

Coverings and Dimensions in Infinite Profinite Groups

Peter Maga

July 12, 2009

MSC Codes: Primary: 22C05; Secondary 03E17, 03E35.

Key words: profinite group, box dimension, Hausdorff dimension, translate, cover, continuum, consistent.

Abstract

Answering a question of Miklós Abért, we prove that an infinite profinite group cannot be the union of less than continuum many translates of a compact subset of box dimension less than 1. Furthermore, we show that it is consistent with the axioms of set theory that in any infinite profinite group there exists a compact subset of Hausdorff dimension 0 such that one can cover the group by less than continuum many translates of it.

Introduction

The following question arose recently. If a compact subset of reals is small in a sense, does it follow that the real line cannot be the union of less than continuum many translates of it?

Of course, the answer depends on our notion "small". U. B. Darji and T. Keleti [4] proved that if a compact subset $C \subseteq \mathbb{R}$ has packing dimension (or box dimension) less than 1, then \mathbb{R} is not the union of less than continuum many translates of C .

Assuming the continuum hypothesis, if C has Lebesgue measure zero, then \mathbb{R} can only be covered by continuum many translates of C . However, M. Elekes and J. Steprans [5] showed that it is consistent with ZFC (axioms of the set theory, including the Axiom of Choice) that there exists a compact set $C \subseteq \mathbb{R}$ of Lebesgue measure zero such that \mathbb{R} can be covered with less than continuum many translates of C . A. Máthé [8] proved that it is consistent that there exists a set with the same properties and Hausdorff dimension 0.

M. Elekes and Á. Tóth [6] proved that the above-mentioned consistency (the existence of a compact subset C of Haar measure 0 such that less than continuum many translates of C can cover the whole space) holds in the case of commutative Polish groups. They also reduced the noncommutative case to the investigation of Lie groups and profinite groups.

M. Abért [1] showed the consistency in infinite profinite groups. In the same article he asked whether similar theorems to the result of Darji and Keleti can hold in the case of infinite profinite groups and box dimension: is it true that if a compact subset has box dimension less than 1, then the group cannot be covered with less than continuum many translates of it?

In this paper we prove the following results. First, applying the method due to Darji and Keleti, we show that if a compact subset of an infinite profinite group has box dimension less than 1, then the group is not the union of less than continuum many translates of it.

Then, we show that it is consistent with ZFC that in any infinite profinite group there exists a set of Hausdorff-dimension 0 such that the group is the union of less than continuum many translates of it. Our technique is based on A. Máthé's [8], who proved the theorem in the case of reals.

Note that although the two notions of dimension differ, they coincide in the case of closed subgroups [2].

The following definition of profinite groups ensures that we work on infinite profinite groups.

Definition. *Let G_1, G_2, \dots be finite groups and let $\varphi_k : G_{k+1} \rightarrow G_k$ be onto, but not one-to-one homomorphisms. Then they determine the profinite group G , whose elements are*

$$\{(g_1, g_2, \dots) \mid \forall k \in \mathbb{N} : g_k \in G_k, \varphi_k(g_{k+1}) = g_k\},$$

while the multiplication of G is defined as the product of the multiplication in coordinates:

$$(g_1, g_2, \dots) \cdot_G (h_1, h_2, \dots) = (g_1 \cdot_{G_1} h_1, g_2 \cdot_{G_2} h_2, \dots).$$

For any $x \in G$, we denote by $x_{|k}$ the k -th coordinate of x . For any $X \subseteq G$, let $X_{|k} = \{x_{|k} \mid x \in X\}$.

Note that other definitions exist. Generally, one can handle finite groups as profinite groups (we can get this by dropping "but not one-to-one"). Throughout this paper, profinite group means infinite profinite group.

Let us make the following remark. If we consider only an infinite subsequence of (G_n) and the compositions of the corresponding homomorphisms, we obtain a new profinite group that is isomorphic to the original one (and we obtain the isomorphism, as well). Applying these subsequences will be proved very useful: without changing group theoretic properties, we will be able to change other properties.

From now on, we follow [2] in terminology. We introduce the natural metric on G :

Definition. *If $x, y \in G$ such that $x \neq y$, then let $d(x, y) = \frac{1}{|G_k|}$, if $x_{|j} = y_{|j}$ for all $j < k$ and $x_{|k} \neq y_{|k}$. Let $d(x, x) = 0$.*

One can readily check that this metric makes G a Polish space. Although this metric can be changed, when we switch to a subsequence of (G_n) , the topology cannot.

1 Box-dimension

Notation. Let $|S|$ denote the cardinality of set S .

Definition. Let $X \subseteq G$. Then let

$$\underline{\dim}_B(X) = \liminf_{k \rightarrow \infty} \frac{\log |X|_k|}{\log |G|_k|}; \quad \overline{\dim}_B(X) = \limsup_{k \rightarrow \infty} \frac{\log |X|_k|}{\log |G|_k|}$$

the lower and upper box-dimension of X , respectively. If $\underline{\dim}_B(X) = \overline{\dim}_B(X)$, then we say that $\dim_B(X)$ exists and its value is $\underline{\dim}_B(X) = \overline{\dim}_B(X)$. Throughout this section, dimension means box-dimension.

If G is a given profinite group, we can consider its n -th power G^n . We can define the lower and upper box-dimensions for the subsets of $G^n = G \times \dots \times G$ as above, except for the denominator, in which we still divide by $\log |G|_k|$. Hence for example, $\dim_B(G^n) = n$. The metric of G^n is defined as follows:

$$d_n((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max(d(x_1, y_1), \dots, d(x_n, y_n)).$$

Let $X, Y \subseteq G$. One can readily check that $\underline{\dim}_B(X) + \underline{\dim}_B(Y) \leq \underline{\dim}_B(X \times Y)$, $\overline{\dim}_B(X) + \overline{\dim}_B(Y) \geq \overline{\dim}_B(X \times Y)$. Hence if $\dim_B(X), \dim_B(Y)$ exist, then $\dim_B(X) + \dim_B(Y) = \dim_B(X \times Y)$.

Lemma 1.1 *If the interior of $X \subseteq G$ is nonempty, then $\dim_B(X) = 1$. If the interior of $X \subseteq G^n$ is nonempty, then $\dim_B(X) = n$.*

Proof. It is sufficient to verify the 1-dimensional statement. Let $x = (x_1, x_2, \dots)$ be an interior point of X . Then for some $k \in \mathbb{N}$, $(x_1, \dots, x_k, g_{k+1}, g_{k+2}, \dots) \in X$ for all $g_{k+1} \in G_{k+1}, g_{k+2} \in G_{k+2}, \dots$. Then for all $l \geq k$, $|X|_l| \geq \frac{|G|_l|}{|G|_k|}$. Hence $\frac{\log |X|_l|}{\log |G|_l|} \rightarrow 1$, since $|G|_l| \rightarrow \infty$. \square

If $X \subseteq G$, then let

$$X_*^n = \{(x_1, \dots, x_n) \in X^n \mid i \neq j \Rightarrow x_i \neq x_j\}.$$

Let $F_n : G^{n+1} \rightarrow G^n$ be the following function:

$$F_n(x_1, \dots, x_n, g) = (x_1g, \dots, x_ng).$$

It is easy to see that $d(F_n(x), F_n(y)) \leq d(x, y)$, which implies that F is continuous. One can see that F_n cannot increase the lower and upper dimensions, since $|X|_k| \geq |F(X)|_k|$.

Lemma 1.2 *Let $X \subseteq G$ such that $F_n(X^n \times G) \cap P_*^n = \emptyset$. Then for all $g \in G$, $|Xg \cap P| \leq n - 1$. If $|P|$ is continuum, then G cannot be covered with less than continuum many right-translates of X . (Right-translate of X is Xg .)*

Proof. Let us suppose that the first claim is false. Then $x_1g = y_1, \dots, x_ng = y_n$ are different elements of P , where $x_1, \dots, x_n \in X$. Hence $F_n(x_1, \dots, x_n, g) \in P_*^n$ and this is a contradiction. The second statement is clear from the first one. \square

Lemma 1.3 *Let $F \subseteq G^n$ be compact, nowhere dense. Then there exists a perfect subset $P \subseteq G$ such that $|P|$ is continuum and $F \cap P_*^n = \emptyset$.*

Proof. First, define a decreasing sequence (U_k) of open sets.

Let $U_0 = G$, which is open. In general, the open set U_k is the union of n^k open component. The components of U_{k+1} will be chosen such that they will satisfy the following properties: each component of U_k contains n open components of U_{k+1} ; every component of U_{k+1} has diameter at most $\frac{1}{2^k}$; the closure of every component of U_{k+1} is in U_k . Denote these components of U_{k+1} by $V_1, \dots, V_{n^{k+1}}$. If x_1, \dots, x_n are elements of pairwise different V_{j_1}, \dots, V_{j_n} , then $(x_1, \dots, x_n) \notin F$.

First, take some $V_1, \dots, V_{n^{k+1}}$ such that they are pairwise disjoint; the diameter of every V_j is at most $\frac{1}{2^k}$; their closure are in U_k ; every component of U_k contains exactly n of $V_1, \dots, V_{n^{k+1}}$. Next shrink these subsets. Let $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n^{k+1}\}$ be an injective function. Let $V'_1 \subseteq V_1, \dots, V'_{n^{k+1}} \subseteq V_{n^{k+1}}$ such that $V'_{\pi(1)} \times \dots \times V'_{\pi(n)} \cap F = \emptyset$ (there exist such subsets, because F is not dense in $V_{\pi(1)} \times \dots \times V_{\pi(n)}$). From this point on, denote $V'_1, \dots, V'_{n^{k+1}}$ by $V_1, \dots, V_{n^{k+1}}$. Repeat this shrinking operation for all injective function $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n^{k+1}\}$. Finally, we obtain a subset of $U_k = V_1 \cup \dots \cup V_{n^{k+1}}$ that satisfies every condition, let it be U_{k+1} .

Let $P = \bigcap_{k=0}^{\infty} U_k$. It is easy to check that P has all the properties we were looking for. \square

Lemma 1.4 *Let $X \subseteq G$ be compact such that $\overline{\dim}_B(X) < 1$. Then G cannot be covered with less than continuum many right-translates of X .*

Proof. Let $n \geq 2$ such that $n \overline{\dim}_B(X) < n - 1$, which implies $\overline{\dim}_B(X^n) \leq n \overline{\dim}_B(X) < n - 1$. Since $X^n \times G \subseteq G^{n+1}$ and $\overline{\dim}_B(X^n \times G) < n$, $\overline{\dim}_B(F_n(X^n \times G)) < n$. By Lemma 1.1, the interior of the compact set $F_n(X^n \times G) \subseteq G^n$ is empty, then $F_n(X^n \times G)$ is nowhere dense. Then by Lemma 1.3, for an appropriate perfect set $P \subseteq G$ of cardinality continuum, $F_n(X^n \times G) \cap P_*^n = \emptyset$. By Lemma 1.2, we are done. \square

By switching to a well-chosen subsequence of (G_n) , we can strengthen our theorem to lower dimension.

Theorem 1.5 *Let $X \subseteq G$ be compact such that $\underline{\dim}_B(X) < 1$. Then G cannot be covered with less than continuum many right-translates of X .*

Proof. Let us choose a subsequence of G_n . The obtained group, denoted by \tilde{G} is isomorphic to G . Let X be a given compact subset of G such that $\underline{\dim}_B(X) < 1$. By definition, for a convenient subsequence, $\widetilde{\underline{\dim}_B(X)} = \underline{\dim}_B(X) < 1$ holds. Since less than continuum many right-translates of \tilde{X} cannot cover \tilde{G} , less than continuum many right-translates of X cannot cover G . \square

2 Hausdorff-dimension

We define the Hausdorff-dimension following [2].

Definition. Let $X \subseteq G$. Then

$$\mu_\delta^s(X) = \inf \left\{ \sum_{n=1}^{\infty} (\text{diam}(A_n))^s \mid \bigcup_{n=1}^{\infty} A_n \supseteq X, \text{diam}(A_n) < \delta \right\},$$

$$\mu^s(X) = \lim_{\delta \rightarrow 0} \mu_\delta^s(X).$$

Furthermore,

$$\dim_{\text{H}}(X) = \sup\{t \mid \mu^t(X) = \infty\} (= \inf\{t \mid \mu^t(X) = 0\})$$

is the Hausdorff-dimension of X . In this section, dimension means Hausdorff-dimension.

We prove the following theorem.

Theorem 2.1 Let G be a profinite group. It is consistent with ZFC that there exists a compact subset X of Hausdorff-dimension 0 such that G can be covered with less than continuum many right-translates of X .

Notation. Let $N_j \triangleleft G$ be the following normal subgroup: the set of those elements, whose j -th coordinate is 1 (the unit element of the corresponding group).

Lemma 2.2 Let $n, k \in \mathbb{N}$. Then there exists a compact set $K_{n,k} \subseteq G$ with the following properties. It is the union of closed subsets F_i such that $\sum_i \text{diam}(F_i)^{\frac{1}{k}} < \frac{1}{k}$; for arbitrary $x_1, \dots, x_n \in G$ there exists a $g \in G$ such that $x_1g, \dots, x_ng \in K_{n,k}$; $K_{n,k}$ is the union of a normal subgroup N_j and some of its cosets.

Proof. The statement clearly holds for $n = 1$. Let $K_{1,k} = N_j$ be a sufficiently small normal subgroup of G . In the following, we define the subsets $K_{n,k}$ by induction.

Fix $k \in \mathbb{N}$. How can we define $K_{n,k}$? First, consider $K_{n-1,k}$. Our plan is to obtain $K_{n,k}$ by adding some subsets to $K_{n-1,k}$. Assume that $K_{n-1,k}$ consists of a normal subgroup N_j and some of its cosets. For some closed subsets F_i , $K_{n-1,k} = \cup_i F_i$ and

$$\sum_i \text{diam}(F_i)^{\frac{1}{k}} < \frac{1}{k} - \varepsilon.$$

Take an element from each coset of N_j , we denote them by h_1, \dots, h_m . Then take some closed balls B_1, \dots, B_m such that $h_1 \in B_1, \dots, h_m \in B_m$ and

$$\sum_{i=1}^m \text{diam}(B_i)^{\frac{1}{k}} < \varepsilon.$$

Clearly, we can choose the balls B_1, \dots, B_m such that they are all cosets of a normal subgroup $N_{j'}$ for some $j' > j$. We claim that $K_{n,k} = \cup_i B_i \cup K_{n-1,k}$ has all the properties we need. First, we clearly have

$$\sum_i \text{diam}(F_i)^{\frac{1}{k}} + \sum_{i=1}^m \text{diam}(B_i)^{\frac{1}{k}} < \frac{1}{k}.$$

Let x_1, \dots, x_n be arbitrary elements of G . For elements x_1, \dots, x_{n-1} , there is a $g \in G$ such that $x_1g, \dots, x_{n-1}g \in K_{n-1,k}$. Then x_ng is in one of the cosets corresponding to N_j , which implies that for an appropriate element $g' \in N_j$, $x_n gg' = h_i \in B_i$. Furthermore, multiplication with g' from the right ($x \mapsto xg'$) maps each coset of N_j to itself, hence gg' is an appropriate element: $x_1 gg', \dots, x_n gg' \in K_{n,k}$. Moreover, $K_{n,k}$ is the union of $N_{j'}$ and some of its cosets. \square

Corollary 2.3 *If $n, k \in \mathbb{N}$, N_j is a given normal subgroup and H is a coset of N_j , then there exists a subset $K_{n,k}^H \subseteq H$ satisfying the following properties. It is the union of closed subsets F_i such that $\sum_i \text{diam}(F_i)^{\frac{1}{k}} < \frac{1}{k}$; for arbitrary $x_1, \dots, x_n \in H$ there exists a $g \in N_j$ such that $x_1g, \dots, x_n g \in K_{n,k}^H$; $K_{n,k}^H$ is the union of some cosets corresponding to a normal subgroup $N_{j'}$ ($j' > j$).*

Proof. Consider $K_{n,k}$. We can translate each coset of N_j into H : $H_1 h_1 = \dots = H_m h_m = H$. Let

$$K_{n,k}^H = \bigcup_{i=1}^m (K_{n,k} \cap H_i) h_i.$$

One can easily check that the defined set satisfies the conditions. \square

Definition. *Let $f : \mathbb{N} \rightarrow \mathbb{N}$. Then we call a set $\prod_{n=1}^{\infty} A_n$ f -slalom, if $|A_n| \leq f(n)$ for all $n \in \mathbb{N}$. Let f and an f -slalom $\prod_{n=1}^{\infty} A_n$ be given. We say that $g : \mathbb{N} \rightarrow \mathbb{N}$ is a sub-slalom-size, if $g(n) \leq f(n)$ holds for all $n \in \mathbb{N}$. We call a set $\prod_{n=1}^{\infty} B_n$ g -sub-slalom, if it is a g -slalom and $B_n \subseteq A_n$ for all $n \in \mathbb{N}$.*

Let $f, g : \mathbb{N} \rightarrow \mathbb{N}$, where $g(i) < f(i)$. Assume that $f(i)$ and $g(i)$ tend to infinity. It is consistent with ZFC that every f -slalom can be covered with less than continuum many g -sub-slaloms: the method is described in [5, page 9 – 10.], where the authors use [7, Theorem 2.10.] and [3, Theorem 2.3.9. and page 388.]

Our plan is the following. We define an f -slalom that corresponds to G and introduce a g sub-slalom-size that tends to infinity. Using the sets $K_{n,k}$, we construct a zero-dimensional compact subset $C \subseteq G$ such that any g -sub-slalom can be right-translated into C . Since it is consistent that G is the union of less than continuum many g -sub-slalom, it is also consistent that G can be covered with less than continuum many right-translates of C .

Construction.

Consider the set $\tilde{K}_{1,1} = K_{1,1}$ given by Lemma 2.2. Assume that it consists of a single normal subgroup N_j .

Assume that we are on the m -th level and we have already defined the subset $\tilde{K}_{(m-1)!, (m-1)!}$ that consists of some cosets of the same size (corresponding to the same normal subgroup). Denote one of these cosets by H . Then $\tilde{K}_{(m-1)!, (m-1)!}$ consists of some appropriate left-translates of H : $H = H_1, \dots, H_{S(m-1)}$. Hence for appropriate elements $1 = h_1, \dots, h_{S(m-1)}$, $H = h_1 H_1 = \dots = h_{S(m-1)} H_{S(m-1)}$. In H , take the subset $K_{m!, m! S(m-1)}^H$ given by Corollary 2.3. Then apply every h_i^{-1} and

let

$$\tilde{K}_{m!,m!} = \bigcup_{i=1}^{S(m-1)} h_i^{-1} K_{m!,m!S(m-1)}^H.$$

By this induction, we obtain a set $\tilde{K}_{m!,m!}$ for all m .

Then $(\tilde{K}_{m!,m!})$ is a sequence of decreasing compact sets. Let

$$C = \bigcap_{m=1}^{\infty} \tilde{K}_{m!,m!}.$$

Lemma 2.4 *For the constructed set C , $\dim_{\mathbb{H}}(C) = 0$ holds.*

Proof. By the definition of $K_{m!,m!S(m-1)}^H$, there exist some closed sets F_i such that $K_{m!,m!S(m-1)}^H = \cup_i F_i$ and

$$\sum_i \text{diam}(F_i)^{\frac{1}{m!S(m-1)}} < \frac{1}{m!S(m-1)}.$$

Since $\tilde{K}_{m!,m!}$ is the union of $S(m-1)$ copies of $K_{m!,m!S(m-1)}^H$, it is covered with the union of closed sets F_i such that

$$\sum_i \text{diam}(F_i)^{\frac{1}{m!S(m-1)}} < \frac{1}{m!}.$$

Then, by increasing the exponent ($\text{diam}(G) \leq 1$),

$$\sum_i \text{diam}(F_i)^{\frac{1}{m!}} < \frac{1}{m!},$$

which implies that C is zero-dimensional. \square

Defining the slalom corresponding to G .

Consider the following subsequence of (G_n) . Let N_{m_k} be a strictly decreasing sequence of the normal subgroups such that $\tilde{K}_{m!,m!}$ consists of cosets corresponding to N_{m_k} . We can switch to any infinite subsequence of (G_n) . Choose a subsequence (denoted by (G_m)) such that the following two conditions hold: $\tilde{K}_{m!,m!}$ consists of cosets corresponding to N_m ; $|G_m/G_{m-1}| > m$ (where G_0 is the trivial group). From this point on, we use this sequence, which defines G , as well. Recall that the topology is the same, however the metric could change (let the new metric be d'). Let $f(m) = |G_m/G_{m-1}|$ and let $g(m) = m$. The f -slalom corresponding to G is the following: $\prod_{m=1}^{\infty} G_m/G_{m-1}$. It is indeed an f -slalom and it represents G as follows: the elements of G are in natural bijection with the elements of the product of the quotient maps φ_m ($m = 1, 2, \dots$).

Lemma 2.5 *Let Z be a g -sub-slalom. There exists a $\bar{y} \in G$ such that $Z\bar{y} \subseteq C$.*

Proof. Recall the notations of the construction. Let $\tilde{K}_{(m-1)!,(m-1)!}$ consists of the cosets $H = H_1, \dots, H_{S(m-1)}$ that correspond to N_{m-1} . Let $1 = h_1, \dots, h_{S(m-1)} \in G$ be such that $H = h_1 H_1 = \dots = h_{S(m-1)} H_{S(m-1)}$.

First, we prove by induction that for all m there exists a $y_m \in G$ such that $(Zy)_m \subseteq (\tilde{K}_{m!,m!})_m$. This is clear for $m = 1$, since $|Z_1| = 1$ and $\tilde{K}_{1!,1!}$ is nonempty. Suppose that it holds for $m - 1$. For all i , $|Z_i| \leq i!$. By induction, there exists $x \in G$ such that

$$(Zx)_{|m-1} \subseteq (\tilde{K}_{(m-1)!,(m-1)!})_{|m-1} = (H_1)_{|m-1} \cup \dots \cup (H_{S(m-1)})_{|m-1}.$$

Since $H_1, \dots, H_{S(m-1)}$ are cosets of N_{m-1} , if they contain a subset restricted to G_{m-1} , then so do in G . Therefore $(Zx)_m \subseteq (H_1)_m \cup \dots \cup (H_{S(m-1)})_m$. For all $b \in Z$, we take a left-translate of $(bx)_m \in (Zx)_m$ that is in the coset H_i the following way: for each element $(bx)_m \in (H_i)_m$, consider $(h_i bx)_m$. How many elements of this form do we have in H_i ? The number of elements of the form $(bx)_m$ is at most $m!$, since Z is a g -slalom. We left-translated all of these elements $(bx)_m$ by exactly one element h_i . Therefore we have at most $m!$ elements of the form $(h_i bx)_m$ in H_i . Then we can right-translate all element of the form $(h_i bx)_m$ into $(K_{m!,m!S(m-1)}^H)_m$ by an appropriate element $h \in G$: continue each element of the form $(h_i bx)_m$ to an element of G , these continuations can be right-translated into $K_{m!,m!S(m-1)}^H$, by its definition. Then xh is a suitable element y_m , because for any $b \in Z$ $((bx)_m \in (H_i)_m)$

$$(bxh)_m = (h_i^{-1} h_i bxh)_m \in (h_i^{-1} K_{m!,m!S(m-1)}^H)_m \subseteq (\tilde{K}_{m!,m!})_m,$$

which completes the proof of the first claim.

For all m , consider the element $y_m \in G$ such that $(Zy_m)_m \subseteq (\tilde{K}_{m!,m!})_m$. Because of the compactness of G , (y_m) has a convergent subsequence. Let its limit be \bar{y} . We claim that $Z\bar{y} \subseteq C$. We prove by contradiction. Suppose that for an element $b \in Z$, $b\bar{y} \notin C$. Let $d'(b\bar{y}, C) = D > 0$. If $m > m_0$, then $d'(by_m, b\bar{y}) < D/2$. Furthermore, if $m > m_1$, then $\tilde{K}_{m!,m!} \subseteq U_{D/2}(C)$, where $U_\varepsilon(C) = \{x \in G \mid d'(x, C) < \varepsilon\}$. Since $(by_m)_m \in (\tilde{K}_{m!,m!})_m$, $by_m \in \tilde{K}_{m!,m!}$ (because of the fact that $\tilde{K}_{m!,m!}$ is the union of complete cosets corresponding to N_m , if it contains an element restricted to G_m , then so do in G). Hence $d'(by_m, C) < D/2$. If $m > m_0, m_1$, then

$$d'(b\bar{y}, C) \leq d'(by_m, b\bar{y}) + d'(by_m, C) < D,$$

which is a contradiction. \square

These lemmas imply the theorem: by Lemma 2.4, C is zero-dimensional and by Lemma 2.5, every g -sub-slalom can be covered with an appropriate right-translate of C . Since it is consistent that G is the union of less than continuum many g -sub-slaloms, it is also consistent that G is the union of less than continuum many right-translates of C . \square

Acknowledgement

I am grateful to my supervisor, Tamás Keleti for his valuable advice and remarks and to Miklós Abért for proposing the problem.

References

- [1] M. Abért: *Less than continuum many translates of a compact nullset may cover any infinite profinite groups*, Journal of Group Theory **11** (2008), No 4., 545-553.
- [2] Y. Barnea, A. Shalev: *Hausdorff Dimension, Pro-p Groups and Kac-Moody Algebras*, Trans. Amer. Math. Soc. **349** (1997), No. 12., 5073-5091.
- [3] T. Bartosynski, H. Judah: *Set Theory: On the Structure of the Real Line*, A. K. Peters, Wellesley, Massachusetts, 1995.
- [4] U. B. Darji, T. Keleti: *Covering the real line with translates of a compact set*, Proc. Amer. Math. Soc. **131** (2003)
- [5] M. Elekes, J. Steprans: *Less than 2^ω many translates of a compact nullset may cover the real line*, Fund. Math. **181** (2004), No. 1., 89-96.
- [6] M. Elekes, Á. Tóth: *Covering locally compact groups by less than continuum many translates of a compact nullset*, Fund. Math. **193** (2007), 243-257.
- [7] G. Gruenhage, R. Levy: *Covering ${}^\omega\omega$ by special Cantor sets*, Comment, Math. Univ. Carolin. **43** (2002), No. 3., 497-509.
- [8] A. Máthé, *Covering the real line with translates of a zero dimensional compact set*, submitted to Fund. Math.

Peter Maga
Nyíregyháza, Számadó u. 1.
HUNGARY-4405