A dimension-depending multiplicity result for the Schrödinger equation

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Abstract

We consider the Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = \lambda K(x)f(u) \text{ in } \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases} \qquad (P_\lambda)$$

where $N \geq 2$, $\lambda \geq 0$ is a parameter, $V, K : \mathbb{R}^N \to \mathbb{R}$ are radially symmetric functions, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with sublinear growth at infinity. We first prove that for λ small enough no non-zero solution exists for (P_λ), while for λ large enough at least two distinct non-zero radially symmetric solutions do exist. By exploiting a Ricceri-type three-critical points theorem, the principle of symmetric criticality and a group-theoretical approach, the existence of at least $N - 3\,(N \bmod 2)$ distinct pairs of non-zero solutions is guaranteed for (P_λ) whenever λ is large enough, $N \neq 3$, and f is odd.

Keywords: Schrödinger equation, sublinear, three-critical points theorem, principle of symmetric criticality.

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1 Introduction

In this paper we consider the Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = \lambda K(x) f(u) \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P_λ}$$

where $N \geq 2$, $V, K : \mathbb{R}^N \to \mathbb{R}$ are some non-negative potentials, $\lambda \geq 0$ is a parameter, while $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function. The interest in this problem comes from mathematical physics; for instance, certain kinds of solitary waves in the nonlinear Klein-Gordon or Schrödinger equations appear as solutions of problem (P_λ).

Problem (P_λ) or its related form has been studied by many authors during the last two decades under various assumptions on the potentials V, K and on the nonlinear function f. Most of these papers address the case when V and K have suitable sign- and growth-properties, and f has a superlinear and subcritical growth. In these papers existence and multiplicity results for (P_λ) are established via various variational arguments, see Rabinowitz [9], Bartsch *et al.* [1, 2], and further subsequent papers. In particular, if f is odd, the existence of infinitely many solutions for (P_λ) is usually guaranteed. A particularly interesting paper is due to Clapp and Weth [3] where the existence of at least $\frac{N}{2} + 1$ pairs of non-zero solutions for (P_λ) is proved for every $\lambda > 0$ by assuming certain one-sided asymptotic estimates for V and K when $f(s) = |s|^{p-2}s$, $p \in (2, 2^*)$. Problem (P_λ) has been also studied in the case when f is odd and has an asymptotically linear growth at infinity; more precisely, Liu, van Heerden and Wang [7] prove a multiplicity result for (P_λ) whenever $V(x) = \mu g(x) + 1$, $\mu > 0$, and the number of solutions for (P_λ) depends on the behavior of the dimension of the eigenspace of a specific Dirichlet eigenvalue problem defined on the bounded domain $\Omega = \text{int}(g^{-1}(0))$.

The aim of the present paper is to supplement the aforementioned contributions by requiring that the non-zero continuous function $f : \mathbb{R} \to \mathbb{R}$ has a *sublinear growth at infinity* and a *superlinear growth near zero*. More precisely, we assume that

(f_1) $f(s) = o(|s|)$ as $|s| \to \infty$;

(f_2) $f(s) = o(|s|)$ as $s \to 0$;

(f_3) there exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$, where $F(s) = \int_0^s f(t)dt$.

In order to avoid technicalities, we assume in the sequel that potentials $V, K : \mathbb{R}^N \to \mathbb{R}$ satisfy

(H_V) $V \in C(\mathbb{R}^N)$ is radially symmetric and $\inf_{\mathbb{R}^N} V > 0$;

(H_K) $K \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ is radially symmetric and $K \geq 0$, $K \not\equiv 0$.

Note that solutions of (P_λ) are being sought in weak form in the space

$$W = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx < \infty \right\}.$$

In fact, under the above conditions on f, V and K which will be assumed throughout in the sequel, every weak solution u of (P_λ) is a classical one. Indeed, we have $\Delta u =: h \in L^2_{\rm loc}(\mathbb{R}^N)$, thus $u \in H^2_{\rm loc}(\mathbb{R}^N)$ (cf. Evans [4, §8.3]) and u satisfies (P_λ) for a.a. $x \in \mathbb{R}^N$.

The hypotheses (f_1), (f_2) and (f_3) guarantee that the number

$$c_f = \max_{s\neq 0}\left|\frac{f(s)}{s}\right|$$

is well-defined, positive and finite. Now, we are in a position to state our main result.

Theorem 1.1 *Assume that $N \geq 2$. Let $V, K : \mathbb{R}^N \to \mathbb{R}$ be two potentials such that both (H_V) and (H_K) hold, and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function verifying $(f_1) - (f_3)$. Then, the following assertions hold:*

(i) *For every $\lambda \in [0, c_f^{-1}\|K\|^{-1}_{L^\infty}\inf_{\mathbb{R}^N} V)$, problem (P_λ) has only the zero solution;*

(ii) *There exists $\Lambda_0 > 0$ such that for every $\lambda > \Lambda_0$, problem (P_λ) has at least two distinct non-zero radially symmetric solutions in W;*

(iii) *If f is odd and $N \neq 3$, there exists $\Lambda_1 > 0$ such that for every $\lambda > \Lambda_1$ problem (P_λ) has at least $s_N = N - 3\,(N \bmod 2)$ distinct pairs of non-zero solutions $\{\pm u_i^\lambda\} \subset W$, $i = 1, ..., s_N$.*

The proof of Theorem 1.1 (i) is direct. In order to prove Theorem 1.1 (ii)-(iii), we find critical points of the energy functional associated with problem (P_λ) by means of a Ricceri-type three-critical points theorem and the well-known Palais' principle of symmetric criticality. In particular, the proof of the multiplicity in Theorem 1.1 (iii) requires special treatment. Our strategy is to apply Ricceri's result to some particular *subspaces* of W which have two main properties:

- they can be compactly embedded into $L^p(\mathbb{R}^N)$, $p \in (2, 2^*)$;
- they cannot be compared from a symmetrical point of view, i.e., their pairwise intersections contain only the 0 element.

After a careful group-theoretical analysis inspired from Bartsch and Willem [2], we are able to construct $s'_N = \left[\frac{N-1}{2}\right] + (-1)^N$ such subspaces of W whenever $N \neq 3$. Further energy-level analysis together with Ricceri's multiplicity result provides at least two pairs of distinct non-zero solutions for (P_λ) belonging to these subspaces separately whenever λ is large enough. Thus, the minimal number of distinct pairs of non-zero solutions for (P_λ) is $s_N = 2s'_N = N - 3\,(N \bmod 2)$. One can also observe that $s_N \geq 2$ for every $N \neq 3$. Furthermore, in each dimension $N \geq 2$, two pairs of solutions are radially symmetric, while if $s_N > 2$ (which occurs for $N = 4$ or $N \geq 6$), the rest of the $(s_N - 2)$ pairs of solutions are sign-changing and non-radially symmetric functions. This statement is based on the aforementioned group-theoretical argument which is described in Section 2.

In Section 2 we recall Ricceri's three-critical point theorem and display the group-theoretical arguments needed for the proof of Theorem 1.1 (iii). In Section 3 we prove our main theorem.

2 Preliminaries

2.1 A Ricceri-type three-critical point theorem

The functional space

$$W=\left\{u\in H^1(\mathbb{R}^N):\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)dx<\infty\right\}$$

is endowed with its natural inner product $\langle u,v\rangle_W=\int_{\mathbb{R}^N}(\nabla u\nabla v+V(x)uv)\,dx$ and norm $\|\cdot\|_W=\sqrt{\langle\cdot,\cdot\rangle_W}$. Due to hypothesis (H_V), it is clear that the embeddings $W\subset H^1(\mathbb{R}^N)\subset L^p(\mathbb{R}^N)$ are continuous, $p\in[2,2^*)$. Here, $2^*=\infty$ if $N=2$, and $2^*=2N/(N-2)$ for $N\geq 3$. Once (f_1) and (f_2) hold, the functional $\mathcal{F}:W\to\mathbb{R}$ defined by

$$\mathcal{F}(u)=\int_{\mathbb{R}^N}K(x)F(u)dx$$

is well-defined, is of class C^1, and the critical points of the functional $E_\lambda:W\to\mathbb{R}$ defined by

$$E_\lambda(u)=\frac{1}{2}\|u\|_W^2-\lambda\mathcal{F}(u)$$

are precisely the weak solutions for problem (P_λ). In order to find critical points for E_λ, we will apply the principle of symmetric criticality together with a recent critical point theorem due to Ricceri [10]. In order to recall Ricceri's result, we need the following definition: if X is a Banach space, we denote by $\mathcal{W}_X$ the class of those functionals $E:X\to\mathbb{R}$ having the property that if $\{u_n\}$ is a sequence in X converging weakly to $u\in X$ and $\liminf_n E(u_n)\leq E(u)$ then $\{u_n\}$ has a subsequence converging strongly to u.

Theorem 2.1 [10, Theorem 2] *Let $(X,\|\cdot\|)$ be a separable, reflexive, real Banach space, let $E_1:X\to\mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; and let $E_2:X\to\mathbb{R}$ be a C^1 functional with compact derivative. Assume that E_1 has a strict local minimum point u_0 with $E_1(u_0)=E_2(u_0)=0$. Assume that $\tau<\chi$, where*

$$\tau:=\max\left\{0,\limsup_{\|u\|\to\infty}\frac{E_2(u)}{E_1(u)},\limsup_{u\to u_0}\frac{E_2(u)}{E_1(u)}\right\},\tag{2.1}$$

$$\chi=\sup_{E_1(u)>0}\frac{E_2(u)}{E_1(u)}.\tag{2.2}$$

Then, for each compact interval $[a,b]\subset(1/\chi,1/\tau)$ (with the conventions $1/0=\infty$ and $1/\infty=0$) there exists $\kappa>0$ with the following property: for every $\lambda\in[a,b]$ and every C^1 functional $E_3:X\to\mathbb{R}$ with compact derivative, there exists $\delta>0$ such that for each $\theta\in[0,\delta]$, the equation

$$E_1'(u)-\lambda E_2'(u)-\theta E_3'(u)=0$$

admits at least three solutions in X having norm less than κ.

Remark 2.1 A close inspection of the proof of Theorem 2.1 provides us with further information on the critical points of $E_1 - \lambda E_2$ whenever $\lambda \in (1/\chi, 1/\tau)$. First, from (2.1) one can easily deduce that u_0 is a strict local minimum point (without being a global minimum point) of $E_1 - \lambda E_2$. Second, since $E_1 - \lambda E_2$ is coercive, bounded from below which satisfies the Palais-Smale condition, there exists a global minimum point $u_1 \in X$ of $E_1 - \lambda E_2$ with $E_1(u_1) - \lambda E_2(u_1) < 0 = E_1(u_0) - \lambda E_2(u_0)$. Third, by applying a mountain pass argument, see Pucci and Serrin [8], one can guarantee the existence of a third critical point $u_2 \in X$ of $E_1 - \lambda E_2$, different from u_0 and u_1, such that $c_\lambda = E_1(u_2) - \lambda E_2(u_2) \geq \max\{E_1(u_0) - \lambda E_2(u_0), E_1(u_1) - \lambda E_2(u_1)\} = 0$.

2.2 Special symmetries

Let $N \geq 2$ be fixed and assume that a closed subgroup of the orthogonal group $\mathbf{O}(N)$ acts on the space W, i.e., $(\phi, u) \mapsto \phi * u \in W$, $\phi \in G$, $u \in W$. We define the set of fixed points of W with respect to the group G which contains the $G-$invariant functions, i.e.,

$$W_G = \{u \in W : \phi * u = u \text{ for every } \phi \in G\}.$$

In particular, if $G = \mathbf{O}(N)$ and $'*'$ is the standard linear and isometric action defined as

$$(\phi * u)(x) = u(\phi^{-1}x) \text{ for } x \in \mathbb{R}^N, \phi \in \mathbf{O}(N), \tag{2.3}$$

the set $W_{\mathbf{O}(N)}$ is exactly the subspace of radially symmetric functions of W. Standard arguments show that $W_{\mathbf{O}(N)} \subset L^p(\mathbb{R}^N)$ is compact for every $p \in (2, 2^*)$, see Lions [6].

In order to prove Theorem 1.1 (iii), more specific groups and actions are needed whose origin can be found in Bartsch and Willem [2]. Let us fix $N = 4$ or $N \geq 6$ and define the number

$$t_N = \left[\frac{N-3}{2}\right] + (-1)^N.$$

Note that $t_N \geq 1$ and for every $i \in \{1, ..., t_N\}$, we may introduce the following subgroups of the orthogonal group $\mathbf{O}(N)$:

$$G_{N,i} = \begin{cases} \mathbf{O}(\frac{N}{2}) \times \mathbf{O}(\frac{N}{2}), & \text{if } \ i = \frac{N-2}{2}, \\ \mathbf{O}(i+1) \times \mathbf{O}(N-2i-2) \times \mathbf{O}(i+1), & \text{if } \ i \neq \frac{N-2}{2}. \end{cases}$$

We introduce the involution function $\tau_i : \mathbb{R}^N \to \mathbb{R}^N$ associated with $G_{N,i}$ by

$$\tau_i(x) = \begin{cases} (x_3, x_1), & \text{if } \ i = \frac{N-2}{2}, \text{ and } x = (x_1, x_3) \text{ with } x_1, x_3 \in \mathbb{R}^{\frac{N}{2}}; \\ (x_3, x_2, x_1), & \text{if } \ i \neq \frac{N-2}{2}, \text{ and } x = (x_1, x_2, x_3) \text{ with } x_1, x_3 \in \mathbb{R}^{i+1}, \ x_2 \in \mathbb{R}^{N-2i-2}. \end{cases}$$

By definition, we clearly have that $\tau_i \notin G_{N,i}$, $\tau_i G_{N,i} \tau_i^{-1} = G_{N,i}$ and $\tau_i^2 = \mathrm{id}_{\mathbb{R}^N}$.

Now, let $G_{N,i}^{\tau_i} = \langle G_{N,i}, \tau_i \rangle = G_{N,i} \cup \tau_i G_{N,i}$. We know from the properties of τ_i that only two types of elements in $G_{N,i}^{\tau_i}$ can be distinguished; namely, $\phi = g \in G_{N,i}$, and $\phi = \tau_i g \in G_{N,i}^{\tau_i} \setminus G_{N,i}$ (with $g \in G_{N,i}$). The action of the compact group $G_{N,i}^{\tau_i}$ on W is defined by

$$(g * u)(x) = u(g^{-1}x), \quad ((\tau_i g) * u)(x) = -u(g^{-1}\tau_i^{-1}x), \tag{2.4}$$

for $g \in G_{N,i}$, $u \in W$ and $x \in \mathbb{R}^N$. Now from Bartsch and Willem [2, pp. 455-457], the embedding $W_{G_{N,i}^{\tau_i}} \subset L^p(\mathbb{R}^N)$ is compact for every $p \in (2, 2^*)$.

The next result is of crucial importance in Theorem 1.1 (iii).

Theorem 2.2 *The following statements hold true:*

(i) *If* $N = 4$ *or* $N \geq 6$, *then* $W_{G^{\tau_i}_{N,i}} \cap W_{\mathbf{O}(N)} = \{0\}$ *for all* $i \in \{1, ..., t_N\}$*;*

(ii) *If* $N = 6$ *or* $N \geq 8$, *then* $W_{G^{\tau_i}_{N,i}} \cap W_{G^{\tau_j}_{N,j}} = \{0\}$ *for every* $i, j \in \{1, ..., t_N\}$ *with* $i \neq j$.

Proof. (i) Let $u \in W_{G^{\tau_i}_{N,i}} \cap W_{\mathbf{O}(N)}$. Due to the $G^{\tau_i}_{N,i}-$invariance of u and (2.4) we have $u(x) = -u(\tau_i^{-1}x)$ for every $x \in \mathbb{R}^N$. Since u is also radial and $|x| = |\tau_i x|$, we necessarily have that $u = 0$.

(ii) Although in [5] a similar property is proved, for the sake of completeness we give the proof here as well. Let $N = 6$ or $N \geq 8$ be fixed; then $t_N \geq 2$. Fix $i, j \in \{1, ..., t_N\}$ with $i < j$ and denote by $\langle G_{N,i}; G_{N,j}\rangle$ the group generated by $G_{N,i}$ and $G_{N,j}$.

We claim that the group $\langle G_{N,i}; G_{N,j}\rangle$ acts transitively on the sphere S^{N-1}. If this fact holds, the conclusion easily follows. Indeed, let $u \in W_{G^{\tau_i}_{N,i}} \cap W_{G^{\tau_j}_{N,j}}$. In particular, u is both $G_{N,i}-$, and $G_{N,j}-$invariant, i.e. $g_i * u = g_j * u = u$ for every $g_i \in G_{N,i}$ and $g_j \in G_{N,j}$, respectively. Consequently, u is also $\langle G_{N,i}, G_{N,j}\rangle-$invariant; thus, $u(x) = u(g_{ij}x)$ for every $g_{ij} \in \langle G_{N,i}, G_{N,j}\rangle$ and $x \in \mathbb{R}^N$. Since $\langle G_{N,i}; G_{N,j}\rangle$ acts transitively on the sphere S^{N-1}, we have that

$$\text{Orbit}\{g_{ij}x : g_{ij} \in \langle G_{N,i}, G_{N,j}\rangle\} = |x|S^{N-1} \text{ for all } x \in \mathbb{R}^N.$$

Therefore, u is radially symmetric, and we can apply (i) thus obtaining that $u = 0$.

The transitivity of $\langle G_{N,i}, G_{N,j}\rangle$ on S^{N-1} will be proved step by step, namely, first on $S^{N-j-2} \times \{0_{j+1}\}$, then on $S^{N-i-2} \times \{0_{i+1}\}$, and finally on the whole S^{N-1}. For simplicity, set $0_k = (0, ..., 0) \in \mathbb{R}^k$, $k \in \{1, ..., N\}$.

First step. $\langle G_{N,i}; G_{N,j}\rangle$ *acts transitively on* $S^{N-j-2} \times \{0_{j+1}\}$.
When $j = \frac{N-2}{2}$, then $G_{N,j} = \mathbf{O}(\frac{N}{2}) \times \mathbf{O}(\frac{N}{2})$, thus the proof is trivial since $\mathbf{O}(\frac{N}{2})$ acts transitively on $S^{\frac{N-2}{2}}$. Assume now that $j \neq (N-2)/2$. We first show that for every $x = (x_1, x_2, x_3) \in S^{N-j-2}$ with $x_1 \in \mathbb{R}^{i+1}$, $x_2 \in \mathbb{R}^{j-i}$, $x_3 \in \mathbb{R}^{N-2j-2}$, and $y \in S^j$ fixed arbitrarily, there exists $g_{ij} \in \langle G_{N,i}; G_{N,j}\rangle$ such that

$$g_{ij}(y, 0_{N-j-1}) = (x, 0_{j+1}). \tag{2.5}$$

Since $\mathbf{O}(j+1)$ acts transitively on S^j, for every $\tilde{x}_2 \in \mathbb{R}^{j-i}$ with the property that $(x_1, \tilde{x}_2) \in S^j$, there exists an element $g_j \in \mathbf{O}(j+1)$ such that

$$g_j y = (x_1, \tilde{x}_2). \tag{2.6}$$

Note that $|x_1|^2 + |\tilde{x}_2|^2 = 1$ and $|x_1|^2 + |x_2|^2 + |x_3|^2 = 1$; thus $|\tilde{x}_2|^2 = |x_2|^2 + |x_3|^2$.

If $\tilde{x}_2 = 0_{j-i}$ then $x_2 = 0_{j-i}$ and $x_3 = 0_{N-2j-2}$; thus, $x = (x_1, 0_{N-j-i-2})$. Let $g_{ij} := g_j \times \text{id}_{\mathbb{R}^{N-j-1}} \in G_{N,j}$. Then, due to (2.6), we have relation (2.5) by

$$g_{ij}(y, 0_{N-j-1}) = (g_j y, 0_{N-j-1}) = (x_1, 0_{j-i}, 0_{N-j-1}) = (x_1, 0_{N-i-1}) = (x, 0_{j+1}).$$

If $\tilde{x}_2 \neq 0_{j-i}$, let $\rho = |\tilde{x}_2| > 0$. Since $\mathbf{O}(N-2i-2)$ acts transitively on S^{N-2i-3} (thus, also on the sphere ρS^{N-2i-3}), there exists $g_i \in \mathbf{O}(N-2i-2)$ such that $g_i(\tilde{x}_2, 0_{N-j-i-2}) = (x_2, x_3, 0_{j-i}) \in \rho S^{N-2i-3}$. Let

$$\tilde{g}_i = \text{id}_{\mathbb{R}^{i+1}} \times g_i \times \text{id}_{\mathbb{R}^{i+1}} \in G_{N,i} \text{ and } \tilde{g}_j = g_j \times \text{id}_{\mathbb{R}^{N-j-1}} \in G_{N,j},$$

the element g_j coming from relation (2.6). Then $g_{ij} := \tilde{g}_i\tilde{g}_j \in \langle G_{N,i}; G_{N,j}\rangle$ and on account of (2.6) and $i+1 < N-j-1$ (since $i < j \leq t_N$), we obtain relation (2.5) by

$$\begin{aligned}\tilde{g}_i\tilde{g}_j(y, 0_{N-j-1}) &= \tilde{g}_i(g_j y, 0_{N-j-1}) = \tilde{g}_i(x_1, \tilde{x}_2, 0_{N-j-1}) \\ &= (x_1, g_i(\tilde{x}_2, 0_{N-j-i-2}), 0_{i+1}) = (x_1, x_2, x_3, 0_{j-i}, 0_{i+1}) \\ &= (x, 0_{j+1}).\end{aligned}$$

Now, let $\overline{x}, \tilde{x} \in S^{N-j-2}$. Then, fixing $y \in S^j$, due to (2.5), there are $g_1, g_2 \in \langle G_{N,i}; G_{N,j}\rangle$ such that $g_1(y, 0_{N-j-1}) = (\overline{x}, 0_{j+1})$ and $g_2(y, 0_{N-j-1}) = (\tilde{x}, 0_{j+1})$. Consequently, $g_2 g_1^{-1} \in \langle G_{N,i}; G_{N,j}\rangle$ and $g_2 g_1^{-1}(\overline{x}, 0_{j+1}) = (\tilde{x}, 0_{j+1})$, i.e., $\langle G_{N,i}; G_{N,j}\rangle$ acts transitively on $S^{N-j-2} \times \{0_{j+1}\}$.

Second step. $\langle G_{N,i}; G_{N,j}\rangle$ *acts transitively on* $S^{N-i-2} \times \{0_{i+1}\}$.

The proof is similar to that of the first step. We now show that for every $x = (x_1, x_2, x_3) \in S^{N-i-2}$ with $x_1 \in \mathbb{R}^{i+1}$, $x_2 \in \mathbb{R}^{N-j-i-2}$, $x_3 \in \mathbb{R}^{j-i}$, and $y \in S^{N-j-2}$ fixed arbitrarily, there is $g_{ij} \in \langle G_{N,i}; G_{N,j}\rangle$ such that

$$g_{ij}(y, 0_{j+1}) = (x, 0_{i+1}). \tag{2.7}$$

Let $\tilde{x}_2 \in \mathbb{R}^{N-j-i-2}$ be such that $|x_1|^2 + |\tilde{x}_2|^2 = 1$. Then, due to the first step, one can find $\tilde{g}_{ij} \in \langle G_{N,i}; G_{N,j}\rangle$ such that $\tilde{g}_{ij}(y, 0_{j+1}) = (x_1, \tilde{x}_2, 0_{j+1})$.

If $\tilde{x}_2 = 0_{N-j-i-2}$ then $x_2 = 0_{N-j-i-2}$ and $x_3 = 0_{j-i}$. Consequently, (2.7) is verified with the choice $g_{ij} := \tilde{g}_{ij} \in \langle G_{N,i}; G_{N,j}\rangle$.

If $\tilde{x}_2 \neq 0_{N-j-i-2}$ then let $\rho = |\tilde{x}_2| > 0$. Since $\mathbf{O}(N-2i-2)$ acts transitively on S^{N-2i-3} (thus, also on the sphere ρS^{N-2i-3}), there exists $g_i \in \mathbf{O}(N-2i-2)$ such that $g_i(\tilde{x}_2, 0_{j-i}) = (x_2, x_3) \in \rho S^{N-2i-3}$. Let $\tilde{g}_i = \mathrm{id}_{\mathbb{R}^{i+1}} \times g_i \times \mathrm{id}_{\mathbb{R}^{i+1}} \in G_{N,i}$. Then

$$\tilde{g}_i\tilde{g}_{ij}(y, 0_{j+1}) = \tilde{g}_i(x_1, \tilde{x}_2, 0_{j+1}) = (x_1, g_i(\tilde{x}_2, 0_{j-i}), 0_{i+1}) = (x, 0_{i+1}).$$

Consequently, $g_{ij} := \tilde{g}_i\tilde{g}_{ij} \in \langle G_{N,i}; G_{N,j}\rangle$ verifies relation (2.7). Now, following the last part of the first step we obtain the required statement.

Third (concluding) step. $\langle G_{N,i}; G_{N,j}\rangle$ *acts transitively on* S^{N-1}.

We first show that for every $x = (x_1, x_2, x_3) \in S^{N-1}$ with $x_1 \in \mathbb{R}^{i+1}$, $x_2 \in \mathbb{R}^{N-j-i-2}$, $x_3 \in \mathbb{R}^{j+1}$, and $y \in S^{N-i-2}$ fixed arbitrarily, there is $g_{ij} \in \langle G_{N,i}; G_{N,j}\rangle$ such that

$$g_{ij}(y, 0_{i+1}) = x. \tag{2.8}$$

Let $\tilde{x}_3 \in \mathbb{R}^{j-i}$ such that $|\tilde{x}_3| = |x_3|$. Then, due to the second step, there exists $\tilde{g}_{ij} \in \langle G_{N,i}; G_{N,j}\rangle$ such that $\tilde{g}_{ij}(y, 0_{i+1}) = (x_1, x_2, \tilde{x}_3, 0_{i+1})$.

If $\tilde{x}_3 = 0_{j-i}$ then $x_3 = 0_{j+1}$ and (2.8) is verified with $g_{ij} := \tilde{g}_{ij} \in \langle G_{N,i}; G_{N,j}\rangle$.

If $\tilde{x}_3 \neq 0_{j-i}$, let $\rho = |\tilde{x}_3| = |x_3| > 0$. Since $\mathbf{O}(j+1)$ acts transitively on S^j, there exists $g_j \in \mathbf{O}(j+1)$ such that $g_j(\tilde{x}_3, 0_{i+1}) = x_3 \in \rho S^j$. Let us fix the element $\tilde{g}_j = \mathrm{id}_{\mathbb{R}^{N-j-1}} \times g_j \in G_{N,j}$. Then the element $g_{ij} := \tilde{g}_j\tilde{g}_{ij} \in \langle G_{N,i}; G_{N,j}\rangle$ verifies relation (2.8); indeed, one has

$$\tilde{g}_j\tilde{g}_{ij}(y, 0_{i+1}) = \tilde{g}_j(x_1, x_2, \tilde{x}_3, 0_{i+1}) = (x_1, x_2, g_j(\tilde{x}_3, 0_{i+1})) = (x_1, x_2, x_3) = x.$$

Now, let $\overline{x}, \tilde{x} \in S^{N-1}$. Then, fixing $y \in S^{N-i-2}$, on account of (2.8), there are $g_1, g_2 \in \langle G_{N,i}; G_{N,j}\rangle$ such that $g_1(y, 0_{i+1}) = \overline{x}$ and $g_2(y, 0_{i+1}) = \tilde{x}$. Consequently, $g_2 g_1^{-1} \in \langle G_{N,i}; G_{N,j}\rangle$ and $g_2 g_1^{-1}(\overline{x}) = \tilde{x}$, i.e., the group $\langle G_{N,i}; G_{N,j}\rangle$ acts transitively on S^{N-1}, which concludes the proof. $\square$

3 Proof of Theorem 1.1

In the sequel we assume that all the assumptions of Theorem 1.1 are fulfilled.

Proof of Theorem 1.1 (i). Assume that $u \in W$ is a solution of (P_λ). Multiplying (P_λ) by the test function u and using the definition of the number $c_f > 0$, we obtain

$$\begin{aligned} \|u\|_W^2 &= \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx \\ &= \lambda \int_{\mathbb{R}^N} K(x)f(u)u \\ &\leq \lambda \frac{\|K\|_{L^\infty}}{\inf_{\mathbb{R}^N} V} c_f \int_{\mathbb{R}^N} V(x)u^2 \\ &\leq \lambda \frac{\|K\|_{L^\infty}}{\inf_{\mathbb{R}^N} V} c_f \|u\|_W^2. \end{aligned}$$

Now, if $0 \leq \lambda < c_f^{-1}\|K\|_{L^\infty}^{-1}\inf_{\mathbb{R}^N} V$, the above estimate implies $u = 0$, which concludes the proof of (i). $\square$

As we pointed out in the Introduction, the solutions of (P_λ) are exactly the critical points for the functional $E_\lambda = \mathcal{E}_1 - \lambda\mathcal{E}_2 : W \to \mathbb{R}$, where

$$\mathcal{E}_1(u) = \frac{1}{2}\|u\|_W^2 \text{ and } \mathcal{E}_2(u) = \mathcal{F}(u),\ u \in W.$$

Before proving (ii) and (iii) of Theorem 1.1, we need the following

Lemma 3.1 (i) $\limsup_{\|u\|_W \to \infty} \frac{\mathcal{F}(u)}{\|u\|_W^2} \leq 0$;

(ii) $\limsup_{u \to 0} \frac{\mathcal{F}(u)}{\|u\|_W^2} \leq 0$;

(iii) *Let X be a closed subspace of W which is compactly embedded into $L^r(\mathbb{R}^N)$, $r \in (2, 2^*)$. Then $\mathcal{F}|_X$ has a compact derivative.*

Proof. Due to (f_1) and (f_2), for every fixed $\varepsilon > 0$ there is a $\delta_\varepsilon \in (0,1)$ such that

$$|f(s)| < \varepsilon \frac{\inf_{\mathbb{R}^N} V}{\|K\|_{L^\infty}} |s| \text{ for all } |s| \leq \delta_\varepsilon \text{ and } |s| \geq \delta_\varepsilon^{-1}.$$

Since $f \in C(\mathbb{R}, \mathbb{R})$, there also exist two constants $M_\varepsilon^1, M_\varepsilon^2 > 0$ such that

$$\frac{|f(s)|}{|s|^{q-1}} \leq M_\varepsilon^1 \text{ and } \frac{|f(s)|}{|s|^{p-1}} \leq M_\varepsilon^2 \text{ for all } |s| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}],$$

where $1 < q < 2 < p < 2^*$. Combining the above two relations, we obtain that

$$|f(s)| \leq \varepsilon \frac{\inf_{\mathbb{R}^N} V}{\|K\|_{L^\infty}} |s| + M_\varepsilon^1 |s|^{q-1} \quad \text{for all } s \in \mathbb{R}; \tag{3.1}$$

$$|f(s)| \leq \varepsilon \frac{\inf_{\mathbb{R}^N} V}{\|K\|_{L^\infty}} |s| + M_\varepsilon^2 |s|^{p-1} \text{ for all } s \in \mathbb{R}. \tag{3.2}$$

On account of (3.1), since $K \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and the embedding $W \subset L^r(\mathbb{R}^N)$ is continuous for $r \in (2, 2^*)$, by the Hölder inequality one can find $C_\varepsilon^1 > 0$ such that

$$\begin{aligned} \mathcal{F}(u) &\leq \int_{\mathbb{R}^N} K(x)|F(u)| \\ &\leq \int_{\mathbb{R}^N} K(x) \left[\varepsilon \frac{\inf_{\mathbb{R}^N} V}{2\|K\|_{L^\infty}} u^2 + \frac{M_\varepsilon^1}{q} |u|^q \right] \\ &\leq \frac{\varepsilon}{2} \|u\|_W^2 + C_\varepsilon^1 \|u\|_W^q. \end{aligned}$$

Consequently, for every $u \in W \setminus \{0\}$, we have that

$$\frac{\mathcal{F}(u)}{\|u\|_W^2} \leq \frac{\varepsilon}{2} + C_\varepsilon^1 \|u\|_W^{q-2}.$$

Since $q < 2$, the arbitrariness of $\varepsilon > 0$ yields (i).

A similar argument based on (3.2) gives the existence of a $C_\varepsilon^2 > 0$ such that for every $u \in W \setminus \{0\}$,

$$\frac{\mathcal{F}(u)}{\|u\|_W^2} \leq \frac{\varepsilon}{2} + C_\varepsilon^2 \|u\|_W^{p-2}.$$

Since $\varepsilon > 0$ is arbitrary and $p > 2$, (ii) follows readily.

The proof of (iii) is standard. □

For any $0 < r_1 < r_2$, let $A[r_1, r_2] = \{x \in \mathbb{R}^N : r_1 \leq |x| \leq r_2\}$ be the closed annulus with radii r_1 and r_2 . Since $K \in L^\infty(\mathbb{R}^N)$ is a radially symmetric function with $K \geq 0$ and $K \not\equiv 0$ (cf. hypothesis (H_K)), one can find real numbers $R > r > 0$ and $K_0 > 0$ such that

$$\operatorname{essinf}_{x \in A[r,R]} K(x) \geq K_0. \tag{3.3}$$

Proof of Theorem 1.1 (ii). Let $s_0 \in \mathbb{R}$ from (f_3). For a fixed $\sigma \in (0, (R - r)/2)$ with r, R from (3.3), we can define a radially symmetric truncation function $u_\sigma \in W_{\mathbf{O}(N)}$ such that

(a) $\operatorname{supp} u_\sigma \subseteq A[r, R]$;

(b) $\|u_\sigma\|_{L^\infty} \leq |s_0|$;

(c) $u_\sigma(x) = s_0$ for every $x \in A[r + \sigma, R - \sigma]$.

Here is an example of such a function $u_\sigma : \mathbb{R}^N \to \mathbb{R}$

$$u_\sigma(x) = \begin{cases} \frac{s_0}{\sigma}(|x| - r)_+ & \text{if } |x| \leq r + \sigma; \\ s_0 & \text{if } r + \sigma < |x| \leq R - \sigma; \\ \frac{s_0}{\sigma}(R - |x|)_+ & \text{if } |x| \geq R - \sigma, \end{cases}$$

where $z_+ = \max(z, 0)$. Denoting by ω_N the volume of the unit ball in $\mathbb{R}^N$, we clearly have from the properties (a)-(c) and relation (3.3) that

$$\|u_\sigma\|_W^2 \geq s_0^2 \omega_N \inf_{\mathbb{R}^N} V \left((R-\sigma)^N - (r+\sigma)^N\right),$$

and

$$\begin{aligned} \mathcal{F}(u_\sigma) \quad &\geq \quad \omega_N [K_0 F(s_0) \left((R-\sigma)^N - (r+\sigma)^N\right) - \|K\|_{L^\infty} \max_{|t|\leq|s_0|} |F(t)| \times \\ &\quad \times \left((r+\sigma)^N - r^N + R^N - (R-\sigma)^N\right)]. \end{aligned}$$

If σ is close enough to 0, the right-hand sides of both inequalities are strictly positive. Therefore, we can define the number

$$\lambda_0 = \inf \left\{ \frac{\|u\|_W^2}{2\mathcal{F}(u)} : u \in W_{\mathbf{O}(N)},\ \mathcal{F}(u) > 0 \right\}. \tag{3.4}$$

Moreover, it is also clear (cf. Lemma 3.1 and the above estimates) that

$$\chi_0 = \sup \left\{ \frac{2\mathcal{F}(u)}{\|u\|_W^2} : u \in W_{\mathbf{O}(N)} \setminus \{0\} \right\} \in (0, \infty)$$

and $\chi_0^{-1} = \lambda_0$.

Now, we are in a position to apply Theorem 2.1 with $X = W_{\mathbf{O}(N)}$ and $E_1, E_2 : W_{\mathbf{O}(N)} \to \mathbb{R}$ defined by $E_1 = \mathcal{E}_1|_{W_{\mathbf{O}(N)}}$ and $E_2 = \mathcal{E}_2|_{W_{\mathbf{O}(N)}}$. On account of Lemma 3.1, the assumptions of Theorem 2.1 are fulfilled with $u_0 = 0 \in W_{\mathbf{O}(N)}$ and $\tau = 0$. Thus, for every $\lambda > \Lambda_0 := \lambda_0 = \chi_0^{-1} > 0$, the functional $E_\lambda|_{W_{\mathbf{O}(N)}}$ has at least three distinct critical points in $W_{\mathbf{O}(N)}$. Since E_λ is $\mathbf{O}(N)-$invariant, i.e., $E_\lambda(\phi * u) = E_\lambda(u)$ for every $\phi \in \mathbf{O}(N)$ and $u \in W$ (cf. relation (2.3) and hypotheses (H_V) and (H_K)), the principle of symmetric criticality implies that the critical points of $E_\lambda|_{W_{\mathbf{O}(N)}}$ are also critical points for E_λ. This concludes the proof. □

Remark 3.1 Theorem 2.1 allows us to assert that problem (P_λ) is stable to small perturbations of subcritical type. To be more precise, let us consider the perturbed problem

$$\begin{cases} -\Delta u + V(x)u = \lambda K(x) f(u) + \mu L(x) g(u) & \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$P_{\lambda,\mu}$}$$

where $\mu \in \mathbb{R}$, $L \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ is a radially symmetric function, while $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that for some $c > 0$ and $2 < p < 2^*$,

$$|g(s)| \leq c(|s| + |s|^{p-1}) \text{ for all } s \in \mathbb{R}.$$

We can easily prove that the function $J : W \to \mathbb{R}$ defined by

$$J(u) = \int_{\mathbb{R}^N} L(x) G(u) dx,$$

is of class C^1 and $E_3 = J|_{W_{\mathbf{O}(N)}}$ has a compact derivative, where $G(s) = \int_0^s g(t)dt$. Consequently, we may apply Theorem 2.1 in its generality providing precise information on the stability of (P_λ); namely, problem $(P_{\lambda,\mu})$ has at least three distinct radially symmetric solutions whenever $\lambda > \Lambda_0$ and μ is small enough. Furthermore, some norm-estimates of the solutions of $(P_{\lambda,\mu})$ are also available on compact intervals of $[\Lambda_0, \infty)$.

Proof of Theorem 1.1 (iii). Let $N \neq 3$. Since f is odd, the energy functional E_λ is even, and its critical points (hence solutions for (P_λ)) appear in symmetric pairs. Consequently, a similar argument as in (ii) shows that there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$, problem (P_λ) has at least two pairs $\{\pm u_{0,1}^\lambda\}$ and $\{\pm u_{0,2}^\lambda\}$ of non-zero distinct radially symmetric solutions for (P_λ) which belong to $W_{\mathbf{O}(N)}$. In the case when $N = 2$ or $N = 5$ we have $s_N = 2$, i.e., the conclusion of (iii) follows from the latter arguments.

Consequently, it remains to consider $N = 4$ or $N \geq 6$. In this case $t_N \geq 1$, so we may fix $i \in \{1, ..., t_N\}$ arbitrarily. Without any loss of generality, we may assume for $0 < r < R$ in relation (3.3) that $r(5 + 4\sqrt{2}) \geq R$. Due to the latter choice, it is clear that the sets

$$Q_1 = \left\{ (x_1, x_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{i+1} : \sqrt{\left(|x_1| - \frac{R+3r}{4}\right)^2 + |x_3|^2} \leq \frac{R-r}{4} \right\};$$

$$Q_2 = \left\{ (x_1, x_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{i+1} : \sqrt{\left(|x_3| - \frac{R+3r}{4}\right)^2 + |x_1|^2} \leq \frac{R-r}{4} \right\}$$

are disjoint. For every $\sigma \in (0, 1]$, we introduce the set

$$\begin{aligned} D_\sigma^i = \Bigg\{ x \in \mathbb{R}^N : & \sqrt{\left(|x_1| - \frac{R+3r}{4}\right)^2 + |x_3|^2} \leq \sigma \frac{R-r}{4}, \\ & \sqrt{\left(|x_3| - \frac{R+3r}{4}\right)^2 + |x_1|^2} \leq \sigma \frac{R-r}{4}, \\ & |x_2| \leq \sigma \frac{R-r}{4} \Bigg\}, \end{aligned}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^N$ with $x_1, x_3 \in \mathbb{R}^{i+1}$, $x_2 \in \mathbb{R}^{N-2i-2}$ whenever $i \neq \frac{N-2}{2}$, and $x = (x_1, x_3) \in \mathbb{R}^N$ with $x_1, x_3 \in \mathbb{R}^{\frac{N}{2}}$ whenever $i = \frac{N-2}{2}$ (and x_2 is considered formally 0). Note that the set $D_\sigma^i \subset \mathbb{R}^N$ is $G_{N,i}^{\tau_i}$-invariant, i.e., $\phi D_\sigma^i \subset D_\sigma^i$ for every $\phi \in G_{N,i}^{\tau_i}$. Moreover, $\text{meas}(D_\sigma^i) > 0$ for every $\sigma \in (0, 1]$ and

$$\lim_{\sigma \to 1} \text{meas}(D_1^i \setminus D_\sigma^i) = 0. \tag{3.5}$$

Let $s_0 \in \mathbb{R}$ from (f_3) and for a fixed number $\sigma \in (0, 1)$, we construct the following special truncation function

$$\begin{aligned} u_\sigma^i(x) \;=\; & \left[\left(\frac{R-r}{4} - \max\left(\sqrt{\left(|x_1| - \frac{R+3r}{4}\right)^2 + |x_3|^2}, \sigma \frac{R-r}{4} \right) \right)_+ \right. \\ & \left. - \left(\frac{R-r}{4} - \max\left(\sqrt{\left(|x_3| - \frac{R+3r}{4}\right)^2 + |x_1|^2}, \sigma \frac{R-r}{4} \right) \right)_+ \right] \times \\ & \times \left(\frac{R-r}{4} - \max\left(|x_2|, \sigma \frac{R-r}{4} \right) \right)_+ \frac{16 s_0}{(R-r)^2 (1-\sigma)^2}. \end{aligned}$$

The special shape of u_σ^i shows that $\phi * u_\sigma^i = u_\sigma^i$ for every $\phi \in G_{N,i}^{\tau_i}$ (see relation (2.4)), thus $u_\sigma^i \in W_{G_{N,i}^{\tau_i}}$. Moreover, the following useful properties hold:

(a') $\mathrm{supp} u_\sigma^i = D_1^i \subseteq A[r,R]$;

(b') $\|u_\sigma^i\|_{L^\infty} \leq |s_0|$;

(c') $|u_\sigma^i(x)| = |s_0|$ for every $x \in D_\sigma^i$.

Since F is even (thus $F(s_0) = F(-s_0)$), by exploiting the properties (a')-(c'), we obtain that

$$\mathcal{F}(u_\sigma^i) \geq K_0 F(s_0)\mathrm{meas}(D_\sigma^i) - \|K\|_{L^\infty} \max_{|t|\leq|s_0|} |F(t)|\mathrm{meas}(D_1^i \setminus D_\sigma^i).$$

If σ is close enough to 1, the right-hand side of the latter term is strictly positive, see (3.5). Consequently, we can introduce the number

$$\lambda_i = \inf\left\{\frac{\|u\|_W^2}{2\mathcal{F}(u)} : u \in W_{G_{N,i}^{\tau_i}},\ \mathcal{F}(u) > 0\right\}. \tag{3.6}$$

As before, one has that

$$\chi_i = \sup\left\{\frac{2\mathcal{F}(u)}{\|u\|_W^2} : u \in W_{G_{N,i}^{\tau_i}} \setminus \{0\}\right\} \in (0,\infty)$$

and $\chi_i^{-1} = \lambda_i$.

We can apply Theorem 2.1 with $X = W_{G_{N,i}^{\tau_i}}$ and $E_1, E_2 : W_{G_{N,i}^{\tau_i}} \to \mathbb{R}$ defined by $E_1 = \mathcal{E}_1|_{W_{G_{N,i}^{\tau_i}}}$ and $E_2 = \mathcal{E}_2|_{W_{G_{N,i}^{\tau_i}}}$. Due to Lemma 3.1, the assumptions of Theorem 2.1 are satisfied with $u_0 = 0 \in W_{G_{N,i}^{\tau_i}}$ and $\tau = 0$. Consequently, for every $\lambda > \chi_i^{-1} = \lambda_i > 0$, the functional $E_\lambda|_{W_{G_{N,i}^{\tau_i}}}$ has at least three distinct critical points in $W_{G_{N,i}^{\tau_i}}$. More precisely (cf. Remark 2.1), one of them is 0 (which is a strict local minimizer of $E_\lambda|_{W_{G_{N,i}^{\tau_i}}}$), while the other two elements $u_{i,1}^\lambda, u_{i,2}^\lambda \in W_{G_{N,i}^{\tau_i}} \setminus \{0\}$ are such that $u_{i,1}^\lambda$ is a global minimum point of $E_\lambda|_{W_{G_{N,i}^{\tau_i}}}$ with $E_\lambda(u_{i,1}^\lambda) < 0$, and $u_{i,2}^\lambda$ is a mountain pass-type critical point of $E_\lambda|_{W_{G_{N,i}^{\tau_i}}}$ with $E_\lambda(u_{i,2}^\lambda) \geq 0$. Since f is odd, the energy functional E_λ is even; in particular, $u_{i,1}^\lambda \neq \pm u_{i,2}^\lambda$, and the pairs $\{\pm u_{i,1}^\lambda\}$ and $\{\pm u_{i,2}^\lambda\}$ are distinct critical points for $E_\lambda|_{W_{G_{N,i}^{\tau_i}}}$.

Due to the evenness of E_λ, relation (2.4), and hypotheses (H_V), (H_K), we have that $E_\lambda(\phi * u) = E_\lambda(u)$ for every $\phi \in G_{N,i}^{\tau_i}$ and $u \in W$, i.e., E_λ is $G_{N,i}^{\tau_i}-$invariant on W. On account of the principle of symmetric criticality, the critical point pairs $\{\pm u_{i,1}^\lambda\}$ and $\{\pm u_{i,2}^\lambda\}$ of $E_\lambda|_{W_{G_{N,i}^{\tau_i}}}$ are also critical point pairs for E_λ whenever $\lambda > \lambda_i$, hence solutions for problem (P_λ).

Now, it remains to count the number of distinct solutions of the above type. Due to Theorem 2.2, there are at least $(1 + t_N)$ subspaces of W whose mutual intersections contain only the 0 element:

(I) the subspace $W_{\mathbf{O}(N)}$ of radially symmetric functions of W, and

(II) t_N subspace(s) of W of the type $W_{G_{N,i}^{\tau_i}}$.

As we pointed out above, each of these subspaces contain two distinct pairs of non-zero solutions for (P_λ) whenever λ is large enough. More precisely, if

$$\lambda > \Lambda_1 := \max\{\lambda_0, \lambda_1, ..., \lambda_{t_N}\},$$

where λ_0 comes from the radial case (see (3.4)), while λ_i is from (3.6), $i \in \{1, ..., t_N\}$, problem (P_λ) has at least

$$s_N = 2(1 + t_N) = N - 3(N \bmod 2)$$

distinct pairs of non-zero solutions. This concludes our proof. □

Remark 3.2 The statement of Theorem 1.1 (iii) is not relevant for $N = 3$ since $s_3 = 0$. However, Theorem 1.1 (ii) gives two distinct (pairs of) non-zero, radially symmetric solutions for (P_λ) whenever λ is large enough (and f is odd).

Remark 3.3 The proof of Theorem 1.1 (iii) shows that in each dimension $N \geq 2$, two pairs of solutions are radially symmetric. Moreover, if $N = 4$ or $N \geq 6$, then $s_N \geq 4$ and the rest of the $(s_N - 2)$ pairs of solutions are sign-changing and non-radially symmetric functions in W.

Remark 3.4 From a Strauss-type estimate (see Lions [6]) we know that every $u \in W$ satisfies $u(x) \to 0$ as $|x| \to \infty$. Thus, all solutions in Theorem 1.1 (ii)-(iii) have this property.

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