Linear quasigroups.I

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Abstract

The article is devoted to linear quasigroups and some of their generalizations. In the first part main definitions and notions of the theory of quasigroups are given. In the second part some elementary properties of linear quasigroups and their generalizations are presented. Finally in the third part endomorphisms and endotopies of linear quasigroups and their generalizations are investigated.

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I. Main definitions and notions

Definition 1.1. A groupoid $(Q,\cdot)$ is called *a quasigroup* if the equations

$$a\cdot x=b, \quad y\cdot a=b$$

are uniquely solvable for any $a,b\in Q$ [1].

A quasigroup can also be defined in another way - this is an algebra $(Q,\cdot,/,\backslash)$ with three binary operations $(\cdot)$, $(/)$ and $(\backslash)$, satisfying the following identities:

$$(xy)/y=x,\ (x/y)y=x,\ y(y\backslash x)=x,\ y\backslash(yx)=x.$$

The quasigroup $(Q,\cdot)$ is called *isotopic* to the quasigroup $(Q,\circ)$, if there exist three permutations α,β,γ of the set Q such that $\gamma(x\circ y)=\alpha x\cdot\beta y$ for any $x,y\in Q$.

Definition 1.2. *The element $f(x)$ $(e(x))$ of a quasigroup $(Q,\cdot)$ is called left (right) local identity element of an element $x\in Q$, if $f(x)\cdot x=x$ $(x\cdot e(x)=x)$.*

If $f(x)\cdot y=y$ $(y\cdot e(x)=y)$ for all $y\in Q$, then $f(x)$ $(e(x))$ is called *left (right) identity element* of a quasigroup $(Q,\cdot)$.

If $f(x)=e(x)$ for all $x\in Q$ i.e. when all left and right local identities of the quasigroup coincide, then $(Q,\cdot)$ has identity element which is denoted as $e:\ e(x)=f(x)=e$. A quasigroup with identity is called *a loop*.

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Theorem 1.1 (A.A. Albert [1]). *Every quasigroup is isotopic to some loop.*

An isotopy of the form $T=(\alpha,\beta,\varepsilon)$ where ε is the identity permutation is called *principal isotopy.*

Theorem 1.2 (A.A. Albert [1]). *If a loop* $(Q,\circ)$ *is isotopic to a group* $(Q,+)$, *then* $(Q,\circ)$ *is a group and* $(Q,\circ)\cong(Q,+)$, *that is the groups* $(Q,\circ)$ *and* $(Q,+)$ *are isomorphic.*

Theorem 1.3 (V.D. Belousov [1]). *If the loop* $(Q,\circ)$ *is principally isotopic to the quasigroup* $(Q,\cdot)$, *then the isotopy must have the form:* $T=(R_a^{-1},L_b^{-1},\varepsilon)$, where $L_a x=ax,\ R_a x=xa,\ R_a^{-1}x=a/x,\ L_b^{-1}x=b\backslash x,$ for all $a,b,x\in Q$.

An isotopy of the form $T=(R_a^{-1},L_b^{-1},\varepsilon)$ is also called LP-isotopy of the quasigroup $(Q,\cdot)$. So any LP-isotopy of a quasigroup is a loop.

In the class of quasigroups that are isotopic to groups the interesting object are the so called *linear quasigroups* that were first introduced by V.D. Belousov in [2] in connection with researching balanced identities in quasigroups.

Definition 1.3. *A quasigroup* $(Q,\cdot)$ *is called linear over a group* $(Q,+)$, *if it has the form:*

$$xy=\varphi x+c+\psi y, \tag{1}$$

where $\varphi,\psi\in Aut(Q,+)$, c *is a fixed element from* Q.

Later G.B. Belyavskaya and A.KH. Tabarov in [3], by analogy with linear quasigroups, defined alinear quasigroups, as well as introduced the classes of left and right linear quasigroups, left and right alinear quasigroups and mixed type of linearity.

Definition 1.4. *A quasigroup* $(Q,\cdot)$ *is called alinear over a group* $(Q,+)$, *if it has the form:*

$$xy=\bar\varphi x+c+\bar\psi y, \tag{2}$$

where $\bar\varphi,\bar\psi$ *are antiautomorphisms of* $(Q,+)$, c *is a fixed element from* Q.

Definition 1.5. *A quasigroup* $(Q,\cdot)$ *is called left (right) linear over a group* $(Q,+)$, *if it has the form:*

$$xy=\varphi x+c+\beta y\ \ (xy=\alpha x+c+\psi y),$$

where β *(accordingly* α*) is a permutation of the set* Q, $\varphi\in Aut(Q,+)$ $(\psi\in Aut(Q,+))$.

Definition 1.6. *A quasigroup* $(Q,\cdot)$ *is called left (right) alinear over a group* $(Q,+)$, *if it has the form:*

$$xy=\bar\varphi x+c+\beta y\quad (xy=\alpha x+c+\bar\psi y),$$

where β *(accordingly* α*) is a permutation of the set* Q, $\bar\varphi$ $\left(\text{accordingly } \bar\psi\right)$ *is an antiautomorphism of the group* $(Q,+)$.

Definition 1.7. *A quasigroup* $(Q,\cdot)$ *is called of mixed type of linearity of first kind (second kind), if it has the form:*

$$xy=\varphi x+c+\bar\psi y\quad (xy=\bar\varphi x+c+\psi y),$$

where $\varphi\in Aut(Q,+)$ $(\psi\in Aut(Q,+))$, $\bar\psi$ *(accordingly* $\bar\varphi$*) is an antiautomorphism of the group* $(Q,+)$.

All these classes shall be named classes of different type of linearity.

An important subclass of the linear quasigroups are *medial quasigroups.* A quasigroup $(Q,\cdot)$ is called *medial*, if the following identity holds: $xy\cdot uv=xu\cdot yv$. By the theorem of Bruck-Toyoda [4-6] any medial quasigroup is linear over an abelian group with the condition $\varphi\psi=\psi\varphi$, where φ, ψ are automorphisms of the abelian group.

Medial quasigroups were researched by many algebraists, namely R.H. Bruck [4], K. Toyoda [5], D.S. Murdoch [6], T. Kepka and P. Nemec [7,8] , K.K. Shchukin [9,10], V.A. Shcherbacov [11] and others, and this class plays special role in the theory of quasigroups.

Another important subclass of linear quasigroups are T*-quasigroups* which were introduced and researched in detail by T. Kepka and P. Nemec in [7,8]. According to their definition, T-quasigroups are quasigroups of the form (1), where $(Q,+)$ is an abelian group and unlike medial quasigroups φ and ψ not necessarily commute. Later G.B. Belyavskaya characterized the class of T-quasigroups by a system of two identities [12].

II. Some properties of linear and alinear quasigroups

In this part some elementary properties of linear and alinear quasigroups are established. Note that some other properties of linear quasigroups are given as needed. Also we give some necessary facts from the theory of quasigroups.

All left $(L_a : L_a x = ax)$ and right $(R_a : R_a x = xa)$ permutations of a quasigroup $(Q,\cdot)$ generate the group which is called the *multiplication group* of $(Q,\cdot)$ and is denoted by $G(\cdot)$ or $M(Q,\cdot)$.

The permutation $\alpha \in M(Q,\cdot)$ is called *inner* with respect to a fixed element $h \in Q$, if $\alpha h = h$. All inner permutations with respect to the element $h \in Q$ generate a group which is called the *group of inner permutations* of the quasigroup $(Q,\cdot)$ and is denoted by I_h or $I_h(\cdot)$.

The action of the group $I_h(\cdot)$ on the quasigroup $(Q,\cdot)$ is well known [10]. For example, if for a quasigroup $(Q,\cdot)$ the group $I_h(\cdot)$ is normal subgroup in the group $M(Q,\cdot)$, then $(Q,\cdot)$ is an abelian group [10]. According to Theorem 4.4 from [2] the group $I_h(\cdot)$ is generated by the following permutations: $R_{x,y}, L_{x,y}$ and T_x, where $R_{x,y} = R^{-1}_{x\bullet y}R_yR_x$, $L_{x,y} = L^{-1}_{x\circ y}L_xL_y$, $T_x = L^{-1}_{\sigma x}R_x, x\bullet y = L^{-1}_h(hx\cdot y)$, $x\circ y = R^{-1}_h(x\cdot yh)$, $\sigma = R^{-1}_h L_h$, $\sigma \in I_h(\cdot)$. If the quasigroup $(Q,\cdot)$ is a group, then $I_h(\cdot) = I_0(\cdot) = Int(Q,\cdot)$, i.e. the group $I_0(\cdot)$ is the group of inner automorphisms of the group $(Q,\cdot)$.

Let $(Q,\cdot)$ be a linear quasigroup:

$$xy = \varphi x + c + \psi y. \tag{3}$$

Here and henceforth the $< \ldots >$ brackets will be replace the word "generated".

Theorem 2.1. *Let $I_h(\cdot)$ be the group of inner permutations of the linear quasigroup $(Q,\cdot)$. Then*

$$I_h(\cdot) =< R_{e_h}, L_{f_h}, T_x >,$$

where $R_{e_h} = \tilde{R}_{-\varphi h + h}\varphi$, $\quad L_{f_h} = \tilde{L}_{h-\psi h}\psi$, $\quad T_x = \{\tilde{L}_{h-(\psi^{-1}c+x)-\psi^{-1}\varphi h}\tilde{R}_{\psi^{-1}c+x}\psi^{-1}\varphi,\ x \in Q\}$, *and* $\tilde{R}_a x = x + a$, $\quad \tilde{L}_a x = a + x$, *for all* $x \in Q$.

Proof. According to Theorem 4.4 from [2],

$$I_h(\cdot) =< R_{x,y}, L_{x,y}, T_x >,$$

where

$$R_{x,y} = R^{-1}_{x\bullet y}R_yR_x, \ \ L_{x,y} = L^{-1}_{x\circ y}L_xL_y, \ \ T_x = L^{-1}_{\sigma x}R_x,$$

$$x\bullet y = L^{-1}_h(hx\cdot y), \quad x\circ y = R^{-1}_h(x\cdot yh), \quad \sigma = R^{-1}_h L_h.$$

First we note that from (3) it follows that

$$L_x = \tilde{L}_{\varphi x + c}\psi, \quad L^{-1}_x = \psi^{-1}\tilde{L}^{-1}_{\varphi x + c}, \quad R_y = \tilde{R}_{c+\psi y}\varphi, \quad R^{-1}_y = \varphi^{-1}\tilde{R}^{-1}_{c+\psi y}.$$

Then

$$x \bullet y = L_h^{-1}(hx \cdot y) = \psi^{-1}\tilde{L}^{-1}_{\varphi h+c}[\varphi(\varphi h + c + \psi x) + c + \psi y] = \psi^{-1}\tilde{L}^{-1}_{\varphi h+c}[\varphi^2 h+$$

$$+\varphi c + \varphi\psi x + c + \psi y] = \psi^{-1}[-c - \varphi h + \varphi^2 h + \varphi c + \varphi\psi x + c + \psi y] = -\psi^{-1}c-$$
$$-\psi^{-1}\varphi h + \psi^{-1}\varphi^2 h + \psi^{-1}\varphi c + \psi^{-1}\varphi\psi x + \psi^{-1}c + y.$$

Whence

$$R_{x,y}(t) = R^{-1}_{x\bullet y}R_y R_x(t) = R^{-1}_{x\bullet y}(tx \cdot y) = \varphi^{-1}\tilde{R}^{-1}_{c+\psi(x\bullet y)}[\varphi(\varphi t + c + \psi x) + c + \psi y] =$$

$$= \varphi^{-1}[\varphi^2 t + \varphi c + \varphi\psi x + c + \psi y - \psi(x \bullet y) - c] = \varphi t + c + \psi x + \varphi^{-1}c+$$
$$+\varphi^{-1}\psi y - \varphi^{-1}\psi[-\psi^{-1}c - \psi^{-1}\varphi h + \psi^{-1}\varphi^2 h + \psi^{-1}\varphi c + \psi^{-1}\varphi\psi x + \psi^{-1}c+$$
$$+y] - \varphi^{-1}c = \varphi t + c + \psi x + \varphi^{-1}c + \varphi^{-1}\psi y - \varphi^{-1}\psi y - \varphi^{-1}c - \psi x - c - \varphi h + h+$$
$$+\varphi^{-1}c - \varphi^{-1}c = \varphi t - \varphi h + h = \tilde{R}_{-\varphi h+h}\varphi(t),$$

i.e. $R_{x,y} = \tilde{R}_{-\varphi h+h}\varphi$ is independent of x, y. But $R_{e_h,e_h} = R_{e_h}$. Therefore

$$R_{x,y} = R_{e_h} = \tilde{R}_{-\varphi h+h}\varphi.$$

$L_{x,y}$ is calculated similarly.

Let us now calculate T_x:

$$T_x(t) = L^{-1}_{\sigma x}R_x(t) = \psi^{-1}\tilde{L}^{-1}_{\varphi(\sigma x)+c}\tilde{R}_{c+\psi x}\varphi(t) = \psi^{-1}(-c - \varphi(\sigma x) + \varphi t + c + \psi x) =$$

$$= -\psi^{-1}c - \psi^{-1}\varphi(\sigma x) + \psi^{-1}\varphi t + \psi^{-1}c + x = -\psi^{-1}c - \psi^{-1}\varphi R_h^{-1}L_h(x)+$$
$$+\psi^{-1}\varphi t + \psi^{-1}c + x = -\psi^{-1}c - \psi^{-1}\varphi\varphi^{-1}\tilde{R}^{-1}_{c+\psi h}\tilde{L}_{\varphi h+c}\psi x + \psi^{-1}\varphi t + \psi^{-1}c + x =$$
$$= -\psi^{-1}c - \psi^{-1}(\varphi h + c + \psi x - \psi h - c) + \psi^{-1}\varphi t + \psi^{-1}c + x =$$
$$= -\psi^{-1}c + \psi^{-1}c + h - x - \psi^{-1}c - \psi^{-1}\varphi h + \psi^{-1}\varphi t + \psi^{-1}c + x =$$
$$= h - x - \psi^{-1}c - \psi^{-1}\varphi h + \psi^{-1}\varphi t + \psi^{-1}c + x = \tilde{L}_{h-(\psi^{-1}c+x)-\psi^{-1}\varphi h}\tilde{R}_{\psi^{-1}c+x}\psi^{-1}\varphi(t),$$

i.e.

$$T_x(t) = \tilde{L}_{h-(\psi^{-1}c+x)-\psi^{-1}\varphi h}\tilde{R}_{\psi^{-1}c+x}\psi^{-1}\varphi(t).$$

□

Theorem 2.2. *Let $(Q, \cdot)$ be an alinear quasigroup:*

$$xy = \bar{\varphi}x + c + \bar{\psi}y, \tag{4}$$

$I_h(\cdot)$ its group of inner permutations. Then

$$I_h(\cdot) =< R_{x,h}, L_{h,x}, T_x >,$$

where

$$R_{x,h} = \tilde{L}_{h-hx}R_x, \quad L_{h,x} = \tilde{R}_{-xh+h}L_x, \quad T_x = \tilde{L}_{x+\bar{\psi}^{-1}c}\tilde{R}_{\bar{\psi}^{-1}\bar{\varphi}h-\bar{\psi}^{-1}c-x+h}\bar{\psi}^{-1}\bar{\varphi}.$$

Proof. From (4) it follows:

$$L_x = \tilde{L}_{\bar{\varphi}x+c}\bar{\psi}, \quad L_x^{-1} = \bar{\psi}^{-1}\tilde{L}^{-1}_{\bar{\varphi}x+c}, \quad R_y = \tilde{R}_{c+\bar{\psi}y}\bar{\varphi}, \quad R_y^{-1} = \bar{\varphi}^{-1}\tilde{R}^{-1}_{c+\bar{\psi}y}.$$

Let us calculate $R_{x,y}$:

$$\begin{aligned} x \bullet y &= L_h^{-1}(hx \cdot y) = \bar{\psi}^{-1}\tilde{L}^{-1}_{\bar{\varphi}h+c}[\bar{\varphi}(\bar{\varphi}h + c + \bar{\psi}x) + c + \bar{\psi}y] = \\ &= \bar{\psi}^{-1}\tilde{L}^{-1}_{\bar{\varphi}h+c}[\bar{\varphi}\bar{\psi}x + \bar{\varphi}c + \varphi^2 h + c + \bar{\psi}y] = \\ &= \bar{\psi}^{-1}[-c - \bar{\varphi}h + \bar{\varphi}\bar{\psi}x + \bar{\varphi}c + \varphi^2 h + c + \bar{\psi}y] = \\ &= y + \bar{\psi}^{-1}c + \bar{\psi}^{-1}\varphi^2 h + \bar{\psi}^{-1}\bar{\varphi}c + \bar{\psi}^{-1}\bar{\varphi}\bar{\psi}x - \bar{\psi}^{-1}\bar{\varphi}h - \bar{\psi}c. \end{aligned}$$

$$R_{x,y}(t) = R^{-1}_{x\bullet y}R_y R_x(t) = \varphi^{-1}\tilde{R}^{-1}_{c+\bar{\psi}(x\bullet y)}\tilde{R}_{c+\bar{\psi}y}\bar{\varphi}\tilde{R}_{c+\bar{\psi}x}\bar{\varphi}(t) =$$

$$= \bar{\varphi}^{-1}\tilde{R}^{-1}_{c+\bar{\psi}(x\bullet y)}\tilde{R}_{c+\bar{\psi}y}\bar{\varphi}(\bar{\varphi}t + c + \bar{\psi}x) = \bar{\varphi}^{-1}\tilde{R}^{-1}_{c+\bar{\psi}(x\bullet y)}(\bar{\varphi}\bar{\psi}x + \bar{\varphi}c + \varphi^2 t + c + \bar{\psi}y) =$$

$$= \bar{\varphi}^{-1}(\bar{\varphi}\bar{\psi}x + \bar{\varphi}c + \varphi^2 t + c + \bar{\psi}y - \bar{\psi}(x \bullet y) - c) =$$

$$= -\bar{\varphi}^{-1}c - \bar{\varphi}^{-1}\bar{\psi}(x \bullet y) + \bar{\varphi}^{-1}\bar{\psi}y + \bar{\varphi}^{-1}c + \bar{\varphi}t + c + \bar{\psi}x =$$

$$= -\bar{\varphi}^{-1}c - \bar{\varphi}^{-1}\bar{\psi}(y + \bar{\psi}^{-1}c + \bar{\psi}^{-1}\varphi^2 h + \bar{\psi}^{-1}\bar{\varphi}c + \bar{\psi}^{-1}\bar{\varphi}\bar{\psi}x-$$

$$-\bar{\psi}^{-1}\bar{\varphi}h - \bar{\psi}c) + \bar{\varphi}^{-1}\bar{\psi}y + \bar{\varphi}^{-1}c + \bar{\varphi}t + c + \bar{\psi}x =$$

$$= -\bar{\varphi}^{-1}c - \bar{\varphi}^{-1}(-c - \bar{\varphi}h + \bar{\varphi}\bar{\psi}x + \bar{\varphi}c + \varphi^2 h + c + \bar{\psi}y)+$$

$$+\bar{\varphi}^{-1}\bar{\psi}y + \bar{\varphi}^{-1}c + \bar{\varphi}t + c + \bar{\psi}x = -\bar{\varphi}^{-1}x - (\bar{\varphi}^{-1}\bar{\psi}y + \bar{\varphi}^{-1}c + \bar{\varphi}h + c + \bar{\psi}x-$$

$$-h - \bar{\varphi}^{-1}c) + \bar{\varphi}^{-1}\bar{\psi}y + \bar{\varphi}^{-1}c + \bar{\varphi}t + c + \bar{\psi}x = -\bar{\varphi}^{-1}c + \bar{\varphi}^{-1}c + h - \bar{\psi}x - c - \bar{\varphi}h-$$

$$-\bar{\varphi}^{-1}c - \bar{\varphi}^{-1}\bar{\psi}y + \bar{\varphi}^{-1}\bar{\psi}y + \bar{\varphi}^{-1}c + \bar{\varphi}t + c + \bar{\psi}x =$$

$$= h - \bar{\psi}x - c - \bar{\varphi}h + \bar{\varphi}t + c + \bar{\psi}x = h - (\bar{\varphi}h + c + \bar{\psi}x) + \bar{\varphi}t + c + \bar{\psi}x =$$

$$= h - hx + tx = \tilde{L}_{h-hx}\tilde{R}_x(t).$$

Hence $R_{x,y} = \tilde{L}_{h-hx}\tilde{R}_x$. Similarly is calculated $L_{x,y}$.

Let us calculate $T_x = L^{-1}_{\sigma x}R_x$, where $\sigma = R_h^{-1}L_h$, considering that

$$\begin{aligned} \sigma x &= R_h^{-1}L_h x = \bar{\varphi}^{-1}\tilde{R}^{-1}_{c+\bar{\psi}h}\tilde{L}_{\bar{\varphi}h+c}\bar{\psi}x = \bar{\varphi}^{-1}\tilde{R}^{-1}_{c+\bar{\psi}h}(\bar{\varphi}h + c + \bar{\psi}x) = \\ &= \bar{\varphi}^{-1}(\bar{\varphi}h + c + \bar{\psi}x - \bar{\psi}h - c) = -\bar{\varphi}^{-1}c - \bar{\varphi}^{-1}\bar{\psi}h + \bar{\varphi}^{-1}\bar{\psi}x + \bar{\varphi}^{-1}c + h, \end{aligned}$$

we have

$$\begin{aligned} T_x(t) &= L^{-1}_{\sigma x}R_x(t) = \bar{\psi}^{-1}\tilde{L}^{-1}_{\bar{\varphi}(\sigma x)+c}\tilde{R}_{c+\bar{\psi}x}\bar{\varphi}(t) = \bar{\psi}^{-1}(-c - \bar{\varphi}(\sigma x) + \bar{\varphi}t + c + \bar{\psi}x) = \\ &= x + \bar{\psi}^{-1}c + \bar{\psi}^{-1}\bar{\varphi}t - \bar{\psi}^{-1}\bar{\varphi}\sigma x - \bar{\psi}^{-1}c = \\ &= x + \bar{\psi}^{-1}c + \bar{\psi}^{-1}\bar{\varphi}t - \bar{\psi}^{-1}\bar{\varphi}(-\bar{\varphi}^{-1}c - \bar{\varphi}^{-1}\bar{\psi}h + \bar{\varphi}^{-1}\bar{\psi}x + \bar{\varphi}^{-1}c + h) - \bar{\psi}^{-1}c = \\ &= x + \bar{\psi}^{-1}c + \bar{\psi}^{-1}\bar{\varphi}t - (-\bar{\psi}^{-1}c - h + x + \bar{\psi}^{-1}c - \bar{\psi}^{-1}\bar{\varphi}h) - \bar{\psi}^{-1}c = \\ &= x + \bar{\psi}^{-1}c + \bar{\psi}^{-1}\bar{\varphi}t + \bar{\psi}^{-1}\bar{\varphi}h - \bar{\psi}^{-1}c - x + h + \bar{\psi}^{-1}c - \bar{\psi}^{-1}c = \\ &= x + \bar{\psi}^{-1}c + \bar{\psi}^{-1}\bar{\varphi}t + \bar{\psi}^{-1}\bar{\varphi}h - \bar{\psi}^{-1}c - x + h = \\ &= \tilde{L}_{x+\bar{\psi}^{-1}c}\tilde{R}_{\bar{\psi}^{-1}\bar{\varphi}h-\bar{\psi}^{-1}c-x+h}\bar{\psi}^{-1}\bar{\varphi}(t). \end{aligned}$$

Thus, $T_x = \tilde{L}_{x+\bar{\psi}^{-1}c}\tilde{R}_{\bar{\psi}^{-1}\bar{\varphi}h-\bar{\psi}^{-1}c-x+h}\bar{\psi}^{-1}\bar{\varphi}$. □

Theorem 2.3. *Let $(Q,\cdot)$ be a linear (alinear) quasigroup of the form:*

$$xy = \varphi x + \psi y \quad (xy = \bar{\varphi}x + \bar{\psi}y),$$

$\varphi, \psi \in Aut(Q,+)$ ($\bar{\varphi}, \bar{\psi}$ are antiautomorphisms of $(Q,+)$). If $(H,\cdot)$ is a subquasigroup of the quasigroup $(Q,\cdot)$, $0 \in H$, where 0 is the zero element of $(Q,+)$, then $(H,+)$ is a subgroup of the group $(Q,+)$, moreover

$$(H,\cdot) \trianglelefteq (Q,\cdot) \Leftrightarrow (H,+) \trianglelefteq (Q,+).$$

Proof. Let $(H,\cdot)$ be a subquasigroup of the quasigroup $(Q,\cdot)$, $xy = \varphi x + \psi y$, then $x + y = \varphi^{-1}x \cdot \psi^{-1}y = R_0^{-1}x \cdot L_0^{-1}y$. If $x, y, 0 \in H$, then $x + y \in H$, $-x \in H$, i.e. $(H,+)$ is a subgroup of the group $(Q,+)$.

According to [1], if $(H,\cdot) \trianglelefteq (Q,\cdot)$, then $I_h H = H$ for any $h \in Q$, i.e. H is invariant with respect to any permutation from I_h. Let $h = 0$, then $R_{e_0}H = R_0 H = \varphi H = H$, $\quad L_{f_0}H = L_0 H = \psi H = H$ and according to Theorem 2.1

$$T_x = \tilde{L}_{-x}\tilde{R}_x \psi^{-1}\varphi.$$

If $(H,\cdot) \trianglelefteq (Q,\cdot)$, then $T_x H = H$ or $-x + \psi^{-1}\varphi H + x = H$. Then $-x + H + x = H$, i.e. $(H,+) \trianglelefteq (Q,+)$.

The converse is easily verified.

Similarly the theorem can be proved for alinear quasigroups. □

We shall denote for convenience a quasigroup as A or (Q,A). It is known [1] that with each quasigroup A the next five quasigroups are connected:

$$A^{-1},\ {}^{-1}A,\ {}^{-1}(A^{-1}),\ ({}^{-1}A)^{-1},\ [{}^{-1}(A^{-1})]^{-1} = {}^{-1}[({}^{-1}A)^{-1}] = A^*$$

These quasigroups are called *inverse quasigroups or parastrophies.*

It is known [7, 8] that T-quasigroups are invariant under parastrophies. Below we shall establish the form of the parastrophies of linear and alinear quasigroups.

For this we shall re-write (3) in the form:

$$A(x,y) = \varphi x + c + \psi y.$$

It turns out that linear quasigroups, unlike T-quasigroups, are not invariant under parastrophies, namely the following holds:

Proposition 2.1. *Let (Q,A) be a linear quasigroup:*

$$A(x,y) = \varphi x + c + \psi y.$$

Then

$$A^{-1}(x,y) = \bar{\varphi}_1 x + c_1 + \psi_1 y, \quad {}^{-1}A(x,y) = \varphi_2 x + c_2 + \bar{\psi}_2 y,$$
$${}^{-1}(A^{-1})(x,y) = \varphi_3 y + c_3 + \bar{\psi}_3 x, \quad ({}^{-1}A)^{-1}(x,y) = \bar{\varphi}_4 y + c_4 + \psi_4 x,$$
$$A^*(x,y) = \varphi y + c + \psi x,$$

where $\varphi_i, \psi_i \in Aut(Q,+)$, $\bar{\varphi}_i, \bar{\psi}_i$ are antiautomorphisms of the group $(Q,+)$, $c_i \in Q, i = 1,2,3,4$.

Proof. Let $A(x,y)=z$. Then $A^{-1}(x,z)=y$,

$$\begin{aligned}
& z=\varphi x+c+\psi y \Rightarrow z=\varphi x+c+\psi A^{-1}(x,z) \Rightarrow \\
& \Rightarrow \psi A^{-1}(x,z)=-c-\varphi x+z \Rightarrow A^{-1}(x,z)= \\
& =-\psi^{-1}c-\psi^{-1}\varphi x+\psi^{-1}z= \\
& =-\psi^{-1}c-\psi^{-1}\varphi x+\psi^{-1}c-\psi^{-1}c+\psi^{-1}z=\bar{\varphi}_1 x+c_1+\psi_1 z,
\end{aligned}$$

where $\bar{\varphi}_1 x=J\psi^{-1}c+J\psi^{-1}\varphi x+\psi^{-1}c$ is an antiautomorphism of the group $(Q,+)$, $c_1=J\psi^{-1}c$, $\psi_1=\psi^{-1}$.

Hence, $A^{-1}(x,y)=\bar{\varphi}_1 x+c_1+\psi_1 y$.

Let us denoted $A^{-1}(x,y)=B(x,y)$. Let $B(x,y)=t$. Then ${}^{-1}B(t,y)=x$ and $\bar{\varphi}_1({}^{-1}B(t,y))+c_1+\psi_1 y=t \Rightarrow \bar{\varphi}_1({}^{-1}B(t,y))=t-\psi_1 y-c_1 \Rightarrow {}^{-1}B(t,y)=-\bar{\varphi}_1^{-1}c-\bar{\varphi}_1^{-1}\psi_1 y+\bar{\varphi}_1^{-1}t=-\bar{\varphi}_1^{-1}c-\bar{\varphi}_1^{-1}\psi_1 y+\bar{\varphi}_1^{-1}c-\bar{\varphi}_1^{-1}c+\bar{\varphi}_1^{-1}t=\varphi_3 y+c_3+\bar{\psi}_3 t$,
that is

$${}^{-1}B(t,y)={}^{-1}(A^{-1})(t,y)=\varphi_3 y+c_3+\bar{\psi}_3 t,$$

where $\varphi_3 y=-\bar{\varphi}_1^{-1}c-\bar{\varphi}_1^{-1}\psi_1 y+\bar{\varphi}_1^{-1}c$ is an automorphism of the group $(Q,+)$, $\bar{\psi}_3=\bar{\varphi}_1^{-1}$ is an antiautomorphism of the group $(Q,+)$, $c_3=-\bar{\varphi}_1^{-1}c$.

All other parastrophies are computed similarly. □

Note that alinear quasigroups are invariant under parastrophies, namely, we have the following:

Proposition 2.2. *Let (Q,A) be an alinear quasigroup:*

$$A(x,y)=\bar{\varphi}x+c+\bar{\psi}y.$$

Then

$$\begin{gathered}
A^{-1}(x,y)=\bar{\psi}_1 y+c_1+\bar{\varphi}_1 x, \quad {}^{-1}A(x,y)=\bar{\psi}_2 y+c+\bar{\varphi}_2 x, \\
{}^{-1}(A^{-1})(x,y)=\bar{\varphi}_3 x+c_3+\bar{\psi}_3 y, \quad ({}^{-1}A)^{-1}(x,y)=\bar{\varphi}_4 x+c_4+\bar{\psi}_4 y, \\
A^*(x,y)=\bar{\varphi}y+c+\bar{\psi}x,
\end{gathered}$$

where $\bar{\varphi}_i$, $\bar{\psi}_i$ are antiautomorphisms of the group $(Q,+)$, $c_i\in Q$, $\quad i=1,2,3,4$.

Proof. We shall prove the condition of the Proposition for the parastrophy $A^{-1}(x,y)$. The remaining cases are proved similarly.

Let $A(x,y)=z \Rightarrow A^{-1}(x,z)=y$. Then

$$\begin{aligned}
& z=\bar{\varphi}x+c+\bar{\psi}A^{-1}(x,z) \Rightarrow \bar{\psi}A^{-1}(x,z)=-c-\bar{\varphi}x+z \Rightarrow A^{-1}(x,z)= \\
& =\bar{\psi}^{-1}z-\bar{\psi}^{-1}\bar{\varphi}x-\bar{\psi}^{-1}c=\bar{\psi}^{-1}z-\bar{\psi}^{-1}c+\bar{\psi}^{-1}c-\bar{\psi}^{-1}\bar{\varphi}x-\bar{\psi}^{-1}c= \\
& =\bar{\psi}_1 z+c_1+\bar{\varphi}_1 x,
\end{aligned}$$

where $\bar{\psi}_1=\bar{\psi}^{-1}$, $\bar{\varphi}_1 x=\bar{\psi}^{-1}c-\bar{\psi}^{-1}\bar{\varphi}x-\bar{\psi}^{-1}c$ are antiautomorphisms of the group $(Q,+)$, $c_1=-\bar{\psi}^{-1}c$,

that is $\quad A^{-1}(x,z)=\bar{\psi}_1 z+c_1+\bar{\varphi}_1 x$.

□

Now we consider the cases when a left and a right linear (alinear) quasigroup are connected between each other. First let $(Q,\cdot)$ be linear over a loop $(Q,+)$ and consider the more common case:

Lemma 2.1. *Let the quasigroup $(Q,\cdot)$ be left linear over the loop $(Q,*)$, $xy=\varphi x*\beta y$ and right linear over the loop $(Q,\circ)$, $xy=\alpha x\circ\psi y$, where α,β are permutations of the set Q, $\varphi\in Aut(Q,*)$, $\psi\in Aut(Q,\circ)$. Then $(Q,\cdot)$ is a linear quasigroup if and only if $\alpha=R^*_{\beta 0}\varphi$, $\beta=L^\circ_{\alpha e}\psi$, where 0 is the identity element of the loop $(Q,\circ)$, e is the identity element of the loop $(Q,*)$, $R^*_a x=x*a$, $L^\circ_a x=a\circ x$.*

Proof. First note that a quasigroup of the form $xy = \alpha x \circ (c \circ \psi y)$ or $xy = (\varphi x \circ c_1) \circ \beta y$, can always be converted to the form $xy = \alpha_1 x \circ \psi y$ or respectively $xy = \varphi x \circ \beta_1 y$, where α_1 and β_1 are some permutations of the set Q. So for convenience we re-write a left or a right linear quasigroup in the form $xy = \alpha x \circ \psi y$ or $xy = \varphi x * \beta y$ respectively. According to the condition of Lemma 2.1, $(Q,\cdot)$ is a left and a right linear quasigroup, that is $xy = \alpha x \circ \psi y = \varphi x * \beta y$. Putting in the last equality first $x = 0$, then $y = e$, we get $\alpha = R^*_{\beta 0}\varphi$, $\beta = L^\circ_{\alpha e}\psi$. The reverse condition is obvious.

□

Proposition 2.3. *Let $(Q,\cdot)$ be a left and right linear (alinear) quasigroup. Then $(Q,\cdot)$ is a linear (an alinear) quasigroup.*

Proof. Let the quasigroup $(Q,\cdot)$ be a left and a right linear quasigroup. Then $(Q,\cdot)$ has the form: $xy = \alpha x + c + \psi y = \varphi x \oplus c_1 \oplus \beta y$, where α, β are permutations of the set $Q, \varphi \in Aut(Q,\oplus), \psi \in Aut(Q,+)$.

But then

$$xy = \alpha_1 x + \psi y = \varphi x \oplus \beta_1 y, \tag{5}$$

where $\alpha_1 = \alpha x + c, \quad \beta_1 y = c_1 \oplus \beta y$, that is the groups $(Q,+)$ and $(Q,\oplus)$ are principally isotopic, so by the Albert's Theorem they are isomorphic. Moreover, from the proof of this theorem it follows that there is such an element $k \in Q$, that $R_k(x \oplus y) = R_k x + R_k y$, where $R_k x = x + k$. Considering this in (5), we find that

$$R_k(\alpha_1 x + \psi y) = R_k\varphi x \oplus R_k\beta y, R_k(\alpha_1 x + \psi y) = R_k\varphi x + R_k\beta_1 y,$$

or

$$\alpha_1 R_k^{-1} x + \psi y = R_k\varphi R_k^{-1} x + \beta_1 y.$$

Put in the last equality $y = 0:\ \alpha_1 R_k^{-1} x = \varphi_1 x + d$, where $d = \beta_1 0$ is some element of the set Q, $\varphi_1 = R_k\varphi R_k^{-1}$ is automorphism of the group $(Q,+)$.

Indeed, since $R_k(x \oplus y) = R_k x + R_k y$, then

$$\begin{aligned} R_k\varphi R_k^{-1}(x+y) &= R_k\varphi R_k^{-1}[R_k(R_k^{-1}x \oplus R_k^{-1}y)] = \\ &= R_k\varphi(R_k^{-1}x \oplus R_k^{-1}y) = R_k[\varphi R_k^{-1}x \oplus \varphi R_k^{-1}y] = \\ &= R_k[R_k^{-1}(R_k\varphi R_k^{-1}x + R_k\varphi R_k^{-1}y)] = R_k\varphi R_k^{-1}x + R_k\varphi R_k^{-1}y. \end{aligned}$$

Hence $R_k\varphi R_k^{-1} \in Aut(Q,+)$. According to Lemma 2.5 from [2] $\alpha_1 R_k^{-1}$, and thus α_1 are quasiautomorphisms of this group. Therefore $\alpha_1 x = \varphi_2 x + s, \quad s \in Q$, and $xy = \varphi_2 x + c + \psi y$, where $\varphi_2, \psi \in Aut(Q,+)$, that is $(Q,\cdot)$ is a linear quasigroup.

Similarly, Proposition 2.3 can be proved for alinear quasigroups. □

Corollary 2.1. *A left (right) linear quasigroup $(Q,\cdot)$ $xy = \varphi x + c + \beta y$ $(xy = \alpha x + c + \psi y)$ is a right (left) linear quasigroup if and only if the permutation β (accordingly α) is a quasiautomorphism of the group $(Q,+)$.*

Corollary 2.2. *A left (right) alinear quasigroup $(Q,\cdot)$ $xy = \bar{\varphi} x + c + \beta y$ $(xy = \alpha x + c + \bar{\psi} y)$ is a right (left) alinear quasigroup if and only if the permutation β (accordingly α) is an antiquasiautomorphism of the group $(Q,+)$.*

Proposition 2.4. *A quasigroup which is left (right) linear and left (right) alinear is a T-quasigroup.*

Proof. Let $(Q, \cdot)$ be a left linear and a left alinear quasigroup:

$$xy = \varphi x + c + \beta y = \bar{\varphi} x \oplus s \oplus \gamma y,$$

where $\varphi \in Aut(Q, +), \bar{\varphi}$ is an antiautomorphism of the group $(Q, \oplus)$, β, γ are permutations of the set Q. Then

$$xy = \varphi x + \beta_1 y = \bar{\varphi} x \oplus \gamma_1 y, \tag{6}$$

where $\beta y = c + \beta y, \quad \gamma_1 y = s \oplus \gamma y$. Then according to Albert's Theorem $(Q, +) \cong (Q, \oplus)$, besides that there is an element $k \in Q$ such that

$$R_k(x \oplus y) = R_k x + R_k y, \tag{7}$$

where $R_k x = x + k$. Considering this in (6), we obtain

$$R_k(\varphi x + \beta_1 y) = R_k \bar{\varphi} x + R_k \gamma_1 y$$

or

$$\varphi x + \beta_1 y = \bar{\varphi} x + k + \gamma_1 y.$$

Put $y = 0: \quad \varphi x + \beta_1 0 = \bar{\varphi} x + k + \gamma_1 0$. Then $\varphi x + \beta_1 0 - \gamma_1 0 - k = \bar{\varphi} x$, $\bar{\varphi} x = \varphi x + p = R_p \varphi x$, where $p = \beta_1 0 - \gamma_1 0 - k$ is some element of the set Q.

Considering (7), note that $R_k \bar{\varphi} R_k^{-1}$ is an antiautomorphism of the group $(Q, +)$. But $R_k \bar{\varphi} R_k^{-1} = R_k R_p \varphi R_k^{-1}$. Hence,

$$\begin{aligned}
&R_k R_p \varphi R_k^{-1}(x + y) = R_k R_p \varphi R_k^{-1} y + R_k R_p \varphi R_k^{-1} x,\\
&\varphi(x + y - k) + p + k = \varphi(y - k) + p + k + \varphi(x - k) + p + k,\\
&\varphi(x + y - k) = \varphi(y - k) + p + k + \varphi(x - k),\\
&\varphi x + \varphi y - \varphi k = \varphi y - \varphi k + p + k + \varphi x - \varphi k,\\
&\varphi x + \varphi y = \varphi y - \varphi k + p + k + \varphi x.
\end{aligned}$$

Put $x = y = 0$: $0 = -\varphi k + p + k$. Then $\varphi x + \varphi y = \varphi y + \varphi x$ or $x + y = y + x$, that is the group $(Q, +)$ is abelian. Hence, $(Q, \cdot)$ is a left T-quasigroup.

The right linear and right alinear case is proved similarly. $\square$

Finally observe that if $(Q, \cdot)$ is a linear and an alinear quasigroup at the same time, then it is a T-quasigroup and from Proposition 2.3 it follows that a left and right T-quasigroup is also a T-quasigroup.

III. Endotopies and endomorphisms of linear quasigroups

Theorem 3.1. *The semigroups of endomorphisms of parastrophic quasigroups coincide:*

$$End\,(Q, \cdot) = End\,(Q, {}^{\sigma}(\cdot)), \tag{8}$$

where $(Q, \cdot)$ is a quasigroup, $(Q, {}^{\sigma}(\cdot))$ is its parastrophy, $End\,(Q, \cdot)$ is the semigroup of endomorphisms of the quasigroup $(Q, \cdot)$ and $End\,(Q, {}^{\sigma}(\cdot))$ is the semigroup of endomorphisms of its parastrophy $(Q, {}^{\sigma}(\cdot))$.

Proof. Let $\gamma \in End\,(Q, \cdot) : \gamma(xy) = \gamma x \cdot \gamma y$. Denote $xy = z$. Then $x = z/y$. Hence $\gamma z = \gamma(z/y) \cdot \gamma y$ or $\gamma(z/y) = \gamma z / \gamma y$.

Consequently, $\gamma \in End\,(Q, {}^{\sigma}(\cdot))$, so $End\,(Q, \cdot) \subseteq End\,(Q, {}^{\sigma}(\cdot))$.

The converse statement is obvious. Then $End\,(Q, \cdot) = End\,(Q, {}^{\sigma}(\cdot))$.

For other parastrophies the equality (8) is proved similarly. $\square$

Corollary 3.1. *The groups of automorphisms of parastrophic quasigroups coincide:*

$$Aut(Q,\cdot) = Aut(Q, {}^{\sigma}(\cdot)).$$

Denote by $S(Q)$ the semigroup of all the transformations of the set Q. Let $(Q,\cdot)$ be a quasigroup and $\varphi \in End\,(Q,\cdot)$. Then

$$\varphi(xy) = \varphi x \cdot \varphi y,$$

$$\varphi L_x = L_{\varphi x}\varphi, \tag{9}$$

$$\varphi R_x = R_{\varphi x}\varphi. \tag{10}$$

The following theorem is a generalization of Theorem 1 from [13].

Theorem 3.2. *Let $(Q,\cdot)$ be a quasigroup and $\varphi \in S\,(Q)$ is such that $\varphi L_x = L_z\varphi$, for any $x \in Q$ and some z, depending on x. Then $z = \varphi x$ (that is $\varphi \in End\,(Q,\cdot)$) if and only if $\varphi e = e\varphi$, where e is a map such that $e : a \to e(a)$ and $a \cdot e(a) = a$.*

The Theorem is proved by analogy with the proof of the Theorem 1 from [13].

Corollary 3.2. *Let i be the right identity element of a quasigroup $(Q,\cdot)$ and φ be a map of the set Q into itself such that $\varphi L_x = L_z\varphi$, for any x and some z, depending on $x \in Q$. Then $z = \varphi x$ (that is $\varphi \in End\,(Q,\cdot)$) if and only if $\varphi\,(i) = i$.*

Now let $\varphi \in S\,(Q)$ and $\varphi\,(xy) = \varphi y \cdot \varphi x$, that is φ is an antiendomorphism of a quasigroup $(Q,\cdot)$. Then in this case instead of (9) and (10) we must consider the following relations:

$$\varphi L_x = R_{\varphi x}\varphi, \tag{11}$$

$$\varphi R_x = L_{\varphi x}\varphi. \tag{12}$$

Theorem 3.3. *Let $(Q,\cdot)$ be a quasigroup and $\varphi \in S\,(Q)$ is such that $\varphi L_x = R_z\varphi$, for any x and some z, depending on $x \in Q$. Then $z = \varphi x$ (that is φ is an antiendomorphism of $(Q,\cdot)$) if and only if $\varphi e = f\varphi$, where e is a map such that $e : a \to e(a)$ and $a \cdot e(a) = a$ and f is a map such that $f :\ a \to f\,(a)$ and $f\,(a) \cdot a = a$.*

Proof. Since $\varphi L_x = R_z\varphi$, we have that $\varphi L_x y = R_z \varphi y$, i.e. $\varphi\,(xy) = \varphi y \cdot z$. In particular, for $y = e\,(x)$: $\varphi\,(x \cdot e\,(x)) = \varphi e\,(x) \cdot z$, whence $\varphi\,(x) = \varphi e\,(x) \cdot z$.

But $\varphi(x) = f\varphi(x) \cdot \varphi(x)$. Then $\varphi e(x) \cdot z = f\varphi(x) \cdot \varphi(x)$.

Assume now that $\varphi e = f\varphi$. After reduction we get $z = \varphi(x)$, that is φ is an antiendomorphism of the quasigroup $(Q,\cdot)$.

Conversely, for an antiendomorphism φ the following holds:

$$f\varphi(x) \cdot \varphi(x) = \varphi(x) = \varphi(x \cdot e(x)) = \varphi e(x) \cdot \varphi(x)$$

hence $f\varphi(x) = \varphi e(x)$ for all $x \in Q$. □

Corollary 3.3. *Let $(Q,\cdot)$ be a loop with the identity element j and $\varphi \in S(Q)$ be such that $\varphi L_x = R_z\varphi$ for any x and some z, depending on $x \in Q$. Then $z = \varphi(x)$ (that is φ is an antiendomorphism of $(Q,\cdot)$) if and only if $\varphi(j) = j$.*

Proof. In a loop $(Q,\cdot)$ we have $e(x) = j = f(x)$ for any $x \in Q$. Let φ be an antiendomorphism of the loop $(Q,\cdot)$. Then

$$\varphi(j) = \varphi(e(x)) = \varphi(f(x)) = f\varphi(x) = j.$$

Conversely, if $\varphi L_x = R_z\varphi$ and $\varphi(j) = j$, then since $j = e(x) = f(x)$, we have $\varphi(e(x)) = e(x) = j = f(x) = f\varphi(x)$, whence $\varphi e = f\varphi$. Then according to Theorem 3.3 φ is an antiendomorphism of the loop $(Q,\cdot)$. □

Recall that the triplet $T = (\alpha, \beta, \gamma)$ of maps of a quasigroup $(Q, \cdot)$ into itself is called an *endotopy* of the quasigroup $(Q, \cdot)$, if the identity:

$$\gamma(xy) = \alpha x \cdot \beta y,$$

is true for any $x, y \in Q$.

In case when $\alpha = \beta = \gamma$, the triplet $T = (\gamma, \gamma, \gamma)$ is called an *endomorphism* of the quasigroup $(Q, \cdot)$.

Obviously the set of all endotopies of the quasigroup $(Q, \cdot)$ is a semigroup with identity. Let us denote this semigroup by $Ent(Q, \cdot)$.

Theorem 3.4. *If the quasigroups $(Q, \cdot)$ and $(Q, \circ)$ are isotopic: $\gamma(x \circ y) = \alpha x \cdot \beta y$, $(\circ) = (\cdot)T$, $T = (\alpha, \beta, \gamma)$, then their semigroups of endotopies are conjugate:*

$$Ent(Q, \cdot) = T^{-1} Ent(Q, \circ) T. \tag{13}$$

Proof. Indeed, let $(\circ) = (\cdot)T$ and $S \in Ent(Q, \circ)$, that is $(\circ)S = (\circ)$. Then $((\cdot)T)S = (\cdot)T$, therefore $(\cdot)TS = (\cdot)T$ and $(\cdot)TST^{-1} = (\cdot)$, so $TEnt(Q, \circ)T^{-1} \subseteq Ent(Q, \cdot)$, whence $Ent(Q, \circ) \subseteq T^{-1}Ent(Q, \cdot)T$.

On the other side, $(\circ)T^{-1} = (\cdot)$. Let $S' \in Ent(Q, \cdot)$, that is $(\cdot)S' = (\cdot)$. Then $(\circ)T^{-1}S' = (\circ)T^{-1}$ and $(\circ)T^{-1}S' = (\circ)$, whence $T^{-1}Ent(Q, \cdot)T \subseteq Ent(Q, \circ)$. Considering the reverse inclusion we get (13). □

Corollary 3.4. [**1**]. *If the quasigroups $(Q, \cdot)$ and $(Q, \circ)$ are isotopic then their groups of automorphisms are isomorphic, namely*

$$A\nu t(Q, \cdot) = T^{-1} A\nu t(Q, \circ) T.$$

Note that the third component γ of an endotopy $T = (\alpha, \beta, \gamma)$ is called *a quasiendomorphism* of the quasigroup $(Q, \cdot)$. The structure of the quasiautomorphisms of a group is well known (see [1]). As in the case of quasiautomorphisms, the quasiendomorphisms of groups have a simple structure, namely:

Proposition 3.1. *Any quasiendomorphism γ of a group $(Q, +)$ has the form:*

$$\gamma = \tilde{R}_s \gamma_\circ, \tag{14}$$

where $\gamma_\circ \in End(Q, +)$, $s \in Q$, and reversely, the map γ, defined by the equality (14), is a quasiendomorphism of the group $(Q, +)$.

Proof. Let γ be a quasiendomorphism of a group $(Q, +)$, that is there is an endotopy $T = (\alpha, \beta, \gamma): \quad \gamma(x + y) = \alpha x + \beta y$. Put $x = 0$ in the last equality, where 0 is the identity element of the group $(Q, +)$. Then $\gamma y = \alpha 0 + \beta y = l + \beta y = \tilde{L}_l \beta y$, $\gamma = \tilde{L}_l \beta$, $\beta = \tilde{L}_l^{-1} \gamma$, $l = \alpha 0$. Similarly taking $y = 0$ we have: $\gamma = \alpha x + \beta 0 = \alpha x + k = \tilde{R}_k \alpha$, $\gamma = \tilde{R}_k \alpha$, $\alpha = \tilde{R}_k^{-1} \gamma$, $k = \beta 0$. But in any group $\tilde{L}_l^{-1} = \tilde{L}_{-l}$, $\tilde{R}_k^{-1} = \tilde{R}_{-k}$.

Hence,

$$\gamma(x + y) = \alpha x + \beta y = \tilde{R}_{-k} \gamma x + \tilde{L}_{-l} \gamma y = (\gamma x + (-k)) + ((-l) + \gamma y),$$

$$\gamma(x + y) = \gamma x + (-\gamma 0) + \gamma y, \tag{15}$$

since $(-k) + (-l) = -(l + k) = -(\alpha 0 + \beta 0) = -\gamma 0$. Let $\gamma x + (-\gamma 0) = \gamma_0 x$. Adding $(-\gamma 0)$ to the both parts of (15), we have $\gamma(x + y) + (-\gamma 0) = \gamma x + (-\gamma 0) + \gamma y + (-\gamma 0)$, $\gamma_0(x + y) = \gamma_0 x + \gamma_0 y$, that is $\gamma_0 \in End(Q, +)$. From $\gamma x + (-\gamma 0) = \gamma_0 x$ follows (14).

Conversely, any endomorphism and any transformation of a group are quasiendomorphisms, so the product $\tilde{R}_s \gamma_0$ is also a quasiendomorphism of that group. □

Corollary 3.5. *Let γ be a quasiendomorphism of a group $(Q,+)$. Then*

$$\gamma \in End(Q,+) \Leftrightarrow \gamma 0 = 0,$$

where 0 is the identity element of the group $(Q,+)$.

Indeed, let γ be a quasiendomorphism and $\gamma 0 = 0$. Then $\gamma 0 = \tilde{R}_s \gamma_0 0 = R_s 0 = s$ and $s = 0$. Hence, $\gamma = \gamma_0$.

As noted by I.A. Golovko in [14], any endotopy of a group $(Q,+)$ has the form:

$$T = (\tilde{L}_a,\ \tilde{R}_b,\ \tilde{L}_a \tilde{R}_b)\theta, \tag{16}$$

where $\theta \in End(Q,+)$, $a, b \in Q$.

Using (13) and (16), it is easy to find a common form of any endotopy of a linear (alinear) quasigroup $(Q,\cdot)$.

Indeed, let $(Q,\cdot)$ be a linear quasigroup: $xy = \varphi x + c + \psi y = \tilde{R}_c \varphi x + \psi y$. Then $(\cdot) = (+)(\tilde{R}_c\varphi, \psi, \varepsilon)$. If $S \in Ent(Q,+)$, then by (16) $S = (\tilde{L}_a,\ \tilde{R}_b,\ \tilde{L}_a\tilde{R}_b)\theta$, where $\theta \in Ent(Q,+)$, $a, b \in Q$. Then according to (13) for any $P \in Ent(Q,\cdot)$ we have:

$$\begin{aligned} &P = TST^{-1} = (\tilde{R}_c\varphi, \psi, \varepsilon)(\tilde{L}_a\theta, \tilde{R}_b\theta, \tilde{L}_a\tilde{R}_b\theta)(\varphi^{-1}R_c^{-1}, \psi^{-1}, \varepsilon) = \\ &= (\tilde{R}_c\varphi\tilde{L}_a\theta\varphi^{-1}\tilde{R}_c^{-1}, \psi\tilde{R}_b\theta\psi^{-1}, \tilde{L}_a R_b\theta) = (\tilde{R}_c\tilde{L}_{\varphi a}\varphi\theta\varphi^{-1}\tilde{R}_c,\ R_{\psi b}\psi\theta\psi^{-1}, \tilde{L}_a\tilde{R}_b\theta), \end{aligned}$$

where $\varphi, \psi \in Aut(Q,+), \theta \in End(Q,+),\ a, b, c \in Q$.

So, have proved:

Theorem 3.5. *Any endotopy of a linear quasigroup $(Q,\cdot)$: $xy = \varphi x + c + \psi y$ has the form:*

$$\mathrm{P} = (\tilde{R}_c\tilde{L}_{\varphi a}\varphi\theta\varphi^{-1}\tilde{R}_{-c}, \tilde{R}_{\psi b}\psi\theta\psi^{-1}, \tilde{L}_a\tilde{R}_b\theta), \tag{17}$$

where $\varphi, \psi \in Aut(Q,+)$, $\theta \in End(Q,+)$, $a, b, c \in Q$.

Similarly can be showed that any endotopy an alinear quasigroup$(Q,\cdot)$: $xy = \tilde{\varphi}x + c + \tilde{\psi}y$ has the form:

$$\overline{P} = \left(\tilde{R}_d\overline{\varphi}\theta\overline{\varphi}^{-1}\tilde{R}_{-c}, \tilde{L}_{\overline{\psi}b}\overline{\psi}\theta\overline{\psi}^{-1}, \tilde{L}_a\tilde{R}_b\theta\right). \tag{18}$$

Corollary 3.6. *Any endotopy of a T-quasigroup (medial quasigroup) $(Q,\cdot)$: $xy = \varphi x + c + \psi y$ has the form:*

$$P = (R_d\varphi\theta\varphi^{-1}R_{-c},\ \overline{R}_{\psi b}\psi\theta\psi^{-1},\ \overline{L}_{a+b}\theta), \tag{19}$$

where $d = c + \varphi a,\ d \in Q, \theta \in End(Q,+)$.

Corollary 3.7. *Any endotopy a quasigroup of mixed type of linearity of first kind: $xy = \varphi x + c + \bar{\psi}y$ (accordingly linearity of second kind: $xy = \bar{\varphi}x + c + \psi y$) has the form:*

$$P = \left(\tilde{R}_c\tilde{L}_{\varphi a}\varphi\theta\varphi^{-1}\tilde{R}_{-c},\ \bar{\psi}\tilde{R}_b\theta\bar{\psi}^{-1},\ \tilde{L}_a\tilde{R}_b\theta\right),$$

$$\left(P = \left(\tilde{R}_c\tilde{L}_{\varphi a}\bar{\varphi}\theta\bar{\varphi}^{-1}\tilde{R}_{-c},\ \ \psi\tilde{R}_b\theta\bar{\psi}^{-1},\ \tilde{L}_a\tilde{R}_b\theta\right)\right),$$

where $\varphi, \psi \in Aut(Q,+)$, $\bar{\varphi}, \bar{\psi}$ are antiautomorphisms of the group $(Q,+)$, a, b, c are fixed elements of Q, $\theta \in End(Q,+)$, $\tilde{R}_a, \tilde{L}_b$ are transformations of the group $(Q,+)$: $\tilde{R}_a x = x + a$, $\tilde{L}_a x = a + x$.

Corollary 3.8. *Any endomorphism γ of a linear quasigroup $(Q,\cdot) : xy = \varphi x + c + \psi y$ can be presented in the form:*

$$\gamma = \tilde{R}_c\tilde{L}_{\varphi a}\varphi\theta\varphi^{-1}\tilde{R}_{-c} = \tilde{R}_{\psi b}\psi\theta\psi^{-1} = \tilde{L}_a\tilde{R}_b\theta,$$

where $\theta \in End(Q,+)$, $a \in Q$.

Proof. It is follows from the coincidence of all components of the endotopy of a linear quasigroup. □

Remark. Similarly, any endomorphism (automorphism) of alinear quasigroups, quasigroups of mixed type of linearity, T-quasigroups and medial quasigroups can be presented by automorphisms, endomorphisms and transformations of respective groups [15].

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References:

1. V.D. Belousov. *Foundations of the theory of quasigroups and loops.* Moscow, Nauka, 1967.(in Russian).

2. V.D. Belousov. *Balanced identities on quasigroups.* Mat. Sb., vol.70(112): no.1, 1966, p.55-97,(in Russian).

3. G.B. Belyavskaya, A.Kh. Tabarov. *Characterization of linear and alinear quasigroups.* Diskretnaya matematika, v.4, no.2, 1992, p.142-147.(in Russian).

4. R.H. Bruck. *Some results in theory of quasigroups.* - Trans. Amer. Math. Soc., 1944, vol. 55, p.19-52.

5. K. Toyoda. *On axsioms of linear functions.* - Proc. Imp. Acad.Tokyo., 1941, vol. 17, p.221-227.

6. D.S. Murdoch. *Quasigroups which satisfy certain generalized associative laws.* - Amer. J.Math., 1939, vol. 61, p.509-522.

7. T. Kepka and P. Nemec. *T-quasigroups. I.* - Acta univ. Carolin. Math.Phys., 1971, vol.12, No. 1, .31-39.

8. T. Kepka and P. Nemec. *T-quasigroups.II.* - Acta univ. Carolin. Math.Phys., 1971, vol. 12, No. 2, .39-49.

9. K.K. Shchukin. *On simple medial quasigroups.* - Math. issled. Kishinev, 1991, vol. 120, p.114-117,(in Russian).

10. K.K. Shchukin. *Action a group on a quasigroup.* Kishinev State University Printing House, Kishinev, 1985, (in Russian).

11. V.A. Shcherbacov. *On linear and inverse quasigroups and their applications in code theory.* Dissertation of Doctor of Science, 247 pages, Chisinau, 2008.

12. G.B. Belyavskaya. *Abelian quasigroups are T-quasigroups.* Quasigroups and Related Systems, 1994, vol.1, no.1, p.1-7.

13. L.V. Safonova, K.K. Shchukin. *Calculation of automorphisms and antiautomorphisms of quasigroups.* Izvest. AS SSR Moldova, Mathematica, 1990, vol. 3, No. 3, p.49-55,(in Russian).

14. I.A. Golovko. *Endotopies in quasigroups.* Abstracts. First Allunion symposium on theory quasigroups and its applications. Sukhumi,1968, p.14-15, (in Russian).

15. A.Kh. Tabarov. *Homomorphisms and endomorphisms of linear and alinear quasigroups.* Discrete Mathematics and Applications, 17 (2007), no.3, p. 253-260.