# Asymptotically periodic solutions of some difference and differential inclusions in Hilbert spaces

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**Abstract**

Let $H$ be a real Hilbert space and let $A : D(A) \subset H \to H$ be a (possibly multivalued) maximal monotone operator. Consider the difference inclusion $\Delta u_n + c_n A u_{n+1} \ni f_n + g_n,\ n = 0, 1, ...,$ where $(c_n) \subset (0, +\infty)$, $(f_n) \subset H$ are $p$-periodic sequences for a positive integer $p$ and $(g_n) \in \ell^1(H)$. We investigate the weak or strong convergence of solutions to $p$-periodic sequences. We also investigate the existence of asymptotically periodic solutions of the continuous analogue of the above difference inclusion $u'(t) + Au(t) \ni f(t) + g(t),\ t > 0$, where $f \in L^2_{loc}(\mathbb{R}_+, H)$ is a $T$-periodic function $(T > 0)$ and $g \in L^1(\mathbb{R}_+, H)$. We show that the previous results due to J. B. Baillon, A. Haraux (1977) and B. Djafari Rouhani, H. Khatibzadeh (2012) corresponding to $g \equiv 0$, respectively $g_n = 0,\ n = 0, 1, ...,$ remain valid for $g \in L^1(\mathbb{R}_+, H)$, respectively $(g_n) \in l^1(H)$.

**Keywords**: difference inclusion; differential inclusion; maximal monotone operator; subdifferential; weak convergence; strong convergence.

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## 1 Introduction

Let $H$ be a real Hilbert space with inner product $(\cdot, \cdot)$ and the induced Hilbertian norm $\|\cdot\|$. Let $A : D(A) \subset H \to H$ be a (possibly multivalued) maximal monotone operator. Consider the difference equation (inclusion)

$$\Delta u_n + c_n A u_{n+1} \ni f_n + g_n, \qquad n = 0, 1, ..., \tag{E}$$

where $(c_n) \subset (0, +\infty)$, $(f_n) \subset H$ are $p$-periodic sequences for a positive integer $p$, $(g_n) \in \ell^1(H) := \{u = (u_1, u_2, \ldots) : \sum_{n=1}^{\infty} \|u_n\| < \infty\}$ and $\Delta$ is the difference operator defined as usual, i.e., $\Delta u_n = u_{n+1} - u_n$. We shall investigate the weak or strong convergence of solutions to $p$-periodic sequences.
Consider also the following differential equation (inclusion)

$$\frac{du}{dt} + Au(t) \ni f(t) + g(t), \qquad t > 0, \tag{E$^c$}$$

where $f \in L^2_{loc}(\mathbb{R}_+, H)$ is a $T$-periodic function for a given $T > 0$ and $g \in L^1(\mathbb{R}_+, H)$. We shall investigate the behavior at infinity of solutions to (E$^c$). More precisely, in this Note we show that the previous results due to Baillon, Haraux [1] and Djafari Rouhani, Khatibzadeh [2] related to the equations (inclusions),

$$\frac{du}{dt} + Au(t) \ni f(t), \qquad t > 0, \tag{E$_0^c$}$$

and

$$\Delta u_n + c_n A u_{n+1} \ni f_n, \qquad n = 0, 1, ..., \tag{E$_0$}$$

respectively, remain valid for (E$^c$) and (E), where $g \in L^1(\mathbb{R}_+, H)$ and $(g_n) \in l^1(H)$.

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## 2 Previous Results

To obtain our main results we recall the following theorems on the existence of asymptotically periodic solutions of the equations $(\mathrm{E}_0)$ and $(\mathrm{E}_0^c)$.

**Lemma 1** ([2]; see also [4] for a different shorter proof)**.** *Assume that $A : D(A) \subset H \to H$ is a maximal monotone operator. Let $c_n > 0$ and $f_n \in H$ be p-periodic sequences, i.e., $c_{n+p} = c_n$, $f_{n+p} = f_n$ $(n = 0, 1, ...)$, for a given positive integer p. Then equation $(\mathrm{E}_0)$ has a bounded solution if and only if it has at least one p-periodic solution. In this case all solutions of $(\mathrm{E}_0)$ are bounded and for every solution $(u_n)$ of $(\mathrm{E}_0)$ there exists a p-periodic solution $(\omega_n)$ of $(\mathrm{E}_0)$ such that*

$$u_n - \omega_n \to 0, \text{ weakly in } H, \text{ as } n \to \infty.$$

*Moreover, every two periodic solutions differ by an additive constant vector.*

**Lemma 2** ([1]; see also [3], p. 169)**.** *Assume that $A$ is the subdifferential of a proper, convex, and lower semicontinuous function $\varphi : H \to (-\infty, +\infty]$, $A = \partial\varphi$. Let $f \in L^2_{loc}(\mathbb{R}_+, H)$ be a $T$-periodic function (for a given $T > 0$). Then, equation $(\mathrm{E}_0^c)$ has a solution bounded on $\mathbb{R}_+$ if and only if it has at least a $T$-periodic solution. In this case all solutions of $(\mathrm{E}_0^c)$ are bounded on $\mathbb{R}_+$ and for every solution $u(t)$, $t \ge 0$, there exists a $T$-periodic solution $q$ of $(\mathrm{E}_0^c)$ such that*

$$u(t) - q(t) \to 0, \text{ as } t \to \infty,$$

*weakly in $H$.*
*Moreover, every two periodic solutions of $(\mathrm{E}_0^c)$ differ by an additive constant, and*

$$\frac{du_n}{dt} \to \frac{dq}{dt}, \text{ as } n \to \infty,$$

*strongly in $L^2(0, T; H)$, where $u_n(t) = u(t + nT)$, $n = 1, 2, ...$*

## 3 New Results on Asymptotically Periodic Solutions

We begin this section with the following result regarding the discrete case, which is an extension of Lemma 1.

**Theorem 1.** *Assume that $A : D(A) \subset H \to H$ is a maximal monotone operator. Let $(g_n) \in \ell^1(H)$ and let $c_n > 0$, $f_n \in H$ be p-periodic sequences, i.e., $c_{n+p} = c_n$, $f_{n+p} = f_n$ $(n = 0, 1, ...)$, for a given positive integer p. Then equation $(\mathrm{E})$ has a bounded solution if and only if equation $(\mathrm{E}_0)$ has at least one p-periodic solution. In this case all solutions of $(\mathrm{E})$ are bounded and for every solution $(u_n)$ of $(\mathrm{E})$ there exists a p-periodic solution $(\omega_n)$ of $(\mathrm{E}_0)$ such that*

$$u_n - \omega_n \to 0, \text{ weakly in } H, \text{ as } n \to \infty.$$

*Proof.* Consider the initial condition

$$u_0 = x, \tag{IC}$$

for a given $x \in H$. We can rewrite equation (E) in the form:

$$u_{n+1} - u_n + c_n A u_{n+1} \ni f_n + g_n.$$

The solution of the problem (E)-(IC) is calculated successively from

$$u_{n+1} = \big(I + c_n A\big)^{-1}\big(u_n + f_n + g_n\big), \qquad n = 0, 1, \ldots,$$

in a unique manner, which will give a unique solution $(u_n)_{n\ge 0}$.
If a solution $(u_n)$ of (E) is bounded, then any other solution $(\tilde{u}_n)$ of (E) is bounded, because

$$\|u_n - \tilde{u}_n\| \le \|u_0 - \tilde{u}_0\| \qquad \forall n = 0, 1, \ldots \tag{1}$$

If a solution $(u_n)$ of (E) is bounded, then any solution $(v_n)$ of $(\mathrm{E}_0)$ is bounded and conversely, because

$$\|u_n - v_n\| \leq \|u_0 - v_0\| + \sum_{k=0}^{n-1} \|g_k\| \leq \|u_0 - v_0\| + \sum_{k=0}^{\infty} \|g_k\| < \infty.$$

According to Lemma 1 the first part of the theorem is proved. For the second part we define $(g_{n,m})_{n,m\geq 0}$ as follows:

$$g_{n,m} = \begin{cases} g_n & \text{if } n < m, \\ 0 & \text{if } n \geq m. \end{cases}$$

Let $(z_n)$ be an arbitrary solution of (E) (which is bounded). For each $m = 0, 1, \ldots$ denote by $(z_{n,m})_{n\geq 0}$ the (unique) solution of the problem

$$z_{n+1,m} - z_{n,m} + c_n A z_{n+1,m} \ni f_n + g_{n,m} \tag{$\mathrm{E}_m$}$$

$$z_{0,m} = z_0. \tag{$\mathrm{IC}_m$}$$

Note that $(z_{n,m})_{n\geq m}$ is a solution of equation $(\mathrm{E}_0)$. By Lemma 1 there is a $p$-periodic (with respect to $n$) solution $(\omega_{n,m})$ of $(\mathrm{E}_0)$ such that

$$z_{n,m} - \omega_{n,m} \to 0, \text{ weakly in } H, \text{ as } n \to \infty. \tag{2}$$

For each $m \geq 0$ we have

$$\begin{gathered} \omega_{1,m} - \omega_{0,m} + c_0 A\omega_{1,m} \ni f_0, \\ \omega_{2,m} - \omega_{1,m} + c_1 A\omega_{2,m} \ni f_1, \\ \vdots \\ \omega_{p,m} - \omega_{p-1,m} + c_{p-1} A\omega_{p,m} \ni f_{p-1}, \end{gathered}$$

where $\omega_{p,m} = \omega_{0,m}$. Since any two periodic solutions of $(\mathrm{E}_0)$ differ by an additive constant, we can write

$$\omega_{t,m} = \zeta_t + a_m \qquad t \in \{0, 1, \ldots, p-1\}, \tag{3}$$

where $(\zeta_t)$ is a an arbitrary but fixed periodic solution of $(\mathrm{E}_0)$, and $(a_m)_{m\geq 0}$ is a sequence in $H$. Thus

$$\begin{gathered} \zeta_1 - \zeta_0 + c_0 A(\zeta_1 + a_m) \ni f_0, \\ \zeta_2 - \zeta_1 + c_1 A(\zeta_2 + a_m) \ni f_1, \\ \vdots \\ \zeta_p - \zeta_{p-1} + c_{p-1} A(\zeta_p + a_m) \ni f_{p-1}, \end{gathered} \tag{4}$$

for all $m \geq 0$, where $\zeta_p = \zeta_0$. Also we can rewrite (2) as

$$z_{kp+t,m} \to \zeta_t + a_m, \text{ weakly in } H, \text{ as } k \to \infty, \tag{5}$$

for all $m \geq 0$ and $t \in \{0, 1, ..., p-1\}$. On the other hand, for $0 \leq m < r$, we have (cf. $(\mathrm{E}_m)$, $(\mathrm{IC}_m)$)

$$\|z_{kp+t,m} - z_{kp+t,r}\| \leq \sum_{j=m}^{r-1} \|g_j\|.$$

According to (5) this implies

$$\|a_m - a_r\| \leq \sum_{j=m}^{r-1} \|g_j\| \leq \sum_{j=m}^{\infty} \|g_j\|, \tag{6}$$

for all $0 \leq m < r$, so there exists an $a \in H$ such that

$$\|a_m - a\| \to 0, \quad \text{as } m \to \infty. \tag{7}$$

Since $A$ is maximal monotone (hence demiclosed), we can pass to the limit in (4) as $m \to \infty$ to obtain

$$\begin{gathered}\zeta_1 - \zeta_0 + c_0 A(\zeta_1 + a) \ni f_0, \\ \zeta_2 - \zeta_1 + c_1 A(\zeta_2 + a) \ni f_1, \\ \vdots \\ \zeta_p - \zeta_{p-1} + c_{p-1} A(\zeta_p + a) \ni f_{p-1},\end{gathered}$$

where $\zeta_p = \zeta_0$. So $\omega_n := \zeta_n + a$ is a $p$-periodic solution of equation ($\mathrm{E}_0$). We can also see that

$$\|z_n - z_{n,m}\| \le \|z_0 - z_{0,m}\| + \sum_{j=m}^{n-1} \|g_j\| \le \sum_{j=m}^{\infty} \|g_j\|. \tag{8}$$

Finally, for all natural $n$, we have $n = kp + t,\ t \in \{0, 1, ..., p-1\}$, and

$$z_n - \omega_n = [z_n - z_{n,m}] + [z_{n,m} - \omega_{t,m}] + [\omega_{t,m} - \omega_n]$$

$$= [z_n - z_{n,m}] + [z_{kp+t,m} - \zeta_t - a_m] + [\zeta_t + a_m - \zeta_t - a],$$

thus the conclusion of the theorem follows by (5), (7) and (8). $\square$

**Remark 1.** If, in addition, $A$ is strongly monotone, then we can easily extend Theorem 2 in [4], as follows:

**Theorem 2.** *Assume that $A : D(A) \subset H \to H$ is a maximal monotone operator, that is also strongly monotone, i.e., there is a constant $b > 0$, such that*

$$(x_1 - x_2, y_1 - y_2) \ge b\|x_1 - x_2\|^2,\ \forall\ x_i \in D(A),\ y_i \in Ax_i,\ i = 1, 2.$$

*Let $c_n > 0$ and $f_n \in H$ be $p$-periodic sequences for a given positive integer $p$ and $(g_n) \in \ell^1(H)$. Then, equation* ($\mathrm{E}_0$) *has a unique $p$-periodic solution $(\omega_n)$ and for every solution $(u_n)$ of* (E) *we have*

$$u_n - \omega_n \to 0,\ \textit{strongly in } H,\ \textit{as } n \to \infty.$$

The proof relies on arguments similar to the above ones.

**Theorem 3.** *Assume that $A : D(A) \subset H \to H$ is the subdifferential of a proper, convex, lower semicontinuous function $\varphi : H \to (-\infty, +\infty]$, $A = \partial\varphi$. Let $f \in L^2_{loc}(\mathbb{R}_+, H)$ be a $T$-periodic function ($T > 0$) and let $g \in L^1(\mathbb{R}_+, H)$. Then equation* ($\mathrm{E}^c$) *has a bounded solution if and only if equation* ($\mathrm{E}^c_0$) *has at least a $T$-periodic solution. In this case all solutions of* ($\mathrm{E}^c$) *are bounded on $\mathbb{R}_+$ and for every solution $u(t)$ of* ($\mathrm{E}^c$) *there exists a $T$-periodic solution $\omega(t)$ of* ($\mathrm{E}^c_0$) *such that*

$$u(t) - \omega(t) \to 0,\ \textit{weakly in } H,\ \textit{as } t \to \infty.$$

*Proof.* If a solution $u(t)$, $t \ge 0$, of equation ($\mathrm{E}^c$) is bounded on $\mathbb{R}_+$, then any other solution $\tilde{u}(t)$, $t \ge 0$, of equation ($\mathrm{E}^c$) is bounded too, because

$$\|u(t) - \tilde{u}(t)\| \le \|u(0) - \tilde{u}(0)\|. \tag{9}$$

If a solution $u(t)$ of ($\mathrm{E}^c$) is bounded, then any solution $v(t)$ of ($\mathrm{E}^c_0$) is bounded and conversely, because

$$\|u(t) - v(t)\| \le \|u(0) - v(0)\| + \int_0^t \|g(s)\| ds \le \|u(0) - v(0)\| + \int_0^\infty \|g(s)\| ds < \infty, \qquad t \ge 0.$$

Thus, the first part of the theorem follows by Lemma 2. In order to prove the second part, we define $g_m : \mathbb{R}_+ \to H$ as follows:

$$g_m(t) = \begin{cases} g(t) & \text{for a.e. } t \in (0, m) \\ 0 & \text{if } t \ge m, \end{cases}$$

where $m = 1, 2, \dots$

Let $u(t)$, $t \geq 0$, be an arbitrary bounded solution of ($\mathrm{E}^c$). For each $m = 1, 2, \dots$ denote by $u_m(t)$, $t \geq 0$, the solution of the Cauchy problem

$$\frac{du_m(t)}{dt} + A(u_m(t)) \ni f(t) + g_m(t), \qquad t > 0, \tag{$\mathrm{E}^c_m$}$$

$$u_m(0) = u(0). \tag{$\mathrm{IC}^c_m$}$$

Since $u_m(t)$, $t \geq m$, is a solution of equation ($\mathrm{E}^c_0$), it follows by Lemma 2 that there is a $T$-periodic solution $q_m(t)$ of ($\mathrm{E}^c_0$), such that

$$u_m(t) - q_m(t) \to 0, \text{ weakly in } H, \text{ as } t \to \infty. \tag{10}$$

In fact, since any two periodic solutions of ($\mathrm{E}^c_0$) differ by an additive constant (cf. Lemma 2), it follows that

$$q_m(t) = q(t) + c_m, \ m = 1, 2, ...,$$

for a fixed periodic solution $q(t)$ of ($\mathrm{E}^c_0$), where $(c_m)$ is a sequence in $H$. Thus, (10) becomes

$$u_m(t) - q(t) \to c_m \text{ as } t \to \infty, \tag{11}$$

weakly in $H$. Moreover,

$$\frac{dq(t)}{dt} + A(q(t) + c_m) \ni f(t). \tag{12}$$

On the other hand, it is easy to see that, for all $m < r$, we have

$$\|[u_m(t) - q(t)] - [u_r(t) - q(t)]\| = \|u_m(t) - u_r(t)\| \leq \|u(0) - u(0)\| + \int_m^r \|g(t)\|\, dt. \tag{13}$$

Therefore, taking the limit as $t \to \infty$, it follows (see (11)),

$$\|c_m - c_r\| \leq \int_m^r \|g(t)\|\, dt, \tag{14}$$

which shows that $(c_m)$ is a convergent sequence, i.e., there exists a point $a \in H$, such that

$$\|c_m - a\| \to 0, \quad \text{as } m \to \infty. \tag{15}$$

Since $A$ is maximal monotone (hence demiclosed), we can pass to the limit in (12), as $m \to \infty$, to deduce that $\omega(t) := q(t) + a$ is a solution of ($\mathrm{E}^c_0$) (which is $T$-periodic). Note also that

$$\|u(t) - u_m(t)\| \leq \int_m^t \|g(s)\|\, ds \leq \int_m^\infty \|g(s)\|\, ds, \ t \geq m. \tag{16}$$

In order to conclude, we use the decomposition

$$u(t) - \omega(t) = [u(t) - u_m(t)] + [u_m(t) - q_m(t)] + [q_m(t) - \omega(t)] = [u(t) - u_m(t)] + [u_m(t) - q(t) - c_m] + [(q(t) + c_m) - (q(t) + a)],$$

which shows that $u(t) - \omega(t)$ converges weakly to zero, as $t \to \infty$ (cf. (11), (15), (16)). In other words, $u(t)$ is asymptotically periodic with respect to the weak topology of $H$. $\square$

**Remark 2.** It is well known that, even in the case $g \equiv 0$, the above result (Theorem 3) is not valid for a general maximal monotone operator $A$, so we cannot expect more in our case.

We close our Note with an example which was discussed in [4] in the particular case $g_n = 0, \ n = 0, 1, \dots$ .

**Example.** Let $H = \mathbb{R}, \ A = \partial\varphi = \varphi', \ \varphi : \mathbb{R} \to \mathbb{R}$,

$$\varphi(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \leq 0, \\ 0 & \text{if } x > 0, \end{cases}$$

$c_n = 1$ and $g_n = \frac{1}{n(n+1)}$ for all $n = 0, 1, \ldots$. If $(f_n)$ is the 3-periodic sequence defined by $f_{3k} = -3,\ f_{3k+1} = 3,\ f_{3k+2} = -1$ for $k = 0, 1, \ldots$, then equation $(\mathrm{E}_0)$ has a 3-periodic solution $(\omega_n)$, $\omega_{3k} = 1,\ \omega_{3k+1} = -1,\ \omega_{3k+2} = 2,\ k = 0, 1, \ldots$. It turns out that this is the unique periodic solution of $(\mathrm{E}_0)$ (see [4]). By Theorem 1, every solution $(u_n)$ of equation (E) tends asymptotically to $(\omega_n)$: $u_{3k} \to 1,\ u_{3k+1} \to -1,\ u_{3k+2} \to 2$ as $k \to \infty$ (see Figure 1).

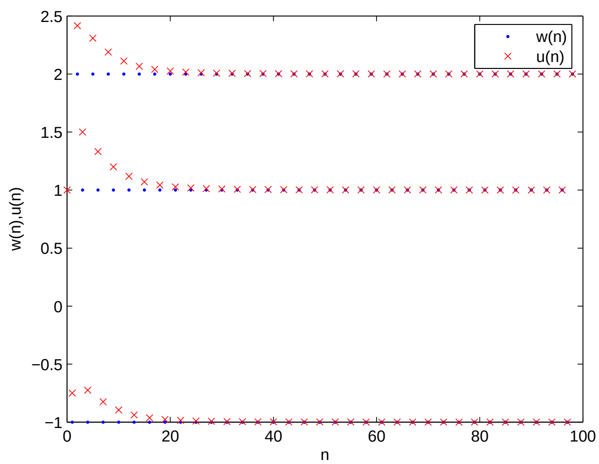
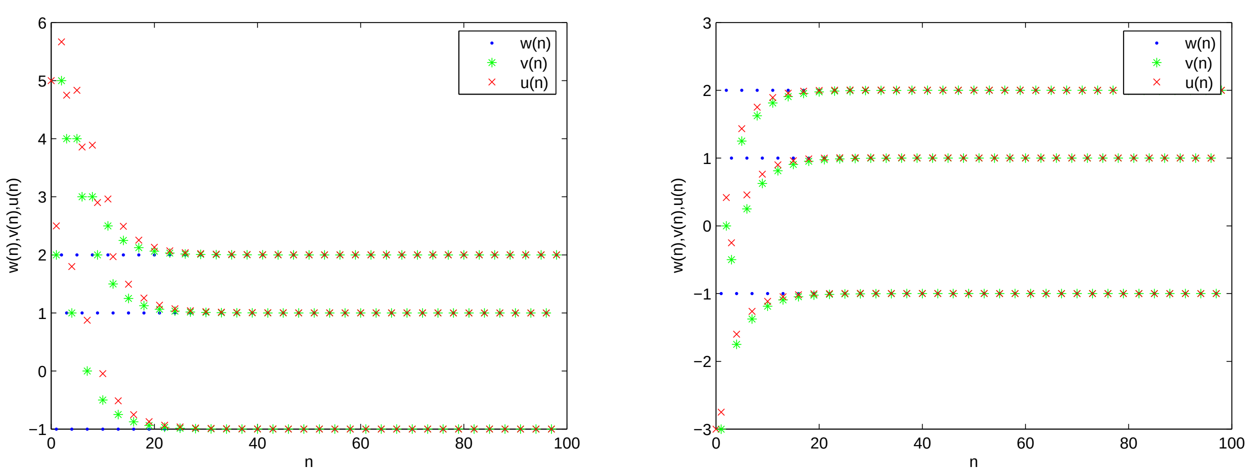



Figure 1: $u_0 = 1$

Next, we consider both the solution $(v_n)$ of $(\mathrm{E}_0)$ and $(u_n)$ of (E), starting from two same points, $u_0 = v_0 = 5$ and $u_0 = v_0 = -3$. Figure 2 below illustrates that they approach $(\omega_n)$.




(a) $u_0 = v_0 = 5$

(b) $u_0 = v_0 = -3$

Figure 2

## References

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