\sigma - \tau)$ is weakly upper semicontinuous for all $\sigma \in \mathcal{S}_m$ and almost every $x \in \Omega$;

(v) there exists $c_2 > 0$ such that $\mathcal{E}(x, \tau) : \tau \geq c\|\tau\|^2_{\mathcal{S}_m}$ for all $\tau \in \mathcal{S}_m$ and almost every $x \in \Omega$.

($\mathcal{H}_\mathcal{P}$) The piezoelectric operator $\mathcal{P} : \Omega \times \mathcal{S}_m \to \mathbb{R}^m$ is such that

(i) $\mathcal{P}(x, \tau) = p(x)\tau$ for all $\tau \in \mathcal{S}_m$ and almost every $x \in \Omega$;

(ii) $p(x) = (p_{ijk}(x))$ with $p_{ijk} = p_{ikj} \in L^\infty(\Omega)$.

($\mathcal{H}_\mathcal{B}$) $\mathcal{B} : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is such that

(i) $\mathcal{B}(x, \zeta) = \beta(x)\zeta$ for all $\zeta \in \mathbb{R}^m$ and almost $x \in \Omega$;

(ii) $\beta(x) = (\beta_{ij}(x))$ with $\beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$;

(iii) there exists $m > 0$ such that $(\beta(x)\zeta) \cdot \zeta \geq m|\zeta|^2$ for all $\zeta \in \mathbb{R}^m$ and almost every $x \in \Omega$.

($\mathcal{H}_j$) $j : \Gamma_3 \times \mathbb{R}^m \to \mathbb{R}$ is such that

(i) $x \rightsquigarrow j(x, \zeta)$ is measurable for all $\zeta \in \mathbb{R}^m$;

(ii) $\zeta \rightsquigarrow j(x, \zeta)$ is locally Lipschitz for almost every $x \in \Gamma_3$;

(iii) there exist $c_3 > 0$ such that $|\partial_2 j(x, \zeta)| \leq c_3(1 + |\zeta|)$ for all $\zeta \in \mathbb{R}^m$ and almost every $x \in \Gamma_3$;

(iv) there exists $c_4 > 0$ such that $j^0_{,2}(x, \zeta; -\zeta) \leq c_4|\zeta|$ for all $\zeta \in \mathbb{R}^m$ and almost every $x \in \Gamma_3$;

(v) $\zeta \rightsquigarrow j(x, \zeta)$ is regular for almost every $x \in \Gamma_3$.

($\mathcal{H}_\phi$) $\phi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $x \rightsquigarrow \phi(x, t)$ is measurable for all $t \in \mathbb{R}$;

(ii) $\zeta \rightsquigarrow \phi(x, \zeta)$ is locally Lipschitz for almost every $x \in \Gamma_3$;

(iii) there exist $c_5 > 0$ such that $|\partial_2 \phi(x, t)| \leq c_5|t|$ for all $t \in \mathbb{R}$ and almost every $x \in \Gamma_3$;

(iv) $t \rightsquigarrow \phi(x, t)$ is regular for almost every $x \in \Gamma_3$.

($\mathcal{H}_{f,q}$) $f_0 \in H$, $f_2 \in L^2(\Gamma_2; \mathbb{R}^m)$, $q_0 \in L^2(\Omega)$, $q_b \in L^2(\Gamma_2)$, $S \in L^\infty(\Gamma_3)$, $S \geq 0$, $\varphi_F \in L^2(\Gamma_3)$.

The main result of this subsection is given by the following theorem.

Theorem 4.1. *Assume fulfilled conditions* ($\mathcal{H}_\mathcal{E}$), ($\mathcal{H}_\mathcal{P}$), ($\mathcal{H}_\mathcal{B}$), ($\mathcal{H}_j$), ($\mathcal{H}_\phi$) *and* ($\mathcal{H}_{f,q}$). *Then problem* ($\mathcal{P}_V$) *admits at least one solution.*

Proof. We observe that problem ($\mathcal{P}_V$) is in fact a system of two coupled hemivariational inequalities. The idea is to apply one of the existence results obtained in Section 2. with suitable choice of ψ_k, J, and F_k ($k \in \{1, 2\}$).

First, let us take $n = 2$ and define $X_1 = V$, $X_2 = W$, $Y_1 = L^2(\Gamma_3; \mathbb{R}^m)$, $Y_2 = L^2(\Gamma_3)$, $K_1 = X_1$ and $K_2 = X_2$. Next we introduce $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$ defined by

$$T_1 = i_\tau \circ \gamma_m \circ i_m|_{\Gamma_3}, \qquad T_2 = \gamma \circ i|_{\Gamma_3},$$

where $i_m : V \to H_1 = H^1(\Omega; \mathbb{R}^m)$ is the embedding operator, $\gamma_m : H_1 \to H^{1/2}(\Gamma; \mathbb{R}^m)$ is the Sobolev trace operator, $i_\tau : H^{1/2}(\Gamma; \mathbb{R}^m) \to L^2(\Gamma_3; \mathbb{R}^m)$ is the operator defined by $i_\tau(v) = v_\tau$, $i : W \to H^1(\Omega)$ is the embedding operator and $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$ is the Sobolev trace operator. Clearly T_1 and T_2 are linear and compact operators. We consider next $\psi_1 : X_1 \times X_2 \times X_1 \to \mathbb{R}$ and $\psi_2 : X_1 \times X_2 \times X_2 \to \mathbb{R}$ defined by

$$\psi_1(u, \varphi, v) = (\mathcal{E}\varepsilon(u), \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} + (\mathcal{P}^{\mathsf{T}}\nabla\varphi, \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} ,$$

$$\psi_2(u, \varphi, \chi) = (\mathcal{B}\nabla\varphi, \nabla\chi - \nabla\varphi)_H - (\mathcal{P}\varepsilon(u), \nabla\chi - \nabla\varphi)_H ,$$

$J : Y_1 \times Y_2 \to \mathbb{R}$ defined by

$$J(w, \eta) = \int_{\Gamma_3} j(x, w(x))\, d\Gamma + \int_{\Gamma_3} \phi(x, \eta(x) - \varphi(x))\, d\Gamma,$$

and $F_1 : X_1 \times X_2 \to X_1^*$ and $F_2 : X_1 \times X_2 \to X_2^*$ defined by

$$F_1(u, \varphi) = f, \qquad F_2(u, \varphi) = q.$$

It is easy to check from the above definitions that if $(\mathcal{H}_{\mathcal{E}})$, $(\mathcal{H}_{\mathcal{P}})$, $(\mathcal{H}_{\mathcal{B}})$, hold, then the functionals ψ_1, ψ_2 satisfy conditions (H1) and (H6). Taking $(\mathcal{H}_j)$ and $(\mathcal{H}_\phi)$ into account we conclude that J is a regular locally Lipschitz functional which satisfies

$$J^0_{,1}(w, \eta; z) = \int_{\Gamma_3} j^0_{,2}(x, w(x); z(x))\, d\Gamma$$

$$J^0_{,2}(w, \eta; \zeta) = \int_{\Gamma_3} \phi^0_{,2}(x, \eta(x) - \varphi(x); \zeta(x))\, d\Gamma.$$

Obviously conditions (H2), (H7), (H8) are fulfilled, therefore we can apply Corollary 3.2 to conclude that problem $(\mathcal{P}_V)$ admits at least one solution.

□

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