STRONG CONVERGENCE OF THE METHOD OF ALTERNATING RESOLVENTS

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Abstract. In this paper, we present a generalization of the method of alternating resolvents introduced in the previous authors' paper [On the method of alternating resolvents, Nonlinear Anal. 74 (2011), 5147-5160]. It is shown that the sequence generated by this method converges strongly under weaker conditions on the control parameters. Concerning the error sequences, many more conditions are used here as compared to the above quoted paper.

1. Introduction

Let K_1 and K_2 be nonempty, closed and convex sets in a real Hilbert space H with nonempty intersection, and consider the convex feasibility problem:

$$\text{find an } x \in K_1 \cap K_2. \tag{1}$$

One of the interesting topics in convex optimization and nonlinear analysis is to solve problem (1) iteratively. In 1933, von Neumann showed that when K_1 and K_2 are vector subspaces of H, the sequence of alternating projections

$$H \ni x_0 \mapsto x_1 = P_{K_1}x_0 \mapsto x_2 = P_{K_2}x_1 \mapsto x_3 = P_{K_1}x_2 \mapsto x_4 = P_{K_2}x_3 \mapsto \cdots,$$

converges strongly to the point in $K_1 \cap K_2$ which is the nearest to the starting point x_0. For proofs of this result, see, e.g., [9, 10] and the references therein. In the case when K_1 and K_2 are arbitrary, closed and convex sets with nonempty intersection, Bregman [5] proved that the sequence (x_n) generated by the method of alternating projections converges weakly to a point in $K_1 \cap K_2$. Recently, Hundal [8] constructed an example in ℓ^2 showing that there is a hyperplane K_1 and a cone K_2 with $K_1 \cap K_2 = \{0\}$ such that given any starting $x_0 \in \ell^2 \setminus \{0\}$, the sequence of alternating projections (x_n) converges weakly to zero, but not strongly; see also Matoušková and Reich [12].

A generalization of the method of alternating projections is the so called method of alternating resolvents

$$H \ni x_0 \mapsto x_1 = J_\lambda^A x_0 \mapsto x_2 = J_\lambda^B x_1 \mapsto x_3 = J_\lambda^A x_2 \mapsto x_4 = J_\lambda^B x_3 \mapsto \cdots,$$

for $\lambda > 0$, where $J_\lambda^A := (I + \lambda A)^{-1}$ is the resolvent of a maximal monotone A (which is the projection operator P_{K_1} if A is the normal cone to K_1), while $J_\lambda^B := (I + \lambda B)^{-1}$ for another maximal monotone operator B. Bauschke et al. [1] showed that the sequence generated from this method converges weakly to a point of Fix $J_\lambda^A J_\lambda^B$ - the fixed point set of the composition $J_\lambda^A J_\lambda^B$ - provided that this set is not empty. A weak convergence result associated with the inexact method of alternating resolvents

$$\begin{aligned} x_{2n+1} &= J_{\beta_n}^A(x_{2n} + e_n) \quad \text{for } n = 0, 1, \dots, \\ x_{2n} &= J_{\mu_n}^B(x_{2n-1} + e_n') \quad \text{for } n = 1, 2, \dots, \end{aligned}$$

was proved in [4], where $x_0 \in H$ is a given starting point, (β_n) and (μ_n) are sequences of positive real numbers, while (e_n) and (e_n') are sequences of computational errors. As pointed out before, the sequence (x_n) generated by this method is not strongly convergent for general

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maximal monotone operators A and B. However, there are additional sufficient conditions that guarantee strong convergence. A general condition in this respect is the so-called *convergence condition*, introduced by A. Pazy in 1978 in the Hilbert space setting, which works for first order difference schemes associated with a single m-accretive operator in a Banach space (cf. [13]). If one of the operators A and B satisfies the convergence condition, then a strong convergence result holds for the method of alternating resolvents associated with A and B, similar to Theorem 2 in [13].

In this paper we follow a different idea which allows us to achieve strong convergence for general maximal monotone operators A and B. More precisely, we consider a modified version of the method of alernating resolvents. Several modifications (following the ideas involved in the case of a single maximal monotone operator, cf., [2, 3, 11, 15, 16]) were introduced in [4]. One such modification was defined as follows

$$x_{2n+1} = J^A_{\beta_n}(\alpha_n u + (1-\alpha_n)x_{2n} + e_n) \quad \text{for } n = 0, 1, \dots, \tag{2}$$

$$x_{2n} = J^B_{\mu_n}(\alpha_n u + (1-\alpha_n)x_{2n-1} + e'_n) \quad \text{for } n = 1, 2, \dots, \tag{3}$$

for any given $x_0, u \in H$. The purpose of this paper is to study the convergence of a sequence (x_n) generated by

$$x_{2n+1} = J^A_{\beta_n}(\alpha_n u + (1-\alpha_n)x_{2n} + e_n) \quad \text{for } n = 0, 1, \dots, \tag{4}$$

$$x_{2n} = J^B_{\mu_n}(\lambda_n u + (1-\lambda_n)x_{2n-1} + e'_n) \quad \text{for } n = 1, 2, \dots, \tag{5}$$

where $\alpha_n, \lambda_n \in (0, 1)$, under weaker conditions than those used in [4]. Note that algorithm (4), (5) contains algorithm (2), (3) as a special case, hence the results of this paper generalize and refine the main results of [4].

2. Preliminary Results

In the sequel, H will be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Recall that a map $T : H \to H$ is called nonexpansive if for every $x, y \in H$ we have $\|Tx - Ty\| \le \|x - y\|$. In the case when $\|Tx - Ty\|^2 \le \langle Tx - Ty, x - y\rangle$ holds for any $x, y \in H$, then T is said to be firmly nonexpansive. Obviously, firmly nonexpansive mappings are nonexpansive. An operator $A : D(A) \subset H \to 2^H$ is said to be monotone if

$$\langle x - x', y - y' \rangle \ge 0, \quad \forall\, (x, y), (x', y') \in G(A).$$

In other words, its graph $G(A) = \{(x, y) \in H \times H : x \in D(A), y \in Ax\}$ is a monotone subset of the product space $H \times H$. An operator A is called maximal monotone if in addition to being monotone, its graph is not properly contained in the graph of any other monotone operator. For a maximal monotone operator A, the resolvent of A, defined by $J^A_\beta := (I + \beta A)^{-1}$, is well defined on the whole space H and is single-valued for every $\beta > 0$. In addition, J^A_β is firmly nonexpansive. Firmly nonexpansive operators are characterized by the following

Lemma 1 (Goebel and Reich [7], p. 42 or Goebel and Kirk [6])**.** *A map $T : H \to H$ is firmly nonexpansive if and only if $2T - I$ (where I is the identity map) is nonexpansive.*

Below we give a list of lemmas which will be useful in proving our main result.

Lemma 2 (Suzuki [14])**.** *Let (x_n) and (y_n) be bounded sequences in a real Banach space and let (ρ_n) be a sequence in $(0, 1)$, with $0 < \liminf_{n\to\infty} \rho_n \le \limsup_{n\to\infty} \rho_n < 1$. Suppose that $x_{n+1} = \rho_n y_n + (1-\rho_n)x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

The next lemma can be proved easily.

Lemma 3 (Resolvent Identity)**.** *For any $\beta, \mu > 0$, and $x \in H$, the identity*

$$J^A_\beta x = J^A_\mu \left(\frac{\mu}{\beta} x + \left(1 - \frac{\mu}{\beta}\right) J^A_\beta x \right)$$

holds true, where $A : D(A) \subset H \to 2^H$ is a maximal monotone operator.

Using similar arguments to those used in proving Lemma 2.5 [15], we can prove the following lemma which will also be useful in proving our main result.

Lemma 4. *Let (s_n) be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)(1-\lambda_n)s_n + \alpha_n b_n + \lambda_n c_n + d_n, \quad n \geq 0, \tag{6}$$

where (α_n), (λ_n), (b_n), (c_n) and (d_n) satisfy the conditions: (i) *$\alpha_n, \lambda_n \in [0,1]$, with $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$,* (ii) *$\limsup_{n\to\infty} b_n \leq 0$,* (iii) *$\limsup_{n\to\infty} c_n \leq 0$, and* (iv) *$d_n \geq 0$ for all $n \geq 0$ with $\sum_{n=0}^{\infty} d_n < \infty$. Then $\lim_{n\to\infty} s_n = 0$.*

Proof. For any $\varepsilon > 0$, let N be an integer big enough so that

$$\sum_{n=N}^{\infty} d_n < \frac{\varepsilon}{3}, \quad b_n < \frac{\varepsilon}{3}, \quad \text{and} \quad c_n < \frac{\varepsilon}{3} \quad \forall \quad n \geq N.$$

Then by induction, we have for $n > N$

$$\begin{aligned} s_{n+1} &\leq \left[\prod_{k=N}^{n}(1-\alpha_k)(1-\lambda_k)\right] s_N + \frac{\varepsilon}{3}\left[1 - \prod_{k=N}^{n}(1-\alpha_k)\right] \\ &+ \frac{\varepsilon}{3}\left[1 - \prod_{k=N}^{n}(1-\lambda_k)\right] + \sum_{k=N}^{n} d_k. \end{aligned}$$

It then follows that $\limsup_{n\to\infty} s_n \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this completes the proof. □

Remark 5. If $\lim_{n\to\infty} \alpha_n = 0$, then $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$ if and only if $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Remark 6. When $\lambda_n = 0$ for all $n \geq 0$, we reobtain Lemma 2.5 [15]. Lemma 4 is thus a generalization of Lemma 2.5 [15].

3. Main Results

We begin by proving a strong convergence result associated with the exact iterative process

$$v_{2n+1} = J_{\beta_n}^{A}(\alpha_n u + (1-\alpha_n)v_{2n}) \quad \text{for } n = 0, 1, \ldots, \tag{7}$$

$$v_{2n} = J_{\mu_n}^{B}(\lambda_n u + (1-\lambda_n)v_{2n-1}) \quad \text{for } n = 1, 2, \ldots, \tag{8}$$

where $\alpha_n, \lambda_n \in (0,1)$ and $v_0, u \in H$ are given.

Theorem 7. *Let $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$ be maximal monotone operators with $A^{-1}(0) \cap B^{-1}(0) =: F \neq \emptyset$. For arbitrary but fixed vectors $v_0, u \in H$, let (v_n) be the sequence generated by (7), (8), where $\alpha_n, \lambda_n \in (0,1)$ and $\beta_n, \mu_n \in (0,\infty)$. Assume that* (i) *$\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \lambda_n = 0$,* (ii) *either $\sum_{n=0}^{\infty} \alpha_n = \infty$ or $\sum_{n=0}^{\infty} \lambda_n = \infty$, and* (iii) *both (β_n) and (μ_n) are bounded from below away from zero, with*

$$\lim_{n\to\infty}\left(1 - \frac{\beta_{n+1}}{\beta_n}\right) = 0 \quad \text{and} \quad \lim_{n\to\infty}\left(1 - \frac{\mu_{n+1}}{\mu_n}\right) = 0.$$

Then (v_n) converges strongly to the point of F nearest to u.

Proof. We begin by showing that (v_n) is bounded. Denote $M := \max\{\|u-p\|, \|v_0-p\|\}$, where $p \in F$. We show by induction that

$$\|v_n - p\| \leq M \quad \text{for all} \quad n \geq 0. \tag{9}$$

If (9) holds for $n := 2k$ for some integer k, then using the fact that the resolvent operator is nonexpansive, we see from (7) that

$$\begin{aligned} \|v_{2k+1} - p\| &\leq \lambda_k \|u-p\| + (1-\lambda_k)\|v_{2k} - p\| \\ &\leq \lambda_k M + (1-\lambda_k)M, \end{aligned}$$

showing that (9) also holds for $n+1$. If (9) holds for $n := 2k+1$, then starting with (8), we can show in a similar way that (9) also holds for $n+1$.

Note that using the resolvent identity (see Lemma 3), we can write (8) as

$$v_{2n} = J_\varepsilon^B \left(\frac{\varepsilon}{\mu_n}(\lambda_n u + (1-\lambda_n)v_{2n-1}) + \left(1 - \frac{\varepsilon}{\mu_n}\right) v_{2n} \right),$$

where $\varepsilon > 0$ is the greatest lower bound of (μ_n). Then by the nonexpansivity of the resolvent operator J_ε^B, we have

$$\begin{aligned} \|v_{2n+2} - v_{2n}\| &\le \left\| \left(\frac{\varepsilon\lambda_{n+1}}{\mu_{n+1}} - \frac{\varepsilon\lambda_n}{\mu_n} \right)(u - v_{2n-1}) + + \frac{\varepsilon}{\mu_{n+1}}(1-\lambda_{n+1})(v_{2n+1} - v_{2n-1}) \right. \\ &+ \left. \left(1 - \frac{\varepsilon}{\mu_{n+1}}\right)(v_{2n+2} - v_{2n}) + \left(\frac{\varepsilon}{\mu_{n+1}} - \frac{\varepsilon}{\mu_n} \right)(v_{2n-1} - v_{2n}) \right\| \\ &\le \frac{\varepsilon}{\mu_{n+1}}(1-\lambda_{n+1}) \|v_{2n+1} - v_{2n-1}\| + \left(1 - \frac{\varepsilon}{\mu_{n+1}}\right) \|v_{2n+2} - v_{2n}\| \\ &+ \left| \frac{\varepsilon\lambda_{n+1}}{\mu_{n+1}} - \frac{\varepsilon\lambda_n}{\mu_n} \right| K + \left| \frac{\varepsilon}{\mu_{n+1}} - \frac{\varepsilon}{\mu_n} \right| L, \end{aligned}$$

which implies that

$$\|v_{2n+2} - v_{2n}\| \le (1-\lambda_{n+1}) \|v_{2n+1} - v_{2n-1}\| + \left| \lambda_{n+1} - \frac{\lambda_n \mu_{n+1}}{\mu_n} \right| K + \left| 1 - \frac{\mu_{n+1}}{\mu_n} \right| L. \tag{10}$$

Since $J_{\beta_n}^A$ is firmly nonexpansive for each $n \in \mathbb{N}$, we know from Lemma 1 that there is a nonexpansive map T_n^A such that $T_n^A = 2J_{\beta_n}^A - I$. Therefore, we have from (4) that

$$v_{2n+1} = \frac{z_n + T_n^A z_n}{2},$$

where $z_n := \alpha_n u + (1-\alpha_n)v_{2n}$. Using the resolvent identity, we see that

$$\begin{aligned} \left\| J_{\beta_{n+1}}^A z_n - J_{\beta_n}^A z_n \right\| &= \left\| J_{\beta_{n+1}}^A z_n - J_{\beta_{n+1}}^A \left(\frac{\beta_{n+1}}{\beta_n} z_n + \left(1 - \frac{\beta_{n+1}}{\beta_n}\right) J_{\beta_n}^A z_n \right) \right\| \\ &\le \left| 1 - \frac{\beta_{n+1}}{\beta_n} \right| \|z_n - J_{\beta_n}^A z_n\| \end{aligned}$$

Moreover, using the definition of z_n and the nonexpansivity of T_{n+1}^A, we derive

$$\begin{aligned} \|T_{n+1}^A z_{n+1} - T_n^A z_n\| &\le \|T_{n+1}^A z_{n+1} - T_{n+1}^A z_n\| + \|T_{n+1}^A z_n - T_n^A z_n\| \\ &\le \|z_{n+1} - z_n\| + 2 \left\| J_{\beta_{n+1}}^A z_n - J_{\beta_n}^A z_n \right\| \\ &\le (\alpha_{n+1} + \alpha_n)C_1 + \|x_{2n+2} - x_{2n}\| + 2 \left| 1 - \frac{\beta_{n+1}}{\beta_n} \right| \|z_n - J_{\beta_n}^A z_n\|, \end{aligned}$$

for some positive constant C_1. Similarly, we write (5) as

$$v_{2n} = \frac{w_n + F_n^B w_n}{2},$$

where $w_n := \lambda_n u + (1-\lambda_n)x_{2n-1}$ and $F_n^B := 2J_{\mu_n}^B - I$. In addition,

$$\begin{aligned} \|F_{n+1}^B w_{n+1} - F_n^B w_n\| &\le \|w_{n+1} - w_n\| + 2 \left\| J_{\mu_{n+1}}^B w_n - J_{\mu_n}^B w_n \right\| \\ &\le (\lambda_{n+1} + \lambda_n)C_1 + \|x_{2n+1} - x_{2n-1}\| + 2 \left| 1 - \frac{\mu_{n+1}}{\mu_n} \right| \|w_n - J_{\mu_n}^B w_n\|. \end{aligned}$$

Now, an elementary computation shows that

$$v_{2n+1} = \frac{1}{4} v_{2n-1} + \frac{3}{4} y_n,$$

where

$$y_n = \frac{1}{3}(\lambda_n(1-\alpha_n)(u-v_{2n-1}) + \alpha_n(2u - v_{2n-1} - F_n^B w_n) + (F_n^B w_n + 2T_n^A z_n))$$

Note that (y_n) is bounded, since (x_n) is so. Therefore, for some positive constants C_2, C_2^*, C_3 and C_4, we have

$$\begin{aligned} 3\|y_{n+1}-y_n\| &\leq (\lambda_{n+1}+\lambda_n)C_1 + (\alpha_{n+1}+\alpha_n)C_2 + \left\|F_{n+1}^B w_{n+1} - F_n^B w_n\right\| \\ &+ 2\left\|T_{n+1}^A z_{n+1} - T_n^A z_n\right\| \\ &\leq 2C_1(\lambda_{n+1}+\lambda_n) + C_2^*(\alpha_{n+1}+\alpha_n) + 3\|v_{2n+1}-v_{2n-1}\| \\ &+ \left|1-\frac{\beta_{n+1}}{\beta_n}\right| C_3 + \left|1-\frac{\mu_{n+1}}{\mu_n}\right| C_4 + 2K\left|\lambda_{n+1} - \frac{\lambda_n\mu_{n+1}}{\mu_n}\right|, \end{aligned}$$

which implies that

$$\limsup_{n\to\infty}\{\|y_{n+1}-y_n\| - \|v_{2n+1}-v_{2n-1}\|\} \leq 0.$$

It then follows from Lemma 2 that

$$\lim_{n\to\infty}\|y_n - v_{2n-1}\| = 0.$$

Consequently,

$$\lim_{n\to\infty}\|v_{2n+1} - v_{2n-1}\| = 0.$$

Passing to the limit in (10), we also see that

$$\lim_{n\to\infty}\|v_{2n+2} - v_{2n}\| = 0.$$

Now multiplying the inclusion relation

$$v_{2n+1} - v_{2n+2} + \beta_n A v_{2n+1} \ni \alpha_n(u - v_{2n}) + v_{2n} - v_{2n+2}$$

scalarly by $v_{2n+1} - p$ (where $p \in F$) and using the monotonicity of A, we get

$$\langle v_{2n+1} - v_{2n+2}, v_{2n+1} - p\rangle \leq \alpha_n K' + L'\|v_{2n+2} - v_{2n}\|, \tag{11}$$

for some positive constants K' and L'. Similarly, multiplying the inclusion

$$v_{2n+2} - v_{2n+1} + \mu_{n+1} B v_{2n+2} \ni \lambda_{n+1}(u - v_{2n+1})$$

scalarly by $v_{2n+2} - p$ and using the monotonicity of B, we arrive at

$$\langle v_{2n+2} - v_{2n+1}, v_{2n+2} - p\rangle \leq \lambda_{n+1} L^*, \tag{12}$$

for some positive constant L^*. Combining (11) and (12), we arrive at

$$\lim_{n\to\infty}\|v_{n+1} - v_n\| = 0. \tag{13}$$

Therefore passing to the limit in

$$A v_{2n+1} \ni \frac{\alpha_n(u - v_{2n}) + v_{2n} - v_{2n+1}}{\beta_n},$$

and noting that (β_n) is bounded below away from zero, we see that $\omega_{\mathtt{w}}((v_{2n+1})) \subset A^{-1}(0)$. Similarly, we derive $\omega_{\mathtt{w}}((v_{2n})) \subset B^{-1}(0)$. It then follows from these two inclusions and equation (13) that $\omega_{\mathtt{w}}((v_n)) \subset F = A^{-1}(0) \cap B^{-1}(0)$. We can now extract a weakly convergent subsequence (v_{n_k}) of (v_n), such that

$$\limsup_{n\to\infty}\langle u - P_F u, v_n - P_F u\rangle = \lim_{k\to\infty}\langle u - P_F u, v_{n_k} - P_F u\rangle = \langle u - P_F u, z - P_F u\rangle \leq 0,$$

where $z \in F$ is a weak limit of (v_{n_k}), and $q = P_F u$ denotes the projection of u on F. We remark that the set F is closed and convex - being the intersection of two closed and convex sets - therefore, the projection operator P_F is well defined. We shall show that $v_n \to q$. For this purpose, we multiply

$$v_{2n+1} - q + \beta_n A v_{2n+1} \ni \alpha_n(u - q) + (1 - \alpha_n)(v_{2n} - q)$$

scalarly by $v_{2n+1}-q$ and use the monotonicity of A to derive

$$\begin{aligned} 2\left\|v_{2n+1}-q\right\|^2 &\leq 2(1-\alpha_n)\langle v_{2n}-q, v_{2n+1}-q\rangle+2\alpha_n\langle u-q, v_{2n+1}-q\rangle \\ &\leq (1-\alpha_n)(\|v_{2n}-q\|^2+\|v_{2n+1}-q\|^2)+2\alpha_n\langle u-q, v_{2n+1}-q\rangle, \end{aligned}$$

which implies that

$$\|v_{2n+1}-q\|^2 \leq (1-\alpha_n)\|v_{2n}-q\|^2+2\alpha_n\left\langle u-q, v_{2n+1}-q\right\rangle. \tag{14}$$

Similarly, starting with (8), we arrive at

$$\|v_{2n}-q\|^2 \leq (1-\lambda_n)\|v_{2n-1}-q\|^2+2\lambda_n\left\langle u-q, v_{2n}-q\right\rangle,$$

which together with (14) gives

$$\begin{aligned} \|v_{2n+1}-q\|^2 &\leq (1-\alpha_n)(1-\lambda_n)\|v_{2n-1}-q\|^2+2\alpha_n\langle u-q, v_{2n+1}-q\rangle \\ &+ 2(1-\alpha_n)\lambda_n\langle u-q, v_{2n}-q\rangle. \end{aligned}$$

Therefore by Lemma 4, we derive $v_{2n+1}\to q=P_Fu$. Since $v_{n+1}-v_n\to 0$, we deduce strong convergence of (v_n) to P_Fu. □

The inexact iterative scheme of (7), (8), namely, the iterative process defined by (4), (5) is constructed in such a way that strong convergence (to the point P_Fu) of the sequence generated by it is obtained under the weaker condition that the sequences of error terms converges to zero strongly. When this condition on the error sequences is satisfied, and in addition, the series on $\|e_n\|$ and $\|e_n'\|$ are divergent, then one chooses appropriate parameters α_n and λ_n such that any of the conditions (f)-(l) of Theorem 8 below is satisfied. Note that if any of the series on $\|e_n\|$ and $\|e_n'\|$ is convergent, then we are able to choose one of the parameters independent of the error sequences.

Theorem 8. *Let $A: D(A)\subset H\to 2^H$ and $B: D(B)\subset H\to 2^H$ be maximal monotone operators with $A^{-1}(0)\cap B^{-1}(0)=: F\neq\emptyset$. For arbitrary but fixed vectors $x_0, u\in H$, let (x_n) be the sequence generated by (4), (5), where $\alpha_n, \lambda_n\in(0,1)$ and $\beta_n, \mu_n\in(0,\infty)$. Assume that* (i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\lim_{n\to\infty}\lambda_n=0$, (ii) *either* $\sum_{n=0}^{\infty}\alpha_n=\infty$ *or* $\sum_{n=0}^{\infty}\lambda_n=\infty$, *and* (iii) *both* (β_n) *and* (μ_n) *are bounded from below away from zero, with*

$$\lim_{n\to\infty}\left(1-\frac{\beta_{n+1}}{\beta_n}\right)=0 \quad \textit{and} \quad \lim_{n\to\infty}\left(1-\frac{\mu_{n+1}}{\mu_n}\right)=0.$$

If any of the following conditions is satisfied,

(a) $\sum_{n=0}^{\infty}\|e_n\|<\infty$ *and* $\sum_{n=1}^{\infty}\|e_n'\|<\infty$*;*
(b) $\sum_{n=0}^{\infty}\|e_n\|<\infty$ *and* $\|e_n'\|/\alpha_n\to 0$*;*
(c) $\sum_{n=0}^{\infty}\|e_n\|<\infty$ *and* $\|e_n'\|/\lambda_n\to 0$*;*
(d) $\|e_n\|/\alpha_n\to 0$ *and* $\sum_{n=1}^{\infty}\|e_n'\|<\infty$*;*
(e) $\|e_n\|/\lambda_n\to 0$ *and* $\sum_{n=1}^{\infty}\|e_n'\|<\infty$*;*
(f) $\|e_n\|/\alpha_n\to 0$ *and* $\|e_n'\|/\alpha_n\to 0$*;*
(g) $\|e_n\|/\alpha_n\to 0$ *and* $\|e_n'\|/\lambda_n\to 0$*;*
(h) $\|e_n\|/\lambda_n\to 0$ *and* $\|e_n'\|/\alpha_n\to 0$*;*
(i) $\|e_n\|/\lambda_n\to 0$ *and* $\|e_n'\|/\lambda_n\to 0$*;*
(j) $\|e_n\|/\alpha_n\to 0$ *and* $\|e_n'\|/\alpha_{n-1}\to 0$*;*
(k) $\|e_{n-1}\|/\lambda_n\to 0$ *and* $\|e_n'\|/\alpha_{n-1}\to 0$*;*
(l) $\|e_{n-1}\|/\lambda_n\to 0$ *and* $\|e_n'\|/\lambda_n\to 0$*;*
(m) $\sum_{n=0}^{\infty}\|e_n\|<\infty$ *and* $\|e_n'\|/\alpha_{n-1}\to 0$*;*
(n) $\|e_{n-1}\|/\lambda_n\to 0$ *and* $\sum_{n=1}^{\infty}\|e_n'\|<\infty$*,*

then (x_n) converges strongly to the point of F nearest to u.

Proof. In view of Theorem 7, it is enough to show that $\|x_n-v_n\|\to 0$. Since the resolvent of B is nonexpansive, we derive from (5) and (8) that

$$\|x_{2n}-v_{2n}\| \leq (1-\lambda_n)\left\|x_{2n-1}-v_{2n-1}\right\|+\left\|e_n'\right\|. \tag{15}$$

Similarly, from (4) and (7), we have

$$\|x_{2n+1} - v_{2n+1}\| \leq (1-\alpha_n)\|x_{2n} - v_{2n}\| + \|e_n\|. \tag{16}$$

These two inequalities imply that

$$\|x_{2n+1} - v_{2n+1}\| \leq (1-\alpha_n)(1-\lambda_n)\|x_{2n-1} - v_{2n-1}\| + \|e_n\| + \|e'_n\|.$$

Therefore if the error sequence satisfies any of the conditions (a)-(i), then it readily follows from Lemma 4 that $\|x_{2n+1} - v_{2n+1}\| \to 0$. Passing to the limit in (15), we derive $\|x_{2n} - v_{2n}\| \to 0$ as well. If the error sequence satisfies any of the conditions (j)-(n), then from (15) and (16), we have

$$\|x_{2n} - v_{2n}\| \leq (1-\alpha_{n-1})(1-\lambda_n)\|x_{2n-2} - v_{2n-2}\| + \|e_{n-1}\| + \|e'_n\|.$$

It then follows from Lemma 4 that $\|x_{2n} - v_{2n}\| \to 0$. Passing to the limit in (16), we derive $\|x_{2n+1} - v_{2n+1}\| \to 0$ as well. This completes the proof of the theorem. □

Remark 9. We point out that when $\lambda_n = \alpha_n$ for all $n \geq 1$, algorithm (4), (5) reduces to algorithm (24), (25) introduced by the authors in [4]. When $\lambda_n = 0$ for all $n \geq 1$, algorithm (4), (5) reduces to algorithm (14), (15) which was also introduced by the authors in [4]. By making use of the firmly nonexpansive property of the resolvent operator, we could drop the conditions

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$$

used in both [4, Theorem 2] and [4, Theorem 3]. In addition, we were able to replace the conditions

$$\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$$

used in both [4, Theorem 2] and [4, Theorem 3] with the weaker conditions

$$\lim_{n\to\infty} \left(1 - \frac{\beta_{n+1}}{\beta_n}\right) = 0 \quad \text{and} \quad \lim_{n\to\infty} \left(1 - \frac{\mu_{n+1}}{\mu_n}\right) = 0.$$

Therefore, Theorem 8 contains [4, Theorem 2] and [4, Theorem 3] as special cases.

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