SECOND-ORDER DIFFERENTIAL EQUATIONS ON $\mathbb{R}_+$ GOVERNED BY MONOTONE OPERATORS

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Abstract. Consider in a real Hilbert space H the differential equation (E): $p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t)$, for a.a. $t \in \mathbb{R}_+ = [0, \infty)$, with the condition $u(0) = x \in \overline{D(A)}$, where $A\colon D(A) \subset H \to H$ is a maximal monotone operator, with $[0, 0] \in A$ (or, more generally, $0 \in R(A)$); $p, q \in L^\infty(\mathbb{R}_+)$, with $\operatorname{ess\,inf}\ p > 0$ and either $\operatorname{ess\,inf}\ q > 0$ or $\operatorname{ess\,sup}\ q < 0$; and $f : \mathbb{R}_+ \to H$ is a given function. Recall that equation (E) in the case $p \equiv 1,\ q \equiv 0,\ f \equiv 0$, subject to $u(0) = x$ and the boundedness condition $\sup_{t\geq 0} \|u(t)\| < \infty$, was investigated in early 1970's by V. Barbu, who derived in particular from his results a definition for the square root of the nonlinear operator A. Subsequently H. Brezis, N. H. Pavel, L. Véron and others have paid much attention to equation (E). In this paper we prove the existence and uniqueness of the solution to equation (E) subject to $u(0) = x \in D(A)$ in the weighted space $X = L^2_b(\mathbb{R}_+; H)$, where $b(t) = a(t)/p(t)$, $a(t) = \exp(\int_0^t q(s)/p(s)\, ds)$, under our weak assumptions on p and q (see above) and $f \in X$. For $x \in \overline{D(A)}$ we prove the existence of a generalized solution in the case of general variable coefficients p, q and a classic solution in the case $p \equiv 1,\ q \equiv c \in \mathbb{R} \setminus \{0\}$. If $p \equiv 1,\ q(t) \equiv c \in \mathbb{R} \setminus \{0\}$, $f \equiv 0$ the solutions give rise to a nonlinear semigroup of contractions. If A is linear its infinitesimal generator G is given by $G = -\frac{c}{2}I - \sqrt{\frac{c^2}{4}I + A}$.

1. Introduction.

Throughout this paper H will be a real Hilbert space with respect to an inner product $(\cdot, \cdot)$ and the induced norm $\|x\| = (x, x)^{1/2}$. Consider the nonlinear second-order equation (inclusion)

$$p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t), \quad \text{for a.a. } t \in \mathbb{R}_+ = [0, \infty), \tag{E}$$

with the condition

$$u(0) = x \in \overline{D(A)}, \tag{B}$$

where
$(H1)$ $A\colon D(A) \subset H \to H$ is a maximal monotone operator, with $0 \in D(A)$ and $0 \in A0$;
$(H2)$ $p, q \in L^\infty(\mathbb{R}_+) := L^\infty(\mathbb{R}_+; \mathbb{R})$, with $\operatorname{ess\,inf} p > 0$ and either $\operatorname{ess\,inf} q > 0$ or $\operatorname{ess\,sup} q < 0$;

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and $f\colon \mathbb{R}_+ \to H$ is a given function which will be described later. In other words, (H2) says that both p and q are measurable and satisfy

$$0 < p_0 \le p(t) \le p_1 < \infty \quad \text{for a.a. } t \in \mathbb{R}_+,$$

and

$$q_0 \le q(t) \le q_1 \quad \text{for a.a. } t \in \mathbb{R}_+,$$

where either $0 < q_0 < q_1 < \infty$ or $-\infty < q_0 < q_1 < 0$.

Remark 1.1. *One can assume that* $0 \in R(A)$ *instead of* $0 \in D(A)$, $0 \in A0$. *Indeed, if* $y \in D(A)$ *and* $0 \in Ay$, *then defining the operator* A_y *by* $D(A_y) = D(A) \setminus \{y\}$, $A_y(z) = A(z+y)$, *and* $v(t) = u(t) - p$, *we have* $0 \in D(A_y)$, $0 \in A_y 0$ *and*

$$p(t)v''(t) + q(t)v'(t) \in A_y v(t) + f(t) \quad \textit{for a.a. } t \in \mathbb{R}_+. \tag{E_y}$$

Problem (E), (B) in the case $p \equiv 1$, $q \equiv 0$, $f \equiv 0$, with the additional condition

$$\sup\nolimits_{t \ge 0} \|u(t)\| < \infty, \tag{C}$$

was investigated for the first time by V. Barbu [3], [4] (see also [5], Chapter V, Sections 2.3 and 2.4), who derived in particular from his results a definition for the square root of the nonlinear operator A. The same problem, with (B') instead of (B), was then studied by H. Brezis [6], where

$$u'(0) \in \partial j(u(0) - x), \tag{B'}$$

with ∂j standing for the subdifferential of a proper, convex, lower semicontinuous function $j\colon H \to (-\infty, +\infty]$. If in particular j is the indicator function of the set $\{0\}$ (i.e., $j(0) = 0$ and $j(z) = +\infty$ for all $z \in H \setminus \{0\}$), then condition (B') becomes $u(0) = x$. Subsequently L. Véron [12], [13] studied problem (E), (B), (C) with $f \equiv 0$, under the following conditions

$$p \in W^{2,\infty}(\mathbb{R}_+),\ q \in W^{1,\infty}(\mathbb{R}_+),\ p(t) \ge p_0 > 0 \ \ \forall t \in \mathbb{R}_+. \tag{1.1}$$

These smoothness assumptions on the coefficients p and q are in fact too strong. Indeed, one can take advantage of the following form of (E), as pointed out by A. R. Aftabizadeh and N. H. Pavel [1]:

$$(a(t)u'(t))' \in b(t)Au(t) + b(t)f(t), \tag{1.2}$$

where

$$a(t) = \exp\left(\int_0^t \frac{q(s)}{p(s)}ds\right), \quad b(t) = \frac{a(t)}{p(t)},$$

that allows significant relaxation in the regularity of p and q. Following the technique of [1], N. C. Apreutesei [2] reconsidered (E) on $\mathbb{R}_+$, with (B'), in the framework of the weighted space $X = L^2_b(\mathbb{R}_+; H)$, under the following assumptions on p, q:

$$p, q \in W^{1,\infty}(\mathbb{R}_+),\ p(t) \ge p_0 > 0,\ q(t) \ge q_0 > 0 \text{ for all } t \in \mathbb{R}_+, \text{ and } x \in D(A). \tag{1.3}$$

In this paper we prove the existence and uniqueness of the solution of problem (E), (B) in X, under our weak assumptions (H2), by using a simpler treatment. The condition $u \in X$ is more appropriate than the boundedness condition (C) that works well if $p \equiv 1$ and $q \equiv 0$ but does not guarantee uniqueness if $q \equiv c > 0$. For example, if $p \equiv 1$, $q \equiv 1$, $f \equiv 0$ and $A = 0$, then problem (E), (B), (C) has infinitely many solutions (and only one belongs to X). Note also that L. Véron [13] derived the condition

$$\int_0^\infty \exp\left(-\int_0^t \frac{q(s)}{p(s)}ds\right)dt = +\infty$$

as a necessary one for the uniqueness of the solution of problem (E), (B), (C). So the case $p \equiv 1$ and $q \equiv c$, where c is a positive constant, is excluded in Véron's paper, but allowed in our present paper.

We obtain classic existence and uniqueness in X to problem (E), (B) for $x \in D(A)$ and $f \in X$. If $x \in \overline{D(A)}$ we prove the existence of a generalized solution (uniform limit on compact intervals of a sequence of classic solutions). If $f \equiv 0$, the solution $u \in X$ is also bounded on $\mathbb{R}_+$. In the case of constant coefficients $p \equiv 1$, $q \equiv c$, $c \in \mathbb{R} \setminus \{0\}$, we are able to prove the existence of classic solutions for $x \in \overline{D(A)}$ and $f \in X$. If in addition $f \equiv 0$ the solutions give rise to a nonlinear contraction semigroup generated by $G = -\frac{c}{2}I - \sqrt{\frac{c^2}{4}I + A}$, where I is the identity operator, as noticed by B. Djafari Rouhani and H. Khatibzadeh [9]. If A is linear we can show that G is indeed the generator of this semigroup. If $p \equiv 1$, $q \equiv c > 0$, $f \equiv 0$ and $0 \in R(A)$, our results combined with the recent results regarding the asymptotic behavior of solutions reported by B. Djafari Rouhani and H. Khatibzadeh [9], lead to the description of the set of all bounded solutions of equation (E): $u(t) = y + v(t)$, $y \in A^{-1}0$, where $v \in X$ is a solution of the equation $v'' + cv' \in A_y v$, $t > 0$, with $A_y z = A(z + y)$. In particular, according to our present results, $\|v(t)\| = \mathbf{o}(e^{-ct/2})$ and $\|v'(t)\| = \mathbf{o}(e^{-ct/2})$, as $t \to \infty$. In fact the behavior of the v's is even better: $\|v(t)\| = \mathbf{O}(e^{-ct})$ and $\|v'(t)\| = \mathbf{O}(e^{-ct})$, as proved by H. Khatibzadeh [10].

2. Notation and Preparatory Lemmas

Let X be the set of all (classes with respect to the a.e. equality of) functions $f\colon \mathbb{R}_+ \to H$ which are measurable, with $\int_0^\infty b(t)\|f(t)\|^2 dt < \infty$, where

$$b(t) = \frac{a(t)}{p(t)}, \quad a(t) = \exp\left(\int_0^t \frac{q(s)}{p(s)}ds\right).$$

In other words, X is the weighted space $L^2_b(\mathbb{R}_+; H)$. Note that (under (H2)) X is a real Hilbert space with respect to the scalar product

$$(f, g)_X = \int_0^\infty b(t)(f(t), g(t))dt,$$

and the corresponding norm

$$\|f\|_X^2 = \int_0^\infty b(t)\|f(t)\|^2 dt.$$

Lemma 2.1. *Suppose assumptions (H2) hold. If f, $f' \in X$ (where f' is the distributional derivative of f), then*

$$\lim_{t\to\infty} a(t)\|f(t)\|^2 = 0, \tag{2.1}$$

where a and X are defined above.

Proof. Denote $g(t) = \sqrt{a(t)}f(t)$. We have

$$g'(t) = \frac{q(t)}{2p(t)}g(t) + \sqrt{a(t)}f'(t),$$

so both $g,\, g' \in L^2(\mathbb{R}_+; H)$. By Hölder's inequality this implies

$$\|g(t) - g(s)\| = \|\int_s^t g'(\tau)d\tau\| \leq (t-s)^{1/2}\|g'\|_{L^2(\mathbb{R}_+;H)},$$

for all $0 \leq s \leq t$. This shows that g is uniformly continuous on $\mathbb{R}_+$ and so is $\|g\|$. A straightforward reasoning shows that $\limsup_{t\to\infty}\|g(t)\| = 0$, otherwise g cannot be a member of X. □

Now, let us define $B\colon D(B) \subset X \to X$ by

$$D(B) = \{u \in X\colon u',\, u'' \in X,\ u(0) = x\},\ Bu = -pu'' - qu'.$$

Lemma 2.2. *If assumptions (H2) hold, then operator B defined above is the subdifferential of $\Psi\colon X \to (-\infty, +\infty]$, $\Psi(u) = (1/2)\|u'\|^2_{L^2_a} + j(u(0) - x)$, which is proper, convex and lower semicontinuous (hence B is maximal monotone), where $L^2_a := L^2_a(\mathbb{R}_+; H)$ and j is the indicator function of the set $\{0\} \subset H$.*

Proof. Note that $Bu = -\frac{p}{a}(au')'$ for all $u \in D(B)$.
Claim 1: B is monotone.

Indeed, for all $u,\, v \in D(B)$, we have

$$\begin{aligned}(Bu - Bv, u - v)_X = &-\int_0^\infty ((a(u'-v'))', u - v)\, dt \\ &= a(u'-v', u-v)|_0^\infty + \int_0^\infty a\|u'-v'\|^2 dt \geq 0,\end{aligned}$$

since, according to Lemma 2.1,

$$\lim_{t\to\infty} a(t)(u'(t) - v'(t), u(t) - v(t)) = 0.$$

Claim 2: Ψ is proper.

For example, the function $\hat{u}$, defined by

$$\hat{u}(t) = (1-t)^2 x, \quad \text{for } 0 \leq x \leq 1 \ \text{ and } \ \hat{u}(t) = 0 \ \text{ for } t > 1,$$

satisfies $\Psi(\hat{u}) < \infty$. The effective domain of Ψ is $D(\Psi) = \{u \in X\colon\ u' \in X,\ u(0) = x\}$.
Claim 3: Ψ is convex.

This is obvious.
Claim 4: Ψ is lower semicontinuous.

It is well known this is equivalent to saying that its level sets $M_\lambda := \{u \in X\colon\ \Psi(u) \leq \lambda\}$ $(\lambda \in \mathbb{R})$ are closed in X. If $\lambda \leq 0$ then M_λ is empty. Let $\lambda > 0$ be an arbitrary but fixed number, and let $u_n \in M_\lambda$, $u_n \to u$ in X. Since

$$\|u_n'\|_X^2 = \int_0^\infty b(t)\|u_n'(t)\|^2 dt \leq \frac{1}{p_0}\|u_n'\|^2_{L^2_a} \leq \frac{2}{p_0}\lambda,$$

we have

$$u' \in X \quad \text{and } u_n' \to u' \text{ weakly in X.}$$

Moreover, making use of the formula

$$u_n(t) = x + \int_0^t u_n'(s)ds,$$

we can easily deduce, by Arzelà's compactness criterion, that $u_n(t)$ converges uniformly to $u(t)$ on every compact interval $[0,T]$. In particular, $u(0) = x$. Therefore,

$$\begin{aligned}\lambda \geq \liminf_{n\to\infty} \Psi(u_n) =& \liminf_{n\to\infty} \frac{1}{2}\|u_n'\|_{L_a^2}^2 \\ \geq& \frac{1}{2}\|u'\|_{L_a^2}^2 = \Psi(u).\end{aligned}$$

Since Ψ is proper, convex and lower semicontinuous, it follows that its subdifferential $\partial\Psi$ is a maximal monotone operator. So, to conclude the proof of Lemma 2.2, it suffices to show that

Claim 5: if $u \in D(\partial\Psi)$ and $w \in \partial\Psi(u)$, then $u \in D(B)$ and $w = Bu$.

For all $\theta \in (0,1)$, $v \in D(\Psi)$,

$$\Psi(u + \theta(v-u)) - \Psi(u) \geq \theta(w, v-u)_X.$$

In other words,

$$(u', v'-u')_{L_a^2} + \frac{\theta}{2}\|v'-u'\|_{L_a^2}^2 \geq (w, v-u)_X.$$

Letting θ tend to 0 in this inequality yields

$$(u', v'-u')_{L_a^2} \geq (w, v-u)_X \quad \forall v \in D(\Psi).$$

Choosing $v(t) = u(t) + \varphi(t)h,\ h \in H,\ \varphi \in C_0^\infty(0,\infty)$ (test functions), we obtain

$$\int_0^\infty a(t)\varphi'(t)(h, u'(t))dt = \int_0^\infty b(t)(h, w(t))\varphi(t)dt,$$

for all $h \in H$, $\varphi \in C_0^\infty(0,\infty)$. Therefore $-(au')' = bw$ in the sense of distributions (i.e., in $D'((0,\infty); H)$). It follows that $u'' \in X$ and $w = Bu$. The proof of the lemma is complete. $\square$

Remark 2.3. *The boundary condition* (B) *(i.e.,* $u(0) = x$*) in Lemma 2.2 (in the definition of* B*) can be generalized to*

$$u'(0) \in \beta(u(0) - x), \tag{B''}$$

where $\beta\colon D(\beta) \subset H \to H$ *is a general maximal monotone operator. In this case* B *is maximal monotone (even a subdifferential if* β *is so). While the monotonicity of* B *is obvious, to prove its maximality, it suffices to show that for all* $f \in X$ *the equation* $u + Bu = f$ *has a solution* $u \in D(B)$ *(cf. Minty's Theorem, see, e.g.,* [11], *p. 19). Let* $v \in X$ *the unique solution of*

$$v - pv'' - qv' = f, \quad v(0) = x,$$

which exists by Lemma 2.2, with $v', v'' \in X$*. Let* $u(t) = v(t) + \zeta(t)y$*, where* $y \in H$ *and* $\zeta = \zeta(t)$ *satisfies*

$$\zeta - p\zeta'' - q\zeta' = 0,\ t > 0;\ \zeta(0) = 1;\ \zeta, \zeta' \in L_b^2(\mathbb{R}_+; \mathbb{R}).$$

The existence of ζ *also follows from Lemma 2.2. Clearly,*

$$u - pu'' - qu' = f \quad \textit{for a.a. } t \geq 0.$$

It remains to prove that $u'(0) \in \beta(u(0) - x)$ *for a convenient* $y \in H$, *i.e.,*

$$v'(0) + \zeta'(0)y \in \beta(y).$$

Since β *is maximal monotone, for the existence of* y *it suffices to show that* $\zeta'(0) < 0$. *Assume by contradiction that* $\zeta'(0) \geq 0$. *We have*

$$(a\zeta')' = b\zeta \geq 0 \quad \text{for a.a. } t \in (0,T),$$

for some $T \in (0,\infty)$, *since* $\zeta(0) = 1$. *It follows that* $t \to a(t)\zeta'(t)$ *is nondecreasing on* $[0,T]$. *In particular,* $a(t)\zeta'(t) \geq \zeta'(0) \geq 0$ *and thus* $\zeta'(t) \geq 0$ *in* $[0,T]$, *so* ζ *is nondecreasing in* $[0,T]$. *On the other hand, integrating over* $[t,\infty)$ *the equation* $(a(\zeta)')'\zeta = b\zeta^2$, *we obtain*

$$-a(t)\zeta'(t)\zeta(t) - \int_t^\infty a(s)\zeta'(s)^2 ds = \int_t^\infty b(s)\zeta(s)^2 ds > 0$$

for $t \in (0,T)$ *since* ζ *is positive in* $[0,T]$. *Therefore* $a(t)\frac{d}{dt}\zeta^2(t) < 0$ *for* $t \in (0,T)$, *which implies that* ζ *is decreasing in* $[0,T]$, *contradiction.*

In what follows we restrict ourselves to the boundary condition (B). *The case of the more general condition* (B'') *is still open, but the above remark could be a good starting step towards the solution of problem* (E), (B'').

3. Existence and uniqueness for $x \in D(A)$ and $f \in X$

Let $\bar{A}$ denote the realization of A in X, that is

$$\bar{A} = \{[u,v] \in X \times X : \, [u(t), v(t)] \in A \ \text{ for a.a. } t \in \mathbb{R}_+\},$$

where X is the space defined in the previous section. Under (H1) $\bar{A}$ is maximal monotone in X. Recall that for all $\lambda > 0$ the realization of the resolvent operator $J_\lambda = (I + \lambda A)^{-1}$ is equal to $(I + \lambda\bar{A})^{-1}$, and the realization of the Yosida approximation $A_\lambda = \lambda^{-1}(I - J_\lambda)$ coincides with the Yosida approximation of $\bar{A}$, i.e., $\overline{A_\lambda} = (\bar{A})_\lambda =: \bar{A}_\lambda$. For details, see, e.g., [11], p. 31.

Theorem 3.1. *If (H1) and (H2) hold,* $x \in D(A)$ *and* $f \in X$, *then there exists a unique* $u \in D(B)$ *satisfying equation (E), where* B *and its domain* $D(B)$ *have been defined in the previous section.*

Proof. We devide the proof into several steps.
Step 1: the case $f = f_n$ for a given positive integer n, where $f_n(t) = f(t)$ for a.a. $t \in (0,n)$, and $f_n(t) = 0$ for a.a. $t > n$.
Obviously, f_n converges in X to f. In a first stage, we assume that n is fixed. For $\lambda > 0$ we denote by $u_\lambda \in D(B)$ the unique solution of the equation

$$Bu_\lambda + \bar{A}_\lambda u_\lambda + \lambda u_\lambda = -f_n. \tag{3.1}$$

The existence of u_λ follows from the maximality of the sum $B + \bar{A}_\lambda$. Obviously, (3.1) can be written as

$$-\left(a(t)u_\lambda'(t)\right)' + b(t)A_\lambda u_\lambda(t) + \lambda b(t)u_\lambda(t) = -b(t)f_n(t) \ \text{ for a.a. } t \in \mathbb{R}_+. \tag{3.2}$$

Now, we multiply (3.2) by $u_\lambda(t)$ and then integrate over $[0,t]$:

$$\begin{aligned}(u_\lambda'(0), x) - a(t)(u_\lambda'(t), u_\lambda(t)) + \int_0^t a(s)\|u_\lambda'(s)\|^2 ds + \lambda\int_0^t b(s)\|u_\lambda(s)\|^2 ds \\ \leq \int_0^n b(s)\|f_n(s)\| \cdot \|u_\lambda(s)\| ds \leq \|f_n\|_X \cdot \|\sqrt{b}u_\lambda\|_{L^2(0,n;H)}.\end{aligned} \tag{3.3}$$

We have used the monotonicity of A_λ and the fact that $A_\lambda 0 = 0$. Since $u_\lambda \in D(B)$, we have by Lemma 2.1

$$\lim_{t\to\infty} a(t)(u_\lambda(t), u'_\lambda(t)) = 0. \tag{3.4}$$

By (3.3) and (3.4) we derive

$$p_0\|u'_\lambda\|_X^2 + \lambda\|u_\lambda\|_X^2 \le -(u'_\lambda(0), x) + \|f_n\|_X\Big(\int_0^n b(s)\|u_\lambda\|^2 ds\Big)^{1/2}. \tag{3.5}$$

Using in (3.5) the identity

$$u_\lambda(s) = x + \int_0^s u'_\lambda(\tau)\,d\tau, \quad s \in [0,n],$$

we obtain

$$p_0\|u'_\lambda\|_X^2 + \lambda\|u_\lambda\|_X^2 \le \|x\|\cdot\|u'_\lambda(0)\| + \|f_n\|_X(C_1 + C_2\|u'_\lambda\|_X), \tag{3.6}$$

where C_1, C_2 are positive constants (depending on n). In what follows we will denote by C_i $(i = 3, 4, \cdots)$ different positive constants. Estimate (3.6) implies

$$\frac{p_0}{2}\|u'_\lambda\|_X^2 + \lambda\|u_\lambda\|_X^2 \le \|x\|\cdot\|u'_\lambda(0)\| + C_3. \tag{3.7}$$

Since A_λ is monotone and Lipschitzian,

$$a(t)\,((A_\lambda u_\lambda(t))', u'_\lambda(t)) \ge 0 \quad \text{for a.a. } t > 0. \tag{3.8}$$

Integration of (3.8) over $[0,t]$ yields

$$\begin{aligned}
&(A_\lambda u_\lambda(t), a(t)u'_\lambda(t)) - (A_\lambda x, u'_\lambda(0))\\
&\quad\ge \int_0^t (A_\lambda u_\lambda(s), (a(s)u'_\lambda(s))')ds\\
&\quad= \int_0^t (A_\lambda u_\lambda(s), b(s)A_\lambda u_\lambda(s) + \lambda b(s)u_\lambda(s) + b(s)f_n(s))ds\\
&\quad\ge \int_0^t b(s)\|A_\lambda u_\lambda(s)\|^2 ds + \int_0^t b(s)(A_\lambda u_\lambda(s), f_n(s))ds.
\end{aligned} \tag{3.9}$$

Note that

$$\|A_\lambda u_\lambda(t)\| \le \frac{1}{\lambda}\|u_\lambda(t)\|,$$

so letting $t \to \infty$ in (3.9), we get (cf. Lemma 2.1)

$$\|\bar{A}_\lambda u_\lambda\|_X^2 \le \|\bar{A}_\lambda u_\lambda\|_X\|f_n\|_X - (A^0 x, u'_\lambda(0)), \tag{3.10}$$

where A^0 is the minimal section of A. Therefore,

$$\|\bar{A}_\lambda u_\lambda\|_X^2 \le \|f_n\|_X + 2\|A^0 x\|\cdot\|u'_\lambda(0)\|. \tag{3.11}$$

Using (3.1), (3.7) and (3.11) we can derive the estimate

$$\begin{aligned}
\|u''_\lambda\|_X \le& \frac{1}{p_0}(\|f_n\|_X + \|q\|_{L^\infty(\mathbb{R}_+)}\|u'_\lambda\|_X + \lambda\|u_\lambda\|_X + \|\bar{A}_\lambda u_\lambda\|_X)\\
\le& C_4 + C_5\|u'_\lambda(0)\|^{1/2} \quad \text{for all } 0 < \lambda \le \lambda_0,
\end{aligned} \tag{3.12}$$

where λ_0 is an arbitrary but fixed constant. On the other hand,

$$\begin{aligned}\|u'_\lambda(0)\|^2 &= -\int_0^\infty (a(t)\|u'_\lambda(t)\|^2)'dt \\ &= -\int_0^\infty q(t)b(t)\|u'_\lambda(t)\|^2 dt - 2\int_0^t a(t)(u'_\lambda(t), u''_\lambda(t))\,dt, \end{aligned}\tag{3.13}$$

which yields

$$\begin{aligned}\|u'_\lambda(0)\|^2 \leq& C_6(\|u'_\lambda\|_X^2 + \|u''_\lambda\|_X^2) \\ \text{(by (3.7) and (3.12))} \ \leq& C_7\|u'_\lambda(0)\| + C_8,\ \forall\lambda\in(0,\lambda_0]\end{aligned}\tag{3.14}$$

which shows that $\{\|u'_\lambda(0)\|,\ 0<\lambda\leq\lambda_0\}$ is a bounded set. This fact combined with (3.7), (3.11), (3.12) implies that u'_λ, u''_λ, $\bar{A}_\lambda u_\lambda$ are all bounded in X for $0<\lambda\leq\lambda_0$. The set $\{u_\lambda : 0<\lambda\leq\lambda_0\}$ is also bounded in X. This follows by integration over $[0,\infty)$ of the identity

$$(a(t)\|u_\lambda(t)\|^2)' = q(t)b(t)\|u_\lambda(t)\|^2 + 2a(t)(u_\lambda(t), u'_\lambda(t)),$$

which leads to

$$-\|x\|^2 = \int_0^t q(t)b(t)\|u_\lambda(t)\|^2 dt + 2\int_0^t a(t)(u_\lambda(t), u'_\lambda(t))dt$$

and therefore

$$\omega\|u_\lambda\|_X^2 \leq \|x\|^2 + 2p_1\|u_\lambda\|_X\|u'_\lambda\|_X,$$

where $\omega = \operatorname{ess\,inf}|q|$ (i.e., $\omega = q_0$ if $q_0>0$ and $\omega=-q_1$ if $q_1<0$). By the facts we have established so far, there exists a $u\in X$ (depending on n but for the moment n is fixed), such that u', $u''\in X$ and

$$u_\lambda\to u,\ u'_\lambda\to u',\ u''_\lambda\to u'' \text{ weakly in } X,\tag{3.15}$$

as $\lambda\to 0$, on a subsequence. Now, for $\lambda,\nu\in(0,\lambda_0]$, we can easily derive from equation (3.2)

$$\begin{aligned}-(a(u'_\lambda - u'_\nu)', u_\lambda - u_\nu) &+ b(A_\lambda u_\lambda - A_\nu u_\nu, u_\lambda - u_\nu) \\ &+ b(\lambda u_\lambda - \nu u_\nu, u_\lambda - u_\nu) = 0 \quad \text{for a.a. } t>0.\end{aligned}$$

Integration over $[0,\infty)$ gives

$$\begin{aligned}&p_0\|u'_\lambda - u'_\nu\|_X^2 + (\bar{A}_\lambda u_\lambda - \bar{A}_\nu u_\nu, \overline{J_\lambda}u_\lambda - \overline{J_\nu}u_\nu)_X \\ &\leq -(\bar{A}_\lambda u_\lambda - \bar{A}_\nu u_\nu, \lambda\bar{A}_\lambda u_\lambda - \nu\bar{A}_\nu u_\nu)_X - (\lambda u_\lambda - \nu u_\nu, u_\lambda - u_\nu)_X.\end{aligned}\tag{3.16}$$

Since u_λ and $\bar{A}_\lambda u_\lambda$ are bounded in X for $0<\lambda\leq\lambda_0$ and $\bar{A}_\lambda u_\lambda\in\bar{A}\,\overline{J_\lambda}u_\lambda$, we can derive from (3.16)

$$\|u'_\lambda - u'_\nu\|_X^2 \leq C_9(\lambda+\nu),\tag{3.17}$$

so, on a subsequence,

$$u'_\lambda\to u' \text{ strongly in } X, \quad \text{as } \lambda\to 0.\tag{3.18}$$

Let $T\in(0,\infty)$ be arbitrary but fixed. We have

$$u'_\lambda\to u' \text{ strongly in } L^2(0,T;H),\tag{3.19}$$

$$u''_\lambda\to u'' \text{ weakly in } L^2(0,T;H).\tag{3.20}$$

Moreover,

$$\begin{aligned}\|u_\lambda(t)-u_\nu(t)\| =& \|\int_0^t [u'_\lambda(s)-u'_\nu(s)]\,ds\| \\ \le& T^{1/2}\|u'_\lambda - u'_\nu\|_{L^2(0,T;H)},\ 0\le t\le T,\end{aligned}$$

which implies

$$u_\lambda \to u \ \text{ in } C([0,T];H), \tag{3.21}$$

thus in particular $u(0)=x$ and $u_\lambda \to u$ in $L^2(0,T;H)$. Note also that

$$\|J_\lambda u_\lambda(\cdot) - u\|_{L^2(0,T;H)} \le \lambda\|A_\lambda u_\lambda(\cdot)\|_{L^2(0,T;H)} + \|u_\lambda - u\|_{L^2(0,T;H)},$$

which implies

$$J_\lambda u_\lambda(\cdot) \to u \ \text{ strongly in } L^2(0,T;H). \tag{3.22}$$

By (3.19), (3.20) and equation (3.1) it follows that

$$A_\lambda u_\lambda(\cdot) \to pu'' + qu' - f_n \ \text{ weakly in } L^2(0,T;H). \tag{3.23}$$

Since $A_\lambda u_\lambda(t) \in AJ_\lambda u_\lambda(t)$ and the realization of A in $L^2(0,T;H)$ is a maximal monotone operator in this space, hence demiclosed, we can deduce from (3.22) and (3.23) that u satisfies equation (E) for a.a. $t\in(0,T)$. Since T was arbitrarily chosen, u satisfies (E) for a.a. $t\in\mathbb{R}_+$.

Step 2: general $f\in X$.

From now on we consider n variable and denote by u_n the solution corresponding to f_n whose existence was proved above, i.e.,

$$-f_n \in Bu_n + \bar{A}u_n. \tag{3.24}$$

It is easily seen that for a.a. $t>0$

$$b(-f_n+f_m) \in -(a(u'_n-u'_m))' + b(\bar{A}u_n - \bar{A}u_m), \tag{3.25}$$

which implies

$$p_0\|u'_n - u'_m\|_X^2 \le \|f_n - f_m\|_X \|u_n - u_m\|_X. \tag{3.26}$$

On the other hand, if we integrate over $[0,\infty)$ the equation

$$(a\|u_n-u_m\|^2)' = qb\|u_n-u_m\|^2 + 2a(u_n-u_m, u'_n-u'_m), \tag{3.27}$$

we derive

$$\|u_n-u_m\|_X \le C_{10}\|u'_n-u'_m\|_X.$$

This inequality combined with (3.26) shows that both (u_n) and (u'_n) are Cauchy sequences in X, so there exists $u\in X$ such that $u'\in X$,

$$u_n\to u,\quad u'_n\to u' \quad \text{strongly in } X, \tag{3.28}$$

and $u(0)=x$.

Note that, for a fixed n, u_n is approximated by the solution of equation (3.1), as $\lambda\to 0^+$, where the term λu_λ is omitted (we do not need this term at this stage). The existence of a solution u_λ for this modified (3.1) follows by a reasoning similar to that used in the proof of Step 1. By (3.19) and (3.20) (that hold true again) we easily see that $u'_\lambda(t)$ converges uniformly to $u'_n(t)$ on every compact interval $[0,T]$ as λ tends to 0. In particular,

$$u'_\lambda(0)\to u'_n(0), \ \text{ as } \lambda\to 0. \tag{3.29}$$

Returning to (3.9) (or (3.10)), where the term $\lambda b u_\lambda$ is omitted, we see that

$$-(A^0x, u'_n(0)) \ge \|w_n\|_X^2 + (w_n, f_n)_X,$$

where w_n is the weak limit in X of $\bar{A}_\lambda u_\lambda$, as $\lambda \to 0^+$, $w_n \in \bar{A}u_n$. Therefore,

$$\|w_n\|_X^2 \le \|f\|_X + 2\|A^0 x\| \cdot \|u_n'(0)\|. \tag{3.30}$$

Using (3.28) and (3.30) in equation (3.24), more precisely, in equation

$$Bu_n + w_n = -f_n,$$

we obtain the analogue of (3.12)

$$\|u_n''\|_X \le \tilde{C}_4 + \tilde{C}_5 \|u_n'(0)\|^{1/2}. \tag{3.31}$$

We can also derive, as we did before, the analogue of (3.14) and thus $\|u_n'(0)\|$ is bounded. By virtue of (3.31), $\|u_n''\|_X$ is bounded too. Hence $u'' \in X$ and u_n'' conveges weakly in X to u'', on a subsequence. Starting from (3.24), using in particular the fact that f_n converges in X to f, we can show by the standard procedure that u satisfies equation (E) for a.a. $t > 0$.

Step 3: Uniqueness.

Let $v \in D(B)$ be another solution of equation (E) that (satisfies $v(0) = x$ and) corresponds to the same $f \in X$. Multiplying by $u(t) - v(t)$ the obvious equation (inclusion)

$$(a(u' - v'))' \in b(Au - Av) \text{ for a.a. } t > 0, \tag{3.32}$$

and integrating the resulting equation over $[t, \infty)$, we obtain

$$\frac{1}{2} a(t) \frac{d}{dt} \|u(t) - v(t)\|^2 + \int_t^\infty a(s) \|u'(s) - v'(s)\|^2 ds \le 0, \tag{3.33}$$

which implies that

$$\frac{d}{dt} \|u(t) - v(t)\|^2 \le 0 \text{ for all } t \ge 0,$$

so $t \to \|u(t) - v(t)\|$ is nonincreasing on $\mathbb{R}_+$. In particular,

$$\|u(t) - v(t)\| \le \|u(0) - v(0)\| = 0 \quad \forall t \ge 0, \tag{3.34}$$

which implies $u \equiv v$. The theorem is completely proved. □

Remark 3.2. *Note that we also have uniqueness for u_λ and u_n above so all the convergences are true for the whole sequences involved.*

4. The case $x \in \overline{D(A)}$ and $f \in X$

Let $x_n \in D(A)$, $\|x_n - x\| \to 0$. Denote by u_n the solution of equation (E) given by Theorem 3.1 satisfying $u_n(0) = x_n$. If we multiply the equation

$$(a(u_n' - u_m'))' \in b(Au_n - Au_m) \quad \text{for a.a. } t > 0, \tag{4.1}$$

by $u_n(t) - u_m(t)$ and integrate over $[t, \infty)$, we get

$$\frac{1}{2} a(t) \frac{d}{dt} \|u_n(t) - u_m(t)\|^2 \le -\int_t^\infty a(s) \|u_n'(s) - u_m'(s)\|^2 ds \le 0 \quad \text{for all } t \ge 0. \tag{4.2}$$

It follows that

$$\frac{d}{dt} \|u_n(t) - u_m(t)\|^2 \le 0 \quad \text{for all } t \ge 0, \tag{4.3}$$

so $t \mapsto \|u_n(t) - u_m(t)\|$ is nonincreasing on $\mathbb{R}_+$. In particular,

$$\|u_n(t) - u_m(t)\| \le \|x_n - x_m\| \quad \text{for all } t \ge 0. \tag{4.4}$$

Thus there exists $u \in C(\mathbb{R}_+; H)$, such that u_n converges to u in $C([0,T]; H)$ for all $T \in (0, \infty)$ and $u(0) = x$. If f is identically zero, then it is easy to see that $t \to \|u_n(t)\|$ is nonincreasing and hence

$$\|u_n(t)\| \leq \|u_n(0)\| = \|x_n\| \quad \text{for all } t \geq 0,$$

hence $u \in L^\infty(\mathbb{R}_+; H)$.

Returning to the nonhomogeneous case $f \in X$, if we integrate over $[0,T]$ the obvious inequality

$$t((a(u_n' - u_m'))', u_n - u_m) \geq 0 \quad \text{for a.a. } t > 0,$$

we get

$$\begin{aligned} \int_0^T ta(t)\|u_n'(t) - u_m'(t)\|^2 dt + \frac{1}{2}\int_0^T a(t)\frac{d}{dt}\|u_n(t) - u_m(t)\|^2 dt \\ \leq \frac{T}{2} a(T)(\frac{d}{dt}\|u_n - u_m\|^2)(T) \leq 0. \end{aligned}$$

due to (4.3). This implies

$$\begin{aligned} \int_0^T ta(t)\|u_n'(t) - u_m'(t)\|^2 dt \leq & \frac{1}{2}\|x_n - x_m\|^2 + \frac{1}{2}\int_0^T \frac{aq}{p}\|u_n - u_m\|^2 dt \\ \text{(according to (4.4))} \leq & \frac{1}{2}\|x_n - x_m\|^2 (1 + \int_0^T \frac{a|q|}{p} dt). \end{aligned} \tag{4.5}$$

By (4.4) and (4.5) we see that $t^{1/2}u' \in L^2(0,T; H)$ and

$$t^{1/2}u_n' \to t^{1/2}u' \quad \text{strongly in } L^2(0,T; H). \tag{4.6}$$

It is easy to see that u does not depend on the choice of the sequence (x_n) approximating x. We can call u a *generalized solution* of equation (E) satisfying $u(0) = x \in \overline{D(A)}$. If $f \equiv 0$, then u also satisfies (C). This is also the case if the condition $0 \in D(A)$, $0 \in A0$ is replaced by $0 \in R(A)$.

We do not have an estimate for u_n'' to prove that u is a classic solution of equation (E). So the existence of a classic solution for $x \in \overline{D(A)}$ is still an open problem. However, if p and q are constant functions we are able to obtain classic existence for all $x \in \overline{D(A)}$ (see the next section).

5. Existence and uniqueness for $x \in \overline{D(A)}$ in the case of constant coefficents

In this section we assume that p and q are both constant functions. Without any loss of generality, we can assume $p \equiv 1$ and $q \equiv c$, with $c \in \mathbb{R} \setminus \{0\}$. Recall that the case $c = 0$ was extensively studied by V. Barbu [3], [4], [5] and by H. Brezis [6]. We are going to prove that in this particular case problem (E), (B) has a unique classical solution for each $x \in \overline{D(A)}$ and $f \in X$. Before stating precisely the result, we need some definitions. For $\varepsilon > 0$ small, define $\zeta_\varepsilon \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\zeta_\epsilon(t) = \begin{cases} t & \text{if } 0 \leq t \leq \epsilon \\ \epsilon & \text{if } t > \epsilon \end{cases}$$

Let X_ε be the weighted space $L^2(\mathbb{R}_+,; H; \zeta_\varepsilon(t)e^{ct}dt)$. For $\alpha < c$, close to c, denote $X_\alpha = L^2(\mathbb{R}_+; H; t^3 e^{\alpha t}dt)$. Obviously, $X \subset X_\varepsilon \subset X_\alpha$, where X is the space defined in Section 2, with the particular weight $b(t) = e^{ct}$, i.e., $X = L^2(\mathbb{R}_+; H; e^{ct}dt)$.

Theorem 5.1. *If (H1) holds, $p \equiv 1$, $q \equiv c$, with $c \in \mathbb{R} \setminus \{0\}$, $x \in \overline{D(A)}$ and $f \in X$, then there exists a unique $u \in C(\mathbb{R}_+; H)$, u, $u' \in X_\varepsilon$, $u'' \in X_\alpha$, such that u satisfies (E) and $u(0) = x$, for all $\varepsilon > 0$ small and $\alpha < c$, close to c. If $f \equiv 0$, then $\|u(t)\| \leq \|x\|$ for all $t \geq 0$.*

Proof. In the present case $a(t) = b(t) = e^{ct}$. Let $x_n \in D(A)$, $\|x_n - x\| \to 0$, and let $u_n \in X$ be the unique solution of (E) with $f = f_n$ given by Theorem 3.1, satisfying $u_n(0) = x_n$:

$$(e^{ct}u_n'(t))' \in e^{ct}(Au_n(t) + f_n(t)) \quad \text{for a.a. } t > 0, \tag{5.1}$$

where f_n is the truncation of f defined in Section 3. In what follows we develop a technique inspired by Bruck's paper [8]. For m, n two fixed positive integers, define

$$g(t) = \frac{e^{ct}}{2}\|u_n(t) - u_m(t)\|^2.$$

We have,

$$g(t) = cg(t) + e^{ct}(u_n'(t) - u_m'(t), u_n(t) - u_m(t)), \tag{5.2}$$

$$\begin{aligned} g''(t) =& cg'(t) + ce^{ct}(u_n' - u_m', u_n - u_m) + e^{ct}(u_n'' - u_m'', u_n - u_m) + e^{ct}\|u_n' - u_m'\|^2 \\ =& c^2 g(t) + 2ce^{ct}(u_n' - u_m', u_n - u_m) \\ &+ e^{ct}(-c(u_n' - u_m') + Au_n - Au_m + f_n - f_m, u_n - u_m) + e^{ct}\|u_n' - u_m'\|^2 \\ \geq& e^{ct}\left\{\frac{c^2}{2}\|u_n - u_m\|^2 + c(u_n' - u_m', u_n - u_m) + \|u_n' - u_m'\|^2\right\} \\ &+ e^{ct}(f_n - f_m, u_n - u_m) \\ \geq& \gamma e^{ct}\left\{\|u_n - u_m\|^2 + \|u_n' - u_m'\|^2\right\} + e^{ct}(f_n - f_m, u_n - u_m) \end{aligned} \tag{5.3}$$

for $\gamma > 0$ a small constant. From (5.2) and (5.3) it follows that

$$\begin{aligned} &\gamma \int_0^\infty \zeta_\varepsilon(t)e^{ct}\left\{\|u_n - u_m\|^2 + \|u_n' - u_m'\|^2\right\} dt \\ &\leq \int_0^\infty \zeta_\varepsilon(t)g''(t)dt + \int_0^\infty \zeta_\varepsilon(t)e^{ct}\|f_n - f_m\| \cdot \|u_n - u_m\|dt \\ &\leq \int_0^\varepsilon tg''(t)\,dt + \varepsilon\int_\varepsilon^\infty g''(t)dt + \frac{1}{2\gamma}\int_0^\infty \zeta_\varepsilon(t)e^{ct}\|f_n - f_m\|^2 dt \\ &+ \frac{\gamma}{2}\int_0^\infty \zeta_\varepsilon(t)e^{ct}\|u_n - u_m\|^2 dt \\ &\leq tg'(t)|_0^\varepsilon - \int_0^\varepsilon g'(t)dt + \varepsilon[g'(\infty) - g'(\varepsilon)] + \frac{1}{2\gamma}\|f_n - f_m\|_X^2 + \frac{\gamma}{2}\|u_n - u_m\|_{X_\varepsilon}^2 \\ &\leq g(0) + \frac{1}{2\gamma}\|f_n - f_m\|_X^2 + \frac{\gamma}{2}\|u_n - u_m\|_{X_\varepsilon}^2. \end{aligned}$$

Therefore,

$$\frac{\gamma}{2}\|u_n - u_m\|_{X_\varepsilon}^2 + \gamma\|u_n' - u_m'\|_{X_\varepsilon}^2 \leq \frac{1}{2}\|x_n - x_m\|^2 + \frac{1}{2\gamma}\|f_n - f_m\|_X^2. \tag{5.4}$$

Thus there exists $u \in X_\varepsilon$, such that $u' \in X_\varepsilon$, and

$$u_n \to u, \quad u_n' \to u' \quad \text{strongly in } X_\varepsilon. \tag{5.5}$$

Now, in order to derive an estimate for u_n'', we recall that, for a fixed positive integer n, the solution $u_{n\lambda} \in X$ of

$$u_{n\lambda}'' + cu_{n\lambda}' = A_\lambda u_{n\lambda} + f_n, \tag{5.6}$$

approximates u_n as $\lambda \to 0+$:

$$u_{n\lambda} \to u_n \text{ in } C([0,T];H)\ \forall T>0,\ u_{n\lambda}' \to u_n' \text{ strongly in } X,\ u_{n\lambda}'' \to u_n'' \text{ weakly in } X. \tag{5.7}$$

Since $(u_{n\lambda}', (A_\lambda u_{n\lambda})') \geq 0$, we have

$$\begin{aligned}\frac{d}{dt}(e^{\alpha t}u_{n\lambda}', A_\lambda u_{n\lambda}) \geq& (e^{\alpha t}u_{n\lambda}')', A_\lambda u_{n\lambda}) \\ =& (A_\lambda u_{n\lambda} + (\alpha - c)u_{n\lambda}' + f_n, e^{\alpha t}A_\lambda u_{n\lambda}).\end{aligned} \tag{5.8}$$

Multiplying (5.8) by t^3 and then integrating over $\mathbb{R}_+$, we obtain

$$\begin{aligned}\|A_\lambda u_{n\lambda}\|_{X_\alpha}^2 \leq& \|A_\lambda u_{n\lambda}\|_{X_\alpha} \cdot \|f_n\|_{X_\alpha} + (c-\alpha)\|A_\lambda u_{n\lambda}\|_{X_\alpha} \cdot \|u_{n\lambda}'\|_{X_\alpha} \\ &- 3\int_0^\infty t^2 e^{\alpha t}(u_{n\lambda}', A_\lambda u_{n\lambda})dt \\ \leq& \|A_\lambda u_{n\lambda}\|_{X_\alpha} \cdot \|f_n\|_{X_\alpha} + (c-\alpha)\|A_\lambda u_{n\lambda}\|_{X_\alpha} \cdot \|u_{n\lambda}'\|_{X_\alpha} \\ &+ 3\|A_\lambda u_{n\lambda}\|_{X_\alpha}\left(\int_0^\infty te^{\alpha t}\|u_{n\lambda}'\|^2 dt\right)^{1/2}.\end{aligned}$$

We have used α instead of c, $\alpha < c$, so for any polynomial P, $|P(t)|e^{\alpha t} \leq C_{11}e^{ct}$, so that all the above computations are permitted. The last estimate yields

$$\|A_\lambda u_{n\lambda}\|_{X_\alpha} \leq \|f\|_{X_\alpha} + (c-\alpha)\|u_{n\lambda}'\|_{X_\alpha} + 3\left(\int_0^\infty te^{\alpha t}\|u_{n\lambda}'\|^2 dt\right)^{1/2}. \tag{5.9}$$

Now, combining (5.6) and (5.9), we find

$$\begin{aligned}\|u_{n\lambda}''\|_{X_\alpha} \leq& |c|\,\|u_{n\lambda}'\|_{X_\alpha} + \|A_\lambda u_{n\lambda}\|_{X_\alpha} + \|f_n\|_{X_\alpha} \\ \leq& 2|c|\,\|u_{n\lambda}'\|_{X_\alpha} + 2\|f\|_{X_\alpha} + 3\left(\int_0^\infty te^{\alpha t}\|u_{n\lambda}'\|^2 dt\right)^{1/2}.\end{aligned} \tag{5.10}$$

Since $u_{n\lambda}' \to_n'$ strongly in X and $u_{n\lambda}'' \to u_n''$ weakly in X, as $\lambda \to 0+$, we derive from (5.10)

$$\|u_n''\|_{X_\alpha} \leq 2|c|\|u_n'\|_{X_\alpha} + 2\|f\|_{X_\alpha} + 3\left(\int_0^\infty te^{\alpha t}\|u_n'\|^2 dt\right)^{1/2}. \tag{5.11}$$

We know that u_n' is convergent (hence bounded) in X_ε so (5.11) shows that u_n'' is bounded in X_α. Thus $u'' \in X_\alpha$ and u_n'' converges weakly in X_α to u''. Letting $n \to \infty$ in

$$u_n'' + cu_n' \in Au_n + f_n,$$

with respect to the topology of $L^2((\delta,T);H)$ for $0 < \delta < T < \infty$ we can show by the standard procedure that u satisfies equation (E) for a.a. $t \in (\delta,T)$, so for a.a. $t > 0$ (since δ and T can be chosen arbitrarily). Note that the weights are not relevant on $[\delta,T]$. Now let us prove that $u \in C(\mathbb{R}_+;H)$ and $u(0) = x$. Of course, we only need to prove continuity at $t = 0+$. From equation (5.2) we easily derive

$$\begin{aligned}
&\frac{1}{2}e^{ct}\frac{d}{dt}\|u_n(t)-u_m(t)\|^2 + \int_t^\infty e^{cs}\|u_n'(s)-u_m'(s)\|^2 ds \\
&\le \int_t^\infty e^{cs}(f_n(s)-f_m(s), u_n(s)-u_m(s))\, ds \\
&\le \int_m^n e^{cs}\|f(s)\|\cdot\|u_n(s)-u_m(s)\|\, ds \\
&\le \|f\|_X\|u_n-u_m\|_{X_\varepsilon} < \eta \quad \text{for } N_\eta < m < n. \qquad (5.12)
\end{aligned}$$

If we multiply (5.12) by e^{-ct} and then integrate over $[0,t]$, we get

$$\frac{1}{2}\|u_n(t)-u_m(t)\|^2 \le \frac{1}{2}\|x_n-x_m\|^2 + \eta t \quad \text{for } N_\eta < m < n.$$

Thus $u_n(t)$ converges uniformly as $n\to\infty$ on a compact interval $[0,t_0]$ to a continuous function $v = v(t)$. In particular, $v(0) = x$. From the previous part of the proof $u_n \to u$ and $u_n' \to u'$ in X_ε, for all $\varepsilon > 0$, so u is continuous on $(0,\infty)$. Obviously, $u(t) = v(t)$ for $t \in (0,t_0]$. It follows that

$$\lim_{t\to 0^+} u(t) = v(0) = x,$$

and $u \in C(\mathbb{R}_+; H)$.

Concerning uniqueness, it follows by a reasoning we have already used. Indeed, if u, v are two solutions corresponding to $x \in \overline{D(A)}$ and $f \in X$ in the class indicated in the statement of the theorem, then

$$\frac{1}{2}e^{ct}\|u(t)-v(t)\|^2 + \int_t^\infty e^{cs}\|u'(s)-v'(s)\|^2 ds \le 0, \quad \text{for all } t > 0.$$

This implies that $t \to \|u(t)-v(t)\|$ is nonincreasing and thus

$$\|u(t)-v(t)\| \le \|u(0)-v(0)\| = 0, \quad \text{for all } t \ge 0, \tag{5.13}$$

i.e., $u \equiv v$. If u is the solution corresponding to $x \in \overline{D(A)}$, $f \equiv 0$ and v is the solution corresponding to $x = 0$, $f \equiv 0$, i.e., $v \equiv 0$, then (5.13) yields

$$\|u(t)\| \le \|x\|, \quad \text{for all } t \ge 0.$$

The proof is now complete. □

Remark 5.2. *If $c < 0$, then condition $f \in X$ allows unbounded f's. In this case the corresponding u's are so, as illustrated by simple examples.*

Remark 5.3. *In Theorem 5.1 we have $u'' \in L^2([\delta,\infty); H; e^{ct}dt)$ for all $\delta > 0$. Indeed, since u satisfies equation (E) for a.a. $t > 0$, for every $\delta > 0$, there is a $t_0 \in (0,\delta)$, such that $u(t_0) \in D(A)$, so we can apply Theorem 3.1 with $x := u(t_0)$ and $[t_0,\infty)$ instead of $\mathbb{R}_+$. Obviously, u, $u' \in L^2([\delta,\infty); H; e^{ct}dt)$ as well, for all $\delta > 0$.*

6. Constant coefficients and $f \equiv 0$

Consider the homogeneous equation

$$u''(t) + cu' \in Au(t), \ t > 0, \tag{E_0}$$

with

$$u(0) = x, \tag{B}$$

where $c \in \mathbb{R} \setminus \{0\}$. Recall that, if $c > 0$, the boundedness condition (C) added to (E) is not enough to guarantee uniqueness of u. However, we have uniqueness if we impose the condition $u \in X$, where $X = L^2_b(\mathbb{R}_+; H)$, with $b(t) = e^{ct}$. It is easy to see that the solutions of problem (E_0), (B) given by Theorems 3.1 and 5.1, generate a nonlinear semigroup of contractions $\{S(t)\colon \overline{D(A)} \to \overline{D(A)};\, t \geq 0\}$, $S(t)x := u(t)$, where $u \in X$ satisfies (E_0) and (B). We also have the properties

$$\forall x \in D(A),\ e^{ct/2}S(t)x,\ e^{ct/2}\frac{d}{dt}S(t)x,\ e^{ct/2}\frac{d^2}{dt^2}S(t)x \in L^2(\mathbb{R}_+; H); \tag{6.1}$$

$$\forall x \in \overline{D(A)},\ e^{ct/2}S(t)x,\ e^{ct/2}\frac{d}{dt}S(t)x,\ e^{ct/2}\frac{d^2}{dt^2}S(t)x \in L^2([\varepsilon, \infty); H)\ \forall \varepsilon > 0. \tag{6.2}$$

The last property in (6.2) follows from the fact that we can use as the initial state $u(t_0) \in D(A)$ for some $t_0 > 0$ instead of $x \in \overline{D(A)}$.

Let $F\colon D(F) \subset H \to H$ be such that $G = -F$ is the generator of the semigroup $\{S(t)\colon \overline{D(A)} \to \overline{D(A)};\, t \geq 0\}$ defined above. Operator F, which is maximal monotone, can be regarded as the formal solution of the operator equation

$$F^2 + cF - A = 0,$$

i.e.,

$$F = \frac{c}{2}I + \sqrt{\frac{c^2}{4}I + A}, \tag{6.3}$$

as remarked by B. Djafari Rouhani and H. Khatibzadeh [9]. Here $\sqrt{}$ represents the square root of $\frac{c^2}{4}I + A$ in Barbu's sense. If A is linear, then $G = -F$, where F is given by (6.3) is indeed the generator of the semigroup $\{S(t)\colon \overline{D(A)} \to \overline{D(A)};\, t \geq 0\}$. To show this, define $v(t) = e^{ct/2}u(t)$, where $u(t) = S(t)x$, $x \in D(A)$. Then,

$$\begin{aligned} v''(t) =& e^{ct/2}(\frac{c^2}{4}I + A)(e^{-ct/2}v(t)) \\ =& \frac{c^2}{4}v(t) + Av(t), \end{aligned}$$

so $v(t) = T(t)x$, where $T(t)$ is the semigroup generated by $-\sqrt{\frac{c^2}{4}I + A}$. Therefore,

$$S(t)x = e^{-ct/2}T(t)x, \quad x \in H,$$

which shows that the generator of $S(t)$ is $G = -F$, where F is given by (6.3), as asserted.

Comments on the asymptotic behavior. By (6.1), (6.2) and Lemma 2.1 (which remains valid if $\mathbb{R}_+$ is replaced by $[\varepsilon, \infty)$, $\varepsilon > 0$), we see that

$$\lim_{t\to\infty} e^{ct/2}\|S(t)x\| = \lim_{t\to\infty} e^{ct/2}\|\frac{d}{dt}S(t)x\| = 0, \tag{6.4}$$

for all $x \in \overline{D(A)}$. On the other hand, Djafari Rouhani and Khatibzadeh [9] have proved that, if $c > 0$ and $0 \in R(A)$, for any bounded solution u of equation (E_0) there exists a $y \in A^{-1}0$, such that

$$\|u(t) - y\| = \mathbf{O}(e^{-ct/2}), \quad \|u'(t)\| = \mathbf{O}(e^{-ct/2}) \tag{6.5}$$

Now are able to describe the set of all bounded solutions of equation (E_0), under the assumption $0 \in R(A)$.

Indeed, if we define $v(t) := u(t) - y$, $A_y x := A(x + y)$, $x \in D(A_y) = D(A) \setminus \{y\}$, then v is the unique solution in our sense of the equation

$$v'' + cv' \in A_y v, \quad t > 0, \tag{6.6}$$

with $v(0) = u(0) - y$. In fact, the set of all bounded solutions of equation (E_0) is

$$Q := \{u(t) = y + v(t) \colon v \text{ is a solution in our sense of (6.6)}, \ y \in A^{-1}0\}.$$

For a given $x \in \overline{D(A)}$ equation (E_0) may have several bounded solutions u satisfying $u(0) = x$, if $A^{-1}0$ is not a singleton.

Note that estimates (6.5) are weaker than (6.4). However, it is worth pointing out that recently Khatibzadeh [10] has shown that, if $c > 0$, then $\mathbf{O}(e^{-ct/2})$ in (6.5) can be replaced by $\mathbf{O}(e^{-ct})$.

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