

SECOND-ORDER DIFFERENTIAL EQUATIONS ON \mathbb{R}_+ GOVERNED BY MONOTONE OPERATORS

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ABSTRACT. Consider in a real Hilbert space H the differential equation $(E): p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t)$, for a.a. $t \in \mathbb{R}_+ = [0, \infty)$, with the condition $u(0) = x \in \overline{D(A)}$, where $A: D(A) \subset H \rightarrow H$ is a maximal monotone operator, with $[0, 0] \in A$ (or, more generally, $0 \in R(A)$); $p, q \in L^\infty(\mathbb{R}_+)$, with $\text{ess inf } p > 0$ and either $\text{ess inf } q > 0$ or $\text{ess sup } q < 0$; and $f: \mathbb{R}_+ \rightarrow H$ is a given function. Recall that equation (E) in the case $p \equiv 1$, $q \equiv 0$, $f \equiv 0$, subject to $u(0) = x$ and the boundedness condition $\sup_{t \geq 0} \|u(t)\| < \infty$, was investigated in early 1970's by V. Barbu, who derived in particular from his results a definition for the square root of the nonlinear operator A . Subsequently H. Brezis, N. H. Pavel, L. Véron and others have paid much attention to equation (E) . In this paper we prove the existence and uniqueness of the solution to equation (E) subject to $u(0) = x \in D(A)$ in the weighted space $X = L^2_b(\mathbb{R}_+; H)$, where $b(t) = a(t)/p(t)$, $a(t) = \exp(\int_0^t q(s)/p(s) ds)$, under our weak assumptions on p and q (see above) and $f \in X$. For $x \in \overline{D(A)}$ we prove the existence of a generalized solution in the case of general variable coefficients p, q and a classic solution in the case $p \equiv 1$, $q \equiv c \in \mathbb{R} \setminus \{0\}$. If $p \equiv 1$, $q(t) \equiv c \in \mathbb{R} \setminus \{0\}$, $f \equiv 0$ the solutions give rise to a nonlinear semigroup of contractions. If A is linear its infinitesimal generator G is given by $G = -\frac{c}{2}I - \sqrt{\frac{c^2}{4}I + A}$.

1. INTRODUCTION.

Throughout this paper H will be a real Hilbert space with respect to an inner product (\cdot, \cdot) and the induced norm $\|x\| = (x, x)^{1/2}$. Consider the nonlinear second-order equation (inclusion)

$$p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t), \quad \text{for a.a. } t \in \mathbb{R}_+ = [0, \infty), \quad (E)$$

with the condition

$$u(0) = x \in \overline{D(A)}, \quad (B)$$

where

(H1) $A: D(A) \subset H \rightarrow H$ is a maximal monotone operator, with $0 \in D(A)$ and $0 \in A0$;

(H2) $p, q \in L^\infty(\mathbb{R}_+) := L^\infty(\mathbb{R}_+; \mathbb{R})$, with $\text{ess inf } p > 0$ and either $\text{ess inf } q > 0$ or $\text{ess sup } q < 0$;

2010 *Mathematics Subject Classification.* 47J25, 47H05, 47H09.

Key words and phrases. second-order differential equation on \mathbb{R}_+ ; maximal monotone operator; existence and uniqueness; constant coefficients; nonlinear semigroup of contractions; description of the set of bounded solutions.

and $f: \mathbb{R}_+ \rightarrow H$ is a given function which will be described later. In other words, (H2) says that both p and q are measurable and satisfy

$$0 < p_0 \leq p(t) \leq p_1 < \infty \quad \text{for a.a. } t \in \mathbb{R}_+,$$

and

$$q_0 \leq q(t) \leq q_1 \quad \text{for a.a. } t \in \mathbb{R}_+,$$

where either $0 < q_0 < q_1 < \infty$ or $-\infty < q_0 < q_1 < 0$.

Remark 1.1. *One can assume that $0 \in R(A)$ instead of $0 \in D(A)$, $0 \in A0$. Indeed, if $y \in D(A)$ and $0 \in A_y$, then defining the operator A_y by $D(A_y) = D(A) \setminus \{y\}$, $A_y(z) = A(z + y)$, and $v(t) = u(t) - p$, we have $0 \in D(A_y)$, $0 \in A_y 0$ and*

$$p(t)v''(t) + q(t)v'(t) \in A_y v(t) + f(t) \quad \text{for a.a. } t \in \mathbb{R}_+. \quad (E_y)$$

Problem (E), (B) in the case $p \equiv 1$, $q \equiv 0$, $f \equiv 0$, with the additional condition

$$\sup_{t \geq 0} \|u(t)\| < \infty, \quad (C)$$

was investigated for the first time by V. Barbu [3], [4] (see also [5], Chapter V, Sections 2.3 and 2.4), who derived in particular from his results a definition for the square root of the nonlinear operator A . The same problem, with (B') instead of (B), was then studied by H. Brezis [6], where

$$u'(0) \in \partial j(u(0) - x), \quad (B')$$

with ∂j standing for the subdifferential of a proper, convex, lower semicontinuous function $j: H \rightarrow (-\infty, +\infty]$. If in particular j is the indicator function of the set $\{0\}$ (i.e., $j(0) = 0$ and $j(z) = +\infty$ for all $z \in H \setminus \{0\}$), then condition (B') becomes $u(0) = x$. Subsequently L. Véron [12], [13] studied problem (E), (B), (C) with $f \equiv 0$, under the following conditions

$$p \in W^{2,\infty}(\mathbb{R}_+), \quad q \in W^{1,\infty}(\mathbb{R}_+), \quad p(t) \geq p_0 > 0 \quad \forall t \in \mathbb{R}_+. \quad (1.1)$$

These smoothness assumptions on the coefficients p and q are in fact too strong. Indeed, one can take advantage of the following form of (E), as pointed out by A. R. Aftabizadeh and N. H. Pavel [1]:

$$(a(t)u'(t))' \in b(t)Au(t) + b(t)f(t), \quad (1.2)$$

where

$$a(t) = \exp\left(\int_0^t \frac{q(s)}{p(s)} ds\right), \quad b(t) = \frac{a(t)}{p(t)},$$

that allows significant relaxation in the regularity of p and q . Following the technique of [1], N. C. Apreutesei [2] reconsidered (E) on \mathbb{R}_+ , with (B'), in the framework of the weighted space $X = L_b^2(\mathbb{R}_+; H)$, under the following assumptions on p, q :

$$p, q \in W^{1,\infty}(\mathbb{R}_+), \quad p(t) \geq p_0 > 0, \quad q(t) \geq q_0 > 0 \quad \text{for all } t \in \mathbb{R}_+, \quad \text{and } x \in D(A). \quad (1.3)$$

In this paper we prove the existence and uniqueness of the solution of problem (E), (B) in X , under our weak assumptions (H2), by using a simpler treatment. The condition $u \in X$ is more appropriate than the boundedness condition (C) that works well if $p \equiv 1$ and $q \equiv 0$ but does not guarantee uniqueness if $q \equiv c > 0$. For example, if $p \equiv 1$, $q \equiv 1$, $f \equiv 0$ and $A = 0$, then problem (E), (B), (C) has

infinitely many solutions (and only one belongs to X). Note also that L. Véron [13] derived the condition

$$\int_0^\infty \exp\left(-\int_0^t \frac{q(s)}{p(s)} ds\right) dt = +\infty$$

as a necessary one for the uniqueness of the solution of problem (E), (B), (C). So the case $p \equiv 1$ and $q \equiv c$, where c is a positive constant, is excluded in Véron's paper, but allowed in our present paper.

We obtain classic existence and uniqueness in X to problem (E), (B) for $x \in D(A)$ and $f \in X$. If $x \in \overline{D(A)}$ we prove the existence of a generalized solution (uniform limit on compact intervals of a sequence of classic solutions). If $f \equiv 0$, the solution $u \in X$ is also bounded on \mathbb{R}_+ . In the case of constant coefficients $p \equiv 1$, $q \equiv c$, $c \in \mathbb{R} \setminus \{0\}$, we are able to prove the existence of classic solutions for $x \in \overline{D(A)}$ and $f \in X$. If in addition $f \equiv 0$ the solutions give rise to a nonlinear contraction semigroup generated by $G = -\frac{c}{2}I - \sqrt{\frac{c^2}{4}I + A}$, where I is the identity operator, as noticed by B. Djafari Rouhani and H. Khatibzadeh [9]. If A is linear we can show that G is indeed the generator of this semigroup. If $p \equiv 1$, $q \equiv c > 0$, $f \equiv 0$ and $0 \in R(A)$, our results combined with the recent results regarding the asymptotic behavior of solutions reported by B. Djafari Rouhani and H. Khatibzadeh [9], lead to the description of the set of all bounded solutions of equation (E): $u(t) = y + v(t)$, $y \in A^{-1}0$, where $v \in X$ is a solution of the equation $v'' + cv' \in A_y v$, $t > 0$, with $A_y z = A(z + y)$. In particular, according to our present results, $\|v(t)\| = \mathbf{o}(e^{-ct/2})$ and $\|v'(t)\| = \mathbf{o}(e^{-ct/2})$, as $t \rightarrow \infty$. In fact the behavior of the v 's is even better: $\|v(t)\| = \mathbf{O}(e^{-ct})$ and $\|v'(t)\| = \mathbf{O}(e^{-ct})$, as proved by H. Khatibzadeh [10].

2. NOTATION AND PREPARATORY LEMMAS

Let X be the set of all (classes with respect to the a.e. equality of) functions $f: \mathbb{R}_+ \rightarrow H$ which are measurable, with $\int_0^\infty b(t)\|f(t)\|^2 dt < \infty$, where

$$b(t) = \frac{a(t)}{p(t)}, \quad a(t) = \exp\left(\int_0^t \frac{q(s)}{p(s)} ds\right).$$

In other words, X is the weighted space $L_b^2(\mathbb{R}_+; H)$. Note that (under (H2)) X is a real Hilbert space with respect to the scalar product

$$(f, g)_X = \int_0^\infty b(t)(f(t), g(t)) dt,$$

and the corresponding norm

$$\|f\|_X^2 = \int_0^\infty b(t)\|f(t)\|^2 dt.$$

Lemma 2.1. *Suppose assumptions (H2) hold. If $f, f' \in X$ (where f' is the distributional derivative of f), then*

$$\lim_{t \rightarrow \infty} a(t)\|f(t)\|^2 = 0, \tag{2.1}$$

where a and X are defined above.

Proof. Denote $g(t) = \sqrt{a(t)}f(t)$. We have

$$g'(t) = \frac{q(t)}{2p(t)}g(t) + \sqrt{a(t)}f'(t),$$

so both $g, g' \in L^2(\mathbb{R}_+; H)$. By Hölder's inequality this implies

$$\|g(t) - g(s)\| = \left\| \int_s^t g'(\tau) d\tau \right\| \leq (t-s)^{1/2} \|g'\|_{L^2(\mathbb{R}_+; H)},$$

for all $0 \leq s \leq t$. This shows that g is uniformly continuous on \mathbb{R}_+ and so is $\|g\|$. A straightforward reasoning shows that $\limsup_{t \rightarrow \infty} \|g(t)\| = 0$, otherwise g cannot be a member of X . \square

Now, let us define $B: D(B) \subset X \rightarrow X$ by

$$D(B) = \{u \in X : u', u'' \in X, u(0) = x\}, \quad Bu = -pu'' - qu'.$$

Lemma 2.2. *If assumptions (H2) hold, then operator B defined above is the sub-differential of $\Psi: X \rightarrow (-\infty, +\infty]$, $\Psi(u) = (1/2)\|u'\|_{L_a^2}^2 + j(u(0) - x)$, which is proper, convex and lower semicontinuous (hence B is maximal monotone), where $L_a^2 := L_a^2(\mathbb{R}_+; H)$ and j is the indicator function of the set $\{0\} \subset H$.*

Proof. Note that $Bu = -\frac{p}{a}(au')'$ for all $u \in D(B)$.

Claim 1: B is monotone.

Indeed, for all $u, v \in D(B)$, we have

$$\begin{aligned} (Bu - Bv, u - v)_X &= - \int_0^\infty ((a(u' - v'))', u - v) dt \\ &= a(u' - v', u - v)|_0^\infty + \int_0^\infty a \|u' - v'\|^2 dt \geq 0, \end{aligned}$$

since, according to Lemma 2.1,

$$\lim_{t \rightarrow \infty} a(t)(u'(t) - v'(t), u(t) - v(t)) = 0.$$

Claim 2: Ψ is proper.

For example, the function \hat{u} , defined by

$$\hat{u}(t) = (1-t)^2 x, \quad \text{for } 0 \leq t \leq 1 \quad \text{and} \quad \hat{u}(t) = 0 \quad \text{for } t > 1,$$

satisfies $\Psi(\hat{u}) < \infty$. The effective domain of Ψ is $D(\Psi) = \{u \in X : u' \in X, u(0) = x\}$.

Claim 3: Ψ is convex.

This is obvious.

Claim 4: Ψ is lower semicontinuous.

It is well known this is equivalent to saying that its level sets $M_\lambda := \{u \in X : \Psi(u) \leq \lambda\}$ ($\lambda \in \mathbb{R}$) are closed in X . If $\lambda \leq 0$ then M_λ is empty. Let $\lambda > 0$ be an arbitrary but fixed number, and let $u_n \in M_\lambda, u_n \rightarrow u$ in X . Since

$$\|u'_n\|_X^2 = \int_0^\infty b(t) \|u'_n(t)\|^2 dt \leq \frac{1}{p_0} \|u'_n\|_{L_a^2}^2 \leq \frac{2}{p_0} \lambda,$$

we have

$$u' \in X \quad \text{and} \quad u'_n \rightarrow u' \quad \text{weakly in } X.$$

Moreover, making use of the formula

$$u_n(t) = x + \int_0^t u'_n(s) ds,$$

we can easily deduce, by Arzelà's compactness criterion, that $u_n(t)$ converges uniformly to $u(t)$ on every compact interval $[0, T]$. In particular, $u(0) = x$. Therefore,

$$\begin{aligned} \lambda &\geq \liminf_{n \rightarrow \infty} \Psi(u_n) = \liminf_{n \rightarrow \infty} \frac{1}{2} \|u'_n\|_{L^2_a}^2 \\ &\geq \frac{1}{2} \|u'\|_{L^2_a}^2 = \Psi(u). \end{aligned}$$

Since Ψ is proper, convex and lower semicontinuous, it follows that its subdifferential $\partial\Psi$ is a maximal monotone operator. So, to conclude the proof of Lemma 2.2, it suffices to show that

Claim 5: if $u \in D(\partial\Psi)$ and $w \in \partial\Psi(u)$, then $u \in D(B)$ and $w = Bu$.

For all $\theta \in (0, 1)$, $v \in D(\Psi)$,

$$\Psi(u + \theta(v - u)) - \Psi(u) \geq \theta(w, v - u)_X.$$

In other words,

$$(u', v' - u')_{L^2_a} + \frac{\theta}{2} \|v' - u'\|_{L^2_a}^2 \geq (w, v - u)_X.$$

Letting θ tend to 0 in this inequality yields

$$(u', v' - u')_{L^2_a} \geq (w, v - u)_X \quad \forall v \in D(\Psi).$$

Choosing $v(t) = u(t) + \varphi(t)h$, $h \in H$, $\varphi \in C_0^\infty(0, \infty)$ (test functions), we obtain

$$\int_0^\infty a(t)\varphi'(t)(h, u'(t)) dt = \int_0^\infty b(t)(h, w(t))\varphi(t) dt,$$

for all $h \in H$, $\varphi \in C_0^\infty(0, \infty)$. Therefore $-(au')' = bw$ in the sense of distributions (i.e., in $D'((0, \infty); H)$). It follows that $u'' \in X$ and $w = Bu$. The proof of the lemma is complete. \square

Remark 2.3. The boundary condition (B) (i.e., $u(0) = x$) in Lemma 2.2 (in the definition of B) can be generalized to

$$u'(0) \in \beta(u(0) - x), \tag{B''}$$

where $\beta: D(\beta) \subset H \rightarrow H$ is a general maximal monotone operator. In this case B is maximal monotone (even a subdifferential if β is so). While the monotonicity of B is obvious, to prove its maximality, it suffices to show that for all $f \in X$ the equation $u + Bu = f$ has a solution $u \in D(B)$ (cf. Minty's Theorem, see, e.g., [11], p. 19). Let $v \in X$ the unique solution of

$$v - pv'' - qv' = f, \quad v(0) = x,$$

which exists by Lemma 2.2, with $v', v'' \in X$. Let $u(t) = v(t) + \zeta(t)y$, where $y \in H$ and $\zeta = \zeta(t)$ satisfies

$$\zeta - p\zeta'' - q\zeta' = 0, \quad t > 0; \quad \zeta(0) = 1; \quad \zeta, \zeta' \in L^2_b(\mathbb{R}_+; \mathbb{R}).$$

The existence of ζ also follows from Lemma 2.2. Clearly,

$$u - pu'' - qu' = f \quad \text{for a.a. } t \geq 0.$$

It remains to prove that $u'(0) \in \beta(u(0) - x)$ for a convenient $y \in H$, i.e.,

$$v'(0) + \zeta'(0)y \in \beta(y).$$

Since β is maximal monotone, for the existence of y it suffices to show that $\zeta'(0) < 0$. Assume by contradiction that $\zeta'(0) \geq 0$. We have

$$(a\zeta)' = b\zeta \geq 0 \text{ for a.a. } t \in (0, T),$$

for some $T \in (0, \infty)$, since $\zeta(0) = 1$. It follows that $t \rightarrow a(t)\zeta'(t)$ is nondecreasing on $[0, T]$. In particular, $a(t)\zeta'(t) \geq \zeta'(0) \geq 0$ and thus $\zeta'(t) \geq 0$ in $[0, T]$, so ζ is nondecreasing in $[0, T]$. On the other hand, integrating over $[t, \infty)$ the equation $(a(\zeta)'\zeta = b\zeta^2$, we obtain

$$-a(t)\zeta'(t)\zeta(t) - \int_t^\infty a(s)\zeta'(s)^2 ds = \int_t^\infty b(s)\zeta(s)^2 ds > 0$$

for $t \in (0, T)$ since ζ is positive in $[0, T]$. Therefore $a(t)\frac{d}{dt}\zeta^2(t) < 0$ for $t \in (0, T)$, which implies that ζ is decreasing in $[0, T]$, contradiction.

In what follows we restrict ourselves to the boundary condition (B). The case of the more general condition (B'') is still open, but the above remark could be a good starting step towards the solution of problem (E), (B'').

3. EXISTENCE AND UNIQUENESS FOR $x \in D(A)$ AND $f \in X$

Let \bar{A} denote the realization of A in X , that is

$$\bar{A} = \{[u, v] \in X \times X : [u(t), v(t)] \in A \text{ for a.a. } t \in \mathbb{R}_+\},$$

where X is the space defined in the previous section. Under (H1) \bar{A} is maximal monotone in X . Recall that for all $\lambda > 0$ the realization of the resolvent operator $J_\lambda = (I + \lambda A)^{-1}$ is equal to $(I + \lambda \bar{A})^{-1}$, and the realization of the Yosida approximation $A_\lambda = \lambda^{-1}(I - J_\lambda)$ coincides with the Yosida approximation of \bar{A} , i.e., $\bar{A}_\lambda = (\bar{A})_\lambda =: \bar{A}_\lambda$. For details, see, e.g., [11], p. 31.

Theorem 3.1. *If (H1) and (H2) hold, $x \in D(A)$ and $f \in X$, then there exists a unique $u \in D(B)$ satisfying equation (E), where B and its domain $D(B)$ have been defined in the previous section.*

Proof. We devide the proof into several steps.

Step 1: the case $f = f_n$ for a given positive integer n , where $f_n(t) = f(t)$ for a.a. $t \in (0, n)$, and $f_n(t) = 0$ for a.a. $t > n$.

Obviously, f_n converges in X to f . In a first stage, we assume that n is fixed. For $\lambda > 0$ we denote by $u_\lambda \in D(B)$ the unique solution of the equation

$$Bu_\lambda + \bar{A}_\lambda u_\lambda + \lambda u_\lambda = -f_n. \quad (3.1)$$

The existence of u_λ follows from the maximality of the sum $B + \bar{A}_\lambda$. Obviously, (3.1) can be written as

$$-(a(t)u'_\lambda(t))' + b(t)A_\lambda u_\lambda(t) + \lambda b(t)u_\lambda(t) = -b(t)f_n(t) \text{ for a.a. } t \in \mathbb{R}_+. \quad (3.2)$$

Now, we multiply (3.2) by $u_\lambda(t)$ and then integrate over $[0, t]$:

$$\begin{aligned} & (u'_\lambda(0), x) - a(t)(u'_\lambda(t), u_\lambda(t)) + \int_0^t a(s)\|u'_\lambda(s)\|^2 ds + \lambda \int_0^t b(s)\|u_\lambda(s)\|^2 ds \\ & \leq \int_0^n b(s)\|f_n(s)\| \cdot \|u_\lambda(s)\| ds \leq \|f_n\|_X \cdot \|\sqrt{b}u_\lambda\|_{L^2(0, n; H)}. \end{aligned} \quad (3.3)$$

We have used the monotonicity of A_λ and the fact that $A_\lambda 0 = 0$. Since $u_\lambda \in D(B)$, we have by Lemma 2.1

$$\lim_{t \rightarrow \infty} a(t)(u_\lambda(t), u'_\lambda(t)) = 0. \quad (3.4)$$

By (3.3) and (3.4) we derive

$$p_0 \|u'_\lambda\|_X^2 + \lambda \|u_\lambda\|_X^2 \leq -(u'_\lambda(0), x) + \|f_n\|_X \left(\int_0^n b(s) \|u_\lambda\|^2 ds \right)^{1/2}. \quad (3.5)$$

Using in (3.5) the identity

$$u_\lambda(s) = x + \int_0^s u'_\lambda(\tau) d\tau, \quad s \in [0, n],$$

we obtain

$$p_0 \|u'_\lambda\|_X^2 + \lambda \|u_\lambda\|_X^2 \leq \|x\| \cdot \|u'_\lambda(0)\| + \|f_n\|_X (C_1 + C_2 \|u'_\lambda\|_X), \quad (3.6)$$

where C_1, C_2 are positive constants (depending on n). In what follows we will denote by C_i ($i = 3, 4, \dots$) different positive constants. Estimate (3.6) implies

$$\frac{p_0}{2} \|u'_\lambda\|_X^2 + \lambda \|u_\lambda\|_X^2 \leq \|x\| \cdot \|u'_\lambda(0)\| + C_3. \quad (3.7)$$

Since A_λ is monotone and Lipschitzian,

$$a(t) ((A_\lambda u_\lambda(t))', u'_\lambda(t)) \geq 0 \quad \text{for a.a. } t > 0. \quad (3.8)$$

Integration of (3.8) over $[0, t]$ yields

$$\begin{aligned} & (A_\lambda u_\lambda(t), a(t) u'_\lambda(t)) - (A_\lambda x, u'_\lambda(0)) \\ & \geq \int_0^t (A_\lambda u_\lambda(s), (a(s) u'_\lambda(s))') ds \\ & = \int_0^t (A_\lambda u_\lambda(s), b(s) A_\lambda u_\lambda(s) + \lambda b(s) u_\lambda(s) + b(s) f_n(s)) ds \\ & \geq \int_0^t b(s) \|A_\lambda u_\lambda(s)\|^2 ds + \int_0^t b(s) (A_\lambda u_\lambda(s), f_n(s)) ds. \end{aligned} \quad (3.9)$$

Note that

$$\|A_\lambda u_\lambda(t)\| \leq \frac{1}{\lambda} \|u_\lambda(t)\|,$$

so letting $t \rightarrow \infty$ in (3.9), we get (cf. Lemma 2.1)

$$\|\bar{A}_\lambda u_\lambda\|_X^2 \leq \|\bar{A}_\lambda u_\lambda\|_X \|f_n\|_X - (A^0 x, u'_\lambda(0)), \quad (3.10)$$

where A^0 is the minimal section of A . Therefore,

$$\|\bar{A}_\lambda u_\lambda\|_X^2 \leq \|f_n\|_X + 2 \|A^0 x\| \cdot \|u'_\lambda(0)\|. \quad (3.11)$$

Using (3.1), (3.7) and (3.11) we can derive the estimate

$$\begin{aligned} \|u''_\lambda\|_X & \leq \frac{1}{p_0} (\|f_n\|_X + \|q\|_{L^\infty(\mathbb{R}_+)} \|u'_\lambda\|_X + \lambda \|u_\lambda\|_X + \|\bar{A}_\lambda u_\lambda\|_X) \\ & \leq C_4 + C_5 \|u'_\lambda(0)\|^{1/2} \quad \text{for all } 0 < \lambda \leq \lambda_0, \end{aligned} \quad (3.12)$$

where λ_0 is an arbitrary but fixed constant. On the other hand,

$$\begin{aligned} \|u'_\lambda(0)\|^2 &= - \int_0^\infty (a(t)\|u'_\lambda(t)\|^2)' dt \\ &= - \int_0^\infty q(t)b(t)\|u'_\lambda(t)\|^2 dt - 2 \int_0^t a(t)(u'_\lambda(t), u''_\lambda(t)) dt, \end{aligned} \quad (3.13)$$

which yields

$$\begin{aligned} \|u'_\lambda(0)\|^2 &\leq C_6(\|u'_\lambda\|_X^2 + \|u''_\lambda\|_X^2) \\ &\text{(by (3.7) and (3.12)) } \leq C_7\|u'_\lambda(0)\| + C_8, \quad \forall \lambda \in (0, \lambda_0] \end{aligned} \quad (3.14)$$

which shows that $\{\|u'_\lambda(0)\|, 0 < \lambda \leq \lambda_0\}$ is a bounded set. This fact combined with (3.7), (3.11), (3.12) implies that $u'_\lambda, u''_\lambda, \bar{A}_\lambda u_\lambda$ are all bounded in X for $0 < \lambda \leq \lambda_0$. The set $\{u_\lambda : 0 < \lambda \leq \lambda_0\}$ is also bounded in X . This follows by integration over $[0, \infty)$ of the identity

$$(a(t)\|u_\lambda(t)\|^2)' = q(t)b(t)\|u_\lambda(t)\|^2 + 2a(t)(u_\lambda(t), u'_\lambda(t)),$$

which leads to

$$-\|x\|^2 = \int_0^t q(t)b(t)\|u_\lambda(t)\|^2 dt + 2 \int_0^t a(t)(u_\lambda(t), u'_\lambda(t)) dt$$

and therefore

$$\omega\|u_\lambda\|_X^2 \leq \|x\|^2 + 2p_1\|u_\lambda\|_X\|u'_\lambda\|_X,$$

where $\omega = \text{ess inf}|q|$ (i.e., $\omega = q_0$ if $q_0 > 0$ and $\omega = -q_1$ if $q_1 < 0$). By the facts we have established so far, there exists a $u \in X$ (depending on n but for the moment n is fixed), such that $u', u'' \in X$ and

$$u_\lambda \rightarrow u, u'_\lambda \rightarrow u', u''_\lambda \rightarrow u'' \text{ weakly in } X, \quad (3.15)$$

as $\lambda \rightarrow 0$, on a subsequence. Now, for $\lambda, \nu \in (0, \lambda_0]$, we can easily derive from equation (3.2)

$$\begin{aligned} -(a(u'_\lambda - u'_\nu)', u_\lambda - u_\nu) + b(A_\lambda u_\lambda - A_\nu u_\nu, u_\lambda - u_\nu) \\ + b(\lambda u_\lambda - \nu u_\nu, u_\lambda - u_\nu) = 0 \quad \text{for a.a. } t > 0. \end{aligned}$$

Integration over $[0, \infty)$ gives

$$\begin{aligned} p_0\|u'_\lambda - u'_\nu\|_X^2 + (\bar{A}_\lambda u_\lambda - \bar{A}_\nu u_\nu, \bar{J}_\lambda u_\lambda - \bar{J}_\nu u_\nu)_X \\ \leq -(\bar{A}_\lambda u_\lambda - \bar{A}_\nu u_\nu, \lambda \bar{A}_\lambda u_\lambda - \nu \bar{A}_\nu u_\nu)_X - (\lambda u_\lambda - \nu u_\nu, u_\lambda - u_\nu)_X. \end{aligned} \quad (3.16)$$

Since u_λ and $\bar{A}_\lambda u_\lambda$ are bounded in X for $0 < \lambda \leq \lambda_0$ and $\bar{A}_\lambda u_\lambda \in \bar{A}\bar{J}_\lambda u_\lambda$, we can derive from (3.16)

$$\|u'_\lambda - u'_\nu\|_X^2 \leq C_9(\lambda + \nu), \quad (3.17)$$

so, on a subsequence,

$$u'_\lambda \rightarrow u' \text{ strongly in } X, \text{ as } \lambda \rightarrow 0. \quad (3.18)$$

Let $T \in (0, \infty)$ be arbitrary but fixed. We have

$$u'_\lambda \rightarrow u' \text{ strongly in } L^2(0, T; H), \quad (3.19)$$

$$u''_\lambda \rightarrow u'' \text{ weakly in } L^2(0, T; H). \quad (3.20)$$

Moreover,

$$\begin{aligned} \|u_\lambda(t) - u_\nu(t)\| &= \left\| \int_0^t [u'_\lambda(s) - u'_\nu(s)] ds \right\| \\ &\leq T^{1/2} \|u'_\lambda - u'_\nu\|_{L^2(0,T;H)}, \quad 0 \leq t \leq T, \end{aligned}$$

which implies

$$u_\lambda \rightarrow u \quad \text{in } C([0, T]; H), \quad (3.21)$$

thus in particular $u(0) = x$ and $u_\lambda \rightarrow u$ in $L^2(0, T; H)$. Note also that

$$\|J_\lambda u_\lambda(\cdot) - u\|_{L^2(0,T;H)} \leq \lambda \|A_\lambda u_\lambda(\cdot)\|_{L^2(0,T;H)} + \|u_\lambda - u\|_{L^2(0,T;H)},$$

which implies

$$J_\lambda u_\lambda(\cdot) \rightarrow u \quad \text{strongly in } L^2(0, T; H). \quad (3.22)$$

By (3.19), (3.20) and equation (3.1) it follows that

$$A_\lambda u_\lambda(\cdot) \rightarrow pu'' + qu' - f_n \quad \text{weakly in } L^2(0, T; H). \quad (3.23)$$

Since $A_\lambda u_\lambda(t) \in AJ_\lambda u_\lambda(t)$ and the realization of A in $L^2(0, T; H)$ is a maximal monotone operator in this space, hence demiclosed, we can deduce from (3.22) and (3.23) that u satisfies equation (E) for a.a. $t \in (0, T)$. Since T was arbitrarily chosen, u satisfies (E) for a.a. $t \in \mathbb{R}_+$.

Step 2: general $f \in X$.

From now on we consider n variable and denote by u_n the solution corresponding to f_n whose existence was proved above, i.e.,

$$-f_n \in Bu_n + \bar{A}u_n. \quad (3.24)$$

It is easily seen that for a.a. $t > 0$

$$b(-f_n + f_m) \in -(a(u'_n - u'_m))' + b(\bar{A}u_n - \bar{A}u_m), \quad (3.25)$$

which implies

$$p_0 \|u'_n - u'_m\|_X^2 \leq \|f_n - f_m\|_X \|u_n - u_m\|_X. \quad (3.26)$$

On the other hand, if we integrate over $[0, \infty)$ the equation

$$(a\|u_n - u_m\|^2)' = qb\|u_n - u_m\|^2 + 2a(u_n - u_m, u'_n - u'_m), \quad (3.27)$$

we derive

$$\|u_n - u_m\|_X \leq C_{10} \|u'_n - u'_m\|_X.$$

This inequality combined with (3.26) shows that both (u_n) and (u'_n) are Cauchy sequences in X , so there exists $u \in X$ such that $u' \in X$,

$$u_n \rightarrow u, \quad u'_n \rightarrow u' \quad \text{strongly in } X, \quad (3.28)$$

and $u(0) = x$.

Note that, for a fixed n , u_n is approximated by the solution of equation (3.1), as $\lambda \rightarrow 0^+$, where the term λu_λ is omitted (we do not need this term at this stage). The existence of a solution u_λ for this modified (3.1) follows by a reasoning similar to that used in the proof of Step 1. By (3.19) and (3.20) (that hold true again) we easily see that $u'_\lambda(t)$ converges uniformly to $u'_n(t)$ on every compact interval $[0, T]$ as λ tends to 0. In particular,

$$u'_\lambda(0) \rightarrow u'_n(0), \quad \text{as } \lambda \rightarrow 0. \quad (3.29)$$

Returning to (3.9) (or (3.10)), where the term λbu_λ is omitted, we see that

$$-(A^0 x, u'_n(0)) \geq \|w_n\|_X^2 + (w_n, f_n)_X,$$

where w_n is the weak limit in X of $\bar{A}_\lambda u_\lambda$, as $\lambda \rightarrow 0^+$, $w_n \in \bar{A}u_n$. Therefore,

$$\|w_n\|_X^2 \leq \|f\|_X + 2\|A^0 x\| \cdot \|u'_n(0)\|. \quad (3.30)$$

Using (3.28) and (3.30) in equation (3.24), more precisely, in equation

$$Bu_n + w_n = -f_n,$$

we obtain the analogue of (3.12)

$$\|u''_n\|_X \leq \tilde{C}_4 + \tilde{C}_5 \|u'_n(0)\|^{1/2}. \quad (3.31)$$

We can also derive, as we did before, the analogue of (3.14) and thus $\|u'_n(0)\|$ is bounded. By virtue of (3.31), $\|u''_n\|_X$ is bounded too. Hence $u'' \in X$ and u''_n converges weakly in X to u'' , on a subsequence. Starting from (3.24), using in particular the fact that f_n converges in X to f , we can show by the standard procedure that u satisfies equation (E) for a.a. $t > 0$.

Step 3: Uniqueness.

Let $v \in D(B)$ be another solution of equation (E) that (satisfies $v(0) = x$ and) corresponds to the same $f \in X$. Multiplying by $u(t) - v(t)$ the obvious equation (inclusion)

$$(a(u' - v'))' \in b(Au - Av) \quad \text{for a.a. } t > 0, \quad (3.32)$$

and integrating the resulting equation over $[t, \infty)$, we obtain

$$\frac{1}{2}a(t)\frac{d}{dt}\|u(t) - v(t)\|^2 + \int_t^\infty a(s)\|u'(s) - v'(s)\|^2 ds \leq 0, \quad (3.33)$$

which implies that

$$\frac{d}{dt}\|u(t) - v(t)\|^2 \leq 0 \quad \text{for all } t \geq 0,$$

so $t \rightarrow \|u(t) - v(t)\|$ is nonincreasing on \mathbb{R}_+ . In particular,

$$\|u(t) - v(t)\| \leq \|u(0) - v(0)\| = 0 \quad \forall t \geq 0, \quad (3.34)$$

which implies $u \equiv v$. The theorem is completely proved. \square

Remark 3.2. Note that we also have uniqueness for u_λ and u_n above so all the convergences are true for the whole sequences involved.

4. THE CASE $x \in \overline{D(A)}$ AND $f \in X$

Let $x_n \in D(A)$, $\|x_n - x\| \rightarrow 0$. Denote by u_n the solution of equation (E) given by Theorem 3.1 satisfying $u_n(0) = x_n$. If we multiply the equation

$$(a(u'_n - u'_m))' \in b(Au_n - Au_m) \quad \text{for a.a. } t > 0, \quad (4.1)$$

by $u_n(t) - u_m(t)$ and integrate over $[t, \infty)$, we get

$$\frac{1}{2}a(t)\frac{d}{dt}\|u_n(t) - u_m(t)\|^2 \leq - \int_t^\infty a(s)\|u'_n(s) - u'_m(s)\|^2 ds \leq 0 \quad \text{for all } t \geq 0. \quad (4.2)$$

It follows that

$$\frac{d}{dt}\|u_n(t) - u_m(t)\|^2 \leq 0 \quad \text{for all } t \geq 0, \quad (4.3)$$

so $t \mapsto \|u_n(t) - u_m(t)\|$ is nonincreasing on \mathbb{R}_+ . In particular,

$$\|u_n(t) - u_m(t)\| \leq \|x_n - x_m\| \quad \text{for all } t \geq 0. \quad (4.4)$$

Thus there exists $u \in C(\mathbb{R}_+; H)$, such that u_n converges to u in $C([0, T]; H)$ for all $T \in (0, \infty)$ and $u(0) = x$. If f is identically zero, then it is easy to see that $t \rightarrow \|u_n(t)\|$ is nonincreasing and hence

$$\|u_n(t)\| \leq \|u_n(0)\| = \|x_n\| \quad \text{for all } t \geq 0,$$

hence $u \in L^\infty(\mathbb{R}_+; H)$.

Returning to the nonhomogeneous case $f \in X$, if we integrate over $[0, T]$ the obvious inequality

$$t((a(u'_n - u'_m)))', u_n - u_m) \geq 0 \quad \text{for a.a. } t > 0,$$

we get

$$\begin{aligned} & \int_0^T ta(t)\|u'_n(t) - u'_m(t)\|^2 dt + \frac{1}{2} \int_0^T a(t) \frac{d}{dt} \|u_n(t) - u_m(t)\|^2 dt \\ & \leq \frac{T}{2} a(T) \left(\frac{d}{dt} \|u_n - u_m\|^2 \right) (T) \leq 0. \end{aligned}$$

due to (4.3). This implies

$$\begin{aligned} & \int_0^T ta(t)\|u'_n(t) - u'_m(t)\|^2 dt \leq \frac{1}{2} \|x_n - x_m\|^2 + \frac{1}{2} \int_0^T \frac{aq}{p} \|u_n - u_m\|^2 dt \\ & \quad (\text{according to (4.4)}) \leq \frac{1}{2} \|x_n - x_m\|^2 \left(1 + \int_0^T \frac{a|q|}{p} dt \right). \end{aligned} \quad (4.5)$$

By (4.4) and (4.5) we see that $t^{1/2}u' \in L^2(0, T; H)$ and

$$t^{1/2}u'_n \rightarrow t^{1/2}u' \quad \text{strongly in } L^2(0, T; H). \quad (4.6)$$

It is easy to see that u does not depend on the choice of the sequence (x_n) approximating x . We can call u a *generalized solution* of equation (E) satisfying $u(0) = x \in \overline{D(A)}$. If $f \equiv 0$, then u also satisfies (C). This is also the case if the condition $0 \in D(A)$, $0 \in A0$ is replaced by $0 \in R(A)$.

We do not have an estimate for u''_n to prove that u is a classic solution of equation (E). So the existence of a classic solution for $x \in \overline{D(A)}$ is still an open problem. However, if p and q are constant functions we are able to obtain classic existence for all $x \in \overline{D(A)}$ (see the next section).

5. EXISTENCE AND UNIQUENESS FOR $x \in \overline{D(A)}$ IN THE CASE OF CONSTANT COEFFICIENTS

In this section we assume that p and q are both constant functions. Without any loss of generality, we can assume $p \equiv 1$ and $q \equiv c$, with $c \in \mathbb{R} \setminus \{0\}$. Recall that the case $c = 0$ was extensively studied by V. Barbu [3], [4], [5] and by H. Brezis [6]. We are going to prove that in this particular case problem (E), (B) has a unique classical solution for each $x \in \overline{D(A)}$ and $f \in X$. Before stating precisely the result, we need some definitions. For $\varepsilon > 0$ small, define $\zeta_\varepsilon: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$\zeta_\varepsilon(t) = \begin{cases} t & \text{if } 0 \leq t \leq \varepsilon \\ \varepsilon & \text{if } t > \varepsilon \end{cases}$$

Let X_ε be the weighted space $L^2(\mathbb{R}_+; H; \zeta_\varepsilon(t)e^{ct} dt)$. For $\alpha < c$, close to c , denote $X_\alpha = L^2(\mathbb{R}_+; H; t^3 e^{\alpha t} dt)$. Obviously, $X \subset X_\varepsilon \subset X_\alpha$, where X is the space defined in Section 2, with the particular weight $b(t) = e^{ct}$, i.e., $X = L^2(\mathbb{R}_+; H; e^{ct} dt)$.

Theorem 5.1. *If (H1) holds, $p \equiv 1$, $q \equiv c$, with $c \in \mathbb{R} \setminus \{0\}$, $x \in \overline{D(A)}$ and $f \in X$, then there exists a unique $u \in C(\mathbb{R}_+; H)$, $u, u' \in X_\varepsilon$, $u'' \in X_\alpha$, such that u satisfies (E) and $u(0) = x$, for all $\varepsilon > 0$ small and $\alpha < c$, close to c . If $f \equiv 0$, then $\|u(t)\| \leq \|x\|$ for all $t \geq 0$.*

Proof. In the present case $a(t) = b(t) = e^{ct}$. Let $x_n \in D(A)$, $\|x_n - x\| \rightarrow 0$, and let $u_n \in X$ be the unique solution of (E) with $f = f_n$ given by Theorem 3.1, satisfying $u_n(0) = x_n$:

$$(e^{ct}u'_n(t))' \in e^{ct}(Au_n(t) + f_n(t)) \quad \text{for a.a. } t > 0, \quad (5.1)$$

where f_n is the truncation of f defined in Section 3. In what follows we develop a technique inspired by Bruck's paper [8]. For m, n two fixed positive integers, define

$$g(t) = \frac{e^{ct}}{2} \|u_n(t) - u_m(t)\|^2.$$

We have,

$$g(t) = cg(t) + e^{ct}(u'_n(t) - u'_m(t), u_n(t) - u_m(t)), \quad (5.2)$$

$$\begin{aligned} g''(t) &= cg'(t) + ce^{ct}(u'_n - u'_m, u_n - u_m) + e^{ct}(u''_n - u''_m, u_n - u_m) + e^{ct}\|u'_n - u'_m\|^2 \\ &= c^2g(t) + 2ce^{ct}(u'_n - u'_m, u_n - u_m) \\ &\quad + e^{ct}(-c(u'_n - u'_m) + Au_n - Au_m + f_n - f_m, u_n - u_m) + e^{ct}\|u'_n - u'_m\|^2 \\ &\geq e^{ct} \left\{ \frac{c^2}{2} \|u_n - u_m\|^2 + c(u'_n - u'_m, u_n - u_m) + \|u'_n - u'_m\|^2 \right\} \\ &\quad + e^{ct}(f_n - f_m, u_n - u_m) \\ &\geq \gamma e^{ct} \left\{ \|u_n - u_m\|^2 + \|u'_n - u'_m\|^2 \right\} + e^{ct}(f_n - f_m, u_n - u_m) \end{aligned} \quad (5.3)$$

for $\gamma > 0$ a small constant. From (5.2) and (5.3) it follows that

$$\begin{aligned} &\gamma \int_0^\infty \zeta_\varepsilon(t) e^{ct} \left\{ \|u_n - u_m\|^2 + \|u'_n - u'_m\|^2 \right\} dt \\ &\leq \int_0^\infty \zeta_\varepsilon(t) g''(t) dt + \int_0^\infty \zeta_\varepsilon(t) e^{ct} \|f_n - f_m\| \cdot \|u_n - u_m\| dt \\ &\leq \int_0^\varepsilon tg''(t) dt + \varepsilon \int_\varepsilon^\infty g''(t) dt + \frac{1}{2\gamma} \int_0^\infty \zeta_\varepsilon(t) e^{ct} \|f_n - f_m\|^2 dt \\ &\quad + \frac{\gamma}{2} \int_0^\infty \zeta_\varepsilon(t) e^{ct} \|u_n - u_m\|^2 dt \\ &\leq tg'(t)|_0^\varepsilon - \int_0^\varepsilon g'(t) dt + \varepsilon[g'(\infty) - g'(\varepsilon)] + \frac{1}{2\gamma} \|f_n - f_m\|_X^2 + \frac{\gamma}{2} \|u_n - u_m\|_{X_\varepsilon}^2 \\ &\leq g(0) + \frac{1}{2\gamma} \|f_n - f_m\|_X^2 + \frac{\gamma}{2} \|u_n - u_m\|_{X_\varepsilon}^2. \end{aligned}$$

Therefore,

$$\frac{\gamma}{2} \|u_n - u_m\|_{X_\varepsilon}^2 + \gamma \|u'_n - u'_m\|_{X_\varepsilon}^2 \leq \frac{1}{2} \|x_n - x_m\|^2 + \frac{1}{2\gamma} \|f_n - f_m\|_X^2. \quad (5.4)$$

Thus there exists $u \in X_\varepsilon$, such that $u' \in X_\varepsilon$, and

$$u_n \rightarrow u, \quad u'_n \rightarrow u' \quad \text{strongly in } X_\varepsilon. \quad (5.5)$$

Now, in order to derive an estimate for u_n'' , we recall that, for a fixed positive integer n , the solution $u_{n\lambda} \in X$ of

$$u_{n\lambda}'' + cu_{n\lambda}' = A_\lambda u_{n\lambda} + f_n, \quad (5.6)$$

approximates u_n as $\lambda \rightarrow 0+$:

$$u_{n\lambda} \rightarrow u_n \text{ in } C([0, T]; H) \quad \forall T > 0, \quad u_{n\lambda}' \rightarrow u_n' \text{ strongly in } X, \quad u_{n\lambda}'' \rightarrow u_n'' \text{ weakly in } X. \quad (5.7)$$

Since $(u_{n\lambda}', (A_\lambda u_{n\lambda})') \geq 0$, we have

$$\begin{aligned} \frac{d}{dt}(e^{\alpha t} u_{n\lambda}', A_\lambda u_{n\lambda}) &\geq (e^{\alpha t} u_{n\lambda}')', A_\lambda u_{n\lambda}) \\ &= (A_\lambda u_{n\lambda} + (\alpha - c)u_{n\lambda}' + f_n, e^{\alpha t} A_\lambda u_{n\lambda}). \end{aligned} \quad (5.8)$$

Multiplying (5.8) by t^3 and then integrating over \mathbb{R}_+ , we obtain

$$\begin{aligned} \|A_\lambda u_{n\lambda}\|_{X_\alpha}^2 &\leq \|A_\lambda u_{n\lambda}\|_{X_\alpha} \cdot \|f_n\|_{X_\alpha} + (c - \alpha) \|A_\lambda u_{n\lambda}\|_{X_\alpha} \cdot \|u_{n\lambda}'\|_{X_\alpha} \\ &\quad - 3 \int_0^\infty t^2 e^{\alpha t} (u_{n\lambda}', A_\lambda u_{n\lambda}) dt \\ &\leq \|A_\lambda u_{n\lambda}\|_{X_\alpha} \cdot \|f_n\|_{X_\alpha} + (c - \alpha) \|A_\lambda u_{n\lambda}\|_{X_\alpha} \cdot \|u_{n\lambda}'\|_{X_\alpha} \\ &\quad + 3 \|A_\lambda u_{n\lambda}\|_{X_\alpha} \left(\int_0^\infty t e^{\alpha t} \|u_{n\lambda}'\|^2 dt \right)^{1/2}. \end{aligned}$$

We have used α instead of c , $\alpha < c$, so for any polynomial P , $|P(t)|e^{\alpha t} \leq C_{11}e^{ct}$, so that all the above computations are permitted. The last estimate yields

$$\|A_\lambda u_{n\lambda}\|_{X_\alpha} \leq \|f\|_{X_\alpha} + (c - \alpha) \|u_{n\lambda}'\|_{X_\alpha} + 3 \left(\int_0^\infty t e^{\alpha t} \|u_{n\lambda}'\|^2 dt \right)^{1/2}. \quad (5.9)$$

Now, combining (5.6) and (5.9), we find

$$\begin{aligned} \|u_{n\lambda}''\|_{X_\alpha} &\leq |c| \|u_{n\lambda}'\|_{X_\alpha} + \|A_\lambda u_{n\lambda}\|_{X_\alpha} + \|f_n\|_{X_\alpha} \\ &\leq 2|c| \|u_{n\lambda}'\|_{X_\alpha} + 2\|f\|_{X_\alpha} + 3 \left(\int_0^\infty t e^{\alpha t} \|u_{n\lambda}'\|^2 dt \right)^{1/2}. \end{aligned} \quad (5.10)$$

Since $u_{n\lambda}' \rightarrow u_n'$ strongly in X and $u_{n\lambda}'' \rightarrow u_n''$ weakly in X , as $\lambda \rightarrow 0+$, we derive from (5.10)

$$\|u_n''\|_{X_\alpha} \leq 2|c| \|u_n'\|_{X_\alpha} + 2\|f\|_{X_\alpha} + 3 \left(\int_0^\infty t e^{\alpha t} \|u_n'\|^2 dt \right)^{1/2}. \quad (5.11)$$

We know that u_n' is convergent (hence bounded) in X_ε so (5.11) shows that u_n'' is bounded in X_α . Thus $u'' \in X_\alpha$ and u_n'' converges weakly in X_α to u'' . Letting $n \rightarrow \infty$ in

$$u_n'' + cu_n' \in Au_n + f_n,$$

with respect to the topology of $L^2((\delta, T); H)$ for $0 < \delta < T < \infty$ we can show by the standard procedure that u satisfies equation (E) for a.a. $t \in (\delta, T)$, so for a.a. $t > 0$ (since δ and T can be chosen arbitrarily). Note that the weights are not relevant on $[\delta, T]$. Now let us prove that $u \in C(\mathbb{R}_+; H)$ and $u(0) = x$. Of course,

we only need to prove continuity at $t = 0+$. From equation (5.2) we easily derive

$$\begin{aligned}
& \frac{1}{2}e^{ct} \frac{d}{dt} \|u_n(t) - u_m(t)\|^2 + \int_t^\infty e^{cs} \|u'_n(s) - u'_m(s)\|^2 ds \\
& \leq \int_t^\infty e^{cs} (f_n(s) - f_m(s), u_n(s) - u_m(s)) ds \\
& \leq \int_m^n e^{cs} \|f(s)\| \cdot \|u_n(s) - u_m(s)\| ds \\
& \leq \|f\|_X \|u_n - u_m\|_{X_\varepsilon} < \eta \quad \text{for } N_\eta < m < n.
\end{aligned} \tag{5.12}$$

If we multiply (5.12) by e^{-ct} and then integrate over $[0, t]$, we get

$$\frac{1}{2} \|u_n(t) - u_m(t)\|^2 \leq \frac{1}{2} \|x_n - x_m\|^2 + \eta t \quad \text{for } N_\eta < m < n.$$

Thus $u_n(t)$ converges uniformly as $n \rightarrow \infty$ on a compact interval $[0, t_0]$ to a continuous function $v = v(t)$. In particular, $v(0) = x$. From the previous part of the proof $u_n \rightarrow u$ and $u'_n \rightarrow u'$ in X_ε , for all $\varepsilon > 0$, so u is continuous on $(0, \infty)$. Obviously, $u(t) = v(t)$ for $t \in (0, t_0]$. It follows that

$$\lim_{t \rightarrow 0^+} u(t) = v(0) = x,$$

and $u \in C(\mathbb{R}_+; H)$.

Concerning uniqueness, it follows by a reasoning we have already used. Indeed, if u, v are two solutions corresponding to $x \in D(A)$ and $f \in X$ in the class indicated in the statement of the theorem, then

$$\frac{1}{2}e^{ct} \|u(t) - v(t)\|^2 + \int_t^\infty e^{cs} \|u'(s) - v'(s)\|^2 ds \leq 0, \quad \text{for all } t > 0.$$

This implies that $t \rightarrow \|u(t) - v(t)\|$ is nonincreasing and thus

$$\|u(t) - v(t)\| \leq \|u(0) - v(0)\| = 0, \quad \text{for all } t \geq 0, \tag{5.13}$$

i.e., $u \equiv v$. If u is the solution corresponding to $x \in \overline{D(A)}$, $f \equiv 0$ and v is the solution corresponding to $x = 0$, $f \equiv 0$, i.e., $v \equiv 0$, then (5.13) yields

$$\|u(t)\| \leq \|x\|, \quad \text{for all } t \geq 0.$$

The proof is now complete. \square

Remark 5.2. *If $c < 0$, then condition $f \in X$ allows unbounded f 's. In this case the corresponding u 's are so, as illustrated by simple examples.*

Remark 5.3. *In Theorem 5.1 we have $u'' \in L^2([\delta, \infty); H; e^{ct} dt)$ for all $\delta > 0$. Indeed, since u satisfies equation (E) for a.a. $t > 0$, for every $\delta > 0$, there is a $t_0 \in (0, \delta)$, such that $u(t_0) \in D(A)$, so we can apply Theorem 3.1 with $x := u(t_0)$ and $[t_0, \infty)$ instead of \mathbb{R}_+ . Obviously, $u, u' \in L^2([\delta, \infty); H; e^{ct} dt)$ as well, for all $\delta > 0$.*

6. CONSTANT COEFFICIENTS AND $f \equiv 0$

Consider the homogeneous equation

$$u''(t) + cu' \in Au(t), \quad t > 0, \tag{E_0}$$

with

$$u(0) = x, \tag{B}$$

where $c \in \mathbb{R} \setminus \{0\}$. Recall that, if $c > 0$, the boundedness condition (C) added to (E) is not enough to guarantee uniqueness of u . However, we have uniqueness if we impose the condition $u \in X$, where $X = L_b^2(\mathbb{R}_+; H)$, with $b(t) = e^{ct}$. It is easy to see that the solutions of problem (E_0) , (B) given by Theorems 3.1 and 5.1, generate a nonlinear semigroup of contractions $\{S(t): \overline{D(A)} \rightarrow \overline{D(A)}; t \geq 0\}$, $S(t)x := u(t)$, where $u \in X$ satisfies (E_0) and (B). We also have the properties

$$\forall x \in D(A), e^{ct/2}S(t)x, e^{ct/2}\frac{d}{dt}S(t)x, e^{ct/2}\frac{d^2}{dt^2}S(t)x \in L^2(\mathbb{R}_+; H); \quad (6.1)$$

$$\forall x \in \overline{D(A)}, e^{ct/2}S(t)x, e^{ct/2}\frac{d}{dt}S(t)x, e^{ct/2}\frac{d^2}{dt^2}S(t)x \in L^2([\varepsilon, \infty); H) \quad \forall \varepsilon > 0. \quad (6.2)$$

The last property in (6.2) follows from the fact that we can use as the initial state $u(t_0) \in D(A)$ for some $t_0 > 0$ instead of $x \in \overline{D(A)}$.

Let $F: D(F) \subset H \rightarrow H$ be such that $G = -F$ is the generator of the semigroup $\{S(t): \overline{D(A)} \rightarrow \overline{D(A)}; t \geq 0\}$ defined above. Operator F , which is maximal monotone, can be regarded as the formal solution of the operator equation

$$F^2 + cF - A = 0,$$

i.e.,

$$F = \frac{c}{2}I + \sqrt{\frac{c^2}{4}I + A}, \quad (6.3)$$

as remarked by B. Djafari Rouhani and H. Khatibzadeh [9]. Here $\sqrt{\quad}$ represents the square root of $\frac{c^2}{4}I + A$ in Barbu's sense. If A is linear, then $G = -F$, where F is given by (6.3) is indeed the generator of the semigroup $\{S(t): \overline{D(A)} \rightarrow \overline{D(A)}; t \geq 0\}$. To show this, define $v(t) = e^{ct/2}u(t)$, where $u(t) = S(t)x$, $x \in D(A)$. Then,

$$\begin{aligned} v''(t) &= e^{ct/2}\left(\frac{c^2}{4}I + A\right)(e^{-ct/2}v(t)) \\ &= \frac{c^2}{4}v(t) + Av(t), \end{aligned}$$

so $v(t) = T(t)x$, where $T(t)$ is the semigroup generated by $-\sqrt{\frac{c^2}{4}I + A}$. Therefore,

$$S(t)x = e^{-ct/2}T(t)x, \quad x \in H,$$

which shows that the generator of $S(t)$ is $G = -F$, where F is given by (6.3), as asserted.

Comments on the asymptotic behavior. By (6.1), (6.2) and Lemma 2.1 (which remains valid if \mathbb{R}_+ is replaced by $[\varepsilon, \infty)$, $\varepsilon > 0$), we see that

$$\lim_{t \rightarrow \infty} e^{ct/2}\|S(t)x\| = \lim_{t \rightarrow \infty} e^{ct/2}\left\|\frac{d}{dt}S(t)x\right\| = 0, \quad (6.4)$$

for all $x \in \overline{D(A)}$. On the other hand, Djafari Rouhani and Khatibzadeh [9] have proved that, if $c > 0$ and $0 \in R(A)$, for any bounded solution u of equation (E_0) there exists a $y \in A^{-1}0$, such that

$$\|u(t) - y\| = \mathbf{O}(e^{-ct/2}), \quad \|u'(t)\| = \mathbf{O}(e^{-ct/2}) \quad (6.5)$$

Now are able to describe the set of all bounded solutions of equation (E_0) , under the assumption $0 \in R(A)$.

Indeed, if we define $v(t) := u(t) - y$, $A_y x := A(x + y)$, $x \in D(A_y) = D(A) \setminus \{y\}$, then v is the unique solution in our sense of the equation

$$v'' + cv' \in A_y v, \quad t > 0, \quad (6.6)$$

with $v(0) = u(0) - y$. In fact, the set of all bounded solutions of equation (E_0) is

$$Q := \{u(t) = y + v(t) : v \text{ is a solution in our sense of (6.6), } y \in A^{-1}0\}.$$

For a given $x \in \overline{D(A)}$ equation (E_0) may have several bounded solutions u satisfying $u(0) = x$, if $A^{-1}0$ is not a singleton.

Note that estimates (6.5) are weaker than (6.4). However, it is worth pointing out that recently Khatibzadeh [10] has shown that, if $c > 0$, then $\mathbf{O}(e^{-ct/2})$ in (6.5) can be replaced by $\mathbf{O}(e^{-ct})$.

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