An existence result for a nonhomogeneous problem in $\mathbb{R}^2$ related to nonlinear Hencky-type materials

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Abstract. This paper investigates a nonlinear and non-homogeneous system of partial differential equations. The motivation comes from the fact that in a particular case the problem discussed here can be used in modeling the behavior of nonlinear Hencky-type materials. The main result of the paper establishes the existence of a nontrivial solution in an adequate functional space of Orlicz-Sobolev type by using Schauder's fixed point theorem combined with adequate variational techniques.

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1 Introduction

Assume $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain with small Lipschitz constants representing the volume occupied by a body. Letting $\overrightarrow{u} = (u_1, u_2)$ be the *displacement vector* we introduce the *strain tensor* $\varepsilon(\overrightarrow{u}) = (\varepsilon_{ij}(\overrightarrow{u}))_{1\leq i,j\leq 2}$ by

$$\varepsilon_{ij}(\overrightarrow{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad \forall\, 1 \leq i,j \leq 2\,.$$

Further, let $\varepsilon^D(\overrightarrow{u})$ be the *deviatoric part* of $\varepsilon(\overrightarrow{u})$, i.e.

$$\varepsilon^D(\overrightarrow{u}) := \varepsilon(\overrightarrow{u}) - \frac{1}{2}\mathrm{trace}(\varepsilon(\overrightarrow{u}))\mathbf{I}\,,$$

where $\mathrm{trace}(\varepsilon(\overrightarrow{u}))$ denotes the trace of tensor $\varepsilon(\overrightarrow{u})$, i.e.

$$\mathrm{trace}(\varepsilon(\overrightarrow{u})) := \varepsilon_{11}(\overrightarrow{u}) + \varepsilon_{22}(\overrightarrow{u})\,,$$

and $\mathbf{I}$ denotes the identity tensor on $\mathbb{R}^2$.

Next, let $a : \Omega \to \mathbb{R}$ be a function such that map $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) := \begin{cases} a(|t|)t & \text{for} \quad t \neq 0 \\ 0 & \text{for} \quad t = 0\,, \end{cases}$$

is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. For each $t \geq 0$ define

$$\Phi(t) := \int_0^t \varphi(s)\, ds\,.$$

For the displacement vector $\overrightarrow{u} = (u_1, u_2)$ let $\sigma(\overrightarrow{u}) = (\sigma_{ij}(\overrightarrow{u}))_{1\leq i,j\leq 2}$ stand for the *stress tensor*. Here we assume that the stress tensor and the strain tensor satisfy the following nonlinear Hooke-type law

$$\sigma(\overrightarrow{u}) = \lambda \mathrm{div}\, \overrightarrow{u}\, \mathbf{I} + 2\mu(x, \overrightarrow{u})\varphi(\varepsilon^D(\overrightarrow{u}))\,, \tag{1.1}$$

where $\lambda \geq 0$ and $\mu(x, \overrightarrow{u}) > 0$ stand as *Lamé coefficients*. This constitutive law describes the properties of the material of which the body Ω is made. We assume that μ depends on both x and $\overrightarrow{u}$. Specifically, we assume that $\mu : \Omega \times \mathbb{R}^2 \to \mathbb{R}^+$ is a Carathéodory function for which there exist two positive constants $0 < \mu_1 < \mu_2$ such that

$$0 < \mu_1 \leq \mu(x, \overrightarrow{t}) \leq \mu_2, \quad \text{a.e. } x \in \Omega,\ \forall\ \overrightarrow{t} \in \mathbb{R}^2\,. \tag{1.2}$$

In components relation (1.1) becomes

$$\sigma_{ij}(\overrightarrow{u}) = \lambda \mathrm{div}\, \overrightarrow{u}\, \delta_{ij} + 2\mu(x, \overrightarrow{u}) a(|\varepsilon^D(\overrightarrow{u})|)\varepsilon^D_{ij}(\overrightarrow{u})\,,$$

where δ_{ij} is the *Kroneker symbol* (that is $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$). Recall also that

$$|\varepsilon^D(\overrightarrow{u})| = \sqrt{\mathrm{trace}(\varepsilon^D(\overrightarrow{u})^T \varepsilon^D(\overrightarrow{u}))}\,.$$

In this paper we are concerned with the investigation of the existence of solutions for the following problem

$$\begin{cases} -\displaystyle\sum_{j=1}^{2} \frac{\partial}{\partial x_j}(\sigma_{ij}(\overrightarrow{u})) = f_i & \text{in} \quad \Omega,\ \ i \in \{1, 2\} \\ u_i = 0 & \text{on} \quad \partial\Omega,\ \ i \in \{1, 2\}\,, \end{cases} \tag{1.3}$$

where $\overrightarrow{f} = (f_1, f_2) : \Omega \to \mathbb{R}^2$ is a given function. In a very simple formulation equations (1.3) describe the fact that the force $\overrightarrow{f}$ acts on the body Ω clamped on the boundary ($\overrightarrow{u} = \overrightarrow{0}$ on $\partial\Omega$).

Problems of type (1.3) arise from the study of nonlinear *Hencky materials*. To be more specific, according to Zeidler [21, p. 202 and p. 256] (see also Bildhauer & Fuchs [4]) in the particular case when the Lamé coefficient μ from relation (1.1) is a constant function the energetic functional associated with the differential operator involved in problem (1.3), i.e.

$$\int_\Omega \left[\frac{\lambda}{2}(\mathrm{div}\, \overrightarrow{u})^2 + 2\mu \Phi(|\varepsilon^D(\overrightarrow{u})|)\right]\, dx\,, \tag{1.4}$$

models nonlinear Hencky material behavior. This model consists in the finite logarithmic strain-based extension of the standard linear elastic material (see, de Souza Neto *et al.* [7] for more details and for a numerical treatment of the model). It was Hencky [13] who in 1933 initally proposed modeling the behavior of vulcanized rubbers. In this paper the nonlinear material behavior occurs for nonlinear functions Φ in (1.4). Particularly, it generalizes the special case of linear elasticity obtained when in (1.4) we take $\Phi(t) = t^2$ and $\varepsilon^D(\overrightarrow{u})$ is replaced by $\varepsilon(\overrightarrow{u})$. We refer to Nečas & Hlaváček [18] for more details and connections regarding the *classical* elasticity system. On the other hand, we note that in the case when μ is not a positive constant but a function depending on de displacement vector $\overrightarrow{u}$ then we cannot associate an energetic functional with the differential operator involved in problem (1.3). Thus, in this more general case the use of variational techniques will not be sufficient and we will combine these techniques with a fixed point theorem in order to prove the existence of a solution for problem (1.3). Finally, we point out that the dependence of μ on the displacement vector is natural. A similar situation can be found in [18] on page 51 where the Lamé coefficient μ depends on the derivatives of the displacement vector $\overrightarrow{u}$ and a relation of type (1.2) is satisfied in the context of Hooke's Law (3.1) from [18] and Hencky's type relation (3.8) from [18] (see also the discussion on page 125 in [18]).

The paper is organized as follows: in Section 2 we offer some preliminary results on Orlicz-Sobolev spaces and we introduce some notations which will be used in the sequel; in Section 3 we point out the main result of this paper; in Section 4 we recall two remarkable inequalities in Orlicz-Sobolev spaces; in Section 5 we prove the main result of the paper.

2 A review on Orlicz-Sobolev spaces

Problems of type (1.3) in their general formulations ask for an Orlicz-Sobolev space framework in order to define an adequate functional space in which to seek solutions for the equation.

With that end in view, define

$$\Phi(t) = \int_0^t \varphi(s)\, ds, \quad \Phi^\star(t) = \int_0^t \varphi^{-1}(s)\, ds. \tag{2.1}$$

Note that Φ is a *Young function*, that is, $\Phi(0) = 0$, Φ is convex, and $\lim_{t\to\infty} \Phi(t) = +\infty$. Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$, $\lim_{t\to 0} \Phi(t)/t = 0$, and $\lim_{t\to\infty} \Phi(t)/t = +\infty$, Φ is an *N–function* (see [1] or [2] for more details). $\Phi^\star$ is called the *complementary* function of Φ, and it satisfies

$$\Phi^\star(t) = \sup\{st - \Phi(s) \ : \ s \geq 0\}, \qquad \text{for all } t \geq 0\,.$$

In addition, $\Phi^\star$ is also an N-function and *Young's inequality* holds:

$$st \leq \Phi(s) + \Phi^\star(t), \qquad \text{for all } s, t \geq 0.$$

Letting

$$\varphi_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \ \text{ and } \ \varphi^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}\,,$$

we will assume that

$$1 < \varphi_0 \leq \frac{t\varphi(t)}{\Phi(t)} \leq \varphi^0 < \infty, \qquad \text{for all } t > 0\,. \tag{2.2}$$

We indicate several examples of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$, and for which (2.2) holds. For more details the reader is referred to [6, Examples 1-3, p. 243].

1) $\varphi(t) = |t|^{p-2}t$, with $p > 1$. It can be showed that $\varphi_0 = \varphi^0 = p$;

2) $\varphi(t) = \log(1 + |t|^r)|t|^{p-2}t$, with $p, r > 1$. In this case $\varphi_0 = p$, and $\varphi^0 = p + r$;

3) $\varphi(t) = \frac{|t|^{p-2}t}{\log(1+|t|)}$, if $t \neq 0$, $\varphi(0) = 0$, with $p > 2$. In this case it turns out that $\varphi_0 = p - 1$ and $\varphi^0 = p$.

Next, we point out some useful results obtained under the assumption that (2.2) holds true. Indeed, in this case the following relations are valid

$$\Phi^\star(\varphi(s)) \leq s\varphi(s) - \Phi(s) \leq (\varphi^0 - 1)\Phi(s), \quad \forall\, s \geq 0\,, \tag{2.3}$$

$$\beta(\rho)\Phi(t) \leq \Phi(\rho t) \leq \gamma(\rho)\Phi(t), \quad \forall\, t > 0,\ \rho > 0\,, \tag{2.4}$$

where,

$$\beta(\rho) := \begin{cases} \rho^{\varphi^0} & \text{if } \rho \in (0,1] \\ \rho^{\varphi_0} & \text{if } \rho \in (1,\infty)\,, \end{cases} \qquad \gamma(\rho) := \begin{cases} \rho^{\varphi_0} & \text{if } \rho \in (0,1] \\ \rho^{\varphi^0} & \text{if } \rho \in (1,\infty)\,, \end{cases}$$

see [9, Lemmas A.2 and 2.1] for the proofs.

To proceed further, we need to recall some basic facts about Orlicz-Sobolev spaces. For more details we refer to the books by Adams & Hedberg [2], Adams [1], and Rao & Ren [19], and to the papers by Clément *et al.* [5, 6], García-Huidobro *et al.* [10] and Gossez [12].

With φ, Φ, and $\Phi^\star$ as defined above, the Orlicz space $L^\Phi(\Omega)$ is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega \Phi(|u|)\, dx < \infty\,.$$

This is a Banach space with respect to the *Luxemburg norm*, defined by

$$\|u\|_\Phi := \inf\left\{\mu > 0 \ :\ \int_\Omega \Phi\left(\frac{|u(x)|}{\mu}\right)\, dx \leq 1\right\}.$$

In the context of Orlicz spaces Hölder's inequality reads as follows (see [19, Inequality 4, p. 79]):

$$\int_\Omega uv dx \leq C\, \|u\|_{L^\Phi}\, \|v\|_{L^{\Phi^\star}} \qquad \text{for all } u \in L^\Phi(\Omega) \text{ and } v \in L^{\Phi^\star}(\Omega)\,, \tag{2.5}$$

where $C > 0$ is a positive constant.

The Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ is defined by

$$W^{1,\Phi}(\Omega) := \left\{u \in L^\Phi(\Omega) \ :\ \frac{\partial u}{\partial x_i} \in L^\Phi(\Omega),\ i = 1, \cdots, N\right\},$$

and is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} := \|u\|_\Phi + \||\nabla u|\|_\Phi .$$

In what follows $W_0^{1,\Phi}(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ in $W^{1,\Phi}(\Omega)$. By [12, Lemma 5.7],

$$\|u\|_{0,\Phi} := \||\nabla u|\|_\Phi$$

is an equivalent norm on $W_0^{1,\Phi}(\Omega)$. In an appropriate context we point out that on $W_0^{1,\Phi}(\Omega)$ we can also consider the equivalent norm

$$\|u\|_{2,\Phi} := \sum_{j=1}^{2} \left\| \frac{\partial u}{\partial x_j} \right\|_\Phi ,$$

see [15, Proposition 1] for details.

Let φ_0 and φ^0 be defined as above, and assume that (2.2) holds. This implies that Φ satisfies the Δ_2-condition:

$$\Phi(2t) \le K\Phi(t), \quad \forall\, t \ge 0\,, \tag{2.6}$$

where K is a positive constant (see [17, Proposition 2.3]). On the other hand (see, e.g., [9, Lemma 2.1] or [17, Proposition 2.1]), we have

$$\|u\|_\Phi^{\varphi^0} \le \int_\Omega \Phi(|u(x)|)\, dx \le \|u\|_\Phi^{\varphi_0}, \quad \forall\, u \in L^\Phi(\Omega),\ \|u\|_\Phi < 1, \tag{2.7}$$

and

$$\|u\|_\Phi^{\varphi_0} \le \int_\Omega \Phi(|u(x)|)\, dx \le \|u\|_\Phi^{\varphi^0}, \quad \forall\, u \in L^\Phi(\Omega),\ \|u\|_\Phi > 1. \tag{2.8}$$

Finally, we assume that Φ is such that the map $[0,\infty) \ni t \to \Phi(\sqrt{t})$ is convex. We note that this, together with (2.6), implies that the Orlicz space $L^\Phi(\Omega)$ is a uniformly convex (and hence reflexive) Banach space (see [17, Proposition 2.2]).

Remark. Let

$$\varphi(t) = |t|^{p-2}t, \quad \forall\, t \in \mathbb{R},$$

with $p > 1$. As we already mentioned in example 1) it can be shown that in this case we have

$$\varphi_0 = \varphi^0 = p.$$

Moreover, in this particular case the corresponding Orlicz space $L^\Phi(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$ while the Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ becomes the classical Sobolev space $W^{1,p}(\Omega)$.

3 The main result

Define now the reflexive Banach space

$$V := W_0^{1,\Phi}(\Omega) \times W_0^{1,\Phi}(\Omega)\,,$$

endowed with the norm

$$\|\overrightarrow{u}\| := \||\nabla u_1|\|_\Phi + \||\nabla u_2|\|_\Phi \,,$$

for $\overrightarrow{u} = (u_1, u_2)$.

Assume $\varphi_0 \geq 2$, where φ_0 is given in relation (2.2). Then $W_0^{1,\Phi}(\Omega)$ is continuously embedded in $W_0^{1,2}(\Omega)$ (see, e.g. [16, Lemma 2]). Thus, solutions of problem (1.3) will be sought in V. We say that $\overrightarrow{u} \in V$ is a *weak solution* for problem (1.3) if

$$\lambda \int_\Omega \mathrm{div}\overrightarrow{u}\,\mathrm{div}\overrightarrow{\psi}\; dx + 2\sum_{i,j=1}^{2} \int_\Omega \mu(x,\overrightarrow{u})a(|\varepsilon^D(\overrightarrow{u})|)\varepsilon_{ij}^D(\overrightarrow{u})\varepsilon_{ij}^D(\overrightarrow{\psi})\; dx - \sum_{i=1}^{2}\int_\Omega f_i\psi_i\; dx = 0,$$

for all $\overrightarrow{\psi} = (\psi_1, \psi_2) \in V$.

The main result of this paper is given by the following theorem.

Theorem 1. *Assume relation (2.2) is fulfilled with $\varphi_0 \geq 2$ and the map $[0,\infty) \ni t \to \Phi(\sqrt{t})$ is convex. Assume also that the inequalities in (1.2) hold. Then for all $\overrightarrow{f} = (f_1, f_2) \in \left(L^{\Phi^\star}(\Omega)\right)^2$ problem (1.3) has a weak solution.*

4 Remarkable inequalities in Orlicz-Sobolev spaces

We point out two remarkable inequalities in Orlicz-Sobolev spaces that will prove to be useful in the study of equation (1.3).

4.1 Korn's inequality in Orlicz spaces

We start by recalling a Korn-type inequality proved in [8, Theorem 1]

Theorem 2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with small Lipschitz constants. Assume condition (2.2) is satisfied. Then there exists a positive constant $C_1 > 0$ such that*

$$\sum_{i=1}^{2} \int_\Omega \Phi(|\nabla u_i|)\; dx \leq C_1 \int_\Omega \Phi(|\varepsilon^D(\overrightarrow{u})|)\; dx, \quad \forall\, \overrightarrow{u} \in V\,.$$

4.2 A Poincaré type inequality in Orlicz spaces

Here we recall a Poincaré type inequality in Orlicz spaces (see, e.g. [10, Lemma 2.1] combined with inequality (2.4)).

Theorem 3. *Assume condition (2.2) is fulfilled. Then, there exists a positive constant $C_2 > 0$ such that*

$$\int_\Omega \Phi\left(|u(x)|\right)\; dx \leq C_2 \int_\Omega \Phi(|\nabla u(x)|)\; dx, \quad \forall\, u \in C_0^\infty(\Omega)\,.$$

5 Proof of Theorem 1

Fix an arbitrary function $\overrightarrow{f} = (f_1, f_2) \in \left(L^{\Phi^\star}(\Omega)\right)^2$. The main tool of our proof of Theorem 1 will be Schauder's fixed point theorem (see [3, Theorem 3.21]):

Schauder's Fixed Point Theorem. *Assume that K is a compact and convex subset of the Banach space B and $S : K \to K$ is a continuous map. Then S possesses a fixed point.*

We start by proving some auxiliary results which will be useful in establishing Theorem 1.

Proposition 1. *For each $\overrightarrow{v} \in L^\Phi(\Omega) \times L^\Phi(\Omega)$ the problem*

$$\begin{cases} -\displaystyle\sum_{j=1}^{2} \frac{\partial}{\partial x_j}(\lambda \operatorname{div}\overrightarrow{u}\delta_{ij} + 2\mu(x, \overrightarrow{v})a(|\varepsilon^D(\overrightarrow{u})|)\varepsilon^D_{ij}(\overrightarrow{u})) = f_i & in \quad \Omega, \ \ i \in \{1, 2\} \\ u_i = 0 & on \quad \partial\Omega, \ \ i \in \{1, 2\}\,, \end{cases} \tag{5.1}$$

has a weak solution $\overrightarrow{u} \in V$, i.e.

$$\lambda \int_\Omega \operatorname{div}\overrightarrow{u} \operatorname{div}\overrightarrow{\psi}\, dx + 2\sum_{i,j=1}^{2}\int_\Omega \mu(x, \overrightarrow{v})a(|\varepsilon^D(\overrightarrow{u})|)\varepsilon^D_{ij}(\overrightarrow{u})\varepsilon^D_{ij}(\overrightarrow{\psi})\, dx - \sum_{i=1}^{2}\int_\Omega f_i\psi_i\, dx = 0\,, \tag{5.2}$$

for all $\overrightarrow{\psi} = (\psi_1, \psi_2) \in V$.

The proof will be provided later.

Fix $\overrightarrow{v} \in (L^\Phi(\Omega))^2$. First, we note that condition (1.2) guarantees that $\mu(x, \overrightarrow{v}) \in L^\infty(\Omega)$.

Consider the energy functional associated with problem (5.1), $J : V \to \mathbb{R}$ defined by

$$J(\overrightarrow{u}) := \Lambda(\overrightarrow{u}) - \sum_{i=1}^{2}\int_\Omega f_i u_i\, dx\,,$$

where

$$\Lambda(\overrightarrow{u}) := \int_\Omega \left[\frac{\lambda}{2}(\operatorname{div}\overrightarrow{u})^2 + 2\mu(x, \overrightarrow{v})\Phi(|\varepsilon^D(\overrightarrow{u})|)\right]\, dx\,.$$

Theorems 2 and 3 show that functional J is well defined. Furthermore, we can show the following result.

Proposition 2. *Functional J is well-defined on V and $J \in C^1(V, \mathbb{R})$ with the derivative given by*

$$\langle J'(\overrightarrow{u}), \overrightarrow{\psi}\rangle = \lambda\int_\Omega \operatorname{div}\overrightarrow{u}\operatorname{div}\overrightarrow{\psi}\, dx + 2\sum_{i,j=1}^{2}\int_\Omega \mu(x, \overrightarrow{v})a(|\varepsilon^D(\overrightarrow{u})|)\varepsilon^D_{ij}(\overrightarrow{u})\varepsilon^D_{ij}(\overrightarrow{\psi})\, dx - \sum_{i=1}^{2}\int_\Omega f_i\psi_i\, dx\,,$$

for all $\overrightarrow{u}$, $\overrightarrow{\psi} \in V$.

To prove Proposition 2 it is enough to show that the following lemma holds true and to combine that fact with inequality (2.5).

Lemma 1. *Functional* Λ *is of class* $C^1(V,\mathbb{R})$ *and*

$$\langle \Lambda'(\overrightarrow{u}),\varphi\rangle = \lambda\int_\Omega \operatorname{div}\overrightarrow{u}\operatorname{div}\overrightarrow{\psi}\,dx + 2\sum_{i,j=1}^{2}\int_\Omega \mu(x,\overrightarrow{v})a(|\varepsilon^D(\overrightarrow{u})|)\varepsilon_{ij}^D(\overrightarrow{u})\varepsilon_{ij}^D(\overrightarrow{\psi})\,dx\,,$$

for all $\overrightarrow{u}$, $\overrightarrow{\psi}\in V$.

Proof. **Existence of the Gâteaux derivative.** Let $\overrightarrow{u}$, $\overrightarrow{\psi}\in V$. Fix $0<|t|<1$. Then, by the mean value theorem, there exists $\theta\in[0,1]$ such that

$$\begin{aligned}&\frac{\lambda}{2}\operatorname{div}(\overrightarrow{u}+t\overrightarrow{\psi})^2+2\mu\Phi(|\varepsilon^D(\overrightarrow{u}+t\overrightarrow{\psi})|)-\frac{\lambda}{2}(\operatorname{div}\overrightarrow{u})^2-2\mu(x,\overrightarrow{v})\Phi(|\varepsilon^D(\overrightarrow{u})|)/t=\\ &\lambda\operatorname{div}(\overrightarrow{u}+\theta t\overrightarrow{\psi})\operatorname{div}\overrightarrow{\psi}+2\mu(x,\overrightarrow{v})\varphi(|\varepsilon^D(\overrightarrow{u}+\theta t\overrightarrow{\psi})|)\frac{\sum_{i,j=1}^{2}\varepsilon_{ij}^D(\overrightarrow{u}+\theta t\overrightarrow{\psi})\varepsilon_{ij}^D(\overrightarrow{\psi})}{|\varepsilon^D(\overrightarrow{u}+\theta t\overrightarrow{\psi})|}\,.\end{aligned}$$

Since $\overrightarrow{u}$, $\overrightarrow{\psi}\in V$ it follows that $\operatorname{div}\overrightarrow{u}$, $\operatorname{div}\overrightarrow{\psi}$, $\varepsilon_{ij}^D(\overrightarrow{u})$, $\varepsilon_{ij}^D(\overrightarrow{\psi})\in L^\Phi(\Omega)$. Combining that fact with relation (2.3) we deduce that $\operatorname{div}\overrightarrow{u}$, $\operatorname{div}\overrightarrow{\psi}$, $\varphi(|\varepsilon^D(\overrightarrow{u}+\theta t\overrightarrow{\psi})|)\in L^{\Phi^\star}(\Omega)$. Moreover, it is obvious that $\frac{\varepsilon_{ij}^D(\overrightarrow{u}+\theta t\overrightarrow{\psi})}{|\varepsilon^D(\overrightarrow{u}+\theta t\overrightarrow{\psi})|}\in L^\infty(\Omega)$ and recall that $\mu(x,\overrightarrow{v})\in L^\infty(\Omega)$. Thus, Hölder's inequality in Orlicz spaces assures that

$$\lambda\operatorname{div}(\overrightarrow{u}+\theta t\overrightarrow{\psi})\operatorname{div}\overrightarrow{\psi}+2\mu(x,\overrightarrow{v})\varphi(|\varepsilon^D(\overrightarrow{u}+\theta t\overrightarrow{\psi})|)\frac{\sum_{i,j=1}^{2}\varepsilon_{ij}^D(\overrightarrow{u}+\theta t\overrightarrow{\psi})\varepsilon_{ij}^D(\overrightarrow{\psi})}{|\varepsilon^D(\overrightarrow{u}+\theta t\overrightarrow{\psi})|}\in L^1(\Omega)\,.$$

It follows from the Lebesgue Dominated Convergence Theorem

$$\begin{aligned}\langle \Lambda'(\overrightarrow{u}),\varphi\rangle &= \lambda\int_\Omega \operatorname{div}\overrightarrow{u}\operatorname{div}\overrightarrow{\psi}\,dx+2\int_\Omega\mu(x,\overrightarrow{v})\varphi(|\varepsilon^D(\overrightarrow{u})|)\frac{\sum_{i,j=1}^{2}\varepsilon_{ij}^D(\overrightarrow{u})\varepsilon_{ij}^D(\overrightarrow{\psi})}{|\varepsilon^D(\overrightarrow{u})|}\,dx\\ &= \lambda\int_\Omega \operatorname{div}\overrightarrow{u}\operatorname{div}\overrightarrow{\psi}\,dx+2\sum_{i,j=1}^{2}\int_\Omega\mu(x,\overrightarrow{v})a(|\varepsilon^D(\overrightarrow{u})|)\varepsilon_{ij}^D(\overrightarrow{u})\varepsilon_{ij}^D(\overrightarrow{\psi})\,dx\,.\end{aligned}$$

Continuity of the Gâteaux derivative. Assume $\overrightarrow{u}_n\to\overrightarrow{u}$ in V. Then it is obvious that $(u_k)_n\to u_k$ in $W_0^{1,\Phi}(\Omega)$ for each $k\in\{1,2\}$. Consequently, $\varepsilon_{ij}^D(\overrightarrow{u}_n)\to\varepsilon_{ij}^D(\overrightarrow{u})$ in $L^\Phi(\Omega)$. That fact and relation (2.3) yield

$$\varphi(|\varepsilon^D(\overrightarrow{u}_n)|)\to\varphi(|\varepsilon^D(\overrightarrow{u})|)\quad\text{in}\quad L^{\Phi^\star}(\Omega)\,.$$

On the other hand, we have

$$\begin{aligned}|\langle\Lambda'(\overrightarrow{u}_n)-\Lambda'(\overrightarrow{u}),\overrightarrow{\psi}\rangle| &\le \lambda\int_\Omega|\operatorname{div}\overrightarrow{u}_n-\operatorname{div}\overrightarrow{u}||\operatorname{div}\overrightarrow{\psi}|\,dx+\\ &\quad 2\sum_{i,j=1}^{2}\int_\Omega\mu(x,\overrightarrow{v})|\varphi(|\varepsilon^D(\overrightarrow{u}_n)|)-\varphi(|\varepsilon^D(\overrightarrow{u})|)|\,|\varepsilon^D(\overrightarrow{\psi})|\,dx\\ &\le \lambda\|\operatorname{div}\overrightarrow{u}_n-\operatorname{div}\overrightarrow{u}\|_\Phi\|\operatorname{div}\overrightarrow{\psi}\|_{\Phi^\star}+\\ &\quad 2\mu_2\|\varphi(|\varepsilon^D(\overrightarrow{u}_n)|)-\varphi(|\varepsilon^D(\overrightarrow{u})|)\|_{\Phi^\star}\|\varepsilon_{ij}^D(\overrightarrow{\psi})\|_\Phi\end{aligned}$$

and so there are two positive constants A_1 and A_2 such that

$$\|\Lambda'(\overrightarrow{u}_n) - \Lambda'(\overrightarrow{u})\| \leq A_1\|\text{div}\,\overrightarrow{u}_n - \text{div}\,\overrightarrow{u}\|_\Phi + A_2\|\varphi(|\varepsilon^D(\overrightarrow{u}_n)|) - \varphi(|\varepsilon^D(\overrightarrow{u})|)\|_{\Phi^\star} \to 0, \quad \text{as } n \to \infty.$$

The proof of Lemma 1 is complete. □

By Proposition 2 it is clear that the critical points of J give the weak solutions of equation (5.1). Thus, in what follows we will focus our attention on finding a critical point for functional J. We start by establishing some auxiliary results.

Lemma 2. *Functional J is coercive.*

Proof. First, we note that Hölder's inequality in Orlicz spaces and assumption (1.2) yield

$$\begin{aligned} J(\overrightarrow{u}) &= \Lambda(\overrightarrow{u}) - \sum_{i=1}^{2}\int_\Omega f_i u_i \, dx \\ &\geq 2\mu_1 \int_\Omega \Phi(|\varepsilon^D(\overrightarrow{u})|)\, dx - \sum_{i=1}^{2}\|f_i\|_{\Phi^\star}\|u_i\|_\Phi \,, \end{aligned}$$

for all $\overrightarrow{u} \in V$. Theorem 2 and the equivalence of different norms in $W_0^{1,\Phi}(\Omega)$ pointed out above assure the existence of two positive constants D_1 and D_2 such that

$$J(\overrightarrow{u}) \geq D_1 \sum_{i=1}^{2}\int_\Omega \Phi(|\nabla u_i|)\, dx - D_2\|\overrightarrow{u}\|\,,$$

for all $\overrightarrow{u} \in V$. For each $\overrightarrow{u} = (u_1, u_2) \in V$ and each $i \in \{1, 2\}$ define

$$a_i(\overrightarrow{u}) := \begin{cases} \varphi_0 & \text{if} \quad \||\nabla u_i|\|_\Phi > 1 \\ \varphi^0 & \text{if} \quad \||\nabla u_i|\|_\Phi \leq 1\,. \end{cases}$$

Thus, the last inequality and relation (2.4) imply

$$\begin{aligned} J(\overrightarrow{u}) &\geq D_1 \sum_{i=1}^{2}\||\nabla u_i|\|_\Phi^{a_i(\overrightarrow{u})} - D_2\|\overrightarrow{u}\| \\ &\geq D_1 \sum_{i=1}^{2}(\||\nabla u_i|\|_\Phi^{\varphi_0} - 1) - D_2\|\overrightarrow{u}\|\,, \end{aligned}$$

or

$$J(\overrightarrow{u}) \geq D_3\|\overrightarrow{u}\|^{\varphi_0} - 2D_1 - D_2\|\overrightarrow{u}\|\,, \tag{5.3}$$

for all $\overrightarrow{u} \in V$, where $D_3 > 0$ is a constant. Since $\varphi_0 \geq 2$ the last inequality assures that the conclusion of Lemma 2 is valid. □

In order to go further let us define $I_0 : V \to \mathbb{R}$ by

$$I_0(\overrightarrow{u}) = \int_\Omega \mu(x, \overrightarrow{v})\Phi(|\varepsilon^D(\overrightarrow{u})|)\, dx\,.$$

By Proposition 2 it is clear that $I_0 \in C^1(V, \mathbb{R})$ and

$$\langle I_0'(\overrightarrow{u}), \overrightarrow{\psi}\rangle = \sum_{i,j=1}^{2} \int_\Omega \mu(x, \overrightarrow{v})a(|\varepsilon^D(\overrightarrow{u})|)\varepsilon^D_{ij}(\overrightarrow{u})\varepsilon^D_{ij}(\overrightarrow{\psi})\, dx\,,$$

for all $\overrightarrow{u},\ \overrightarrow{\psi} \in V$.

On the other hand, relation (2.6) and the fact that the function $[0, \infty) \ni t \to \Phi(\sqrt{t})$ is convex enable us to apply [14, Lemma 2.1] in order to obtain

$$\frac{1}{2}I_0(\overrightarrow{u}) + \frac{1}{2}I_0(\overrightarrow{w}) \geq I_0\left(\frac{\overrightarrow{u}+\overrightarrow{w}}{2}\right) + I_0\left(\frac{\overrightarrow{u}-\overrightarrow{w}}{2}\right), \quad \forall\, \overrightarrow{u},\ \overrightarrow{w} \in V\,,$$

while simple computations show that

$$\frac{1}{2}\left(\int_\Omega (\mathrm{div}(\overrightarrow{u}))^2\, dx + \int_\Omega (\mathrm{div}(\overrightarrow{w}))^2\, dx\right) = \int_\Omega \left(\mathrm{div}\left(\frac{\overrightarrow{u}+\overrightarrow{w}}{2}\right)\right)^2 dx + \int_\Omega \left(\mathrm{div}\left(\frac{\overrightarrow{u}-\overrightarrow{w}}{2}\right)\right)^2 dx\,,$$

for all $\overrightarrow{u},\ \overrightarrow{w} \in V$. Consequently, functional J is convex on V and therefore it is weakly lower semi-continuous. Particularly, the above inequalities show also that functional Λ is also convex and

$$\frac{1}{2}\Lambda(\overrightarrow{u}) + \frac{1}{2}\Lambda(\overrightarrow{w}) \geq \Lambda\left(\frac{\overrightarrow{u}+\overrightarrow{w}}{2}\right) + \Lambda\left(\frac{\overrightarrow{u}-\overrightarrow{w}}{2}\right), \quad \forall\, \overrightarrow{u},\ \overrightarrow{w} \in V\,, \tag{5.4}$$

Proof of Proposition 1 (concluded). Since J is coercive and weakly lower semi-continuous we conclude via the Direct Method of the Calculus of Variations (see, e.g., [20, Theorem 1.2]), that there exists a global minimum point of J, $\overrightarrow{u} \in V$ and consequently a weak solution of problem (5.1). The proof of Proposition 1 is thus complete. □

In the sequel we will use the following notation

$$L := (L^\Phi(\Omega))^2\,.$$

Function space L will be endowed with the following norm

$$\|\overrightarrow{u}\|_L := \|u_1\|_\Phi + \|u_2\|_\Phi\,,$$

for all $\overrightarrow{u} = (u_1, u_2) \in L$. From the properties of Orlicz-Sobolev spaces it is clear that V is compactly embedded in L.

Next, for each $\overrightarrow{v} \in L$ let $\overrightarrow{u} = T(\overrightarrow{v}) \in V$ be the weak solution of problem (5.1) given by Proposition 1. Thus, we can actually introduce an application $T : L \to V$, associating with each $\overrightarrow{v} \in L$, the solution of problem (5.1), $T(\overrightarrow{v}) \in V$.

Lemma 3. *There exists $C > 0$ a universal constant such that*

$$\int_\Omega \Phi(|\epsilon^D(T(\overrightarrow{v}))|)\, dx \leq C, \quad \forall\, \overrightarrow{v} \in L\,. \tag{5.5}$$

Proof. Taking $\varphi = T(\overrightarrow{v})$ in (5.2) we find

$$\begin{aligned}\lambda \int_\Omega (\mathrm{div} T(\overrightarrow{v}))^2\, dx \;&+\; 2\sum_{i,j=1}^{2} \int_\Omega \mu(x,\overrightarrow{v}) a(|\varepsilon^D(T(\overrightarrow{v}))|)(\varepsilon^D_{ij}(T(\overrightarrow{v})))^2\, dx \\ &-\; \sum_{i=1}^{2} \int_\Omega f_i (T(\overrightarrow{v}))_i\, dx = 0, \quad \forall\, \overrightarrow{v} \in L\,.\end{aligned}$$

Taking into account relations (1.2) and (2.2), the above equality yields

$$\begin{aligned} 2\mu_1\varphi_0 \int_\Omega \Phi(|\varepsilon^D(T(\overrightarrow{v}))|)\, dx \;&\leq\; 2\sum_{i,j=1}^{2} \int_\Omega \mu(x,\overrightarrow{v}) a(|\varepsilon^D(T(\overrightarrow{v}))|)(\varepsilon^D_{ij}(T(\overrightarrow{v})))^2\, dx \\ &\leq\; \sum_{i=1}^{2} \int_\Omega f_i (T(\overrightarrow{v}))_i\, dx, \quad \forall\, \overrightarrow{v} \in L\,. \end{aligned} \tag{5.6}$$

Let now $\epsilon > 0$ be such that $\epsilon < \min\{1, \mu_1, 2\mu_1\varphi_0/C_1C_2\}$ (here C_1 and C_2 are the constants given in Theorems 2 and 3). Then, by Young's inequality, relation (2.4) and Theorems 2 and 3 we deduce

$$\begin{aligned} \sum_{i=1}^{2} \int_\Omega f_i (T(\overrightarrow{v}))_i\, dx \;&\leq\; \sum_{i=1}^{2} \int_\Omega \Phi(\epsilon^{1/\varphi_0}(T(\overrightarrow{v}))_i)\, dx + \sum_{i=1}^{2} \int_\Omega \Phi^\star\left(\frac{1}{\epsilon^{1/\varphi_0}} f_i\right)\, dx \\ &\leq\; \epsilon \sum_{i=1}^{2} \int_\Omega \Phi((T(\overrightarrow{v}))_i)\, dx + \sum_{i=1}^{2} \int_\Omega \Phi^\star\left(\frac{1}{\epsilon^{1/\varphi_0}} f_i\right)\, dx \\ &\leq\; \epsilon C_2 \sum_{i=1}^{2} \int_\Omega \Phi(|\nabla (T(\overrightarrow{v}))_i|)\, dx + C_\epsilon \\ &\leq\; \epsilon C_1 C_2 \int_\Omega \Phi(|\epsilon^D(T(\overrightarrow{v})|)\, dx + C_\epsilon, \quad \forall\, \overrightarrow{v} \in L\,, \end{aligned}$$

where $C_\epsilon := \sum_{i=1}^{2} \int_\Omega \Phi^\star\left(\frac{1}{\epsilon^{1/\varphi_0}} f_i\right)\, dx$ is a positive constant depending only on ϵ, f_1 and f_2.

The above pieces of information imply

$$(2\mu_1\varphi_0 - \epsilon C_1 C_2) \int_\Omega \Phi(|\varepsilon^D(T(\overrightarrow{v}))|)\, dx \leq C_\epsilon, \quad \forall\, \overrightarrow{v} \in L\,.$$

Consequently, taking

$$C := \frac{C_\epsilon}{2\mu_1\varphi_0 - \epsilon C_1 C_2}\,,$$

we infer that relation (5.5) holds true. The proof of Lemma 3 is thus complete. □

Remark. By Lemma 3 and Theorems 2 and 3 it clearly follows that there exists a universal constant $C_0 > 0$ such that

$$\int_\Omega \Phi(|T(\overrightarrow{v})|)\, dx \leq C_0, \quad \forall\, \overrightarrow{v} \in L\,.$$

Define now, $I : V \to \mathbb{R}$ by

$$I(\overrightarrow{u}) := \int_\Omega \left[\frac{\lambda}{2}(\operatorname{div}\overrightarrow{u})^2 + 2\Phi(|\varepsilon^D(\overrightarrow{u})|)\right]\, dx\,.$$

Functional I is of class $C^1(V, \mathbb{R})$ and

$$\langle I'(\overrightarrow{u}), \varphi\rangle = \lambda \int_\Omega \operatorname{div}\overrightarrow{u}\operatorname{div}\overrightarrow{\psi}\, dx + 2\sum_{i,j=1}^{2}\int_\Omega a(|\varepsilon^D(\overrightarrow{u})|)\varepsilon^D_{ij}(\overrightarrow{u})\varepsilon^D_{ij}(\overrightarrow{\psi})\, dx\,,$$

for all $\overrightarrow{u}$, $\overrightarrow{\psi} \in V$.

Lemma 4. *Assume that the sequence* $\{\overrightarrow{u}_n\}$ *converges weakly to* $\overrightarrow{u}$ *in* V *and*

$$\limsup_{n\to\infty}\langle I'(\overrightarrow{u}_n), \overrightarrow{u}_n - \overrightarrow{u}\rangle \leq 0\,.$$

Then $\{\overrightarrow{u}_n\}$ *converges strongly to* $\overrightarrow{u}$ *in* V.

Proof. Since $\overrightarrow{u}_n$ converges weakly to $\overrightarrow{u}$ in V it follows that $\{\|\overrightarrow{u}_n\|\}$ is a bounded sequence. That fact and relations (2.7) and (2.8) imply that sequence $\{I(\overrightarrow{u}_n)\}$ is bounded. Then, up to a subsequence, we deduce that $I(\overrightarrow{u}_n) \to c$. Furthermore, the convexity of I (which can be obtained similarly as the convexity of functional Λ) implies its weak lower semi-continuity which yields

$$I(\overrightarrow{u}) \leq \liminf_{n\to\infty} I(\overrightarrow{u}_n) = c\,.$$

On the other hand, since I is convex we have

$$I(\overrightarrow{u}) \geq I(\overrightarrow{u}_n) + \langle I'(\overrightarrow{u}_n), \overrightarrow{u} - \overrightarrow{u}_n\rangle\,. \tag{5.7}$$

Next, by hypothesis $\limsup_{n\to\infty}\langle I'(\overrightarrow{u}_n), \overrightarrow{u}_n - \overrightarrow{u}\rangle \leq 0$, we conclude that $I(\overrightarrow{u}) = c$.

Taking into account that $(\overrightarrow{u}_n + \overrightarrow{u})/2$ converges weakly to $\overrightarrow{u}$ in V and using again the weak lower semi-continuity of I we find

$$c = I(\overrightarrow{u}) \leq \liminf_{n\to\infty} I\left(\frac{\overrightarrow{u}_n + \overrightarrow{u}}{2}\right)\,. \tag{5.8}$$

We assume by contradiction that $\overrightarrow{u}_n$ does not converge to $\overrightarrow{u}$ in V. Then by (2.7) it follows that there exist $\epsilon > 0$ and a subsequence $(\overrightarrow{u}_{n_m})$ of $(\overrightarrow{u}_n)$ such that

$$\sum_{i=1}^{2}\int_\Omega \Phi\left(\frac{|\nabla(u_i)_{n_m} - \nabla u_i|}{2}\right) \geq \frac{\epsilon C_1}{2}, \quad \forall\, m\,, \tag{5.9}$$

where C_1 is the constant given in Theorem 2, or, by Theorem 2

$$\int_\Omega \Phi\left(\left|\epsilon^D\left(\frac{\overrightarrow{u}_{n_m} - \overrightarrow{u}}{2}\right)\right|\right)\, dx \geq \frac{\epsilon}{2}\,, \tag{5.10}$$

and consequently,

$$I\left(\frac{\overrightarrow{u}_{n_m}-\overrightarrow{u}}{2}\right)=\frac{\lambda}{2}\int_{\Omega}\left(\text{div}\left(\frac{\overrightarrow{u}_{n_m}-\overrightarrow{u}}{2}\right)\right)^2\,dx+2\int_{\Omega}\Phi\left(\left|\epsilon^D\left(\frac{\overrightarrow{u}_{n_m}-\overrightarrow{u}}{2}\right)\right|\right)\,dx\geq\epsilon\,. \tag{5.11}$$

Similar arguments as in the proof of relation (5.4) can be used in order to get

$$\frac{1}{2}I(\overrightarrow{u})+\frac{1}{2}I(\overrightarrow{u}_{n_m})-I\left(\frac{\overrightarrow{u}+\overrightarrow{u}_{n_m}}{2}\right)\geq I\left(\frac{\overrightarrow{u}-\overrightarrow{u}_{n_m}}{2}\right),\quad\forall\, m\,. \tag{5.12}$$

Combining (5.11) and (5.12) we deduce

$$\frac{1}{2}I(\overrightarrow{u})+\frac{1}{2}I(\overrightarrow{u}_{n_m})-I\left(\frac{\overrightarrow{u}+\overrightarrow{u}_{n_m}}{2}\right)\geq\epsilon,\quad\forall\, m\,.$$

Letting $m\to\infty$ in the above inequality we obtain

$$c-\epsilon\geq\limsup_{m\to\infty}I\left(\frac{\overrightarrow{u}+\overrightarrow{u}_{n_m}}{2}\right)\,dx\,,$$

and that is a contradiction with (5.8). It follows that $\overrightarrow{u}_n$ converges strongly to $\overrightarrow{u}$ in V and Lemma 4 is proved. □

Lemma 5. *The map $T:L\to V$ is continuous.*

Proof. Let $(\overrightarrow{v}_n)$ be a sequence in L converging strongly to $\overrightarrow{v}$. Set

$$\overrightarrow{u}_n:=T(\overrightarrow{v}_n),\quad\forall\, n\,.$$

By Lemma 3 we have

$$\int_{\Omega}\Phi(|\epsilon^D(\overrightarrow{u}_n)|)\,dx=\int_{\Omega}\Phi(|\epsilon^D(T(\overrightarrow{v}_n))|)\,dx\leq C,\quad\forall\, n\,,$$

i.e. $(\overrightarrow{u}_n)$ is bounded in V. It follows that a subsequence again denoted $(\overrightarrow{u}_n)$ convergence weakly to $\overrightarrow{u}$ in V. Furthermore, since $\varphi_0\geq 2$ we deduce that V is continuously embedded in $W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)$. Combining that result with the fact that $W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)$ is compactly embedded in $L^{\varphi^0}(\Omega)\times L^{\varphi^0}(\Omega)$, while $L^{\varphi^0}(\Omega)\times L^{\varphi^0}(\Omega)$ is continuously embedded in L, we infer that V is compactly embedded in L. Consequently, $\overrightarrow{u}_n$ converges strongly to $\overrightarrow{u}$ in L.

On the other hand, for each n we have

$$\lambda\int_{\Omega}\text{div}\overrightarrow{u}_n\text{div}\overrightarrow{\psi}\,dx+2\sum_{i,j=1}^{2}\int_{\Omega}\mu(x,\overrightarrow{v}_n)a(|\varepsilon^D(\overrightarrow{u}_n)|)\varepsilon_{ij}^D(\overrightarrow{u}_n)\varepsilon_{ij}^D(\overrightarrow{\psi})\,dx-\sum_{i=1}^{2}\int_{\Omega}f_i\psi_i\,dx=0\,, \tag{5.13}$$

for all $\overrightarrow{\psi}\in V$. Taking $\overrightarrow{\psi}=\overrightarrow{u}_n-\overrightarrow{u}$ in the above equality and taking into account relation (1.2) it follows that

$$\langle I^{'}(\overrightarrow{u}_n),\overrightarrow{u}_n-\overrightarrow{u}\rangle=o(1)\,,$$

where I was defined before Lemma 4. By Lemma 4 we find that $(\overrightarrow{u}_n)$ converges (strongly) to $\overrightarrow{u}$ in V, so $T : L \to V$ is continuous. The proof of Lemma 5 is complete. $\square$

Remark. Since V is compactly embedded in L (i.e. the inclusion operator $i : V \to L$ is compact), it follows by Lemma 5 that the operator $S : L \to L$, $S = i \circ T$ is compact.

Proof of Theorem 1. Let C_1 be the constant given in the Remark pointed out after Lemma 3, i.e.,

$$\int_\Omega \Phi(|S(\overrightarrow{v})|)\, dx \leq C_0, \quad \forall\, \overrightarrow{v} \in L\,.$$

Consider the ball

$$B_{C_0}(\overrightarrow{0}) := \{\overrightarrow{v} \in L : \int_\Omega \Phi(|\overrightarrow{v}|)\, dx \leq C_0\}\,.$$

Clearly, $B_{C_0}(\overrightarrow{0})$ is a convex closed subset of L and $S(B_{C_0}(\overrightarrow{0})) \subset B_{C_0}(\overrightarrow{0})$. Moreover, by the Remark pointed out after Lemma 5, $S(B_{C_0}(\overrightarrow{0}))$ is relatively compact in $B_{C_0}(\overrightarrow{0})$.

Finally, by Lemma 5 and the Remark pointed out after it, $S : B_{C_0}(\overrightarrow{0}) \to B_{C_0}(\overrightarrow{0})$ is a continuous map. By Schauder's Fixed Point Theorem, S has a fixed point. This gives us a weak solution to problem (1.3) and thus the proof of Theorem 1 is finally complete. $\square$

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