SECOND-ORDER DIFFERENTIAL EQUATIONS ON $\mathbb{R}_+$ GOVERNED BY MONOTONE OPERATORS

GHEORGHE MOROŞANU

ABSTRACT. Consider in a real Hilbert space $H$ the differential equation

$E_1: p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t)$, for a.a. $t \in \mathbb{R}_+ = [0, \infty)$, with the condition $u(0) = x \in D(A)$, where $A: D(A) \subset H \to H$ is a maximal monotone operator, with $[0, 0] \in A$ (or, more generally, $0 \in R(A)$); $p, q \in L^\infty(\mathbb{R}_+)$, with ess inf $p > 0$ and either ess inf $q > 0$ or ess sup $q < 0$; and $f: \mathbb{R}_+ \to H$ is a given function. Recall that equation $E_1$ in the case $p \equiv 1$, $q \equiv 0$, subject to $u(0) = x$ and the boundedness condition $\sup_{t \geq 0} \|u(t)\| < \infty$, was investigated in early 1970’s by V. Barbu, who derived in particular from his results a definition for the square root of the nonlinear operator $A$. Subsequently H. Brezis, N. H. Pavel, L. Véron and others have paid much attention to equation $E_1$. In this paper we prove the existence and uniqueness of the solution to equation $E_1$ subject to $u(0) = x \in D(A)$ in the weighted space $X = L^2_0(\mathbb{R}_+; H)$, where $b(t) = a(t)/p(t)$, $a(t) = \exp(\int_0^t q(s)/p(s) ds)$, under our weak assumptions on $p$ and $q$ (see above) and $f \in X$. For $x \in D(A)$ we prove the existence of a generalized solution in the case of general variable coefficients $p, q$ and a classic solution in the case $p \equiv 1, q \equiv c \in \mathbb{R} \setminus \{0\}$. If $p \equiv 1$, $q(t) \equiv c \in \mathbb{R} \setminus \{0\}$, $f \equiv 0$ the solutions give rise to a nonlinear semigroup of contractions. If $A$ is linear its infinitesimal generator $G$ is given by

$G = -\frac{1}{2}I - \sqrt{\frac{1}{2}}I + A.$

1. Introduction.

Throughout this paper $H$ will be a real Hilbert space with respect to an inner product $(\cdot, \cdot)$ and the induced norm $\|x\| = (x, x)^{1/2}$. Consider the nonlinear second-order equation (inclusion)

$p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t)$, for a.a. $t \in \mathbb{R}_+ = [0, \infty)$, \hspace{1cm} (E)

with the condition

$u(0) = x \in D(A)$, \hspace{1cm} (B)

where

$H_1$: $A: D(A) \subset H \to H$ is a maximal monotone operator, with $0 \in D(A)$ and $0 \in A0$;

$H_2$: $p, q \in L^\infty(\mathbb{R}_+) := L^\infty(\mathbb{R}_+; \mathbb{R})$, with ess inf $p > 0$ and either ess inf $q > 0$ or ess sup $q < 0$;

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and $f : \mathbb{R}_+ \to H$ is a given function which will be described later. In other words, (H2) says that both $p$ and $q$ are measurable and satisfy
\[ 0 < p_0 \leq p(t) \leq p_1 < \infty \quad \text{for a.a. } t \in \mathbb{R}_+, \]
and
\[ q_0 \leq q(t) \leq q_1 \quad \text{for a.a. } t \in \mathbb{R}_+, \]
where either $0 < q_0 < q_1 < \infty$ or $-\infty < q_0 < q_1 < 0$.

**Remark 1.1.** One can assume that $0 \in R(A)$ instead of $0 \in D(A)$, $0 \in A0$. Indeed, if $y \in D(A)$ and $0 \in Ay$, then defining the operator $Ay$ by $D(Ay) = D(A) \setminus \{y\}$, $Ay(z) = A(z + y)$, and $v(t) = u(t) - p$, we have $0 \in D(Ay)$, $0 \in Ay0$ and
\[ p(t)v''(t) + q(t)v'(t) \in Ayv(t) + f(t) \quad \text{for a.a. } t \in \mathbb{R}_+. \quad (E_y) \]
Problem $(E)$, $(B)$ in the case $p \equiv 1$, $q \equiv 0$, $f \equiv 0$, with the additional condition
\[ \sup_{t \geq 0} \|u(t)\| < \infty, \quad (C) \]
was investigated for the first time by V. Barbu [3], [4] (see also [5], Chapter V, Sections 2.3 and 2.4), who derived in particular from his results a definition for the square root of the nonlinear operator $A$. The same problem, with $(B')$ instead of $(B)$, was then studied by H. Brezis [6], where
\[ u'(0) \in \partial j(u(0) - x), \quad (B') \]
with $\partial j$ standing for the subdifferential of a proper, convex, lower semicontinuous function $j : H \to (-\infty, +\infty]$. If in particular $j$ is the indicator function of the set $\{0\}$ (i.e., $j(0) = 0$ and $j(z) = +\infty$ for all $z \in H \setminus \{0\}$), then condition $(B')$ becomes $u(0) = x$. Subsequently L. Véron [12], [13] studied problem $(E)$, $(B)$, $(C)$ with $f \equiv 0$, under the following conditions
\[ p \in W^{2,\infty}(\mathbb{R}_+), \quad q \in W^{1,\infty}(\mathbb{R}_+), \quad p(t) \geq p_0 > 0 \quad \forall t \in \mathbb{R}_+. \quad (1.1) \]
These smoothness assumptions on the coefficients $p$ and $q$ are in fact too strong. Indeed, one can take advantage of the following form of $(E)$, as pointed out by A. R. Aftabizadeh and N. H. Pavel [1]:
\[ (a(t)u'(t))' \in b(t)Au(t) + b(t)f(t), \quad (1.2) \]
where
\[ a(t) = \exp \left( \int_0^t \frac{q(s)}{p(s)} ds \right), \quad b(t) = \frac{a(t)}{p(t)}, \]
that allows significant relaxation in the regularity of $p$ and $q$. Following the technique of [1], N. C. Apreutesei [2] reconsidered $(E)$ on $\mathbb{R}_+$, with $(B')$, in the framework of the weighted space $X = L^2_{\rho}(\mathbb{R}_+; H)$, under the following assumptions on $p, q$:
\[ p, q \in W^{1,\infty}(\mathbb{R}_+), \quad p(t) \geq p_0 > 0, \quad q(t) \geq q_0 > 0 \quad \text{for all } t \in \mathbb{R}_+, \quad \text{and } x \in D(A). \quad (1.3) \]
In this paper we prove the existence and uniqueness of the solution of problem $(E)$, $(B)$ in $X$, under our weak assumptions (H2), by using a simpler treatment. The condition $u \in X$ is more appropriate than the boundedness condition $(C)$ that works well if $p \equiv 1$ and $q \equiv 0$ but does not guarantee uniqueness if $q \equiv c > 0$. For example, if $p \equiv 1$, $q \equiv 1$, $f \equiv 0$ and $A = 0$, then problem $(E)$, $(B)$, $(C)$ has
infinite solutions (and only one belongs to X). Note also that L. Véron [14] derived the condition

$$\int_0^\infty \exp \left( - \int_0^t \frac{q(s)}{p(s)} \, ds \right) \, dt = +\infty$$

as a necessary one for the uniqueness of the solution of problem (E), (B), (C). So the case \( p \equiv 1 \) and \( q \equiv c \), where \( c \) is a positive constant, is excluded in Véron’s paper, but allowed in our present paper.

We obtain classic existence and uniqueness in X to problem (E), (B) for \( x \in \mathcal{D}(A) \) and \( f \in X \). If \( x \in \mathcal{D}(A) \) we prove the existence of a generalized solution (uniform limit on compact intervals of a sequence of classic solutions). If \( f \equiv 0 \), the solution \( u \in X \) is also bounded on \( \mathbb{R}_+ \). In the case of constant coefficients \( p \equiv 1, q \equiv c, c \in \mathbb{R} \setminus \{0\} \), we are able to prove the existence of classic solutions for \( x \in \mathcal{D}(A) \) and \( f \in X \). If in addition \( f \equiv 0 \) the solutions give rise to a nonlinear contraction semigroup generated by \( G = -\frac{c}{2}I - \sqrt{\frac{c^2}{4}I + A} \), where \( I \) is the identity operator, as noticed by B. Djafari Rouhani and H. Khatibzadeh [9]. If \( A \) is linear we can show that \( G \) is indeed the generator of this semigroup. If \( p \equiv 1, q \equiv c > 0, f \equiv 0 \) and \( 0 \in \mathcal{R}(A) \), our results combined with the recent results regarding the asymptotic behavior of solutions reported by B. Djafari Rouhani and H. Khatibzadeh [9], lead to the description of the set of all bounded solutions of equation \( (E) : u(t) = y + v(t), \ y \in A^{-1}0, \) where \( v \in X \) is a solution of the equation \( v'' + cv' \in A_y v, \ t > 0, \) with \( A_y z = A(z + y) \). In particular, according to our present results, \( \|v(t)\| = o(e^{-c t/2}) \) and \( \|v'(t)\| = o(e^{-c t/2}) \), as \( t \to \infty \). In fact the behavior of the v’s is even better: \( \|v(t)\| = O(e^{-ct}) \) and \( \|v'(t)\| = O(e^{-ct}) \), as proved by H. Khatibzadeh [10].

2. Notation and Preparatory Lemmas

Let \( X \) be the set of all (classes with respect to the a.e. equality of) functions \( f: \mathbb{R}_+ \to H \) which are measurable, with \( \int_0^\infty b(t)\|f(t)\|^2 \, dt < \infty \), where

$$b(t) = \frac{a(t)}{p(t)}, \quad a(t) = \exp \left( \int_0^t \frac{q(s)}{p(s)} \, ds \right).$$

In other words, \( X \) is the weighted space \( L^2_0(\mathbb{R}_+; H) \). Note that (under (H2)) \( X \) is a real Hilbert space with respect to the scalar product

$$\langle f, g \rangle_X = \int_0^\infty b(t)(f(t), g(t)) \, dt,$$

and the corresponding norm

$$\|f\|_X^2 = \int_0^\infty b(t)\|f(t)\|^2 \, dt.$$

**Lemma 2.1.** Suppose assumptions (H2) hold. If \( f, f' \in X \) (where \( f' \) is the distributional derivative of \( f \)), then

$$\lim_{t \to \infty} a(t)\|f(t)\|^2 = 0,$$

where \( a \) and \( X \) are defined above.
Proof. Denote \( g(t) = \sqrt{a(t)}f(t) \). We have
\[
g'(t) = \frac{q(t)}{2p(t)}g(t) + \sqrt{a(t)}f'(t),
\]
so both \( g, g' \in L^2(\mathbb{R}_+; H) \). By Hölder’s inequality this implies
\[
\|g(t) - g(s)\| = \left\| \int_s^t g'(\tau)d\tau \right\| \leq (t-s)^{1/2}\|g'\|_{L^2(\mathbb{R}_+; H)},
\]
for all \( 0 \leq s \leq t \). This shows that \( g \) is uniformly continuous on \( \mathbb{R}_+ \) and so is \( \|g\| \).
A straightforward reasoning shows that \( \limsup_{t \to \infty} \|g(t)\| = 0 \), otherwise \( g \) cannot be a member of \( X \).

Now, let us define \( B: D(B) \subset X \to X \) by
\[
D(B) = \{ u \in X: u', u'' \in X, u(0) = x \}, \quad Bu = -pu'' - qu'.
\]

**Lemma 2.2.** If assumptions (H2) hold, then operator \( B \) defined above is the subdifferential of \( \Psi: X \to (-\infty, +\infty] \), \( \Psi(u) = (1/2)\|u''\|_{L^2}^2 + j(u(0) - x) \), which is proper, convex and lower semicontinuous (hence \( B \) is maximal monotone), where \( L^2_a := L^2_a(\mathbb{R}_+; H) \) and \( j \) is the indicator function of the set \( \{0\} \subset H \).

**Proof.** Note that \( Bu = -\frac{p}{2}(au')' \) for all \( u \in D(B) \).

**Claim 1:** \( B \) is monotone.

Indeed, for all \( u, v \in D(B) \), we have
\[
(Bu - Bv, u - v)_X = -\int_0^\infty ((a(u' - v'))', u - v) dt
= a(u' - v', u - v)|_0^\infty + \int_0^\infty a\|u' - v'\|^2 dt \geq 0,
\]
since, according to Lemma 2.1
\[
\lim_{t \to \infty} a(t)(u'(t) - v'(t), u(t) - v(t)) = 0.
\]

**Claim 2:** \( \Psi \) is proper.

For example, the function \( \hat{u} \), defined by
\[
\hat{u}(t) = (1 - t)^2 x, \quad \text{for } 0 \leq x \leq 1 \quad \text{and} \quad \hat{u}(t) = 0 \quad \text{for } t > 1,
\]
satisfies \( \Psi(\hat{u}) < \infty \). The effective domain of \( \Psi \) is \( D(\Psi) = \{ u \in X: u' \in X, u(0) = x \} \).

**Claim 3:** \( \Psi \) is convex.

This is obvious.

**Claim 4:** \( \Psi \) is lower semicontinuous.

It is well known this is equivalent to saying that its level sets \( M_\lambda := \{ u \in X: \Psi(u) \leq \lambda \} (\lambda \in \mathbb{R}) \) are closed in \( X \). If \( \lambda \leq 0 \) then \( M_\lambda \) is empty. Let \( \lambda > 0 \) be an arbitrary but fixed number, and let \( u_n \in M_\lambda, u_n \to u \) in \( X \). Since
\[
\|u_n''\|_X^2 = \int_0^\infty b(t)\|u_n'(t)\|^2 dt \leq \frac{1}{p_0}\|u_n''\|_{L^2}^2 \leq \frac{2}{p_0}\lambda,
\]
we have
\[
\quad u' \in X \quad \text{and} \quad u_n' \to u' \text{ weakly in } X.
\]
Moreover, making use of the formula
\[ u_n(t) = x + \int_0^t u'_n(s) ds, \]
we can easily deduce, by Arzelà’s compactness criterion, that \( u_n(t) \) converges uniformly to \( u(t) \) on every compact interval \([0, T]\). In particular, \( u(0) = x \). Therefore,
\[
\lambda \geq \liminf_{n \to \infty} \Psi(u_n) = \liminf_{n \to \infty} \frac{1}{2} \|u'_n\|_{L^2_2}^2 \\
\geq \frac{1}{2} \|u'\|_{L^2_2}^2 = \Psi(u).
\]

Since \( \Psi \) is proper, convex and lower semicontinuous, it follows that its subdifferential \( \partial \Psi \) is a maximal monotone operator. So, to conclude the proof of Lemma 2.2, it suffices to show that Claim 5: if \( u \in D(\partial \Psi) \) and \( w \in \partial \Psi(u) \), then \( u \in D(B) \) and \( w = Bu \).

For all \( \theta \in (0, 1) \), \( v \in D(\Psi) \),
\[
\Psi(u + \theta(v - u)) - \Psi(u) \geq \theta(w, v - u)_X.
\]
In other words,
\[
(u', v' - u')_{L^2_2} + \frac{\theta}{2} \|v' - u'\|_{L^2_2}^2 \geq (w, v - u)_X.
\]
Letting \( \theta \) tend to 0 in this inequality yields
\[
(u', v' - u')_{L^2_2} \geq (w, v - u)_X \quad \forall v \in D(\Psi).
\]
Choosing \( v(t) = u(t) + \varphi(t)h, \ h \in H, \ \varphi \in C^\infty_0(0, \infty) \) (test functions), we obtain
\[
\int_0^\infty a(t) \varphi'(t)(h, u'(t)) dt = \int_0^\infty b(t)(h, w(t)) \varphi(t) dt,
\]
for all \( h \in H, \ \varphi \in C^\infty_0(0, \infty) \). Therefore \(- (au')' = bw \) in the sense of distributions (i.e., in \( D'(([0, \infty); H)) \)). It follows that \( u'' \in X \) and \( w = Bu \). The proof of the lemma is complete. \( \square \)

Remark 2.3. The boundary condition (B) (i.e., \( u(0) = x \)) in Lemma 2.2 (in the definition of \( B \)) can be generalized to
\[
u'(0) \in \beta(u(0) - x), \quad (B'')
\]
where \( \beta : D(\beta) \subset H \to H \) is a general maximal monotone operator. In this case \( B \) is maximal monotone (even a subdifferential if \( \beta \) is so). While the monotonicity of \( B \) is obvious, to prove its maximality, it suffices to show that for all \( f \in X \) the equation \( u + Bu = f \) has a solution \( u \in D(B) \) (cf. Minty’s Theorem, see, e.g., [11], p. 19). Let \( v \in X \) the unique solution of
\[
v - pv'' - qv' = f, \quad v(0) = x,
\]
which exists by Lemma 2.2 with \( v', v'' \in X \). Let \( u(t) = v(t) + \zeta(t)y, \) where \( y \in H \) and \( \zeta = \zeta(t) \) satisfies
\[
\zeta - p\zeta'' - q\zeta' = 0, \ t > 0; \ \zeta(0) = 1; \ \zeta, \zeta' \in L^2_2(\mathbb{R}_+; \mathbb{R}).
\]
The existence of \( \zeta \) also follows from Lemma 2.2. Clearly,
\[
u - pu'' - qu' = f \quad \text{for a.a.} \ t \geq 0.
\]
It remains to prove that \( w'(0) \in \beta(u(0) - x) \) for a convenient \( y \in H \), i.e.,
\[
v'(0) + \zeta'(0)y \in \beta(y).
\]

Since \( \beta \) is maximal monotone, for the existence of \( y \) it suffices to show that \( \zeta'(0) < 0 \). Assume by contradiction that \( \zeta'(0) \geq 0 \). We have
\[
(a\zeta')' = b \zeta \geq 0 \quad \text{for a.a. } t \in (0, T),
\]
for some \( T \in (0, \infty) \), since \( \zeta(0) = 1 \). It follows that \( t \to a(t)\zeta'(t) \) is nondecreasing on \([0, T]\). In particular, \( a(t)\zeta'(t) \geq \zeta'(0) \geq 0 \) and thus \( \zeta'(t) \geq 0 \) in \([0, T]\), so \( \zeta \) is nondecreasing in \([0, T]\). On the other hand, integrating over \([t, \infty)\) the equation
\[
(a\zeta')' = b \zeta,
\]
we obtain
\[
-a(t)\zeta'(t)\zeta(t) - \int_t^\infty a(s)\zeta'(s)^2 \, ds = \int_t^\infty b(s)\zeta(s)^2 \, ds > 0
\]
for \( t \in (0, T) \) since \( \zeta \) is positive in \([0, T]\). Therefore \( a(t)\frac{d}{dt}\zeta^2(t) < 0 \) for \( t \in (0, T) \), which implies that \( \zeta \) is decreasing in \([0, T]\), contradiction.

In what follows we restrict ourselves to the boundary condition \( B \). The case of the more general condition \( B' \) is still open, but the above remark could be a good starting step towards the solution of problem \( (E), (B') \).

3. Existence and uniqueness for \( x \in D(A) \) and \( f \in X \)

Let \( \tilde{A} \) denote the realization of \( A \) in \( X \), that is
\[
\tilde{A} = \{ [u, v] \in X \times X : [u(t), v(t)] \in A \text{ for a.a. } t \in \mathbb{R}_+ \},
\]
where \( X \) is the space defined in the previous section. Under \( (H1) \) \( \tilde{A} \) is maximal monotone in \( X \). Recall that for all \( \lambda > 0 \) the realization of the resolvent operator \( J_\lambda = (I + \lambda A)^{-1} \) is equal to \( (I + \lambda \tilde{A})^{-1} \), and the realization of the Yosida approximation \( \Lambda_\lambda = \lambda^{-1}(I - J_\lambda) \) coincides with the Yosida approximation of \( \tilde{A} \), i.e., \( \tilde{\Lambda}_\lambda = (\tilde{A})_\lambda = \tilde{A}_\lambda \). For details, see, e.g., [11], p. 31.

**Theorem 3.1.** If \( (H1) \) and \( (H2) \) hold, \( x \in D(A) \) and \( f \in X \), then there exists a unique \( u \in D(B) \) satisfying equation \( (E) \), where \( B \) and its domain \( D(B) \) have been defined in the previous section.

**Proof.** We divide the proof into several steps.

**Step 1:** the case \( f = f_n \) for a given positive integer \( n \), where \( f_n(t) = f(t) \) for a.a. \( t \in (0, n) \), and \( f_n(t) = 0 \) for a.a. \( t > n \).

Obviously, \( f_n \) converges in \( X \) to \( f \). In a first stage, we assume that \( n \) is fixed. For \( \lambda > 0 \) we denote by \( u_\lambda \in D(B) \) the unique solution of the equation
\[
Bu_\lambda + \tilde{A}_\lambda u_\lambda + \lambda u_\lambda = -f_n.
\]

The existence of \( u_\lambda \) follows from the maximality of the sum \( B + \tilde{A}_\lambda \). Obviously, (3.1) can be written as
\[
- (a(t)u_\lambda'(t))' + b(t)A_\lambda u_\lambda(t) + \lambda b(t)u_\lambda(t) = -b(t)f_n(t) \quad \text{for a.a. } t \in \mathbb{R}_+.
\]

Now, we multiply \( u_\lambda(t) \) by \( u_\lambda(t) \) and then integrate over \([0, t]\):
\[
(u_\lambda'(0), x) - a(t)(u_\lambda'(t), u_\lambda(t)) + \int_0^t a(s)||u_\lambda(s)||^2 \, ds + \lambda \int_0^t b(s)||u_\lambda(s)||^2 \, ds
\]
\[
\leq \int_0^t b(s)||f_n(s)|| \cdot ||u_\lambda(s)|| \, ds \leq ||f_n||_{X'} \cdot ||\sqrt{b}u_\lambda||_{L^2(0,n;H)}.
\]
We have used the monotonicity of $A_{\lambda}$ and the fact that $A_{\lambda}0 = 0$. Since $u_{\lambda} \in D(B)$, we have by Lemma 2.1
\[
\lim_{t \to \infty} a(t)(u_{\lambda}(t), u'_{\lambda}(t)) = 0. \tag{3.4}
\]
By (3.3) and (3.4) we derive
\[
p_{0}\|u'_{\lambda}\|_{X}^{2} + \lambda\|u_{\lambda}\|_{X}^{2} \leq -(u'_{\lambda}(0), x) + \|f_{n}\|_{X}(\int_{0}^{t} b(s)\|u_{\lambda}\|^{2}ds)^{1/2}. \tag{3.5}
\]
Using (3.5) the identity
\[
u_{\lambda}(s) = x + \int_{0}^{s} u'_{\lambda}(\tau) d\tau, \quad s \in [0, n],
\]
we obtain
\[
p_{0}\|u'_{\lambda}\|_{X}^{2} + \lambda\|u_{\lambda}\|_{X}^{2} \leq \|x\| \cdot \|u'_{\lambda}(0)\| + \|f_{n}\|_{X}(C_{1} + C_{2}\|u'_{\lambda}\|_{X}), \tag{3.6}
\]
where $C_{1}$, $C_{2}$ are positive constants (depending on $n$). In what follows we will denote by $C_{i}$ ($i = 3, 4, \cdots$) different positive constants. Estimate (3.6) implies
\[
p_{0}\|u'_{\lambda}\|_{X}^{2} + \lambda\|u_{\lambda}\|_{X}^{2} \leq \|x\| \cdot \|u'_{\lambda}(0)\| + C_{3}. \tag{3.7}
\]
Since $A_{\lambda}$ is monotone and Lipschitzian,
\[
a(t)((A_{\lambda}u_{\lambda}(t))', u'_{\lambda}(t)) \geq 0 \quad \text{for a.a. } t > 0. \tag{3.8}
\]
Integration of (3.8) over $[0, t]$ yields
\[
(A_{\lambda}u_{\lambda}(t), a(t)u'_{\lambda}(t)) - (A_{\lambda}x, u'_{\lambda}(0))
\]
\[
\geq \int_{0}^{t} (A_{\lambda}u_{\lambda}(s), (a(s)u'_{\lambda}(s))) ds
\]
\[
\geq \int_{0}^{t} (A_{\lambda}u_{\lambda}(s), b(s)A_{\lambda}u_{\lambda}(s) + \lambda b(s)u_{\lambda}(s) + b(s)f_{n}(s)) ds
\]
\[
\geq \int_{0}^{t} b(s)\|A_{\lambda}u_{\lambda}(s)\|^{2} ds + \int_{0}^{t} b(s)(A_{\lambda}u_{\lambda}(s), f_{n}(s)) ds. \tag{3.9}
\]
Note that
\[
\|A_{\lambda}u_{\lambda}(t)\| \leq \frac{1}{\lambda^{\prime}}\|u_{\lambda}(t)\|
\]
so letting $t \to \infty$ in (3.9), we get (cf. Lemma 2.1)
\[
\|A_{\lambda}u_{\lambda}\|_{X}^{2} \leq \|A_{\lambda}u_{\lambda}\|_{X}\|f_{n}\|_{X} - (A^{0}x, u'_{\lambda}(0)), \tag{3.10}
\]
where $A^{0}$ is the minimal section of $A$. Therefore,
\[
\|A_{\lambda}u_{\lambda}\|_{X}^{2} \leq \|f_{n}\|_{X} + 2\|A^{0}x\| \cdot \|u'_{\lambda}(0)\|. \tag{3.11}
\]
Using (3.1), (3.7) and (3.11) we can derive the estimate
\[
\|u_{\lambda}\|_{X} \leq \frac{1}{p_{0}}(\|f_{n}\|_{X} + \|q\|_{L^{\infty}(\mathbb{R}^{+})}\|u'_{\lambda}\|_{X} + \lambda\|u_{\lambda}\|_{X} + \|A_{\lambda}u_{\lambda}\|_{X})
\]
\[
\leq C_{4} + C_{5}\|u'_{\lambda}(0)\|^{1/2} \quad \text{for all } 0 < \lambda \leq \lambda_{0}, \tag{3.12}
\]
where \( \lambda_0 \) is an arbitrary but fixed constant. On the other hand,

\[
\|u_\lambda'(0)\|^2 = -\int_0^\infty (a(t)\|u_\lambda'(t)\|^2)'\,dt
\]

\[
= -\int_0^\infty q(t)b(t)\|u_\lambda'(t)\|^2\,dt - 2\int_0^t a(t)(u_\lambda'(t), u_\lambda'(t))\,dt,
\]

which yields

\[
\|u_\lambda'(0)\|^2 \leq C_6(\|u_\lambda'\|^2_X + \|u_\lambda''\|^2_X)
\]

(by (3.7) and (3.12)) \( \leq C_7\|u_\lambda'(0)\| + C_8, \forall \lambda \in (0, \lambda_0) \)

which shows that \( \{\|u_\lambda'(0)\|, \, 0 < \lambda \leq \lambda_0\} \) is a bounded set. This fact combined with

(3.7), (3.11), (3.12) implies that \( u_\lambda', u_\lambda'', A_\lambda u_\lambda \) are all bounded in \( X \) for \( 0 < \lambda \leq \lambda_0 \).

The set \( \{u_\lambda : \, 0 < \lambda \leq \lambda_0\} \) is also bounded in \( X \). This follows by integration over \( [0, \infty) \) of the identity

\[
(a(t)\|u_\lambda(t)\|^2)' = q(t)b(t)\|u_\lambda(t)\|^2 + 2a(t)(u_\lambda(t), u_\lambda'(t)),
\]

which leads to

\[
-x^2 = \int_0^t q(t)b(t)\|u_\lambda(t)\|^2\,dt + 2\int_0^t a(t)(u_\lambda(t), u_\lambda'(t))\,dt
\]

and therefore

\[
\omega\|u_\lambda\|^2_X \leq \|x\|^2 + 2p_1\|u_\lambda\|_X\|u_\lambda'\|_X,
\]

where \( \omega = \text{ess inf} |q| \) (i.e., \( \omega = q_0 \) if \( q_0 > 0 \) and \( \omega = -q_1 \) if \( q_1 < 0 \)). By the facts we have established so far, there exists a \( u \in X \) (depending on \( n \) but for the moment \( n \) is fixed), such that \( u', u'' \in X \) and

\[
u_\lambda \to u, \quad u'_\lambda \to u', \quad u''_\lambda \to u'' \text{ weakly in } X,
\]

as \( \lambda \to 0 \), on a subsequence. Now, for \( \lambda, \nu \in (0, \lambda_0] \), we can easily derive from equation (3.2)

\[
-(a(u'_\lambda - u'_\nu)', u_\lambda - u_\nu) + b(A_\lambda u_\lambda - A_\nu u_\nu, u_\lambda - u_\nu)
\]

\[
+ b(\lambda u_\lambda - \nu u_\nu, u_\lambda - u_\nu) = 0 \quad \text{for a.a. } t > 0.
\]

Integration over \( [0, \infty) \) gives

\[
p_0\|u_\lambda' - u_\nu'\|^2_X + (\bar{A}_\lambda u_\lambda - \bar{A}_\nu u_\nu, \bar{T}_u u_\lambda - \bar{T}_u u_\nu)_X
\]

\[
\leq - (\bar{A}_\lambda u_\lambda - \bar{A}_\nu u_\nu, A_\lambda u_\lambda - \nu A_\nu u_\nu)_X - (\lambda u_\lambda - \nu u_\nu, u_\lambda - u_\nu)_X.
\]

(3.16)

Since \( u_\lambda \) and \( \bar{A}_\lambda u_\lambda \) are bounded in \( X \) for \( 0 < \lambda \leq \lambda_0 \) and \( \bar{A}_\lambda u_\lambda \in \bar{T}_u u_\lambda \), we can derive from (3.16)

\[
\|u_\lambda' - u_\nu'\|^2_X \leq C_9(\lambda + \nu),
\]

so, on a subsequence,

\[
u_\lambda' \to u' \text{ strongly in } X, \quad \text{as } \lambda \to 0.
\]

(3.18)

Let \( T \in (0, \infty) \) be arbitrary but fixed. We have

\[
u_\lambda' \to u' \text{ strongly in } L^2(0, T; H),
\]

(3.19)

\[
u_\lambda'' \to u'' \text{ weakly in } L^2(0, T; H).
\]

(3.20)
Moreover,
\[
\|u_\lambda(t) - u_\nu(t)\| = \| \int_0^t [u_\lambda'(s) - u_\nu'(s)] \, ds \|
\]
\[
\leq T^{1/2} \| u_\lambda' - u_\nu' \|_{L^2(0,T;H)}, \quad 0 \leq t \leq T,
\]
which implies
\[
u \rightarrow u \quad \text{in} \quad C([0,T];H),
\]
(3.21)
thus in particular \( u(0) = x \) and \( u_\lambda \rightarrow u \) in \( L^2(0,T;H) \). Note also that
\[
\|J_\lambda u_\lambda(\cdot) - u\|_{L^2(0,T;H)} \leq \lambda \| A\lambda u_\lambda(\cdot)\|_{L^2(0,T;H)} + \| u_\lambda - u\|_{L^2(0,T;H)},
\]
(3.22)
which implies
\[
J_\lambda u_\lambda(\cdot) \rightarrow u \quad \text{strongly in} \quad L^2(0,T;H).
\]
By (3.19), (3.20) and equation (3.1) it follows that
\[
A_\lambda u_\lambda(\cdot) \rightarrow pu'' + qu' - f_n \quad \text{weakly in} \quad L^2(0,T;H).
\]
(3.23)
Since \( A_\lambda u_\lambda(t) \in A J_\lambda u_\lambda(t) \) and the realization of \( A \) in \( L^2(0,T;H) \) is a maximal monotone operator in this space, hence demiclosed, we can deduce from (3.22) and (3.23) that \( u \) satisfies equation \([E]\) for a.a. \( t \in (0,T) \). Since \( T \) was arbitrarily chosen, \( u \) satisfies \([E]\) for a.a. \( t \in \mathbb{R}_+ \).

**Step 2:** general \( f \in X \).

From now on we consider \( n \) variable and denote by \( u_n \) the solution corresponding to \( f_n \) whose existence was proved above, i.e.,
\[
-f_n \in Bu_n + Au_n.
\]
(3.24)
It is easily seen that for a.a. \( t > 0 \)
\[
b(-f_n + f_m) - (a(u_n - u_m'))' + b(Au_n - Au_m),
\]
(3.25)
which implies
\[
p_0 \| u_n' - u_m' \|_X^2 \leq \| f_n - f_m \|_X \| u_n - u_m \|_X.
\]
(3.26)
On the other hand, if we integrate over \([0, \infty)\) the equation
\[
(a \| u_n - u_m \|^2)'' + qb \| u_n - u_m \|^2 + 2a(u_n - u_m, u_n' - u_m'),
\]
(3.27)
we derive
\[
\| u_n - u_m \|_X \leq C_{10} \| u_n' - u_m' \|_X.
\]
This inequality combined with (3.26) shows that both \((u_n)\) and \((u_n')\) are Cauchy sequences in \( X \), so there exists \( u \in X \) such that \( u' \in X \),
\[
u_n \rightarrow u, \quad u_n' \rightarrow u' \quad \text{strongly in} \quad X,
\]
(3.28)
and \( u(0) = x \).

Note that, for a fixed \( n \), \( u_n \) is approximated by the solution of equation \([3.1]\), as \( \lambda \rightarrow 0^+ \), where the term \( \lambda u_\lambda \) is omitted (we do not need this term at this stage).

The existence of a solution \( u_\lambda \) for this modified \([3.1]\) follows by a reasoning similar to that used in the proof of Step 1. By (3.19) and (3.20) (that hold true again) we easily see that \( u_\lambda(t) \) converges uniformly to \( u_n(t) \) on every compact interval \([0,T]\) as \( \lambda \) tends to 0. In particular,
\[
u_\lambda'(0) \rightarrow u_n'(0), \quad \text{as} \quad \lambda \rightarrow 0.
\]
(3.29)
Returning to \([3.9]\) (or \([3.10]\)), where the term \( \lambda bu_\lambda \) is omitted, we see that
\[
-(A^0_x, u_n'(0)) \geq \| w_n \|_X^2 + (w_n, f_n)_X,
\]
where \( w_n \) is the weak limit in \( X \) of \( \bar{A}_\lambda u_\lambda \), as \( \lambda \to 0^+ \), \( w_n \in \bar{A}u_n \). Therefore,
\[
\|w_n\|^2_X \leq \|f\|_X + 2\|A^0x\| \cdot ||u'_n(0)||.  
\]  
(3.30)

Using (3.28) and (3.30) in equation (3.24), more precisely, in equation
\[
Bu_n + w_n = -f_n, 
\]
we obtain the analogue of (3.12)
\[
\|u''\|_X \leq \hat{C}_4 + \hat{C}_5\|u'_n(0)\|^{1/2}.  
\]  
(3.31)

We can also derive, as we did before, the analogue of (3.14) and thus \( ||u'_n(0)|| \) is bounded. By virtue of (3.31), \( \|u''_n\|_X \) is bounded too. Hence \( u''_n \in X \) and \( u''_n \) converges weakly in \( X \) to \( w'' \), on a subsequence. Starting from (3.24), using in particular the fact that \( f_n \) converges in \( X \) to \( f \), we can show by the standard procedure that \( u \) satisfies equation (E) for a.a. \( t > 0 \).

**Step 3:** Uniqueness.

Let \( v \in D(B) \) be another solution of equation (E) that (satisfies \( v(0) = x \) and) corresponds to the same \( f \in X \). Multiplying by \( u(t) - v(t) \) the obvious equation (inclusion)
\[
(a(u' - v'))' \in b(Au - Av) \quad \text{for a.a. } t > 0,  
\]  
(3.32)

and integrating the resulting equation over \( [t, \infty) \), we obtain
\[
\frac{1}{2}a(t) \frac{d}{dt}\|u(t) - v(t)\|^2 + \int_t^{\infty} a(s)\|u'(s) - v'(s)\|^2 \, ds \leq 0,  
\]  
(3.33)

which implies that
\[
\frac{d}{dt}\|u(t) - v(t)\|^2 \leq 0 \quad \text{for all } t \geq 0,  
\]
so \( t \to \|u(t) - v(t)\| \) is nonincreasing on \( \mathbb{R}_+ \). In particular,
\[
\|u(t) - v(t)\| \leq \|u(0) - v(0)\| = 0 \quad \forall t \geq 0,  
\]  
(3.34)

which implies \( u \equiv v \). The theorem is completely proved.

**Remark 3.2.** Note that we also have uniqueness for \( u_\lambda \) and \( u_n \) above so all the convergences are true for the whole sequences involved.

---

4. The case \( x \in \overline{D(A)} \) and \( f \in X \)

Let \( x_n \in D(A) \), \( \|x_n - x\| \to 0 \). Denote by \( u_n \) the solution of equation (E) given by Theorem 3.1 satisfying \( u_n(0) = x_n \). If we multiply the equation
\[
(a(u'_n - u'_m))' \in b(Au_n - Au_m) \quad \text{for a.a. } t > 0,  
\]  
(4.1)

by \( u_n(t) - u_m(t) \) and integrate over \( [t, \infty) \), we get
\[
\frac{1}{2}a(t) \frac{d}{dt}\|u_n(t) - u_m(t)\|^2 \leq - \int_t^{\infty} a(s)\|u'_n(s) - u'_m(s)\|^2 \, ds \leq 0 \quad \text{for all } t \geq 0.  
\]  
(4.2)

It follows that
\[
\frac{d}{dt}\|u_n(t) - u_m(t)\|^2 \leq 0 \quad \text{for all } t \geq 0,  
\]  
(4.3)

so \( t \to \|u_n(t) - u_m(t)\| \) is nonincreasing on \( \mathbb{R}_+ \). In particular,
\[
\|u_n(t) - u_m(t)\| \leq \|x_n - x_m\| \quad \text{for all } t \geq 0.  
\]  
(4.4)
Thus there exists \( u \in C([R_+; H]) \), such that \( u_n \) converges to \( u \) in \( C([0, T]; H) \) for all \( T \in (0, \infty) \) and \( u(0) = x \). If \( f \) is identically zero, then it is easy to see that 
\[
(1 + t) = \left( \frac{a}{1 + t} \right) \left( \frac{t}{1 + t} \right)
\]
deepth 2.2em\( \frac{a}{1 + t} \left( \frac{t}{1 + t} \right) \left( \frac{1}{p} \right) \right) dt \) (4.3). This implies 
\[
\int_0^T t(a(t) \frac{d}{dt} \|u_n - u_m\|^2) = \frac{T}{2} a(T) \frac{d}{dt} \|u_n - u_m\|^2(T) \leq 0.
\]
Due to (4.3). This implies 
\[
\int_0^T t(a(t) \frac{d}{dt} \|u_n - u_m\|^2 dt \leq \frac{1}{2} \|x_n - x_m\|^2 + \frac{1}{2} \int_0^T \frac{p}{q} \|u_n - u_m\|^2 dt
\]
(4.5) By (4.4) and (4.5) we see that \( t^{1/2} u' \in L^2(0, T; H) \) and 
\[
t^{1/2} u_n' \rightarrow t^{1/2} u' \text{ strongly in } L^2(0, T; H).
\]
By (4.4) and (4.5) we see that \( t^{1/2} u_n' \rightarrow t^{1/2} u' \) strongly in \( L^2(0, T; H) \). (4.6)

It is easy to see that \( u \) does not depend on the choice of the sequence \( (x_n) \) approximating \( x \). We can call \( u \) a generalized solution of equation \( (E) \) satisfying 
\[
u \in D(A) \). If \( f \equiv 0 \), then \( u \) also satisfies \( (C) \). This is also the case if the condition \( 0 \in D(A), 0 \in A0 \) is replaced by \( 0 \in R(A) \).

We do not have an estimate for \( u_n' \) to prove that \( u \) is a classic solution of equation \( (E) \). So the existence of a classic solution for \( x \in D(A) \) is still an open problem.

However, if \( p \) and \( q \) are constant functions we are able to obtain classic existence for all \( x \in D(A) \) (see the next section).

5. Existence and uniqueness for \( x \in D(A) \) in the case of constant coefficients

In this section we assume that \( p \) and \( q \) are both constant functions. Without any loss of generality, we can assume \( p \equiv 1 \) and \( q \equiv c \), with \( c \in R \setminus \{0\} \). Recall that the case \( c = 0 \) was extensively studied by V. Barbu [3], [4], [5] and by H. Brezis [6]. We are going to prove that in this particular case problem \( (E), (B) \) has a unique classical solution for each \( x \in D(A) \) and \( f \in X \). Before stating precisely the result, we need some definitions. For \( \varepsilon > 0 \) small, define \( \varphi : R_+ \rightarrow R_+ \) by 
\[
\varphi(t) = \cases{ t & if \( 0 \leq t \leq \varepsilon \\
\varepsilon & if \( t > \varepsilon \)
\end{cases}
\]
Let \( X_\varepsilon \) be the weighted space \( L^2(R_+; H; \varphi(t) e^{\varepsilon dt}) \). For \( \alpha < c \), close to \( c \), denote \( X_\alpha = L^2(R_+; H); t^\alpha e^{\varepsilon dt} \). Obviously, \( X \subset X_\varepsilon \subset X_\alpha \), where \( X \) is the space defined in Section 2, with the particular weight \( b(t) = e^{\varepsilon t} \), i.e., \( X = L^2(R_+; H; e^{\varepsilon dt}) \).
Theorem 5.1. If (H1) holds, \( p \equiv 1, q \equiv c, \) with \( c \in \mathbb{R} \setminus \{0\}, \) \( x \in D(A) \) and \( f \in X, \) then there exists a unique \( u \in C(\mathbb{R}^+; H), \) \( u, u' \in X, \) \( u'' \in X, \) such that \( u \) satisfies (E) and \( u(0) = x, \) for all \( \varepsilon > 0 \) small and \( \alpha < c, \) close to \( c. \) If \( f \equiv 0, \) then \( \|u(t)\| \leq \|x\| \) for all \( t \geq 0. \)

Proof. In the present case \( a(t) = b(t) = e^{ct}. \) Let \( x_n \in D(A), \) \( \|x_n - x\| \to 0, \) and let \( u_n \in X \) be the unique solution of (E) with \( f = f_n \) given by Theorem 3.1 satisfying \( u_n(0) = x_n: \)

\[
(e^{ct}u'_n(t))' \in e^{ct}(Au_n(t) + f_n(t)) \text{ for a.a. } t > 0,
\]

where \( f_n \) is the truncation of \( f \) defined in Section 3. In what follows we develop a technique inspired by Bruck’s paper [8]. For \( m,n \) two fixed positive integers, define

\[
g(t) = \frac{e^{ct}}{2}\|u_n(t) - u_m(t)\|^2.
\]

We have,

\[
g(t) = cg(t) + e^{ct}(u'_n(t) - u'_m(t)), \quad u_n(t) - u_m(t), \quad (5.2)
\]

\[
g''(t) = c g'(t) + c e^{ct}(u'_n(t) - u'_m(t)) + e^{ct}(u''_n - u''_m, n - m) + e^{ct}\|u'_n - u'_m\|^2
\]

\[
= c g(t) + 2c e^{ct}(u'_n(t) - u'_m(t)) + e^{ct}(u'_n(t) - u'_m(t)) + e^{ct}(u''_n - u''_m(t)) + e^{ct}(u'_n(t) - u'_m(t))
\]

\[
\geq c e^{ct}\left\{\frac{c^2}{2}\|u_n - u_m\|^2 + c(u'_n(t) - u'_m(t)) + u'_n(t) - u'_m(t)\}
\]

\[
+ e^{ct}(f_n - f_m, u_n - u_m)
\]

\[
\geq c e^{ct}\left\{\|u_n - u_m\|^2 + \|u'_n(t) - u'_m(t)\|^2\} + e^{ct}(f_n - f_m, u_n - u_m)
\]

(5.3)

for \( \gamma > 0 \) a small constant. From (5.2) and (5.3) it follows that

\[
\gamma \int_0^\infty \zeta_\varepsilon(t) e^{ct}\left\{\|u_n - u_m\|^2 + \|u'_n - u'_m\|^2\right\} dt
\]

\[
\leq \int_0^\infty \zeta_\varepsilon(t) g''(t) dt + \int_0^\infty \zeta_\varepsilon(t) e^{ct}\|f_n - f_m\| \cdot \|u_n - u_m\| dt
\]

\[
\leq \int_0^\varepsilon t g''(t) dt + \varepsilon \int_0^\infty g''(t) dt + \frac{1}{2\gamma} \int_0^\infty \zeta_\varepsilon(t) e^{ct}\|f_n - f_m\|^2 dt
\]

\[
+ \frac{\gamma}{2} \int_0^\infty \zeta_\varepsilon(t) e^{ct}\|u_n - u_m\|^2 dt
\]

\[
\leq t g'(t) + \varepsilon \int_0^\varepsilon g'(t) dt + \varepsilon [g'(\infty) - g'(\varepsilon)] + \frac{1}{2\gamma} |f_n - f_m|^2_X + \frac{\gamma}{2} |u_n - u_m|^2_X
\]

\[
\leq g(0) + \frac{1}{2\gamma} |f_n - f_m|^2_X + \frac{\gamma}{2} |u_n - u_m|^2_X.
\]

Therefore,

\[
\frac{\gamma}{2} |u_n - u_m|^2_X + \frac{\gamma}{2} |u'_n(t) - u'_m(t)|^2_X \leq \frac{1}{2} |x_n - x_m|^2 + \frac{1}{2\gamma} |f_n - f_m|^2_X.
\]

(5.4)

Thus there exists \( u \in X, \) such that \( u' \in X, \) and

\[
u_n \to u, \quad u'_n \to u' \text{ strongly in } X.
\]

(5.5)
Now, in order to derive an estimate for \( u''_n \), we recall that, for a fixed positive integer \( n \), the solution \( u_{n\lambda} \in X \) of
\[
u''_{n\lambda} + cu'_{n\lambda} = A\lambda u_{n\lambda} + f_n,
\]  
(5.6)
approximates \( u_n \) as \( \lambda \to 0^+ \):
\[\begin{align*}
u_{n\lambda} & \to u_n \text{ in } C([0, T]; H) \forall T > 0, \quad u'_{n\lambda} \to u'_n \text{ strongly in } X, \quad u''_{n\lambda} \to u''_n \text{ weakly in } X. 
\end{align*}\]
(5.7)
Since \((u'_{n\lambda}, (A\lambda u_{n\lambda})') \geq 0\), we have
\[
\frac{d}{dt}(e^{\alpha t}u'_{n\lambda}, A\lambda u_{n\lambda}) \geq (e^{\alpha t}u'_{n\lambda})', A\lambda u_{n\lambda})
\]
\[= (A\lambda u_{n\lambda} + (\alpha - c)u'_{n\lambda} + f_n, e^{\alpha t}A\lambda u_{n\lambda}).
\]
(5.8)
Multiplying \((5.8)\) by \( t^3 \) and then integrating over \( \mathbb{R}_+ \), we obtain
\[
\|A\lambda u_{n\lambda}\|^2_{X_\alpha} \leq \|A\lambda u_{n\lambda}\|_{X_\alpha} \cdot \|f_n\|_{X_\alpha} + (c - \alpha)\|A\lambda u_{n\lambda}\|_{X_\alpha} \cdot \|u'_{n\lambda}\|_{X_\alpha}
\]
\[+ 3\int_0^\infty t^2 e^{\alpha t}(u'_{n\lambda}, A\lambda u_{n\lambda})dt \leq \|A\lambda u_{n\lambda}\|_{X_\alpha} \cdot \|f_n\|_{X_\alpha} + (c - \alpha)\|A\lambda u_{n\lambda}\|_{X_\alpha} \cdot \|u'_{n\lambda}\|_{X_\alpha}
\]
\[+ 3\|A\lambda u_{n\lambda}\|_{X_\alpha} \left(\int_0^\infty te^{\alpha t}\|u'_{n\lambda}\|^2 dt\right)^{1/2}.\]

We have used \( \alpha \) instead of \( c, \alpha < c \), so for any polynomial \( P(t)e^{\alpha t} \leq C_1 e^{\epsilon t} \), so that all the above computations are permitted. The last estimate yields
\[
\|A\lambda u_{n\lambda}\|_{X_\alpha} \leq \|f\|_{X_\alpha} + (c - \alpha)\|u'_{n\lambda}\|_{X_\alpha} + \left(\int_0^\infty te^{\alpha t}\|u'_{n\lambda}\|^2 dt\right)^{1/2}. \quad (5.9)
\]
Now, combining \((5.6)\) and \((5.9)\), we find
\[
\|u''_{n\lambda}\|_{X_\alpha} \leq |c|\|u'_{n\lambda}\|_{X_\alpha} + \|A\lambda u_{n\lambda}\|_{X_\alpha} + \|f_n\|_{X_\alpha}
\]
\[\leq |c|\|u'_{n\lambda}\|_{X_\alpha} + 2\|f\|_{X_\alpha} + \left(\int_0^\infty te^{\alpha t}\|u'_{n\lambda}\|^2 dt\right)^{1/2}. \quad (5.10)
\]
Since \( u'_{n\lambda} \to u'_n \) strongly in \( X \) and \( u''_{n\lambda} \to u''_n \) weakly in \( X \), as \( \lambda \to 0^+ \), we derive from \((5.10)\)
\[
\|u''_{n}\|_{X_\alpha} \leq |c|\|u'_{n}\|_{X_\alpha} + 2\|f\|_{X_\alpha} + \left(\int_0^\infty te^{\alpha t}\|u'_{n}\|^2 dt\right)^{1/2}. \quad (5.11)
\]
We know that \( u'_{n} \) is convergent (hence bounded) in \( X_\epsilon \) so \((5.11)\) shows that \( u''_{n} \) is bounded in \( X_\alpha \). Thus \( u'' \in X_\alpha \) and \( u''_{n} \) converges weakly in \( X_\alpha \) to \( u'' \). Letting \( n \to \infty \) in
\[
u'' + cu' = Au_n + f_n,
\]
with respect to the topology of \( L^2(\delta, T; H) \) for \( 0 < \delta < T < \infty \) we can show by the standard procedure that \( u \) satisfies equation \( (E) \) for \( a.a. \ t \in (\delta, T) \), so for \( a.a. \ t > 0 \) (since \( \delta \) and \( T \) can be chosen arbitrarily). Note that the weights are not relevant on \([\delta, T]\). Now let us prove that \( u \in C(\mathbb{R}_+; H) \) and \( u(0) = x \). Of course,
Indeed, since \( u \delta > t \) and \[ \text{Remark 5.3.} \]

the corresponding solution corresponding to \( x \) with \( t > 0 \), there is a

we only need to prove continuity at \( t = 0^+ \). From equation (5.2) we easily derive

\[
\frac{1}{2} e^{ct} \frac{d}{dt} \| u_n(t) - u_m(t) \|^2 + \int_{t}^{\infty} e^{cs} \| u_n'(s) - u_m'(s) \|^2 ds
\]

\[
\leq \int_{t}^{\infty} e^{cs} (f_n(s) - f_m(s), u_n(s) - u_m(s)) ds
\]

\[
\leq \int_{m}^{n} e^{cs} \| f(s) \| \cdot \| u_n(s) - u_m(s) \| ds
\]

\[
\leq \| f \|_X \| u_n - u_m \|_X < \eta \quad \text{for } N_\eta < m < n. \quad (5.12)
\]

If we multiply (5.12) by \( e^{-ct} \) and then integrate over \([0, t]\), we get

\[
\frac{1}{2} \| u_n(t) - u_m(t) \|^2 \leq \frac{1}{2} \| x_n - x_m \|^2 + \eta t \quad \text{for } N_\eta < m < n.
\]

Thus \( u_n(t) \) converges uniformly as \( n \to \infty \) on a compact interval \([0, t_0]\) to a continuous function \( v = v(t) \). In particular, \( v(0) = x \). From the previous part of the proof \( u_n \to u \) and \( u_n' \to u' \) in \( X \), for all \( \varepsilon > 0 \), so \( u \) is continuous on \((0, \infty)\). Obviously, \( u(t) = v(t) \) for \( t \in (0, t_0] \). It follows that

\[
\lim_{t \to 0^+} u(t) = v(0) = x,
\]

and \( u \in C(\mathbb{R}_+; H) \).

Concerning uniqueness, it follows by a reasoning we have already used. Indeed, if \( u, v \) are two solutions corresponding to \( x \in D(A) \) and \( f \in X \) in the class indicated in the statement of the theorem, then

\[
\frac{1}{2} e^{ct} \| u(t) - v(t) \|^2 + \int_{t}^{\infty} e^{cs} \| u'(s) - v'(s) \|^2 ds \leq 0, \quad \text{for all } t > 0.
\]

This implies that \( t \to \| u(t) - v(t) \| \) is nonincreasing and thus

\[
\| u(t) - v(t) \| \leq \| u(0) - v(0) \| = 0, \quad \text{for all } t \geq 0, \quad (5.13)
\]

i.e., \( u \equiv v \). If \( u \) is the solution corresponding to \( x \in D(A) \), \( f \equiv 0 \) and \( v \) is the solution corresponding to \( x = 0 \), \( f \equiv 0 \), i.e., \( v \equiv 0 \), then (5.13) yields

\[
\| u(t) \| \leq \| x \|, \quad \text{for all } t \geq 0.
\]

The proof is now complete. \( \square \)

**Remark 5.2.** If \( c < 0 \), then condition \( f \in X \) allows unbounded \( f \)'s. In this case the corresponding \( u \)'s are so, as illustrated by simple examples.

**Remark 5.3.** In Theorem 5.3, we have \( u'' \in L^2([\delta, \infty); H; e^{ct} dt) \) for all \( \delta > 0 \). Indeed, since \( u \) satisfies equation (E) for a.a. \( t > 0 \), for every \( \delta > 0 \), there is a \( t_0 \in (0, \delta) \), such that \( u(t_0) \in D(A) \), so we can apply Theorem 3.1 with \( x := u(t_0) \) and \([t_0, \infty)\) instead of \( \mathbb{R}_+ \). Obviously, \( u, u' \in L^2([\delta, \infty); H; e^{ct} dt) \) as well, for all \( \delta > 0 \).

### 6. CONSTANT COEFFICIENTS AND \( f \equiv 0 \)

Consider the homogeneous equation

\[
u''(t) + cu' \in Au(t), \quad t > 0, \quad (E_0)
\]

with

\[
u(0) = x, \quad (B)
\]
where $c \in \mathbb{R} \setminus \{0\}$. Recall that, if $c > 0$, the boundedness condition $[E]$ added to $[B]$ is not enough to guarantee uniqueness of $u$. However, we have uniqueness if we impose the condition $u \in X$, where $X = L^2_0(\mathbb{R}_+; H)$, with $b(t) = e^{ct}$. It is easy to see that the solutions of problem $[E_0]$, $[B]$ given by Theorems 3.1 and 5.1 generate a nonlinear semigroup of contractions $\{S(t): \overline{D(A)} \to \overline{D(A)}; t \geq 0\}$, $S(t)x := u(t)$, where $u \in X$ satisfies $[E_0]$ and $[B]$. We also have the properties

$$\forall x \in D(A), \ e^{ct/2}S(t)x, \ e^{ct/2} \frac{dt}{dt}S(t)x, \ e^{ct/2} \frac{dt^2}{dt^2}S(t)x \in L^2(\mathbb{R}_+; H); \quad (6.1)$$

$$\forall x \in \overline{D(A)}, \ e^{ct/2}S(t)x, \ e^{ct/2} \frac{dt}{dt}S(t)x, \ e^{ct/2} \frac{dt^2}{dt^2}S(t)x \in L^2([\varepsilon, \infty); H) \forall \varepsilon > 0. \quad (6.2)$$

The last property in $[6.2]$ follows from the fact that we can use as the initial state $u(t_0) \in D(A)$ for some $t_0 > 0$ instead of $x \in \overline{D(A)}$.

Let $F: D(F) \subset H \to H$ be such that $G = -F$ is the generator of the semigroup $\{S(t): \overline{D(A)} \to \overline{D(A)}; t \geq 0\}$ defined above. Operator $F$, which is maximal monotone, can be regarded as the formal solution of the operator equation

$$F^2 + cf - A = 0,$$

i.e.,

$$F = \frac{c}{2} I + \sqrt{\frac{c^2}{4} I + A}, \quad (6.3)$$

as remarked by B. Djafari Rouhani and H. Khatibzadeh [9]. Here $\sqrt{\cdot}$ represents the square root of $\frac{c^2}{4} I + A$ in Barbu’s sense. If $A$ is linear, then $G = -F$, where $F$ is given by $[6.3]$ is indeed the generator of the semigroup $\{S(t): \overline{D(A)} \to \overline{D(A)}; t \geq 0\}$. To show this, define $v(t) = e^{ct/2}u(t)$, where $u(t) = S(t)x$, $x \in D(A)$. Then,

$$v'(t) = e^{ct/2}(\frac{c^2}{4} I + A)(e^{-ct/2}v(t))$$

$$= \frac{c^2}{4}v(t) + Av(t),$$

so $v(t) = T(t)x$, where $T(t)$ is the semigroup generated by $-\sqrt{\frac{c^2}{4} I + A}$. Therefore,

$$S(t)x = e^{-ct/2}T(t)x, \quad x \in H,$$

which shows that the generator of $S(t)$ is $G = -F$, where $F$ is given by $[6.3]$, as asserted.

**Comments on the asymptotic behavior.** By $[6.1]$, $[6.2]$ and Lemma 2.1 (which remains valid if $\mathbb{R}_+$ is replaced by $[\varepsilon, \infty)$, $\varepsilon > 0$), we see that

$$\lim_{t \to \infty} e^{ct/2}\|S(t)x\| = \lim_{t \to \infty} e^{ct/2}\|\frac{dt}{dt}S(t)x\| = 0, \quad (6.4)$$

for all $x \in \overline{D(A)}$. On the other hand, Djafari Rouhani and Khatibzadeh [9] have proved that, if $c > 0$ and $0 \in R(A)$, for any bounded solution $u$ of equation $[E_0]$ there exists a $y \in A^{-1}0$, such that

$$\|u(t) - y\| = O(e^{-ct/2}), \quad \|u'(t)\| = O(e^{-ct/2}) \quad (6.5)$$
Now are able to describe the set of all bounded solutions of equation $E_0$, under the assumption $0 \in R(A)$.

Indeed, if we define $v(t) := u(t) - y$, $A_y x := A(x + y)$, $x \in D(A_y) = D(A) \setminus \{y\}$, then $v$ is the unique solution in our sense of the equation

$$v'' + cv' \in A_y v, \quad t > 0,$$

with $v(0) = u(0) - y$. In fact, the set of all bounded solutions of equation $E_0$ is

$$\mathcal{Q} := \{u(t) = y + v(t) : v \text{ is a solution in our sense of (6.6), } y \in A^{-1}0\}.$$ 

For a given $x \in D(A)$ equation $E_0$ may have several bounded solutions $u$ satisfying $u(0) = x$, if $A^{-1}0$ is not a singleton.

Note that estimates (6.5) are weaker than (6.4). However, it is worth pointing out that recently Khatibzadeh [10] has shown that, if $c > 0$, then $O(e^{-ct/2})$ in (6.5) can be replaced by $O(e^{-ct})$.

References


Department of Mathematics and its Applications, Central European University, 1051 Budapest, Hungary

E-mail address: morosanug@ceu.hu