Modified Rockafellar's algorithms

O. A. Boikanyo and G. Moroşanu

Department of Mathematics and its Applications

Central European University

Nador u. 9, H-1051 Budapest, Hungary

E-mail addresses: boikanyo_oganeditse@phd.ceu.hu / boikanyoa@yahoo.co.uk, morosanug@ceu.hu

Abstract

In this paper, proximal point algorithms for nonexpansive (sequences of nonexpansive) maps and maximal monotone operators are studied. A modification of Xu's algorithm is given and a strong convergence result associated with it is proved when the error sequence is in ℓ^p for $1 \leq p < 2$. We also propose some other modifications of the celebrated Rockafellar's algorithm which generate weak or strong convergent sequences.

Keywords: Proximal point algorithms, monotone operator, nonexpansive map, weak and strong convergence.

2000 Mathematics Subject Classification: 47J25, 47H05, 47H09.

1 Introduction

In what follows, H will be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and the Hilbertian norm $\| \cdot \|$. Recall that a mapping (possibly multivalued) $A : D(A) \subset H \to H$ is said to be a monotone operator if

$$\langle x - x', y - y' \rangle \geq 0, \quad \forall\, (x, y), (x', y') \in A.$$

Viewed as a subset of $H \times H$, a monotone operator A is said to be maximal monotone if it is not properly contained in any other monotone operator on H. According to Minty's result, this is equivalent to saying that the map $I + \lambda A$ is a surjection for some $\lambda > 0$ (equivalently for all $\lambda > 0$), where I denotes the identity operator on H. If A is maximal monotone, and $\lambda > 0$, then the operator $J_\lambda : H \to D(A)$ defined by $J_\lambda(x) = (I + \lambda A)^{-1}(x)$ is called the resolvent of A. It is well known that J_λ is single-valued and nonexpansive; meaning that for all $x, y \in H$, $\|J_\lambda(x) - J_\lambda(y)\| \leq \|x - y\|$. The *Yosida approximation* $A_\lambda : H \to H$ can also be defined by the formula $A_\lambda(x) = \lambda^{-1}(I - J_\lambda)(x)$ for $x \in H$. This operator is Lipschitz continuous with constant λ^{-1}, that is, for $x, y \in H$, $\|A_\lambda(x) - A_\lambda(y)\| \leq \lambda^{-1}\|x - y\|$. From the literature, we know that $A_\lambda(x) \in A(J_\lambda(x))$ for all $x \in H$, and $\|A_\lambda(x)\| \leq |A(x)|$ for all $x \in D(A)$, where $|A(x)| = \inf\{\|y\| : y \in A(x)\}$.

Notation: We adopt the notation: For a given sequence $\{x_n\}$, $s - \lim_{n\to\infty} x_n$ denotes the strong limit of $\{x_n\}$ and $\omega_{\mathtt{w}}(\{x_n\})$ will denote the weak ω-limit set of $\{x_n\}$, that is,

$$\omega_{\mathtt{w}}(\{x_n\}) := \{x \in H \mid x_{n_k} \rightharpoonup x \text{ for some subsequence } \{x_{n_k}\} \text{ of } \{x_n\}\}.$$

Here "$\rightharpoonup$" denotes weak convergence.

Denote $F := A^{-1}(0) = F(J_\lambda)$, the set of all fixed points of J_λ for all $\lambda > 0$, and for a given function f, the set of all its fixed points will be denoted by $F(f)$. If $\varphi : H \to (-\infty, +\infty]$ is a proper and convex function, then the subdifferential of φ is the operator (possibly multivalued) $\partial\varphi : H \to H$ defined by

$$\partial\varphi(x) = \{w \in H \mid \varphi(x) - \varphi(v) \leq \langle w, x - v\rangle, \quad \forall\, v \in H\}.$$

An interesting and important topic in nonlinear analysis and convex optimization concerns the following problem:

$$\text{find an } x \text{ such that } 0 \in A(x), \tag{1}$$

where A is a maximal monotone operator. In fact many problems that involve convexity can be formulated as finding the zeros of maximal monotone operators. In particular, $A = \partial\varphi$, where φ is a proper, convex and lower semi-continuous function, is a maximal monotone operator and a point $p \in H$ minimizes φ if and only if $0 \in \partial\varphi(p)$. One method for finding the zeros of (1), devised by Rockafellar [5], is the proximal point algorithm (PPA), which starts at an arbitrary initial point $x_0 \in H$ and generates recursively a sequence of points

$$x_{n+1} = (I + \beta_n A)^{-1} x_n + e_n,\ \forall\, n \geq 0, \tag{2}$$

where $\{\beta_n\} \subset (0, \infty)$ and $\{e_n\}$ is considered to be the error sequence. He proved the weak convergence of the sequence $\{x_n\}$ defined by (2) above, provided the sequence $\{\beta_n\}$ is bounded away from zero and the error sequence $\{e_n\}$ is summable. From the numerical point of view, weak convergence is not enough for an efficient algorithm. In this regard, Rockafellar's open question of whether the PPA converges strongly in general, became of central importance. Güler [3] constructed an example in ℓ^2 showing that there exists a proper, convex and lower semi-continuous function φ such that given any bounded positive sequence $\{\beta_n\}$, there exists a point $x_0 \in D(\varphi)$ for which the PPA given by $x_{n+1} = (I + \beta_n A)^{-1} x_n$ and starting at x_0 converges weakly, but not strongly to a minimizing point of φ. Knowing that the PPA does not converge strongly in general motivated the following question: Can we modify Rockafellar's algorithm in such a way that strong convergence is guaranteed? Several attempts have been made in this direction by different authors, among them, Solodov and Svaiter [6], and Xu [8]. In particular, Xu's algorithm requires that the error sequence be summable, which is too strong from the computational point of view. Again we ask: Can we replace the summability of $\{e_n\}$ by a weaker condition and still get strong convergence results? We answer affirmatively this question (see Section

3).

In a recent paper of Takahashi [7], a strong convergence theorem of a modified PPA was proved for resolvents of accretive operators in Banach spaces by the so called viscosity approximation method without taking into account the error terms. The $(n+1)$th iterate was given as

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)(I + \beta_n A)^{-1}(x_n),$$

where $f : C \to C$ is a strict contraction (a-contraction with $0 < a < 1$) defined on a nonempty closed convex subset C of a reflexive Banach space. In a Hilbert space setting, an analogue of the above mentioned theorem can also be proved even when one takes into account the error terms in ℓ^1. The result will in turn be a generalization of Xu's Theorem 5.1 [8]. Motivated by Takahashi's result, we explore the case when $f : H \to H$ is a nonexpansive map. In fact, we consider more, namely the case when $f := f_n$ is any sequence of nonexpansive maps. More precisely, we propose two main algorithms. In the first algorithm, the $(n+1)$th iterate takes the form

$$x_{n+1} = \alpha_n f_n(x_n) + (1 - \alpha_n)(I + \beta_n A)^{-1}(x_n) + e_n. \tag{3}$$

As expected, our algorithm as given by (3) guarantees weak convergence result. What is more, for a special choice of f_n, strong convergence is obtained. Even more, we show that one can always find a sequence of regularization parameters such that strong convergence is obtained. This leads to a new algorithm whose $(n+1)$th iterate is given by

$$x_{n+1} = \alpha_n (I + \lambda_n A)^{-1}(x_n) + (1 - \alpha_n) f_n(x_n) + e_n. \tag{4}$$

Another algorithm is defined by

$$x_{n+1} = \alpha_n (I + \lambda_n A)^{-1} u + (1 - \alpha_n) f_n(x_n) + e_n, \tag{5}$$

where u is any point of H (not necessarily the starting point x_0 of the PPA). Of course the algorithm given by (4) is just the relaxed form of this algorithm. It is then shown that the trajectory of the PPA given by algorithm 5 and starting at any point x_0 approaches a zero of A (that is, a fixed point of J_{λ_n} for all $\lambda_n > 0$) if λ_n grows large without bound as n does. Our results generalize, improve and extend the results in Section 5 of [8].

2 Remarks on the PPA

Throughout this paper $A : D(A) \subset H \to H$ is assumed to be a maximal monotone operator. We begin this section with a result that says that the boundedness of a sequence generated by the PPA necessarily implies the existence of zeros of A provided $\{\beta_n\} \notin \ell^1$.

Theorem 1 *Assume $e_n = 0$ for $n = 0, 1, \ldots,$ and $\sum_{n=0}^{\infty} \beta_n = \infty$. If there exists $x_0 \in H$ such that the sequence $\{x_n\}$ generated by (2) is bounded, then $F := A^{-1}(0)$ is nonempty.*

Proof:

Let $x_0 \in H$ be such that the sequence $\{x_n\}$ generated by (2) is bounded. Let $(x, y) \in A$. We have

$$x_{n+1} - x + \beta_n(Ax_{n+1} - y) + \beta_n y \ni x_n - x.$$

Multiplying this equation scalarly by $x_{n+1} - x$ yields,

$$\begin{aligned}\|x_{n+1} - x\|^2 + \beta_n\langle Ax_{n+1} - y, x_{n+1} - x\rangle + \beta_n\langle y, x_{n+1} - x\rangle &= \langle x_n - x, x_{n+1} - x\rangle \\ &\leq \frac{1}{2}\|x_n - x\|^2 + \frac{1}{2}\|x_{n+1} - x\|^2.\end{aligned}$$

Therefore,

$$\frac{1}{2}\|x_{n+1} - x\|^2 + \beta_n\langle y, x_{n+1} - x\rangle \leq \frac{1}{2}\|x_n - x\|^2.$$

By summing from $n = 0$ to $n = N$, we get,

$$\langle y, \frac{\sum_{n=0}^{N} \beta_n x_{n+1}}{\sum_{n=0}^{N} \beta_n} - x\rangle \leq \frac{\|x_0 - x\|^2}{2\sum_{n=0}^{N} \beta_n}\ . \tag{6}$$

Since $\{x_n\}$ is bounded, so is $\{w_n\}$, where

$$w_n := (\sum_{k=0}^{n} \beta_k)^{-1} \sum_{k=0}^{n} \beta_k x_{k+1}. \tag{7}$$

Let p be a weak cluster point of $\{w_n\}$. Then passing to the limit in (6), we obtain,

$$\langle y, p - x\rangle \leq 0 \tag{8}$$

for all $(x, y) \in A$ since $\sum_{n=0}^{\infty} \beta_n = \infty$. By (8) and the maximality of A, it follows that $(p, 0) \in A$, which implies that $F \neq \emptyset$. ■

Remarks:

1). Conversely, if $F \neq \emptyset$, then for all $x_0 \in H$, $\{\beta_n\} \subset (0, \infty)$, $\{e_n\} \subset H$, and $\sum_{n=0}^{\infty} \|e_n\| < \infty$, the sequence $\{x_n\}$ generated by (2) is bounded. Indeed, for $p \in F$, we have

$$\|x_{n+1} - p\| \leq \|x_n - p\| + \|e_n\|,$$

and therefore, by summing from $n = 0$ to $n = N$, we get

$$\|x_{N+1} - p\| \leq \|x_0 - p\| + \sum_{n=0}^{N} \|e_n\| \leq \|x_0 - p\| + \sum_{n=0}^{\infty} \|e_n\| < \infty.$$

If in addition, $\sum_{n=0}^{\infty} \beta_n = \infty$, then the average w_n defined by (7) converges weakly to some point $p \in F$, (see [4], p. 139).

If we assume more, that is, $\sum_{n=0}^{\infty} \beta_n^2 = \infty$, then the sequence $\{x_n\}$ itself converges weakly to some point $p \in F$, (see [4], p. 142).

2). In the case when $A = \partial\varphi$ where φ is proper, convex and lower semi-continuous, the weaker additional condition $\sum_{n=0}^{\infty} \beta_n = \infty$ is enough to ensure weak convergence of $\{x_n\}$, (see again [4], p. 142).

The interested reader is referred to Remark 3.1 ([4], p. 145) to see what happens for the case when $\{\beta_n\} \in \ell^1$.

Recall that an operator $A : D(A) \subset H \to H$ is said to be coercive if it satisfies the following condition

$$\lim_{\|\xi\|\to\infty,\ (\xi,\eta)\in A} \frac{(\eta, \xi - v_0)}{\|\xi\|} = \infty, \tag{9}$$

for some $v_0 \in H$.

We know that if A is coercive, then its range $R(A) = H$, thus $F \neq \emptyset$, (see for example [4]). We now show that if $\{e_n\}$ is bounded and $\{\beta_n\}$ is bounded below away from zero, then the sequence $\{x_n\}$ generated by (2) is bounded, provided A is assumed to be coercive.

Theorem 2 *Assume A is coercive and maximal monotone. Let $\|e_n\| \leq C$ and $\beta_n \geq \varepsilon > 0$ for $n \geq 0$, where C and ε are given constants. Then for any $x_0 \in H$ the sequence $\{x_n\}$ generated by (2) is bounded.*

Proof: The proof is essentially done in, [4, p. 152]. We just adapt the old proof to the present framework.

Since A is coercive, the set $F := A^{-1}(0)$ is nonempty. Now setting $u_n := x_n - e_{n-1}$, (2) becomes

$$u_n + e_{n-1} \in u_{n+1} + \beta_n A(u_{n+1}), \quad \text{for} \quad n \geq 1,$$

which implies that

$$\|u_{n+1} - p\|^2 \leq \langle u_n - p, u_{n+1} - p\rangle + \langle e_{n-1}, u_{n+1} - p\rangle, \quad \text{for} \quad n \geq 1,$$

for every $p \in F$. Therefore,

$$\|u_{n+1}\| \leq \|u_n\| + C + 2dist(0, F), \quad \text{for} \quad n \geq 1. \tag{10}$$

Denote $C_1 := C + 2\|v_0\|$. By (9), there exists $K > 0$ such that

$$(\xi, \eta) \in A, \quad \|\xi\| > K \quad \text{implies} \quad \frac{(\eta, \xi - v_0)}{\|\xi - v_0\|} \geq \frac{C_1}{\varepsilon}. \tag{11}$$

Suppose that there exists k such that $\|u_{k+1}\| > K$. Then multiplying

$$u_k - v_0 + e_{k-1} \in u_{k+1} - v_0 + \beta_k A(u_{k+1})$$

by $(u_{k+1} - v_0)/\|u_{k+1} - v_0\|$, where v_0 is the vector associated with the coercivity of A, and making use of (11), we get,

$$\|u_{k+1} - v_0\| + C_1 \leq \|u_k - v_0\| + \|e_{k-1}\| \leq \|u_k\| + \|v_0\| + C,$$

or

$$\|u_{k+1}\| \leq \|u_{k+1} - v_0\| + \|v_0\| \leq \|u_k\| + 2\|v_0\| + C - C_1,$$

which implies that

$$\|u_{k+1}\| \leq \|u_k\|.$$

Therefore we have

$$\|u_{n+1}\| \leq \max\{K + C + 2dist(0, F), \|u_n\|\} \quad \text{for} \quad n \geq 1. \tag{12}$$

Setting $\rho_n = \max\{K + C + 2dist(0, F), \|u_n\|\}$, we deduce from (12) that

$$\rho_{n+1} \leq \rho_n, \quad \text{for} \quad n \geq 1.$$

Hence

$$\|x_n\| \leq \|u_n\| + C \leq \rho_n + C \leq \max\{K + C + 2dist(0, F), \|x_1 - e_0\|\} + C, \quad \text{for} \quad n \geq 1,$$

showing that $\{x_n\}$ is bounded. ∎

Remark: If A is maximal monotone, strongly monotone (hence coercive), $\beta_n \to \infty$, $\|e_n\| \to 0$, then for any $x_0 \in H$ the sequence $\{x_n\}$ generated by (2) converges strongly to $p = A^{-1}(0)$. Indeed, if we denote $u_n := x_n - e_{n-1}$, and on multiplying

$$u_n - p + e_{n-1} \in u_{n+1} - p + \beta_n A(u_{n+1})$$

scalarly by $u_{n+1} - p$ yields,

$$(1 + c\beta_n)\|u_{n+1} - p\|^2 \leq \|u_n - p\|\|u_{n+1} - p\| + \|e_{n-1}\|\|u_{n+1} - p\|,$$

where c is the strong monotonicity constant. Therefore if we set $M := \sup_n\{\|u_n - p\| + \|e_{n-1}\|\}$, then we have

$$\|u_{n+1} - p\| \leq \frac{M}{(1 + c\beta_n)} \to 0.$$

Hence $\{x_n\}$ converges strongly to p.

When A is the subdifferential, coercivity of A is equivalent to the conditions given in the following result which is due to Brézis [2, p. 42] (see also [1, p. 56]).

Proposition 1 [2] *Let $\varphi : H \to (-\infty, +\infty]$ be a proper, convex, lower semi-continuous function, and let $A = \partial\varphi$. Then the following conditions are equivalent;*

(i) $\displaystyle\lim_{\|\xi\|\to\infty,\, (\xi,\eta)\in A} \frac{\varphi(\xi)}{\|\xi\|} = \infty$;

(ii) *A is coercive, that is, there exists a $v_0 \in H$ such that*

$$\lim_{\|\xi\|\to\infty,\, (\xi,\eta)\in A} \frac{(\eta, \xi - v_0)}{\|\xi\|} = \infty \ ;$$

(iii) *$R(A) = H$ and A^{-1} is bounded.*

We remark that coercivity of $A = \partial\varphi$ is stronger than the condition

$$\lim_{\|\xi\|\to\infty} \varphi(\xi) = \infty. \tag{13}$$

In the case of the subdifferential, Theorem 2 can be proved under the weaker coercivity condition (13). More precisely, we have:

Theorem 3 *Assume $A = \partial\varphi$, where $\varphi : H \to (-\infty, \infty]$ is a proper, convex and lower semicontinuous function satisfying condition (13). Let $\sum_{n=0}^{\infty} \|e_n\|^2 < \infty$ and $\beta_n \geq \varepsilon > 0$ for a given constant ε and $n \geq 0$. Then for any $x_0 \in H$ the sequence $\{x_n\}$ generated by (2) is bounded.*

Proof:

According to Theorem 1.10 of [4], there exists a $p \in D(\varphi)$ such that $\varphi(p) = \inf_{x\in H} \varphi(x)$, that is, $F := A^{-1}(0)$ is nonempty. Denote $u_n := x_n - e_{n-1}$. Then by definition of the subdifferential and on multiplying

$$u_n + e_{n-1} \in u_{n+1} + \beta_n A(u_{n+1})$$

scalarly by $u_{n+1} - u_n$ yields

$$\begin{aligned} \|u_{n+1} - u_n\|^2 + \beta_n(\varphi(u_{n+1}) - \varphi(u_n)) &\leq \|u_{n+1} - u_n\|^2 + \beta_n\langle Au_{n+1}, u_{n+1} - u_n\rangle \\ &= \langle e_{n-1}, u_{n+1} - u_n\rangle \leq \frac{1}{2}\|e_{n-1}\|^2 + \frac{1}{2}\|u_{n+1} - u_n\|^2, \end{aligned}$$

for $n \geq 1$. Therefore,

$$\frac{1}{2}\|u_{n+1} - u_n\|^2 + \beta_n(\varphi(u_{n+1}) - \varphi(u_n)) \leq \frac{1}{2}\|e_{n-1}\|^2 \quad \text{for} \quad n \geq 1,$$

which implies that

$$\varphi(u_{n+1}) \leq \varphi(u_n) + \frac{1}{2\varepsilon}\|e_{n-1}\|^2 \quad \text{for} \quad n \geq 1.$$

By summing,

$$\varphi(u_{n+1}) \leq \varphi(u_1) + \frac{1}{2\varepsilon}\sum_{j=0}^{n-1} \|e_j\|^2 \leq \varphi(u_1) + \frac{1}{2\varepsilon}\sum_{j=0}^{\infty} \|e_j\|^2 < \infty.$$

It follows from (13) that $\{u_n\}$ is bounded, and so is $\{x_n\}$, since $e_n \to 0$. ■

Remark: According to Theorem 3.6 of [4], we have under the assumptions of Theorem 3,

$$\varphi(u_{n+1}) = \varphi(x_{n+1} - e_n) \to \inf_{x\in H} \varphi(x).$$

This implies that every weak limit point of $\{x_n\}$ belongs to $F := A^{-1}(0)$ (the set of all minimum points of φ). Therefore if F is singletone (which happens if, for example, φ is strictly convex), then

$$x_n \rightharpoonup p,$$

where p is the (unique) minimum point of φ. If in addition, we assume that the "level sets" $\{v \in H : \varphi(v) \leq \lambda\}$, $\lambda \in \mathbf{R}$, are compact (which happens in many cases), then we have strong convergence: $\|x_n - p\| \to 0$.

3 Remarks on Xu's algorithm

In the present section we consider the following algorithm which is a slight modification of Xu's algorithm 5.1 [8].

Algorithm 1 Let A be a maximal monotone operator.
Step 1. Choose $x_0, u \in H$ arbitrarily.
Step 2. For each $n \geq 0$, choose a regularization parameter $\beta_n > 0$ and compute

$$y_n = (I + \beta_n A)^{-1}(x_n) + e_n. \tag{14}$$

Step 3. For each $n \geq 0$, choose the relaxation parameter $\alpha_n \in (0,1)$ and compute the $(n+1)$th iterate:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)y_n + e'_n, \tag{15}$$

where $\{e_n\}$ and $\{e'_n\}$ can be interpreted as sequences of computational errors. Here e_n is considered to be the "main error" while e'_n is a "smaller error". More precisely, we assume that there exists $K > 0$ such that $\|e'_n\| \leq K\|e_n\|$, for all $n \geq 0$.

Note that if the sequences $\{e_n\}$ and $\{x_n\}$ generated by the above algorithm are bounded, then $F := A^{-1}(0)$ is nonempty, provided $\beta_n \to \infty$.

Indeed, if we denote $u_n := y_n - e_n$, then by (15), $\{y_n\}$ is bounded, and so is $\{u_n\}$. Hence there exists a subsequence $\{u_{n_k}\}$ which converges weakly to some $p \in H$. From (14) and (15), we derive

$$u_n + \beta_n A(u_n) \ni \alpha_{n-1}u + (1-\alpha_{n-1})(u_{n-1} + e_{n-1}) + e'_n$$

which is equivalent to

$$A(u_n) \ni \frac{1}{\beta_n}(\alpha_{n-1}u - u_n + (1-\alpha_{n-1})(u_{n-1} + e_{n-1}) + e'_n). \tag{16}$$

Since A is demiclosed, $u_{n_k} \rightharpoonup p$ and the right hand side of (16) converges strongly to zero, it follows that $(p, 0) \in A$, hence $F \neq \emptyset$. We have therefore proved that:

Proposition 2 *Assume that $\beta_n \to \infty$. If the sequences $\{e_n\}$ and $\{x_n\}$ defined by algorithm 1 are bounded, then $F := A^{-1}(0)$ is nonempty.*

On the other hand, for a coercive operator A, it is immediate that the set $F := A^{-1}(0)$ is nonempty. If in addition, we assume that the sequence $\{e_n\}$ is bounded and the sequence $\{\beta_n\}$ is bounded below away from zero, then we can show that the sequence $\{x_n\}$ generated by algorithm 1 is itself bounded. We state this fact more formally in the following theorem whose proof is similar to the proof of Theorem 2.

Theorem 4 *Assume that $A : D(A) \subset H \to H$ is maximal monotone and coercive. Let $\|e_n\| \leq C$ and $\beta_n \geq \varepsilon > 0$ for $n \geq 0$, where C and ε are given constants. Then for any $x_0, u \in H$, the sequence $\{x_n\}$ generated by algorithm 1 is bounded.*

Taking $u = x_0$ and $e'_n = 0$ for all $n \geq 0$, the above algorithm reduces to algorithm 5.1 [8]. On proving strong convergence in this case, the error sequence in Theorem 5.1 [8] is required to be summable, a condition too strong for computational purposes. We now address the question of whether or not can the summability of $\{e_n\}$ be replaced by a weaker condition and still get strong convergence results. For this purpose, we recall the following lemma due to Xu [8].

Lemma 1 [8]. *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - a_n)s_n + a_n b_n + c_n, \quad n \geq 0,$$

where $\{a_n\}$, $\{b_n\}$ $\{c_n\}$ satisfy the conditions;

(i) $\{a_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} a_n = \infty$,

(ii) $\limsup_{n\to\infty} b_n \leq 0$,

(iii) $c_n \geq 0$ *for all* $n \geq 0$ *with* $\sum_{n=0}^{\infty} c_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

More often, we shall use the following inequality which is usually referred to as the sub-differential inequality.

$$\|x + y\|^2 \leq \|y\|^2 + 2\langle x, x + y\rangle \quad \text{for all} \quad x, y \in H.$$

The following theorem extends and improves Theorem 5.1 [8].

Theorem 5 *Assume that*

(i) *Either* $\{\|e_n\|\} \in \ell^1$, $\alpha_n \in (0, 1)$ *with* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) *Or* $\{\|e_n\|\} \in \ell^p \backslash \ell^1$, $p \in (1, 2)$, $\alpha_n \in (0, 1)$ *with* $\alpha_n \geq \varepsilon \|e_n\|^{2-p}$ *for some* $\varepsilon > 0$, *and* $\alpha_n \to 0$.

If A is maximal monotone and $F := A^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}$ generated by algorithm 1 converges strongly to $q = P_F u$ provided $\beta_n \to \infty$.

Proof: (The first part of the proof is analogous to the proof of Theorem 5.1 [8])

(i). Case $\{\|e_n\|\} \in \ell^1$, and $\alpha_n \in (0, 1)$ satisfying, $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. We divide the proof into steps.

Step 1: Note that for $q \in F$, we have

$$\begin{aligned} \|x_{n+1} - q\| &\leq \alpha_n \|u - q\| + (1 - \alpha_n)\|y_n - q\| + \|e'_n\| \\ &\leq \alpha_n \|u - q\| + (1 - \alpha_n)\|x_n - q\| + (1 - \alpha_n)\|e_n\| + \|e'_n\|, \end{aligned}$$

which implies that

$$\|x_{n+1} - q\| \leq \prod_{k=0}^{n}(1 - \alpha_k)\|x_0 - q\| + \left[1 - \prod_{k=0}^{n}(1 - \alpha_k)\right] \|u - q\| + \sum_{k=0}^{n}(\|e_k\| + \|e'_k\|),$$

showing that $\{x_n\}$ is bounded, and so is $\{y_n\}$.

Step 2: We want to show that $\overline{\lim}_{n\to\infty}\langle u-q, x_n-q\rangle \le 0$. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ so that

$$\overline{\lim}_{n\to\infty}\langle u-q, x_n-q\rangle = \lim_{k\to\infty}\langle u-q, x_{n_k}-q\rangle.$$

Since $\{x_n\}$ is bounded, $\{x_{n_k}\}$ converges weakly on a subsequence, again denoted by $\{x_{n_k}\}$, to some x_∞. Then it follows that

$$\overline{\lim}_{n\to\infty}\langle u-q, x_n-q\rangle = \langle u-q, x_\infty-q\rangle.$$

It remains to show that $x_\infty \in F$. Note that

$$\|x_{n+1}-y_n\| \le \alpha_n\|u-y_n\| + \|e'_n\| \le M\alpha_n + \|e'_n\| \to 0,$$

which implies that $y_{n_k-1}-e_{n_k-1} \rightharpoonup x_\infty$. On the other hand, since

$$A(y_{n_k-1}-e_{n_k-1}) \ni \frac{1}{\beta_{n_k-1}}(x_{n_k-1}-y_{n_k-1}+e_{n_k-1}) \to 0,$$

and because A is demiclosed, we have $x_\infty \in F$.

Step 3: Now we show that $\{x_n\}$ converges strongly to $q = P_F u$. Applying the subdifferential inequality, we get

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\alpha_n(u-q+e'_n) + (1-\alpha_n)(y_n-q+e'_n)\|^2 \\
&\le (1-\alpha_n)\|y_n-q+e'_n\|^2 + 2\alpha_n\langle u-q+e'_n, x_{n+1}-q\rangle \\
&\le (1-\alpha_n)(\|x_n-q\| + \|e_n\| + \|e'_n\|)^2 + 2\alpha_n\langle u-q+e'_n, x_{n+1}-q\rangle \\
&\le (1-\alpha_n)\|x_n-q\|^2 + \alpha_n b_n + c_n,
\end{aligned}$$

where $c_n = (1+K)\|e_n\|(2\|x_n-q\| + (1+K)\|e_n\|)$ with $\sum_{n=0}^\infty c_n < \infty$ and from Step 2, $\overline{\lim}_{n\to\infty} b_n \le 0$ where $b_n = 2\langle u-q+e'_n, x_{n+1}-q\rangle$. Hence it follows from Lemma 1 that $x_n \to q$.

(ii). Suppose that $\sum_{n=0}^\infty \|e_n\| = \infty$ and $\sum_{n=0}^\infty \|e_n\|^p < \infty$, for some $p \in (1,2)$. Denote $z_n := y_n - e_n$ and $\tilde{e}_n := (1-\alpha_n)e_n + e'_n$. Then we have from (14) and (15),

$$z_n = (I+\beta_n A)^{-1}(x_n) \quad \text{and} \quad x_{n+1} = \alpha_n u + (1-\alpha_n)z_n + \tilde{e}_n. \tag{17}$$

Let $p \in F$. We have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\alpha_n)^2\|z_n-p+\tilde{e}_n\|^2 + 2\alpha_n\langle u-p+\tilde{e}_n, x_{n+1}-p\rangle \\
&\le (1-\alpha_n)^2(\|x_n-p\| + \|\tilde{e}_n\|)^2 + 2\alpha_n\langle u-p+\tilde{e}_n, x_{n+1}-p\rangle \\
&\le (1-\alpha_n)^2(\|x_n-p\| + (1+K)\|e_n\|)^2 + 2M\alpha_n\|x_{n+1}-p\|,
\end{aligned}$$

where $M > 0$ is a constant such that $\|x_0-p\| \le M$, and $\|u-p\| + \|\tilde{e}_n\| \le M$ for all $n \ge 0$.

Assume that $\|e_n\|$ is small enough for all $n \geq 0$, otherwise one can consider algorithm 1 for $n \geq N$, with $x_0 := x_N$. We want to prove that for $C := 2M$, we have

$$\|x_n - p\| \leq C \quad \text{for all} \quad n \geq 0. \tag{18}$$

Inequality (18) is clearly true for $n = 0$. Assume that it is true for some n. Then, we have from the previous estimate,

$$\|x_{n+1} - p\|^2 \leq (1 - \alpha_n)^2 (C + (1 + K)\|e_n\|)^2 + 2M\alpha_n \|x_{n+1} - p\|.$$

Therefore,

$$(\|x_{n+1} - p\| - M\alpha_n)^2 \leq M^2\alpha_n^2 + (1 - \alpha_n)^2 (C + (1 + K)\|e_n\|)^2,$$

which implies that

$$\|x_{n+1} - p\| \leq M\alpha_n + \sqrt{M^2\alpha_n^2 + (1 - \alpha_n)^2 (C + (1 + K)\|e_n\|)^2}. \tag{19}$$

Now let us prove that

$$M\alpha_n + \sqrt{M^2\alpha_n^2 + (1 - \alpha_n)^2 (C + (1 + K)\|e_n\|)^2} \leq C,$$

or equivalently,

$$M^2\alpha_n^2 + (1 - \alpha_n)^2 (C + (1 + K)\|e_n\|)^2 \leq C^2 - 2MC\alpha_n + M^2\alpha_n^2,$$

or equivalently,

$$(1 - \alpha_n)(C + (1 + K)\|e_n\|)^2 \leq C^2. \tag{20}$$

Since $\alpha_n \geq \varepsilon \|e_n\|^{2-p}$, to prove (20), it suffices to show that

$$(1 - \varepsilon \|e_n\|^{2-p})(C + (1 + K)\|e_n\|)^2 \leq C^2,$$

or equivalently,

$$C^2 - \varepsilon C^2 \|e_n\|^{2-p} + 2C(1+K)\|e_n\| - 2C(1+K)\varepsilon \|e_n\|^{3-p} + (1+K)^2 \|e_n\|^2 - (1+K)^2 \varepsilon \|e_n\|^{4-p} \leq C^2,$$

or equivalently,

$$-\varepsilon C^2 + 2C(1 + K)\|e_n\|^{p-1} - 2C(1 + K)\varepsilon \|e_n\| + (1 + K)^2 \|e_n\|^p - (1 + K)^2 \varepsilon \|e_n\|^2 \leq 0,$$

which holds true because $\|e_n\|$ is small and $p > 1$. Therefore from (19) and (20) we see that (18) holds true for $n + 1$.

Step 2: As in Step 2 of the first part, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\overline{\lim}_{n\to\infty} \langle u - q, x_n - q\rangle = \langle u - q, x_\infty - q\rangle.$$

On the other hand, for any $p \in F$, we have from (17)

$$\begin{aligned}\|x_{n+1}-z_n\| &\leq \alpha_n(\|u\|+\|z_n\|)+\|\tilde{e}_n\| \leq \alpha_n(\|u\|+\|z_n-p\|)+\|\tilde{e}_n\| \\ &\leq \alpha_n(\|u\|+\|x_n-p\|)+\|\tilde{e}_n\| \leq \alpha_n M+\|\tilde{e}_n\| \to 0,\end{aligned}$$

implies that

$$z_{n_k-1}=(I+\beta_{n_k-1}A)^{-1}(x_{n_k-1}) \rightharpoonup x_\infty.$$

Therefore $(x_\infty, 0) \in A$, which implies that $x_\infty \in F$.

Step 3: Finally, we show that $\{x_n\}$ converges strongly to $q = P_F u$. We have

$$\begin{aligned}\|x_{n+1}-q\|^2 &\leq (1-\alpha_n)\|z_n-q+\tilde{e}_n\|^2+2\alpha_n\langle u-q+\tilde{e}_n, x_{n+1}-q\rangle \\ &\leq (1-\alpha_n)(\|x_n-q\|+\|\tilde{e}_n\|)^2+2\alpha_n\langle u-q+\tilde{e}_n, x_{n+1}-q\rangle \\ &\leq (1-\alpha_n)(\|x_n-q\|+(1+K)\|e_n\|)^2+\frac{\alpha_n}{2}b_n \\ &\leq (1-\alpha_n)\|x_n-q\|^2+(1+K)^2\|e_n\|^2+\frac{\alpha_n}{2}b_n+ \\ &\quad 2(1-\alpha_n)\left[\left(\sqrt{\frac{\varepsilon}{2}}\|x_n-q\|\|e_n\|^{1-\frac{p}{2}}\right)\left(\sqrt{\frac{2}{\varepsilon}}(1+K)\|e_n\|^{\frac{p}{2}}\right)\right] \\ &\leq (1-\alpha_n)\left(1+\frac{\varepsilon}{2}\|e_n\|^{2-p}\right)\|x_n-q\|^2+\frac{\alpha_n}{2}b_n+\frac{2}{\varepsilon}(1+K)^2\|e_n\|^p+ \\ &\quad (1+K)^2\|e_n\|^2, \end{aligned} \tag{21}$$

where $b_n = 4\langle u-q+\tilde{e}_n, x_{n+1}-q\rangle$ with $\overline{\lim}_{n\to\infty} b_n \leq 0$. Set $a_n := \frac{\alpha_n}{2}$. Then $\sum_{n=0}^{\infty} a_n = \infty$, $a_n \to 0$ and

$$\alpha_n = a_n + \frac{1}{2}\alpha_n \geq a_n + \frac{\varepsilon}{2}\|e_n\|^{2-p}.$$

Therefore, we have from (21)

$$\|x_{n+1}-q\|^2 \leq (1-a_n)\|x_n-q\|^2 + a_n b_n + c_n,$$

where $c_n = (1+K)^2(\|e_n\|^2 + \frac{2}{\varepsilon}\|e_n\|^p)$ with $\sum_{n=0}^{\infty} c_n < \infty$ because $1 < p < 2$ and $\sum_{n=0}^{\infty}\|e_n\|^p < \infty$ implies that $\sum_{n=0}^{\infty}\|e_n\|^2 < \infty$. Hence it follows from Lemma 1 that $\{x_n\}$ converges strongly to $q = P_F u$. ■

Example:

Let $\|e_n\| = \frac{1}{(n+1)^{2/3}}$, $e'_n = 0$ for all $n \geq 0$, $p = \frac{5}{3}$, and $\alpha_n = \|e_n\|^{2-p} = \frac{1}{(n+1)^{1/3}}$. This case is not covered in Theorem 5.1 [8]. However according to Theorem 5, $\{x_n\}$ as defined in algorithm 1 converges strongly to $q = P_F u$, if $\beta_n \to \infty$.

4 Other Modified Rockafellar's Algorithms

Let $f_n : H \to H$ be a sequence of nonexpansive maps. We define the following algorithm:
Step 1. Choose $x_0 \in H$ arbitrarily.
Step 2. For each $n \geq 0$, choose the regularization parameter $\lambda_n > 0$ and compute

$$\overline{y}_n = f_n(x_n) + e'_n \quad \text{and} \quad \overline{z}_n = (I+\lambda_n A)^{-1}u + e''_n.$$

Step 3. For each $n \geq 0$, choose the relaxation parameter $\alpha_n \in (0,1)$ and compute $(n+1)$th iterate:

$$x_{n+1} = \alpha_n \overline{z}_n + (1-\alpha_n)\overline{y}_n + e_n''',$$

where $\{e_n'\}$, $\{e_n''\}$ and $\{e_n'''\}$ are interpreted as sequences of computational errors.

The error terms e_n' and e_n'' are considered to be the "main errors" whereas e_n''' as being "smaller". While this algorithm takes into account all the possible errors at each step, its disadvantage is that it might not be so convenient to work with in theory. There is therefore a need to present it in a simplified version. Notice that the $(n+1)$th iterate can be written as

$$x_{n+1} = \alpha_n (I + \lambda_n A)^{-1} u + (1-\alpha_n) f_n(x_n) + e_n,$$

where $e_n = \alpha_n e_n'' + (1-\alpha_n)e_n' + e_n'''$. If $\{\|e_n'\|\}, \{\|e_n''\|\}, \{\|e_n'''\|\} \in \ell^p$ for $1 \leq p \leq \infty$, then $\{\|e_n\|\} \in \ell^p$ also. So we can redefine this algorithm as follows:

Algorithm 2 Let $f_n : H \to H$ be a sequence of nonexpansive maps.
Step 1. Choose $x_0, u \in H$ arbitrarily.
Step 2. For each $n \geq 0$, choose the regularization parameter $\lambda_n > 0$ and compute

$$y_n = f_n(x_n) \quad \text{and} \quad z_n = (I + \lambda_n A)^{-1} u.$$

Step 3. For each $n \geq 0$, choose the relaxation parameter $\alpha_n \in (0,1)$ and compute the $(n+1)$th iterate:

$$x_{n+1} = \alpha_n z_n + (1-\alpha_n) y_n + e_n,$$

where $\{e_n\}$ is a sequence of computational errors.

From now on, we will always assume our algorithms have already been converted into a form similar to the one given above, that is, the errors are present in Step 3 only.

Theorem 6 *Assume $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $f_n : H \to H$ is a sequence of nonexpansive maps. If A is maximal monotone, $\emptyset \neq F := A^{-1}(0) \subset \bigcap_n F(f_n)$, $\alpha_n \in (0,1)$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\beta_n > 0$ and $\lambda_n \to \infty$, then for every $x_0, u \in H$, the sequence $\{x_n\}$ generated by algorithm 2 converges strongly to $q = P_F u$. Here $F(f_n)$ denotes the set of all fixed points of f_n.*

Proof:

For $p \in F$, we have

$$\|x_{n+1} - p\| \leq \alpha_n \|u - p\| + (1-\alpha_n)\|x_n - p\| + \|e_n\|,$$

which implies that

$$\|x_{n+1} - p\| \leq \prod_{k=0}^{n}(1-\alpha_k)\|x_0 - p\| + \left[1 - \prod_{k=0}^{n}(1-\alpha_k)\right] \|u - p\| + \sum_{k=0}^{n} \|e_k\|,$$

showing that $\{x_n\}$ is bounded, and so is $\{y_n\}$.

Set $q = P_F u$. Then,

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|(1-\alpha_n)(y_n - q + e_n) + \alpha_n((I+\lambda_n A)^{-1}u - q + e_n)\|^2 \\ &\le (1-\alpha_n)\|y_n - q + e_n\|^2 + 2\alpha_n\langle (I+\lambda_n A)^{-1}u - q + e_n, x_{n+1} - q\rangle \\ &\le (1-\alpha_n)(\|x_n - q\| + \|e_n\|)^2 + 2M\alpha_n\|(I+\lambda_n A)^{-1}u - q + e_n\| \\ &= (1-\alpha_n)\|x_n - q\|^2 + \alpha_n b_n + c_n, \end{aligned}$$

where $b_n = 2M\|(I+\lambda_n A)^{-1}u - q + e_n\| \to 0$ and $c_n = \|e_n\|(2\|x_n - q\| + \|e_n\|) \to 0$. Hence by Lemma 1, $\|x_n - q\| \to 0$. ■

In the case when $f_n = (I+\beta_n B)^{-1}$, we can show that if $\{e_n\}$ is bounded and $\{\beta_n\}$ is bounded below away from zero, then the sequence $\{x_n\}$ generated by algorithm 2 with $f_n = (I+\beta_n B)^{-1}$ is bounded, provided B is assumed to be coercive. (See Section 2).

Theorem 7 *Assume A, B are maximal monotone operators and B is coercive with $\emptyset \neq F := A^{-1}(0) \subset B^{-1}(0)$. Let $\|e_n\| \le C$ and $\beta_n \ge \varepsilon > 0$ for $n \ge 0$, where C and ε are given constants. Then for any $x_0, u \in H$ the sequence $\{x_n\}$ generated by algorithm 2 with $f_n = (I+\beta_n B)^{-1}$ is bounded.*

Corollary 1 *Assume A is maximal monotone operator and coercive. Let $\|e_n\| \le C$ and $\beta_n \ge \varepsilon > 0$ for $n \ge 0$, where C and ε are given constants. Then for any $x_0, u \in H$ the sequence $\{x_n\}$ generated by algorithm 2 with $f_n = (I+\beta_n A)^{-1}$ is bounded.*

We now discuss in details the following relaxed algorithm.

Algorithm 3 Let $f_n : H \to H$ be a sequence of nonexpansive maps.
Step 1. Choose $x_0 \in H$ arbitrarily.
Step 2. For each $n \ge 0$, choose the regularization parameter $\beta_n > 0$ and compute

$$y_n = (I+\beta_n A)^{-1}(x_n) \quad \text{and} \quad z_n = f_n(x_n).$$

Step 3. For each $n \ge 0$, choose the relaxation parameter $\alpha_n \in (0,1)$ and compute $(n+1)$th iterate:

$$x_{n+1} = \alpha_n z_n + (1-\alpha_n)y_n + e_n,$$

where $\{e_n\}$ is a sequence of computational errors.

In the next theorem we generalize Theorem 5.2 [8]. Note that if $f_n = 0$ for all $n \ge 0$, we are in the case (14), (15) with $u = 0$.

Claim: If $p \in \bigcap_n F(f_n)$ and $p \in F$, then $\{x_n\}$ is bounded.

$$\begin{aligned} \|x_{n+1} - p\| &\le \alpha_n\|f_n(x_n) - p\| + (1-\alpha_n)\|y_n - p\| + \|e_n\| \\ &\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|x_n - p\| + \|e_n\| \\ &= \|x_n - p\| + \|e_n\|, \end{aligned}$$

which implies that

$$\|x_{n+1} - p\| \leq \|x_0 - p\| + \sum_{n=0}^{\infty} \|e_n\| < \infty,$$

showing that $\{x_n\}$ is bounded. Moreover,

$$\|x_{n+1} - p\| - \sum_{k=0}^{n} \|e_k\| \leq \|x_n - p\| - \sum_{k=0}^{n-1} \|e_k\|.$$

Hence $\{\|x_n - p\|\}$ converges.

If in addition, $\bigcap_n F(f_n) \supset F$, then $\{\|x_n - p\|\}$ converges for all $p \in F$. Moreover, if $\beta_n \to \infty$, then $\omega_{\mathtt{w}}(\{x_n\}) \subset F$ so that Opial's lemma (see for instance, [4], p. 5) guarantees the weak convergence of $\{x_n\}$ to a point of F.

We have thus proved the following result.

Theorem 8 *Let A be a maximal monotone operator with $\emptyset \neq F := A^{-1}(0) \subset \bigcap_n F(f_n)$, where $f_n : H \to H$ is a sequence of nonexpansive maps. Assume that $\sum_{n=0}^{\infty} \|e_n\| < \infty$, $\alpha_n \in (0,1)$ with $\alpha_n \to 0$, and $\beta_n \to \infty$. Then for every $x_0 \in H$, the sequence $\{x_n\}$ generated by algorithm 3 converges weakly to some point $q \in F$.*

Remarks:
1). If $f_n := f = I$ for all $n \geq 0$, then $F(f) \supset F$, and hence Algorithm 3 reduces to Algorithm 5.2 of [8].
2). Note that A_λ is nonexpansive for $\lambda_n := \lambda = 1$ for all $n \geq 0$. In this case, if $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\beta_n \to \infty$ and $F = A^{-1}(0) \neq \emptyset$, then we again get weak convergence for the sequence generated by algorithm 3 with $f_n = A_1$. However, this result is weaker than Theorem 8.

Indeed, for $q \in F$

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n(A_1(x_n) - A_1 q - q) + (1 - \alpha_n)(y_n - q)\| \\
&\leq \alpha_n\|x_n - q\| + \alpha_n\|q\| + (1 - \alpha_n)\|x_n - q\| + \|e_n\| \\
&= \|x_n - q\| + \alpha_n\|q\| + \|e_n\|,
\end{aligned}$$

which implies that

$$\|x_{n+1} - q\| - \sum_{k=0}^{n}(\alpha_k\|q\| + \|e_k\|) \leq \|x_n - q\| - \sum_{k=0}^{n-1}(\alpha_k\|q\| + \|e_k\|),$$

showing that $\lim_{n\to\infty} \|x_n - q\| = \rho(q)$ for all $q \in F$. Therefore $\{x_n\}$ is bounded, and so is $\{y_n\}$. Moreover,

$$\begin{aligned}
\|x_{n+1} - J_{\beta_n}(x_n)\| &\leq \alpha_n\|A_1(x_n) - A_1(q)\| + \alpha_n\|q\| + \alpha_n\|q - J_{\beta_n}(x_n)\| + \|e_n\| \\
&\leq 2\alpha_n\|x_n - q\| + \alpha_n\|q\| + \|e_n\| \to 0,
\end{aligned}$$

where $J_{\beta_n}(x_n) = (I + \beta_n A)^{-1}(x_n)$. Consequently, $x_\infty \in A^{-1}(0)$ if $x_{n_k} \rightharpoonup x_\infty$. Hence by Opial's lemma, there exists $p \in F$ such that $x_n \rightharpoonup p$.

It is easy to see that this result holds for any sequence of nonexpansive maps, hence the following theorem.

Theorem 9 *Let A be a maximal monotone operator with $F := A^{-1}(0) \neq \emptyset$, and $f_n : H \to H$ be a sequence of nonexpansive maps. Assume that $\sum_{n=0}^{\infty} \|e_n\| < \infty$, $\alpha_n \in (0,1)$, with $\sum_{n=0}^{\infty} \alpha_n < \infty$, and $\beta_n \to \infty$. Then for every $x_0 \in H$, the sequence $\{x_n\}$ generated by algorithm 3 converges weakly to some point $q \in F$.*

3). **Special Case:** $f_n := f = P_F$, where $F = A^{-1}(0) \neq \emptyset$. In this case we have strong convergence.

Theorem 10 *Let A be a maximal monotone operator with $F := A^{-1}(0) \neq \emptyset$. Assume that $\sum_{n=0}^{\infty} \|e_n\| < \infty$, $\alpha_n \in (0,1)$, $\alpha_n \to 0$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\beta_n \to \infty$. Then for every $x_0 \in H$, the sequence $\{x_n\}$ generated by algorithm 3 with $f_n = P_F$ (for all $n \geq 0$) converges strongly to some point $q \in F$.*

We first show that $v_n = P_F x_n$ is strongly convergent (to some $q \in F$).

$$\begin{aligned} \|x_{n+m} - v_n\| &= \|\alpha_{n+m-1}(P_F x_{n+m-1} - v_n + e_{n+m-1}) + (1-\alpha_{n+m-1})(y_{n+m-1} - v_n + e_{n+m-1})\| \\ &\leq \alpha_{n+m-1}\|x_{n+m-1} - v_n\| + (1-\alpha_{n+m-1})\|x_{n+m-1} - v_n\| + \|e_{n+m-1}\| \\ &= \|x_{n+m-1} - v_n\| + \|e_{n+m-1}\| \\ &\leq \|x_n - v_n\| + \sum_{k=n}^{n+m-1} \|e_k\|, \end{aligned} \tag{22}$$

which implies that

$$\|x_{n+m} - v_{n+m}\| \leq \|x_n - v_n\| + \sum_{k=n}^{n+m-1} \|e_k\|.$$

In particular, $\{\|x_n - v_n\|\}$ is convergent. By the parallelogram law applied to $v_{n+m} - x_{n+m}$ and $v_n - x_{n+m}$,

$$\|v_{n+m} - v_n\|^2 + \|2x_{n+m} - (v_n + v_{n+m})\|^2 = 2(\|x_{n+m} - v_{n+m}\|^2 + \|x_{n+m} - v_n\|^2).$$

Therefore, using (22), we have

$$\begin{aligned} \|v_{n+m} - v_n\|^2 + 4\|x_{n+m} - v_{n+m}\|^2 &\leq 2\left(\|x_{n+m} - v_{n+m}\|^2 + \|x_{n+m} - v_n\|^2\right) \\ &\leq 2\|x_{n+m} - v_{n+m}\|^2 + 2\left(\|x_n - v_n\| + \sum_{k=n}^{\infty} \|e_k\|\right)^2, \end{aligned}$$

which implies that

$$\|v_{n+m} - v_n\|^2 \leq -2\|x_{n+m} - v_{n+m}\|^2 + 2\left(\|x_n - v_n\| + \sum_{k=n}^{\infty} \|e_k\|\right)^2.$$

Thus $\{v_n\}$ is Cauchy, hence converges strongly to some $q \in F$.

For $q = s - \lim P_F x_n$, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\alpha_n(P_F x_n - q + e_n) + (1-\alpha_n)(y_n - q + e_n)\|^2 \\ &\leq (1-\alpha_n)^2\|y_n - q + e_n\|^2 + 2\alpha_n\langle P_F x_n - q + e_n, x_{n+1} - q\rangle \\ &\leq (1-\alpha_n)(\|x_n - q\| + \|e_n\|)^2 + 2M\alpha_n\|P_F x_n - q + e_n\| \\ &= (1-\alpha_n)\|x_n - q\|^2 + \alpha_n b_n + c_n, \end{aligned}$$

where $b_n = 2M\|P_F x_n - q + e_n\| \to 0$ and $c_n = \|e_n\|(2\|x_n - q\| + \|e_n\|) \to 0$. Hence by Lemma 1, $\{x_n\}$ converges strongly to q. $\blacksquare$

4). If $f_n = f = (I + \lambda A)^{-1}$ for all $n \geq 0$ and $\lambda > 0$, then $F(f) = F$, and again we obtain weak convergence of $\{x_n\}$ under the assumptions of Theorem 8.
5). If $f_n = (I + \lambda_n A)^{-1}$, for $\lambda_n > 0$, then $F(f_n) = F$ for all $n \geq 0$, so we have the following algorithm:

Special case of algorithm 3: Let A be a maximal monotone operator.
Step 1. Choose $x_0 \in H$ arbitrarily.
Step 2. For each $n \geq 0$, choose the regularization parameters $\beta_n, \lambda_n > 0$ and compute

$$y_n = (I + \beta_n A)^{-1}(x_n) \quad \text{and} \quad z_n = (I + \lambda_n A)^{-1}(x_n).$$

Step 3. For each $n \geq 0$, choose the relaxation parameter $\alpha_n \in (0, 1)$ and compute $(n+1)$th iterate:

$$x_{n+1} = \alpha_n z_n + (1 - \alpha_n) y_n + e_n,$$

where $\{e_n\}$ is a sequence of computational errors.

Note that for $q \in F$, we have

$$\|x_{n+1} - q\| \leq \|x_n - q\| + \|e_n\|.$$

So if $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $\alpha_n \to 0$ we deduce that $\lim_{n\to\infty} \|x_n - q\| = \rho(q)$ for all $q \in F$. Since $\omega_{\mathtt{w}}(\{x_n\}) \subset F$ for $\beta_n \to \infty$, Opial's lemma guarantees the weak convergence of $\{x_n\}$ to a point of F.

Having in mind that

$$(I + \lambda A)^{-1} x \to P_F x \quad \text{as} \quad \lambda \to \infty,$$

it is expected that the sequence $\{x_n\}$ generated by the above algorithm converges strongly, if both $\lambda_n, \beta_n \to \infty$. However, it turns out that only the assumption $\lambda_n \to \infty$ is enough to guarantee strong convergence. Our aim now is to construct a sequence of parameters $\{\lambda_n\}$ such that for very large n, the corresponding sequence $\{x_n\}$ as given by the algorithm in question converges strongly to a point of F. We then have the following modified algorithm.

Let A be a maximal monotone operator.
Step 1. Choose $x_0 \in H$ arbitrarily.
Step 2. For each $n \geq 0$, choose the regularization parameter $\beta_n > 0$ and compute

$$y_n = (I + \beta_n A)^{-1}(x_n).$$

Step 3. Choose another regularization parameter λ_n large enough such that

$$\|(I + \lambda_n A)^{-1}(x_n) - P_F x_n\| < \frac{1}{n}, \quad \text{and compute} \quad z_n = (I + \lambda_n A)^{-1}(x_n). \tag{23}$$

Step 4. For $n \geq 0$, choose a relaxation parameter $\alpha_n \in (0, 1)$ and compute the $(n + 1)$th iterate:

$$x_{n+1} = \alpha_n z_n + (1 - \alpha_n) y_n + e_n,$$

where $\{e_n\}$ is a sequence of computational errors.

Theorem 11 *Let $f_n : H \to H$ be a sequence of nonexpansive maps. If A is maximal monotone with $\emptyset \neq F = A^{-1}(0)$, $\alpha_n \in (0,1)$, $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} \|e_n\| < \infty$, then for every $x_0 \in H$, the sequence $\{x_n\}$ constructed by the above algorithm converges strongly to some $p \in F$.*

Proof:

For $q \in F$, we have

$$\|x_{n+1} - q\| \leq \|x_n - q\| + \|e_n\|,$$

showing that $\{\|x_n - q\|\}$ is convergent, (and hence bounded).

Now denote $v_n := P_F x_n$. As in Theorem 10, we derive strong convergence of $\{v_n\}$ to some point $p \in F$.

Now

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)(y_n - p + e_n) + \alpha_n(z_n - p + e_n)\|^2 \\
&\leq (1-\alpha_n)\|y_n - p + e_n\|^2 + 2\alpha_n\langle z_n - p + e_n, x_{n+1} - p\rangle \\
&\leq (1-\alpha_n)(\|x_n - p\| + \|e_n\|)^2 + 2\alpha_n\langle z_n - v_n, x_{n+1} - p\rangle + \\
&\quad 2\alpha_n\langle v_n - p + e_n, x_{n+1} - p\rangle \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + 2M\alpha_n(\|z_n - v_n\| + \|v_n - p + e_n\|) + c_n \\
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n b_n + c_n,
\end{aligned}$$

where $b_n := 2M(\|z_n - v_n\| + \|v_n - p + e_n\|) \to 0$ and $c_n = \|e_n\|(2\|x_n - p\| + \|e_n\|)$ with $\sum_{n=0}^{\infty} c_n < \infty$. Hence by Lemma 1, $\|x_n - p\| \to 0$. ■

We observe that in proving the above result, we only required β_n to be positive, so we can actually replace y_n in the above algorithm by any nonexpansive map f (and hence by a sequence of nonexpansive maps $\{f_n\}$) satisfying the condition $F \subset F(f)$ ($F \subset \bigcap_n F(f_n)$). Therefore we can generalize at once the above algorithm and hence the result as in the following theorem. Observe the shift in the position of f from the previous results!

Algorithm 4 Let A be a maximal monotone operator and $f_n : H \to H$ be a sequence of nonexpansive maps.
Step 1. Choose $x_0 \in H$ arbitrarily.
Step 2. For each $n \geq 0$, compute

$$y_n = f_n(x_n).$$

Step 3. Choose a regularization parameter λ_n large enough such that

$$\|(I + \lambda_n A)^{-1}(x_n) - P_F x_n\| < \frac{1}{n}, \quad \text{and compute} \quad z_n = (I + \lambda_n A)^{-1}(x_n).$$

Step 4. For $n \geq 0$, choose a relaxation parameter $\alpha_n \in (0,1)$ and compute the $(n+1)$th iterate:

$$x_{n+1} = \alpha_n z_n + (1-\alpha_n) y_n + e_n,$$

where $\{e_n\}$ is a sequence of computational errors.

Theorem 12 *Let $f_n : H \to H$ be a sequence of nonexpansive maps. If A is maximal monotone with $\emptyset \neq F \subset \bigcap_n F(f_n)$, $\alpha_n \in (0,1)$, $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} \|e_n\| < \infty$, then for every $x_0 \in H$, the sequence $\{x_n\}$ constructed by algorithm 4 converges strongly to some $p \in F$.*

6). If $f_n = (I + \lambda_n B)^{-1}$, for $\lambda_n > 0$, we have the following algorithm:

Algorithm 5 Let A and B be maximal monotone operators.
Step 1. Choose $x_0 \in H$ arbitrarily.
Step 2. For each $n \geq 0$, choose the regularization parameters $\beta_n, \lambda_n > 0$ and compute

$$y_n = (I + \beta_n A)^{-1}(x_n) \quad \text{and} \quad z_n = (I + \lambda_n B)^{-1}(x_n).$$

Step 3. For each $n \geq 0$, choose the relaxation parameter $\alpha_n \in (0,1)$ and compute $(n+1)$th iterate:

$$x_{n+1} = \alpha_n z_n + (1 - \alpha_n) y_n + e_n,$$

where $\{e_n\}$ is a sequence of computational errors.

Theorem 13 *Assume that A and B are maximal monotone operators with $\emptyset \neq F := A^{-1}(0) = B^{-1}(0)$. If $\sum_{n=0}^{\infty} \|e_n\| < \infty$, $\alpha_n \in (0,1)$, $\alpha_n \to 0$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\beta_n \to \infty$, then for every $x_0 \in H$, the sequence $\{x_n\}$ generated by algorithm 5 converges weakly to some $q \in F$.*

Proof:

For $p \in F$, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|z_n - p\| + (1 - \alpha_n)\|x_n - p\| + \|e_n\| \\ &\leq \|x_n - p\| + \|e_n\|, \end{aligned}$$

which implies that $\lim_{n\to\infty} \|x_n - p\|$ exists for every $p \in F$. Moreover,

$$\begin{aligned} \|x_{n+1} - (I + \beta_n A)^{-1} x_n\| &= \|\alpha_n[(I + \lambda_n B)^{-1}(x_n) - (I + \beta_n A)^{-1}(x_n)] + e_n\| \\ &\leq \alpha_n(\|(I + \lambda_n B)^{-1}(x_n) - p\| + \|(I + \beta_n A)^{-1}(x_n) - p\|) + \|e_n\| \\ &\leq 2\alpha_n \|x_n - p\| + \|e_n\| \to 0. \end{aligned}$$

Consequently, $x_\infty \in A^{-1}(0) = F$ if $x_{n_k} \rightharpoonup x_\infty$. Hence by Opial's lemma, there exists a point, say $q \in F$ such that $x_n \rightharpoonup q$. ■

The case when f is a strict contraction was discussed in [7]. However, it is worth mentioning that the author only considered algorithm 5 (with $z_n = f(x_n)$ where $f : H \to H$ for all $n \geq 0$) without any error terms and proved strong convergence of $\{x_n\}$ (under appropriate conditions) in a Banach space setting. For the sake of completeness of our discussion, we prove a strong convergence result of $\{x_n\}$ given by algorithm 5 in the case when f is a strict contraction. We adapt the proof of Theorem 4.2 [7] to this situation.

Theorem 14 *Assume that* $\sum_{n=0}^{\infty}\|e_n\| < \infty$ *and* $f : H \to H$ *is a strict contraction with Lipschitz constant* $a \in (0,1)$*. If* A *is maximal monotone,* $F := A^{-1}(0) \neq \emptyset$*,* $\alpha_n \in (0,1)$*,* $\alpha_n \to 0$ *with* $\sum_{n=0}^{\infty}\alpha_n = \infty$*, and* $\beta_n \to \infty$*, then for every* $x_0 \in H$*, the sequence* $\{x_n\}$ *generated by algorithm 5 (with* $z_n = f(x_n)$ *for all* $n \geq 0$*) converges strongly to the unique fixed point* z *of* $P_F \circ f$*, that is* $z = P_F f(z)$*.*

Proof:

Step 1. Fix $p \in A^{-1}(0)$ and set $M = \max\{\|x_0 - p\|, \frac{1}{1-a}\|f(p) - p\|\}$. We show by induction that for any $n \geq 0$,

$$\|x_n - p\| \leq M + \sum_{k=0}^{n-1}\|e_k\|. \tag{24}$$

For n=0, (24) is clearly true. Assume that (24) holds for some $n \geq 0$. We show that it also holds for $n+1$. For $p \in F$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n\|f(x_n) - p\| + (1-\alpha_n)\|y_n - p\| + \|e_n\| \\
&\leq \alpha_n(\|f(x_n) - f(p)\| + \|f(p) - p\|) + (1-\alpha_n)\|y_n - p\| + \|e_n\| \\
&\leq (1 - \alpha_n(1-a))\|x_n - p\| + \alpha_n\|f(p) - p\| + \|e_n\| \\
&= (1 - \alpha_n(1-a))\|x_n - p\| + \alpha_n(1-a)\frac{1}{1-a}\|f(p) - p\| + \|e_n\| \\
&\leq (1 - \alpha_n(1-a))\left[M + \sum_{k=0}^{n-1}\|e_k\|\right] + \alpha_n(1-a)\frac{1}{1-a}M + \|e_n\| \\
&\leq M + \sum_{k=0}^{n}\|e_k\|.
\end{aligned}$$

Step 2. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\overline{\lim}_{n\to\infty}\langle f(z) - z, x_{n+1} - z\rangle = \lim_{k\to\infty}\langle f(z) - z, x_{n_k+1} - z\rangle.$$

We assume that $x_{n_k+1} \rightharpoonup v$. Then it follows that

$$\overline{\lim}_{n\to\infty}\langle f(z) - z, x_{n+1} - z\rangle = \langle f(z) - z, v - z\rangle.$$

So it only remains to show that $v \in F$. For this purpose, we note that

$$\|x_{n+1} - (I + \beta_n A)^{-1}(x_n)\| \leq \alpha_n\|f(x_n) - (I + \beta_n A)^{-1}(x_n)\| + \|e_n\| \to 0.$$

As in the proof of Theorem 5, we deduce that $v \in F$, hence $\overline{\lim}_{n\to\infty}\langle f(z) - z, x_n - z\rangle \leq 0$.

Step 3. Finally, we show that $\|x_n - z\| \to 0$. Applying the subdifferential inequality, we have,

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1-\alpha_n)^2\|(I + \beta_n A)^{-1}(x_n) - z + e_n\|^2 + 2\alpha_n\langle f(x_n) - z + e_n, x_{n+1} - z\rangle \\
&\leq (1-\alpha_n)^2(\|x_n - z\|^2 + \|e_n\|(\|e_n\| + 2\|x_n - z\|)) + \\
&\quad 2\alpha_n\langle f(x_n) - f(z), x_{n+1} - z\rangle + 2\alpha_n\langle f(z) - z + e_n, x_{n+1} - z\rangle.
\end{aligned}$$

Since $\{\|e_n\|\}$ and $\{x_n\}$ are bounded, we have $\|e_n\| + 2\|x_n - z\| \le K$ for some constant K. Therefore,

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le (1-\alpha_n)^2\|x_n - z\|^2 + K\|e_n\| + 2\alpha_n\langle f(z) - z + e_n, x_{n+1} - z\rangle + \\ &\quad 2a\alpha_n\|x_n - z\|\|x_{n+1} - z\| \\ &\le (1-\alpha_n)^2\|x_n - z\|^2 + K\|e_n\| + 2\alpha_n\langle f(z) - z + e_n, x_{n+1} - z\rangle + \\ &\quad a\alpha_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2),\end{aligned}$$

which implies that

$$\begin{aligned}(1 - a\alpha_n)\|x_{n+1} - z\|^2 &\le (1 - 2\alpha_n + a\alpha_n)\|x_n - z\|^2 + K\|e_n\| + \alpha_n^2\|x_n - z\|^2 + \\ &\quad 2\alpha_n\langle f(z) - z + e_n, x_{n+1} - z\rangle,\end{aligned}$$

or equivalently,

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \left(1 - \frac{2(1-a)\alpha_n}{1 - a\alpha_n}\right)\|x_n - z\|^2 + \frac{K\|e_n\|}{1 - a\alpha_n} + \\ &\quad \frac{2(1-a)\alpha_n}{1 - a\alpha_n}\left(\frac{\alpha_n M'}{2(1-a)} + \frac{1}{1-a}\langle f(z) - z + e_n, x_{n+1} - z\rangle\right) \\ &\le (1 - a_n)\|x_n - z\|^2 + a_n b_n + c_n,\end{aligned}$$

where

$$c_n = \frac{K\|e_n\|}{1 - a\alpha_n},\ a_n = \frac{2(1-a)\alpha_n}{1 - a\alpha_n},\ b_n = \frac{\alpha_n M'}{2(1-a)} + \frac{1}{1-a}\langle f(z) - z + e_n, x_{n+1} - z\rangle,$$

and $M' = \sup_n \|x_n - z\|^2$, with $\sum_{n=0}^{\infty} a_n = \infty$ and $\lim_{n\to\infty} a_n = 0$. Note also that $\sum_{n=0}^{\infty} c_n < \infty$ and $\overline{\lim}_{n\to\infty} b_n \le 0$. Hence from Lemma 1, we have $\|x_n - z\| \to 0$. ■

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