Asymptotically periodic solutions of some difference and differential inclusions in Hilbert spaces

Gheorghe Moroşanu[a]*and Figen Özpınar[b]†

[a] Department of Mathematics and its Applications, Central European University, Budapest, Hungary
[b] Bolvadin Vocational School, Afyon Kocatepe University, Afyonkarahisar, Turkey

Abstract

Let H be a real Hilbert space and let $A : D(A) \subset H \to H$ be a (possibly multivalued) maximal monotone operator. Consider the difference inclusion $\Delta u_n + c_n A u_{n+1} \ni f_n + g_n,\ n = 0, 1, ...,$ where $(c_n) \subset (0, +\infty)$, $(f_n) \subset H$ are p-periodic sequences for a positive integer p and $(g_n) \in \ell^1(H)$. We investigate the weak or strong convergence of solutions to p-periodic sequences. We also investigate the existence of asymptotically periodic solutions of the continuous analogue of the above difference inclusion $u'(t) + Au(t) \ni f(t) + g(t),\ t > 0$, where $f \in L^2_{loc}(\mathbb{R}_+, H)$ is a T-periodic function $(T > 0)$ and $g \in L^1(\mathbb{R}_+, H)$. We show that the previous results due to J. B. Baillon, A. Haraux (1977) and B. Djafari Rouhani, H. Khatibzadeh (2012) corresponding to $g \equiv 0$, respectively $g_n = 0,\ n = 0, 1, ...,$ remain valid for $g \in L^1(\mathbb{R}_+, H)$, respectively $(g_n) \in l^1(H)$.

Keywords: difference inclusion; differential inclusion; maximal monotone operator; subdifferential; weak convergence; strong convergence.

2010 Mathematics Subject Classification: 39A10, 39A11, 47H05, 34G25

1 Introduction

Let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and the induced Hilbertian norm $\|\cdot\|$. Let $A : D(A) \subset H \to H$ be a (possibly multivalued) maximal monotone operator. Consider the difference equation (inclusion)

$$\Delta u_n + c_n A u_{n+1} \ni f_n + g_n, \qquad n = 0, 1, ..., \tag{E}$$

where $(c_n) \subset (0, +\infty)$, $(f_n) \subset H$ are p-periodic sequences for a positive integer p, $(g_n) \in \ell^1(H) := \{u = (u_1, u_2, \ldots) : \sum_{n=1}^{\infty} \|u_n\| < \infty\}$ and Δ is the difference operator defined as usual, i.e., $\Delta u_n = u_{n+1} - u_n$. We shall investigate the weak or strong convergence of solutions to p-periodic sequences.
Consider also the following differential equation (inclusion)

$$\frac{du}{dt} + Au(t) \ni f(t) + g(t), \qquad t > 0, \tag{E^c}$$

where $f \in L^2_{loc}(\mathbb{R}_+, H)$ is a T-periodic function for a given $T > 0$ and $g \in L^1(\mathbb{R}_+, H)$. We shall investigate the behavior at infinity of solutions to (E^c). More precisely, in this Note we show that the previous results due to Baillon, Haraux [1] and Djafari Rouhani, Khatibzadeh [2] related to the equations (inclusions),

$$\frac{du}{dt} + Au(t) \ni f(t), \qquad t > 0, \tag{E$^c_0$}$$

and

$$\Delta u_n + c_n A u_{n+1} \ni f_n, \qquad n = 0, 1, ..., \tag{E$_0$}$$

respectively, remain valid for (E^c) and (E), where $g \in L^1(\mathbb{R}_+, H)$ and $(g_n) \in l^1(H)$.

* **Email**: morosanug@ceu.hu
† **Email**: fozpinar@aku.edu.tr

2 Previous Results

To obtain our main results we recall the following theorems on the existence of asymptotically periodic solutions of the equations (E_0) and (E_0^c).

Lemma 1 ([2]; see also [4] for a different shorter proof)**.** *Assume that $A : D(A) \subset H \to H$ is a maximal monotone operator. Let $c_n > 0$ and $f_n \in H$ be p-periodic sequences, i.e., $c_{n+p} = c_n$, $f_{n+p} = f_n$ $(n = 0, 1, ...)$, for a given positive integer p. Then equation (E_0) has a bounded solution if and only if it has at least one p-periodic solution. In this case all solutions of (E_0) are bounded and for every solution (u_n) of (E_0) there exists a p-periodic solution (ω_n) of (E_0) such that*

$$u_n - \omega_n \to 0, \text{ weakly in } H, \text{ as } n \to \infty.$$

Moreover, every two periodic solutions differ by an additive constant vector.

Lemma 2 ([1]; see also [3], p. 169)**.** *Assume that A is the subdifferential of a proper, convex, and lower semicontinuous function $\varphi : H \to (-\infty, +\infty]$, $A = \partial\varphi$. Let $f \in L^2_{loc}(\mathbb{R}_+, H)$ be a T-periodic function (for a given $T > 0$). Then, equation (E_0^c) has a solution bounded on $\mathbb{R}_+$ if and only if it has at least a T-periodic solution. In this case all solutions of (E_0^c) are bounded on $\mathbb{R}_+$ and for every solution $u(t)$, $t \geq 0$, there exists a T-periodic solution q of (E_0^c) such that*

$$u(t) - q(t) \to 0, \text{ as } t \to \infty,$$

weakly in H.
Moreover, every two periodic solutions of (E_0^c) differ by an additive constant, and

$$\frac{du_n}{dt} \to \frac{dq}{dt}, \text{ as } n \to \infty,$$

strongly in $L^2(0, T; H)$, where $u_n(t) = u(t + nT)$, $n = 1, 2, ...$

3 New Results on Asymptotically Periodic Solutions

We begin this section with the following result regarding the discrete case, which is an extension of Lemma 1.

Theorem 1. *Assume that $A : D(A) \subset H \to H$ is a maximal monotone operator. Let $(g_n) \in \ell^1(H)$ and let $c_n > 0$, $f_n \in H$ be p-periodic sequences, i.e., $c_{n+p} = c_n$, $f_{n+p} = f_n$ $(n = 0, 1, ...)$, for a given positive integer p. Then equation (E) has a bounded solution if and only if equation (E_0) has at least one p-periodic solution. In this case all solutions of (E) are bounded and for every solution (u_n) of (E) there exists a p-periodic solution (ω_n) of (E_0) such that*

$$u_n - \omega_n \to 0, \text{ weakly in } H, \text{ as } n \to \infty.$$

Proof. Consider the initial condition

$$u_0 = x, \tag{IC}$$

for a given $x \in H$. We can rewrite equation (E) in the form:

$$u_{n+1} - u_n + c_n A u_{n+1} \ni f_n + g_n.$$

The solution of the problem (E)-(IC) is calculated successively from

$$u_{n+1} = \big(I + c_n A\big)^{-1}\big(u_n + f_n + g_n\big), \qquad n = 0, 1, \ldots,$$

in a unique manner, which will give a unique solution $(u_n)_{n\geq 0}$.
If a solution (u_n) of (E) is bounded, then any other solution $(\tilde{u}_n)$ of (E) is bounded, because

$$\|u_n - \tilde{u}_n\| \leq \|u_0 - \tilde{u}_0\| \qquad \forall n = 0, 1, \ldots \tag{1}$$

If a solution (u_n) of (E) is bounded, then any solution (v_n) of (E_0) is bounded and conversely, because

$$\|u_n - v_n\| \le \|u_0 - v_0\| + \sum_{k=0}^{n-1} \|g_k\| \le \|u_0 - v_0\| + \sum_{k=0}^{\infty} \|g_k\| < \infty.$$

According to Lemma 1 the first part of the theorem is proved. For the second part we define $(g_{n,m})_{n,m\ge 0}$ as follows:

$$g_{n,m} = \begin{cases} g_n & \text{if } n < m, \\ 0 & \text{if } n \ge m. \end{cases}$$

Let (z_n) be an arbitrary solution of (E) (which is bounded). For each $m = 0, 1, \ldots$ denote by $(z_{n,m})_{n\ge 0}$ the (unique) solution of the problem

$$z_{n+1,m} - z_{n,m} + c_n A z_{n+1,m} \ni f_n + g_{n,m} \tag{E_m}$$

$$z_{0,m} = z_0. \tag{IC_m}$$

Note that $(z_{n,m})_{n\ge m}$ is a solution of equation (E_0). By Lemma 1 there is a p-periodic (with respect to n) solution $(\omega_{n,m})$ of (E_0) such that

$$z_{n,m} - \omega_{n,m} \to 0, \text{ weakly in } H, \text{ as } n \to \infty. \tag{2}$$

For each $m \ge 0$ we have

$$\begin{aligned} \omega_{1,m} - \omega_{0,m} + c_0 A\omega_{1,m} &\ni f_0, \\ \omega_{2,m} - \omega_{1,m} + c_1 A\omega_{2,m} &\ni f_1, \\ &\vdots \\ \omega_{p,m} - \omega_{p-1,m} + c_{p-1} A\omega_{p,m} &\ni f_{p-1}, \end{aligned}$$

where $\omega_{p,m} = \omega_{0,m}$. Since any two periodic solutions of (E_0) differ by an additive constant, we can write

$$\omega_{t,m} = \zeta_t + a_m \qquad t \in \{0, 1, \ldots, p-1\}, \tag{3}$$

where (ζ_t) is a an arbitrary but fixed periodic solution of (E_0), and $(a_m)_{m\ge 0}$ is a sequence in H. Thus

$$\begin{aligned} \zeta_1 - \zeta_0 + c_0 A(\zeta_1 + a_m) &\ni f_0, \\ \zeta_2 - \zeta_1 + c_1 A(\zeta_2 + a_m) &\ni f_1, \\ &\vdots \\ \zeta_p - \zeta_{p-1} + c_{p-1} A(\zeta_p + a_m) &\ni f_{p-1}, \end{aligned} \tag{4}$$

for all $m \ge 0$, where $\zeta_p = \zeta_0$. Also we can rewrite (2) as

$$z_{kp+t,m} \to \zeta_t + a_m, \text{ weakly in } H, \text{ as } k \to \infty, \tag{5}$$

for all $m \ge 0$ and $t \in \{0, 1, ..., p-1\}$. On the other hand, for $0 \le m < r$, we have (cf. (E_m), (IC_m))

$$\|z_{kp+t,m} - z_{kp+t,r}\| \le \sum_{j=m}^{r-1} \|g_j\|.$$

According to (5) this implies

$$\|a_m - a_r\| \le \sum_{j=m}^{r-1} \|g_j\| \le \sum_{j=m}^{\infty} \|g_j\|, \tag{6}$$

for all $0 \le m < r$, so there exists an $a \in H$ such that

$$\|a_m - a\| \to 0, \quad \text{as } m \to \infty. \tag{7}$$

Since A is maximal monotone (hence demiclosed), we can pass to the limit in (4) as $m \to \infty$ to obtain

$$\begin{gathered}\zeta_1 - \zeta_0 + c_0 A(\zeta_1 + a) \ni f_0,\\ \zeta_2 - \zeta_1 + c_1 A(\zeta_2 + a) \ni f_1,\\ \vdots\\ \zeta_p - \zeta_{p-1} + c_{p-1} A(\zeta_p + a) \ni f_{p-1},\end{gathered}$$

where $\zeta_p = \zeta_0$. So $\omega_n := \zeta_n + a$ is a p-periodic solution of equation (E_0). We can also see that

$$\|z_n - z_{n,m}\| \le \|z_0 - z_{0,m}\| + \sum_{j=m}^{n-1} \|g_j\| \le \sum_{j=m}^{\infty} \|g_j\|. \tag{8}$$

Finally, for all natural n, we have $n = kp + t,\ t \in \{0, 1, ..., p-1\}$, and

$$z_n - \omega_n = [z_n - z_{n,m}] + [z_{n,m} - \omega_{t,m}] + [\omega_{t,m} - \omega_n]$$

$$= [z_n - z_{n,m}] + [z_{kp+t,m} - \zeta_t - a_m] + [\zeta_t + a_m - \zeta_t - a],$$

thus the conclusion of the theorem follows by (5), (7) and (8). □

Remark 1. If, in addition, A is strongly monotone, then we can easily extend Theorem 2 in [4], as follows:

Theorem 2. *Assume that $A : D(A) \subset H \to H$ is a maximal monotone operator, that is also strongly monotone, i.e., there is a constant $b > 0$, such that*

$$(x_1 - x_2, y_1 - y_2) \ge b\|x_1 - x_2\|^2,\ \forall\ x_i \in D(A),\ y_i \in Ax_i,\ i = 1, 2.$$

Let $c_n > 0$ and $f_n \in H$ be p-periodic sequences for a given positive integer p and $(g_n) \in \ell^1(H)$. Then, equation (E_0) has a unique p-periodic solution (ω_n) and for every solution (u_n) of (E) we have

$$u_n - \omega_n \to 0,\ \textit{strongly in } H,\ \textit{as } n \to \infty.$$

The proof relies on arguments similar to the above ones.

Theorem 3. *Assume that $A : D(A) \subset H \to H$ is the subdifferential of a proper, convex, lower semicontinuous function $\varphi : H \to (-\infty, +\infty]$, $A = \partial\varphi$. Let $f \in L^2_{loc}(\mathbb{R}_+, H)$ be a T-periodic function ($T > 0$) and let $g \in L^1(\mathbb{R}_+, H)$. Then equation ($\mathrm{E}^c$) has a bounded solution if and only if equation (E^c_0) has at least a T-periodic solution. In this case all solutions of (E^c) are bounded on $\mathbb{R}_+$ and for every solution $u(t)$ of (E^c) there exists a T-periodic solution $\omega(t)$ of (E^c_0) such that*

$$u(t) - \omega(t) \to 0,\ \textit{weakly in } H,\ \textit{as } t \to \infty.$$

Proof. If a solution $u(t)$, $t \ge 0$, of equation (E^c) is bounded on $\mathbb{R}_+$, then any other solution $\tilde{u}(t)$, $t \ge 0$, of equation (E^c) is bounded too, because

$$\|u(t) - \tilde{u}(t)\| \le \|u(0) - \tilde{u}(0)\|. \tag{9}$$

If a solution $u(t)$ of (E^c) is bounded, then any solution $v(t)$ of (E^c_0) is bounded and conversely, because

$$\|u(t) - v(t)\| \le \|u(0) - v(0)\| + \int_0^t \|g(s)\| ds \le \|u(0) - v(0)\| + \int_0^\infty \|g(s)\| ds < \infty, \qquad t \ge 0.$$

Thus, the first part of the theorem follows by Lemma 2. In order to prove the second part, we define $g_m : \mathbb{R}_+ \to H$ as follows:

$$g_m(t) = \begin{cases} g(t) & \text{for a.e. } t \in (0, m) \\ 0 & \text{if } t \ge m, \end{cases}$$

where $m = 1, 2, \ldots$
Let $u(t)$, $t \geq 0$, be an arbitrary bounded solution of (E^c). For each $m = 1, 2, \ldots$ denote by $u_m(t)$, $t \geq 0$, the solution of the Cauchy problem

$$\frac{du_m(t)}{dt} + A(u_m(t)) \ni f(t) + g_m(t), \qquad t > 0, \tag{E^c_m}$$

$$u_m(0) = u(0). \tag{IC^c_m}$$

Since $u_m(t)$, $t \geq m$, is a solution of equation (E^c_0), it follows by Lemma 2 that there is a T-periodic solution $q_m(t)$ of (E^c_0), such that

$$u_m(t) - q_m(t) \to 0, \text{ weakly in } H, \text{ as } t \to \infty. \tag{10}$$

In fact, since any two periodic solutions of (E^c_0) differ by an additive constant (cf. Lemma 2), it follows that

$$q_m(t) = q(t) + c_m, \ m = 1, 2, ...,$$

for a fixed periodic solution $q(t)$ of (E^c_0), where (c_m) is a sequence in H. Thus, (10) becomes

$$u_m(t) - q(t) \to c_m \text{ as } t \to \infty, \tag{11}$$

weakly in H. Moreover,

$$\frac{dq(t)}{dt} + A(q(t) + c_m) \ni f(t). \tag{12}$$

On the other hand, it is easy to see that, for all $m < r$, we have

$$\|[u_m(t) - q(t)] - [u_r(t) - q(t)]\| = \|u_m(t) - u_r(t)\| \leq \|u(0) - u(0)\| + \int_m^r \|g(t)\|\, dt. \tag{13}$$

Therefore, taking the limit as $t \to \infty$, it follows (see (11)),

$$\|c_m - c_r\| \leq \int_m^r \|g(t)\|\, dt, \tag{14}$$

which shows that (c_m) is a convergent sequence, i.e., there exists a point $a \in H$, such that

$$\|c_m - a\| \to 0, \quad \text{as } m \to \infty. \tag{15}$$

Since A is maximal monotone (hence demiclosed), we can pass to the limit in (12), as $m \to \infty$, to deduce that $\omega(t) := q(t) + a$ is a solution of (E^c_0) (which is T-periodic). Note also that

$$\|u(t) - u_m(t)\| \leq \int_m^t \|g(s)\|\, ds \leq \int_m^\infty \|g(s)\|\, ds, \ t \geq m. \tag{16}$$

In order to conclude, we use the decomposition

$$u(t) - \omega(t) = [u(t) - u_m(t)] + [u_m(t) - q_m(t)] + [q_m(t) - \omega(t)] = [u(t) - u_m(t)] + [u_m(t) - q(t) - c_m] + [(q(t) + c_m) - (q(t) + a)],$$

which shows that $u(t) - \omega(t)$ converges weakly to zero, as $t \to \infty$ (cf. (11), (15), (16)). In other words, $u(t)$ is asymptotically periodic with respect to the weak topology of H. $\square$

Remark 2. It is well known that, even in the case $g \equiv 0$, the above result (Theorem 3) is not valid for a general maximal monotone operator A, so we cannot expect more in our case.

We close our Note with an example which was discussed in [4] in the particular case $g_n = 0, \ n = 0, 1, \ldots$.

Example. Let $H = \mathbb{R}, \ A = \partial\varphi = \varphi', \ \varphi : \mathbb{R} \to \mathbb{R}$,

$$\varphi(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \leq 0, \\ 0 & \text{if } x > 0, \end{cases}$$

$c_n = 1$ and $g_n = \frac{1}{n(n+1)}$ for all $n = 0, 1, \ldots$. If (f_n) is the 3-periodic sequence defined by $f_{3k} = -3$, $f_{3k+1} = 3$, $f_{3k+2} = -1$ for $k = 0, 1, \ldots$, then equation (E$_0$) has a 3-periodic solution (ω_n), $\omega_{3k} = 1$, $\omega_{3k+1} = -1$, $\omega_{3k+2} = 2$, $k = 0, 1, \ldots$. It turns out that this is the unique periodic solution of (E$_0$) (see [4]). By Theorem 1, every solution (u_n) of equation (E) tends asymptotically to (ω_n): $u_{3k} \to 1$, $u_{3k+1} \to -1$, $u_{3k+2} \to 2$ as $k \to \infty$ (see Figure 1).

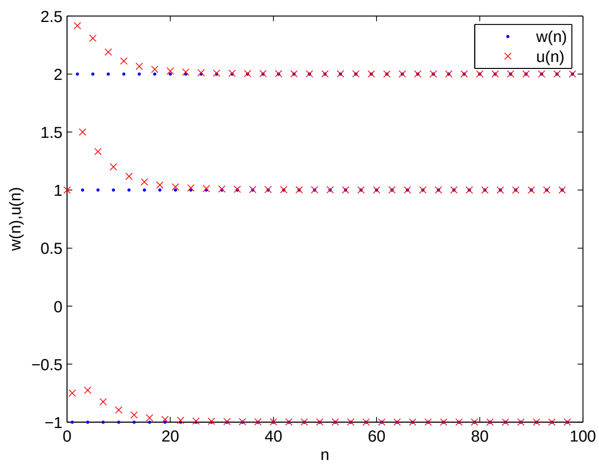
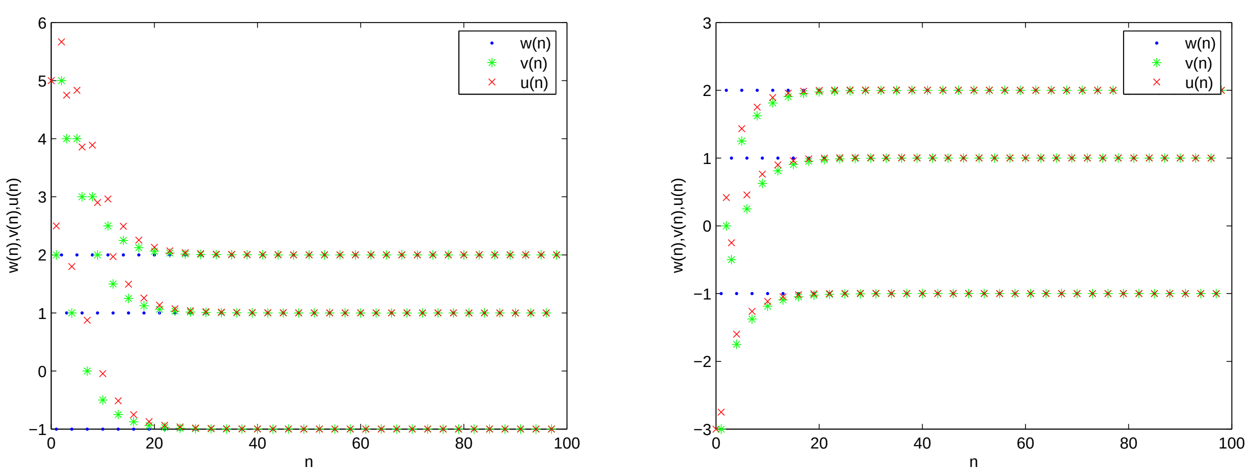


Figure 1: $u_0 = 1$

Next, we consider both the solution (v_n) of (E$_0$) and (u_n) of (E), starting from two same points, $u_0 = v_0 = 5$ and $u_0 = v_0 = -3$. Figure 2 below illustrates that they approach (ω_n).



(a) $u_0 = v_0 = 5$

(b) $u_0 = v_0 = -3$

Figure 2

References

[1] J. B. Baillon and A. Haraux, *Comportement á l'infini pour les équations d'évolution avec forcing périodique*, Archive Rat. Mech. Anal. 67 (1977), 101–109.

[2] B. Djafari Rouhani and H. Khatibzadeh, *Existence and asymptotic behaviour of solutions to first- and second-order difference equations with periodic forcing*, J. Difference Eqns Appl., DOI:10.1080/10236198.2012.658049.

[3] G. Moroşanu, *Nonlinear Evolution Equations and Applications*, D.Reidel, Dordrecht–Boston–Lancaster–Tokyo, 1988.

[4] G. Moroşanu and F. Özpınar, *Periodic forcing for some difference equations in Hilbert spaces*, Bull. Belgian Math. Soc. (Simon Stevin), to appear.