Linear quasigroups. II

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Abstract

The article is a continuation of the author's work "Linear quasigroups. I" and devoted to linear quasigroups and some of their generalizations. In the second part identities and linearity of quasigroups are investigated, in particular, the approach of finding quasigroup identities from some variety of loops is presented. Besides that we give short chronology of development the theory of quasigroups and loops and some historical aspects.

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I. Identities and a linearity of quasigroups

First note that all necessary definitions and notions can be found in [1, 2].

It is well known that the theory of identities in various algebraic systems plays an important role. In the theory of the quasigroups many classes are given by identities (for example, medial quasigroups, distributive quasigroups, Steiner quasigroups, CH-quasigroups, F-quasigroups etc.).

The theory of identities in the algebras has two interrelated aspects: *identities and algebras.* Accordingly, there are two global questions (see [3]):

(1) describe algebras with identities;

(2) describe identities in algebras.

In the first case we have a structure of algebra, in which an identity or systems of identities of a special form is satisfied. Often, on the structure of algebras noticeably affected only execution of a nontrivial identity, for example in the situation of the associative rings. There are excellent examples when the algebra of a certain natural class is completely determined by its identities.

The second case is about finding the specific identities of algebras or a class of algebras. It should be noted that in this formulation the problem is extremely difficult. For example, in many cases there is no clarity in question of existence of the finite basis of a system of identities (*the problem of finite basis*). Moreover, the problem of identities, as a rule, is separated from themselves algebras, in which these identities are satisfied, although, of course, for each system of

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identities a class of algebras can be chosen for which this system is defined. Here we will discuss varieties of algebras. The language of varieties, in fact erases a difference between algebras and identities, or rather, it allows easily move from one of these concepts to another. Precisely in this language, as a rule, the work in the theory of identities is provided.

To this we shall add that namely in the second part of the question G.B. Belyavskaya and A.Kh. Tabarov in [4-6] characterized the classes of linear, alinear, mixed type of linearity, onesided linear and alinear quasigroups, T-quasigroups and some of their generalizations by identities and system of identities.

One of the properties of the quasigroup which characterizes its nearness to groups, is availability of its isotopy to groups. The class of quasigroups which are isotopic to groups was researched by many algebraists. This class is near to groups and so their construction largely is easy for investigation. However, in this class there are many difficult problems which are still unresolved.

The class of quasigroups are isotopic to groups, first was investigated by V.D. Belousov [7]. In particular, V.D. Belousov proved that the class of quasigroups isotopic to groups is characterized by the following identity of five variables:

$$x\,(y\backslash\,((z/u\,)\,v\,)\,) = ((x\,(y\backslash z\,\,))\,/u\,)\,v\ .$$

In other words a variety of quasigroups are isotopic to groups characterized by this identity. Later, F.N.Sokhatsky noticed that quasigroups which are isotopic to groups, can be characterized by the identity with four variables [8]:

$$[(x(u\backslash y))/u]z = x[u\backslash((y/x)z)].$$

A variety are also all quasigroups isotopic to abelian groups. It also was first noted by V.D.Belousov [7]. This class of quasigroups characterized by the identity of the four variables:

$$x\backslash(y(u\backslash v)) = u\backslash(y(x\backslash v)).$$

In [2] we have given some information about linear quasigroups and their generalizations. Recall that a quasigroup $(Q,\cdot)$ is called *linear* over a group $(Q,+)$, if it has the form:

$$xy = \varphi x + c + \psi y, \tag{1}$$

where $\varphi,\psi \in Aut(Q,+)$, c is a fixed element from Q.

By analogy with linear quasigroups, G.B. Belyavskaya and A.KH. Tabarov in [4,5] defined alinear quasigroups, as well as introduced the classes of left and right linear quasigroups, left and right alinear quasigroups and mixed type of linearity.

In any algebra identities are closely connected with algorithmic problems. Articles of T. Ivens [9,10], M.M. Glukhov and A.A. Gvaramiya [11] were devoted to the algorithmic problems of the theory of quasigroups. In 1951 T. Ivens [9] proved an assertion about positive solvability of word problem for finitely-presented algebras of all variety algebras $V(\Sigma)$, in which the theorem about embedding of such extremity partial (Σ)-algebra in algebra from V has a place. Further, by the way of carried words to canonical type T. Ivens proved the isomorphism problem for some classes of multiplicative system, in particular for quasigroups and loops [10].

In 1970-1971 M.M. Glukhov [12] formulated the condition which is stronger then the theorem of embedding and such that in variety not only quasigroups, but in universal algebras the algorithmic problems of word, isomorphism and embedding are positively solved. M.M. Glukhov called this condition R. The varieties on algebra in which the condition R is satisfied, were called R-varieties.

In the research of algebras of one variety, sometimes it should be helpful to use algebras of other variety concerned with it. Exactly this method was used in a series of works during the research of quasigroups isotopic to groups. In this connection it is important to consider the approach know as equivalence and rational equivalence of a class of algebras presented in [13] which we have used in proving the main theorem. Using the notions of an equivalence and a rational equivalence in [14] was proved that class primitive linear quasigroups, and in particular, T-quasigroups are varieties. G.B. Belyavskaya and A.Kh. Tabarov in [5,6] established the various systems of identities which characterized some varieties of linear quasigroups and T-quasigroups.

We present here the meaning of the equivalence class of algebras introduced in [13].

Definition 1. *The class of algebras K_1, K_2 with accordingly to signatures Δ_1, Δ_2 are called equivalent if the bijective map*

$$f : K_1 \to K_2,$$

exists, satisfying the conditions:

1) For any algebra $A \in K_1$ the basis algebras A and $f(A)$ coincide;

2) For any algebras $A, B \in K_1$ the map of A in B is a homomorphism if and only if it is a homomorphism of algebra $f(A)$ in $f(B)$.

At the same time the map f is called *an equivalence* between classes K_1 and K_2.

If f is an equivalence between the varieties of algebras K_1 and K_2, then the following assertions are hold:

- Subset $A_1 \subset A$ is the subalgebra of algebra A, if and only if $f(A_1)$ is the subalgebra of algebra $f(A)$.

- Subset $M \subset A$ generates the algebra A if and only if M generates the algebra $f(A)$.

- An equivalent relation in A is a congruence of algebra A if and only if it is a congruence of algebra $f(A)$.

- Algebra $A \in K_1$ is free in variety K_1 with the basis M if and only if $f(A)$ is free in K_2 with the basis M.

- A class of algebras L from K_1 is a subvariety in K_1 if and only if the class $\{f(A) : A \in L\}$ is a subvariety in K_2.

Among all the classes of equivalences of algebras, rational equivalences particulary stand out.

Let K be the class of algebras of signature Δ and X be the alphabet of variables. Then on each Δ-word P in alphabet X containing in its entries exactly n variables, for example $x_1, \ldots, x_n$, the n-ary operation w_P on every algebra A from K could be determined. The meaning of this operation on elements $a_1, \ldots, a_n \in A$ is equal to the meaning in A of a word received from P replacing $x_1, \ldots, x_n$ respectively with the elements $a_1, \ldots, a_n$. A such determined operation is called *a derivative operation* in signature Δ corresponding to the Δ-word P.

The transfer of the signature Δ_1 into signature Δ_2 is called any reflection $\tau\colon \Delta_1 \to \Delta_2$, such that every operation $h \in \Delta_1$ is reflected in some derivative operation in the same arity in signature Δ_2. If τ is such a transfer then on any algebra B of signature Δ_2 the algebra $A = T_\tau(B)$ of signature Δ_1 could be determined with the same main set if an operation $h \in \Delta_1$ which coincides with τ(h) could also be determined.

The class of algebras K_1, K_2 with the respective signatures Δ_1, Δ_2 is called rationally equivalent if the transfer of signatures $\tau\colon \Delta_1 \to \Delta_2$ and $\sigma\colon \Delta_2 \to \Delta_1$ and the bijective mapping $f : K_1 \to K_2$ exists such that $T_\tau T_\sigma$ and $T_\sigma T_\tau$ are identity mappings.

It is clear that a rational equivalence class of the algebras is an equivalence and therefore for rational equivalence classes the above introduced assertion is true.

In variety of quasigroups as well as for all algebras, a free quasigroups of this variety plays the important role. However for all variety of algebras to constructively describe free quasigroup with the basis A in this or that variety of quasigroup it is necessary to have an algorithm of recognition of equivalence of words. In the general case this problem is very difficult. The solution of this problem in given variety $Q(\Omega,\Sigma)$, naturally depends on determination of its system of identities Σ. There are varieties of quasigroups with solved problem of words. All R-varieties [11,12] are related to these varieties. At the same time there exists varieties with unsolved problem of words in free quasigroups [15].

For any variety of quasigroups there is an interesting question about a connection of its free quasigroups with the free groups of its isotopy closing group. First this problem for the variety of T-quasigroups was considered in [16,17]. In these works the construction of free T-quasigroups, were offered based on the use of the concept rational equivalence of varieties of algebra with different signatures were introduced in [14](see also [18]).

The class of quasigroups $Q(\Gamma)$ in the signature $\Omega_1 = \{\cdot, /, \backslash, u\}$, where u is symbol of 0-ary operation and variety of U algebras of signature $\Omega_2 = \{+, -, 0,\ \alpha, \beta, \gamma, \delta, c\}$, where $\alpha, \beta, \gamma, \delta$ symbols of unary and c symbol of 0-ary operation are given by the system of group identities

$$(x+y)+z = x+(y+z), x+0 = x, 0+x = x, x+(-x) = 0, (-x)+x = 0$$

in signature $\{+, -, 0\}$ and identities:

$$x\alpha\gamma = x\gamma\alpha = x,\ x\beta\delta = x\delta\beta = x,\ 0\alpha = 0\beta = 0\gamma = 0\delta = 0.$$

was considered in [14].

Below we substituted in the last system γ into α^{-1} and δ into β^{-1}, and will write it in the next form:

$$x\alpha\alpha^{-1} = x\alpha^{-1}\alpha = x,\ x\beta\beta^{-1} = x\beta^{-1}\beta = x,\ 0\alpha = 0\beta = 0\alpha^{-1} = 0\beta^{-1} = 0.$$

As transfers $\tau\colon \Omega_1 \to \Omega_2$ and $\sigma : \Omega_2 \to \Omega_1$ were taken mappings:

$$\tau(\cdot)(x,y) = x\alpha + d + y\beta,$$

$$\tau(/)(x,y) = (x - y\beta - c)\gamma,$$

$$\tau(\backslash)(x,y) = (-c - x\alpha + y)\delta,$$

$$\tau(u) = 0,$$

$$\sigma(+)(x,y) = (x/u)((u/u)\backslash y),$$

$$\sigma(-)(x) = (u/u)((x/u)\backslash u),$$

$$\sigma(\alpha) = x(u\backslash u),\ \ \sigma(\beta) = (u/u)x,$$

$$\sigma(\gamma) = x/(u\backslash u),\ \ \sigma(\delta) = (u/u)\backslash x,$$

$$\sigma(0) = u, \sigma(c) = uu.$$

It was proved, that the classes of algebras $Q(\Gamma)$ and U are rationally equivalence. From here it follows, in particular, that $Q(\Gamma)$ are variety quasigroups. Analogously it was proved that the varieties are the classes of right-linear, left-linear, the linear quasigroups and T-quasigroups.

The authors observe that the received results about rational equivalence allows many questions about quasigroups could be formulated in the more usual language of algebras from the varieties U which are near to groups. In particular it is noted that in this way one can construct free quasigroups from the variety quasigroups $Q(\Gamma)$,$Q(A\Gamma)$ ($Q(A\Gamma)$ is the variety of quasigroups isotopic to abelian groups). This idea, with implicit use of the equivalence of varieties, was realized in [16,17] for T-quasigroups. They constructed a free algebra with basis X in some different terms in the signature which received a broader signature by the unary operation symbols $\alpha, \alpha^{-1}, \beta, \beta^{-1}$, on which were determined the quasigroup operations and in result a free T-quasigroup was received. Following attentive analysis of the works [16,17] it became clear that some of the analogies of this construction should be present for the variety of all linear quasigroups [19].

As it was noted above the constructive description of the free quasigroups requires the decision problem of word in the free quasigroups in accordance of variety or as well as problem of identities relations for the quasigroups of the considered varieties. In this way M.M. Glukhov in the report on the algebraic conference devoted to 100-anniversary of A.G. Kurosh announced the next result: if for free groups some varieties of group solvable the word problem, then analogous fact is true for its isotopic closure [20]. However from this result does not follows the decision of word problem for varieties of different types of linear quasigroups, as far as in these varieties essential role plays limitations on the permutations which are components of isotopy. Therefore the word problem for free quasigroups from varieties of different types of linear quasigroups remains open.

Nevertheless for some subclasses of linear quasigroups, namely T-quasigroups and medial quasigroups the word problem for a free algebras is solveable [21].

Theorem 1.1. *In a variety of all T-quasigroups the word problem for free algebras is solved.*

Proof: To prove this theorem we will need use the auxiliary variety of algebras, namely the variety $U(\Delta_1, S)$ algebras of signature $\Delta_1 = \Delta \cup \Delta_0$ with system of identities S, where $\Delta_0 = \{\alpha, \alpha^{-1}, \beta, \beta^{-1}\}$ is the system of 0-ary operations, $\Delta = \{+, -, 0\}$ is the group signature, and S is the system of identities:

$$\left.\begin{array}{l}(x+y)+z = x+(y+z), x+y = y+x,\\ x+0 = x, x+(-x) = 0, -(x+y) = -x-y,\end{array}\right\} \tag{2}$$

$$\gamma^{-1}\gamma x = x, 0\gamma = 0, (x+y)\gamma = x\gamma + y\gamma, \gamma \in \Delta_0. \tag{3}$$

Besides that we will use the free group $G =< \alpha, \beta >$ with the basis $\{\alpha, \beta\}$. Let's remember that every element of group G is presented by the uniquely introduced word, that is a word which does not have sub-words of type $\gamma\gamma^{-1}, \gamma \in \Delta_0$. At the same time it is supposed that $(\gamma^{-1})^{-1} = \gamma$.

Lemma. *In variety algebras $U(\Delta_1, S)$ the problem of the equality of words for the free algebras is solved.*

Proof of Lemma. Let $F(A)$ be a free algebra of variety $U(\Delta_1, S)$ with the basis A. Lets introduce the meaning of the canonical Δ_1-word in alphabet A: the Δ_1-word R in alphabet A we call canonical if $R = 0$ or has the form

$$R = c_{11}a_1\gamma_{11} + \ldots + c_{1k_1}a_1\gamma_{1k_1} + \ldots + c_{nk_1}a_n\gamma_{nk_1} + \ldots + c_{nk_n}a_n\gamma_{nk_n}, \tag{4}$$

where $n > 0$, $c_{ij} \in Z\backslash\{0\}$, for $k_i \geq 0$, $\gamma_{ij} \in G$; and γ_{ij}, for $j = 1, \ldots, k_i$, are various pairs of introduced words from the group G for any fixed $i \in \{1, \ldots, n\}$.

Here under $a_i\gamma$ it is necessary to understand the sum of c items $a_i\gamma$ for $c > 0$ and $(-c)$ items for $c < 0$. The items of type $c_1a_i\gamma$, $c_2a_i\gamma$ will be called similar and the changing of its sum by the words $(c_1 + c_2)a_i\gamma$ will be called *introducing of similar*. It is evident that the *introducing of similar* should be realized by the elementary transformations of words on identities (2), (3).

Let's note that in sum (4) brackets which determine the order of operations were not used. Anyway the equality of (4) is correct due to the law of associativity of addition. The law of associativity will be used with this meaning in future.

Let's prove that for any Δ_1-word P in alphabet A there exists the equivalent of its canonical Δ_1-word R and this word is unique within all permutations of items.

The existence of the word R we prove by induction with the rank of the word P.

If $rankP = 0$, than $P = 0$ or $P = a_i$ and the assertion is obvious. Let's assume that it is true for all words of rank $r < m$ and we will consider the case where $rankP = m > 0$.

According to the definition of the Δ_1-word it is possible is three cases:

1) $P = (P_1) + (P_2)$. In this case to find the unknown word R it is enough to find the sum of the canonical words for P_1, P_2, and in it, using identities (2), to group items by the same factors from A in each resulting sum and reduce similar items and delete zero items, if they exist.

2) $P = (P_1)\gamma, \gamma \in \Delta_0$. In this case it is enough using identities (3) to add to every item of the canonical word for P the letter γ and make a reduction of type $\gamma^{-1}\gamma$, if it exists.

3) $P = -(P_1)$. In this case it is enough in the canonical word for P_1 to change all the coefficients from Z to the opposite numbers.

From this result the existence of the canonical word equivalent to the word P is proved.

Now we will prove its uniqueness, within the permutations of items. Firstly, from the demonstrated proof of existence, we will obtain the procedure of introducing any given Δ_1-word P to a canonical type.

Step 1. Using identities

$$(x+y)+z = x+(y+z), -(x+y) = -x-y, \ \ (x+y)\gamma = x\gamma + y\gamma, \gamma \in \Delta_0,$$

we open all brackets in word P and bring it to the sum of words of type $c_i a_i \gamma_i$.

Step 2. Using identities $\gamma^{-1}\gamma x = x, \gamma \in \Delta_0$, in every word of type $c_i a_i \gamma_i$ change the word γ_i from the group G by using the introduced word.

Step 3. In the received sum reduce similar words.

Step 4. Using the identities $0 + x = x$, $x + 0 = x$, we delete the zero items.

Step 5. We group the items with factors from A in the remaining items.

We notice that the described procedure is not an algorithm due to its ambiguity which may appear in steps 3, 5. Therefore the resulting canonical words should be different but it will be easy to see, that they differ only in the re-arrangement of their items.

The set of all such received canonical words will be denoted by $\Phi(P)$. In this way, all words from $\Phi(P)$ are canonical and equivalent to the word R and they differ from each other only by the re-arrangement of items in sums with the same factors from A. In this case such a permutation may be arbitrary.

Now we will prove that any canonical word which is equivalent to P is contained in $\Phi(P)$. Let $R_1 \in \Phi(P)$ and R_2 be any canonical word equivalent to P (which is received, possibly by another method). Then the words R_1, R_2 are equivalent and so it is possible to pass from R_1 to R_2 with the help of a finite number of elementary transformations:

$$R_1 \to T_1 \to T_2 \to \ldots \to T_m = R_2.$$

We next prove that $\Phi(T_i) = \Phi(T_{i+1})$, $i = 1, \ldots, m-1$. For this it is necessary to consider all possible elementary transformations $g : T_i \to T_{i+1}$. At the same time this transformation by the identity of associativity should not be considered, because it has been used implicitly in the recording of sums of many items without the re-arrangement of brackets.

a) Let the transformation $g : T_i \to T_{i+1}$ be carried out by the identity $x + y = y + x$. Then the sums received from words T_i, T_{i+1}, following step 1 of procedure Φ will differ only by the order of items so after steps 2-3 we receive words, which are differed only in the order of items. Hence in this case $\Phi(T_i) = \Phi(T_{i+1})$.

b) Let the transformation $g : T_i \to T_{i+1}$ be carried out by the identity $-(x+y) = -x-y$. Then using the words T_i, T_{i+1} by the procedure Φ, after the first step we get the same words and hence $\Phi(T_i) = \Phi(T_{i+1})$.

c) Let the transformation $g : T_i \to T_{i+1}$ be carried out by the identity $\gamma^{-1}\gamma x = x$. Then, using the words T_i, T_{i+1} by the procedure Φ, after the first step we accordingly get the sums S_1, S_2, moreover, some items in S_2 will be obtained from the corresponding items of the sum S_1 by deleting $\gamma^{-1}\gamma$, and so after step 2 we will obtain the same sums. Hence in this case we will have the equality $\Phi(T_i) = \Phi(T_{i+1})$.

This equality by analogy proves those cases where g carried out on other identities from (2), (3).

From the proved equalities $\Phi(T_i) = \Phi(T_{i+1})$, $i = 1, \ldots, m-1$ it follows that

$$\Phi(R_1) = \Phi(R_2).$$

This way all the canonical words which are equivalent to word P differ only by the rearrangement of items in sums with the same factors from A. From this we have received the proof of the lemma.

For recognition of the equivalence of the words P_1, P_2 it is enough to find and compare the sets $\Phi(P_1)$, $\Phi(P_2)$. P_1 and P_2 are equivalents in the algebra $F(A)$ if and only if $\Phi(P_1) = \Phi(P_2)$. Evidently for testing of the equality $\Phi(P_1) = \Phi(P_2)$ it is enough to compare only one representative from $\Phi(P_1)$ and $\Phi(P_2)$.

Let's return now to the proof of theorem 1.

Following [13,14,18] let's establish the connection between the varieties $Q(\Omega, \Sigma)$ of all T-quasigroups and the variety of algebras $U(\Delta_1, S)$. Since the signature Δ_1 contains the 0-ary operation O, then to establish the equivalence of the varieties it is necessary to expand Ω by way of introducing the 0-ary operation O. In this connection, we will be considering the variety $Q(\Omega_1, \Sigma)$, where $\Omega_1 = \{\cdot, /, \backslash, u\}$, and u is a symbol of 0-ary operation.

It is known (and evident) that if a quasigroup is isotopic to group G then it is principally isotopic to some other group which is isomorphic to G. Since the variety of groups is closed compared to isomorphisms of the groups, then it should be clear that the considered quasigroups are principally isotopic to groups from U. So in contrast with [13,14], we will not introduce in signature Δ_1 the additional symbol of unary operation c. The transfers of signature $\tau : \Omega_1 \to \Delta_1$ and $\sigma : \Delta_1 \to \Omega_1$ will be determined by on the following formulas:

$$\tau(\cdot)(x, y) = x\alpha + y\beta,$$

$$\tau(/)(x, y) = (x - y\beta)\alpha^{-1},$$

$$\tau(\backslash)(x, y) = (-x\alpha + y)\beta^{-1},$$

$$\tau(u) = 0,$$

$$\sigma(+)(x, y) = (x/u)(u/u)\backslash y),$$

$$\sigma(-)(x) = (u/u)((x/u)\backslash u),$$

$$\sigma(\alpha) = x(u\backslash u), \sigma(\beta) = (u/u)x,$$

$$\sigma(\alpha^{-1}) = x/(u\backslash u), \sigma(\beta^{-1}) = (u/u)\backslash x,$$

$$\sigma(0) = u.$$

The transfers τ and σ induce one-to-one maps between Ω_1-words and Δ_1-words and vice versa. For the Ω_1-word P, the Δ_1-word $\tau(P)$ is obtained by the use of all operations from

Ω_1 via its τ-transfers. Analogously for the Δ_1-word R the Ω_1-word $\sigma(R)$ is determined. Let's note that according to the result of F.N.Sokhatsky [8], the system of identities of varieties of quasigroups $Q(\Omega_1, \Sigma)$ which are isotopic to abelian groups is obtained by addition to the system of identities Σ_0 which are received from (2) by changing the group operation by its transfers.

From [13,14] it follows that the transfers τ and σ also induce the bijective and one-to-one maps T_τ, T_σ of the algebras of varieties $U(\Delta_1, S)$ and $Q(\Omega_1, \Sigma)$ and the above mentioned varieties are rationally equivalent. Hence, two Δ_1-words are equivalent in the free algebra $F()$ of variety $U(\Delta_1, S)$ if and only if its transfers are equivalent in free quasigroup $Q(A_1)$ with the basis $A_1 = A \cup \{u\}$ of the variety $Q(\Omega_1, \Sigma)$. Evidently, the classes of words which form elements of the free quasigroup $Q(A)$ of variety $Q(\Omega, \Sigma)$ are subclasses of some classes of quasigroup $Q(A_1)$. Hence the elements from $Q(A)$ are equal if and only if the corresponding elements in algebra $F(A)$ are equal. From here and the above lemma follows the proof of theorem 1.

Corollary 1.1. *In a variety of all medial quasigroups the word problem for free algebras is solved.*

Remark. There is also a more general approach to the concept of a linear quasigroup, namely, quasigroups which are linear over some loop are considered (T. Kepka [22,23], P. Nemec [24], V.A. Shcherbacov [25] and others). A quasigroup $(Q, \cdot)$ is *called linear over loop* $(Q, +)$, if it has the form $xy = (\varphi x + \psi y) + d$, where $\varphi, \psi \in Aut\,(Q, +)\,, d \in Q$, assuming that the loop $(Q, +)$ will be fairly well-known and examined loop, for example Moufang loops, i.e. loops with the identity $x + (y + (x + z)) = ((x + y) + x) + z$ for all $x, y, z \in Q$.

The general idea of a quasigroup is linear over loop has been crystallized in the works of Prague's algebraic school (T.Kepka, J.Jezek, [18,22-24]). Recently in literature the term *generalized linear quasigroups* [25] has appeared.

As V.A. Shcherbacov [25] noted, many well known (classical) objects belonged to the class of generalized linear quasigroups. For example, medial quasigroups (*Toyoda Theorem* [26]), distributive quasigroups (*Belousov Theorem* [27], distributive Steiner quasigroups, leftdistributive quasigroups (*Belousov-Onoi Theorem* [28], CH-quasigroups (*Manin Theorem* [29], T-quasigroups (*Belyavskaya Theorem* [30], n-ary groups *(Gluskin-Hosszu Theorem* [31], n-ary medial quasigroups (*Evans Theorem* [32] *and Belousov Theorem*), F-quasigroups (*Kepka-Kinyon-Phillips Theorem*, [33] are quasigroups of such kind. It is necessary to note also the new results of V.A. Shcherbacov on the structure of left and right F-, SM- and E-quasigroups (*Shcherbacov Theorem)* [34].

The generalization was suggested in view of several theorems about connections between some classes of quasigroups and loops. The first in this series is Toyoda-Mudoch Theorem on medial quasigroups. Any medial quasigroup can be obtained as follows: $xy = \varphi x + \psi y + d$, where $\varphi, \psi \in Aut(Q, +)$, $\varphi\psi = \psi\varphi$, $d \in Q$, $(Q, +)$ is abelian group. A quasigroup $(Q, \cdot)$ with identities $x \cdot yz = xy \cdot xz, \quad xy \cdot z = xz \cdot yz$ is called *distributive*. If a quasigroup satisfies only the first identity, then it is called *leftdistributive*. In 1958 V.D. Belousov proved that every distributive quasigroup can be obtained as follows: $xy = \varphi x + \psi y$, where φ and ψ some automorphisms commutative Moufang loop (CML) $(Q, +)$.

A quasigroup with identities $xy = yx, \quad x(xy) = y$ is called *CH-quasigroup* such that in it any three elements generate medial subquasigroup. CH-quasigroups were introduced by Yu.I. Manin in connection with solving the problem in algebraic geometry, namely researching cubic hypersurfaces. Yu.I. Manin proved that any CH-quasigroup can be obtained by using the following construction: $xy = (-x-y)+d$, where $d \in Z(Q, +)$ and $Z(Q, +) = \{a \in Q | a+(x+y) = (a+x)+y,$ $x, y \in Q\}$ is the center of CML. Later study of linear quasigroups over Moufang loops, CML, groups, abelian groups was also carried out by other mathematicians.

By a known Stein Theorem [35] any leftdistributive quasigroup $(Q, \cdot)$ which is isotopic to the group $(Q,+)$, can be obtained using the following construction: $xy = x + \varphi(-x + y)$, where φ is automorphism of $(Q,+)$. Due to the associativity of the group operation, we obtain: $xy = (x - \varphi x) + \varphi y = \psi x + \varphi y$. An automorphism φ such that ψ is substitution. Thus, a leftdistributive quasigroups, isotopic to groups, in fact, is right linear over groups.

After well know work of V.D.Belousov [36] czech algebraists – T. Kepka, P. Nemec, J. Jezek and representatives Belousov's quasigroup school - G.B. Belyavskaya, V.A. Shcherbacov, V.I. Izbash, K.K. Shchukin, F.N. Sokhatsky, P.N. Sirbu, A. KH. Tabarov, W.A. Dudek comprehensively and intensively studied a linear quasigroups and some of their generalizations. They investigated the algebraic (morphisms, congruences, core, center, associator, commutator, multiplication group) and the combinatorial (orthogonality, numerical estimates, Latin squares) aspects of generalized linear quasigroups , also n-ary linear quasigroups.

It is important to note that Belousov's quasigroup school has become a world center for the development of the theory of quasigroups and loops. In addition, in the development of the theory of quasigroups and loops enormous contribution have been made by the representatives of the Czech algebraic school. Sufficiently detailed historical overview of the theory of quasigroups is contained in doctoral dissertations of H. Kiechle [37] and V.A. Shcherbacov [25].

II. About one approach of finding quasigroup identities from some variety of loops

This part is devoted to an approach of finding the identities in the class of quasigroups isotopic to known classes of loops from some variety of loops. It is assumed that the class of loops from some variety of loops are given by an identity or system of identities. For this, the notion a derived identity is introduced and it is proved that for any identity from some of a variety of quasigroups (loops) there is derivative identity. The introduced notion of derived identities allows to find an arbitrary identity for the class of quasigroups which are isotopic not only to groups but also to loops from some variety of loops and generalizes the method of A.A. Gvaramiya [38], where for the class of quasigroups are isotopic to a groups, it is possible to obtain any quasigroup identity from the group identities. Also by this way it is easy to obtain V.D. Belousov's identities which characterized the class of quasigroups is isotopic to a group (an abelian group)[36]. As an illustration we show that the class of quasigroups which is isotopic to the nilpotent group is characterized by identity. It should be noted that the notion of a derived identity in terms of free quasigroup and the theory of automats also can be found in [38]. However, our proposed approach does not require using free objects, in particular a free quasigroups. Sufficiently confined by the methods of theory of quasigroups. It should be noted that henceforth we shall consider a quasigroup as an algebra $(Q,\cdot,/,\backslash)$ with three binary operations $(\cdot)$, $(/)$ and $(\backslash)$, satisfying the following identities:

$$(xy)/y = x,\ (x/y)y = x,\ y(y\backslash x) = x,\ y\backslash(yx) = x.$$

Definition 2.1. *Let $(Q,\cdot)$ be a quasigroup with the identity $w_1 = w_2$. The identity $w_1' = w_2'$ is called derived from the identity $w_1 = w_2$, if it is received from $w_1 = w_2$ by adding two new variables u, v in the left and the right sides of this identity using the quasigroup operations $(\cdot), (\backslash)$, $(/)$.*

In this case the identity $w_1 = w_2$ is considered to be main. We give some examples of derived identities: the identity $(x\backslash z)\backslash(y/t) = (x/z)/(y/t)$ is received from the identity $x\backslash z = x/z$ by adding two new variables y, t and the operations $(\backslash)$, $(/)$. In some cases the operations $(\cdot)$, $(\backslash)$, $(/)$ may be coincided. In this case we have the identity from one operation, for example the identity $(xt \cdot y)\cdot uv = xu \cdot (y \cdot zv)$ which is received from the identity $(xy)\cdot(uv) = (xu)\cdot(yv)$ (the identity of mediality) by replacing the variables x on xt, v on zv.

As in the case of balanced identities (see [36]), the derived identity $w_1' = w_2'$ is called *the first kind* if variables are ordered identically and *the second kind* otherwise. For example, the identity $(x\backslash z)\backslash(y/t) = (x/z)/(y/t)$ is the derived identity of first kind, the identity $(xy)\cdot(uv) = (xu)\cdot(yv)$ is the derived identity of second kind.

Lemma 2.1. *For any identity $w_1 = w_2$ of some variety quasigroups there is a derived identity $w_1' = w_2'$.*

Proof: Let $(Q,\cdot)$ be a quasigroup. By the Albert's theorem [1] the quasigroup $(Q,\cdot)$ is isotopic to some loop $(Q,\ +)$. It is sufficiently to consider the principal isotopy $x + y = R_a^{-1}x \cdot L_b^{-1}y$. Suppose that in the loop is satisfied the identity $w_1 = w_2$. Now we transfer the quasigroup using the isotopy $T = (R_a, L_b, \varepsilon)$. We obtain an identity containing the permutations R_a and L_b (or R_a^{-1} and L_b^{-1}). Since the elements $a, b \in Q$ may run over all Q, so replacing a and b on any $u, v \in Q$, we have the identity $w_1' = w_2'$ which is obtained from the identity $w_1 = w_2$ by adding two new variables u and v.

Remark 2.1. It is obvious that forming an isotopy we can obtain various derived identities. So for some identity there are indefinite derived identities, in the case when a quasigroup is an indefinite.

Remark 2.2. It is possible to obtain also such is called an identity with permutations for some identity [39].

Example. Let $(Q,\cdot)$ be a quasigroup isotopic to the Moufang loop $(Q,+)$: $x + y = R_a^{-1}x \cdot L_b^{-1}y$. A loop $(Q,+)$ is called Moufang loop if in it is satisfied the identity:

$$(x + (y + z)) + x = (x + y) + (z + x),$$

for any $x, y, z \in Q$. At transition to quasigroup operation and replacing the elements $a, b \in Q$ by $u, v \in Q$ we have the following identity:

$$((x \cdot (v\backslash((y \cdot (u\backslash zu)))/u) \cdot (u\backslash xu) = (((x \cdot (u\backslash yu))/u)) \cdot (v\backslash(z \cdot (u\backslash xu))).$$

Lemma 2.2. *For any derived identity $w_1' = w_2'$ there is a main identity $w_1 = w_2$ and vice versa.*

Denote by $|w|$ the length of an identity, that is number of variables in it.

Lemma 2.3. Suppose that for a quasigroup $(Q,\cdot)$ an identity $w_1 = w_2$ is the main and the identity $w_1' = w_2'$ is its derived identity, $|w_1| = m$ and $|w_2| = n$. Then $|w| = m + n$ and the length of derived identity is equal to $k = |w'| = |w_1'| + |w_2'| \leq m + n + 2$.

The proofs of lemmas 2.2 and 2.3 are obvious.

Now let us show how to obtain the identities which characterize the class of quasigroups isotopic to the nilpotent group of level $\leq$n. This class is described by identity (4) (see Theorem 2.1). The corollary of this result is theorem V.D. Belousov [36], where quasigroups which are isotopic to an abelian groups are characterized by the identity. In [36] this identity are given in other form and by the operations $(\cdot)$ and $(\backslash)$, namely

$$x\backslash(y(u\backslash v)) = u\backslash(y(x\backslash v)). \tag{5}$$

Let us show that in case when a quasigroup is isotopic to an abelian group we can obtain other identity from Belousov's identity using transformations. Indeed, we can re-write the identity (5) in the following form:

$$L_x^{-1}(y \cdot L_u^{-1}v) = L_u^{-1}(y \cdot L_x^{-1}v)$$

$$L_u L_x^{-1}(y \cdot L_u^{-1}v) = (y \cdot L_x^{-1}v).$$

Now v can be replaced by $L_x^{-1}v$. Then

$$L_u L_x^{-1}(y \cdot L_u^{-1} L_x v) = (y \cdot v),$$

$$y \cdot L_u^{-1} L_x v = L_x L_u^{-1}(y \cdot v),$$
$$y \cdot L_u^{-1}(x \cdot v) = x \cdot L_u^{-1}(y \cdot v),$$
$$y \cdot L_u^{-1}(R_v x) = x \cdot L_u^{-1}(R_v y).$$

In the last equality x and y can be replaced by $R_v^{-1}x$ and $R_v^{-1}y$ accordingly. Then $R_v^{-1}y \cdot L_u^{-1}x = R_v^{-1}x \cdot L_u^{-1}y, \quad (y/v) \cdot (u\backslash x) = (x/v) \cdot (u\backslash y)$, or

$$(x/v) \cdot (u\backslash y) = (y/v) \cdot (u\backslash x).$$

Below we show that the last identity is (7) when $n = 1$.

Also note that quasigroups are isotopic to abelian groups can be characterized by other identity which is "symmetrical"to Belousov's identity [36] which includes the operations $(\cdot)$ and $(/)$, namely

$$((u/v)x)/y = ((u/y)x)/v.$$

It is easy to show that from the last identity we can obtaine Belousov's identity by such transformations.

Now for convenience we will use the additive form of a group. As well known [40] a commutator of the elements x_1 and x_2 in a group $(Q, +)$ is $[x_1, x_2] = x_1 + x_2 - x_1 - x_2$. A commutator of the elements $x_1, x_2, ..., x_{n+1}$ is defined recursively:

$$[x_1, x_2, ..., x_{n+1}] = [... \ [x_1, x_2], x_3 \ ], ..., x_{n+1} \ ].$$

A group $(Q, \ +)$ is called nilpotent of class $\leq$n, if

$$[x_1, x_2, ..., x_{n+1}] = 0,$$

where 0 is zero element of the group $(Q, +)$.

Let us denote $[x_1, x_2] = x_1 + x_2 - x_1 - x_2 = t$, where $t \in Q$. Then

$$x_1 + x_2 = t + (x_2 + x_1). \tag{6}$$

From $xy = R_a x + L_b y$ it follows that $x + y = R_a^{-1}x \cdot L_b^{-1}y$. Hence we can write (6) in the following form:

$$R_a^{-1}x_1 \cdot L_b^{-1}x_2 = R_a^{-1}t \cdot L_b^{-1}(R_a^{-1}x_2 \cdot L_b^{-1}x_1).$$

Hence

$$R_a^{-1}t = (R_a^{-1}x_1 \cdot L_b^{-1}x_2)/L_b^{-1}(R_a^{-1}x_2 \cdot L_b^{-1}x_1)$$

or

$$t = R_a((R_a^{-1}x_1 \cdot L_b^{-1}x_2)/L_b^{-1}(R_a^{-1}x_2 \cdot L_b^{-1}x_1)).$$

So, to commutator $[x_1, x_2] = x_1 + x_2 - x_1 - x_2$ of the elements x_1 and x_2 of a group $(Q, +)$ corresponds the element $t = R_a((R_a^{-1}x_1 \cdot L_b^{-1}x_2)/L_b^{-1}(R_a^{-1}x_2 \cdot L_b^{-1}x_1))$ of a quasigroup $(Q, \cdot)$.

For convenience denote the element t by $\{x_1, x_2\}$ and is called a quasicommutator of the elements x_1 and x_2 in a quasigroup $(Q, \cdot)$:

$$\{x_1, x_2\} = R_a((R_a^{-1}x_1 \cdot L_b^{-1}x_2)/L_b^{-1}(R_a^{-1}x_2 \cdot L_b^{-1}x_1)).$$

Similarly to commutator $[[x_1, x_2], x_3]$ of the elements x_1, x_2 and x_3 of a group $(Q, +)$ corresponds the element $R_a((R_a^{-1}\{x_1, x_2\} \cdot L_b^{-1}x_3)/L_b^{-1}(R_a^{-1}x_3 \cdot L_b^{-1}\{x_1, x_2\}))$ which is denote by $\{x_1, x_2, x_3\}$ and is called a quasicommutator of the elements x_1, x_2, x_3. So a quasicommutator of the elements $x_1, x_2, ..., x_{n+1}$ is defined recursively:

$$\{x_1, x_2, ..., x_{n+1}\} =$$

$$= R_a((R_a^{-1}\{x_1, x_2, ..., x_n\} \cdot L_b^{-1}x_{n+1})/L_b^{-1}(R_a^{-1}x_{n+1} \cdot L_b^{-1}\{x_1, x_2, ..., x_n\})).$$

Theorem 2.1. *A quasigroup $(Q, \cdot)$ is isotopic to a nilpotent group of class $\leq n$ if and only if in it the following identity is satisfied:*

$$(\{x_1, x_2, ..., x_n\}/u) \cdot (v \backslash x_{n+1}) = (x_{n+1}/u) \cdot (v \backslash \{x_1, x_2, ..., x_n\}). \tag{7}$$

Proof: Let $(Q, \cdot)$ be quasigroup is isotopic to a nilpotent group of class $\leq n$. We show that in quasigroup $(Q, \cdot)$ the identity (7) is satisfied. As was noted above it is enough to consider a principle isotopy of the form $x + y = R_a^{-1}x \cdot L_b^{-1}y$. The proof can be shown by induction.

If $n = 1$, then the group is abelian and identity (7) has the form

$$(x_1/u) \cdot (v \backslash x_2) = (x_2/u) \cdot (v \backslash x_1). \tag{8}$$

Let us show that in this case (8) is satisfied. For this first note that from $xy = R_a x + L_b y$ follows $x/y = R_a^{-1}(x + JL_b y)$ and $x \backslash y = L_b^{-1}(JR_a x + y)$, where $Jx = -x$. Then we have:

$$(x_1/u) \cdot (v \backslash x_2) = R_a R_a^{-1}(x_1 + JL_b u) + L_b L_b^{-1}(JR_a v + x_2) = x_1 + JL_b u + JR_a v + x_2,$$

$$(x_2/u) \cdot (v \backslash x_1) = R_a R_a^{-1}(x_2 + JL_b u) + L_b L_b^{-1}(JR_a v + x_1) = x_2 + JL_b u + JR_a v + x_1.$$

Comparing the right sides of last equalities and accordingly commutativity of a group concluded that the equality (8) is satisfied.

Now by induction suppose that the identity (7) in the quasigroup $(Q, \cdot)$ is satisfied which is isotopic to a nilpotent group $(Q, +)$ of class $n = k$, i.e.

$$(\{x_1, x_2, ..., x_{k-1}\}/u) \cdot (v \backslash x_k) = (x_k/u) \cdot (v \backslash \{x_1, x_2, ..., x_{k-1}\}).$$

Let us show the case when $n = k + 1$. In this case the identity (7) has the form:

$$(\{x_1, x_2, ..., x_k\}/u) \cdot (v \backslash x_{k+1}) = (x_{k+1}/u) \cdot (v \backslash \{x_1, x_2, ..., x_k\}).$$

Replacing the operation $(\cdot)$ by $(+)$ in the left and right sides of the last identity we obtain:

$$(\{x_1, x_2, ..., x_k\}/u) \cdot (v \backslash x_{k+1}) = R_a R_a^{-1}(\{x_1, x_2, ..., x_k\} +$$

$$+JL_b u) + L_b L_b^{-1}(JR_a v + x_{k+1}) = \{x_1, x_2, ..., x_k\} + JL_b u + JR_a v + x_{k+1},$$

$$(x_{k+1}/u) \cdot (v \backslash \{x_1, x_2, ..., x_k\}) = R_a R_a^{-1}(x_{k+1} + JL_b u) + L_b L_b^{-1}(JR_a v +$$

$$+\{x_1, x_2, ..., x_k\}) = x_{k+1} + JL_b u + JR_a v + \{x_1, x_2, ..., x_k\}.$$

On other side in a nilpotent group of class $n = k$:

$$[x_1, x_2, ..., x_{k+1}] = [[x_1, x_2, ..., _k], x_{k+1}] = 0$$

or

$$[x_1, x_2, ..., x_k] + x_{k+1} = x_{k+1} + [x_1, x_2, ..., x_k].$$

But the commutator $[x_1, x_2, ..., x_k]$ corresponds to the quasicommutator $\{x_1, x_2, ..., x_k\}$. Then

$$\{x_1, x_2, ..., x_k\} + x_{k+1} = x_{k+1} + \{x_1, x_2, ..., x_k\}.$$

As the element x_{k+1} is arbitrary, then suppose $x_{k+1} = JL_b u + JR_a v$ we get that the element $JL_b u + JR_a v$ commutes with the quasicommutator $\{x_1, x_2, ..., x_k\}$:

$$\{x_1, x_2, ..., x_k\} + JL_b u + JR_a v = JL_b u + JR_a v + \{x_1, x_2, ..., x_k\}.$$

Then we have

$$\{x_1, x_2, ..., x_k\} + JL_b u + JR_a v + x_{k+1} = x_{k+1} + JL_b u + JR_a v + \{x_1, x_2, ..., x_k\}.$$

The last equality shows that the identity (7) is true for $n = k + 1$.

Now let in a quasigroup $(Q, \cdot)$ the identity (7) is satisfied

$$(\{x_1, x_2, ..., x_n\} / u) \cdot (v \backslash x_{n+1}) = (_{n+1}/u) \cdot (v \backslash \{x_1, x_2, ..., x_n\}).$$

It is necessary to prove that a quasigroup is isotopic to a nilpotent group of class $\leq n$.

By induction for $n = 1$ have:

$$(x_1/u) \cdot (v \backslash x_2) = (x_2/u) \cdot (v \backslash x_1)$$

As we have shown in above the last identity characterizes the class of quasigroups which are isotopic to abelian groups.

For $n = 2$ we have the following identity:

$$(\{x_1, x_2\} / u) \cdot (v \backslash x_3) = (x_3/u) \cdot (v \backslash \{x_1, x_2\}),$$

which characterizes a quasigroups isotopic to a metabelian groups (a nilpotent group of class 2). Continuing this process, we find that a quasigroup is isotopic to a nilpotent group of class $\leq n$.

Corollary 2.1. A quasigroup $(Q, \cdot)$ is isotopic to a nilpotent group of class $\leq n$ if and only if in it the following equality is satisfied:

$$\{x_1, x_2, ..., x_{n+1}\} = ba,$$

where $ba = 0$ is zero element of the group $(Q, +)$.

A quasigroup is isotopic to a commutative group, can also be characterized by an identity with 4 variables, namely

Corollary 2.2. (Belousov's Theorem [2]) *A quasigroup $(Q, \cdot)$ is isotopic to an abelian group if and only if in it the following equality is satisfied:*

$$x \backslash (y(u \backslash v)) = u \backslash (y(x \backslash v)).$$

Corollary 2.3. *A quasigroup $(Q, \cdot)$ is isotopic to an abelian group if and only if in it the following equality is satisfied:*

$$((u/v)\, x) / y = ((u/y)\, x) / v. \tag{9}$$

The variety of quasigroups isotopic to abelian groups was also investigated by M.M. Glukhov, A. Drapal, etc. In particular, M.M. Glukhov has described an identities of length 4, which characterizes class of quasigroups which are isotopic to abelian groups:

$$((x/y)/u)/v = ((x/v)/u)/y, \tag{10}$$

$$(x(y\backslash(uv)) = u(y\backslash(xv)) \tag{11}$$

A. Drapal [41] shows that this class can be characterized by the following identity:

$$((xy)/u)v = ((xv)/u)y \tag{12}$$

Note also that independently from others M.M. Glukhov has obtained the identities (7), (10-12).

As known [40] a group is called *engel* if in it the following identity is satisfied:

$$[x, \underbrace{y, ..., y}_{n \text{ copies}}] = 0.$$

Using Theorem 2.1 we can get criterion when a quasigroup is isotopic to engel group.

Corollary 2.4. *A quasigroup $(Q, \cdot)$ is isotopic to an engel group if and only if in it the identity:*

$$(\{x, \underbrace{y, ..., y}_{(n-1) \text{ copies}}\}/u) \cdot (v\backslash y) = (y/u) \cdot (v\backslash\{x, \underbrace{y, ..., y}_{(n-1) \text{ copies}}\}). \tag{13}$$

is satisfied.

Remark 2.3. Quasigroups are isotopic to abelian groups also can be characterized by the identities which contain 5 variables.

Theorem 2.2. *A quasigroup $(Q, \cdot)$ is isotopic to an abelian group if and only if in it at least one from the following equalities is satisfied:*

$$(x/u) \cdot (v\backslash yz) = (y(v\backslash x))/u \cdot z, \tag{14}$$

$$(x/u) \cdot (v\backslash(y(v\backslash z)) = (z/u) \cdot (v\backslash(y(v\backslash x))), \tag{15}$$

$$(x/u) \cdot (v\backslash yz) = (yz/u) \cdot (v\backslash x), \tag{16}$$

$$x(v\backslash((y/u)z) = ((xz)/u) \cdot (v\backslash y), \tag{17}$$

$$x(v\backslash((y/u) \cdot (v\backslash z)) = (z/u) \cdot (v\backslash(x(v\backslash y))). \tag{18}$$

Proof: *Necessity.* Suppose that a quasigroup $(Q, \cdot)$ is isotopic to an abelian group $(Q, +)$. Let us show that in a quasigroup the identity (14) is satisfied. For this it is enough to consider a principle isotopy of the form

$$x + y = R_a^{-1}x \cdot L_b^{-1}y. \tag{19}$$

From (19) follows that

$$xy = R_a x + L_b y. \tag{20}$$

Then it is evident that $x/y = R_a^{-1}(x + JL_b y)$ and $x\backslash y = L_b^{-1}(JR_a x + y)$, where $Jx = -x$, $x \in Q$.

Indeed, denoted $xy = z$. Then $x = z/y$, $z = R_a(x/y) + L_b y$, $R_a(z/y) = z + JL_b y$, or $z/y = R_a^{-1}(z + JL_b y)$.

Similarly if $xy = z$, then $y = x\backslash z$, $z = R_a x + L_b(x\backslash z)$, $L_b(x\backslash z) = JR_a x + z$, $x\backslash z = L_b^{-1}(JR_a x + z)$.

Using (20) and expressions for enviers operations $(/)$ and $(\backslash)$, we have:

$$\begin{aligned}(x/u) \cdot (v\backslash yz) =& R_a(x/u) + L_b(v\backslash yz) = \\ =& R_a R_a^{-1}(x + JL_b u) + L_b L_b^{-1}(JR_a v + yz) = \\ =& (x + JL_b u) + (JR_a v + R_a y + L_b z) = \\ =& x + JL_b u + JR_a v + R_a y + L_b z.\end{aligned}$$

$$\begin{aligned}(y(v\backslash x))/u \cdot z &= R_a((y(v\backslash x))/u) + L_b z = \\ &= R_a R_a^{-1}(y(v\backslash x) + JL_b u) + L_b z = \\ &= (y \cdot L_b^{-1}(JR_a v + x) + JL_b u) + L_b z \\ &= (R_a y + (JR_a v + x) + JL_b u) + L_b z = \\ &= R_a y + JR_a v + x + JL_b u + L_b z.\end{aligned}$$

Hence,

$$\begin{aligned}(x/u)(v\backslash yz) &= x + JL_b u + JR_a v + R_a y + L_b z. \\ (y(v\backslash x))/u \cdot z &= R_a y + JR_a v + x + JL_b u + L_b z.\end{aligned}$$

As the group $(Q,+)$ is an abelian so the right sides of the identities are equal. Therefore in a quasigroup $(Q,\cdot,\backslash,/)$ the identity (14) is satisfied.

Sufficiency. Suppose that in a quasigroup $(Q,\cdot,/,\backslash)$ the identity (14) is satisfied . Re-write the identity (14) in the following form:

$$R_u^{-1}x \cdot L_u^{-1}(yz) = R_u^{-1}(y \cdot L_u^{-1}x) \cdot z.$$

Put $u = a,\ v = b$, then

$$R_a^{-1}x \cdot L_b^{-1}(yz) = R_a^{-1}(y \cdot L_b^{-1}x) \cdot z. \tag{21}$$

The last equality can be written in the form:

$$R_a^{-1}x \cdot L_b^{-1}(yz) = R_a^{-1}(L_b^{-1}x * y) \cdot z,$$

where $L_b^{-1}x * y = y \cdot L_b^{-1}x$.

Hence according to Belousov's theorem about four quasigroups, the quasigroup $(Q,*)$ and also the quasigroup $(Q,\cdot)$ are isotopic to some group. Therefore, the principle isotopy $(Q,+)$: $x + y = R_a^{-1}x \cdot L_b^{-1}y$ is a group. Using this in (21) we have:

$$R_a^{-1}R_a x + L_b L_b^{-1}(R_a y + L_b z) = R_a R_a^{-1}(R_a y + L_b L_b^{-1}x) + L_b z$$

or

$$x + (R_a y + L_b z) = (R_a y + x) + L_b z$$

Replace y by $R_a^{-1}y$, and z by $L_b^{-1}z$;

$$x + (y + z) = (y + x) + z.$$

Hence, the group $(Q,+)$ is abelian.

The condition of theorem for the identities (15) - (18) are showed analogously.

III. Chronology development theory of quasigroups and loops

Below we show a short chronology of development of the theory of quasigroups and loops, also a list of those who made a contribution in this theory.

L. Euler (1783). *Latin squares.*

R. Moufang (1935). *Moufang loops.*

A. K. Suschkewitsch (1929,1937). *The generalized groups.*

B. A. Hausman, O. Ore (1937), G. N. Garrison (1940). *Some aspects of the theory of quasigroups.*

K. Toyoda, D.S. Murdoch (1939,1941). *Medial quasigroups.*

A. A. Albert (1943-1944), R. H. Bruck (1944-1946), R. Artzy (1955-1965), Sh.K. Stein (1957), A. Sade (1960). *Basics algebraic theory quasigoups and loops.*

V. D. Belousov (1950-1989). *Foundations of the theory of quasigroups and loops, Moscow, Nauka, 1967.(in Russian). (First monograph on structure theory of quasigroups).*

I. Mal'tsev (1970). *Connection between Moufang loops and Mal'tsev's algebras.*

T. Evans (1950-1960), M. M. Glukhov, A. A. Gvaramiya (1969-1971). *The algorithmic problems of the theory of quasigroups and loops.*

Y. I. Manin (1972). CH*-quasigroups.*

J. Denes and A. D. Keedwel. (1974). *Latin squares and their applications. Academiai Kiado, Budaest.*

J. Smith, O. Chein, H.O. Pflugfelder, T. Kepka, J. Jezek, P. Nemec, A. Drapal, G.B. Belyavskaya, V.A. Shcherbacov, K.K. Shchukin, V.I. Izbash, P.N. Sirbu, F.N. Sokhatsky, E. Kuznetsov, I.P. Shestakov, A. Grishkov, A. Romanowska, W. Dudek, A. Krapez, G.P. Nagy, M.K. Kinyon, K. Kunen, J.D. Philips, P. Vojtechovsky. *The modern a structure theory of quasigroups.*

M.A. Akivis, V. V. Goldberg, L.V. Sabinin, P.O. Mikheev. *A quasigroups and a geometry.*

Presently the theory of quasigroups, as well as other algebraic structures, is developed in several directions, but among them, in our opinion, there are three main, namely:

1) *Research the inner nature itself quasigroups;*

2) *A tendency to obtain analogues of known results and theorems from other algebraic structures;*

3) *Applications of the theory of quasigroups (cryptology, differential geometry, theoretical physics (Poincare's quasigroups, Lorentz's quasigroups).*

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References:

1. V.D. Belousov. *Foundations of the theory of quasigroups and loops.* Moscow, Nauka, 1967.(in Russian).

2. A.Kh. Tabarov. *Linear quasigroups.I* . Preprint, Central European University, Department Mathematics and its Applications, Hungary, May 2010, 14 pages.

3. Yu.A. Bakhturin, A.Yu. Olshanskii. *Identical relations.* Sovremen. problemy matem. Fundament. napravl., 117-240: in Itogi Nauki i Tekhniki VINITI AN SSSR, 18, Moscow, 1988, (in Russian). Translated in Encyclopaedia of Mathematical Sciences, Algebra II, Noncommutative Rings. Identities Vol. 18, Springer, Heidelberg, 1991 Review.

4. G.B. Belyavskaya, A.Kh. Tabarov. *Characterization of linear and alinear quasigroups.* Diskretnaya matematika, v.4, no.2, 1992, p.142-147.(in Russian).

5. G.B. Belyavskaya, A.Kh. Tabarov. *Nuclei and center of linear quasigroups.* Bul. Acad. Stiinte, Republic Moldova, Matematica, Kishinev (1991), No. 3(6), p.37-42,(in Russian).

6. G.B. Belyavskaya, A.Kh. Tabarov. *One-sided T-quasigroups and irreducible balanced identities.* Quasigroups and Related Systems. Kishinev, 1994, No. 1, p.8-21.

7. V.D. Belousov. *Balanced identities on quasigroups.* Mat. Sb., vol.70(112): no.1, 1966, p.55-97,(in Russian).

8. F.N. Sokhatski. *Associates and decompositions of multiary.* Diss. of Doctor of Sciences, Kiev, 2006.(in Ukrainian).

9. T. Evans. *The word problem for absract algebras.* J.London Math.Soc. 1951. 28. No 1, p.64-67.

10. T. Evans. *The isomorphism problem for some classes of multiplicative systems.* Trans. Amer. Math. Soc. 109(1963), p. 303-312.

11. M.M. Glukhov, A.A. Gvaramiya. *On algorithmic problems for some classes of quasigroups.* DAN USSR, 1967, vol.177, No 1, p.14-16.(in Russian).

12. M.M. Glukhov. *On free products and algorithmic problems in R-varieties of universal algebras.* Soviet Math. Dokl., vol.11 (1970), No.4,p.957-960.

13. B. Csacany. *On the equivalence of certain classes of algebraic systems.* Acta Sci. Math. Szeged, 1962, vol.23, p.46-57.

14. J. Jezek and T. Kepka. *Quasigroups, isotopic to a group.* Commentationes math. Universitatis Carolinae, 1975, vol.16, No.1, p.59-76.

15. A.I. Mal'tsev. *Identical relations on varieties of quasigroups.* Mat. Sb., 69(1):p.3-12, 1966,(in Russian).

16. T. Kepka and P. Nemec. *T-quasigroups. I.* - Acta univ. Carolin. Math.Phys., 1971, vol.12, No. 1, p.31-39.

17. T. Kepka and P. Nemec. *T-quasigroups. II.* - Acta univ. Carolin. Math.Phys., 1971, vol. 12, No. 2, p.39-49.

18. J. Jezek and T. Kepka. *Varieties of abelian quasigroups.* Czech. Math. Journal, 1977, vol.27, p.473-503.

19. A.Kh. Tabarov. *A construction of free linear quasigroups.* Dokl. Akad. of Sciences of RT, 2005, vol. XLVIII, No. 11-12, p.22-28.(in Russian).

20. M.M. Glukhov. *On free quasigroups of some varieties and their multiplicative groups.* International conference devoted to 100-anniversary of A.G.Kurosh. Abstracts.Moscow, MSU, 2008, p.68.(in Russian).

21. A.Kh. Tabarov. *Algorithmic solvability of the words problem for some variety of linear quasigroups.* International Conference. Mal'tsev Meeting'09, dedicated to the 100th anniversary of A.I. Mal'tsev, August 24-28, 2009. Collectin of Abstracts, p.168 (in Russian).

22. T. Kepka. *Structure of weakly of abelian quasigroups.* - Czech. Math. Journal, 1978, vol.28, p.181-188.

23. T. Kepka. *Structure of triabelian quasigroups.* - Comment. math. Univ. Carolinae, 1976, vol.17, p.229-240.

24. P. Nemec. *Commutative Moufang loops corresponding to linear quasigroups.* - Comment. math. Univ. Carolinae, 1988, vol. 2, p.303-308.

25. V.A. Shcherbacov. *On linear and inverse quasigroups and their applications in code theory.* Dissertation of Doctor of Sciences, 247 pages, Chisinau, 2008.

26. K. Toyoda. *On axsioms of linear functions.* - Proc. Imp. Acad.Tokyo., 1941, vol. 17, p.221-227.

27. V.D. Belousov. *On the structure of distributive quasigroups.* - Uspekhi Mat. Nauk, 1958, vol.13, No 3, p.235-236 (in Russian).

28. V.D. Belousov, V.I. Onoi *On loops, isotopic to leftdistributive quasigroups.* - Math. issled. Kishinev, 1972, vol. 3(25), c. 135-152.(in Russian).

29. Yu.I. Manin. *Cubic forms.* Moscow, Nauka , 1972.(in Russian).

30. G.B. Belyavskaya. *Abelian quasigroups are T-quasigroups.* Quasigroups and Related Systems, 1994, vol.1, no.1, p.1-7.

31. V.D. Belousov. *n-Ary quasigroups.* Shtiinsa, Kishinev, 1971.(in Russian).

32. T. Evans. *Abstract mean values.* - Duke math. J., 1963, vol.30, p.331-347.

33. T. Kepka, M.K. Kinyon and J.D. Phillips. *The structure of F-quasigroups.* http://arxiv.org/ abs/math/0510298(2005), 24 pages.

34. V.A. Shcherbacov *On the structure of left and right F-, SM- and E-quasigroups.* Journal of Generalized Lie Theory and Applications, vol.3 (2009), no.3, p.197-259.

35. S.K. Stein. *Left distributive quasigroups.* - Proc.Amer.Math.Soc., 1959, vol.10, No.4, p.577-578.

36. V.D. Belousov. *Balanced identities on quasigroups.* Mat. Sb., vol.70(112): no.1, 1966, p.55-97,(in Russian).

37. H. Kiechle. *Theory of K-loops.* Habilitationsschrift. Fachbereich mathematik der Universitat Hamburg. Hamburg, 1998, (Habilitation Dissertation).

38. A.A. Gvaramiya. *Aksiomatiziruemie klassi kvazigrupp i mnogosortnaya algebra*. Dissertation of Doctor of Sciences, Sukhumi, 1985, (in Russian).

39. G.B. Belyavskaya, A.Kh. Tabarov. *Identities with permutations leading to linearity of quasigroups.* Discrete Mathematics and Applications, 2009, vol. 19, iss. 2, p.173-190.

40. M.I. Kargapolov, Yu.I. Merzlyakov. *Foundations of group theory.* Moscow, Nauka, 1977, (in Russian).

41. A. Drapal. *On multiplication groups of relatively free quasigroups isotopic to abelian groups.* Czech. Math. J. 55 (130)(2005), p.61-86.