NASH-STAMPACCHIA EQUILIBRIUM POINTS ON MANIFOLDS

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Abstract. Motivated by Nash equilibrium problems on 'curved' strategy sets, the concept of Nash-Stampacchia equilibrium points is introduced for a finite family of non-smooth functions defined on geodesic convex sets of certain Riemannian manifolds. Characterization, existence, and stability of Nash-Stampacchia equilibria are studied when the strategy sets are compact/non-compact subsets of certain Hadamard manifolds, exploiting two well-known geometrical features of these spaces both involving the metric projection operator. These two properties actually characterize the non-positivity of the sectional curvature of complete and simply connected Riemannian spaces, delimiting the Hadamard manifolds as the optimal geometrical framework of Nash-Stampacchia equilibrium problems. Our analytical approach exploits various elements from set-valued analysis, dynamical systems, and non-smooth calculus on Riemannian manifolds developed by Yu. S. Ledyaev and Q. J. Zhu [Trans. Amer. Math. Soc. 359 (2007), 3687-3732].

1. Introduction

After the seminal paper of Nash [14] there has been considerable interest in the theory of Nash equilibria due to its applicability in various real-life phenomena (game theory, price theory, networks, etc). Appreciating Nash's contributions, R. B. Myerson states that "Nash's theory of noncooperative games should now be recognized as one of the outstanding intellectual advances of the twentieth century". The Nash equilibrium problem involves n players such that each player know the equilibrium strategies of the partners, but moving away from his/her own strategy alone a player has nothing to gain. Formally, if the sets K_i denote the strategies of the players and $f_i : K_1 \times ... \times K_n \to \mathbb{R}$ are their loss-functions, $i \in \{1, ..., n\}$, the problem is to find an n-tuple $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{K} = K_1 \times ... \times K_n$ such that $f_i(\mathbf{p}) \leq f_i(\mathbf{p}; q_i)$ for every $q_i \in K_i$ and $i \in \{1, ..., n\}$, where $(\mathbf{p}; q_i) = (p_1, ..., p_{i-1}, q_i, p_{i+1}, ..., p_n) \in \mathbf{K}$. Such point $\mathbf{p}$ is called a *Nash equilibrium point for* $(\mathbf{f}, \mathbf{K}) = (f_1, ..., f_n; K_1, ..., K_n)$, the set of these points being denoted by $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$.

While most of the known developments in the Nash equilibrium theory deeply exploit the usual convexity of the sets K_i together with the vector space structure of their ambient spaces M_i (i.e., $K_i \subset M_i$), it is nevertheless true that these results are in large part *geometrical* in nature. The main purpose of this paper is to enhance those geometrical and analytical structures which serve as a basis of a systematic study of Nash-type equilibrium problems in a general setting as presently possible. In the light of these facts our contribution to the Nash equilibrium theory should be considered rather intrinsical and analytical than game-theoretical.

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We assume *a priori* that the strategy sets K_i are *geodesic convex* subsets of certain finite-dimensional Riemannian manifolds (M_i, g_i), i.e., for any two points of K_i there exists a unique geodesic in (M_i, g_i) connecting them which belongs entirely to K_i. This approach can be widely applied when the strategy sets are 'curved'. Note that the choice of such Riemannian structures does not influence the Nash equilibrium points for $(\mathbf{f}, \mathbf{K})$. As far as we know, the first step into this direction was made recently in [10] via a McClendon-type minimax inequality for acyclic ANRs, guaranteeing the existence of at least one Nash equilibrium point for $(\mathbf{f}, \mathbf{K})$ whenever $K_i \subset M_i$ are compact and geodesic convex sets of certain finite-dimensional Riemannian manifolds (M_i, g_i) while the functions f_i have certain regularity properties, $i \in \{1, ..., n\}$.

In [10] we introduced and studied for a wide class of non-smooth functions the set of *Nash-Clarke points for* $(\mathbf{f}, \mathbf{K})$, denoted in the sequel as $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$; for details, see Section 3. Note that $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ is larger than $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$; thus, a promising way to find the elements of $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$ is to determine the set $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$. In spite of the naturalness of this approach, we already pointed out its limited applicability due to the involved structure of $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$, conjecturing a more appropriate concept in order to locate the elements of $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$.

Motivated by the latter problem, we observe that the Fréchet and limiting subdifferential calculus of lower semicontinuous functions on Riemannian manifolds developed by Ledyaev and Zhu [11] and Azagra, Ferrera and López-Mesas [1] provides a very satisfactory approach. The idea is to consider the following system of variational inequalities: find $\mathbf{p} \in \mathbf{K}$ and $\xi_C^i \in \partial_C^i f_i(\mathbf{p})$ such that

$$\langle \xi_C^i, \exp_{p_i}^{-1}(q_i) \rangle_{g_i} \geq 0 \text{ for all } q_i \in K_i, \; i \in \{1, ..., n\},$$

where $\partial_C^i f_i(\mathbf{p})$ denotes the Clarke subdifferential of the locally Lipschitz function $f_i(\mathbf{p}; \cdot)$ at the point $p_i \in K_i$; for details, see Section 3. The solutions of this system form the set of *Nash-Stampacchia equilibrium points for* $(\mathbf{f}, \mathbf{K})$, denoted by $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$, which is the main concept of the present paper.

One of the advantages of the new concept is that the set $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ is 'closer' to $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$ than $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$. More precisely, we state that $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ for the same class of non-smooth functions $\mathbf{f} = (f_1, ..., f_n)$ as in [10] (see Theorem 3.1 (i)-(ii)). To establish these inclusions we give an explicit characterization of the Fréchet and limiting normal cones of geodesic convex sets in arbitrarily Riemannian manifolds by exploiting some fundamental results from [1] and [11]. Moreover, if $\mathbf{f} = (f_1, ..., f_n)$ verifies a suitable convexity assumption then the three Nash-type equilibria coincide (see Theorem 3.1 (iii)).

Having these inclusions in mind, the main purpose of the present paper is to establish existence, location and stability of Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ in different settings. While a Nash equilibrium point is obtained precisely as the fixed point of a suitable function (see for instance Nash's original proof via Kakutani fixed-point theorem), Nash-Stampacchia equilibrium points are expected to be characterized in a similar way as fixed points of a special map defined on the product Riemannian manifold $\mathbf{M} = M_1 \times ... \times M_n$ endowed with its natural Riemannian metric $\mathbf{g}$ inherited from the metrics g_i, $i \in \{1, ..., n\}$. In order to achieve this aim, certain curvature and topological restrictions are needed on the manifolds (M_i, g_i). By assuming that the ambient Riemannian manifolds (M_i, g_i) for the geodesic convex strategy sets K_i are *Hadamard manifolds*, the key observation (see

Theorem 4.1) is that $\mathbf{p} \in \mathbf{K}$ is a Nash-Stampacchia equilibrium point for $(\mathbf{f}, \mathbf{K})$ if and only if $\mathbf{p}$ is a fixed point of the set-valued map $A_\alpha^{\mathbf{f}} : \mathbf{K} \to 2^{\mathbf{K}}$ defined by

$$A_\alpha^{\mathbf{f}}(\mathbf{p}) = P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha \partial_C^\Delta \mathbf{f}(\mathbf{p}))).$$

Here, $P_{\mathbf{K}}$ is the metric projection operator associated to the geodesic convex set $\mathbf{K} \subset \mathbf{M}$, $\alpha > 0$ is a fixed number, and $\partial_C^\Delta \mathbf{f}(\mathbf{p})$ denotes the diagonal Clarke subdifferential at point $\mathbf{p}$ of $\mathbf{f} = (f_1, ..., f_n)$; see Section 3.

Within this geometrical framework, two cases are discussed. First, when $\mathbf{K} \subset \mathbf{M}$ is *compact*, one can prove via the Begle's fixed point theorem for set-valued maps the existence of at least one Nash-Stampacchia equilibrium point for $(\mathbf{f}, \mathbf{K})$ (see Theorem 4.2). Second, we consider the case when $\mathbf{K} \subset \mathbf{M}$ is *not* necessarily *compact*. By requiring more regularity on $\mathbf{f}$ in order to avoid technicalities, we consider two dynamical systems; a discrete one

$$(DDS)_\alpha \qquad \mathbf{p}_{k+1} = A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\mathbf{p}_k)), \quad \mathbf{p}_0 \in \mathbf{M};,$$

and a continuous one

$$(CDS)_\alpha \qquad \begin{cases} \dot{\eta}(t) = \exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))) \\ \eta(0) = \mathbf{p}_0 \in \mathbf{M}. \end{cases}$$

The main result (see Theorem 4.3) proves that the set of Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ is a *singleton* and the orbits of both dynamical systems exponentially converge to this unique point whenever a Lipschitz-type condition holds on $\partial_C^\Delta \mathbf{f}$. Here, we exploit some arguments from the theory of differential equations on manifolds as well as careful comparison results of Rauch-type. It is clear by construction that the orbit of $(DDS)_\alpha$ is viable relative to the set $\mathbf{K}$, i.e., $\mathbf{p}_k \in \mathbf{K}$ for every $k \geq 1$. By using a recent result of Ledyaev and Zhu [11], one can also prove an invariance property of the set $\mathbf{K}$ with respect to the orbit of $(CDS)_\alpha$. Note that the aforementioned results concerning the 'projected' dynamical system $(CDS)_\alpha$ are new even in the Euclidean setting studied by Cavazzuti, Pappalardo and Passacantando [4], and Xia and Wang [17].

Since the manifolds (M_i, g_i) are assumed to be of Hadamard type (see Theorems 4.1-4.3), so is the product manifold $(\mathbf{M}, \mathbf{g})$. Our analytical arguments concerning Nash-Stampacchia equilibrium problems deeply exploit two geometrical features of closed, geodesic convex sets of the product *Hadamard manifold* $(\mathbf{M}, \mathbf{g})$:

- (A) *Validity of the obtuse-angle property*, see Proposition 2.1 (i). This fact is exploited in the characterization of Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ via the fixed points of the map $A_\alpha^{\mathbf{f}}$, see Theorem 4.1.
- (B) *Non-expansiveness of the projection operator*, see Proposition 2.1 (ii). This property is applied several times in the proof of Theorems 4.2-4.3.

It is natural to ask to what extent the Riemannian structures of (M_i, g_i) are determined when the properties (A) and (B) simultaneously hold on the product manifold $(\mathbf{M}, \mathbf{g})$. A constructive proof combined with the parallelogramoid of Levi-Civita and a result of C.-H. Chen [6] shows that if (M_i, g_i) are complete, simply connected Riemannian manifolds then (A) and (B) are both verified on $(\mathbf{M}, \mathbf{g})$ if and only if (M_i, g_i) are Hadamard manifolds (see Theorem 5.1). Consequently, we may assert that Hadamard manifolds are the optimal geometrical framework to elaborate a fruitful theory of Nash-Stampacchia equilibrium problems on manifolds.

The paper is divided as follows. In §2 we recall (or even prove) those notions and results which will be used throughout the paper: basic elements from Riemannian geometry, the parallelogramoid of Levi-Civita; properties of the metric projection; non-smooth calculus, dynamical systems and viability results on Riemannian manifolds. In §3 we compare the three Nash-type equilibria; namely, the set of Nash equilibrium points, the set of Nash-Clarke points, and the set of Nash-Stampacchia points for $(\mathbf{f}, \mathbf{K})$, respectively. Simultaneously, we also recall some results from [10]. In §4, we prove the main results of this paper. First, we are dealing with the existence of Nash-Stampacchia points for $(\mathbf{f}, \mathbf{K})$ in the compact case. Then, the uniqueness and exponential stability of Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ is proved whenever $\mathbf{K}$ is not necessarily compact in the Hadamard manifold $(\mathbf{M}, \mathbf{g})$. We present an example in both cases. Finally, in §5 we characterize the geometric properties (A) and (B) on $(\mathbf{M}, \mathbf{g})$ by the Hadamard structures of the complete and simply connected Riemannian manifolds (M_i, g_i), $i \in \{1, ..., n\}$.

2. Preliminaries: metric projections, non-smooth calculus and dynamical systems on Riemannian Manifolds

2.1. Elements from Riemannian geometry.

We first recall those elements from Riemannian geometry which will be used throughout the paper. We mainly follow Cartan [3] and do Carmo [8].

In this subsection, (M, g) is a connected m-dimensional Riemannian manifold. Let $TM = \cup_{p \in M}(p, T_pM)$ and $T^*M = \cup_{p \in M}(p, T_p^*M)$ be the tangent and cotangent bundles to M. For every $p \in M$, the Riemannian metric induces a natural Riesz-type isomorphism between the tangent space T_pM and its dual T_p^*M; in particular, if $\xi \in T_p^*M$ then there exists a unique $W_\xi \in T_pM$ such that

$$\langle \xi, V \rangle_{g,p} = g_p(W_\xi, V) \text{ for all } V \in T_pM. \tag{2.1}$$

Instead of $g_p(W_\xi, V)$ and $\langle \xi, V \rangle_{g,p}$ we shall write simply $g(W_\xi, V)$ and $\langle \xi, V \rangle_g$ when no confusion arises. Due to (2.1), the elements ξ and W_ξ are identified. With the above notations, the norms on T_pM and T_p^*M are defined by

$$\|\xi\|_g = \|W_\xi\|_g = \sqrt{g(W_\xi, W_\xi)}.$$

Moreover, the generalized Cauchy-Schwartz inequality is also valid, saying that for every $V \in T_pM$ and $\xi \in T_p^*M$,

$$|\langle \xi, V \rangle_g| \leq \|\xi\|_g \|V\|_g. \tag{2.2}$$

Let $\xi_k \in T_{p_k}^*M$, $k \in \mathbb{N}$, and $\xi \in T_p^*M$. The sequence $\{\xi_k\}$ converges to ξ, denoted by $\lim_k \xi_k = \xi$, when $p_k \to p$ and $\langle \xi_k, W(p_k) \rangle_g \to \langle \xi, W(p) \rangle_g$ as $k \to \infty$, for every C^∞ vector field W on M.

Let $h : M \to \mathbb{R}$ be a C^1 functional at $p \in M$; the differential of h at p, denoted by $dh(p)$, belongs to T_p^*M and is defined by

$$\langle dh(p), V \rangle_g = g(\text{grad}h(p), V) \text{ for all } V \in T_pM.$$

If $(x^1, ..., x^m)$ is the local coordinate system on a coordinate neighborhood (U_p, ψ) of $p \in M$, and the local components of dh are denoted $h_i = \frac{\partial h}{\partial x_i}$, then the local components of grad$h$ are $h^i = g^{ij} h_j$. Here, g^{ij} are the local components of g^{-1}.

Let $\gamma : [0, r] \to M$ be a C^1 path, $r > 0$. The length of γ is defined by

$$L_g(\gamma) = \int_0^r \|\dot{\gamma}(t)\|_g dt.$$

For any two points $p, q \in M$, let

$$d_g(p,q) = \inf\{L_g(\gamma) : \gamma \text{ is a } C^1 \text{ path joining } p \text{ and } q \text{ in } M\}.$$

The function $d_g : M \times M \to \mathbb{R}$ is a metric which generates the same topology on M as the underlying manifold topology. For every $p \in M$ and $r > 0$, we define the open ball of center $p \in M$ and radius $r > 0$ by

$$B_g(p,r) = \{q \in M : d_g(p,q) < r\}.$$

Let us denote by ∇ the unique natural covariant derivative on (M,g), also called the Levi-Civita connection. A vector field W along a C^1 path γ is called parallel when $\nabla_{\dot\gamma} W = 0$. A C^∞ parameterized path γ is a geodesic in (M,g) if its tangent $\dot\gamma$ is parallel along itself, i.e., $\nabla_{\dot\gamma}\dot\gamma = 0$. The geodesic segment $\gamma : [a,b] \to M$ is called minimizing if its length is not larger than the length of any other piecewise differentiable curve joining $\gamma(a)$ and $\gamma(b)$.

Standard ODE theory implies that for every $V \in T_pM$, $p \in M$, there exists an open interval $I_V \ni 0$ and a unique geodesic $\gamma_V : I_V \to M$ with $\gamma_V(0) = p$ and $\dot\gamma_V(0) = V$. Due to the 'homogeneity' property of the geodesics (see [8, p. 64]), we may define the exponential map $\exp_p : T_pM \to M$ as $\exp_p(V) = \gamma_V(1)$. Moreover,

$$d\exp_p(0) = \mathrm{id}_{T_pM}. \tag{2.3}$$

Note that there exists an open (starlike) neighborhood $\mathcal{U}$ of the zero vectors in TM and an open neighborhood $\mathcal{V}$ of the diagonal $M \times M$ such that the exponential map $V \mapsto \exp_{\pi(V)}(V)$ is smooth and the map $\pi \times \exp : \mathcal{U} \to \mathcal{V}$ is a diffeomorphism, where π is the canonical projection of TM onto M. Moreover, for every $p \in M$ there exists a number $r_p > 0$ and a neighborhood $\tilde U_p$ such that for every $q \in \tilde U_p$, the map $\exp_q$ is a C^∞ diffeomorphism on $B(0,r_p) \subset T_qM$ and $\tilde U_p \subset \exp_q(B(0,r_p))$; the set $\tilde U_p$ is called a *totally normal neighborhood* of $p \in M$. In particular, it follows that every two points $q_1, q_2 \in \tilde U_p$ can be joined by a minimizing geodesic of length less than r_p. Moreover, for every $q_1, q_2 \in \tilde U_p$ we have

$$\| \exp_{q_1}^{-1}(q_2)\|_g = d_g(q_1,q_2). \tag{2.4}$$

We conclude this subsection by recalling a less used form of the *sectional curvature* by the so-called Levi-Civita parallelogramoid. Let $p \in M$ and $V_0, W_0 \in T_pM$ two vectors with $g(V_0,W_0) = 0$. Let $\sigma : [-\delta, 2\delta] \to M$ be the geodesic segment $\sigma(t) = \exp_p(tV_0)$ and W be the unique parallel vector field along σ with the initial data $W(0) = W_0$, the number $\delta > 0$ being small enough. For any $t \in [0,\delta]$, let $\gamma_t : [0,\delta] \to M$ be the geodesic $\gamma_t(u) = \exp_{\sigma(t)}(uW(t))$. Then, the sectional curvature of the two-dimensional subspace S =span$\{W_0, V_0\} \subset T_pM$ at the point $p \in M$ is given by

$$K_p(S) = \lim_{u,t\to 0} \frac{d_g^2(p,\sigma(t)) - d_g^2(\gamma_0(u),\gamma_t(u))}{d_g(p,\gamma_0(u)) \cdot d_g(p,\sigma(t))},$$

see Cartan [3, p. 244-245]. The infinitesimal geometrical object determined by the four points p, $\sigma(t)$, $\gamma_0(u)$, $\gamma_t(u)$ (with t, u small enough) is called the parallelogramoid of Levi-Civita.

2.2. **Metric projections.** Let (M,g) be an m-dimensional Riemannian manifold ($m \geq 2$), $K \subset M$ be a non-empty set. Let

$$P_K(q) = \{p \in K : d_g(q,p) = \inf_{z\in K} d_g(q,z)\}$$

be the set of *metric projections* of the point $q \in M$ to the set K. Due to the theorem of Hopf-Rinow, if (M, g) is complete, then any closed set $K \subset M$ is *proximinal*, i.e., $P_K(q) \neq \emptyset$ for all $q \in M$. In general, P_K is a set-valued map. When $P_K(q)$ is a singleton for every $q \in M$, we say that K is a *Chebyshev set*. The map P_K is *non-expansive* if

$$d_g(P_K(q_1), P_K(q_2)) \leq d_g(q_1, q_2) \text{ for all } q_1, q_2 \in M.$$

In particular, K is a Chebyshev set whenever the map P_K is non-expansive.

The set $K \subset M$ is *geodesic convex* if every two points $q_1, q_2 \in K$ can be joined by a unique geodesic whose image belongs to K. Note that (2.4) is also valid for every $q_1, q_2 \in K$ in a geodesic convex set K since $\exp_{q_i}^{-1}$ is well-defined on K, $i \in \{1, 2\}$. The function $f : K \to \mathbb{R}$ is *convex*, if $f \circ \gamma : [0, 1] \to \mathbb{R}$ is convex in the usual sense for every geodesic $\gamma : [0, 1] \to K$ provided that $K \subset M$ is a geodesic convex set.

A non-empty closed set $K \subset M$ verifies the *obtuse-angle property* if for fixed $q \in M$ and $p \in K$ the following two statements are equivalent:

- (OA_1) $p \in P_K(q)$;
- (OA_2) If $\gamma : [0, 1] \to M$ is the unique minimal geodesic from $\gamma(0) = p \in K$ to $\gamma(1) = q$, then for every geodesic $\sigma : [0, \delta] \to K$ $(\delta \geq 0)$ emanating from the point p, we have $g(\dot{\gamma}(0), \dot{\sigma}(0)) \leq 0$.

Remark 1. (a) The first variational formula of Riemannian geometry shows that (OA_1) implies (OA_2) for every closed set $K \subset M$ in a complete Riemannian manifold (M, g).

(b) In the Euclidean case $(\mathbb{R}^m, \langle \cdot, \cdot \rangle_{\mathbb{R}^m})$, (here, $\langle \cdot, \cdot \rangle_{\mathbb{R}^m}$ is the standard inner product in $\mathbb{R}^m$), every non-empty closed convex set $K \subset \mathbb{R}^m$ verifies the obtuse-angle property, see Moskovitz-Dines [13], which reduces to the well-known geometric form:

$$p \in P_K(q) \Leftrightarrow \langle q - p, z - p \rangle_{\mathbb{R}^m} \leq 0 \text{ for all } z \in K.$$

A Riemannian manifold (M, g) is a *Hadamard manifold* if it is complete, simply connected and its sectional curvature is non-positive. It is well-known that on a Hadamard manifold (M, g) every geodesic convex set is a Chebyshev set. Moreover, we have

Proposition 2.1. *Let (M, g) be a finite-dimensional Hadamard manifold, $K \subset M$ be a closed set. The following statements hold true:*

- (i) (Walter [16]) *If $K \subset M$ is geodesic convex, it verifies the obtuse-angle property;*
- (ii) (Grognet [9]) *P_K is non-expansive if and only if $K \subset M$ is geodesic convex.*

2.3. **Non-smooth calculus on manifolds.** We first recall some basic notions and results from the subdifferential calculus on Riemannian manifolds, developed by Azagra, Ferrera and López-Mesas [1], Ledyaev and Zhu [11]. Then, we establish an analytical characterization of the limiting/Fréchet normal cone on Riemannian manifolds (see Theorem 2.1) which plays a crucial role in the study of Nash-Stampacchia equilibrium points.

Let (M, g) be an m-dimensional Riemannian manifold and let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function with $\text{dom}(f) \neq \emptyset$. The *Fréchet-subdifferential* of f at $p \in \text{dom}(f)$ is the set

$$\partial_F f(p) = \{dh(p) : h \in C^1(M) \text{ and } f - h \text{ attains a local minimum at } p\}.$$

Proposition 2.2. [1, Theorem 4.3] *Let (M, g) be an m-dimensional Riemannian manifold and let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, $p \in$dom$(f) \neq \emptyset$ and $\xi \in T_p^* M$. The following statements are equivalent:*

(i) $\xi \in \partial_F f(p)$;
(ii) *For every chart $\psi : U_p \subset M \to \mathbb{R}^m$ with $p \in U_p$, if $\zeta = \xi \circ d\psi^{-1}(\psi(p))$, we have that*

$$\liminf_{v \to 0} \frac{(f \circ \psi^{-1})(\psi(p) + v) - f(p) - \langle \zeta, v \rangle_g}{\|v\|} \geq 0;$$

(iii) *There exists a chart $\psi : U_p \subset M \to \mathbb{R}^m$ with $p \in U_p$, if $\zeta = \xi \circ d\psi^{-1}(\psi(p))$, then*

$$\liminf_{v \to 0} \frac{(f \circ \psi^{-1})(\psi(p) + v) - f(p) - \langle \zeta, v \rangle_g}{\|v\|} \geq 0.$$

In addition, if f is locally bounded from below, i.e., for every $q \in M$ there exists a neighborhood U_q of q such that f is bounded from below on U_q, the above conditions are also equivalent to

(iv) *There exists a function $h \in C^1(M)$ such that $f - h$ attains a global minimum at p and $\xi = dh(p)$.*

Now, we recall two further notions of subdifferential. Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function; the *limiting subdifferential* and *singular subdifferential of f at $p \in M$* are the sets

$$\partial_L f(p) = \{\lim_k \xi_k : \xi_k \in \partial_F f(p_k),\ (p_k, f(p_k)) \to (p, f(p))\}$$

and

$$\partial_\infty f(p) = \{\lim_k t_k \xi_k : \xi_k \in \partial_F f(p_k),\ (p_k, f(p_k)) \to (p, f(p)), t_k \to 0^+\}.$$

Proposition 2.3. [11] *Let (M, g) be a finite-dimensional Riemannian manifold and let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Then, we have*

(i) $\partial_F f(p) \subset \partial_L f(p)$, $p \in$ dom(f);
(ii) $0 \in \partial_\infty f(p)$, $p \in M$;
(iii) *If $p \in$dom(f) is a local minimum of f, then $0 \in \partial_F f(p) \subset \partial_L f(p)$.*

Let $K \subset M$ be a closed set. Following [11], the *Fréchet-normal cone* and *limiting normal cone* of K at $p \in K$ are the sets

$$N_F(p; K) = \partial_F \delta_K(p)$$

and

$$N_L(p; K) = \partial_L \delta_K(p),$$

where δ_K is the indicator function of the set K, i.e., $\delta_K(q) = 0$ if $q \in K$ and $\delta_K(q) = +\infty$ if $q \notin K$. The following result - which is one of our key tools to study Nash-Stampacchia equilibrium points on manifolds - is probably know, but since we have not found an explicit reference, we give its complete proof.

Theorem 2.1. *Let (M, g) be an m-dimensional Riemannian manifold. For any closed, geodesic convex set $K \subset M$ and $p \in K$, we have*

$$N_F(p; K) = N_L(p; K) = \{\xi \in T_p^* M : \langle \xi, \exp_p^{-1}(q) \rangle_g \leq 0 \ for\ all\ q \in K\}.$$

Proof. We first prove that

$$N_F(p;K) \subset \{\xi \in T_p^*M : \langle \xi, \exp_p^{-1}(q)\rangle_g \leq 0 \text{ for all } q \in K\}. \tag{2.5}$$

To see this, let us fix $\xi \in N_F(p;K) = \partial_F \delta_K(p)$, i.e., on account of Proposition 2.2 (i) $\Leftrightarrow$ (iv), there exists $h \in C^1(M)$ such that $\xi = dh(p)$ and $\delta_K - h$ attains a global minimum at p. In particular, the latter fact implies that

$$h(q) \leq h(p) \text{ for all } q \in K. \tag{2.6}$$

Fix $q \in K$. Since K is geodesic convex, the unique geodesic $\gamma : [0,1] \to M$ joining the points p and q, defined by $\gamma(t) = \exp_p(t \exp_p^{-1}(q))$, belongs entirely to K. Therefore, in view of (2.6), we have that $(h \circ \gamma)(t) \leq (h \circ \gamma)(0) = h(p)$ for every $t \in [0,1]$. Consequently,

$$(h \circ \gamma)'(0) = \lim_{t \to 0^+} \frac{(h \circ \gamma)(t) - (h \circ \gamma)(0)}{t} \leq 0.$$

On the other hand, we have that

$$(h \circ \gamma)'(0) = \langle dh(\gamma(0)), \dot\gamma(0)\rangle_g = \langle \xi, \exp_p^{-1}(q)\rangle_g,$$

which concludes the proof of relation (2.5).

Now, we prove that

$$N_L(p;K) \subset \{\xi \in T_p^*M : \langle \xi, \exp_p^{-1}(q)\rangle_g \leq 0 \text{ for all } q \in K\}. \tag{2.7}$$

Indeed, let $\xi \in N_L(p;K) = \partial_L \delta_K(p)$. Thus, there exists a sequence $\{p_k\} \subset M$ such that $(p_k, \delta_K(p_k)) \to (p, \delta_K(p))$ with $\xi_k \in \partial_F \delta_K(p_k)$ and $\lim_k \xi_k = \xi$. Note that $\delta_K(p) = 0$, thus we necessarily have $\{p_k\} \subset K$. By relation (2.5) and $\xi_k \in \partial_F \delta_K(p_k) = N_F(p_k;K)$ we have that $\langle \xi_k, \exp_{p_k}^{-1}(q)\rangle_g \leq 0$ for all $q \in K$ and $k \in \mathbb{N}$. Letting $k \to \infty$ in the last inequality and taking into account that $\lim_k \xi_k = \xi$, we conclude that $\langle \xi, \exp_p^{-1}(q)\rangle_g \leq 0$ for all $q \in K$, i.e., (2.7) is proved. Now, according to Proposition 2.3 (i) and relation (2.7), we have that

$$N_F(p;K) \subset N_L(p;K) \subset \{\xi \in T_p^*M : \langle \xi, \exp_p^{-1}(q)\rangle_g \leq 0 \text{ for all } q \in K\}.$$

To conclude the proof, it remains to show that

$$\{\xi \in T_p^*M : \langle \xi, \exp_p^{-1}(q)\rangle_g \leq 0 \text{ for all } q \in K\} \subset N_F(p;K).$$

Let us fix $\xi \in T_p^*M$ with

$$\langle \xi, \exp_p^{-1}(q)\rangle_g \leq 0 \text{ for all } q \in K. \tag{2.8}$$

We show that (iii) from Proposition 2.2 holds true with the choices $f = \delta_K$ and $\psi = \exp_p^{-1} : \tilde U_p \to T_pM = \mathbb{R}^m$ where $\tilde U_p \subset M$ is a totally normal ball centered at p. Due to these choices, the inequality from Proposition 2.2 (iii) reduces to

$$\liminf_{v \to 0} \frac{\delta_K(\exp_p(v)) - \langle \xi, v\rangle_g}{\|v\|} \geq 0, \tag{2.9}$$

since we have $\delta_K(p) = 0$, $\psi(p) = 0$ and $d\psi^{-1}(\psi(p)) = d\exp_p(0) = \mathrm{id}_{T_pM}$, see (2.3). To verify (2.9), two subcases are considered ($\|v\|$ is assumed to be small enough):

(a) $\exp_p(v) \notin K$. Then $\delta_K(\exp_p(v)) = +\infty$, thus the inequality (2.9) is proved.

(b) $\exp_p(v) \in K$. Then $\delta_K(\exp_p(v)) = 0$ and there exists a unique $q \in K \cap \tilde U_p$ such that $v = \exp_p^{-1}(q)$. Thus, (2.9) follows at once from (2.8).

Consequently, from Proposition 2.2 (i) $\Leftrightarrow$ (iii), we have that $\xi \in \partial_F \delta_K(p)$, i.e., $\xi \in N_F(p;K)$. □

Proposition 2.4. [11, Theorem 4.13 (Sum rule)] *Let (M,g) be an m-dimensional Riemannian manifold and let $f_1,...,f_H : M \to \mathbb{R}\cup\{+\infty\}$ be lower semicontinuous functions. Then, for every $p \in M$ we have either $\partial_L(\sum_{l=1}^H f_l)(p) \subset \sum_{l=1}^H \partial_L f_l(p)$, or there exist $\xi_l^\infty \in \partial_\infty f_l(p)$, $l = 1,...,H$, not all zero such that $\sum_{l=1}^H \xi_l^\infty = 0$.*

Let $U \subset M$ be an open subset of the Riemannian manifold (M,g). We say that a function $f : U \to \mathbb{R}$ is *locally Lipschitz at* $p \in U$ if there exist an open neighborhood $U_p \subset U$ of p and a number $C_p > 0$ such that for every $q_1, q_2 \in U_p$,

$$|f(q_1) - f(q_2)| \le C_p d_g(q_1, q_2).$$

The function $f : U \to \mathbb{R}$ is *locally Lipschitz* on (U,g) if it is locally Lipschitz at every $p \in U$.

Fix $p \in U$, $v \in T_pM$, and let $\tilde{U}_p \subset U$ be a totally normal neighborhood of p. If $q \in \tilde{U}_p$, following [1, Section 5], for small values of $|t|$, we may introduce

$$\sigma_{q,v}(t) = \exp_q(tw),\ w = d(\exp_q^{-1} \circ \exp_p)_{\exp_p^{-1}(q)} v.$$

If the function $f : U \to \mathbb{R}$ is locally Lipschitz on (U,g), then

$$f^0(p,v) = \limsup_{q\to p,\ t\to 0^+} \frac{f(\sigma_{q,v}(t)) - f(q)}{t}$$

is called the *Clarke generalized derivative of f at $p \in U$ in direction $v \in T_pM$*, and

$$\partial_C f(p) = \mathrm{co}(\partial_L f(p))$$

is the *Clarke subdifferential of f at $p \in U$*, where 'co' stands for the convex hull. When $f : U \to \mathbb{R}$ is a C^1 functional at $p \in U$ then $\partial_C f(p) = \partial_L f(p) = \partial_F f(p) = \{df(p)\}$, see [1, Proposition 4.6]. Moreover, when (M,g) is the standard Euclidean space, the Clarke subdifferential and the Clarke generalized gradient do coincide, see Clarke [7].

One can easily prove that the function $f^0(\cdot,\cdot)$ is upper-semicontinuous on $TU = \cup_{p\in U} T_pM$ and $f^0(p,\cdot)$ is positive homogeneous. In addition, if $U \subset M$ is geodesic convex and $f : U \to \mathbb{R}$ is convex, then

$$f^0(p,v) = \lim_{t\to 0^+} \frac{f(\exp_p(tv)) - f(p)}{t}, \tag{2.10}$$

see Claim 5.4 and the first relation on p. 341 of [1].

Proposition 2.5. [11, Corollary 5.3] *Let (M,g) be a complete Riemannian manifold and let $f : M \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function. Then the following statements are equivalent:*

(i) *f is locally Lipschitz at $p \in M$;*
(ii) *$\partial_C f$ is bounded in a neighborhood of $p \in M$;*
(iii) *$\partial_\infty f(p) = \{0\}$.*

2.4. **Dynamical systems on manifolds.** In this subsection we recall the existence of a local solution for a Cauchy-type problem defined on Riemannian manifolds and its viability relative to a closed set.

Let (M,g) be a finite-dimensional Riemannian manifold and $G : M \to TM$ be a vector field on M, i.e., $G(p) \in T_pM$ for every $p \in M$. We assume in the sequel that

$G : M \to TM$ is a C^{1-0} vector field (i.e., locally Lipschitz); then the dynamical system

$$(DS)_G \qquad \begin{cases} \dot{\eta}(t) = G(\eta(t)), \\ \eta(0) = p_0, \end{cases}$$

has a unique maximal semiflow $\eta : [0, T) \to M$, see Chang [5, p. 15]. In particular, η is an absolutely continuous function such that $[0, T) \ni t \mapsto \dot{\eta}(t) \in T_{\eta(t)}M$ and it verifies $(DS)_G$ for a.e. $t \in [0, T)$.

A set $K \subset M$ is *invariant with respect to the solutions of* $(DS)_G$ if for every initial point $p_0 \in K$ the unique maximal semiflow/orbit $\eta : [0, T) \to M$ of $(DS)_G$ fulfills the property that $\eta(t) \in K$ for every $t \in [0, T)$. We introduce the Hamiltonian function as

$$H_G(p, \xi) = \langle \xi, G(p) \rangle_g, \ (p, \xi) \in M \times T_p^*M.$$

Note that $H_G(p, dh(p)) < \infty$ for every $p \in M$ and $h \in C^1(M)$. Therefore, after a suitable adaptation of the results from Ledyaev and Zhu [11, Subsection 6.2] we may state

Proposition 2.6. *Let $G : M \to TM$ be a C^{1-0} vector field and $K \subset M$ be a non-empty closed set. The following statements are equivalent:*

(i) *K is invariant with respect to the solutions of $(DS)_G$;*
(ii) *$H_G(p, \xi) \leq 0$ for any $p \in K$ and $\xi \in N_F(p; K)$.*

3. Nash-type equilibria on Riemannian manifolds: comparisons

Let $K_1, ..., K_n$ ($n \geq 2$) be non-empty sets, corresponding to the strategies of n players and $f_i : K_1 \times ... \times K_n \to \mathbb{R}$ ($i \in \{1, ..., n\}$) be the payoff functions, respectively. Throughout the paper, the following notations will be used:

- $\mathbf{K} = K_1 \times ... \times K_n$;
- $\mathbf{f} = (f_1, ..., f_n)$;
- $(\mathbf{f}, \mathbf{K}) = (f_1, ..., f_n; K_1, ..., K_n)$;
- $\mathbf{p} = (p_1, ..., p_n)$;
- $(\mathbf{p}; q_i) = (p_1, ..., p_{i-1}, q_i, p_{i+1}, ..., p_n)$;
- $(\mathbf{K}; U_i) = K_1 \times ... \times K_{i-1} \times U_i \times K_{i+1} \times ... \times K_n$, for some $U_i \supset K_i$.

Definition 3.1. The set of *Nash equilibrium points for* $(\mathbf{f}, \mathbf{K})$ is

$$\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = \{\mathbf{p} \in \mathbf{K} : f_i(\mathbf{p}; q_i) \geq f_i(\mathbf{p}) \ \text{ for all } q_i \in K_i, \ i \in \{1, ..., n\}\}.$$

The main result of the paper [10] states that in a quite general framework the set of Nash equilibrium points for $(\mathbf{f}, \mathbf{K})$ is not empty. More precisely, we have

Proposition 3.1. [10] *Let (M_i, g_i) be finite-dimensional Riemannian manifolds; $K_i \subset M_i$ be non-empty, compact, geodesic convex sets; and $f_i : \mathbf{K} \to \mathbb{R}$ be continuous functions such that $K_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$ is convex on K_i for every $\mathbf{p} \in \mathbf{K}$, $i \in \{1, ..., n\}$. Then there exists at least one Nash equilibrium point for $(\mathbf{f}, \mathbf{K})$, i.e., $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) \neq \emptyset$.*

Similarly to Proposition 3.1, let us assume that for every $i \in \{1, ..., n\}$, one can find a finite-dimensional Riemannian manifold (M_i, g_i) such that the strategy set

K_i is closed and geodesic convex in (M_i, g_i). Let $\mathbf{M} = M_1 \times ... \times M_n$ be the product manifold with its standard Riemannian product metric

$$\mathbf{g}(\mathbf{V}, \mathbf{W}) = \sum_{i=1}^{n} g_i(V_i, W_i)$$

for every $\mathbf{V} = (V_1, ..., V_n), \mathbf{W} = (W_1, ..., W_n) \in T_{p_1}M_1 \times ... \times T_{p_n}M_n = T_{\mathbf{p}}\mathbf{M}$. Let $\mathbf{U} = U_1 \times ... \times U_n \subset \mathbf{M}$ be an open set such that $\mathbf{K} \subset \mathbf{U}$; we always mean that U_i inherits the Riemannian structure of (M_i, g_i). Let

$$\mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})} = \{\mathbf{f} = (f_1, ..., f_n) \in C^0(\mathbf{K}, \mathbb{R}^n) \;:\; \begin{array}{l} f_i : (\mathbf{K}; U_i) \to \mathbb{R} \text{ is continuous and} \\ f_i(\mathbf{p}; \cdot) \text{ is locally Lipschitz on } (U_i, g_i) \\ \text{for all } \mathbf{p} \in \mathbf{K},\; i \in \{1, ..., n\}\}. \end{array}$$

The next notion has been introduced in [10].

Definition 3.2. Let $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$. The set of *Nash-Clarke points for* $(\mathbf{f}, \mathbf{K})$ is

$$\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K}) = \{\mathbf{p} \in \mathbf{K} : f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i)) \geq 0 \text{ for all } q_i \in K_i,\; i \in \{1, ..., n\}\}.$$

Here, $f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i))$ denotes the Clarke generalized derivative of $f_i(\mathbf{p}; \cdot)$ at point $p_i \in K_i$ in direction $\exp_{p_i}^{-1}(q_i) \in T_{p_i}M_i$. More precisely,

$$f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i)) = \limsup_{q \to p_i, q \in U_i,\; t \to 0^+} \frac{f_i(\mathbf{p}; \sigma_{q,\exp_{p_i}^{-1}(q_i)}(t)) - f_i(\mathbf{p}; q)}{t}, \tag{3.1}$$

where $\sigma_{q,v}(t) = \exp_q(tw)$, and $w = d(\exp_q^{-1} \circ \exp_{p_i})_{\exp_{p_i}^{-1}(q)} v$ for $v \in T_{p_i}M_i$, and $|t|$ is small enough. By exploiting a minimax result of McClendon [12], the following existence result is available concerning the Nash-Clarke points for $(\mathbf{f}, \mathbf{K})$.

Proposition 3.2. [10] *Let (M_i, g_i) be complete finite-dimensional Riemannian manifolds; $K_i \subset M_i$ be non-empty, compact, geodesic convex sets; and $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$ such that for every $\mathbf{p} \in \mathbf{K}$, $i \in \{1, ..., n\}$, $K_i \ni q_i \mapsto f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i))$ is convex and f_i^0 is upper semicontinuous on its domain of definition. Then $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K}) \neq \emptyset$.*

Remark 2. Although Proposition 3.2 gives a possible approach to locate Nash equilibrium points on Riemannian manifolds, its applicability is quite reduced. As far as we know, only two special cases can be described which imply the convexity of $K_i \ni q_i \mapsto f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i))$; namely, (a) (M_i, g_i) is Euclidean, $i \in I_1$; (b) $K_i = \text{Im}\gamma_i$ where $\gamma_i : [0, 1] \to M_i$ is a minimal geodesic and $f_i^0(\mathbf{p}, \dot{\gamma}_i(t_i)) \geq -f_i^0(\mathbf{p}, -\dot{\gamma}_i(t_i))$, $i \in I_2$ for every $\mathbf{p} \in \mathbf{K}$ with $p_i = \gamma_i(t_i)$ $(0 \leq t_i \leq 1)$. Note that the sets $I_1, I_2 \subset \{1, ..., n\}$ are such that $I_1 \cup I_2 = \{1, ..., n\}$. The limited applicability of Proposition 3.2 motivates the introduction and study of the following concept which plays the central role in the present paper.

Definition 3.3. Let $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$. The set of *Nash-Stampacchia equilibrium points for* $(\mathbf{f}, \mathbf{K})$ is

$$\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \{\mathbf{p} \in \mathbf{K} \;:\; \begin{array}{l} \exists \xi_C^i \in \partial_C^i f_i(\mathbf{p}) \text{ such that } \langle \xi_C^i, \exp_{p_i}^{-1}(q_i) \rangle_{g_i} \geq 0, \\ \text{for all } q_i \in K_i,\; i \in \{1, ..., n\}\}. \end{array}$$

Here, $\partial_C^i f_i(\mathbf{p})$ denotes the Clarke subdifferential of the function $f_i(\mathbf{p}; \cdot)$ at point $p_i \in K_i$, i.e., $\partial_C f_i(\mathbf{p}; \cdot)(p_i) = \text{co}(\partial_L f_i(\mathbf{p}; \cdot)(p_i))$.

Our first aim is to compare the three Nash-type points introduced in Definitions 3.1-3.3. Before to do that, we introduce another class of functions. If $U_i \subset M_i$ is geodesic convex for every $i \in \{1, ..., n\}$, we may define

$$\mathcal{K}_{(\mathbf{K},\mathbf{U},\mathbf{M})} = \{\mathbf{f} \in C^0(\mathbf{K}, \mathbb{R}^n) \quad : \quad f_i : (\mathbf{K}; U_i) \to \mathbb{R} \text{ is continuous and } f_i(\mathbf{p}; \cdot) \text{ is convex on } (U_i, g_i) \text{ for all } \mathbf{p} \in \mathbf{K},\ i \in \{1, ..., n\}\}.$$

Remark 3. Due to Azagra, Ferrera and López-Mesas [1, Proposition 5.2], one has $\mathcal{K}_{(\mathbf{K},\mathbf{U},\mathbf{M})} \subset \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$ whenever $U_i \subset M_i$ is geodesic convex for every $i \in \{1, ..., n\}$.

The main result of this section reads as follows.

Theorem 3.1. *Let (M_i, g_i) be finite-dimensional Riemannian manifolds; $K_i \subset M_i$ be non-empty, closed, geodesic convex sets; $U_i \subset M_i$ be open sets containing K_i; and $f_i : \mathbf{K} \to \mathbb{R}$ be some functions, $i \in \{1, ..., n\}$. Then, we have*

(i) $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ *whenever* $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$;
(ii) $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ *whenever* $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$ *and $U_i \subset M_i$ are geodesic convex for every $i \in \{1, ..., n\}$*;
(iii) $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ *whenever* $\mathbf{f} \in \mathcal{K}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$.

Proof. (i) Let $\mathbf{p} \in \mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$ and fix $i \in \{1, ..., n\}$. Since $f_i(\mathbf{p}; q_i) \geq f_i(\mathbf{p})$ for all $q_i \in K_i$, then

$$f_i(\mathbf{p}; q_i) + \delta_{K_i}(q_i) - f_i(\mathbf{p}) - \delta_{K_i}(p_i) \geq 0 \ \text{ for all } q_i \in U_i,$$

which means that $p_i \in K_i$ is a global minimum of $f_i(\mathbf{p}; \cdot) + \delta_{K_i}$ on U_i. According to Proposition 2.3 (iii), one has $0 \in \partial_L(f_i(\mathbf{p}; \cdot) + \delta_{K_i})(p_i)$. Since $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$, conform Proposition 2.5 (i)⇔(iii), we have that $\partial_\infty f_i(\mathbf{p}; \cdot)(p_i) = \{0\}$. Thus, considering the functions $f_i(\mathbf{p}; \cdot)$ and δ_{K_i} in Proposition 2.4, we may exclude its second alternative, obtaining

$$\begin{aligned} 0 &\in \partial_L f_i(\mathbf{p}; \cdot)(p_i) + \partial_L \delta_{K_i}(p_i) = \partial_L f_i(\mathbf{p}; \cdot)(p_i) + N_L(p_i; K_i) \\ &\subset \partial_C f_i(\mathbf{p}; \cdot)(p_i) + N_L(p_i; K_i) = \partial_C^i f_i(\mathbf{p}) + N_L(p_i; K_i). \end{aligned}$$

Consequently, there exists $\xi_C^i \in \partial_C^i f_i(\mathbf{p})$ with $-\xi_C^i \in N_L(p_i; K_i)$. On account of Theorem 2.1, we obtain $\langle \xi_C^i, \exp_{p_i}^{-1}(q_i) \rangle_{g_i} \geq 0$ for all $q_i \in K_i$, i.e., $\mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$.

(ii) Let $\mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$. Fix also arbitrarily $i \in \{1, ..., n\}$ and $q_i \in K_i$. It follows that there exists $\xi_C^i \in \partial_C^i f_i(\mathbf{p}) = \partial_C f_i(\mathbf{p}; \cdot)(p_i)$ such that

$$\langle \xi_C^i, \exp_{p_i}^{-1}(q_i) \rangle_{g_i} \geq 0. \tag{3.2}$$

By definition, there exist some $\lambda_l \geq 0$, $l \in J$, with $\text{card} J < \infty$ and $\sum_{l \in J} \lambda_l = 1$ such that $\xi_{L,l}^i \in \partial_L f_i(\mathbf{p}; \cdot)(p_i)$ and $\xi_C^i = \sum_l \lambda_l \xi_{L,l}^i$. Consequently, for each $l \in J$, there exists a sequence $\{p_{i,l}^k\} \subset U_i$ and $\xi_{i,l}^k \in \partial_F f_i(\mathbf{p}; \cdot)(p_{i,l}^k)$ with

$$\lim_k p_{i,l}^k = p_i, \ \lim_k \xi_{i,l}^k = \xi_{L,l}^i. \tag{3.3}$$

We may assume that $p_{i,l}^k \neq q_i$ for each $k \in \mathbb{N}$ and $l \in J$. In view of Proposition 2.2 (i)⇔(ii), we have in particular that

$$\liminf_{t \to 0^+} \frac{f_i(\mathbf{p}; \exp_{p_{i,l}^k}(t \exp_{p_{i,l}^k}^{-1}(q_i))) - f_i(\mathbf{p}; p_{i,l}^k) - \langle \xi_{i,l}^k, t \exp_{p_{i,l}^k}^{-1}(q_i) \rangle_{g_i}}{t d_{g_i}(p_{i,l}^k, q_i)} \geq 0. \tag{3.4}$$

Indeed, since $U_i \subset M_i$ is convex, we may choose $\psi = \exp^{-1}_{p^k_{i,l}} : U_i \to T_{p^k_{i,l}} M_i = \mathbb{R}^{\dim M_i}$ and $v = t \exp^{-1}_{p^k_{i,l}}(q_i)$ with $t \to 0^+$; consequently, $\psi(p^k_{i,l}) = 0$, $d\psi^{-1}(\psi(p^k_{i,l})) = d\exp_{p^k_{i,l}}(0) = \mathrm{id}_{T_{p^k_{i,l}} M_i}$, and $\| \exp^{-1}_{p^k_{i,l}}(q_i)\|_{g_i} = d_{g_i}(p^k_{i,l}, q_i)$.

Now, by (3.1) and (3.4) it follows that for every $k \in \mathbb{N}$,

$$f^0_i((\mathbf{p}; p^k_{i,l}), \exp^{-1}_{p^k_{i,l}}(q_i)) \geq \langle \xi^k_{i,l}, \exp^{-1}_{p^k_{i,l}}(q_i)\rangle_{g_i}.$$

By the upper-semicontinuity of $f^0_i((\mathbf{p}; \cdot), \cdot)$ and relation (3.3), we have that

$$\begin{aligned} f^0_i(\mathbf{p}, \exp^{-1}_{p_i}(q_i)) &= f^0_i((\mathbf{p}; p_i), \exp^{-1}_{p_i}(q_i)) \\ &\geq \limsup_k f^0_i((\mathbf{p}; p^k_{i,l}), \exp^{-1}_{p^k_{i,l}}(q_i)) \\ &\geq \limsup_k \langle \xi^k_{i,l}, \exp^{-1}_{p^k_{i,l}}(q_i)\rangle_{g_i} \\ &= \langle \xi^i_{L,l}, \exp^{-1}_{p_i}(q_i)\rangle_{g_i}. \end{aligned}$$

Multiplying by λ_l the above inequality and adding them for each $l \in J$, from relation (3.2) we obtain that

$$f^0_i(\mathbf{p}, \exp^{-1}_{p_i}(q_i)) \geq \langle \sum_{l \in J} \lambda_l \xi^i_{L,l}, \exp^{-1}_{p_i}(q_i)\rangle_{g_i} = \langle \xi^i_C, \exp^{-1}_{p_i}(q_i)\rangle_{g_i} \geq 0.$$

In conclusion, we have that $\mathbf{p} \in \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$.

(iii) Due to (i)-(ii) and Remark 3, it is enough to prove that $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$. Let $\mathbf{p} \in \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$, i.e., for every $i \in \{1, ..., n\}$ and $q_i \in K_i$,

$$f^0_i(\mathbf{p}, \exp^{-1}_{p_i}(q_i)) \geq 0. \tag{3.5}$$

Fix $i \in \{1, ..., n\}$ and $q_i \in K_i$ arbitrary. Since $f_i(\mathbf{p}; \cdot)$ is convex on (U_i, g_i), on account of (2.10), we have

$$f^0_i(\mathbf{p}, \exp^{-1}_{p_i}(q_i)) = \lim_{t \to 0^+} \frac{f_i(\mathbf{p}; \exp_{p_i}(t \exp^{-1}_{p_i}(q_i))) - f_i(\mathbf{p})}{t}. \tag{3.6}$$

Note that the function

$$R(t) = \frac{f_i(\mathbf{p}; \exp_{p_i}(t \exp^{-1}_{p_i}(q_i))) - f_i(\mathbf{p})}{t}$$

is well-defined on the whole interval $(0, 1]$; indeed, $t \mapsto \exp_{p_i}(t \exp^{-1}_{p_i}(q_i))$ is the minimal geodesic joining the points $p_i \in K_i$ and $q_i \in K_i$ which belongs to $K_i \subset U_i$. Moreover, it is well-known that $t \mapsto R(t)$ is non-decreasing on $(0, 1]$. Consequently,

$$f_i(\mathbf{p}; q_i) - f_i(\mathbf{p}) = f_i(\mathbf{p}; \exp_{p_i}(\exp^{-1}_{p_i}(q_i))) - f_i(\mathbf{p}) = R(1) \geq \lim_{t \to 0^+} R(t).$$

On the other hand, (3.5) and (3.6) give that $\lim_{t \to 0^+} R(t) \geq 0$, which concludes the proof. □

Remark 4. In [10] we considered the sets $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$ and $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$. Note however that the set of Nash-Stampacchia equilibrium points $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$, which is between the former ones, seems to be the most appropriate concept to find Nash equilibrium points in very general contexts: (a) the set of Nash-Stampacchia equilibria is larger than those of Nash equilibrium points; (b) an efficient theory of Nash-Stampacchia equilibria can be developed whenever the sets K_i, $i \in \{1, ..., n\}$, are subsets of certain Hadamard manifolds. In the next section we fully develop this theory.

4. Nash-Stampacchia equilibria on Hadamard manifolds: existence, uniqueness and exponential stability

Let (M_i, g_i) be finite-dimensional Hadamard manifolds, $i \in \{1, ..., n\}$. Standard arguments show that $(\mathbf{M}, \mathbf{g})$ is also a Hadamard manifold, see Ballmann [2, Example 4, p.147] and O'Neill [15, Lemma 40, p. 209]. Moreover, on account of the characterization of (warped) product geodesics, see O'Neill [15, Proposition 38, p. 208], if $\exp_{\mathbf{p}}$ denotes the usual exponential map on $(\mathbf{M}, \mathbf{g})$ at $\mathbf{p} \in \mathbf{M}$, then for every $\mathbf{V} = (V_1, ..., V_n) \in T_{\mathbf{p}}\mathbf{M}$, we have

$$\exp_{\mathbf{p}}(\mathbf{V}) = (\exp_{p_1}(V_1), ..., \exp_{p_n}(V_n)).$$

We consider that $K_i \subset M_i$ are non-empty, closed, geodesic convex sets and $U_i \subset M_i$ are open sets containing K_i, $i \in \{1, ..., n\}$.

Let $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$. The *diagonal Clarke subdifferential of* $\mathbf{f} = (f_1, ..., f_n)$ at $\mathbf{p} \in \mathbf{K}$ is

$$\partial_C^{\Delta}\mathbf{f}(\mathbf{p}) = (\partial_C^1 f_1(\mathbf{p}), ..., \partial_C^n f_n(\mathbf{p})).$$

From the definition of the metric $\mathbf{g}$, for every $\mathbf{p} \in \mathbf{K}$ and $\mathbf{q} \in \mathbf{M}$ it turns out that

$$\langle \xi_C^{\Delta}, \exp_{\mathbf{p}}^{-1}(\mathbf{q})\rangle_{\mathbf{g}} = \sum_{i=1}^{n} \langle \xi_C^i, \exp_{p_i}^{-1}(q_i)\rangle_{g_i}, \quad \xi_C^{\Delta} = (\xi_C^1, ..., \xi_C^n) \in \partial_C^{\Delta}\mathbf{f}(\mathbf{p}). \tag{4.1}$$

4.1. **Nash-Stampacchia equilibrium points versus fixed points of $A_\alpha^{\mathbf{f}}$.** For each $\alpha > 0$ and $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$, we define the set-valued map $A_\alpha^{\mathbf{f}} : \mathbf{K} \to 2^{\mathbf{K}}$ by

$$A_\alpha^{\mathbf{f}}(\mathbf{p}) = P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha \partial_C^{\Delta}\mathbf{f}(\mathbf{p}))), \quad \mathbf{p} \in \mathbf{K}.$$

Note that for each $\mathbf{p} \in \mathbf{K}$, the set $A_\alpha^{\mathbf{f}}(\mathbf{p})$ is non-empty and compact. The following result plays a crucial role in our further investigations.

Theorem 4.1. *Let (M_i, g_i) be finite-dimensional Hadamard manifolds; $K_i \subset M_i$ be non-empty, closed, geodesic convex sets; $U_i \subset M_i$ be open sets containing K_i, $i \in \{1, ..., n\}$; and $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$. Then the following statements are equivalent:*

(i) $\mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$;
(ii) $\mathbf{p} \in A_\alpha^{\mathbf{f}}(\mathbf{p})$ *for all* $\alpha > 0$;
(iii) $\mathbf{p} \in A_\alpha^{\mathbf{f}}(\mathbf{p})$ *for some* $\alpha > 0$.

Proof. In view of relation (4.1) and the identification between $T_{\mathbf{p}}\mathbf{M}$ and $T_{\mathbf{p}}^*\mathbf{M}$, see (2.1), we have that

$$\begin{aligned} \mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \quad &\Leftrightarrow \quad \exists \xi_C^{\Delta} = (\xi_C^1, ..., \xi_C^n) \in \partial_C^{\Delta}\mathbf{f}(\mathbf{p}) \text{ such that} \\ &\qquad \langle \xi_C^{\Delta}, \exp_{\mathbf{p}}^{-1}(\mathbf{q})\rangle_{\mathbf{g}} \geq 0 \text{ for all } \mathbf{q} \in \mathbf{K} \\ &\Leftrightarrow \quad \exists \xi_C^{\Delta} = (\xi_C^1, ..., \xi_C^n) \in \partial_C^{\Delta}\mathbf{f}(\mathbf{p}) \text{ such that} \\ &\qquad \mathbf{g}(-\alpha\xi_C^{\Delta}, \exp_{\mathbf{p}}^{-1}(\mathbf{q})) \leq 0 \text{ for all } \mathbf{q} \in \mathbf{K} \text{ and} \\ &\qquad \text{for all/some } \alpha > 0. \end{aligned} \tag{4.2}$$

On the other hand, let $\gamma, \sigma : [0, 1] \to \mathbf{M}$ be the unique minimal geodesics defined by $\gamma(t) = \exp_{\mathbf{p}}(-t\alpha\xi_C^{\Delta})$ and $\sigma(t) = \exp_{\mathbf{p}}(t \exp_{\mathbf{p}}^{-1}(\mathbf{q}))$ for any fixed $\alpha > 0$ and $\mathbf{q} \in \mathbf{K}$. Since $\mathbf{K}$ is geodesic convex in $(\mathbf{M}, \mathbf{g})$, then $\text{Im}\sigma \subset \mathbf{K}$ and

$$\mathbf{g}(\dot\gamma(0), \dot\sigma(0)) = \mathbf{g}(-\alpha\xi_C^{\Delta}, \exp_{\mathbf{p}}^{-1}(\mathbf{q})). \tag{4.3}$$

Taking into account relation (4.3) and Proposition 2.1 (i), i.e., the validity of the obtuse-angle property on the Hadamard manifold $(\mathbf{M}, \mathbf{g})$, (4.2) is equivalent to

$$\mathbf{p} = \gamma(0) = P_{\mathbf{K}}(\gamma(1)) = P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha\xi_C^{\Delta})),$$

which is nothing but $\mathbf{p} \in A_\alpha^{\mathbf{f}}(\mathbf{p})$. □

Remark 5. Note that the implications (ii) ⇒ (i) and (iii) ⇒ (i) hold for arbitrarily Riemannian manifolds, see Remark 1 (a). These implications are enough to find Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ via fixed points of the map $A_\alpha^{\mathbf{f}}$. However, in the sequel we exploit further aspects of the Hadamard manifolds as non-expansiveness of the projection operator of geodesic convex sets and a Rauch-type comparison property. Moreover, in the spirit of Nash's original idea that Nash equilibria appear exactly as fixed points of a specific map, Theorem 4.1 provides a full characterization of Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ via the fixed points of the set-valued map $A_\alpha^{\mathbf{f}}$ when (M_i, g_i) are Hadamard manifolds.

In the sequel, two cases will be considered to guarantee Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$, depending on the compactness of the strategy sets K_i.

4.2. **Existence of Nash-Stampacchia equilibrium points; compact case.** Our first result guarantees the existence of a Nash-Stampacchia equilibrium point for $(\mathbf{f}, \mathbf{K})$ whenever the sets K_i are compact; the proof is based on Begle's fixed point theorem for set-valued maps. More precisely, we have

Theorem 4.2. *Let (M_i, g_i) be finite-dimensional Hadamard manifolds; $K_i \subset M_i$ be non-empty, compact, geodesic convex sets; and $U_i \subset M_i$ be open sets containing K_i, $i \in \{1, ..., n\}$. Assume that $\mathbf{f} \in \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$ and $\mathbf{K} \ni \mathbf{p} \mapsto \partial_C^{\Delta}\mathbf{f}(\mathbf{p})$ is upper semicontinuous. Then there exists at least one Nash-Stampacchia equilibrium point for $(\mathbf{f}, \mathbf{K})$, i.e., $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \neq \emptyset$.*

Proof. Fix $\alpha > 0$ arbitrary. We prove that the set-valued map $A_\alpha^{\mathbf{f}}$ has closed graph. Let $(\mathbf{p}, \mathbf{q}) \in \mathbf{K} \times \mathbf{K}$ and the sequences $\{\mathbf{p}_k\}, \{\mathbf{q}_k\} \subset \mathbf{K}$ such that $\mathbf{q}_k \in A_\alpha^{\mathbf{f}}(\mathbf{p}_k)$ and $(\mathbf{p}_k, \mathbf{q}_k) \to (\mathbf{p}, \mathbf{q})$ as $k \to \infty$. Then, for every $k \in \mathbb{N}$, there exists $\xi_{C,k}^{\Delta} \in \partial_C^{\Delta}\mathbf{f}(\mathbf{p}_k)$ such that $\mathbf{q}_k = P_{\mathbf{K}}(\exp_{\mathbf{p}_k}(-\alpha\xi_{C,k}^{\Delta}))$. On account of Proposition 2.5 (i)⇔(ii), the sequence $\{\xi_{C,k}^{\Delta}\}$ is bounded on the cotangent bundle $T^*\mathbf{M}$. Using the identification between elements of the tangent and cotangent fibers, up to a subsequence, we may assume that $\{\xi_{C,k}^{\Delta}\}$ converges to an element $\xi_C^{\Delta} \in T_{\mathbf{p}}^*\mathbf{M}$. Since the set-valued map $\partial_C^{\Delta}\mathbf{f}$ is upper semicontinuous on $\mathbf{K}$ and $\mathbf{p}_k \to \mathbf{p}$ as $k \to \infty$, we have that $\xi_C^{\Delta} \in \partial_C^{\Delta}\mathbf{f}(\mathbf{p})$. The non-expansiveness of $P_{\mathbf{K}}$ (see Proposition 2.1 (ii)) gives that

$$\begin{aligned}
&\mathbf{d_g}(\mathbf{q}, P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha\xi_C^{\Delta}))) \leq \\
&\quad\leq \mathbf{d_g}(\mathbf{q}, \mathbf{q}_k) + \mathbf{d_g}(\mathbf{q}_k, P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha\xi_C^{\Delta}))) \\
&\quad= \mathbf{d_g}(\mathbf{q}, \mathbf{q}_k) + \mathbf{d_g}(P_{\mathbf{K}}(\exp_{\mathbf{p}_k}(-\alpha\xi_{C,k}^{\Delta})), P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha\xi_C^{\Delta}))) \\
&\quad\leq \mathbf{d_g}(\mathbf{q}, \mathbf{q}_k) + \mathbf{d_g}(\exp_{\mathbf{p}_k}(-\alpha\xi_{C,k}^{\Delta}), \exp_{\mathbf{p}}(-\alpha\xi_C^{\Delta}))
\end{aligned}$$

Letting $k \to \infty$, both terms in the last expression tend to zero. Indeed, the former follows from the fact that $\mathbf{q}_k \to \mathbf{q}$ as $k \to \infty$, while the latter is a simple consequence of the local behaviour of the exponential map recalled in Subsection 2.1. Thus,

$$\mathbf{q} = P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha\xi_C^{\Delta})) \in P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha\partial_C^{\Delta}\mathbf{f}(\mathbf{p}))) = A_\alpha^{\mathbf{f}}(\mathbf{p}),$$

i.e., the graph of $A_\alpha^{\mathbf{f}}$ is closed.

By definition, for each $\mathbf{p} \in \mathbf{K}$ the set $\partial_C^{\Delta}\mathbf{f}(\mathbf{p})$ is convex, so contractible. Since both $P_{\mathbf{K}}$ and the exponential map are continuous, $A_\alpha^{\mathbf{f}}(\mathbf{p})$ is contractible as well for each $\mathbf{p} \in \mathbf{K}$, so acyclic.

Now, we are in position to apply Begle's fixed point theorem, see for instance McClendon [12, Proposition 1.1]. Consequently, there exists $\mathbf{p} \in \mathbf{K}$ such that $\mathbf{p} \in A_\alpha^{\mathbf{f}}(\mathbf{p})$. On account of Theorem 4.1, $\mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$. □

Example 4.1. Let

$$K_1 = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1^2 + x_2^2 \leq 4 \leq (x_1 - 1)^2 + x_2^2\},\ K_2 = [-1, 1],$$

and the functions $f_1, f_2 : K_1 \times K_2 \to \mathbb{R}$ defined for $(x_1, x_2) \in K_1$ and $y \in K_2$ by

$$f_1((x_1, x_2), y) = y(x_1^3 + y(1 - x_2)^3); \quad f_2((x_1, x_2), y) = -y^2 x_2 + 4|y|(x_1 + 1).$$

It is clear that $K_1 \subset \mathbb{R}^2$ is not convex in the usual sense while $K_2 \subset \mathbb{R}$ is. However, if we consider the Poincaré upper-plane model $(\mathbb{H}^2, g_{\mathbb{H}})$, the set $K_1 \subset \mathbb{H}^2$ is geodesic convex (and compact) with respect to the metric $g_{\mathbb{H}} = (\frac{\delta_{ij}}{x_2^2})$. Therefore, we embed the set K_1 into the Hadamard manifold $(\mathbb{H}^2, g_{\mathbb{H}})$, and K_2 into the standard Euclidean space $(\mathbb{R}, g_0)$. After natural extensions of $f_1(\cdot, y)$ and $f_2((x_1, x_2), \cdot)$ to the whole $U_1 = \mathbb{H}^2$ and $U_2 = \mathbb{R}$, respectively, we clear have that $f_1(\cdot, y)$ is a C^1 function on $\mathbb{H}^2$ for every $y \in K_2$, while $f_2((x_1, x_2), \cdot)$ is a locally Lipschitz function on $\mathbb{R}$ for every $(x_1, x_2) \in K_1$. Therefore, $\mathbf{f} = (f_1, f_2) \in \mathcal{L}_{(K_1 \times K_2, \mathbb{H}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R})}$ and for every $((x_1, x_2), y) \in \mathbf{K} = K_1 \times K_2$, we have

$$\partial_C^1 f_1((x_1, x_2), y) = \mathrm{grad} f_1(\cdot, y)(x_1, x_2) = \big(g_{\mathbb{H}}^{ij} \frac{\partial f_1(\cdot, y)}{\partial x_j}\big)_i = 3yx_2^2(x_1^2, -y(1 - x_2)^2);$$

$$\partial_C^2 f_2((x_1, x_2), y) = \begin{cases} -2yx_2 - 4(x_1 + 1) & \text{if} \quad y < 0, \\ 4(x_1 + 1)[-1, 1] & \text{if} \quad y = 0, \\ -2yx_2 + 4(x_1 + 1) & \text{if} \quad y > 0. \end{cases}$$

It is now clear that the map $\mathbf{K} \ni ((x_1, x_2), y) \mapsto \partial_C^{\Delta}\mathbf{f}(((x_1, x_2), y))$ is upper semicontinuous. Consequently, on account of Theorem 4.2, $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \neq \emptyset$, and its elements are precisely the solutions $((\tilde{x}_1, \tilde{x}_2), \tilde{y}) \in \mathbf{K}$ of the system

$$\begin{cases} \langle \partial_C^1 f_1((\tilde{x}_1, \tilde{x}_2), \tilde{y}), \exp^{-1}_{(\tilde{x}_1, \tilde{x}_2)}(q_1, q_2)\rangle_{g_{\mathbb{H}}} \geq 0 & \text{for all} \quad (q_1, q_2) \in K_1, \\ \xi_C^2(q - \tilde{y}) \geq 0 \quad \text{for some } \xi_C^2 \in \partial_C^2 f_2((\tilde{x}_1, \tilde{x}_2), \tilde{y}) & \text{for all} \quad q \in K_2. \end{cases} \tag{S_1}$$

In order to solve (S_1) we first observe that

$$K_1 \subset \{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{3} \leq x_2 \leq 2(x_1 + 1)\}. \tag{4.4}$$

We distinguish some cases:

(a) If $\tilde{y} = 0$ then both inequalities of (S_1) hold for every $(\tilde{x}_1, \tilde{x}_2) \in K_1$ by choosing $\xi_C^2 = 0 \in \partial_C^2 f_2((\tilde{x}_1, \tilde{x}_2), 0)$ in the second relation. Thus, $((\tilde{x}_1, \tilde{x}_2), 0) \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ for every $(\tilde{x}_1, \tilde{x}_2) \in \mathbf{K}$.

(b) Let $0 < \tilde{y} < 1$. The second inequality of (S_1) gives that $-2\tilde{y}\tilde{x}_2 + 4(\tilde{x}_1 + 1) = 0$; together with (4.4) it yields $0 = \tilde{y}\tilde{x}_2 - 2(\tilde{x}_1 + 1) < \tilde{x}_2 - 2(\tilde{x}_1 + 1) \leq 0$, a contradiction.

(c) Let $\tilde{y} = 1$. The second inequality of (S_1) is true if and only if $-2\tilde{x}_2 + 4(\tilde{x}_1 + 1) \leq 0$. Due to (4.4), we necessarily have $\tilde{x}_2 = 2(\tilde{x}_1 + 1)$; this Euclidean line intersects the set K_1 in the unique point $(\tilde{x}_1, \tilde{x}_2) = (0, 2) \in K_1$. By the geometrical meaning of the exponential map one can conclude that

$$\{t \exp^{-1}_{(0,2)}(q_1, q_2) : (q_1, q_2) \in K_1, t \geq 0\} = \{(x, -y) \in \mathbb{R}^2 : x, y \geq 0\}.$$

Taking into account this relation and $\partial_C^1 f_1((0,2),1) = (0,-12)$, the first inequality of (S_1) holds true as well. Therefore, $((0,2),1) \in \mathcal{S}_{NS}(\mathbf{f},\mathbf{K})$.

(d) Similar reason as in (b) (for $-1 < \tilde{y} < 0$) and (c) (for $\tilde{y} = -1$) gives that $((0,2),-1) \in \mathcal{S}_{NS}(\mathbf{f},\mathbf{K})$.

Thus, from (a)-(d) we have that $\mathcal{S}_{NS}(\mathbf{f},\mathbf{K}) = (K_1 \times \{0\}) \cup \{((0,2),1),((0,2),-1)\}$.

Now, on account of Theorem 3.1 (i) we may choose the Nash equilibrium points for $(\mathbf{f},\mathbf{K})$ among the elements of $\mathcal{S}_{NS}(\mathbf{f},\mathbf{K})$ obtaining that $\mathcal{S}_{NE}(\mathbf{f},\mathbf{K}) = K_1 \times \{0\}$.

4.3. **Uniqueness of the Nash-Stampacchia equilibrium point; non-compact case.** In the sequel, we are focusing to the location of Nash-Stampacchia equilibrium points for $(\mathbf{f},\mathbf{K})$ in the case when K_i are *not* necessarily compact on the Hadamard manifolds (M_i, g_i). In order to avoid technicalities in our further calculations, we introduce the class of functions

$$\mathcal{C}_{(\mathbf{K},\mathbf{U},\mathbf{M})} = \{\mathbf{f} \in C^0(\mathbf{K},\mathbb{R}^n) \ : \ f_i : (\mathbf{K}; U_i) \to \mathbb{R} \text{ is continuous and } f_i(\mathbf{p};\cdot) \text{ is of class } C^1 \text{ on } (U_i, g_i) \text{ for all } \mathbf{p} \in \mathbf{K},\ i \in \{1,...,n\}\}.$$

If is clear that $\mathcal{C}_{(\mathbf{K},\mathbf{U},\mathbf{M})} \subset \mathcal{L}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$. Moreover, when $\mathbf{f} \in \mathcal{C}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$ then $\partial_C^{\triangle}\mathbf{f}(\mathbf{p})$ and $A_\alpha^{\mathbf{f}}(\mathbf{p})$ are singletons for every $\mathbf{p} \in \mathbf{K}$ and $\alpha > 0$.

For $\mathbf{f} \in \mathcal{C}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$, $\alpha > 0$ and $0 < \rho < 1$ we assume the Lipschitz-type condition:

$(H_{\mathbf{K}}^{\alpha,\rho})$ $\mathbf{d_g}(\exp_{\mathbf{p}}(-\alpha\partial_C^{\triangle}\mathbf{f}(\mathbf{p})), \exp_{\mathbf{q}}(-\alpha\partial_C^{\triangle}\mathbf{f}(\mathbf{q}))) \leq (1-\rho)\mathbf{d_g}(\mathbf{p},\mathbf{q})$ for all $\mathbf{p},\mathbf{q} \in \mathbf{K}$.

Finding fixed points for $A_\alpha^{\mathbf{f}}$, one could expect to apply *dynamical systems*; we consider both *discrete* and *continuous* ones. First, for some $\alpha > 0$ and $\mathbf{p}_0 \in \mathbf{M}$ fixed, we consider the discrete dynamical system

$$(DDS)_\alpha \qquad \mathbf{p}_{k+1} = A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\mathbf{p}_k)).$$

Second, according to Theorem 4.1, we clearly have that

$$\mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f},\mathbf{K}) \Leftrightarrow 0 = \exp_{\mathbf{p}}^{-1}(A_\alpha^{\mathbf{f}}(\mathbf{p})) \ \text{ for all/some } \alpha > 0.$$

Consequently, for some $\alpha > 0$ and $\mathbf{p}_0 \in \mathbf{M}$ fixed, the above equivalence motivates the study of the continuous dynamical system

$$(CDS)_\alpha \qquad \begin{cases} \dot{\eta}(t) = \exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))) \\ \eta(0) = \mathbf{p}_0. \end{cases}$$

The next result describes the exponential stability of the orbits in both cases.

Theorem 4.3. *Let (M_i, g_i) be finite-dimensional Hadamard manifolds; $K_i \subset M_i$ be non-empty, closed geodesics convex sets; $U_i \subset M_i$ be open sets containing K_i; and $f_i : \mathbf{K} \to \mathbb{R}$ be functions, $i \in \{1,...,n\}$ such that $\mathbf{f} \in \mathcal{C}_{(\mathbf{K},\mathbf{U},\mathbf{M})}$. Assume that $(H_{\mathbf{K}}^{\alpha,\rho})$ holds true for some $\alpha > 0$ and $0 < \rho < 1$. Then the set of Nash-Stampacchia equilibrium points for $(\mathbf{f},\mathbf{K})$ is a singleton, i.e., $\mathcal{S}_{NS}(\mathbf{f},\mathbf{K}) = \{\tilde{\mathbf{p}}\}$. Moreover, for each $\mathbf{p}_0 \in \mathbf{M}$,*

(i) *the orbit $\{\mathbf{p}_k\}$ of $(DDS)_\alpha$ converges exponentially to $\tilde{\mathbf{p}} \in \mathbf{K}$ and*

$$\mathbf{d_g}(\mathbf{p}_k, \tilde{\mathbf{p}}) \leq \frac{(1-\rho)^k}{\rho}\mathbf{d_g}(\mathbf{p}_1,\mathbf{p}_0) \ \textit{for all } k \in \mathbb{N};$$

(ii) *the orbit η of $(CDS)_\alpha$ is globally defined on $[0,\infty)$ and it converges exponentially to $\tilde{\mathbf{p}} \in \mathbf{K}$ and*

$$\mathbf{d_g}(\eta(t), \tilde{\mathbf{p}}) \leq e^{-\rho t}\mathbf{d_g}(\mathbf{p}_0, \tilde{\mathbf{p}}) \ \textit{for all } t \geq 0.$$

Proof. Let $\mathbf{p},\mathbf{q}\in\mathbf{M}$ be arbitrarily fixed. On account of the non-expansiveness of the projection $P_{\mathbf{K}}$ (see Proposition 2.1 (ii)) and hypothesis $(H_{\mathbf{K}}^{\alpha,\rho})$, we have that

$$\begin{aligned}
&\mathbf{d_g}((A_\alpha^{\mathbf{f}}\circ P_{\mathbf{K}})(\mathbf{p}),(A_\alpha^{\mathbf{f}}\circ P_{\mathbf{K}})(\mathbf{q}))\\
&\quad= \mathbf{d_g}(P_{\mathbf{K}}(\exp_{P_{\mathbf{K}}(\mathbf{p})}(-\alpha\partial_C^{\triangle}\mathbf{f}(P_{\mathbf{K}}(\mathbf{p})))),P_{\mathbf{K}}(\exp_{P_{\mathbf{K}}(\mathbf{q})}(-\alpha\partial_C^{\triangle}\mathbf{f}(P_{\mathbf{K}}(\mathbf{q})))))\\
&\quad\le \mathbf{d_g}(\exp_{P_{\mathbf{K}}(\mathbf{p})}(-\alpha\partial_C^{\triangle}\mathbf{f}(P_{\mathbf{K}}(\mathbf{p}))),\exp_{P_{\mathbf{K}}(\mathbf{q})}(-\alpha\partial_C^{\triangle}\mathbf{f}(P_{\mathbf{K}}(\mathbf{q}))))\\
&\quad\le (1-\rho)\mathbf{d_g}(P_{\mathbf{K}}(\mathbf{p}),P_{\mathbf{K}}(\mathbf{q}))\\
&\quad\le (1-\rho)\mathbf{d_g}(\mathbf{p},\mathbf{q}),
\end{aligned}$$

which means that the map $A_\alpha^{\mathbf{f}}\circ P_{\mathbf{K}}:\mathbf{M}\to\mathbf{M}$ is a $(1-\rho)$-contraction on $\mathbf{M}$.

(i) Since $(\mathbf{M},\mathbf{d_g})$ is a complete metric space, a standard Banach fixed point argument shows that $A_\alpha^{\mathbf{f}}\circ P_{\mathbf{K}}$ has a unique fixed point $\tilde{\mathbf{p}}\in M$. Since $\mathrm{Im}A_\alpha^{\mathbf{f}}\subset\mathbf{K}$, then $\tilde{\mathbf{p}}\in\mathbf{K}$. Therefore, we have that $A_\alpha^{\mathbf{f}}(\tilde{\mathbf{p}})=\tilde{\mathbf{p}}$. Due to Theorem 4.1, $\mathcal{S}_{NS}(\mathbf{f},\mathbf{K})=\{\tilde{\mathbf{p}}\}$ and the estimate for $\mathbf{d_g}(\mathbf{p}_k,\tilde{\mathbf{p}})$ yields in a usual manner.

(ii) Since $A_\alpha^{\mathbf{f}}\circ P_{\mathbf{K}}:\mathbf{M}\to\mathbf{M}$ is a $(1-\rho)$-contraction on $\mathbf{M}$ (thus locally Lipschitz in particular), the map $\mathbf{M}\ni\mathbf{p}\mapsto G(\mathbf{p}):=\exp_{\mathbf{p}}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\mathbf{p})))$ is of class C^{1-0}. Now, due to the arguments from Subsection 2.4, we may guarantee the existence of a unique maximal orbit $\eta:[0,T_{\max})\to M$ of $(CDS)_\alpha$.

We assume that $T_{\max}<\infty$. Let us define the continuous function $h:[0,T_{\max})\to\mathbb{R}$ by

$$h(t)=\frac{1}{2}\mathbf{d}_{\mathbf{g}}^2(\eta(t),\tilde{\mathbf{p}}).$$

The function h is differentiable for a.e. $t\in[0,T_{\max})$ and in the differentiable points of η we have

$$\begin{aligned}
h'(t) &= -\mathbf{g}(\dot\eta(t),\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}))\\
&= -\mathbf{g}(\exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))),\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})) \qquad \text{(see } (CDS)_\alpha)\\
&= -\mathbf{g}(\exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))))-\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}),\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}))\\
&\quad -\mathbf{g}(\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}),\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}))\\
&\le \|\exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))))-\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})\|_{\mathbf{g}}\cdot\|\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}))\|_{\mathbf{g}}-\|\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}))\|_{\mathbf{g}}^2.
\end{aligned}$$

In the last estimate we used the Cauchy-Schwartz inequality (2.2). From (2.4) we have that

$$\|\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}))\|_{\mathbf{g}}=\mathbf{d_g}(\eta(t),\tilde{\mathbf{p}}). \tag{4.5}$$

We claim that for every $t\in[0,T_{\max})$ one has

$$\|\exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))))-\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})\|_{\mathbf{g}}\le\mathbf{d_g}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))),\tilde{\mathbf{p}}). \tag{4.6}$$

To see this, fix a differentiable point $t\in[0,T_{\max})$ of η, and let $\gamma:[0,1]\to\mathbf{M}$, $\tilde\gamma:[0,1]\to T_{\eta(t)}\mathbf{M}$ and $\overline{\gamma}:[0,1]\to T_{\eta(t)}\mathbf{M}$ be three curves such that

- γ is the unique minimal geodesic joining the two points $\gamma(0)=\tilde{\mathbf{p}}\in\mathbf{K}$ and $\gamma(1)=A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))$;
- $\tilde\gamma(s)=\exp_{\eta(t)}^{-1}(\gamma(s))$, $s\in[0,1]$;
- $\overline{\gamma}(s)=(1-s)\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})+s\exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))))$, $s\in[0,1]$.

By the definition of γ, we have that

$$L_{\mathbf{g}}(\gamma)=\mathbf{d_g}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))),\tilde{\mathbf{p}}). \tag{4.7}$$

Moreover, since $\overline{\gamma}$ is a segment of the straight line in $T_{\eta(t)}\mathbf{M}$ that joins the endpoints of $\tilde{\gamma}$, we have that

$$l(\overline{\gamma}) \leq l(\tilde{\gamma}). \tag{4.8}$$

Here, l denotes the length function on $T_{\eta(t)}\mathbf{M}$. Moreover, since the curvature of $(\mathbf{M}, \mathbf{g})$ is non-positive, we may apply a Rauch-type comparison result for the lengths of γ and $\tilde{\gamma}$, see do Carmo [8, Proposition 2.5, p.218], obtaining that

$$l(\tilde{\gamma}) \leq L_{\mathbf{g}}(\gamma). \tag{4.9}$$

Combining relations (4.7), (4.8) and (4.9) with the fact that

$$l(\overline{\gamma}) = \| \exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))) - \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})\|_{\mathbf{g}},$$

relation (4.6) holds true.

Coming back to $h'(t)$, in view of (4.5) and (4.6), it turns out that

$$h'(t) \leq \mathbf{d}_{\mathbf{g}}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))), \tilde{\mathbf{p}}) \cdot \mathbf{d}_{\mathbf{g}}(\eta(t), \tilde{\mathbf{p}}) - \mathbf{d}_{\mathbf{g}}^2(\eta(t), \tilde{\mathbf{p}}). \tag{4.10}$$

On the other hand, note that $\tilde{\mathbf{p}} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$, i.e., $A_\alpha^{\mathbf{f}}(\tilde{\mathbf{p}}) = \tilde{\mathbf{p}}$. By exploiting the non-expansiveness of the projection operator $P_{\mathbf{K}}$, see Proposition 2.1 (ii) and $(H_{\mathbf{K}}^{\alpha,\rho})$, we have that

$$\begin{aligned}
&\mathbf{d}_{\mathbf{g}}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))), \tilde{\mathbf{p}}) = \\
&\quad = \mathbf{d}_{\mathbf{g}}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))), A_\alpha^{\mathbf{f}}(\tilde{\mathbf{p}})) \\
&\quad = \mathbf{d}_{\mathbf{g}}(P_{\mathbf{K}}(\exp_{P_{\mathbf{K}}(\eta(t))}(-\alpha\partial_C^{\triangle}\mathbf{f}(P_{\mathbf{K}}(\eta(t))))), P_{\mathbf{K}}(\exp_{\tilde{\mathbf{p}}}(-\alpha\partial_C^{\triangle}\mathbf{f}(\tilde{\mathbf{p}})))) \\
&\quad \leq \mathbf{d}_{\mathbf{g}}(\exp_{P_{\mathbf{K}}(\eta(t))}(-\alpha\partial_C^{\triangle}\mathbf{f}(P_{\mathbf{K}}(\eta(t)))), \exp_{\tilde{\mathbf{p}}}(-\alpha\partial_C^{\triangle}\mathbf{f}(\tilde{\mathbf{p}}))) \\
&\quad \leq (1-\rho)\mathbf{d}_{\mathbf{g}}(P_{\mathbf{K}}(\eta(t)), \tilde{\mathbf{p}}) \\
&\quad = (1-\rho)\mathbf{d}_{\mathbf{g}}(P_{\mathbf{K}}(\eta(t)), P_{\mathbf{K}}(\tilde{\mathbf{p}})) \\
&\quad \leq (1-\rho)\mathbf{d}_{\mathbf{g}}(\eta(t), \tilde{\mathbf{p}}).
\end{aligned}$$

Combining the above relation with (4.10), for a.e. $t \in [0, T_{\max})$ it yields

$$h'(t) \leq (1-\rho)\mathbf{d}_{\mathbf{g}}^2(\eta(t), \tilde{\mathbf{p}}) - \mathbf{d}_{\mathbf{g}}^2(\eta(t), \tilde{\mathbf{p}}) = -\rho\mathbf{d}_{\mathbf{g}}^2(\eta(t), \tilde{\mathbf{p}}),$$

which is nothing but

$$h'(t) \leq -2\rho h(t) \quad \text{for a.e. } t \in [0, T_{\max}).$$

Due to the latter inequality, we have that

$$\frac{d}{dt}[h(t)e^{2\rho t}] = [h'(t) + 2\rho h(t)]e^{2\rho t} \leq 0 \quad \text{for a.e. } t \in [0, T_{\max}).$$

After integration, one gets

$$h(t)e^{2\rho t} \leq h(0) \quad \text{for all } t \in [0, T_{\max}). \tag{4.11}$$

According to (4.11), the function h is bounded on $[0, T_{\max})$; thus, there exists $\overline{\mathbf{p}} \in \mathbf{M}$ such that $\lim_{t \nearrow T_{\max}} \eta(t) = \overline{\mathbf{p}}$. The last limit means that η can be extended toward the value $T_{\max}$, which contradicts the maximality of $T_{\max}$. Thus, $T_{\max} = \infty$.

Now, relation (4.11) leads to the required estimate; indeed, we have

$$\mathbf{d}_{\mathbf{g}}(\eta(t), \tilde{\mathbf{p}}) \leq e^{-\rho t}\mathbf{d}_{\mathbf{g}}(\eta(0), \tilde{\mathbf{p}}) = e^{-\rho t}\mathbf{d}_{\mathbf{g}}(\mathbf{p}_0, \tilde{\mathbf{p}}) \quad \text{for all } t \in [0, \infty),$$

which concludes our proof. □

Remark 6. We assume the hypotheses of Theorem 4.3 are still verified and $\mathbf{p}_0 \in \mathbf{K}$.
(i) *Discrete case.* Since $\mathrm{Im}A_\alpha^{\mathbf{f}} \subset \mathbf{K}$, then the orbit of $(DDS)_\alpha$ belongs to the set $\mathbf{K}$, i.e., $\mathbf{p}_k \in \mathbf{K}$ for every $k \in \mathbb{N}$.
(ii) *Continuous case.* We shall prove that $\mathbf{K}$ is invariant with respect to the solutions of $(CDS)_\alpha$, i.e., the image of the global solution $\eta : [0, \infty) \to \mathbf{M}$ of $(CDS)_\alpha$ with $\eta(0) = \mathbf{p}_0 \in \mathbf{K}$, entirely belongs to the set $\mathbf{K}$. To show the latter fact, we are going to apply Proposition 2.6 by choosing $M := \mathbf{M}$ and $G : \mathbf{M} \to T\mathbf{M}$ defined by $G(\mathbf{p}) := \exp_{\mathbf{p}}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\mathbf{p})))$.

Fix $\mathbf{p} \in \mathbf{K}$ and $\xi \in N_F(\mathbf{p}; \mathbf{K})$. Since $\mathbf{K}$ is geodesic convex in $(\mathbf{M}, \mathbf{g})$, on account of Theorem 2.1, we have that $\langle \xi, \exp_{\mathbf{p}}^{-1}(\mathbf{q})\rangle_{\mathbf{g}} \leq 0$ for all $\mathbf{q} \in \mathbf{K}$. In particular, if we choose $\mathbf{q}_0 = A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\mathbf{p})) \in \mathbf{K}$, it turns out that

$$H_G(\mathbf{p}, \xi) = \langle \xi, G(\mathbf{p})\rangle_{\mathbf{g}} = \langle \xi, \exp_{\mathbf{p}}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\mathbf{p})))\rangle_{\mathbf{g}} = \langle \xi, \exp_{\mathbf{p}}^{-1}(\mathbf{q}_0)\rangle_{\mathbf{g}} \leq 0.$$

Our claim is proved by applying Proposition 2.6.

Example 4.2. (a) Assume that K_i is closed and convex in the Euclidean space $(M_i, g_i) = (\mathbb{R}^{m_i}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{m_i}})$, $i \in \{1, ..., n\}$, and let $\mathbf{f} \in \mathcal{C}_{(\mathbf{K}, \mathbf{U}, \mathbb{R}^m)}$ where $m = \sum_{i=1}^n m_i$. If $\partial_C^\triangle \mathbf{f}$ is $L-$globally Lipschitz and κ-strictly monotone on $\mathbf{K} \subset \mathbb{R}^m$, then the function $\mathbf{f}$ verifies $(H_{\mathbf{K}}^{\alpha,\rho})$ with $\alpha = \frac{\kappa}{L^2}$ and $\rho = \frac{\kappa^2}{2L^2}$. (Note that the above facts imply that $\kappa \leq L$, thus $0 < \rho < 1$.) Indeed, for every $\mathbf{p}, \mathbf{q} \in \mathbf{K}$ we have that

$$\begin{aligned}
&\mathbf{d}_{\mathbf{g}}^2(\exp_{\mathbf{p}}(-\alpha \partial_C^\triangle \mathbf{f}(\mathbf{p})), \exp_{\mathbf{q}}(-\alpha \partial_C^\triangle \mathbf{f}(\mathbf{q}))) \\
&= \|\mathbf{p} - \alpha \partial_C^\triangle \mathbf{f}(\mathbf{p}) - (\mathbf{q} - \alpha \partial_C^\triangle \mathbf{f}(\mathbf{q}))\|_{\mathbb{R}^m}^2 = \|\mathbf{p} - \mathbf{q} - (\alpha \partial_C^\triangle \mathbf{f}(\mathbf{p}) - \alpha \partial_C^\triangle \mathbf{f}(\mathbf{q}))\|_{\mathbb{R}^m}^2 \\
&= \|\mathbf{p} - \mathbf{q}\|_{\mathbb{R}^m}^2 - 2\alpha \langle \mathbf{p} - \mathbf{q}, \partial_C^\triangle \mathbf{f}(\mathbf{p}) - \partial_C^\triangle \mathbf{f}(\mathbf{q})\rangle_{\mathbb{R}^m} + \alpha^2 \|\partial_C^\triangle \mathbf{f}(\mathbf{p}) - \alpha \partial_C^\triangle \mathbf{f}(\mathbf{q})\|_{\mathbb{R}^m}^2 \\
&\leq (1 - 2\alpha\kappa + \alpha^2 L^2)\|\mathbf{p} - \mathbf{q}\|_{\mathbb{R}^m}^2 = (1 - \frac{\kappa^2}{L^2})\mathbf{d}_{\mathbf{g}}^2(\mathbf{p}, \mathbf{q}) \\
&\leq (1 - \rho)^2 \mathbf{d}_{\mathbf{g}}^2(\mathbf{p}, \mathbf{q}).
\end{aligned}$$

(b) Let $K = K_1 = K_2 = \mathbb{R}_+^2$ and for $x = (x_1, x_2) \in K$ and $y = (y_1, y_2) \in K$, we consider the functions $f_1, f_2 : K \times K \to \mathbb{R}$ defined by

$$f_1(x, y) = (c_{11}x_1 - h_{11}(y))^2 + (c_{12}x_2 - h_{12}(y))^2;$$

$$f_2(x, y) = (c_{21}y_1 - h_{21}(x))^2 + (c_{22}y_2 - h_{22}(x))^2,$$

where $c_{ij} > 0$ are fixed numbers and $h_{ij} : K \to \mathbb{R}$ are L_{ij}-globally Lipschitz functions, $i, j \in \{1, 2\}$. Assume that

$$2 \min_{i,j} c_{ij} > 3 \max_{i,j} L_{ij}.$$

We may prove that there exists a unique Nash equilibrium point for $(\mathbf{f}, \mathbf{K}) = (f_1, f_2; K, K)$. Indeed, we first consider Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$. Extending in a natural way $f_1(\cdot, y)$ and $f_2(x, \cdot)$ to the whole $U_1 = U_2 = \mathbb{R}^2$ for every $x, y \in K$, it yields that $\mathbf{f} \in \mathcal{C}_{(\mathbf{K}, \mathbb{R}^4, \mathbb{R}^4)}$. Moreover, for every $(x, y) \in \mathbf{K}$, we have

$$\partial_C^\triangle \mathbf{f}(x, y) = 2(c_{11}x_1 - h_{11}(y), c_{12}x_2 - h_{12}(y), c_{21}y_1 - h_{21}(x), c_{22}y_2 - h_{22}(x)).$$

A simple calculation shows that $\partial_C^\triangle \mathbf{f}$ is $L-$globally Lipschitz and κ-strictly monotone on $\mathbf{K} \subset \mathbb{R}^4$ with

$$L = 2\sqrt{3} \max_{i,j} c_{ij} > 0; \quad \kappa = 2 \min_{i,j} c_{ij} - 3 \max_{i,j} L_{ij} > 0.$$

According to (a), $\mathbf{f}$ verifies $(H_{\mathbf{K}}^{\alpha,\rho})$ with $\alpha = \frac{\kappa}{L^2}$ and $\rho = \frac{\kappa^2}{2L^2}$. On account of Theorem 4.3, the set of Nash-Stampacchia equilibrium points for $(\mathbf{f}, \mathbf{K})$ contains exactly one point, i.e., the system

$$\begin{cases} (c_{1i}\tilde{x}_i - h_{1i}(\tilde{y}))(x - \tilde{x}_i) \geq 0 & \text{for all} \quad x \in [0,\infty),\ i \in \{1,2\}, \\ (c_{2j}\tilde{y}_j - h_{2j}(\tilde{x}))(y - \tilde{y}_j) \geq 0 & \text{for all} \quad y \in [0,\infty),\ j \in \{1,2\}, \end{cases} \tag{S_2}$$

has a unique solution $(\tilde{x}, \tilde{y}) \in \mathbf{K}$. Moreover, the orbits of both dynamical systems $(DDS)_\alpha$ and $(CDS)_\alpha$ exponentially converge to $(\tilde{x}, \tilde{y})$. Since $f_1(\cdot, y)$ and $f_2(x, \cdot)$ are convex functions on K for every $x, y \in K$, then $\mathbf{f} \in \mathcal{K}_{(\mathbf{K},\mathbb{R}^4,\mathbb{R}^4)}$ as well. Due to Theorem 3.1 (iii), we have that $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \{(\tilde{x}, \tilde{y})\}$.

5. Metric projections vs Hadamard manifolds: curvature rigidity

The obtuse-angle property and the non-expansiveness of $P_{\mathbf{K}}$ for the closed, geodesic convex set $\mathbf{K} \subset \mathbf{M}$ played indispensable roles in the proof of Theorems 4.1-4.3, which are well-known features of Hadamard manifolds (see Proposition 2.1). In Section 4 the product manifold $(\mathbf{M}, \mathbf{g})$ is considered to be a Hadamard one due to the fact that (M_i, g_i) are Hadamard manifolds themselves for each $i \in \{1, ..., n\}$. We actually have the following characterization which is also of geometric interests in its own right and entitles us to assert that Hadamard manifolds are the natural framework to develop the theory of Nash-Stampacchia equilibria on manifolds.

Theorem 5.1. *Let (M_i, g_i) be complete, simply connected Riemannian manifolds, $i \in \{1, ..., n\}$, and $(\mathbf{M}, \mathbf{g})$ their product manifold. The following statements are equivalent:*

(i) *Any non-empty, closed, geodesic convex set $\mathbf{K} \subset \mathbf{M}$ verifies the obtuse-angle property and $P_{\mathbf{K}}$ is non-expansive;*

(ii) *(M_i, g_i) are Hadamard manifolds for every $i \in \{1, ..., n\}$.*

Proof. (ii)⇒(i). As mentioned before, if (M_i, g_i) are Hadamard manifolds for every $i \in \{1, ..., n\}$, then $(\mathbf{M}, \mathbf{g})$ is also a Hadamard manifold, see Ballmann [2, Example 4, p.147] and O'Neill [15, Lemma 40, p. 209]. We apply Proposition 2.1 for $(\mathbf{M}, \mathbf{g})$.

(i)⇒(ii). We first prove that $(\mathbf{M}, \mathbf{g})$ is a Hadamard manifold. Since (M_i, g_i) are complete and simply connected Riemannian manifolds for every $i \in \{1, ..., n\}$, the same is true for $(\mathbf{M}, \mathbf{g})$. We now show that the sectional curvature of $(\mathbf{M}, \mathbf{g})$ is non-positive. To see this, let $\mathbf{p} \in \mathbf{M}$ and $\mathbf{W}_0, \mathbf{V}_0 \in T_{\mathbf{p}}\mathbf{M} \setminus \{\mathbf{0}\}$. We claim that the sectional curvature of the two-dimensional subspace S =span$\{\mathbf{W}_0, \mathbf{V}_0\} \subset T_{\mathbf{p}}\mathbf{M}$ at the point $\mathbf{p}$ is non-positive, i.e., $K_{\mathbf{p}}(S) \leq 0$. We assume without loosing the generality that $\mathbf{V_0}$ and $\mathbf{W_0}$ are $\mathbf{g}$-perpendicular, i.e., $\mathbf{g}(\mathbf{W}_0, \mathbf{V}_0) = 0$.

Let us fix $r_{\mathbf{p}} > 0$ and $\delta > 0$ such that $B_{\mathbf{g}}(\mathbf{p}, r_{\mathbf{p}})$ is a totally normal ball of $\mathbf{p}$ and

$$\delta\left(\|\mathbf{W}_0\|_{\mathbf{g}} + 2\|\mathbf{V}_0\|_{\mathbf{g}}\right) < r_{\mathbf{p}}. \tag{5.1}$$

Let $\sigma : [-\delta, 2\delta] \to \mathbf{M}$ be the geodesic segment $\sigma(t) = \exp_{\mathbf{p}}(t\mathbf{V}_0)$ and $\mathbf{W}$ be the unique parallel vector field along σ with the initial data $\mathbf{W}(0) = \mathbf{W}_0$. For any $t \in [0, \delta]$, let $\gamma_t : [0, \delta] \to \mathbf{M}$ be the geodesic segment $\gamma_t(u) = \exp_{\sigma(t)}(u\mathbf{W}(t))$.

Let us fix $t, u \in [0, \delta]$ arbitrarily, $u \neq 0$. Due to (5.1), the geodesic segment $\gamma_t|_{[0,u]}$ belongs to the totally normal ball $B_{\mathbf{g}}(\mathbf{p}, r_{\mathbf{p}})$ of $\mathbf{p}$; thus, $\gamma_t|_{[0,u]}$ is the unique minimal geodesic joining the point $\gamma_t(0) = \sigma(t)$ to $\gamma_t(u)$. Moreover, since $\mathbf{W}$ is the parallel transport of $\mathbf{W}(0) = \mathbf{W}_0$ along σ, we have $\mathbf{g}(\mathbf{W}(t), \dot{\sigma}(t)) = \mathbf{g}(\mathbf{W}(0), \dot{\sigma}(0)) = \mathbf{g}(\mathbf{W}_0, \mathbf{V}_0) = 0$; therefore,

$$\mathbf{g}(\dot{\gamma}_t(0), \dot{\sigma}(t)) = \mathbf{g}(\mathbf{W}(t), \dot{\sigma}(t)) = 0.$$

Consequently, the minimal geodesic segment $\gamma_t|_{[0,u]}$ joining $\gamma_t(0) = \sigma(t)$ to $\gamma_t(u)$, and the set $\mathbf{K} = \text{Im}\sigma = \{\sigma(t) : t \in [-\delta, 2\delta]\}$ fulfill hypothesis (OA_2). Note that Imσ is a closed, geodesic convex set in $\mathbf{M}$; thus, from hypothesis (i) it follows the set Imσ verifies the obtuse-angle property and the map $P_{\text{Im}\sigma}$ is non-expansive. Therefore, (OA_2) implies (OA_1), i.e., for every $t, u \in [0, \delta]$, we have $\sigma(t) \in P_{\text{Im}\sigma}(\gamma_t(u))$. Since Im$\sigma$ is a Chebyshev set (cf. the non-expansiveness of $P_{\text{Im}\sigma}$), for every $t, u \in [0, \delta]$, we have

$$P_{\text{Im}\sigma}(\gamma_t(u)) = \{\sigma(t)\}. \tag{5.2}$$

In particular, for every $t, u \in [0, \delta]$, relation (5.2) and the non-expansiveness of $P_{\text{Im}\sigma}$ imply

$$\begin{aligned} \mathbf{d_g}(\mathbf{p}, \sigma(t)) &= \mathbf{d_g}(\sigma(0), \sigma(t)) \\ &= \mathbf{d_g}(P_{\text{Im}\sigma}(\gamma_0(u)), P_{\text{Im}\sigma}(\gamma_t(u))) \\ &\leq \mathbf{d_g}(\gamma_0(u), \gamma_t(u)). \end{aligned} \tag{5.3}$$

The above construction (i.e., the parallel transport of $\mathbf{W}(0) = \mathbf{W}_0$ along σ) and the formula of the sectional curvature in the parallelogramoid of Levi-Civita defined by the points p, $\sigma(t)$, $\gamma_0(u)$, $\gamma_t(u)$, see Subsection 2.1, give

$$K_{\mathbf{p}}(S) = \lim_{u,t \to 0} \frac{\mathbf{d}^2_{\mathbf{g}}(\mathbf{p}, \sigma(t)) - \mathbf{d}^2_{\mathbf{g}}(\gamma_0(u), \gamma_t(u))}{\mathbf{d_g}(\mathbf{p}, \gamma_0(u)) \cdot \mathbf{d_g}(\mathbf{p}, \sigma(t))}.$$

According to (5.3), the latter limit is non-positive, so $K_{\mathbf{p}}(S) \leq 0$, which concludes the first part, namely, $(\mathbf{M}, \mathbf{g})$ is a Hadamard manifold.

Now, the main result of Chen [6, Theorem 1] implies that the metric spaces (M_i, d_{g_i}) are Aleksandrov NPC spaces for every $i \in \{1, ..., n\}$. Consequently, for each $i \in \{1, ..., n\}$, the Riemannian manifolds (M_i, g_i) have non-positive sectional curvature, thus they are Hadamard manifolds. The proof is complete. □

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