A dimension-depending multiplicity result for the Schrödinger equation

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Abstract

We consider the Schrödinger equation
\[
\begin{cases}
-\Delta u + V(x)u = \lambda K(x)f(u) \text{ in } \mathbb{R}^N; \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\]

where $N \geq 2$, $\lambda \geq 0$ is a parameter, $V, K : \mathbb{R}^N \to \mathbb{R}$ are radially symmetric functions, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with sublinear growth at infinity. We first prove that for $\lambda$ small enough no non-zero solution exists for $(P_\lambda)$, while for $\lambda$ large enough at least two distinct non-zero radially symmetric solutions do exist. By exploiting a Ricceri-type three-critical points theorem, the principle of symmetric criticality and a group-theoretical approach, the existence of at least $N - 3 \, (N \mod 2)$ distinct pairs of non-zero solutions is guaranteed for $(P_\lambda)$ whenever $\lambda$ is large enough, $N \neq 3$, and $f$ is odd.

Keywords: Schrödinger equation, sublinear, three-critical points theorem, principle of symmetric criticality.
1 Introduction

In this paper we consider the Schrödinger equation
\[
\begin{align*}
-\Delta u + V(x)u &= \lambda K(x)f(u) \quad \text{in } \mathbb{R}^N, \\
u &\in H^1(\mathbb{R}^N),
\end{align*}
\]

where \( N \geq 2, V, K : \mathbb{R}^N \to \mathbb{R} \) are some non-negative potentials, \( \lambda \geq 0 \) is a parameter, while \( f : \mathbb{R} \to \mathbb{R} \) is a nonlinear continuous function. The interest in this problem comes from mathematical physics; for instance, certain kinds of solitary waves in the nonlinear Klein-Gordon or Schrödinger equations appear as solutions of problem \( (P_\lambda) \).

Problem \( (P_\lambda) \) or its related form has been studied by many authors during the last two decades under various assumptions on the potentials \( V, K \) and on the nonlinear function \( f \). Most of these papers address the case when \( V \) and \( K \) have suitable sign- and growth-properties, and \( f \) has a superlinear and subcritical growth. In these papers existence and multiplicity results for \( (P_\lambda) \) are established via various variational arguments, see Rabinowitz [9], Bartsch et al. [1, 2], and further subsequent papers. In particular, if \( f \) is odd, the existence of infinitely many solutions for \( (P_\lambda) \) is usually guaranteed. A particularly interesting paper is due to Clapp and Weth [3] where the existence of at least \( \frac{N}{2} + 1 \) pairs of non-zero solutions for \( (P_\lambda) \) is proved for every \( \lambda > 0 \) by assuming certain one-sided asymptotic estimates for \( V \) and \( K \) when \( f(s) = |s|^{p-2} s, p \in (2, 2^*) \). Problem \( (P_\lambda) \) has been also studied in the case when \( f \) is odd and has an asymptotically linear growth at infinity; more precisely, Liu, van Heerden and Wang [7] prove a multiplicity result for \( (P_\lambda) \) whenever \( V(x) = \mu g(x) + 1, \mu > 0 \), and the number of solutions for \( (P_\lambda) \) depends on the behavior of the dimension of the eigenspace of a specific Dirichlet eigenvalue problem defined on the bounded domain \( \Omega = \text{int}(g^{-1}(0)) \).

The aim of the present paper is to supplement the aforementioned contributions by requiring that the non-zero continuous function \( f : \mathbb{R} \to \mathbb{R} \) has a sublinear growth at infinity and a superlinear growth near zero. More precisely, we assume that
\[
\begin{align*}
(f_1) \quad &f(s) = o(|s|) \text{ as } |s| \to \infty; \\
(f_2) \quad &f(s) = o(|s|) \text{ as } s \to 0; \\
(f_3) \quad &\text{there exists } s_0 \in \mathbb{R} \text{ such that } F(s_0) > 0, \text{ where } F(s) = \int_0^s f(t)dt.
\end{align*}
\]

In order to avoid technicalities, we assume in the sequel that potentials \( V, K : \mathbb{R}^N \to \mathbb{R} \) satisfy
\[
\begin{align*}
(H_V) \quad &V \in C(\mathbb{R}^N) \text{ is radially symmetric and } \inf_{\mathbb{R}^N} V > 0; \\
(H_K) \quad &K \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \text{ is radially symmetric and } K \geq 0, K \not\equiv 0.
\end{align*}
\]

Note that solutions of \( (P_\lambda) \) are being sought in weak form in the space
\[
W = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx < \infty \right\}.
\]
In fact, under the above conditions on $f$, $V$ and $K$ which will be assumed throughout in the sequel, every weak solution $u$ of $(P_\lambda)$ is a classical one. Indeed, we have $\Delta u =: h \in L^2_{\text{loc}}(\mathbb{R}^N)$, thus $u \in H^2_{\text{loc}}(\mathbb{R}^N)$ (cf. Evans [4, §8.3]) and $u$ satisfies $(P_\lambda)$ for a.a. $x \in \mathbb{R}^N$.

The hypotheses $(f_1)$, $(f_2)$ and $(f_3)$ guarantee that the number $c_f = \max_{s \neq 0} \left| \frac{f(s)}{s} \right|$ is well-defined, positive and finite. Now, we are in a position to state our main result.

**Theorem 1.1** Assume that $N \geq 2$. Let $V, K : \mathbb{R}^N \to \mathbb{R}$ be two potentials such that both $(H_V)$ and $(H_K)$ hold, and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function verifying $(f_1) - (f_3)$. Then, the following assertions hold:

(i) For every $\lambda \in [0, c_f^{-1}\|K\|_{L^\infty}^{-1}\inf_{\mathbb{R}^N} V)$, problem $(P_\lambda)$ has only the zero solution;

(ii) There exists $\Lambda_0 > 0$ such that for every $\lambda > \Lambda_0$, problem $(P_\lambda)$ has at least two distinct non-zero radially symmetric solutions in $W$;

(iii) If $f$ is odd and $N \neq 3$, there exists $\Lambda_1 > 0$ such that for every $\lambda > \Lambda_1$ problem $(P_\lambda)$ has at least $s_N = N - 3 (N \mod 2)$ distinct pairs of non-zero solutions $\{\pm u_i^\lambda\} \subset W$, $i = 1, \ldots, s_N$.

The proof of Theorem 1.1 (i) is direct. In order to prove Theorem 1.1 (ii)-(iii), we find critical points of the energy functional associated with problem $(P_\lambda)$ by means of a Ricceri-type three-critical points theorem and the well-known Palais’ principle of symmetric criticality. In particular, the proof of the multiplicity in Theorem 1.1 (iii) requires special treatment. Our strategy is to apply Ricceri’s result to some particular subspaces of $W$ which have two main properties:

- they can be compactly embedded into $L^p(\mathbb{R}^N)$, $p \in (2, 2^*)$;
- they cannot be compared from a symmetrical point of view, i.e., their pairwise intersections contain only the 0 element.

After a careful group-theoretical analysis inspired from Bartsch and Willem [2], we are able to construct $s'_N = \left\lceil \frac{N-1}{2} \right\rceil + (-1)^N$ such subspaces of $W$ whenever $N \neq 3$. Further energy-level analysis together with Ricceri’s multiplicity result provides at least two pairs of distinct non-zero solutions for $(P_\lambda)$ belonging to these subspaces separately whenever $\lambda$ is large enough. Thus, the minimal number of distinct pairs of non-zero solutions for $(P_\lambda)$ is $s_N = 2s'_N = N - 3 (N \mod 2)$. One can also observe that $s_N \geq 2$ for every $N \neq 3$. Furthermore, in each dimension $N \geq 2$, two pairs of solutions are radially symmetric, while if $s_N > 2$ (which occurs for $N = 4$ or $N \geq 6$), the rest of the $(s_N - 2)$ pairs of solutions are sign-changing and non-radially symmetric functions. This statement is based on the aforementioned group-theoretical argument which is described in Section 2.

In Section 2 we recall Ricceri’s three-critical point theorem and display the group-theoretical arguments needed for the proof of Theorem 1.1 (iii). In Section 3 we prove our main theorem.
2 Preliminaries

2.1 A Ricceri-type three-critical point theorem

The functional space

\[ W = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx < \infty \right\} \]

is endowed with its natural inner product \( \langle u, v \rangle_W = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx \) and norm \( \|\cdot\|_W = \sqrt{\langle \cdot, \cdot \rangle_W} \). Due to hypothesis (\( H_V \)), it is clear that the embeddings \( W \subset H^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N) \) are continuous, \( p \in [2, 2^*) \). Here, \( 2^* = \infty \) if \( N = 2 \), and \( 2^* = 2N/(N-2) \) for \( N \geq 3 \). Once \( (f_1) \) and \( (f_2) \) hold, the functional \( F : W \to \mathbb{R} \) defined by

\[ F(u) = \int_{\mathbb{R}^N} K(x)F(u)dx \]

is well-defined, is of class \( C^1 \), and the critical points of the functional \( E_\lambda : W \to \mathbb{R} \) defined by

\[ E_\lambda(u) = \frac{1}{2} \|u\|_W^2 - \lambda F(u) \]

are precisely the weak solutions for problem \( (P_\lambda) \). In order to find critical points for \( E_\lambda \), we will apply the principle of symmetric criticality together with a recent critical point theorem due to Ricceri [10]. In order to recall Ricceri’s result, we need the following definition: if \( X \) is a Banach space, we denote by \( W_X \) the class of those functionals \( E : X \to \mathbb{R} \) having the property that if \( \{u_n\} \) is a sequence in \( X \) converging weakly to \( u \in X \) and \( \liminf_n E(u_n) \leq E(u) \) then \( \{u_n\} \) has a subsequence converging strongly to \( u \).

**Theorem 2.1** [10, Theorem 2] Let \( (X, \| \cdot \|) \) be a separable, reflexive, real Banach space, let \( E_1 : X \to \mathbb{R} \) be a coercive, sequentially weakly lower semicontinuous \( C^1 \) functional belonging to \( W_X \), bounded on each bounded subset of \( X \) and whose derivative admits a continuous inverse on \( X^* \), and let \( E_2 : X \to \mathbb{R} \) be a \( C^1 \) functional with compact derivative. Assume that \( E_1 \) has a strict local minimum point \( u_0 \) with \( E_1(u_0) = E_2(u_0) = 0 \). Assume that \( \tau < \chi \), where

\[
\tau := \max \left\{ 0, \limsup_{\|u\| \to \infty} \frac{E_2(u)}{E_1(u)}, \limsup_{u \to u_0} \frac{E_2(u)}{E_1(u)} \right\}, \quad (2.1)
\]

\[
\chi = \sup_{E_1(u_0) > 0} \frac{E_2(u)}{E_1(u)}. \quad (2.2)
\]

Then, for each compact interval \( [a, b] \subset (1/\chi, 1/\tau) \) (with the conventions \( 1/0 = \infty \) and \( 1/\infty = 0 \)) there exists \( \kappa > 0 \) with the following property: for every \( \lambda \in [a, b] \) and every \( C^1 \) functional \( E_3 : X \to \mathbb{R} \) with compact derivative, there exists \( \delta > 0 \) such that for each \( \theta \in [0, \delta] \), the equation

\[ E_1'(u) - \lambda E_2'(u) - \theta E_3'(u) = 0 \]

admits at least three solutions in \( X \) having norm less than \( \kappa \).
Remark 2.1 A close inspection of the proof of Theorem 2.1 provides us with further information on the critical points of $E_1 - \lambda E_2$ whenever $\lambda \in (1/\tau, 1/\tau)$. First, from (2.1) one can easily deduce that $u_0$ is a strict local minimum point (without being a global minimum point) of $E_1 - \lambda E_2$. Second, since $E_1 - \lambda E_2$ is coercive, bounded from below which satisfies the Palais-Smale condition, there exists a global minimum point $u_1 \in X$ of $E_1 - \lambda E_2$ with $E_1(u_1) - \lambda E_2(u_1) < 0 = E_1(u_0) - \lambda E_2(u_0)$. Third, by applying a mountain pass argument, see Pucci and Serrin [8], one can guarantee the existence of a third critical point $u_2 \in X$ of $E_1 - \lambda E_2$, different from $u_0$ and $u_1$, such that $\phi = E_1(u_2) - \lambda E_2(u_2) \geq \max\{E_1(u_0) - \lambda E_2(u_0), E_1(u_1) - \lambda E_2(u_1)\} = 0$.

2.2 Special symmetries

Let $N \geq 2$ be fixed and assume that a closed subgroup of the orthogonal group $O(N)$ acts on the space $W$, i.e., $(\phi, u) \mapsto \phi * u \in W, \phi \in G, u \in W$. We define the set of fixed points of $W$ with respect to the group $G$ which contains the $G$-invariant functions, i.e.,

$$W_G = \{u \in W : \phi * u = u \text{ for every } \phi \in G\}.$$ 

In particular, if $G = O(N)$ and 's' is the standard linear and isometric action defined as

$$(\phi * u)(x) = u(\phi^{-1} x) \text{ for } x \in \mathbb{R}^N, \phi \in O(N),$$

the set $W_{O(N)}$ is exactly the subspace of radially symmetric functions of $W$. Standard arguments show that $W_{O(N)} \subset L^p(\mathbb{R}^N)$ is compact for every $p \in (2, 2^*)$, see Lions [6].

In order to prove Theorem 1.1 (iii), more specific groups and actions are needed whose origin can be found in Bartsch and Willem [2]. Let us fix $N = 4$ or $N \geq 6$ and define the number

$$t_N = \frac{N - 3}{2} + (-1)^N.$$ 

Note that $t_N \geq 1$ and for every $i \in \{1, ..., t_N\}$, we may introduce the following subgroups of the orthogonal group $O(N)$:

$$G_{N,i} = \begin{cases} O(\frac{N}{2}) \times O(\frac{N}{2}), & \text{if } i = \frac{N-2}{2}, \\
O(i + 1) \times O(N - 2i - 2) \times O(i + 1), & \text{if } i \neq \frac{N-2}{2}. \end{cases}$$

We introduce the involution function $\tau_i : \mathbb{R}^N \rightarrow \mathbb{R}^N$ associated with $G_{N,i}$ by

$$\tau_i(x) = \begin{cases} (x_3, x_1), & \text{if } i = \frac{N+2}{2}, \text{ and } x = (x_1, x_3) \text{ with } x_1, x_3 \in \mathbb{R}^{\frac{N}{2}}; \\
(x_3, x_2, x_1), & \text{if } i \neq \frac{N+2}{2}, \text{ and } x = (x_1, x_2, x_3) \text{ with } x_1, x_3 \in \mathbb{R}^{i+1}, x_2 \in \mathbb{R}^{N-2i-2}. \end{cases}$$

By definition, we clearly have that $\tau_i \notin G_{N,i}$, $\tau_i G_{N,i} \tau_i^{-1} = G_{N,i}$ and $\tau_i^2 = \text{id}_{\mathbb{R}^N}$.

Now, let $G_{N,i}^\tau = (G_{N,i}, \tau_i) = G_{N,i} \cup \tau_i G_{N,i}$. We know from the properties of $\tau_i$ that only two types of elements in $G_{N,i}^\tau$ can be distinguished; namely, $g, g \in G_{N,i}$, and $\phi = \tau_i g \in G_{N,i}^\tau \setminus G_{N,i}$ (with $g \in G_{N,i}$). The action of the compact group $G_{N,i}^\tau$ on $W$ is defined by

$$(g * u)(x) = u(g^{-1} x), \quad ((\tau_i g) * u)(x) = -u(g^{-1} \tau_i^{-1} x),$$

for $g \in G_{N,i}, u \in W$ and $x \in \mathbb{R}^N$. Now from Bartsch and Willem [2, pp. 455-457], the embedding $W_{G_{N,i}^\tau} \subset L^p(\mathbb{R}^N)$ is compact for every $p \in (2, 2^*)$.

The next result is of crucial importance in Theorem 1.1 (iii).
Theorem 2.2 The following statements hold true:

(i) If \( N = 4 \) or \( N \geq 6 \), then \( W_{G_{N,i}^j} \cap W_0(N) = \{ 0 \} \) for all \( i \in \{ 1, ..., t_N \} \);

(ii) If \( N = 6 \) or \( N \geq 8 \), then \( W_{G_{N,i}^j} \cap W_{G_{N,i}^j} = \{ 0 \} \) for every \( i, j \in \{ 1, ..., t_N \} \) with \( i \neq j \).

Proof. (i) Let \( u \in W_{G_{N,i}^j} \cap W_0(N) \). Due to the \( G_{N,i}^j \)-invariance of \( u \) and (2.4) we have \( u(x) = -u(\tau_{i}^{-1}x) \) for every \( x \in \mathbb{R}^{N} \). Since \( u \) is also radial and \( |x| = |\tau_{i}x| \), we necessarily have that \( u = 0 \).

(ii) Although in [5] a similar property is proved, for the sake of completeness we give the proof here as well. Let \( N = 6 \) or \( N \geq 8 \) be fixed; then \( t_N \geq 2 \). Fix \( i, j \in \{ 1, ..., t_N \} \) with \( i < j \) and denote by \( \langle G_{N,i}; G_{N,j} \rangle \) the group generated by \( G_{N,i} \) and \( G_{N,j} \).

We claim that the group \( \langle G_{N,i}; G_{N,j} \rangle \) acts transitively on the sphere \( S^{N-1} \). If this fact holds, the conclusion easily follows. Indeed, let \( u \in W_{G_{N,i}} \cap W_{G_{N,j}} \). In particular, \( u \) is both \( G_{N,i} \)- and \( G_{N,j} \)-invariant, i.e. \( g_i \ast u = g_j \ast u = u \) for every \( g_i \in G_{N,i} \) and \( g_j \in G_{N,j} \), respectively. Consequently, \( u \) is also \( \langle G_{N,i}; G_{N,j} \rangle \)-invariant; thus, \( u(x) = u(g_{ij}x) \) for every \( g_{ij} \in \langle G_{N,i}; G_{N,j} \rangle \) and \( x \in \mathbb{R}^{N} \). Since \( \langle G_{N,i}; G_{N,j} \rangle \) acts transitively on the sphere \( S^{N-1} \), we have that

\[
\text{Orbit}\{g_{ij}x : g_{ij} \in \langle G_{N,i}; G_{N,j} \rangle\} = |x|S^{N-1} \text{ for all } x \in \mathbb{R}^{N}.
\]

Therefore, \( u \) is radially symmetric, and we can apply (i) thus obtaining that \( u = 0 \).

The transitivity of \( \langle G_{N,i}; G_{N,j} \rangle \) on \( S^{N-1} \) will be proved step by step, namely, first on \( S^{N-j-2} \times \{ 0_{j+1} \} \), then on \( S^{N-i-2} \times \{ 0_{i+1} \} \), and finally on the whole \( S^{N-1} \). For simplicity, set \( 0_k = (0, ..., 0) \in \mathbb{R}^k, k \in \{ 1, ..., N \} \).

First step. \( \langle G_{N,i}; G_{N,j} \rangle \) acts transitively on \( S^{N-j-2} \times \{ 0_{j+1} \} \).

When \( j = \frac{N-2}{2} \), then \( G_{N,j} = O(\frac{N}{2}) \times O(\frac{N}{2}) \), thus the proof is trivial since \( O(\frac{N}{2}) \) acts transitively on \( S^{\frac{N-2}{2}} \). Assume now that \( j \neq (N - 2)/2 \). We first show that for every \( x = (x_1, x_2, x_3) \in S^{N-j-2} \) with \( x_1 \in \mathbb{R}^{i+1}, x_2 \in \mathbb{R}^{j-i}, x_3 \in \mathbb{R}^{N-2j-2} \), and \( y \in S^j \) fixed arbitrarily, there exists \( g_{ij} \in \langle G_{N,i}; G_{N,j} \rangle \) such that

\[
g_{ij}(y, 0_{N-j-1}) = (x, 0_{j+1}). \tag{2.5}
\]

Since \( O(j+1) \) acts transitively on \( S^j \), for every \( \tilde{x}_2 \in \mathbb{R}^{j-i} \) with the property that \( (x_1, \tilde{x}_2) \in S^j \), there exists an element \( g_j \in O(j+1) \) such that

\[
g_jy = (x_1, \tilde{x}_2). \tag{2.6}
\]

Note that \( |x_1|^2 + |\tilde{x}_2|^2 = 1 \) and \( |x_1|^2 + |x_2|^2 + |x_3|^2 = 1 \); thus \( |\tilde{x}_2|^2 = |x_2|^2 + |x_3|^2 \).

If \( \tilde{x}_2 = 0_{j-i} \) then \( x_2 = 0_{j-i} \) and \( x_3 = 0_{N-2j-2} \); thus, \( x = (x_1, 0_{N-j-i-2}) \). Let \( g_{ij} := g_j \times id_{\mathbb{R}^{N-j-1}} \in G_{N,j} \). Then, due to (2.6), we have relation (2.5) by

\[
g_{ij}(y, 0_{N-j-1}) = (g_{ij}y, 0_{N-j-1}) = (x_1, 0_{j-i}, 0_{N-j-1}) = (x_1, 0_{N-j-i-2}) = (x, 0_{j+1}).
\]

If \( \tilde{x}_2 \neq 0_{j-i} \), let \( \rho = |\tilde{x}_2| > 0 \). Since \( O(N - 2i - 2) \) acts transitively on \( S^{N-2i-3} \) (thus, also on the sphere \( \rho S^{N-2i-3} \)), there exists \( g_i \in O(N - 2i - 2) \) such that \( g_i(\tilde{x}_2, 0_{N-j-i-2}) = (x_2, x_3, 0_{j-i}) \in \rho S^{N-2i-3} \). Let

\[
\tilde{g}_i = id_{\mathbb{R}^{i+1}} \times g_i \times id_{\mathbb{R}^{i+1}} \in G_{N,j} \text{ and } \tilde{g}_j = g_j \times id_{\mathbb{R}^{N-j-1}} \in G_{N,j},
\]
the element $g_j$ coming from relation (2.6). Then $g_{ij} := \tilde{g}_i \tilde{g}_j \in \langle G_{N;i}; G_{N;j} \rangle$ and on account of (2.6) and $i + 1 < N - j - 1$ (since $i < j \leq t_N$), we obtain relation (2.5) by

$$\tilde{g}_i \tilde{g}_j (y, 0_{N-j-1}) = \tilde{g}_i (g_j y, 0_{N-j-1}) = \tilde{g}_i (x_2, 0_{N-j-1}) = (x_1, g_i (x_2, 0_{N-j-1}), 0_{i+1}) = (x, 0_{i+1}).$$

Now, let $\bar{x}, \tilde{x} \in S^{N-j-2}$. Then, fixing $y \in S^j$, due to (2.5), there are $g_1, g_2 \in \langle G_{N;i}; G_{N;j} \rangle$ such that $g_1 (y, 0_{N-j-1}) = (\bar{x}, 0_{i+1})$ and $g_2 (y, 0_{N-j-1}) = (\tilde{x}, 0_{i+1})$. Consequently, $g_2 g_1^{-1} \in \langle G_{N;i}; G_{N;j} \rangle$ and $g_2 g_1^{-1} (\bar{x}, 0_{i+1}) = (\tilde{x}, 0_{i+1})$, i.e., $\langle G_{N;i}; G_{N;j} \rangle$ acts transitively on $S^{N-j-2} \times \{0_{i+1}\}$.

Second step. $(G_{N;i}; G_{N;j})$ acts transitively on $S^{N-j-2} \times \{0_{i+1}\}$.

The proof is similar to that of the first step. We now show that for every $x = (x_1, x_2, x_3) \in S^{N-j-2}$ with $x_1 \in R^{i+1}$, $x_2 \in R^{N-j-i-2}$, $x_3 \in R^j$, and $y \in S^{N-j-2}$ fixed arbitrarily, there is $g_{ij} \in \langle G_{N;i}; G_{N;j} \rangle$ such that

$$g_{ij} (y, 0_{i+1}) = (x, 0_{i+1}).$$

Let $\tilde{x}_2 \in R^{N-j-i-2}$ be such that $|x_1|^2 + |\tilde{x}_2|^2 = 1$. Then, due to the first step, one can find $\tilde{g}_{ij} \in \langle G_{N;i}; G_{N;j} \rangle$ such that $\tilde{g}_{ij} (y, 0_{i+1}) = (x_1, \tilde{x}_2, 0_{i+1})$.

If $\tilde{x}_2 = 0_{N-j-i-2}$ then $x_2 = 0_{N-j-i-2}$ and $x_3 = 0_{i-j}$. Consequently, (2.7) is verified with the choice $g_{ij} := \tilde{g}_{ij} \in \langle G_{N;i}; G_{N;j} \rangle$.

If $\tilde{x}_2 \neq 0_{N-j-i-2}$ then let $\rho = |\tilde{x}_2| > 0$. Since $O(N-2i-2)$ acts transitively on $S^{N-2i-3}$ (thus, also on the sphere $\rho S^{N-2i-3}$), there exists $g_i \in O(N-2i-2)$ such that $g_i (\tilde{x}_2, 0_{i-j}) = (x_2, x_3) \in \rho S^{N-2i-3}$. Let $\tilde{g}_i = id_{R^{i+1}} \times g_i \times id_{R^{i+1}} \in G_{N;i}$. Then

$$\tilde{g}_i \tilde{g}_{ij} (y, 0_{i+1}) = \tilde{g}_i (x_1, \tilde{x}_2, 0_{i+1}) = (x_1, g_i (\tilde{x}_2, 0_{i-j}), 0_{i+1}) = (x, 0_{i+1}).$$

Consequently, $g_{ij} := \tilde{g}_i \tilde{g}_{ij} \in \langle G_{N;i}; G_{N;j} \rangle$ verifies relation (2.7). Now, following the last part of the first step we obtain the required statement.

Third (concluding) step. $(G_{N;i}; G_{N;j})$ acts transitively on $S^{N-1}$.

We first show that for every $x = (x_1, x_2, x_3) \in S^{N-1}$ with $x_1 \in R^{i+1}$, $x_2 \in R^{N-j-i-2}$, $x_3 \in R^j$, and $y \in S^{N-j-1}$ fixed arbitrarily, there is $g_{ij} \in \langle G_{N;i}; G_{N;j} \rangle$ such that

$$g_{ij} (y, 0_{i+1}) = x.$$  

Let $\tilde{x}_3 \in R^{j}$ such that $|\tilde{x}_3| = |x_3|$. Then, due to the second step, there exists $\tilde{g}_{ij} \in \langle G_{N;i}; G_{N;j} \rangle$ such that $\tilde{g}_{ij} (y, 0_{i+1}) = (x_1, x_2, \tilde{x}_3, 0_{i+1})$.

If $\tilde{x}_3 = 0_{j-i}$ then $x_3 = 0_{i+1}$ and (2.8) is verified with $g_{ij} := \tilde{g}_{ij} \in \langle G_{N;i}; G_{N;j} \rangle$.

If $\tilde{x}_3 \neq 0_{j-i}$, let $\rho = |\tilde{x}_3| = |x_3| > 0$. Since $O(j+1)$ acts transitively on $S^j$, there exists $g_j \in O(j+1)$ such that $g_j (\tilde{x}_3, 0_{i+1}) = x_3 \in \rho S^j$. Let us fix the element $\tilde{g}_j = id_{R^{N-j-i}} \times g_j \in G_{N;j}$. Then the element $g_{ij} := \tilde{g}_j \tilde{g}_{ij} \in \langle G_{N;i}; G_{N;j} \rangle$ verifies relation (2.8); indeed, one has

$$\tilde{g}_j \tilde{g}_{ij} (y, 0_{i+1}) = \tilde{g}_j (x_1, x_2, \tilde{x}_3, 0_{i+1}) = (x_1, x_2, g_j (\tilde{x}_3, 0_{i+1})) = (x_1, x_2, x_3) = x.$$  

Now, let $\bar{x}, \tilde{x} \in S^{N-1}$. Then, fixing $y \in S^{N-j-2}$, on account of (2.8), there are $g_1, g_2 \in \langle G_{N;i}; G_{N;j} \rangle$ such that $g_1 (y, 0_{i+1}) = \bar{x}$ and $g_2 (y, 0_{i+1}) = \tilde{x}$. Consequently, $g_2 g_1^{-1} \in \langle G_{N;i}; G_{N;j} \rangle$ and $g_2 g_1^{-1} (\bar{x}) = \tilde{x}$, i.e., the group $\langle G_{N;i}; G_{N;j} \rangle$ acts transitively on $S^{N-1}$, which concludes the proof. \[ \square \]
3 Proof of Theorem 1.1

In the sequel we assume that all the assumptions of Theorem 1.1 are fulfilled.

Proof of Theorem 1.1 (i). Assume that \( u \in W \) is a solution of \((P_\lambda)\). Multiplying \((P_\lambda)\) by the test function \( u \) and using the definition of the number \( c_f > 0 \), we obtain

\[
\|u\|^2_W = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx
\]

\[
= \lambda \int_{\mathbb{R}^N} K(x)f(u)u
\]

\[
\leq \lambda \frac{\|K\|_{L^\infty}}{\inf_{\mathbb{R}^N} V} c_f \int_{\mathbb{R}^N} V(x)u^2
\]

\[
\leq \lambda \frac{\|K\|_{L^\infty}}{\inf_{\mathbb{R}^N} V} c_f \|u\|^2_W.
\]

Now, if \( 0 \leq \lambda < c_f^{-1} \|K\|_{L^\infty} \inf_{\mathbb{R}^N} V \), the above estimate implies \( u = 0 \), which concludes the proof of (i). \( \square \)

As we pointed out in the Introduction, the solutions of \((P_\lambda)\) are exactly the critical points for the functional \( E_\lambda = E_1 - \lambda E_2 : W \to \mathbb{R} \), where

\[
E_1(u) = \frac{1}{2} \|u\|^2_W \text{ and } E_2(u) = \mathcal{F}(u), \ u \in W.
\]

Before proving (ii) and (iii) of Theorem 1.1, we need the following

Lemma 3.1 (i) \( \limsup_{\|u\|_W \to \infty} \frac{\mathcal{F}(u)}{\|u\|^2_W} \leq 0; \)

(ii) \( \limsup_{u \to 0} \frac{\mathcal{F}(u)}{\|u\|^2_W} \leq 0; \)

(iii) Let \( X \) be a closed subspace of \( W \) which is compactly embedded into \( L^r(\mathbb{R}^N) \), \( r \in (2,2^*). \) Then \( \mathcal{F}|_X \) has a compact derivative.

Proof. Due to \((f_1)\) and \((f_2)\), for every fixed \( \varepsilon > 0 \) there is a \( \delta_\varepsilon \in (0,1) \) such that

\[
|f(s)| < \varepsilon \frac{\inf_{\mathbb{R}^N} V}{\|K\|_{L^\infty}} |s| \text{ for all } |s| \leq \delta_\varepsilon \text{ and } |s| \geq \delta_\varepsilon^{-1}.
\]

Since \( f \in C(\mathbb{R}, \mathbb{R}) \), there also exist two constants \( M_1^\varepsilon, M_2^\varepsilon > 0 \) such that

\[
\frac{|f(s)|}{|s|^{q-1}} \leq M_1^\varepsilon \text{ and } \frac{|f(s)|}{|s|^{p-1}} \leq M_2^\varepsilon \text{ for all } |s| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}],
\]

where \( 1 < q < 2 < p < 2^* \). Combining the above two relations, we obtain that

\[
|f(s)| \leq \varepsilon \frac{\inf_{\mathbb{R}^N} V}{\|K\|_{L^\infty}} |s| + M_1^\varepsilon |s|^{q-1} \text{ for all } s \in \mathbb{R}; \tag{3.1}
\]
\[ |f(s)| \leq \varepsilon \inf_{\mathbb{R}^N} V |s| + M_2^2 |s|^{p-1} \quad \text{for all } s \in \mathbb{R}. \] (3.2)

On account of (3.1), since \( K \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \) and the embedding \( W \subset L^r(\mathbb{R}^N) \) is continuous for \( r \in (2, 2^*) \), by the Hölder inequality one can find \( C_\varepsilon^1 > 0 \) such that

\[
\mathcal{F}(u) \leq \frac{\varepsilon}{2} \|u\|_{W}^2 + C_\varepsilon^1 \|u\|_{W}^q.
\]

Consequently, for every \( u \in W \setminus \{0\} \), we have that

\[
\frac{\mathcal{F}(u)}{\|u\|_{W}^2} \leq \frac{\varepsilon}{2} + C_\varepsilon^1 \|u\|_{W}^{q-2}.
\]

Since \( q < 2 \), the arbitrariness of \( \varepsilon > 0 \) yields (i).

A similar argument based on (3.2) gives the existence of a \( C_\varepsilon^2 > 0 \) such that for every \( u \in W \setminus \{0\} \),

\[
\frac{\mathcal{F}(u)}{\|u\|_{W}^2} \leq \frac{\varepsilon}{2} + C_\varepsilon^2 \|u\|_{W}^{p-2}.
\]

Since \( \varepsilon > 0 \) is arbitrary and \( p > 2 \), (ii) follows readily.

The proof of (iii) is standard. □

For any \( 0 < r_1 < r_2 \), let \( A[r_1, r_2] = \{x \in \mathbb{R}^N : r_1 \leq |x| \leq r_2\} \) be the closed annulus with radii \( r_1 \) and \( r_2 \). Since \( K \in L^\infty(\mathbb{R}^N) \) is a radially symmetric function with \( K \geq 0 \) and \( K \neq 0 \) (cf. hypothesis \((H_K)\)), one can find real numbers \( R > r > 0 \) and \( K_0 > 0 \) such that

\[
\text{essinf}_{x \in A[r, R]} K(x) \geq K_0. \quad (3.3)
\]

**Proof of Theorem 1.1 (ii).** Let \( s_0 \in \mathbb{R} \) from \((f_3)\). For a fixed \( \sigma \in (0, (R - r)/2) \) with \( r, R \) from (3.3), we can define a radially symmetric truncation function \( u_\sigma \in W_0(\mathbb{R}^N) \) such that

(a) \( \text{supp } u_\sigma \subseteq A[r, R] \);

(b) \( \|u_\sigma\|_{L^\infty} \leq |s_0| \);

(c) \( u_\sigma(x) = s_0 \) for every \( x \in A[r + \sigma, R - \sigma] \).

Here is an example of such a function \( u_\sigma : \mathbb{R}^N \to \mathbb{R} \)

\[
u_\sigma(x) = \begin{cases} 
\frac{s_0}{\sigma} (|x| - r)_+ & \text{if } |x| \leq r + \sigma; \\
\frac{s_0}{\sigma} (R - |x|)_+ & \text{if } |x| \geq R - \sigma; \\
s_0 & \text{if } r + \sigma < |x| \leq R - \sigma.
\end{cases}
\]
where \( z_+ = \max(z, 0) \). Denoting by \( \omega_N \) the volume of the unit ball in \( \mathbb{R}^N \), we clearly have from the properties (a)-(c) and relation (3.3) that
\[
\|u_\sigma\|^2_W \geq s_0^2 \omega_N \inf_{\mathbb{R}^N} V ((R - \sigma)^N - (r + \sigma)^N),
\]
and
\[
\mathcal{F}(u_\sigma) \geq \omega_N [K_0 F(s_0) ((R - \sigma)^N - (r + \sigma)^N) - \|K\|_{L^\infty} \max_{|t| \leq |s_0|} |F(t)| \times
\]
\[
\times ((r + \sigma)^N + R^N - (R - \sigma)^N).\]

If \( \sigma \) is close enough to 0, the right-hand sides of both inequalities are strictly positive. Therefore, we can define the number
\[
\lambda_0 = \inf \left\{ \frac{\|u\|^2_W}{2\mathcal{F}(u)} : u \in W_{O(N)}, \mathcal{F}(u) > 0 \right\}.
\] (3.4)

Moreover, it is also clear (cf. Lemma 3.1 and the above estimates) that
\[
\chi_0 = \sup \left\{ \frac{2\mathcal{F}(u)}{\|u\|^2_W} : u \in W_{O(N)} \setminus \{0\} \right\} \in (0, \infty)
\]
and \( \chi_0^{-1} = \lambda_0 \).

Now, we are in a position to apply Theorem 2.1 with \( X = W_{O(N)} \) and \( E_1, E_2 : W_{O(N)} \to \mathbb{R} \) defined by \( E_1 = \mathcal{E}_1|_{\mathcal{E}_{O(N)}} \) and \( E_2 = \mathcal{E}_2|_{\mathcal{E}_{O(N)}} \). On account of Lemma 3.1, the assumptions of Theorem 2.1 are fulfilled with \( u_0 = 0 \in W_{O(N)} \) and \( \tau = 0 \). Thus, for every \( \lambda > \Lambda_0 := \lambda_0 = \chi_0^{-1} > 0 \), the functional \( E_\lambda|_{\mathcal{E}_{O(N)}} \) has at least three distinct critical points in \( W_{O(N)} \). Since \( E_\lambda \) is \( O(N) \)-invariant, i.e., \( E_\lambda(\phi * u) = E_\lambda(u) \) for every \( \phi \in O(N) \) and \( u \in W \) (cf. relation (2.3) and hypotheses \((H_V)\) and \((H_{R'})\)), the principle of symmetric criticality implies that the critical points of \( E_\lambda|_{\mathcal{E}_{O(N)}} \) are also critical points for \( E_\lambda \). This concludes the proof.

\[\square\]

**Remark 3.1** Theorem 2.1 allows us to assert that problem \((P_\lambda)\) is stable to small perturbations of subcritical type. To be more precise, let us consider the perturbed problem
\[
\begin{cases}
-\Delta u + V(x)u = \lambda K(x)f(u) + \mu L(x)g(u) & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\]
\((P_{\lambda, \mu})\)

where \( \mu \in \mathbb{R} \), \( L \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \) is a radially symmetric function, while \( g : \mathbb{R} \to \mathbb{R} \) is a continuous function such that for some \( c > 0 \) and \( 2 < p < 2^* \),
\[
|g(s)| \leq c(|s| + |s|^{p-1}) \text{ for all } s \in \mathbb{R}.
\]

We can easily prove that the function \( J : W \to \mathbb{R} \) defined by
\[
J(u) = \int_{\mathbb{R}^N} L(x)G(u)dx,
\]
is of class \( C^1 \) and \( E_3 = J|_{W_{O(N)}} \) has a compact derivative, where \( G(s) = \int_0^s g(t)dt \).
Consequently, we may apply Theorem 2.1 in its generality providing precise information on the stability of \((P_\lambda)\); namely, problem \((P_{\lambda, \mu})\) has at least three distinct radially symmetric solutions whenever \( \lambda > \Lambda_0 \) and \( \mu \) is small enough. Furthermore, some norm-estimates of the solutions of \((P_{\lambda, \mu})\) are also available on compact intervals of \([\Lambda_0, \infty)\).
\textbf{Proof of Theorem 1.1 (iii).} Let $N \neq 3$. Since $f$ is odd, the energy functional $E_\lambda$ is even, and its critical points (hence solutions for $(P_\lambda)$) appear in symmetric pairs. Consequently, a similar argument as in (ii) shows that there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$, problem $(P_\lambda)$ has at least two pairs $\{\pm u_{0,1}^\lambda\}$ and $\{\pm u_{0,2}^\lambda\}$ of non-zero distinct radially symmetric solutions for $(P_\lambda)$ which belong to $W_0^2(N)$. In the case when $N = 2$ or $N = 5$ we have $s_N = 2$, i.e., the conclusion of (iii) follows from the latter arguments.

Consequently, it remains to consider $N = 4$ or $N \geq 6$. In this case $t_N \geq 1$, so we may fix $i \in \{1, ..., t_N\}$ arbitrarily. Without any loss of generality, we may assume for $0 < r < R$ in relation (3.3) that $r(5 + 4\sqrt{2}) \geq R$. Due to the latter choice, it is clear that the sets

\[
Q_1 = \left\{ (x_1, x_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{i+1} : \sqrt{\left( |x_1| - \frac{R + 3r}{4} \right)^2 + |x_3|^2} \leq \frac{R - r}{4} \right\};
\]

\[
Q_2 = \left\{ (x_1, x_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{i+1} : \sqrt{\left( |x_3| - \frac{R + 3r}{4} \right)^2 + |x_1|^2} \leq \frac{R - r}{4} \right\}
\]

are disjoint. For every $\sigma \in (0, 1]$, we introduce the set

\[
D^i_\sigma = \left\{ x \in \mathbb{R}^N : \sqrt{\left( |x_1| - \frac{R + 3r}{4} \right)^2 + |x_3|^2} \leq \frac{R - r}{4}, \sqrt{\left( |x_3| - \frac{R + 3r}{4} \right)^2 + |x_1|^2} \leq \frac{R - r}{4}, \right. \\
\left. |x_2| \leq \frac{R - r}{4} \right\},
\]

where $x = (x_1, x_2, x_3) \in \mathbb{R}^N$ with $x_1, x_3 \in \mathbb{R}^{i+1}$, $x_2 \in \mathbb{R}^{N-2i-2}$ whenever $i \neq \frac{N-2}{2}$, and $x = (x_1, x_3) \in \mathbb{R}^N$ with $x_1, x_3 \in \mathbb{R}^\frac{N}{2}$ whenever $i = \frac{N-2}{2}$ (and $x_2$ is considered formally 0). Note that the set $D^i_\sigma \subset \mathbb{R}^N$ is $G^\sigma_{N,i}$-invariant, i.e., $\phi D^i_\sigma \subset D^i_\sigma$ for every $\phi \in G^\sigma_{N,i}$. Moreover, $\text{meas}(D^i_\sigma) > 0$ for every $\sigma \in (0, 1]$ and

\[
\lim_{\sigma \to 1} \text{meas}(D^i_\sigma \setminus D^j_\sigma) = 0. \tag{3.5}
\]

Let $s_0 \in \mathbb{R}$ from $(f_3)$ and for a fixed number $\sigma \in (0, 1)$, we construct the following special truncation function

\[
u^i_\sigma(x) = \left[ \left( \frac{R - r}{4} - \max \left( \sqrt{\left( |x_1| - \frac{R + 3r}{4} \right)^2 + |x_3|^2}, \frac{R - r}{4} \right) \right)^+ \right. \\
\left. - \left( \frac{R - r}{4} - \max \left( \sqrt{\left( |x_3| - \frac{R + 3r}{4} \right)^2 + |x_1|^2}, \frac{R - r}{4} \right) \right)^+ \right] \times \\
\left( \frac{R - r}{4} - \max \left( |x_2|, \frac{R - r}{4} \right) \right)^+ \times \frac{16s_0}{(R - r)^2(1 - \sigma)^2}.
\]

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The special shape of $u^i_\sigma$ shows that $\phi * u^i_\sigma = u^i_\sigma$ for every $\phi \in G^*_N$, (see relation (2.4)), thus $u^i_\sigma \in W^*_{G^*_N}$. Moreover, the following useful properties hold:

(a') $\text{supp} u^i_\sigma = D^i_1 \subseteq A[r, R]$;

(b') $\|u^i_\sigma\|_{L^\infty} \leq |s_0|$;

(c') $|u^i_\sigma(x)| = |s_0|$ for every $x \in D^i_\sigma$.

Since $F$ is even (thus $F(s_0) = F(-s_0)$), by exploiting the properties (a')-(c'), we obtain that

$$\mathcal{F}(u^i_\sigma) \geq K_0 F(s_0) \text{meas}(D^i_\sigma) - \|K\|_{L^\infty} \max_{|t|\leq |s_0|} |F(t)| \text{meas}(D^i_\sigma \setminus D^i_\sigma)$.

If $\sigma$ is close enough to 1, the right-hand side of the latter term is strictly positive, see (3.5). Consequently, we can introduce the number

$$\lambda_i = \inf \left\{ \frac{\|u\|^2_W}{2\mathcal{F}(u)} : u \in W^*_{G^*_N}, \mathcal{F}(u) > 0 \right\}. \quad (3.6)$$

As before, one has that

$$\chi_i = \sup \left\{ \frac{2\mathcal{F}(u)}{\|u\|^2_W} : u \in W^*_{G^*_N} \setminus \{0\} \right\} \in (0, \infty)$$

and $\chi_i^{-1} = \lambda_i$.

We can apply Theorem 2.1 with $X = W^*_{G^*_N}$ and $E_1, E_2 : W^*_{G^*_N} \to \mathbb{R}$ defined by $E_1 = \mathcal{E}_1|_{W^*_{G^*_N}}$ and $E_2 = \mathcal{E}_2|_{W^*_{G^*_N}}$. Due to Lemma 3.1, the assumptions of Theorem 2.1 are satisfied with $u_0 = 0 \in W^*_{G^*_N}$, and $\tau = 0$. Consequently, for every $\lambda > \chi_i^{-1} = \lambda_i > 0$, the functional $E_\lambda|_{W^*_{G^*_N}}$ has at least three distinct critical points in $W^*_{G^*_N}$. More precisely (cf. Remark 2.1), one of them is 0 (which is a strict local minimizer of $E_\lambda|_{W^*_{G^*_N}}$), while the other two elements $u^\lambda_{i,1}, u^\lambda_{i,2} \in W^*_{G^*_N} \setminus \{0\}$ are such that $u^\lambda_{i,1}$ is a global minimum point of $E_\lambda|_{W^*_{G^*_N}}$ with $E_\lambda(u^\lambda_{i,1}) < 0$, and $u^\lambda_{i,2}$ is a mountain pass-type critical point of $E_\lambda|_{W^*_{G^*_N}}$ with $E_\lambda(u^\lambda_{i,2}) \geq 0$. Since $f$ is odd, the energy functional $E_\lambda$ is even; in particular, $u^\lambda_{i,1} \neq \pm u^\lambda_{i,2}$, and the pairs $\{\pm u^\lambda_{i,1}\}$ and $\{\pm u^\lambda_{i,2}\}$ are distinct critical points for $E_\lambda|_{W^*_{G^*_N}}$.

Due to the evenness of $E_\lambda$, relation (2.4), and hypotheses $(H_V)$, $(H_K)$, we have that $E_\lambda(\phi * u) = E_\lambda(u)$ for every $\phi \in G^*_N$ and $u \in W$, i.e., $E_\lambda$ is $G^*_N$-invariant on $W$. On account of the principle of symmetric criticality, the critical point pairs $\{\pm u^\lambda_{i,1}\}$ and $\{\pm u^\lambda_{i,2}\}$ of $E_\lambda|_{W^*_{G^*_N}}$ are also critical point pairs for $E_\lambda$ whenever $\lambda > \lambda_i$, hence solutions for problem (P$_i$).

Now, it remains to count the number of distinct solutions of the above type. Due to Theorem 2.2, there are at least $(1 + t_N)$ subspaces of $W$ whose mutual intersections contain only the 0 element:

(I) the subspace $W_{O(N)}$ of radially symmetric functions of $W$, and

(II) $t_N$ subspace(s) of $W$ of the type $W^*_{G^*_N}$. 

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As we pointed out above, each of these subspaces contain two distinct pairs of non-zero solutions for \((P_\lambda)\) whenever \(\lambda\) is large enough. More precisely, if
\[
\lambda > \Lambda_1 := \max\{\lambda_0, \lambda_1, \ldots, \lambda_{t_N}\},
\]
where \(\lambda_0\) comes from the radial case (see (3.4)), while \(\lambda_i\) is from (3.6), \(i \in \{1, \ldots, t_N\}\), problem \((P_\lambda)\) has at least
\[
s_N = 2(1 + t_N) = N - 3(N \mod 2)
\]
distinct pairs of non-zero solutions. This concludes our proof. □

**Remark 3.2** The statement of Theorem 1.1 (iii) is not relevant for \(N = 3\) since \(s_3 = 0\). However, Theorem 1.1 (ii) gives two distinct (pairs of) non-zero, radially symmetric solutions for \((P_\lambda)\) whenever \(\lambda\) is large enough (and \(f\) is odd).

**Remark 3.3** The proof of Theorem 1.1 (iii) shows that in each dimension \(N \geq 2\), two pairs of solutions are radially symmetric. Moreover, if \(N = 4\) or \(N \geq 6\), then \(s_N \geq 4\) and the rest of the \((s_N - 2)\) pairs of solutions are sign-changing and non-radially symmetric functions in \(W\).

**Remark 3.4** From a Strauss-type estimate (see Lions [6]) we know that every \(u \in W\) satisfies \(u(x) \to 0\) as \(|x| \to \infty\). Thus, all solutions in Theorem 1.1 (ii)-(iii) have this property.

**References**


