Equations involving a variable exponent Grushin-type operator

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Abstract. In this paper we define a Grushin-type operator with a variable exponent growth and establish existence results for an equation involving such an operator. A suitable function space setting is introduced. Regarding the tools used in proving the existence of solutions for the equation analyzed here they rely on the critical point theory combined with adequate variational techniques.

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1 Introduction

The study of PDE's involving variable exponents becomes more and more attractive in the last decades since differential operators involving variable exponent growth conditions can serve in describing non-homogeneous phenomena which can occur in different branches of science. In this context we remember that the first model where such kind of operators was considered comes from fluid mechanics, more exactly from the study of electrorheological fluids [19], [18], [20]. After this pioneering model many other applications of differential operators with variable exponents appeared in a large range of fields, such as image restoration [5], mathematical biology [11], the study of dielectric breakdown, electrical resistivity, and polycrystal plasticity [1], [3] or in the study of some models for growth of heterogeneous sandpiles [2]. On the other hand, important results in the context of potential theory have been obtained by the *Research group on variable exponent Lebesgue and Sobolev spaces* from Finland (see `http://www.helsinki.fi/ pharjule/varsob/index.shtml`). In particular, we note that a survey paper regarding the most important results in the field of partial differential equations involving variable exponent growth conditions can be found in the section *Publications* on the web-site quoted above, more exactly [13].

In the applications pointed out above most models involve the $p(x)$-Laplace operator, that is

$$\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)\,,$$

where $p(x)$ is a continuous function satisfying $p(x) > 1$ for each x. Unfortunately, this operator can not cover the case of models describing phenomena in which a certain degeneracy is present. To our best knowledge there are not too many papers in the literature devoted to the study of degenerate equations with variable exponents (actually, the only example that we know is [16]). In the present paper we will be concerned with the analysis of a class of elliptic problems involving variable exponent growth conditions and the presence of a degeneracy in the differential operator considered in the equation (of a different type than the one investigated in [16]). To be more specific, in this article our aim is to study an equation in which is present a variable exponent Grushin type operator (which will be introduced in the next section). The basic idea in investigating such type of operators goes back to the definition of the *classical* Grushin operator, that is a differential operator defined in a domain $\Omega \subset \mathbb{R}^{n+m}$ by

$$\Delta_G u := \Delta_x u + |x|^\gamma \Delta_y u \,,$$

where Δ_x and Δ_y stand for the standard Laplace operators on $\mathbb{R}^n$ respectively $\mathbb{R}^m$, $\gamma > 0$ is a constant and $u = u(x, y),\ (x, y) \in \Omega$. The Grushin operator is not an *elliptic* operator on domains (in $\mathbb{R}^{n+m}$) intersecting the plane $x = 0$, but is a standard example of a *hypoelliptic* operator. The lack of ellipticity is due to the presence of the degeneracy $|x|^\gamma$. It was introduced by the Russian mathematician V. V. Grushin in [12]. Problems involving the Grushin operator have been extensively studied over the years. We just remember the papers [21], [6], [7], [17]. In an appropriate context, results on Grushin's operator were obtained in the framework of Heisenberg groups. Specifically, in [17] it is noticed that the Heisenberg sub-Laplacian is, actually, a Grushin operator (with $\gamma = 2$ and $m = 1$). Operator Δ_G represents the basic source of inspiration in defining a Grushin type operator with a variable exponent in the next section of this paper.

The paper is organized as follows: in Section 2 we define a Grushin-type operator involving a variable exponent; in Section 3 we collect some preliminary results and we introduce some notation which will be used in the sequel; in Section 4 we establish some auxiliary results that will be used later in the analysis of a PDE involving a Grushin-type operator with a variable exponent; in Section 5 we establish existence results for an elliptic PDE involving a Grushin-type operator with a variable exponent; Section 6 is an appendix where the equivalence of some norms is established.

2 A Grushin-type operator involving a variable exponent

Let $\Omega \subset \mathbb{R}^N$ be a bounded and smooth domain, $N = n+m$ with $n,\ m \geq 2$ and assume that Ω intersects the plane $x = 0$, i. e. the set $\{(0_n, y) :\ y \in \mathbb{R}^m\}$, where 0_n is the null vector of $\mathbb{R}^n$. We denote by $\partial\Omega$ the boundary of Ω.

If $(x, y) \in \Omega$ then we denote $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_m)$.

Consider $\gamma > 0$ is a given real number and define the matrix

$$A(x) = \begin{bmatrix} I_n & O_{n,m} \\ O_{m,n} & |x|^\gamma I_m \end{bmatrix} \in M_{N\times N}(\mathbb{R}),$$

where $O_{n,m}$ respectively $O_{m,n}$ are the null matrices in $M_{n\times m}(\mathbb{R})$ respectively $M_{m\times n}(\mathbb{R})$ while I_n respectively I_m stand for the unit matrices in $M_{n\times n}(\mathbb{R})$, respectively $M_{m\times m}(\mathbb{R})$.

Let $G(x,y) : \overline{\Omega} \to (1,\infty)$ be a continuous function. We define the degenerate operator

$$\begin{aligned}
\Delta_{G(x,y)}\cdot &= \operatorname{div}(\nabla_{G(x,y)}\cdot) \\
&= \operatorname{div}_x(|\nabla_x \cdot|^{G(x,y)-2}\nabla_x\cdot) + \operatorname{div}_y(|x|^\gamma |\nabla_y \cdot|^{G(x,y)-2}\nabla_y\cdot) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla_x \cdot|^{G(x,y)-2}\frac{\partial\cdot}{\partial x_i}\right) + |x|^\gamma \sum_{j=1}^{m} \frac{\partial}{\partial y_j}\left(|\nabla_y \cdot|^{G(x,y)-2}\frac{\partial\cdot}{\partial y_j}\right),
\end{aligned}$$

where

$$\nabla_{G(x,y)}\cdot = A(x)\begin{bmatrix} |\nabla_x \cdot|^{G(x,y)-2}\nabla_x\cdot \\ |\nabla_y \cdot|^{G(x,y)-2}\nabla_y\cdot \end{bmatrix}.$$

This is a Grushin-type operator since in the particular case when $G(x,y) = 2$ for each $(x,y) \in \Omega$ we recover the classical Grushin operator. Obviously, this operator is also anisotropic but of a different type than the operators analyzed in [15].

In this paper we will study operators of type $\Delta_{G(x,y)}$ in the particular case when the function $G(x,y)$ is of class C^1 on Ω and has the particular form

$$G(x,y) = p(x) + q(y).$$

In fact, $p \in C^1(\Omega_1)$ and $q \in C^1(\Omega_2)$, where Ω_1 and Ω_2 are the projections of Ω on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Assume that $1 < \inf_{\Omega_1} p(x)$, $1 < \inf_{\Omega_2} q(y)$, $\sup_{\Omega_1} p(x) < N$, $\sup_{\Omega_2} q(y) < N$.

Assume also that there exist two vector functions

$$\overrightarrow{f}(x) = (f_1(x), ..., f_n(x)) \quad \text{and} \quad \overrightarrow{g}(y) = (g_1(y), ..., g_m(y)),$$

of class C^1 on Ω_1 and Ω_2, respectively, such that

$$\operatorname{div}_x \overrightarrow{f}(x) \geq f_0 \quad \text{in} \quad \Omega_1 \quad \text{and} \quad \operatorname{div}_y \overrightarrow{g}(y) \geq g_0, \quad \text{in} \quad \Omega_2, \tag{1}$$

for some constants $f_0 > 0$ and $g_0 > 0$, and

$$\overrightarrow{f}(x) \cdot \nabla_x p(x) = \overrightarrow{g}(y) \cdot \nabla_y q(y) = 0, \quad \forall\, x \in \Omega_1, \quad \forall\, y \in \Omega_2. \tag{2}$$

Remark. If $\overrightarrow{f}$ is the gradient of a scalar function $h = h(x)$, then the first part of (2) becomes $\nabla h \cdot \nabla p = 0$ in Ω_1. From a geometrical point of view, this means that the equations $p(x) = C_1$, $h(x) = C_2$ represent *orthogonal surfaces* in $\mathbb{R}^n$.

The goal of this paper is to prove the existence of solutions for some equations of type

$$\begin{cases} -\Delta_{G(x,y)} u(x,y) = h((x,y), u(x,y)), & \text{for} \quad (x,y) \in \Omega \\ u(x,y) = 0, & \text{for} \quad (x,y) \in \partial\Omega\,. \end{cases} \tag{3}$$

3 Preliminaries

In this section we point out certain results regarding the Lebesgue and Sobolev variable exponent spaces. We refer to [14] and [8] for more details. We will also introduce some notations which will be used throughout, and we present some preliminary results. Finally, we will introduce the suitable functional framework where we can study equations of type (3).

Given $\Omega \subseteq \mathbb{R}^N$ (with $N = n + m$) a bounded and smooth domain an element of Ω will be denoted by $(x,y) \in \Omega$ where

$$x = (x_1, ..., x_n) \quad \text{and} \quad y = (y_1, ..., y_m)\,.$$

Given $s : \overline{\Omega} \to (1, \infty)$ a continuous function the variable exponent Lebesgue space $L^{s(x,y)}(\Omega)$ is defined as follows:

$$L^{s(x,y)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u(x,y)|^{s(x,y)}\, dxdy < +\infty \right\},$$

and it is endowed with the *Luxemburg* norm

$$|u|_{s(x,y)} := \inf \left\{ \tau > 0 \;:\; \int_\Omega \left| \frac{u(x,y)}{\tau} \right|^{s(x,y)} dxdy \le 1 \right\}.$$

The variable exponent Sobolev space $W^{1,s(x,y)}(\Omega)$ is given by

$$W^{1,s(x,y)}(\Omega) := \left\{ u \in L^{s(x,y)}(\Omega) \;:\; |\nabla u| \in L^{s(x,y)}(\Omega) \right\},$$

and

$$|u|_{1,s(x,y)} := |u|_{s(x,y)} + |\, |\nabla u|\, |_{s(x,y)}$$

is a norm on this space, where $\nabla u = (\nabla_x u, \nabla_y u)$. We denote by $W_0^{1,s(x,y)}(\Omega)$ the closure of $C_0^1(\Omega)$ in $W^{1,s(x,y)}(\Omega)$. On this space we can consider the equivalent norm

$$\|u\|_1 := |\, |(\nabla_x u, \nabla_y u)|\, |_{s(x,y)}\,.$$

For each function s defined as above, we define

$$s^- := \min_{(x,y)\in\overline{\Omega}} s(x,y), \qquad \text{and} \qquad s^+ := \max_{(x,y)\in\overline{\Omega}} s(x,y)\,.$$

As we have already pointed out, in this paper we will only consider the case of functions s which satisfy

$$1 < s^- \leq s^+ < \infty\,.$$

Under this assumption it is well-known the spaces $\left(L^{s(x,y)}(\Omega),\ |\cdot|_{s(x,y)}\right)$, $\left(W^{1,\,s(x,y)}(\Omega),\ |\cdot|_{1,s(x,y)}\right)$ and $\left(W_0^{1,\,s(x,y)}(\Omega),\ \|\cdot\|_1\right)$ are separable and reflexive Banach spaces. Moreover, the Hölder type inequality holds true

$$\int_\Omega |uv|\,dx \leq \left(\frac{1}{s^-} + \frac{1}{(s^{'})^-}\right)|u|_{s(x,y)}|v|_{s^{'}(x,y)}, \quad \forall\ u \in L^{s(x,y)}(\Omega),\ \forall\ v \in L^{s^{'}(x,y)}(\Omega),$$

where $s^{'}(x,y) := \frac{s(x,y)}{s(x,y)-1}$. Furthermore, if $u \in L^{s(x,y)}(\Omega)$ then the following implications hold true

$$|u|_{s(x,y)} > 1 \quad \Rightarrow \quad |u|_{s(x,y)}^{s^-} \leq \int_\Omega |u|^{s(x,y)}\ dxdy \leq |u|_{s(x,y)}^{s^+}\,; \tag{4}$$

$$|u|_{s(x,y)} < 1 \quad \Rightarrow \quad |u|_{s(x,y)}^{s^+} \leq \int_\Omega |u|^{s(x,y)}\ dxdy \leq |u|_{s(x,y)}^{s^-}\,; \tag{5}$$

$$|u|_{s(x,y)} = 1 \Leftrightarrow \int_\Omega |u|^{s(x,y)}\ dxdy = 1\,. \tag{6}$$

Obviously, if Ω intersects the plane $x = 0$ the space $\left(W_0^{1,\,s(x,y)}(\Omega),\ \|\cdot\|_1\right)$ is no longer adequate for seeking solutions of problem (3). In this new context the natural space where we can investigate the existence of solutions of equation (3) is defined as the closure of $C_0^1(\Omega)$ under the norm

$$\|u\| := |\ |\nabla_x u| + |x|^{\frac{\gamma}{G(x,y)}}\ |\nabla_y u|\ |_{G(x,y)}\,,$$

where G is given in the above section. Let us denote this Sobolev type space by $W_{0,\gamma}^{1,G(x,y)}(\Omega)$. Standard arguments can be used in order to show that $\left(W_{0,\gamma}^{1,G(x,y)}(\Omega), \|\cdot\|\right)$ is a reflexive Banach space. We point out that the above norm is equivalent with the norm

$$\|u\|_1 = |\ |(\nabla_x u, \nabla_y u)|\ |_{G(x,y)}\,,$$

provided that there exists a positive constant $m > 0$ such that for each $(x,y) \in \Omega$ we have $|x| \geq m$ (see the Appendix).

4 Auxiliary results

The following theorem is essential in our analysis.

Theorem 1. *Assume (1) and (2) are fulfilled. Then there exists a constant $C > 0$ such that*

$$\int_\Omega (1 + |x|^\gamma)|u|^{G(x,y)}\ dxdy \leq C\int_\Omega [|\nabla_x u|^{G(x,y)} + |x|^\gamma|\nabla_y u|^{G(x,y)}]\ dxdy\,,$$

for all $u \in C_c^1(\Omega)$.

Proof. Let $\overrightarrow{0}_n$ and $\overrightarrow{0}_m$ denote the null vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then the flux-divergence theorem implies that for each $u \in C_c^1(\Omega)$ we have

$$\begin{aligned}\int_\Omega \operatorname{div}[|u|^{G(x,y)}(\overrightarrow{f}(x), \overrightarrow{0}_m)]\, dxdy \quad &+ \quad \int_\Omega \operatorname{div}[|x|^\gamma |u|^{G(x,y)}(\overrightarrow{0}_n, \overrightarrow{g}(y))]\, dxdy = \\ &= \int_{\partial\Omega} |u|^{G(x,y)}[(\overrightarrow{f}(x), \overrightarrow{0}_m) + |x|^\gamma(\overrightarrow{0}_n, \overrightarrow{g}(y))] \cdot \overrightarrow{n}(x,y)\, d\sigma(x,y) \\ &= 0\,,\end{aligned}$$

where $\overrightarrow{n}(x,y)$ stands for the unit outward normal at $\partial\Omega$.

Simple computations show that

$$\begin{aligned} I_1 \quad &:= \int_\Omega \operatorname{div}[|u|^{G(x,y)}(\overrightarrow{f}(x), \overrightarrow{0}_m)]\, dxdy \\ &= \int_\Omega |u|^{G(x,y)} \operatorname{div}_x \overrightarrow{f}(x)\, dxdy + \int_\Omega \sum_{i=1}^n |u|^{G(x,y)} f_i(x) \left[\log(|u|)\frac{\partial p}{\partial x_i} + G\frac{1}{|u|}\frac{\partial u}{\partial x_i}\right]\, dxdy\,. \end{aligned}$$

In view of relation (2) we find

$$I_1 = \int_\Omega |u|^{G(x,y)} \operatorname{div}_x \overrightarrow{f}(x)\, dxdy + \int_\Omega G(x,y)|u|^{G(x,y)-2} u \overrightarrow{f}(x) \cdot \nabla_x u\, dxdy\,.$$

Similarly, we have

$$\begin{aligned} I_2 \quad &:= \int_\Omega \operatorname{div}[|x|^\gamma |u|^{G(x,y)}(\overrightarrow{0}_n, \overrightarrow{g}(y))]\, dxdy \\ &= \int_\Omega |x|^\gamma |u|^{G(x,y)} \operatorname{div}_y \overrightarrow{g}(y)\, dxdy + \int_\Omega |x|^\gamma G(x,y)|u|^{G(x,y)-2} u \overrightarrow{g}(y) \cdot \nabla_y u\, dxdy\,. \end{aligned}$$

Combining all the above pieces of information and taking into account relation (1) we get

$$\begin{aligned} f_0 \int_\Omega |u|^{G(x,y)}\, dxdy \quad &+ \quad g_0 \int_\Omega |x|^\gamma |u|^{G(x,y)}\, dxdy \\ &\le G^+ \int_\Omega |u|^{G(x,y)-1} |\overrightarrow{f}(x)||\nabla_x u|\, dxdy + G^+ \int_\Omega |x|^\gamma |u|^{G(x,y)-1} |\overrightarrow{g}(y)||\nabla_y u|\, dxdy\,. \end{aligned}$$

Next, we point out that a Young-type inequality implies that for all $\epsilon > 0$, $(x,y) \in \Omega$, A, $B > 0$, we have

$$AB \le \epsilon A^{\frac{G(x,y)}{G(x,y)-1}} + \frac{1}{\epsilon^{G(x,y)-1}} B^{G(x,y)}\,.$$

We fix $\epsilon > 0$ such that

$$G^+ \epsilon < \frac{f_0}{2}\,.$$

We find

$$\begin{aligned} f_0 \int_\Omega |u|^{G(x,y)}\, dxdy \quad &+ \quad g_0 \int_\Omega |x|^\gamma |u|^{G(x,y)}\, dxdy \\ &\le G^+ \left[\epsilon \int_\Omega |u|^{G(x,y)}\, dxdy + \int_\Omega \left(\frac{1}{\epsilon}\right)^{G(x,y)-1} |\overrightarrow{f}(x)|^{G(x,y)} |\nabla_x u|^{G(x,y)}\, dxdy\right] + \\ &\quad G^+ \left[\epsilon \int_\Omega |u|^{G(x,y)}\, dxdy + \int_\Omega \left(\frac{1}{\epsilon}\right)^{G(x,y)-1} |x|^{\gamma G(x,y)} |\overrightarrow{g}(y)|^{G(x,y)} |\nabla_y u|^{G(x,y)}\, dxdy\right], \end{aligned}$$

and thus, there exists a $C_\epsilon > 0$ such that

$$\begin{aligned}(f_0 - 2G^+\epsilon)\int_\Omega |u|^{G(x,y)}\,dxdy &+ g_0\int_\Omega |x|^\gamma |u|^{G(x,y)}\,dxdy \\ &\leq C_\epsilon \int_\Omega (|\nabla_x u|^{G(x,y)} + |x|^{\gamma G(x,y)}|\nabla_y u|^{G(x,y)})\,dxdy \\ &\leq C_\epsilon (1+\mathrm{diam}(\Omega))^{\gamma G^+}\int_\Omega (|\nabla_x u|^{G(x,y)} + |x|^{\gamma}|\nabla_y u|^{G(x,y)})\,dxdy\,,\end{aligned}$$

where $\mathrm{diam}(\Omega)$ is the diameter of Ω. The proof of Theorem 1 is complete. □

Remark. We point out the fact that the conclusion of Theorem 1 does not hold true in the general case. More exactly, there are examples of functions $G(x,y)$ when

$$\inf_{u\in C_c^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega [|\nabla_x u|^{G(x,y)} + |x|^\gamma |\nabla_y u|^{G(x,y)}]\,dxdy}{\displaystyle\int_\Omega (1+|x|^\gamma)|u|^{G(x,y)}\,dxdy} = 0\,. \tag{7}$$

Indeed, assuming that there exists an open set $U \subset \Omega$ and a point $(x_0, y_0) \in U$ such that $G(x_0, y_0) < G(x,y)$ (or $G(x_0,y_0) > G(x,y)$) for all $(x,y) \in \partial U$ then by [10, Theorem 3.1] we get

$$\inf_{u\in C_c^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^{G(x,y)}\,dxdy}{\displaystyle\int_\Omega |u|^{G(x,y)}\,dxdy} = 0\,,$$

where $\nabla u = (\nabla_x u, \nabla_y u)$. Combining the above result with the inequality

$$2\max\left\{1, \sup_{(x,y)\in\Omega} |x|^\gamma\right\} \frac{\displaystyle\int_\Omega |\nabla u|^{G(x,y)}\,dxdy}{\displaystyle\int_\Omega |u|^{G(x,y)}\,dxdy} \geq \frac{\displaystyle\int_\Omega [|\nabla_x u|^{G(x,y)} + |x|^\gamma |\nabla_y u|^{G(x,y)}]\,dxdy}{\displaystyle\int_\Omega (1+|x|^\gamma)|u|^{G(x,y)}\,dxdy}\,,$$

for all $u \in C_c^1(\Omega) \setminus \{0\}$, we deduce that relation (7) holds true provided that the conditions in [10, Theorem 3.1] are fulfilled. On the other hand, relation (7) shows how important are conditions (1) and (2) in establishing the conclusion of Theorem 1. Unfortunately, it is hard to establish if (1) and (2) are also necessary conditions in Theorem 1.

We point out some examples of functions $\overrightarrow{f}(x)$ and $p(x)$ (or $\overrightarrow{g}(y)$ and $q(y)$) satisfying conditions (1) and (2). The examples are inspired from [16] and [15].

Example 1. Let $n \geq 3$ and $\Omega = B_R(0)$, the ball centered in the origin of radius $R > 0$ in $\mathbb{R}^n$. We define $\overrightarrow{f}(x) : \Omega \to \mathbb{R}^n$ by

$$\overrightarrow{f}(x) = (-x_1, x_2, x_3, ..., x_{n-1}, x_n)\,,$$

Clearly, $\overrightarrow{f}(x)$ is of class C^1 and we have

$$\operatorname{div}\overrightarrow{f}(x) = n - 2 \geq 1, \quad \forall\, x \in \Omega\,.$$

Thus, condition (1) is satisfied.

Next, we define $p : \overline{\Omega} \to (1, \infty)$ by

$$p(x) = x_1(x_2 + x_3 + ... + x_{n-1} + x_n) + 2, \quad \forall\, x \in \overline{\Omega}\,.$$

Obviously, p is of class C^1 and some elementary computations show that

$$\nabla p(x) \cdot \overrightarrow{f}(x) = (x_2 + ... + x_n)(-x_1) + x_1x_2 + ... + x_1x_n = 0, \quad \forall\, x \in \Omega\,.$$

It means that condition (2) is satisfied, too.

Example 2. Another example can obtained in the case when $m \geq 2$ and $q : \overline{\Omega} \to (1, \infty)$ does not depend on at least one of the variables y_1, ..., y_m, say y_1 for example. In that case it is enough to choose $\overrightarrow{g}(y) = (y_1, 0, ..., 0)$ in order to have relations (1) and (2) fulfilled.

Remark. Actually, we can obtain examples of $\overrightarrow{f}(x)$ and $p(x)$, $\overrightarrow{g}(y)$ and $q(y)$ satisfying conditions (1) and (2) by combining the examples pointed out above and considering that $\Omega \subset \mathbb{R}^N$ (with $N = n + m$) is, for instance, a ball centered in the origin.

The next result establishes some compact embeddings of $W_{0,\gamma}^{1,G(x,y)}(\Omega)$ into suitable Lebesgue spaces.

Theorem 2. *Assume that the hypotheses of Theorem 1 are fulfilled and the domain Ω intersects the plane $x = 0$. Furthermore, assume that $s \in (1, G^-)$ and $0 < \gamma < n(G^- - s)$. Then $W_{0,\gamma}^{1,G(x,y)}(\Omega)$ is compactly embedded in $L^s(\Omega)$.*

Proof. Let $\{u_k\}$ be a bounded sequence in $W_{0,\gamma}^{1,G(x,y)}(\Omega)$. Since Ω intersects the plane $x = 0$ it follows that there exists $y_0 \in \mathbb{R}^m$ and $R > 0$ such that $B_R((0_n, y_0)) \subset \Omega$, where 0_n designs the origin in $\mathbb{R}^n$ and $B_R((0_n, y_0)) := \{(x, y) \in \mathbb{R}^N;\ |(x, y) - (0_n, y_0)| < R\}$. Then it is clear that there exists $\epsilon_0 \in (0, \min\{1, R\})$ such that

$$\overline{C}_{\epsilon_0} \subset \overline{B}_R((0_n, y_0)) \subset \overline{\Omega}\,,$$

where

$$\begin{aligned} C_{\epsilon_0} \ &:= \ \{(x, y) \in B_R((0_n, y_0));\ |x| < \epsilon_0\} \\ &= \ \{(x, y) \in \mathbb{R}^{n+m};\ |x| < \epsilon_0,\ |(x, y) - (0_n, y_0)| < R\}\,. \end{aligned}$$

Let $\epsilon \in (0, \epsilon_0)$ be arbitrary but fixed and define

$$C_\epsilon := \{(x, y) \in \mathbb{R}^{n+m};\ |x| < \epsilon,\ |(x, y) - (0_n, y_0)| < R\}\,.$$

By Theorem 1 it follows that $\{u_k\}$ is bounded in $L^{G(x,y)}(\Omega)$. Consequently, $\{u_k\} \subset W^{1,G(x,y)}(\Omega \setminus \overline{C}_\epsilon)$ is a bounded sequence. Since $W^{1,G(x,y)}(\Omega \setminus \overline{C}_\epsilon) \subset W^{1,G^-}(\Omega \setminus \overline{C}_\epsilon)$ we deduce that $\{u_k\}$ is bounded in $W^{1,G^-}(\Omega \setminus \overline{C}_\epsilon)$. The classical compact embedding theorem shows that there exists a convergent subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, in $L^s(\Omega \setminus \overline{C}_\epsilon)$. Thus, for any k, l large enough we have

$$\int_{\Omega\setminus\overline{C}_\epsilon} |u_k - u_l|^s \, dxdy < \epsilon \, .$$

On the other hand, Hölder's inequality for variable exponent spaces implies

$$\begin{aligned}
\int_{C_\epsilon} |u_k - u_l|^s \, dxdy &= \int_{C_\epsilon} \frac{1}{|x|^\gamma} |x|^\gamma |u_k - u_l|^s \, dxdy \\
&\leq D_1 \left| \frac{1}{|x|^{\gamma/G(x,y)}} \chi_{C_\epsilon} \right|_{\frac{G(x,y)}{s}'} \left| |x|^{\gamma/G(x,y)} |u_k - u_l|^s \right|_{\frac{G(x,y)}{s}} ,
\end{aligned}$$

where D_1 is a positive constant.

Taking into account that for each $(x,y) \in C_\epsilon$ we have $|x| < \epsilon_0 \leq 1$, the relations between the Luxemburg norm and modular and the conclusion of Theorem 1 we deduce the estimates

$$\begin{aligned}
\left| |x|^{\gamma/G(x,y)} |u_k - u_l|^s \right|_{\frac{G(x,y)}{s}} &\leq \left(\int_\Omega |x|^{\frac{\gamma}{s}} |u_k - u_l|^{G(x,y)} \, dxdy \right)^{1/G^-} + \left(\int_\Omega |x|^{\frac{\gamma}{s}} |u_k - u_l|^{G(x,y)} \, dxdy \right)^{1/G^+} \\
&\leq \sup_{(x,y)\in\Omega} |x|^{\frac{\gamma}{sG^-}} \left(\int_\Omega |x|^\gamma |u_k - u_l|^{G(x,y)} \, dxdy \right)^{1/G^-} + \\
&\quad \sup_{(x,y)\in\Omega} |x|^{\frac{\gamma}{sG^+}} \left(\int_\Omega |x|^\gamma |u_k - u_l|^{G(x,y)} \, dxdy \right)^{1/G^+} \\
&\leq M < \infty \, ,
\end{aligned}$$

where M is a positive constant.

Finally, we compute

$$\begin{aligned}
\left| \frac{1}{|x|^{\gamma/G(x,y)}} \chi_{C_\epsilon} \right|_{\frac{G(x,y)}{s}'} &\leq \left[\rho_{\frac{G(x,y)}{s}'} \left(\frac{1}{|x|^{\gamma/G(x,y)}} \chi_{C_\epsilon} \right) \right]^{\frac{G(x,y)}{s}'^{+}} + \\
&\quad \left[\rho_{\frac{G(x,y)}{s}'} \left(\frac{1}{|x|^{\gamma/G(x,y)}} \chi_{C_\epsilon} \right) \right]^{\frac{G(x,y)}{s}'^{-}} ,
\end{aligned}$$

where $\left(\frac{G(x,y)}{s}\right)' = \frac{G(x,y)}{G(x,y)-s}$.

Now, we compute

$$I_\epsilon := \int_{C_\epsilon} \left(\frac{1}{|x|^{\gamma/G(x,y)}} \right)^{\frac{G(x,y)}{G(x,y)-s}} \, dxdy \, .$$

Clearly,

$$I_\epsilon \leq \int_{C_\epsilon} |x|^{\frac{-\gamma}{G^- - s}} \, dxdy \, .$$

Define

$$\Omega_1 := \{x \in \mathbb{R}^n;\ |x| < \epsilon\}$$

and

$$\Omega_2 := \{y \in \mathbb{R}^m; |y - y_0| < R\}\,.$$

It is easy to check that

$$\Omega_1 \times \Omega_2 = \{(x,y) \in \mathbb{R}^{n+m};\ |x| < \epsilon,\ |y - y_0| < R\} \supset C_\epsilon\,.$$

Then it is clear that

$$I_\epsilon \le \int_{\Omega_1 \times \Omega_2} |x|^{\frac{-\gamma}{G^- - s}}\ dxdy\,.$$

Define $F : \Omega_1 \times \Omega_2 \to \mathbb{R}$, by $F(x,y) = |x|^{\frac{-\gamma}{G^- - s}}$. For each $x \in \Omega_1 \setminus \{0\}$ we have

$$\int_{\Omega_2} |F(x,y)|\ dy = |x|^{\frac{-\gamma}{G^- - s}} |\Omega_2| < \infty\,.$$

On the other hand, the co-area formula yields

$$\int_{\Omega_1} |x|^{\frac{-\gamma}{G^- - s}}\ dx = \int_0^\epsilon \omega_n t^{n-1-\frac{\gamma}{G^- - s}}\ dt = \text{const.} \epsilon^{n - \frac{\gamma}{G^- - s}}\,,$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$.

Thus,

$$\int_{\Omega_1} dx \int_{\Omega_2} |x|^{\frac{-\gamma}{G^- - s}}\ dy = |\Omega_2| \int_{\Omega_1} |x|^{\frac{-\gamma}{G^- - s}}\ dx < \infty\,,$$

and we can apply Tonelli's Theorem (see, [4, Théorème IV.4]) in order to find $F \in L^1(\Omega_1 \times \Omega_2)$. Next, we apply Fubini's Theorem (see, [4, Théorème IV.5]) and we deduce that

$$\int_{\Omega_1 \times \Omega_2} |x|^{\frac{-\gamma}{G^- - s}}\ dxdy \le \text{const.} \epsilon^{n - \frac{\gamma}{G^- - s}}\,.$$

We infer that

$$I_\epsilon \le M_1(\epsilon^{\alpha_1} + \epsilon^{\alpha_2})\,,$$

where α_1, α_2 and M_1 are positive constants.

Thus,

$$\int_\Omega |u_k - u_l|^s\ dxdy \le M_2(\epsilon + \epsilon^{\alpha_1} + \epsilon^{\alpha_2})\,,$$

where $M_2 > 0$ is a constant.

We conclude that $\{u_k\}$ is a Cauchy sequence in $L^s(\Omega)$ and the conclusion is now obvious. □

Corollary 1. *Assume that the hypotheses of Theorem 1 are fulfilled and the domain Ω intersects the plane $x = 0$. Furthermore, $s : \overline{\Omega} \to (1, \infty)$ is a continuous function and $0 < \gamma < n(G^- - s^+)$, $s^+ \in (1, G^-)$. Then $W_{0,\gamma}^{1,G(x,y)}(\Omega)$ is compactly embedded in $L^{s(x,y)}(\Omega)$.*

5 A PDE involving a Grushin-type operator with a variable exponent

In this section we are working under conditions introduced in the previous sections and furthermore, we assume that the hypotheses of Corollary 1 are fulfilled. We analyze the following equation

$$\begin{cases} -\Delta_{G(x,y)}u = \lambda(1+|x|^\gamma)|u|^{G(x,y)-2}u + \mu|u|^{s(x,y)-2}u, & \text{for} \quad (x,y) \in \Omega \\ u = 0, & \text{for} \quad (x,y) \in \partial\Omega\,, \end{cases} \tag{8}$$

where $\Omega \subset \mathbb{R}^N$ intersects the plane $x = 0$ and $s : \overline{\Omega} \to (1,\infty)$ is a continuous function satisfying the conditions from Corollary 1.

Define

$$\lambda_1 := \inf_{u \in C_c^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \frac{1}{G(x,y)}[|\nabla_x u|^{G(x,y)} + |x|^\gamma |\nabla_y u|^{G(x,y)}]\,dxdy}{\displaystyle\int_\Omega \frac{1+|x|^\gamma}{G(x,y)}|u|^{G(x,y)}\,dxdy}\,.$$

By Theorem 1 we infer that $\lambda_1 > 0$.

We say that $u \in W_{0,\gamma}^{1,G(x,y)}(\Omega)$ is a *weak solution* of problem (8) if

$$\begin{aligned}\int_\Omega (|\nabla_x u|^{G(x,y)-2}\nabla_x u \nabla_x v \quad &+ \quad |x|^\gamma |\nabla_y u|^{G(x,y)-2}\nabla_y u \nabla_y v)\,dxdy \\ &- \quad \lambda\int_\Omega (1+|x|^\gamma)|u|^{G(x,y)-2}uv\,dxdy - \mu\int_\Omega |u|^{s(x,y)-2}uv\,dxdy = 0\end{aligned}$$

for all $v \in W_{0,\gamma}^{1,G(x,y)}(\Omega)$.

The main result of this section is given by the following theorem.

Theorem 3. *For any $\lambda \in (0,\lambda_1)$ and $\mu > 0$ problem (8) has a nontrivial weak solution.*

In order to prove Theorem 3 we define the functional $I : W_{0,\gamma}^{1,G(x,y)}(\Omega) \to \mathbb{R}$ by

$$\begin{aligned}I(u) = \int_\Omega \frac{1}{G(x,y)}[|\nabla_x u|^{G(x,y)} \quad &+ \quad |x|^\gamma |\nabla_y u|^{G(x,y)}]\,dxdy \\ &- \quad \lambda\int_\Omega \frac{1+|x|^\gamma}{G(x,y)}|u|^{G(x,y)}\,dxdy - \mu\int_\Omega \frac{1}{s(x,y)}|u|^{s(x,y)}\,dxdy\,.\end{aligned}$$

Standard arguments show that $I \in C^1(W_{0,\gamma}^{1,G(x,y)}(\Omega),\mathbb{R})$ and

$$\begin{aligned}\langle I'(u),v\rangle = \int_\Omega (|\nabla_x u|^{G(x,y)-2}\nabla_x u \nabla_x v \quad &+ \quad |x|^\gamma |\nabla_y u|^{G(x,y)-2}\nabla_y u \nabla_y v)\,dxdy \\ &- \quad \lambda\int_\Omega (1+|x|^\gamma)|u|^{G(x,y)-2}uv\,dxdy - \mu\int_\Omega |u|^{s(x,y)-2}uv\,dxdy\,,\end{aligned}$$

for all $u,\ v \in W_{0,\gamma}^{1,G(x,y)}(\Omega)$. Thus, in order to find weak solutions of equation (8) it is enough to find *critical points* for functional I.

Lemma 1. *For any $\lambda \in (0, \lambda_1)$ and $\mu > 0$ there exist $\xi > 0$ and $r > 0$ such that*

$$I(u) \geq r, \quad \forall\, u \in W_{0,\gamma}^{1,G(x,y)}(\Omega) \text{ with } \|u\| = \xi\,.$$

Proof. Let $\lambda \in (0, \lambda_1)$ and $\mu > 0$ be arbitrary but fixed. Since $s^+ < G^-$ by Corollary 1 it follows that $W_{0,\gamma}^{1,G(x,y)}(\Omega)$ is continuously embedded in $L^{s^\pm}(\Omega)$. That fact combined with the simple remark that

$$|u(x,y)|^{s(x,y)} \leq |u(x,y)|^{s^+} + |u(x,y)|^{s^-}, \quad \forall\, u \in W_{0,\gamma}^{1,G(x,y)}(\Omega),\ (x,y) \in \Omega\,,$$

and Theorem 1 yield that there exists a constant $c_1 > 0$ such that for any $u \in W_{0,\gamma}^{1,G(x,y)}(\Omega)$ the following inequality holds true

$$I(u) \geq \left(1 - \frac{\lambda}{\lambda_1}\right) \frac{1}{G^+} \int_\Omega [|\nabla_x u|^{G(x,y)} + |x|^\gamma |\nabla_y u|^{G(x,y)}]\, dxdy - \mu c_1 [\|u\|^{s^-} + \|u\|^{s^+}]\,.$$

On the other hand, recalling relation (4) it is clear that for each $u \in W_{0,\gamma}^{1,G(x,y)}(\Omega)$ with $\|u\| > 1$ we have

$$\int_\Omega [|\nabla_x u| + |x|^{\gamma/G(x,y)} |\nabla_y u|]^{G(x,y)}\, dxdy \geq \|u\|^{G^-}\,.$$

Since $G(x,y) \geq 2$ for each $(x,y) \in \Omega$ we deduce that for any $(x,y) \in \Omega$ we have

$$(|\nabla_x u| + |x|^{\gamma/G(x,y)} |\nabla_y u|)^{G(x,y)} \leq 2^{G^+ - 1} (|\nabla_x u|^{G(x,y)} + |x|^\gamma |\nabla_y u|^{G(x,y)})\,.$$

Combining the last three relations we infer that

$$I(u) \geq \left(1 - \frac{\lambda}{\lambda_1}\right) \frac{1}{2^{G^+ - 1} G^+} \|u\|^{G^-} - \mu c_1 [\|u\|^{s^-} + \|u\|^{s^+}]\,, \tag{9}$$

for each $u \in W_{0,\gamma}^{1,G(x,y)}(\Omega)$ with $\|u\| > 1$. (Actually, it is easy to check that a inequality of type (9) holds true for each $u \in W_{0,\gamma}^{1,G(x,y)}(\Omega)$ but replacing $\|u\|^{G^-}$ by $\|u\|^{G^+}$ when $\|u\| \leq 1$.)

Inequality (9) and the fact that $G^- > s^+ \geq s^-$ imply that there exists $\xi > 1$ large enough for which we can choose $r > 0$ such that

$$I(u) \geq r, \quad \forall\, u \in W_{0,\gamma}^{1,G(x,y)}(\Omega) \text{ with } \|u\| = \xi\,.$$

The proof of Lemma 1 is complete. □

Lemma 2. *There exists $\varphi \in W_{0,\gamma}^{1,G(x,y)}(\Omega) \setminus \{0\}$ such that $I(t\varphi) < 0$, for each $t > 0$ small enough.*

Proof. Fix $\varphi \in W_{0,\gamma}^{1,G(x,y)}(\Omega) \setminus \{0\}$ For any $t \in (0,1)$ we have

$$\begin{aligned} I(t\varphi) &= \int_\Omega \frac{1}{G(x,y)} [|\nabla_x (t\varphi)|^{G(x,y)} + |x|^\gamma |\nabla_y (t\varphi)|^{G(x,y)}]\, dxdy - \lambda \int_\Omega \frac{1 + |x|^\gamma}{G(x,y)} |t\varphi|^{G(x,y)}\, dxdy - \\ &\quad \mu \int_\Omega \frac{1}{s(x,y)} |t\varphi|^{s(x,y)}\, dxdy \\ &\leq t^{G^-} \int_\Omega \frac{1}{G(x,y)} [|\nabla_x \varphi|^{G(x,y)} + |x|^\gamma |\nabla_y \varphi|^{G(x,y)}]\, dxdy - t^{s^+} \mu \int_\Omega \frac{1}{s(x,y)} |\varphi|^{s(x,y)}\, dxdy\,. \end{aligned}$$

Since $G^- > s^+$ it is clear that

$$I(t\varphi) < 0\,,$$

provided that

$$0 < t < \min\left\{1, \left(\frac{\mu \cdot \int_\Omega \frac{1}{s(x,y)}|\varphi|^{s(x,y)}\, dxdy}{\int_\Omega \frac{1}{G(x,y)}[|\nabla_x\varphi|^{G(x,y)} + |x|^\gamma|\nabla_y\varphi|^{G(x,y)}]\, dxdy}\right)^{1/(G^- - s^+)}\right\}.$$

The proof of Lemma 2 is complete. □

Proof of Theorem 3. By inequality (9) we obtain that I is bounded from below on $\overline{B_\xi(0)}$ (In this proof $B_\xi(0) := \{v \in W_{0,\gamma}^{1,G(x,y)}(\Omega);\ \ \|v\| < \xi\}$ stands for the ball centered in the origin and of radius ξ in $W_{0,\gamma}^{1,G(x,y)}(\Omega)$). Thus, using Ekeland's variational principle (see [9] or [22]) to the functional $I : \overline{B_\xi(0)} \to \mathbb{R}$, it follows that there exists $u_\epsilon \in \overline{B_\xi(0)}$ such that

$$\begin{aligned} I(u_\epsilon) &< \inf_{\overline{B_\xi(0)}} I + \epsilon \\ I(u_\epsilon) &< I(u) + \epsilon \cdot \|u - u_\epsilon\|, \quad u \neq u_\epsilon. \end{aligned}$$

Using Lemmas 1 and 2 we find

$$\inf_{\partial B_\xi(0)} I \geq r > 0 \quad \text{and} \quad \inf_{\overline{B_\xi(0)}} I < 0\,.$$

We choose $\epsilon > 0$ such that

$$0 < \epsilon \leq \inf_{\partial B_\xi(0)} I - \inf_{\overline{B_\xi(0)}} I\,.$$

Therefore, $I(u_\epsilon) < \inf_{\partial B_\xi(0)} I$ and thus, $u_\epsilon \in B_\xi(0)$.

We define $T : \overline{B_\xi(0)} \to \mathbb{R}$ by $T(u) = I(u) + \epsilon \cdot \|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of T and thus

$$\frac{T(u_\epsilon + \delta \cdot v) - T(u_\epsilon)}{\delta} \geq 0$$

for a small $\delta > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{I(u_\epsilon + \delta \cdot v) - I(u_\epsilon)}{\delta} + \epsilon\|v\| \geq 0.$$

Letting $\delta \to 0$ it follows that $\langle I'(u_\epsilon), v\rangle + \epsilon \cdot \|v\| > 0$ and we infer that $\|I'(u_\epsilon)\| \leq \epsilon$.

We deduce that there exists a sequence $\{u_n\} \subset B_\xi(0)$ such that

$$I(u_n) \to c = \inf_{\overline{B_\xi(0)}} I < 0 \quad \text{and} \quad I'(u_n) \to 0. \tag{10}$$

It is clear that $\{u_n\}$ is bounded in $W_{0,\gamma}^{1,G(x,y)}(\Omega)$. Thus, there exists $u \in W_{0,\gamma}^{1,G(x,y)}(\Omega)$ such that, up to a subsequence, $\{u_n\}$ converges weakly to u in $W_{0,\gamma}^{1,G(x,y)}(\Omega)$. Then Theorems 2 and 1 imply that $\{u_n\}$ converges strongly to u in $L^{s(x,y)}(\Omega)$ and weakly to u in $L^{G(x,y)}(\Omega)$. Thus,

$$\lim_{n\to\infty}\int_\Omega |u_n|^{s(x,y)-2}u_n v\; dxdy = \int_\Omega |u|^{s(x,y)-2}uv\; dxdy\,,$$

and

$$\lim_{n\to\infty}\int_\Omega |u_n|^{G(x,y)-2}u_n v\; dxdy = \int_\Omega |u|^{G(x,y)-2}uv\; dxdy\,,$$

for any $v \in W_{0,\gamma}^{1,G(x,y)}(\Omega)$.

On the other hand, relation (10) implies

$$\lim_{n\to\infty}\langle I^{'}(u_n), v\rangle = 0\,,$$

for all $v \in W_{0,\gamma}^{1,G(x,y)}(\Omega)$.

The above information implies

$$I^{'}(u) = 0\,,$$

and thus, u is a weak solution of equation (8).

We prove now that $u \neq 0$. Assume by contradiction that $u \equiv 0$ and

$$\lim_{n\to\infty}\int_\Omega (|\nabla_x u_n|^{G(x,y)} + |x|^\gamma |\nabla_y u_n|^{G(x,y)})\; dxdy = l \geq 0\,.$$

Since by relation (10) we have $\lim_{n\to\infty}\langle I^{'}(u_n), u_n\rangle = 0$ and $\{u_n\}$ converges strongly to 0 in $L^{s(x,y)}(\Omega)$ and the above relation holds true we obtain

$$\begin{aligned}
0 > c + o(1) \;&=\; \int_\Omega \frac{1}{G(x,y)}[|\nabla_x u_n|^{G(x,y)} + |x|^\gamma |\nabla_y u_n|^{G(x,y)}]\; dxdy - \lambda\int_\Omega \frac{1+|x|^\gamma}{G(x,y)}|u_n|^{G(x,y)}\; dxdy - \\
&\quad\; \mu\int_\Omega \frac{1}{s(x,y)}|u_n|^{s(x,y)}\; dxdy \\
&\geq\; \left(1-\frac{\lambda}{\lambda_1}\right)\int_\Omega \frac{1}{G(x,y)}[|\nabla_x u_n|^{G(x,y)} + |x|^\gamma |\nabla_y u_n|^{G(x,y)}]\; dxdy - \\
&\quad\; \mu\int_\Omega \frac{1}{s(x,y)}|u_n|^{s(x,y)}\; dxdy \\
&\geq\; \left(1-\frac{\lambda}{\lambda_1}\right)\frac{1}{G^+}\int_\Omega [|\nabla_x u_n|^{G(x,y)} + |x|^\gamma |\nabla_y u_n|^{G(x,y)}]\; dxdy - \\
&\quad\; \frac{\mu}{s^-}\int_\Omega |u_n|^{s(x,y)}\; dxdy \to \left(1-\frac{\lambda}{\lambda_1}\right)\frac{1}{G^+}l \geq 0
\end{aligned}$$

and that is a contradiction. We conclude that $u \neq 0$.

Thus, Theorem 3 is completely proved. □

6 Appendix

In this section we show that the norms

$$\|u\| := |\ |\nabla_x u| + |x|^{\frac{\gamma}{G(x,y)}}\ |\nabla_y u|\ |_{G(x,y)}\,,$$

and

$$\|u\|_1 := |\ |(\nabla_x u, \nabla_y u)|\ |_{G(x,y)}\,,$$

are equivalent on the Sobolev type space $W_{0,\gamma}^{1,G(x,y)}(\Omega)$, provided that there exists a constant $m > 0$ such that for each $(x,y) \in \Omega$ we have $|x| \geq m$. Particularly, that fact shows that the function space $W_{0,\gamma}^{1,G(x,y)}(\Omega)$ represents a natural generalization of the *classical* variable exponent Sobolev space $W_0^{1,G(x,y)}(\Omega)$.

More exactly, we will prove

Proposition 1. *Assume that $\Omega \subset \mathbb{R}^N$ $(N = n+m)$ is a bounded and smooth domain for which there exists a constant $m > 0$ such that for each $(x,y) \in \Omega$ we have $|x| \geq m$. Then the norms:*

$$\|u\| := |\ |\nabla_x u| + |x|^{\frac{\gamma}{G(x,y)}}\ |\nabla_y u|\ |_{G(x,y)}\,,$$

$$\|u\|_1 := |\ |(\nabla_x u, \nabla_y u)|\ |_{G(x,y)}\,,$$

$$\|u\|_2 := |\ |(\nabla_x u, |x|^{\frac{\gamma}{G(x,y)}} \nabla_y u)|\ |_{G(x,y)}\,,$$

are equivalent.

Proof. • First, we prove that the norms $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent. Let $u \in W_{0,\gamma}^{1,G(x,y)}(\Omega)$ be arbitrary but fixed. By the definition of the Luxemburg norm we have

$$\int_\Omega \left(\frac{\sqrt{|\nabla_x u|^2 + |x|^{\frac{2\gamma}{G(x,y)}}\ |\nabla_y u|^2}}{\|u\|_2} \right)^{G(x,y)} dxdy \leq 1\,.$$

On the other hand, it is obvious that for each $(x,y) \in \Omega$ we have

$$|\nabla_x u| + |x|^{\frac{\gamma}{G(x,y)}}\ |\nabla_y u| \leq \sqrt{2} \cdot \sqrt{|\nabla_x u|^2 + |x|^{\frac{2\gamma}{G(x,y)}}\ |\nabla_y u|^2}\,.$$

Combining the above two relations we get

$$\int_\Omega \left(\frac{|\nabla_x u| + |x|^{\frac{\gamma}{G(x,y)}}\ |\nabla_y u|}{\sqrt{2} \cdot \|u\|_2} \right)^{G(x,y)} dxdy \leq 1\,.$$

Therefore,

$$\|u\| \leq \sqrt{2} \cdot \|u\|_2, \quad \forall\, u \in W_{0,\gamma}^{1,G(x,y)}(\Omega)\,.$$

Next, we turn back once more to the definition of the Luxemburg norm and deduce

$$\int_\Omega \left(\frac{|\nabla_x u| + |x|^{\frac{\gamma}{G(x,y)}} \, |\nabla_y u|}{\|u\|} \right)^{G(x,y)} dxdy \le 1 \, .$$

On the other hand, simple computations show that for each $(x,y) \in \Omega$ we have

$$|(\nabla_x u, |x|^{\frac{\gamma}{G(x,y)}} \nabla_y u)|^{G(x,y)} = \sqrt{|\nabla_x u|^2 + |x|^{\frac{2\gamma}{G(x,y)}} \, |\nabla_y u|^2}^{\,G(x,y)} \le (|\nabla_x u| + |x|^{\frac{\gamma}{G(x,y)}} |\nabla_y u|)^{G(x,y)}$$

Combining the above two inequalities we get

$$\int_\Omega \left(\frac{|(\nabla_x u, |x|^{\frac{\gamma}{G(x,y)}} \nabla_y u)|}{\|u\|} \right)^{G(x,y)} dxdy \le 1 \, ,$$

or

$$\|u\|_2 \le \|u\|, \quad \forall \, u \in W_{0,\gamma}^{1,G(x,y)}(\Omega) \, .$$

Thus, the norms $\| \cdot \|$ and $\| \cdot \|_2$ are equivalent.

• Second, we prove that the norms $\| \cdot \|_2$ and $\| \cdot \|_1$ are equivalent, provided that there exists a constant $m > 0$ such that for each $(x,y) \in \Omega$ we have $|x| \ge m$. We start by pointing out that taking $a = \sqrt{\min_{(x,y)\in\Omega}\{1, |x|^{2\gamma}\}}$ and $b = \sqrt{\max_{(x,y)\in\Omega}\{1, |x|^{2\gamma}\}}$ we have for all $u \in W_{0,\gamma}^{1,G(x,y)}(\Omega)$, $(x,y) \in \Omega$ and $\mu > 0$ that

$$a \left(\frac{|(\nabla_x u, \nabla_y u)|}{\mu} \right)^{G(x,y)} \le \left(\frac{\sqrt{|\nabla_x u|^2 + |x|^{\frac{2\gamma}{G(x,y)}} \, |\nabla_y u|^2}}{\mu} \right)^{G(x,y)} \le b \left(\frac{|(\nabla_x u, \nabla_y u)|}{\mu} \right)^{G(x,y)} .$$

Integrating over Ω yields

$$\begin{aligned} a \int_\Omega \left(\frac{|(\nabla_x u, \nabla_y u)|}{\mu} \right)^{G(x,y)} dxdy &\le \int_\Omega \left(\frac{\sqrt{|\nabla_x u|^2 + |x|^{\frac{2\gamma}{G(x,y)}} \, |\nabla_y u|^2}}{\mu} \right)^{G(x,y)} dxdy \\ &\le b \int_\Omega \left(\frac{|(\nabla_x u, \nabla_y u)|}{\mu} \right)^{G(x,y)} dxdy \, , \end{aligned} \tag{11}$$

for all $u \in W_{0,\gamma}^{1,G(x,y)}(\Omega)$ and $\mu > 0$.

Next, let $u \in W_{0,\gamma}^{1,G(x,y)}(\Omega)$ be arbitrary but fixed.

Let $\mu = \|u\|_1$ in (11). Then, since $\int_\Omega \left(\frac{|(\nabla_x u, \nabla_y u)|}{\|u\|_1} \right)^{G(x,y)} dxdy = 1$ we have

$$a \le \int_\Omega \left(\frac{\sqrt{|\nabla_x u|^2 + |x|^{\frac{2\gamma}{G(x,y)}} \, |\nabla_y u|^2}}{\|u\|_1} \right)^{G(x,y)} dxdy \le b \, .$$

If $b \le 1$ then it is clear that $\|u\|_2 \le \|u\|_1$.

If $b > 1$ then we have

$$\frac{1}{b}\int_\Omega \left(\frac{\sqrt{|\nabla_x u|^2 + |x|^{\frac{2\gamma}{G(x,y)}}\ |\nabla_y u|^2}}{\|u\|_1}\right)^{G(x,y)} dxdy \leq 1\,.$$

Combining that fact with the inequality $\frac{1}{b^{G(x,y)}} \leq \frac{1}{b}$ for each $(x,y) \in \Omega$ we deduce

$$\int_\Omega \left(\frac{\sqrt{|\nabla_x u|^2 + |x|^{\frac{2\gamma}{G(x,y)}}\ |\nabla_y u|^2}}{b\|u\|_1}\right)^{G(x,y)} dxdy \leq 1\,,$$

i.e.,

$$\|u\|_2 \leq b\|u\|_1\,.$$

Let $\mu = \|u\|_2$ in (11). Then, since $\int_\Omega \left(\frac{|(\nabla_x u, |x|^{\frac{\gamma}{G(x,y)}}\nabla_y u)|}{\|u\|_2}\right)^{G(x,y)} dxdy = 1$ we have

$$a\int_\Omega \left(\frac{|(\nabla_x u, \nabla_y u)|}{\|u\|_2}\right)^{G(x,y)} dxdy \leq 1\,.$$

If $a \geq 1$ then

$$\int_\Omega \left(\frac{|(\nabla_x u, \nabla_y u)|}{\|u\|_2}\right)^{G(x,y)} dxdy \leq \frac{1}{a} \leq 1\,,$$

i.e.,

$$\|u\|_1 \leq \|u\|_2\,.$$

If $a < 1$ then $a^{G(x,y)} \leq a$ for each $(x,y) \in \Omega$ and consequently

$$\int_\Omega \left(\frac{|(\nabla_x u, \nabla_y u)|}{\frac{\|u\|_2}{a}}\right)^{G(x,y)} dxdy \leq 1\,,$$

i.e.,

$$\|u\|_1 \leq \frac{\|u\|_2}{a}\,.$$

In brief, we found

- if $a > 1$ then $\frac{\|u\|_2}{b} \leq \|u\|_1 \leq \|u\|_2$;
- if $b < 1$ then $\|u\|_2 \leq \|u\|_1 \leq \frac{\|u\|_2}{a}$;
- if $a < 1 < b$ then $\frac{\|u\|_2}{b} \leq \|u\|_1 \leq \frac{\|u\|_2}{a}$.

Consequently, the norms $\|\cdot\|_2$ and $\|\cdot\|_1$ are equivalent, provided that there exists a constant $m > 0$ such that for each $(x,y) \in \Omega$ we have $|x| \geq m$.

The proof of Proposition 1 is complete. □

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References

[1] M. Bocea and M. Mihăilescu, Γ-convergence of power-law functionals with variable exponents. *Nonlinear Analysis* **73** (2010), 110-121.

[2] M. Bocea, M. Mihăilescu, M. Pérez-Llanos and J. D. Rossi, Models for growth of heterogeneous sandpiles via Mosco convergence, submitted.

[3] M. Bocea, M. Mihăilescu and C. Popovici, On the asymptotic behavior of variable exponent power-law functionals and applications, *Ricerche di Matematica* **59** No. 2 (2010), 207-238.

[4] H. Brezis, *Analyse fonctionnelle. Théorie, Méthodes et Applications*, Masson, Paris, 1992.

[5] Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image processing, *SIAM J. Appl. Math.* **66** No. 4 (2006), 1383-1406.

[6] L. D'Ambrosio, Hardy inequalities related to Grushin type operators, *Proc. Amer. Math. Soc.* **132** (2) (2004), 725-734.

[7] L. D'Ambrosio, Hardy-type inequalities related to degenerate elliptic differential operators, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **4** (3) (2005), 451-486.

[8] L. Diening, P. Harjulehto, P. Hästö, M. Ružička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, Berlin, 2011.

[9] I. Ekeland, On the variational principle, *J. Math. Anal. App.* **47** (1974), 324-353.

[10] X. Fan, Q. Zhang and D. Zhao, Eigenvalues of $p(x)$-Laplacian Dirichlet problem, *Journal of Mathematical Analysis and Applications* **302** (2005), 306-317.

[11] G. Fragnelli, Positive periodic solutions for a system of anisotropic parabolic equations, *J. Math. Anal. Appl.* **367** (2010), 204-228.

[12] V. V. Grushin, A certain class of elliptic pseudo differential operators that are degenerated on a submanifold, *Mat. Sb.* **84** (126) (1971), 163-195; English transl. in *Math. USSR Sbornik* **13** (1971), 155-183.

[13] P. Harjulehto, P. Hästö, U. V. Le and M. Nuortio, Overview of differential equations with non-standard growth, *Nonlinear Anal.* **72** (2010), 4551-4574.

[14] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.* **41** (116) (1991), 592-618.

[15] M. Mihăilescu and G. Moroşanu, On an eigenvalue problem for an anisotropic elliptic equation involving variable exponents, *Glasgow Mathematical Journal* **52** (2010), 517-527.

[16] M. Mihăilescu, V. Rădulescu and D. Stancu-Dumitru, On a Caffarelli-Kohn-Nirenberg type inequality in bounded domains involving variable exponent growth conditions and applications to PDE's, *Complex Variables-Elliptic Equations*, in press.

[17] R. Monti and D. Morbidelli, Kelvin transform for Grushin operator and critical semilinear equations, *Duke Math. J.* **131** (1) (2006), 167-202.

[18] K.R. Rajagopal and M. Ružička, Mathematical modeling of electrorheological materials, *Contin. Mech. Thermodyn.* **13** (2001), 59-78.

[19] M. Ružička, Flow of shear dependent electrorheological fluids: unsteady space periodic case, *Applied Nonlinear Analysis*, 485-504, Kluwer/Plenum, New York, 1999.

[20] M. Ružička, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics 1748, Springer-Verlag, Berlin, 2000.

[21] N. M. Tri, On Grushin's Equation, *Math. Notes* **63** (1) (1998), 84-93; English transl. from *Mmat. Zametki* **63** (1) (1998), 95-105.

[22] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.