

# A dimension-dependent multiplicity result for the Schrödinger equation

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## Abstract

We consider the Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = \lambda K(x)f(u) & \text{in } \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases} \quad (P_\lambda)$$

where  $N \geq 2$ ,  $\lambda \geq 0$  is a parameter,  $V, K : \mathbb{R}^N \rightarrow \mathbb{R}$  are radially symmetric functions, and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function with sublinear growth at infinity. We first prove that for  $\lambda$  small enough no non-zero solution exists for  $(P_\lambda)$ , while for  $\lambda$  large enough at least two distinct non-zero radially symmetric solutions do exist. By exploiting a Ricceri-type three-critical points theorem, the principle of symmetric criticality and a group-theoretical approach, the existence of at least  $N - 3(N \bmod 2)$  distinct pairs of non-zero solutions is guaranteed for  $(P_\lambda)$  whenever  $\lambda$  is large enough,  $N \neq 3$ , and  $f$  is odd.

*Keywords:* Schrödinger equation, sublinear, three-critical points theorem, principle of symmetric criticality.

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# 1 Introduction

In this paper we consider the Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = \lambda K(x)f(u) \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \quad (P_\lambda)$$

where  $N \geq 2$ ,  $V, K : \mathbb{R}^N \rightarrow \mathbb{R}$  are some non-negative potentials,  $\lambda \geq 0$  is a parameter, while  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a nonlinear continuous function. The interest in this problem comes from mathematical physics; for instance, certain kinds of solitary waves in the nonlinear Klein-Gordon or Schrödinger equations appear as solutions of problem  $(P_\lambda)$ .

Problem  $(P_\lambda)$  or its related form has been studied by many authors during the last two decades under various assumptions on the potentials  $V, K$  and on the nonlinear function  $f$ . Most of these papers address the case when  $V$  and  $K$  have suitable sign- and growth-properties, and  $f$  has a superlinear and subcritical growth. In these papers existence and multiplicity results for  $(P_\lambda)$  are established via various variational arguments, see Rabinowitz [9], Bartsch *et al.* [1, 2], and further subsequent papers. In particular, if  $f$  is odd, the existence of infinitely many solutions for  $(P_\lambda)$  is usually guaranteed. A particularly interesting paper is due to Clapp and Weth [3] where the existence of at least  $\frac{N}{2} + 1$  pairs of non-zero solutions for  $(P_\lambda)$  is proved for every  $\lambda > 0$  by assuming certain one-sided asymptotic estimates for  $V$  and  $K$  when  $f(s) = |s|^{p-2}s$ ,  $p \in (2, 2^*)$ . Problem  $(P_\lambda)$  has been also studied in the case when  $f$  is odd and has an asymptotically linear growth at infinity; more precisely, Liu, van Heerden and Wang [7] prove a multiplicity result for  $(P_\lambda)$  whenever  $V(x) = \mu g(x) + 1$ ,  $\mu > 0$ , and the number of solutions for  $(P_\lambda)$  depends on the behavior of the dimension of the eigenspace of a specific Dirichlet eigenvalue problem defined on the bounded domain  $\Omega = \text{int}(g^{-1}(0))$ .

The aim of the present paper is to supplement the aforementioned contributions by requiring that the non-zero continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  has a *sublinear growth at infinity* and a *superlinear growth near zero*. More precisely, we assume that

$$(f_1) \quad f(s) = o(|s|) \text{ as } |s| \rightarrow \infty;$$

$$(f_2) \quad f(s) = o(|s|) \text{ as } s \rightarrow 0;$$

$$(f_3) \quad \text{there exists } s_0 \in \mathbb{R} \text{ such that } F(s_0) > 0, \text{ where } F(s) = \int_0^s f(t)dt.$$

In order to avoid technicalities, we assume in the sequel that potentials  $V, K : \mathbb{R}^N \rightarrow \mathbb{R}$  satisfy

$$(H_V) \quad V \in C(\mathbb{R}^N) \text{ is radially symmetric and } \inf_{\mathbb{R}^N} V > 0;$$

$$(H_K) \quad K \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \text{ is radially symmetric and } K \geq 0, K \not\equiv 0.$$

Note that solutions of  $(P_\lambda)$  are being sought in weak form in the space

$$W = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx < \infty \right\}.$$

In fact, under the above conditions on  $f$ ,  $V$  and  $K$  which will be assumed throughout in the sequel, every weak solution  $u$  of  $(P_\lambda)$  is a classical one. Indeed, we have  $\Delta u =: h \in L^2_{\text{loc}}(\mathbb{R}^N)$ , thus  $u \in H^2_{\text{loc}}(\mathbb{R}^N)$  (cf. Evans [4, §8.3]) and  $u$  satisfies  $(P_\lambda)$  for a.a.  $x \in \mathbb{R}^N$ .

The hypotheses  $(f_1)$ ,  $(f_2)$  and  $(f_3)$  guarantee that the number

$$c_f = \max_{s \neq 0} \left| \frac{f(s)}{s} \right|$$

is well-defined, positive and finite. Now, we are in a position to state our main result.

**Theorem 1.1** *Assume that  $N \geq 2$ . Let  $V, K : \mathbb{R}^N \rightarrow \mathbb{R}$  be two potentials such that both  $(H_V)$  and  $(H_K)$  hold, and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function verifying  $(f_1) - (f_3)$ . Then, the following assertions hold:*

- (i) *For every  $\lambda \in [0, c_f^{-1} \|K\|_{L^\infty}^{-1} \inf_{\mathbb{R}^N} V)$ , problem  $(P_\lambda)$  has only the zero solution;*
- (ii) *There exists  $\Lambda_0 > 0$  such that for every  $\lambda > \Lambda_0$ , problem  $(P_\lambda)$  has at least two distinct non-zero radially symmetric solutions in  $W$ ;*
- (iii) *If  $f$  is odd and  $N \neq 3$ , there exists  $\Lambda_1 > 0$  such that for every  $\lambda > \Lambda_1$  problem  $(P_\lambda)$  has at least  $s_N = N - 3(N \bmod 2)$  distinct pairs of non-zero solutions  $\{\pm u_i^\lambda\} \subset W$ ,  $i = 1, \dots, s_N$ .*

The proof of Theorem 1.1 (i) is direct. In order to prove Theorem 1.1 (ii)-(iii), we find critical points of the energy functional associated with problem  $(P_\lambda)$  by means of a Ricceri-type three-critical points theorem and the well-known Palais' principle of symmetric criticality. In particular, the proof of the multiplicity in Theorem 1.1 (iii) requires special treatment. Our strategy is to apply Ricceri's result to some particular *subspaces* of  $W$  which have two main properties:

- they can be compactly embedded into  $L^p(\mathbb{R}^N)$ ,  $p \in (2, 2^*)$ ;
- they cannot be compared from a symmetrical point of view, i.e., their pairwise intersections contain only the 0 element.

After a careful group-theoretical analysis inspired from Bartsch and Willem [2], we are able to construct  $s'_N = \lfloor \frac{N-1}{2} \rfloor + (-1)^N$  such subspaces of  $W$  whenever  $N \neq 3$ . Further energy-level analysis together with Ricceri's multiplicity result provides at least two pairs of distinct non-zero solutions for  $(P_\lambda)$  belonging to these subspaces separately whenever  $\lambda$  is large enough. Thus, the minimal number of distinct pairs of non-zero solutions for  $(P_\lambda)$  is  $s_N = 2s'_N = N - 3(N \bmod 2)$ . One can also observe that  $s_N \geq 2$  for every  $N \neq 3$ . Furthermore, in each dimension  $N \geq 2$ , two pairs of solutions are radially symmetric, while if  $s_N > 2$  (which occurs for  $N = 4$  or  $N \geq 6$ ), the rest of the  $(s_N - 2)$  pairs of solutions are sign-changing and non-radially symmetric functions. This statement is based on the aforementioned group-theoretical argument which is described in Section 2.

In Section 2 we recall Ricceri's three-critical point theorem and display the group-theoretical arguments needed for the proof of Theorem 1.1 (iii). In Section 3 we prove our main theorem.

## 2 Preliminaries

### 2.1 A Ricceri-type three-critical point theorem

The functional space

$$W = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx < \infty \right\}$$

is endowed with its natural inner product  $\langle u, v \rangle_W = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv) dx$  and norm  $\|\cdot\|_W = \sqrt{\langle \cdot, \cdot \rangle_W}$ . Due to hypothesis  $(H_V)$ , it is clear that the embeddings  $W \subset H^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$  are continuous,  $p \in [2, 2^*)$ . Here,  $2^* = \infty$  if  $N = 2$ , and  $2^* = 2N/(N - 2)$  for  $N \geq 3$ . Once  $(f_1)$  and  $(f_2)$  hold, the functional  $\mathcal{F} : W \rightarrow \mathbb{R}$  defined by

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} K(x)F(u)dx$$

is well-defined, is of class  $C^1$ , and the critical points of the functional  $E_\lambda : W \rightarrow \mathbb{R}$  defined by

$$E_\lambda(u) = \frac{1}{2}\|u\|_W^2 - \lambda\mathcal{F}(u)$$

are precisely the weak solutions for problem  $(P_\lambda)$ . In order to find critical points for  $E_\lambda$ , we will apply the principle of symmetric criticality together with a recent critical point theorem due to Ricceri [10]. In order to recall Ricceri's result, we need the following definition: if  $X$  is a Banach space, we denote by  $\mathcal{W}_X$  the class of those functionals  $E : X \rightarrow \mathbb{R}$  having the property that if  $\{u_n\}$  is a sequence in  $X$  converging weakly to  $u \in X$  and  $\liminf_n E(u_n) \leq E(u)$  then  $\{u_n\}$  has a subsequence converging strongly to  $u$ .

**Theorem 2.1** [10, Theorem 2] *Let  $(X, \|\cdot\|)$  be a separable, reflexive, real Banach space, let  $E_1 : X \rightarrow \mathbb{R}$  be a coercive, sequentially weakly lower semicontinuous  $C^1$  functional belonging to  $\mathcal{W}_X$ , bounded on each bounded subset of  $X$  and whose derivative admits a continuous inverse on  $X^*$ ; and let  $E_2 : X \rightarrow \mathbb{R}$  be a  $C^1$  functional with compact derivative. Assume that  $E_1$  has a strict local minimum point  $u_0$  with  $E_1(u_0) = E_2(u_0) = 0$ . Assume that  $\tau < \chi$ , where*

$$\tau := \max \left\{ 0, \limsup_{\|u\| \rightarrow \infty} \frac{E_2(u)}{E_1(u)}, \limsup_{u \rightarrow u_0} \frac{E_2(u)}{E_1(u)} \right\}, \quad (2.1)$$

$$\chi = \sup_{E_1(u) > 0} \frac{E_2(u)}{E_1(u)}. \quad (2.2)$$

*Then, for each compact interval  $[a, b] \subset (1/\chi, 1/\tau)$  (with the conventions  $1/0 = \infty$  and  $1/\infty = 0$ ) there exists  $\kappa > 0$  with the following property: for every  $\lambda \in [a, b]$  and every  $C^1$  functional  $E_3 : X \rightarrow \mathbb{R}$  with compact derivative, there exists  $\delta > 0$  such that for each  $\theta \in [0, \delta]$ , the equation*

$$E_1'(u) - \lambda E_2'(u) - \theta E_3'(u) = 0$$

*admits at least three solutions in  $X$  having norm less than  $\kappa$ .*

**Remark 2.1** A close inspection of the proof of Theorem 2.1 provides us with further information on the critical points of  $E_1 - \lambda E_2$  whenever  $\lambda \in (1/\chi, 1/\tau)$ . First, from (2.1) one can easily deduce that  $u_0$  is a strict local minimum point (without being a global minimum point) of  $E_1 - \lambda E_2$ . Second, since  $E_1 - \lambda E_2$  is coercive, bounded from below which satisfies the Palais-Smale condition, there exists a global minimum point  $u_1 \in X$  of  $E_1 - \lambda E_2$  with  $E_1(u_1) - \lambda E_2(u_1) < 0 = E_1(u_0) - \lambda E_2(u_0)$ . Third, by applying a mountain pass argument, see Pucci and Serrin [8], one can guarantee the existence of a third critical point  $u_2 \in X$  of  $E_1 - \lambda E_2$ , different from  $u_0$  and  $u_1$ , such that  $c_\lambda = E_1(u_2) - \lambda E_2(u_2) \geq \max\{E_1(u_0) - \lambda E_2(u_0), E_1(u_1) - \lambda E_2(u_1)\} = 0$ .

## 2.2 Special symmetries

Let  $N \geq 2$  be fixed and assume that a closed subgroup of the orthogonal group  $\mathbf{O}(N)$  acts on the space  $W$ , i.e.,  $(\phi, u) \mapsto \phi * u \in W$ ,  $\phi \in G$ ,  $u \in W$ . We define the set of fixed points of  $W$  with respect to the group  $G$  which contains the  $G$ -invariant functions, i.e.,

$$W_G = \{u \in W : \phi * u = u \text{ for every } \phi \in G\}.$$

In particular, if  $G = \mathbf{O}(N)$  and  $'*$  is the standard linear and isometric action defined as

$$(\phi * u)(x) = u(\phi^{-1}x) \text{ for } x \in \mathbb{R}^N, \phi \in \mathbf{O}(N), \quad (2.3)$$

the set  $W_{\mathbf{O}(N)}$  is exactly the subspace of radially symmetric functions of  $W$ . Standard arguments show that  $W_{\mathbf{O}(N)} \subset L^p(\mathbb{R}^N)$  is compact for every  $p \in (2, 2^*)$ , see Lions [6].

In order to prove Theorem 1.1 (iii), more specific groups and actions are needed whose origin can be found in Bartsch and Willem [2]. Let us fix  $N = 4$  or  $N \geq 6$  and define the number

$$t_N = \left\lfloor \frac{N-3}{2} \right\rfloor + (-1)^N.$$

Note that  $t_N \geq 1$  and for every  $i \in \{1, \dots, t_N\}$ , we may introduce the following subgroups of the orthogonal group  $\mathbf{O}(N)$ :

$$G_{N,i} = \begin{cases} \mathbf{O}(\frac{N}{2}) \times \mathbf{O}(\frac{N}{2}), & \text{if } i = \frac{N-2}{2}, \\ \mathbf{O}(i+1) \times \mathbf{O}(N-2i-2) \times \mathbf{O}(i+1), & \text{if } i \neq \frac{N-2}{2}. \end{cases}$$

We introduce the involution function  $\tau_i : \mathbb{R}^N \rightarrow \mathbb{R}^N$  associated with  $G_{N,i}$  by

$$\tau_i(x) = \begin{cases} (x_3, x_1), & \text{if } i = \frac{N-2}{2}, \text{ and } x = (x_1, x_3) \text{ with } x_1, x_3 \in \mathbb{R}^{\frac{N}{2}}; \\ (x_3, x_2, x_1), & \text{if } i \neq \frac{N-2}{2}, \text{ and } x = (x_1, x_2, x_3) \text{ with } x_1, x_3 \in \mathbb{R}^{i+1}, x_2 \in \mathbb{R}^{N-2i-2}. \end{cases}$$

By definition, we clearly have that  $\tau_i \notin G_{N,i}$ ,  $\tau_i G_{N,i} \tau_i^{-1} = G_{N,i}$  and  $\tau_i^2 = \text{id}_{\mathbb{R}^N}$ .

Now, let  $G_{N,i}^{\tau_i} = \langle G_{N,i}, \tau_i \rangle = G_{N,i} \cup \tau_i G_{N,i}$ . We know from the properties of  $\tau_i$  that only two types of elements in  $G_{N,i}^{\tau_i}$  can be distinguished; namely,  $\phi = g \in G_{N,i}$ , and  $\phi = \tau_i g \in G_{N,i}^{\tau_i} \setminus G_{N,i}$  (with  $g \in G_{N,i}$ ). The action of the compact group  $G_{N,i}^{\tau_i}$  on  $W$  is defined by

$$(g * u)(x) = u(g^{-1}x), \quad ((\tau_i g) * u)(x) = -u(g^{-1}\tau_i^{-1}x), \quad (2.4)$$

for  $g \in G_{N,i}$ ,  $u \in W$  and  $x \in \mathbb{R}^N$ . Now from Bartsch and Willem [2, pp. 455-457], the embedding  $W_{G_{N,i}^{\tau_i}} \subset L^p(\mathbb{R}^N)$  is compact for every  $p \in (2, 2^*)$ .

The next result is of crucial importance in Theorem 1.1 (iii).

**Theorem 2.2** *The following statements hold true:*

(i) *If  $N = 4$  or  $N \geq 6$ , then  $W_{G_{N,i}^{\tau_i}} \cap W_{\mathbf{O}(N)} = \{0\}$  for all  $i \in \{1, \dots, t_N\}$ ;*

(ii) *If  $N = 6$  or  $N \geq 8$ , then  $W_{G_{N,i}^{\tau_i}} \cap W_{G_{N,j}^{\tau_j}} = \{0\}$  for every  $i, j \in \{1, \dots, t_N\}$  with  $i \neq j$ .*

*Proof.* (i) Let  $u \in W_{G_{N,i}^{\tau_i}} \cap W_{\mathbf{O}(N)}$ . Due to the  $G_{N,i}^{\tau_i}$ -invariance of  $u$  and (2.4) we have  $u(x) = -u(\tau_i^{-1}x)$  for every  $x \in \mathbb{R}^N$ . Since  $u$  is also radial and  $|x| = |\tau_i x|$ , we necessarily have that  $u = 0$ .

(ii) Although in [5] a similar property is proved, for the sake of completeness we give the proof here as well. Let  $N = 6$  or  $N \geq 8$  be fixed; then  $t_N \geq 2$ . Fix  $i, j \in \{1, \dots, t_N\}$  with  $i < j$  and denote by  $\langle G_{N,i}; G_{N,j} \rangle$  the group generated by  $G_{N,i}$  and  $G_{N,j}$ .

We claim that the group  $\langle G_{N,i}; G_{N,j} \rangle$  acts transitively on the sphere  $S^{N-1}$ . If this fact holds, the conclusion easily follows. Indeed, let  $u \in W_{G_{N,i}^{\tau_i}} \cap W_{G_{N,j}^{\tau_j}}$ . In particular,  $u$  is both  $G_{N,i}$ - and  $G_{N,j}$ -invariant, i.e.  $g_i * u = g_j * u = u$  for every  $g_i \in G_{N,i}$  and  $g_j \in G_{N,j}$ , respectively. Consequently,  $u$  is also  $\langle G_{N,i}, G_{N,j} \rangle$ -invariant; thus,  $u(x) = u(g_{ij}x)$  for every  $g_{ij} \in \langle G_{N,i}, G_{N,j} \rangle$  and  $x \in \mathbb{R}^N$ . Since  $\langle G_{N,i}; G_{N,j} \rangle$  acts transitively on the sphere  $S^{N-1}$ , we have that

$$\text{Orbit}\{g_{ij}x : g_{ij} \in \langle G_{N,i}, G_{N,j} \rangle\} = |x|S^{N-1} \text{ for all } x \in \mathbb{R}^N.$$

Therefore,  $u$  is radially symmetric, and we can apply (i) thus obtaining that  $u = 0$ .

The transitivity of  $\langle G_{N,i}, G_{N,j} \rangle$  on  $S^{N-1}$  will be proved step by step, namely, first on  $S^{N-j-2} \times \{0_{j+1}\}$ , then on  $S^{N-i-2} \times \{0_{i+1}\}$ , and finally on the whole  $S^{N-1}$ . For simplicity, set  $0_k = (0, \dots, 0) \in \mathbb{R}^k$ ,  $k \in \{1, \dots, N\}$ .

*First step.*  $\langle G_{N,i}; G_{N,j} \rangle$  acts transitively on  $S^{N-j-2} \times \{0_{j+1}\}$ .

When  $j = \frac{N-2}{2}$ , then  $G_{N,j} = \mathbf{O}(\frac{N}{2}) \times \mathbf{O}(\frac{N}{2})$ , thus the proof is trivial since  $\mathbf{O}(\frac{N}{2})$  acts transitively on  $S^{\frac{N-2}{2}}$ . Assume now that  $j \neq (N-2)/2$ . We first show that for every  $x = (x_1, x_2, x_3) \in S^{N-j-2}$  with  $x_1 \in \mathbb{R}^{i+1}$ ,  $x_2 \in \mathbb{R}^{j-i}$ ,  $x_3 \in \mathbb{R}^{N-2j-2}$ , and  $y \in S^j$  fixed arbitrarily, there exists  $g_{ij} \in \langle G_{N,i}; G_{N,j} \rangle$  such that

$$g_{ij}(y, 0_{N-j-1}) = (x, 0_{j+1}). \quad (2.5)$$

Since  $\mathbf{O}(j+1)$  acts transitively on  $S^j$ , for every  $\tilde{x}_2 \in \mathbb{R}^{j-i}$  with the property that  $(x_1, \tilde{x}_2) \in S^j$ , there exists an element  $g_j \in \mathbf{O}(j+1)$  such that

$$g_j y = (x_1, \tilde{x}_2). \quad (2.6)$$

Note that  $|x_1|^2 + |\tilde{x}_2|^2 = 1$  and  $|x_1|^2 + |x_2|^2 + |x_3|^2 = 1$ ; thus  $|\tilde{x}_2|^2 = |x_2|^2 + |x_3|^2$ .

If  $\tilde{x}_2 = 0_{j-i}$  then  $x_2 = 0_{j-i}$  and  $x_3 = 0_{N-2j-2}$ ; thus,  $x = (x_1, 0_{N-j-i-2})$ . Let  $g_{ij} := g_j \times \text{id}_{\mathbb{R}^{N-j-1}} \in G_{N,j}$ . Then, due to (2.6), we have relation (2.5) by

$$g_{ij}(y, 0_{N-j-1}) = (g_j y, 0_{N-j-1}) = (x_1, 0_{j-i}, 0_{N-j-1}) = (x_1, 0_{N-i-1}) = (x, 0_{j+1}).$$

If  $\tilde{x}_2 \neq 0_{j-i}$ , let  $\rho = |\tilde{x}_2| > 0$ . Since  $\mathbf{O}(N-2i-2)$  acts transitively on  $S^{N-2i-3}$  (thus, also on the sphere  $\rho S^{N-2i-3}$ ), there exists  $g_i \in \mathbf{O}(N-2i-2)$  such that  $g_i(\tilde{x}_2, 0_{N-j-i-2}) = (x_2, x_3, 0_{j-i}) \in \rho S^{N-2i-3}$ . Let

$$\tilde{g}_i = \text{id}_{\mathbb{R}^{i+1}} \times g_i \times \text{id}_{\mathbb{R}^{i+1}} \in G_{N,i} \quad \text{and} \quad \tilde{g}_j = g_j \times \text{id}_{\mathbb{R}^{N-j-1}} \in G_{N,j},$$

the element  $g_j$  coming from relation (2.6). Then  $g_{ij} := \tilde{g}_i \tilde{g}_j \in \langle G_{N,i}; G_{N,j} \rangle$  and on account of (2.6) and  $i + 1 < N - j - 1$  (since  $i < j \leq t_N$ ), we obtain relation (2.5) by

$$\begin{aligned} \tilde{g}_i \tilde{g}_j(y, 0_{N-j-1}) &= \tilde{g}_i(g_j y, 0_{N-j-1}) = \tilde{g}_i(x_1, \tilde{x}_2, 0_{N-j-1}) \\ &= (x_1, g_i(\tilde{x}_2, 0_{N-j-i-2}), 0_{i+1}) = (x_1, x_2, x_3, 0_{j-i}, 0_{i+1}) \\ &= (x, 0_{j+1}). \end{aligned}$$

Now, let  $\bar{x}, \tilde{x} \in S^{N-j-2}$ . Then, fixing  $y \in S^j$ , due to (2.5), there are  $g_1, g_2 \in \langle G_{N,i}; G_{N,j} \rangle$  such that  $g_1(y, 0_{N-j-1}) = (\bar{x}, 0_{j+1})$  and  $g_2(y, 0_{N-j-1}) = (\tilde{x}, 0_{j+1})$ . Consequently,  $g_2 g_1^{-1} \in \langle G_{N,i}; G_{N,j} \rangle$  and  $g_2 g_1^{-1}(\bar{x}, 0_{j+1}) = (\tilde{x}, 0_{j+1})$ , i.e.,  $\langle G_{N,i}; G_{N,j} \rangle$  acts transitively on  $S^{N-j-2} \times \{0_{j+1}\}$ .

*Second step.*  $\langle G_{N,i}; G_{N,j} \rangle$  acts transitively on  $S^{N-i-2} \times \{0_{i+1}\}$ .

The proof is similar to that of the first step. We now show that for every  $x = (x_1, x_2, x_3) \in S^{N-i-2}$  with  $x_1 \in \mathbb{R}^{i+1}$ ,  $x_2 \in \mathbb{R}^{N-j-i-2}$ ,  $x_3 \in \mathbb{R}^{j-i}$ , and  $y \in S^{N-j-2}$  fixed arbitrarily, there is  $g_{ij} \in \langle G_{N,i}; G_{N,j} \rangle$  such that

$$g_{ij}(y, 0_{j+1}) = (x, 0_{i+1}). \quad (2.7)$$

Let  $\tilde{x}_2 \in \mathbb{R}^{N-j-i-2}$  be such that  $|x_1|^2 + |\tilde{x}_2|^2 = 1$ . Then, due to the first step, one can find  $\tilde{g}_{ij} \in \langle G_{N,i}; G_{N,j} \rangle$  such that  $\tilde{g}_{ij}(y, 0_{j+1}) = (x_1, \tilde{x}_2, 0_{j+1})$ .

If  $\tilde{x}_2 = 0_{N-j-i-2}$  then  $x_2 = 0_{N-j-i-2}$  and  $x_3 = 0_{j-i}$ . Consequently, (2.7) is verified with the choice  $g_{ij} := \tilde{g}_{ij} \in \langle G_{N,i}; G_{N,j} \rangle$ .

If  $\tilde{x}_2 \neq 0_{N-j-i-2}$  then let  $\rho = |\tilde{x}_2| > 0$ . Since  $\mathbf{O}(N - 2i - 2)$  acts transitively on  $S^{N-2i-3}$  (thus, also on the sphere  $\rho S^{N-2i-3}$ ), there exists  $g_i \in \mathbf{O}(N - 2i - 2)$  such that  $g_i(\tilde{x}_2, 0_{j-i}) = (x_2, x_3) \in \rho S^{N-2i-3}$ . Let  $\tilde{g}_i = \text{id}_{\mathbb{R}^{i+1}} \times g_i \times \text{id}_{\mathbb{R}^{i+1}} \in G_{N,i}$ . Then

$$\tilde{g}_i \tilde{g}_{ij}(y, 0_{j+1}) = \tilde{g}_i(x_1, \tilde{x}_2, 0_{j+1}) = (x_1, g_i(\tilde{x}_2, 0_{j-i}), 0_{i+1}) = (x, 0_{i+1}).$$

Consequently,  $g_{ij} := \tilde{g}_i \tilde{g}_{ij} \in \langle G_{N,i}; G_{N,j} \rangle$  verifies relation (2.7). Now, following the last part of the first step we obtain the required statement.

*Third (concluding) step.*  $\langle G_{N,i}; G_{N,j} \rangle$  acts transitively on  $S^{N-1}$ .

We first show that for every  $x = (x_1, x_2, x_3) \in S^{N-1}$  with  $x_1 \in \mathbb{R}^{i+1}$ ,  $x_2 \in \mathbb{R}^{N-j-i-2}$ ,  $x_3 \in \mathbb{R}^{j+1}$ , and  $y \in S^{N-i-2}$  fixed arbitrarily, there is  $g_{ij} \in \langle G_{N,i}; G_{N,j} \rangle$  such that

$$g_{ij}(y, 0_{i+1}) = x. \quad (2.8)$$

Let  $\tilde{x}_3 \in \mathbb{R}^{j-i}$  such that  $|\tilde{x}_3| = |x_3|$ . Then, due to the second step, there exists  $\tilde{g}_{ij} \in \langle G_{N,i}; G_{N,j} \rangle$  such that  $\tilde{g}_{ij}(y, 0_{i+1}) = (x_1, x_2, \tilde{x}_3, 0_{i+1})$ .

If  $\tilde{x}_3 = 0_{j-i}$  then  $x_3 = 0_{j+1}$  and (2.8) is verified with  $g_{ij} := \tilde{g}_{ij} \in \langle G_{N,i}; G_{N,j} \rangle$ .

If  $\tilde{x}_3 \neq 0_{j-i}$ , let  $\rho = |\tilde{x}_3| = |x_3| > 0$ . Since  $\mathbf{O}(j+1)$  acts transitively on  $S^j$ , there exists  $g_j \in \mathbf{O}(j+1)$  such that  $g_j(\tilde{x}_3, 0_{i+1}) = x_3 \in \rho S^j$ . Let us fix the element  $\tilde{g}_j = \text{id}_{\mathbb{R}^{N-j-1}} \times g_j \in G_{N,j}$ . Then the element  $g_{ij} := \tilde{g}_j \tilde{g}_{ij} \in \langle G_{N,i}; G_{N,j} \rangle$  verifies relation (2.8); indeed, one has

$$\tilde{g}_j \tilde{g}_{ij}(y, 0_{i+1}) = \tilde{g}_j(x_1, x_2, \tilde{x}_3, 0_{i+1}) = (x_1, x_2, g_j(\tilde{x}_3, 0_{i+1})) = (x_1, x_2, x_3) = x.$$

Now, let  $\bar{x}, \tilde{x} \in S^{N-1}$ . Then, fixing  $y \in S^{N-i-2}$ , on account of (2.8), there are  $g_1, g_2 \in \langle G_{N,i}; G_{N,j} \rangle$  such that  $g_1(y, 0_{i+1}) = \bar{x}$  and  $g_2(y, 0_{i+1}) = \tilde{x}$ . Consequently,  $g_2 g_1^{-1} \in \langle G_{N,i}; G_{N,j} \rangle$  and  $g_2 g_1^{-1}(\bar{x}) = \tilde{x}$ , i.e., the group  $\langle G_{N,i}; G_{N,j} \rangle$  acts transitively on  $S^{N-1}$ , which concludes the proof.  $\square$

### 3 Proof of Theorem 1.1

In the sequel we assume that all the assumptions of Theorem 1.1 are fulfilled.

**Proof of Theorem 1.1 (i).** Assume that  $u \in W$  is a solution of  $(P_\lambda)$ . Multiplying  $(P_\lambda)$  by the test function  $u$  and using the definition of the number  $c_f > 0$ , we obtain

$$\begin{aligned} \|u\|_W^2 &= \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx \\ &= \lambda \int_{\mathbb{R}^N} K(x)f(u)u \\ &\leq \lambda \frac{\|K\|_{L^\infty}}{\inf_{\mathbb{R}^N} V} c_f \int_{\mathbb{R}^N} V(x)u^2 \\ &\leq \lambda \frac{\|K\|_{L^\infty}}{\inf_{\mathbb{R}^N} V} c_f \|u\|_W^2. \end{aligned}$$

Now, if  $0 \leq \lambda < c_f^{-1} \|K\|_{L^\infty}^{-1} \inf_{\mathbb{R}^N} V$ , the above estimate implies  $u = 0$ , which concludes the proof of (i).  $\square$

As we pointed out in the Introduction, the solutions of  $(P_\lambda)$  are exactly the critical points for the functional  $E_\lambda = \mathcal{E}_1 - \lambda \mathcal{E}_2 : W \rightarrow \mathbb{R}$ , where

$$\mathcal{E}_1(u) = \frac{1}{2} \|u\|_W^2 \text{ and } \mathcal{E}_2(u) = \mathcal{F}(u), \quad u \in W.$$

Before proving (ii) and (iii) of Theorem 1.1, we need the following

**Lemma 3.1** (i)  $\limsup_{\|u\|_W \rightarrow \infty} \frac{\mathcal{F}(u)}{\|u\|_W^2} \leq 0$ ;

(ii)  $\limsup_{u \rightarrow 0} \frac{\mathcal{F}(u)}{\|u\|_W^2} \leq 0$ ;

(iii) *Let  $X$  be a closed subspace of  $W$  which is compactly embedded into  $L^r(\mathbb{R}^N)$ ,  $r \in (2, 2^*)$ . Then  $\mathcal{F}|_X$  has a compact derivative.*

*Proof.* Due to  $(f_1)$  and  $(f_2)$ , for every fixed  $\varepsilon > 0$  there is a  $\delta_\varepsilon \in (0, 1)$  such that

$$|f(s)| < \varepsilon \frac{\inf_{\mathbb{R}^N} V}{\|K\|_{L^\infty}} |s| \text{ for all } |s| \leq \delta_\varepsilon \text{ and } |s| \geq \delta_\varepsilon^{-1}.$$

Since  $f \in C(\mathbb{R}, \mathbb{R})$ , there also exist two constants  $M_\varepsilon^1, M_\varepsilon^2 > 0$  such that

$$\frac{|f(s)|}{|s|^{q-1}} \leq M_\varepsilon^1 \text{ and } \frac{|f(s)|}{|s|^{p-1}} \leq M_\varepsilon^2 \text{ for all } |s| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}],$$

where  $1 < q < 2 < p < 2^*$ . Combining the above two relations, we obtain that

$$|f(s)| \leq \varepsilon \frac{\inf_{\mathbb{R}^N} V}{\|K\|_{L^\infty}} |s| + M_\varepsilon^1 |s|^{q-1} \text{ for all } s \in \mathbb{R}; \quad (3.1)$$



$$|f(s)| \leq \varepsilon \frac{\inf_{\mathbb{R}^N} V}{\|K\|_{L^\infty}} |s| + M_\varepsilon^2 |s|^{p-1} \quad \text{for all } s \in \mathbb{R}. \quad (3.2)$$

On account of (3.1), since  $K \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$  and the embedding  $W \subset L^r(\mathbb{R}^N)$  is continuous for  $r \in (2, 2^*)$ , by the Hölder inequality one can find  $C_\varepsilon^1 > 0$  such that

$$\begin{aligned} \mathcal{F}(u) &\leq \int_{\mathbb{R}^N} K(x) |F(u)| \\ &\leq \int_{\mathbb{R}^N} K(x) \left[ \varepsilon \frac{\inf_{\mathbb{R}^N} V}{2\|K\|_{L^\infty}} u^2 + \frac{M_\varepsilon^1}{q} |u|^q \right] \\ &\leq \frac{\varepsilon}{2} \|u\|_W^2 + C_\varepsilon^1 \|u\|_W^q. \end{aligned}$$

Consequently, for every  $u \in W \setminus \{0\}$ , we have that

$$\frac{\mathcal{F}(u)}{\|u\|_W^2} \leq \frac{\varepsilon}{2} + C_\varepsilon^1 \|u\|_W^{q-2}.$$

Since  $q < 2$ , the arbitrariness of  $\varepsilon > 0$  yields (i).

A similar argument based on (3.2) gives the existence of a  $C_\varepsilon^2 > 0$  such that for every  $u \in W \setminus \{0\}$ ,

$$\frac{\mathcal{F}(u)}{\|u\|_W^2} \leq \frac{\varepsilon}{2} + C_\varepsilon^2 \|u\|_W^{p-2}.$$

Since  $\varepsilon > 0$  is arbitrary and  $p > 2$ , (ii) follows readily.

The proof of (iii) is standard. □

For any  $0 < r_1 < r_2$ , let  $A[r_1, r_2] = \{x \in \mathbb{R}^N : r_1 \leq |x| \leq r_2\}$  be the closed annulus with radii  $r_1$  and  $r_2$ . Since  $K \in L^\infty(\mathbb{R}^N)$  is a radially symmetric function with  $K \geq 0$  and  $K \not\equiv 0$  (cf. hypothesis  $(H_K)$ ), one can find real numbers  $R > r > 0$  and  $K_0 > 0$  such that

$$\operatorname{ess\,inf}_{x \in A[r, R]} K(x) \geq K_0. \quad (3.3)$$

**Proof of Theorem 1.1 (ii).** Let  $s_0 \in \mathbb{R}$  from  $(f_3)$ . For a fixed  $\sigma \in (0, (R - r)/2)$  with  $r, R$  from (3.3), we can define a radially symmetric truncation function  $u_\sigma \in W_{\mathbf{O}(N)}$  such that

- (a)  $\operatorname{supp} u_\sigma \subseteq A[r, R]$ ;
- (b)  $\|u_\sigma\|_{L^\infty} \leq |s_0|$ ;
- (c)  $u_\sigma(x) = s_0$  for every  $x \in A[r + \sigma, R - \sigma]$ .

Here is an example of such a function  $u_\sigma : \mathbb{R}^N \rightarrow \mathbb{R}$

$$u_\sigma(x) = \begin{cases} \frac{s_0}{\sigma} (|x| - r)_+ & \text{if } |x| \leq r + \sigma; \\ s_0 & \text{if } r + \sigma < |x| \leq R - \sigma; \\ \frac{s_0}{\sigma} (R - |x|)_+ & \text{if } |x| \geq R - \sigma, \end{cases}$$

where  $z_+ = \max(z, 0)$ . Denoting by  $\omega_N$  the volume of the unit ball in  $\mathbb{R}^N$ , we clearly have from the properties (a)-(c) and relation (3.3) that

$$\|u_\sigma\|_W^2 \geq s_0^2 \omega_N \inf_{\mathbb{R}^N} V \left( (R - \sigma)^N - (r + \sigma)^N \right),$$

and

$$\begin{aligned} \mathcal{F}(u_\sigma) &\geq \omega_N [K_0 F(s_0) \left( (R - \sigma)^N - (r + \sigma)^N \right) - \|K\|_{L^\infty} \max_{|t| \leq |s_0|} |F(t)| \times \\ &\quad \times \left( (r + \sigma)^N - r^N + R^N - (R - \sigma)^N \right)]. \end{aligned}$$

If  $\sigma$  is close enough to 0, the right-hand sides of both inequalities are strictly positive. Therefore, we can define the number

$$\lambda_0 = \inf \left\{ \frac{\|u\|_W^2}{2\mathcal{F}(u)} : u \in W_{\mathbf{O}(N)}, \mathcal{F}(u) > 0 \right\}. \quad (3.4)$$

Moreover, it is also clear (cf. Lemma 3.1 and the above estimates) that

$$\chi_0 = \sup \left\{ \frac{2\mathcal{F}(u)}{\|u\|_W^2} : u \in W_{\mathbf{O}(N)} \setminus \{0\} \right\} \in (0, \infty)$$

and  $\chi_0^{-1} = \lambda_0$ .

Now, we are in a position to apply Theorem 2.1 with  $X = W_{\mathbf{O}(N)}$  and  $E_1, E_2 : W_{\mathbf{O}(N)} \rightarrow \mathbb{R}$  defined by  $E_1 = \mathcal{E}_1|_{W_{\mathbf{O}(N)}}$  and  $E_2 = \mathcal{E}_2|_{W_{\mathbf{O}(N)}}$ . On account of Lemma 3.1, the assumptions of Theorem 2.1 are fulfilled with  $u_0 = 0 \in W_{\mathbf{O}(N)}$  and  $\tau = 0$ . Thus, for every  $\lambda > \Lambda_0 := \lambda_0 = \chi_0^{-1} > 0$ , the functional  $E_\lambda|_{W_{\mathbf{O}(N)}}$  has at least three distinct critical points in  $W_{\mathbf{O}(N)}$ . Since  $E_\lambda$  is  $\mathbf{O}(N)$ -invariant, i.e.,  $E_\lambda(\phi * u) = E_\lambda(u)$  for every  $\phi \in \mathbf{O}(N)$  and  $u \in W$  (cf. relation (2.3) and hypotheses  $(H_V)$  and  $(H_K)$ ), the principle of symmetric criticality implies that the critical points of  $E_\lambda|_{W_{\mathbf{O}(N)}}$  are also critical points for  $E_\lambda$ . This concludes the proof.  $\square$

**Remark 3.1** Theorem 2.1 allows us to assert that problem  $(P_\lambda)$  is stable to small perturbations of subcritical type. To be more precise, let us consider the perturbed problem

$$\begin{cases} -\Delta u + V(x)u = \lambda K(x)f(u) + \mu L(x)g(u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \quad (P_{\lambda,\mu})$$

where  $\mu \in \mathbb{R}$ ,  $L \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$  is a radially symmetric function, while  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function such that for some  $c > 0$  and  $2 < p < 2^*$ ,

$$|g(s)| \leq c(|s| + |s|^{p-1}) \text{ for all } s \in \mathbb{R}.$$

We can easily prove that the function  $J : W \rightarrow \mathbb{R}$  defined by

$$J(u) = \int_{\mathbb{R}^N} L(x)G(u)dx,$$

is of class  $C^1$  and  $E_3 = J|_{W_{\mathbf{O}(N)}}$  has a compact derivative, where  $G(s) = \int_0^s g(t)dt$ . Consequently, we may apply Theorem 2.1 in its generality providing precise information on the stability of  $(P_\lambda)$ ; namely, problem  $(P_{\lambda,\mu})$  has at least three distinct radially symmetric solutions whenever  $\lambda > \Lambda_0$  and  $\mu$  is small enough. Furthermore, some norm-estimates of the solutions of  $(P_{\lambda,\mu})$  are also available on compact intervals of  $[\Lambda_0, \infty)$ .

**Proof of Theorem 1.1 (iii).** Let  $N \neq 3$ . Since  $f$  is odd, the energy functional  $E_\lambda$  is even, and its critical points (hence solutions for  $(P_\lambda)$ ) appear in symmetric pairs. Consequently, a similar argument as in (ii) shows that there exists  $\lambda_0 > 0$  such that for every  $\lambda > \lambda_0$ , problem  $(P_\lambda)$  has at least two pairs  $\{\pm u_{0,1}^\lambda\}$  and  $\{\pm u_{0,2}^\lambda\}$  of non-zero distinct radially symmetric solutions for  $(P_\lambda)$  which belong to  $W_{\mathbf{O}(N)}$ . In the case when  $N = 2$  or  $N = 5$  we have  $s_N = 2$ , i.e., the conclusion of (iii) follows from the latter arguments.

Consequently, it remains to consider  $N = 4$  or  $N \geq 6$ . In this case  $t_N \geq 1$ , so we may fix  $i \in \{1, \dots, t_N\}$  arbitrarily. Without any loss of generality, we may assume for  $0 < r < R$  in relation (3.3) that  $r(5 + 4\sqrt{2}) \geq R$ . Due to the latter choice, it is clear that the sets

$$Q_1 = \left\{ (x_1, x_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{i+1} : \sqrt{\left(|x_1| - \frac{R+3r}{4}\right)^2 + |x_3|^2} \leq \frac{R-r}{4} \right\};$$

$$Q_2 = \left\{ (x_1, x_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{i+1} : \sqrt{\left(|x_3| - \frac{R+3r}{4}\right)^2 + |x_1|^2} \leq \frac{R-r}{4} \right\}$$

are disjoint. For every  $\sigma \in (0, 1]$ , we introduce the set

$$D_\sigma^i = \left\{ x \in \mathbb{R}^N : \sqrt{\left(|x_1| - \frac{R+3r}{4}\right)^2 + |x_3|^2} \leq \sigma \frac{R-r}{4}, \right. \\ \left. \sqrt{\left(|x_3| - \frac{R+3r}{4}\right)^2 + |x_1|^2} \leq \sigma \frac{R-r}{4}, \right. \\ \left. |x_2| \leq \sigma \frac{R-r}{4} \right\},$$

where  $x = (x_1, x_2, x_3) \in \mathbb{R}^N$  with  $x_1, x_3 \in \mathbb{R}^{i+1}$ ,  $x_2 \in \mathbb{R}^{N-2i-2}$  whenever  $i \neq \frac{N-2}{2}$ , and  $x = (x_1, x_3) \in \mathbb{R}^N$  with  $x_1, x_3 \in \mathbb{R}^{\frac{N}{2}}$  whenever  $i = \frac{N-2}{2}$  (and  $x_2$  is considered formally 0). Note that the set  $D_\sigma^i \subset \mathbb{R}^N$  is  $G_{N,i}^{\tau_i}$ -invariant, i.e.,  $\phi D_\sigma^i \subset D_\sigma^i$  for every  $\phi \in G_{N,i}^{\tau_i}$ . Moreover,  $\text{meas}(D_\sigma^i) > 0$  for every  $\sigma \in (0, 1]$  and

$$\lim_{\sigma \rightarrow 1} \text{meas}(D_1^i \setminus D_\sigma^i) = 0. \quad (3.5)$$

Let  $s_0 \in \mathbb{R}$  from  $(f_3)$  and for a fixed number  $\sigma \in (0, 1)$ , we construct the following special truncation function

$$u_\sigma^i(x) = \left[ \left( \frac{R-r}{4} - \max \left( \sqrt{\left(|x_1| - \frac{R+3r}{4}\right)^2 + |x_3|^2}, \sigma \frac{R-r}{4} \right) \right)_+ \right. \\ \left. - \left( \frac{R-r}{4} - \max \left( \sqrt{\left(|x_3| - \frac{R+3r}{4}\right)^2 + |x_1|^2}, \sigma \frac{R-r}{4} \right) \right)_+ \right] \times \\ \times \left( \frac{R-r}{4} - \max \left( |x_2|, \sigma \frac{R-r}{4} \right) \right)_+ \frac{16s_0}{(R-r)^2(1-\sigma)^2}.$$

The special shape of  $u_\sigma^i$  shows that  $\phi * u_\sigma^i = u_\sigma^i$  for every  $\phi \in G_{N,i}^{\tau_i}$  (see relation (2.4)), thus  $u_\sigma^i \in W_{G_{N,i}^{\tau_i}}$ . Moreover, the following useful properties hold:

- (a')  $\text{supp} u_\sigma^i = D_1^i \subseteq A[r, R]$ ;
- (b')  $\|u_\sigma^i\|_{L^\infty} \leq |s_0|$ ;
- (c')  $|u_\sigma^i(x)| = |s_0|$  for every  $x \in D_\sigma^i$ .

Since  $F$  is even (thus  $F(s_0) = F(-s_0)$ ), by exploiting the properties (a')-(c'), we obtain that

$$\mathcal{F}(u_\sigma^i) \geq K_0 F(s_0) \text{meas}(D_\sigma^i) - \|K\|_{L^\infty} \max_{|t| \leq |s_0|} |F(t)| \text{meas}(D_1^i \setminus D_\sigma^i).$$

If  $\sigma$  is close enough to 1, the right-hand side of the latter term is strictly positive, see (3.5). Consequently, we can introduce the number

$$\lambda_i = \inf \left\{ \frac{\|u\|_W^2}{2\mathcal{F}(u)} : u \in W_{G_{N,i}^{\tau_i}}, \mathcal{F}(u) > 0 \right\}. \quad (3.6)$$

As before, one has that

$$\chi_i = \sup \left\{ \frac{2\mathcal{F}(u)}{\|u\|_W^2} : u \in W_{G_{N,i}^{\tau_i}} \setminus \{0\} \right\} \in (0, \infty)$$

and  $\chi_i^{-1} = \lambda_i$ .

We can apply Theorem 2.1 with  $X = W_{G_{N,i}^{\tau_i}}$  and  $E_1, E_2 : W_{G_{N,i}^{\tau_i}} \rightarrow \mathbb{R}$  defined by  $E_1 = \mathcal{E}_1|_{W_{G_{N,i}^{\tau_i}}}$  and  $E_2 = \mathcal{E}_2|_{W_{G_{N,i}^{\tau_i}}}$ . Due to Lemma 3.1, the assumptions of Theorem 2.1 are satisfied with  $u_0 = 0 \in W_{G_{N,i}^{\tau_i}}$  and  $\tau = 0$ . Consequently, for every  $\lambda > \chi_i^{-1} = \lambda_i > 0$ , the functional  $E_\lambda|_{W_{G_{N,i}^{\tau_i}}}$  has at least three distinct critical points in  $W_{G_{N,i}^{\tau_i}}$ . More precisely (cf. Remark 2.1), one of them is 0 (which is a strict local minimizer of  $E_\lambda|_{W_{G_{N,i}^{\tau_i}}}$ ), while the other two elements  $u_{i,1}^\lambda, u_{i,2}^\lambda \in W_{G_{N,i}^{\tau_i}} \setminus \{0\}$  are such that  $u_{i,1}^\lambda$  is a global minimum point of  $E_\lambda|_{W_{G_{N,i}^{\tau_i}}}$  with  $E_\lambda(u_{i,1}^\lambda) < 0$ , and  $u_{i,2}^\lambda$  is a mountain pass-type critical point of  $E_\lambda|_{W_{G_{N,i}^{\tau_i}}}$  with  $E_\lambda(u_{i,2}^\lambda) \geq 0$ . Since  $f$  is odd, the energy functional  $E_\lambda$  is even; in particular,  $u_{i,1}^\lambda \neq \pm u_{i,2}^\lambda$ , and the pairs  $\{\pm u_{i,1}^\lambda\}$  and  $\{\pm u_{i,2}^\lambda\}$  are distinct critical points for  $E_\lambda|_{W_{G_{N,i}^{\tau_i}}}$ .

Due to the evenness of  $E_\lambda$ , relation (2.4), and hypotheses  $(H_V)$ ,  $(H_K)$ , we have that  $E_\lambda(\phi * u) = E_\lambda(u)$  for every  $\phi \in G_{N,i}^{\tau_i}$  and  $u \in W$ , i.e.,  $E_\lambda$  is  $G_{N,i}^{\tau_i}$ -invariant on  $W$ . On account of the principle of symmetric criticality, the critical point pairs  $\{\pm u_{i,1}^\lambda\}$  and  $\{\pm u_{i,2}^\lambda\}$  of  $E_\lambda|_{W_{G_{N,i}^{\tau_i}}}$  are also critical point pairs for  $E_\lambda$  whenever  $\lambda > \lambda_i$ , hence solutions for problem  $(P_\lambda)$ .

Now, it remains to count the number of distinct solutions of the above type. Due to Theorem 2.2, there are at least  $(1 + t_N)$  subspaces of  $W$  whose mutual intersections contain only the 0 element:

- (I) the subspace  $W_{\mathbf{O}(N)}$  of radially symmetric functions of  $W$ , and
- (II)  $t_N$  subspace(s) of  $W$  of the type  $W_{G_{N,i}^{\tau_i}}$ .

As we pointed out above, each of these subspaces contain two distinct pairs of non-zero solutions for  $(P_\lambda)$  whenever  $\lambda$  is large enough. More precisely, if

$$\lambda > \Lambda_1 := \max\{\lambda_0, \lambda_1, \dots, \lambda_{t_N}\},$$

where  $\lambda_0$  comes from the radial case (see (3.4)), while  $\lambda_i$  is from (3.6),  $i \in \{1, \dots, t_N\}$ , problem  $(P_\lambda)$  has at least

$$s_N = 2(1 + t_N) = N - 3(N \bmod 2)$$

distinct pairs of non-zero solutions. This concludes our proof.  $\square$

**Remark 3.2** The statement of Theorem 1.1 (iii) is not relevant for  $N = 3$  since  $s_3 = 0$ . However, Theorem 1.1 (ii) gives two distinct (pairs of) non-zero, radially symmetric solutions for  $(P_\lambda)$  whenever  $\lambda$  is large enough (and  $f$  is odd).

**Remark 3.3** The proof of Theorem 1.1 (iii) shows that in each dimension  $N \geq 2$ , two pairs of solutions are radially symmetric. Moreover, if  $N = 4$  or  $N \geq 6$ , then  $s_N \geq 4$  and the rest of the  $(s_N - 2)$  pairs of solutions are sign-changing and non-radially symmetric functions in  $W$ .

**Remark 3.4** From a Strauss-type estimate (see Lions [6]) we know that every  $u \in W$  satisfies  $u(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Thus, all solutions in Theorem 1.1 (ii)-(iii) have this property.

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