THE METHOD OF ALTERNATING RESOLVENTS REVISITED

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Abstract. The purpose of this paper is to prove a strong convergence result associated with a generalization of the method of alternating resolvents introduced by the authors in [Strong convergence of the method of alternating resolvents, J. Nonlinear Convex Anal., to appear] under minimal assumptions on the control parameters involved. Thus this paper represents a significant improvement of the paper mentioned above.

1. Introduction

In the sequel, H will be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. We recall that a map $T : H \to H$ is called nonexpansive if for every $x, y \in H$ we have $\|Tx - Ty\| \leq \|x - y\|$. An operator $A : D(A) \subset H \to 2^H$ is said to be monotone if

$$\langle x - x', y - y' \rangle \geq 0, \quad \forall\, (x, y), (x', y') \in G(A).$$

In other words, its graph $G(A) = \{(x, y) \in H \times H : x \in D(A), y \in Ax\}$ is a monotone subset of the product space $H \times H$. An operator A is called maximal monotone if in addition to being monotone, its graph is not properly contained in the graph of any other monotone operator. Note that if A is maximal monotone, then so is A^{-1}. For a maximal monotone operator A, the resolvent of A, defined by $J_\beta^A := (I + \beta A)^{-1}$, is well defined on the whole space H, is single-valued and nonexpansive for every $\beta > 0$. It is known that the Yosida approximation of A, an operator defined by $A_\beta := \beta^{-1}(I - J_\beta^A)$ (where I is the identity operator) is maximal monotone for every $\beta > 0$.

The method of alternating resolvents is an iterative procedure for finding a point in the intersection $A^{-1}(0) \cap B^{-1}(0) =: F$ (where A and B are maximal monotone operators), and it is defined as follows:

$$x_0 \mapsto x_1 = J_\lambda^A x_0 \mapsto x_2 = J_\lambda^B x_1 \mapsto x_3 = J_\lambda^A x_2 \mapsto x_4 = J_\lambda^B x_3 \mapsto \cdots,$$

for some $\lambda > 0$ and any starting point $x_0 \in H$. Bauscheke et al. [1] showed that such a method converges weakly to some point of F. In fact, they showed that the method of alternating resolvents given above converges weakly to some point of $\mathrm{Fix} J_\lambda^A J_\lambda^B$ - the fixed point set of the composition mapping $J_\lambda^A J_\lambda^B$ - provided that this set is not empty. For a generalized version of this method, see [3]. In the particular case when A and B are normal cones, the method of alternating resolvents reduces to the well known method of alternating projections which was introduced by von Neumann in 1933. The latter method converges weakly [7] to some point in F, but not strongly in general [8]. This fact prompted the current authors to construct sequences generated from the method of alternating resolvents which converge strongly, see [3, 6]. One such sequence was defined

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[6] as

$$x_{2n+1} = J^A_{\beta_n}(\alpha_n u + (1-\alpha_n)x_{2n} + e_n) \quad \text{for } n = 0, 1, \dots, \tag{1}$$

$$x_{2n} = J^B_{\mu_n}(\lambda_n u + (1-\lambda_n)x_{2n-1} + e'_n) \quad \text{for } n = 1, 2, \dots, \tag{2}$$

for any given $u, x_0 \in H$, where (e_n) and (e'_n) are sequences of computational errors, $\alpha_n, \lambda_n \in (0,1)$ and $\beta_n, \mu_n \in (0,\infty)$. In the current paper, we investigate if the method initiated in [12] can be extended to the case of two operators, namely, to the scheme (1), (2) in order to obtain strong convergence results of sequences generated by it under minimal assumptions on the control parameters $\alpha_n, \lambda_n, \beta_n$, and μ_n, thereby refining the previously obtained results associated with the strong convergence of the method of alternating resolvents [3, 6].

2. Preliminary Results

The following two lemmas will be crucial in proving our main results.

Lemma 1 (Boikanyo and Moroşanu [6])**.** *Let (s_n) be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \le (1-\alpha_n)(1-\lambda_n)s_n + \alpha_n b_n + \lambda_n c_n + d_n, \quad n \ge 0, \tag{3}$$

where (α_n), (λ_n), (b_n), (c_n) and (d_n) satisfy the conditions: (i) $\alpha_n, \lambda_n \in [0,1]$, *with* $\prod_{n=0}^\infty (1-\alpha_n) = 0$, (ii) $\limsup_{n\to\infty} b_n \le 0$, (iii) $\limsup_{n\to\infty} c_n \le 0$, *and* (iv) $d_n \ge 0$ *for all $n \ge 0$ with $\sum_{n=0}^\infty d_n < \infty$. Then $\lim_{n\to\infty} s_n = 0$.*

Remark 2. If $\lim_{n\to\infty} a_n = 0$, then $\prod_{n=0}^\infty (1-a_n) = 0$ if and only if $\sum_{n=0}^\infty a_n = \infty$.

Lemma 3 (Maingé [10])**.** *Let (s_n) be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence (s_{n_j}) of (s_n) such that $s_{n_j} \le s_{n_j+1}$ for all $j \ge 0$. For every $n \ge n_0$, define an integer sequence $(\tau(n))$ as*

$$\tau(n) = \max\{k \le n : s_{n_j} < s_{n_j+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \ge n_0$

$$\max\{s_{\tau(n)}, s_n\} \le s_{\tau(n)+1}. \tag{4}$$

The next lemma is well known, it can be found for example in [11, p. 20].

Lemma 4. *Any maximal monotone operator $A : D(A) \subset H \to 2^H$ satisfies the demicloseness principle. In other words, given any two sequences (x_n) and (y_n) satisfying $x_n \to x$ and $y_n \rightharpoonup y$ with $(x_n, y_n) \in G(A)$, then $(x, y) \in G(A)$.*

Lemma 5 (Xu [14])**.** *For any $x \in H$ and $\mu \ge \beta > 0$,*

$$\|x - J_\beta x\| \le 2\,\|x - J_\mu x\|,$$

where $A : D(A) \subset H \to 2^H$ is a maximal monotone operator.

3. Main Results

We begin by proving a strong convergence result associated with the exact iterative process

$$v_{2n+1} = J^A_{\beta_n}(\alpha_n u + (1-\alpha_n)v_{2n}) \quad \text{for } n = 0, 1, \dots, \tag{5}$$

$$v_{2n} = J^B_{\mu_n}(\lambda_n u + (1-\lambda_n)v_{2n-1}) \quad \text{for } n = 1, 2, \dots, \tag{6}$$

where $\alpha_n, \lambda_n \in (0,1)$ and $v_0, u \in H$ are given. The proof of the following theorem makes use of the ideas of the papers [5, 6, 10, 12]

Theorem 6. *Let $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$ be maximal monotone operators with $A^{-1}(0) \cap B^{-1}(0) =: F \neq \emptyset$. For arbitrary but fixed vectors $v_0, u \in H$, let (v_n) be the sequence generated by (5), (6), where $\alpha_n, \lambda_n \in (0,1)$ and $\beta_n, \mu_n \in (0, \infty)$. Assume that* (i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \lambda_n = 0$, (ii) *either* $\sum_{n=0}^{\infty} \alpha_n = \infty$ *or* $\sum_{n=0}^{\infty} \lambda_n = \infty$, *and* (iii) *both* (β_n) *and* (μ_n) *are bounded from below away from zero. Then (v_n) converges strongly to the point of F nearest to u.*

Proof. We already know that (v_n) is bounded [6]. Next, we show that for any $p \in F$

$$\begin{aligned} (1+\alpha_n)\|v_{2n+1} - p\|^2 &\leq (1-\alpha_n)\|v_{2n} - p\|^2 + 2\alpha_n\langle u - p, v_{2n+1} - p\rangle \\ &- (1-\alpha_n)\|v_{2n+1} - v_{2n}\|^2 \end{aligned} \tag{7}$$

holds. Indeed, multiplying

$$v_{2n+1} - p + \beta_n A v_{2n+1} \ni \alpha_n(u-p) + (1-\alpha_n)(v_{2n} - p)$$

scalarly by $v_{2n+1} - p$ and using the monotonicity of A, we obtain

$$\begin{aligned} 2\|v_{2n+1} - p\|^2 &\leq 2\alpha_n\langle u-p, v_{2n+1} - p\rangle + 2(1-\alpha_n)\langle v_{2n} - p, v_{2n+1} - p\rangle \\ &= (1-\alpha_n)(\|v_{2n} - p\|^2 + \|v_{2n+1} - p\|^2 - \|v_{2n+1} - v_{2n}\|^2) \\ &+ 2\alpha_n\langle u - p, v_{2n+1} - p\rangle. \end{aligned}$$

Rearranging terms, we readily get (7). Using similar arguments as above, one can prove that for any $p \in F$

$$\begin{aligned} (1+\lambda_n)\|v_{2n} - p\|^2 &\leq (1-\lambda_n)\|v_{2n-1} - p\|^2 + 2\lambda_n\langle u - p, v_{2n} - p\rangle \\ &- (1-\lambda_n)\|v_{2n} - v_{2n-1}\|^2. \end{aligned}$$

Using this inequality in (7), we get

$$\begin{aligned} (1+\alpha_n)\|v_{2n+1} - p\|^2 &\leq (1-\alpha_n)(1-\lambda_n)\|v_{2n-1} - p\|^2 + 2\alpha_n\langle u-p, v_{2n+1} - p\rangle \\ &- (1-\alpha_n)\|v_{2n+1} - v_{2n}\|^2 + 2\lambda_n(1-\alpha_n)\langle u - p, v_{2n} - p\rangle \\ &- (1-\alpha_n)(1-\lambda_n)\|v_{2n} - v_{2n-1}\|^2. \end{aligned} \tag{8}$$

Denote $s_n := \|v_{2n-1} - P_F u\|^2$. Then it follows from (8) and the boundedness of (v_n) that

$$s_{n+1} - s_n + \|v_{2n} - v_{2n-1}\|^2 + \|v_{2n+1} - v_{2n}\|^2 \leq (\alpha_n + \lambda_n)M, \tag{9}$$

for some positive constant M. On the other hand, we have from (5)

$$\begin{aligned} \|v_{2n+1} - J_\beta^A v_{2n+1}\| &\leq 2\|v_{2n+1} - J_{\beta_n}^A v_{2n+1}\| \\ &\leq 2\|\alpha_n(u - v_{2n}) + (v_{2n} - v_{2n+1})\| \\ &\leq (\alpha_n + \|v_{2n} - v_{2n+1}\|)M', \end{aligned} \tag{10}$$

where $\beta > 0$ is the greatest lower bound of β_n, M' is a positive constant and the first inequality follows from Lemma 5. Similarly, starting from (6), we arrive at

$$\|v_{2n} - J_\mu^B v_{2n}\| \leq (\lambda_n + \|v_{2n} - v_{2n-1}\|)M', \tag{11}$$

where $\mu > 0$ is the greatest lower bound of μ_n. We next consider two possible cases on the sequence (s_n).

CASE 1: (s_n) is eventually decreasing (i.e., there exists $N \geq 0$ such that (s_n) is decreasing for all $n \geq N$). In this case, (s_n) is convergent. Passing to the limit in (9), we get

$$\|v_{n+1} - v_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{12}$$

Therefore, it follows from (10) that

$$\left\|v_{2n+1} - J_\beta^A v_{2n+1}\right\| \to 0 \quad \text{as} \quad n \to \infty.$$

Using the fact that A_β^{-1} is demiclosed, where A_β is the Yosida approximation of A, we get $\omega_{\mathtt{w}}((v_{2n+1})) \subset A^{-1}(0)$. Recall that $\omega_{\mathtt{w}}((x_n))$ denotes the set of all weak cluster points of the sequence (x_n). Similarly, from (11) and (12) we derive $\omega_{\mathtt{w}}((v_{2n})) \subset B^{-1}(0)$. These two inclusions and (12) imply that $\omega_{\mathtt{w}}((v_n)) \subset A^{-1}(0) \cap B^{-1}(0)$. Extract a subsequence (v_{2n_k+1}) of (v_{2n+1}) converging weakly to some $y \in F$ such that

$$\begin{aligned}\limsup_{n\to\infty} \langle u - P_F u, v_{2n+1} - P_F u\rangle &= \lim_{k\to\infty} \langle u - P_F u, v_{2n_k+1} - P_F u\rangle \\ &= \langle u - P_F u, y - P_F u\rangle \leq 0,\end{aligned}$$

where $P_F u$ denotes the projection of u on the set F (which is closed and convex). On the other hand, from (8), we derive

$$\begin{aligned}\|v_{2n+1} - P_F u\|^2 &\leq (1-\alpha_n)(1-\lambda_n)\|v_{2n-1} - P_F u\|^2 + 2\alpha_n\langle u - P_F u, v_{2n+1} - P_F u\rangle \\ &+ 2\lambda_n(1-\alpha_n)\langle u - P_F u, (v_{2n} - v_{2n+1}) + (v_{2n+1} - P_F u)\rangle,\end{aligned}$$

and hence from Lemma 1, we get $v_{2n+1} \to P_F u$ as $n \to \infty$. By virtue of (12), we also have $v_{2n} \to P_F u$ as $n \to \infty$. Hence $v_n \to P_F u$ as desired.

CASE 2: (s_n) is not eventually decreasing, that is, there is a subsequence (s_{n_j}) of (s_n) such that $s_{n_j} \leq s_{n_j+1}$ for all $j \geq 0$. We therefore define an integer sequence $(\tau(n))$ as in Lemma 3 so that for all $n \geq n_0$, $s_{\tau(n)} \leq s_{\tau(n)+1}$ holds. Note that from (9), we have

$$\lim_{n\to\infty} \left\|v_{\tau(n)+1} - v_{\tau(n)}\right\| = 0.$$

It then follows from (10) that $\omega_{\mathtt{w}}((v_{2\tau(n)+1})) \subset A^{-1}(0)$. Similarly, from (11), we have $\omega_{\mathtt{w}}((v_{2\tau(n)})) \subset B^{-1}(0)$. On the other hand, the above limit implies that $\omega_{\mathtt{w}}((v_{2\tau(n)+1})) \subset B^{-1}(0)$. Therefore, $\omega_{\mathtt{w}}((v_{2\tau(n)+1})) \subset F$. Consequently,

$$\limsup_{n\to\infty} \langle u - P_F u, v_{2\tau(n)+1} - P_F u\rangle \leq 0.$$

On the other hand, since $s_{\tau(n)} \leq s_{\tau(n)+1}$ holds, it follows from (8) that

$$\begin{aligned}(2\alpha_{\tau(n)} + \lambda_{\tau(n)} - \alpha_{\tau(n)}\lambda_{\tau(n)})s_{\tau(n)+1} &\leq 2\alpha_{\tau(n)}\langle u - P_F u, v_{2\tau(n)+1} - P_F u\rangle \\ &+ 2\lambda_{\tau(n)}(1-\alpha_{\tau(n)})\langle u - P_F u, v_{2\tau(n)} - v_{2\tau(n)+1}\rangle \\ &+ 2\lambda_{\tau(n)}(1-\alpha_{\tau(n)})\langle u - P_F u, v_{2\tau(n)+1} - P_F u\rangle,\end{aligned}$$

which implies, according to the above pieces of information, that $s_{\tau(n)+1} \to 0$ as $n \to \infty$. Hence, from (4) it follows that $s_n \to 0$ as $n \to \infty$. That is, $v_{2n+1} \to P_F u$ as $n \to \infty$. Now, using again (9), we get $v_{2n} \to P_F u$ as $n \to \infty$. This shows that $v_n \to P_F u$ as $n \to \infty$, and the proof is complete. □

Remark 7. Theorem 6 is a refinement of Theorem 7 [6].

Theorem 8. *Let $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$ be maximal monotone operators with $A^{-1}(0) \cap B^{-1}(0) =: F \neq \emptyset$. For arbitrary but fixed vectors $x_0, u \in H$, let (x_n) be the sequence generated by (1), (2), where $\alpha_n, \lambda_n \in (0,1)$ and $\beta_n, \mu_n \in (0,\infty)$. Assume that* (i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \lambda_n = 0$, (ii) *either* $\sum_{n=0}^\infty \alpha_n = \infty$ *or* $\sum_{n=0}^\infty \lambda_n = \infty$, *and* (iii) *both* (β_n) *and* (μ_n) *are bounded from below away from zero. In addition, if any of the following conditions is satisfied,*

(a) $\sum_{n=0}^\infty \|e_n\| < \infty$ *and* $\sum_{n=1}^\infty \|e'_n\| < \infty$*;*
(b) $\sum_{n=0}^\infty \|e_n\| < \infty$ *and* $\|e'_n\|/\alpha_n \to 0$*;*

(c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *and* $\|e'_n\|/\lambda_n \to 0$*;*
(d) $\|e_n\|/\alpha_n \to 0$ *and* $\sum_{n=1}^{\infty} \|e'_n\| < \infty$*;*
(e) $\|e_n\|/\lambda_n \to 0$ *and* $\sum_{n=1}^{\infty} \|e'_n\| < \infty$*;*
(f) $\|e_n\|/\alpha_n \to 0$ *and* $\|e'_n\|/\alpha_n \to 0$*;*
(g) $\|e_n\|/\alpha_n \to 0$ *and* $\|e'_n\|/\lambda_n \to 0$*;*
(h) $\|e_n\|/\lambda_n \to 0$ *and* $\|e'_n\|/\alpha_n \to 0$*;*
(i) $\|e_n\|/\lambda_n \to 0$ *and* $\|e'_n\|/\lambda_n \to 0$*;*
(j) $\|e_n\|/\alpha_n \to 0$ *and* $\|e'_n\|/\alpha_{n-1} \to 0$*;*
(k) $\|e_{n-1}\|/\lambda_n \to 0$ *and* $\|e'_n\|/\alpha_{n-1} \to 0$*;*
(l) $\|e_{n-1}\|/\lambda_n \to 0$ *and* $\|e'_n\|/\lambda_n \to 0$*;*
(m) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *and* $\|e'_n\|/\alpha_{n-1} \to 0$*;*
(n) $\|e_{n-1}\|/\lambda_n \to 0$ *and* $\sum_{n=1}^{\infty} \|e'_n\| < \infty$*,*

then (x_n) *converges strongly to the point of* F *nearest to* u*.*

Proof. In view of Theorem 6, it is enough to show that $\|x_n - v_n\| \to 0$ as $n \to \infty$. Since the resolvent of A is nonexpansive, we derive from (1) and (5) that

$$\|x_{2n+1} - v_{2n+1}\| \leq (1-\alpha_n)\|x_{2n} - v_{2n}\| + \|e_n\| . \tag{13}$$

Similarly, from (2) and (6), we have

$$\|x_{2n} - v_{2n}\| \leq (1-\lambda_n)\|x_{2n-1} - v_{2n-1}\| + \|e'_n\| . \tag{14}$$

These two inequalities imply that

$$\|x_{2n+1} - v_{2n+1}\| \leq (1-\alpha_n)(1-\lambda_n)\|x_{2n-1} - v_{2n-1}\| + \|e_n\| + \|e'_n\| .$$

Therefore if the error sequence satisfy any of the conditions (a)-(i), then it readily follows from Lemma 1 that $\|x_{2n+1} - v_{2n+1}\| \to 0$ as $n \to \infty$. Passing to the limit in (14), we derive $\|x_{2n} - v_{2n}\| \to 0$ as well. If the error sequence satisfy any of the conditions (j)-(n), then from (13) and (14), we have

$$\|x_{2n} - v_{2n}\| \leq (1-\alpha_{n-1})(1-\lambda_n)\|x_{2n-2} - v_{2n-2}\| + \|e_{n-1}\| + \|e'_n\| .$$

It then follows from Lemma 1 that $\|x_{2n} - v_{2n}\| \to 0$ as $n \to \infty$. Passing to the limit in (13), we derive $\|x_{2n+1} - v_{2n+1}\| \to 0$ as well. This completes the proof of the theorem. □

Remark 9. Theorem 8 improves Theorem 8 [6]. It also contains Theorems 2-3 [6] as special cases since for $\lambda_n = 0$ for all $n \in \mathbb{N}$, algorithm (1), (2) reduces to algorithm (14), (15) which was introduced in [3], and for $\lambda_n = \alpha_n$ for all $n \in \mathbb{N}$, algorithm (1), (2) reduces to algorithm (24), (25) which was introduced in [3].

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