# Spectrum consisting in a continuous family plus an isolated point for a Dirichlet type problem *

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Abstract. Eigenvalue problems involving the Laplace operator on bounded domains lead to a discrete or a continuous set of eigenvalues. In this paper we highlight the case of an eigenvalue problem involving the Laplace operator which possesses, on the one hand, a continuous family of eigenvalues and, on the other hand, at least one more eigenvalue which is isolated in the set of eigenvalues of that problem.

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## 1 Introduction and the main result

In this paper we consider that $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary. By an eigenvalue problem involving the Laplace operator we understand a problem of the type

$$\begin{cases} -\Delta u = \lambda f(x,u), & \text{in} \quad \Omega \\ u = 0, & \text{on} \quad \partial\Omega\,, \end{cases} \tag{1}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given function and $\lambda \in \mathbb{R}$ is a real number. We will say that $\lambda$ is an *eigenvalue* of problem (1) if there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla u \nabla v \; dx - \lambda \int_\Omega f(x,u)v \; dx = 0\,,$$

for any $v \in H_0^1(\Omega)$. Moreover, if $\lambda$ is an eigenvalue of problem (1) then $u \in H_0^1(\Omega) \setminus \{0\}$ given in the above definition is called the *eigenfunction* corresponding to the eigenvalue $\lambda$. In this paper we will be interested in finding positive eigenvalues for problems of type (1), that means $\lambda \in (0, \infty)$.

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The study of eigenvalue problems involving the Laplace operator guides our mind back to a basic result in the elementary theory of partial differential equations which asserts that the problem (which represents a particular case of problem (1), obtained when $f(x,u) = u$)

$$\begin{cases} -\Delta u = \lambda u, & \text{in} \quad \Omega \\ u = 0, & \text{on} \quad \partial\Omega\,, \end{cases} \tag{2}$$

possesses an unbounded sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq ... \leq \lambda_n \leq ....$ This celebrated result goes back to the Riesz-Fredholm theory of self-adjoint and compact operators on Hilbert spaces.

In what concerns $\lambda_1$, the lowest eigenvalue of problem (2), we remember that it can be characterized from a variational point of view as the minimum of the Rayleigh quotient, i.e.

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\;dx}{\displaystyle\int_\Omega u^2\;dx}\,. \tag{3}$$

Moreover, it is known that $\lambda_1$ is simple, i.e. all the associated eigenfunctions are merely multiples of each other (see, e.g. Gilbarg and Trudinger [5]). Furthermore, the corresponding eigenfunctions of $\lambda_1$ never change signs in $\Omega$.

Going further, another type of eigenvalue problems involving the Laplace operator (obtained in the case when we take in (1), $f(x,u) = |u|^{p(x)-2}u$) is given by the model equation

$$\begin{cases} -\Delta u = \lambda |u|^{p(x)-2}u, & \text{in} \quad \Omega \\ u = 0, & \text{on} \quad \partial\Omega\,, \end{cases} \tag{4}$$

where $p(x) : \overline{\Omega} \to (1, 2N/(N-2))$ is a given continuous function. (Obviously, the case when $p(x)$ is a constant function on $\overline{\Omega}$ is allowed but we avoid the case when $p \equiv 2$ since this case is the object of problem (2), discussed above.) For this problem the growth rate of the function $p(x)$ will be essential in the description of the set of eigenvalues. First, assuming that $\min_{\overline{\Omega}} p > 2$ it can be proved (by using a mountain-pass argument) that any $\lambda > 0$ is an eigenvalue of problem (4). Next, in the case when $\min_{\overline{\Omega}} p < 2$ it can be proved (by using Ekeland's variational principle) that the problem has a continuous family of eigenvalues which lies in a neighborhood of the origin (see, e.g., Mihăilescu and Rădulescu [8] or Fan [2] for some extensions). Finally, we point out that the above result can be completed in the particular case when $\max_{\overline{\Omega}} p < 2$. More exactly, in this situation it can be proved that the energy functional associated to problem (4) has a nontrivial (global) minimum point for any positive $\lambda$ large enough. In other words, if $\max_{\overline{\Omega}} p < 2$ then there exist two positive constants $\mu_1$ and $\mu_2$ such that any $\lambda \in (0, \mu_1) \cup (\mu_2, \infty)$ is an eigenvalue of problem (4).

We notice that in all the situations presented above on (4) the set of eigenvalues is not completely described, excepting the case when $\min_{\overline{\Omega}} p > 2$. However, in all the cases the set of eigenvalues possesses a continuous subfamily.

In what concerns the eigenvalue problems involving quasilinear operators we remember, in the case of homogeneous elliptic operators, the contributions of Anane [1], de Thélin [12, 13], Lindqvist [6] and Filippucci-Pucci-Rădulescu [4], while in the case of nonhomogeneous elliptic operators we point out the recent advances of Fan-Zhang-Zhao [3], Mihăilescu-Rădulescu [8, 9, 10], Mihăilescu-Pucci-Rădulescu [7] and Fan [2].

Motivated by the above results on problems (2) and (4) which show that the eigenvalue problems involving the Laplace operator lead to a discrete spectrum (see the case of problem (2)) or a continuous spectrum (see the case of problem (4) in the different forms pointed out above) we consider important to supplement the above situations by studying a new eigenvalue problem involving the Laplace operator which possesses, on the one hand, a continuous family of eigenvalues and, on the other hand, at least one more eigenvalue which is isolated in the set of eigenvalues of that problem.

More exactly, we will study problem (1) in the case when

$$f(x,t)=\begin{cases} h(x,t), & \text{if} \quad t\geq 0\\ t, & \text{if} \quad t\leq 0\,, \end{cases} \tag{5}$$

where $h:\Omega\times[0,\infty)\to\mathbb{R}$ is a Carathéodory function satisfying the following hypotheses

(H1) there exists a positive constant $C\in(0,1)$ such that $|h(x,t)|\leq Ct$ for any $t\geq 0$ and a.e. $x\in\Omega$;

(H2) there exists $t_0>0$ such that $H(x,t_0):=\int_0^{t_0} h(x,s)\;ds>0$, for a.e. $x\in\Omega$;

(H3) $\lim_{t\to\infty}\frac{h(x,t)}{t}=0$, uniformly in $x$.

**Examples.** We point out certain examples of functions $h$ which satisfies the hypotheses (H1)-(H3):

1. $h(x,t)=\sin(t/2)$, for any $t\geq 0$ and any $x\in\Omega$;
2. $h(x,t)=k\log(1+t)$, for any $t\geq 0$ and any $x\in\Omega$, where $k\in(0,1)$ is a constant;
3. $h(x,t)=g(x)(t^{q(x)-1}-t^{p(x)-1})$, for any $t\geq 0$ and any $x\in\Omega$, where $p(x)$, $q(x):\overline{\Omega}\to(1,2)$ are two continuous functions satisfying $\max_{\overline{\Omega}} p<\min_{\overline{\Omega}} q$, and $g\in L^\infty(\Omega)$ satisfies $0<\inf_\Omega g\leq\sup_\Omega g<1$.

The main result of our paper is the following:

**Theorem 1.** *Assume that function $f$ is given by relation (5) and conditions (H1), (H2) and (H3) are fulfilled. Assume that $\lambda_1$ is defined by relation (3). Then $\lambda_1$ is an eigenvalue of problem (1) which is isolated in the set of eigenvalues of this problem. Moreover, the set of eigenvectors corresponding to $\lambda_1$ is a cone. Furthermore, any $\lambda\in(0,\lambda_1)$ is not an eigenvalue of problem (1) but there exists $\mu_1>\lambda_1$ such that any $\lambda\in(\mu_1,\infty)$ is an eigenvalue of problem (1).*

## 2 Proof of the main result

For any $u\in H_0^1(\Omega)$ we denote

$$u_\pm(x)=\max\{\pm u(x),0\},\quad\forall\,x\in\Omega\,.$$

Then $u_+$, $u_- \in H_0^1(\Omega)$ and

$$\nabla u_+ = \begin{cases} 0, & \text{if} \quad [u \le 0] \\ \nabla u, & \text{if} \quad [u > 0]\,, \end{cases} \qquad \nabla u_- = \begin{cases} 0, & \text{if} \quad [u \ge 0] \\ \nabla u, & \text{if} \quad [u < 0]\,, \end{cases}$$

(see, e.g. [5, Theorem 7.6]). Thus, problem (1) with $f$ given by relation (5) becomes

$$\begin{cases} -\Delta u = \lambda[h(x,u_+) - u_-], & \text{in} \quad \Omega \\ u = 0, & \text{on} \quad \partial\Omega\,, \end{cases} \tag{6}$$

and $\lambda > 0$ is an eigenvalue of problem (6) if there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla u_+ \nabla v\; dx - \int_\Omega \nabla u_- \nabla v\; dx - \lambda \int_\Omega [h(x,u_+) - u_-]v\; dx = 0\,, \tag{7}$$

for any $v \in H_0^1(\Omega)$.

**Lemma 1.** *Any $\lambda \in (0, \lambda_1)$ is not an eigenvalue of problem (6).*

*Proof.* Assume that $\lambda > 0$ is an eigenvalue of problem (6) with the corresponding eigenfunction $u$. Letting $v = u_+$ and $v = u_-$ in the definition of the eigenvalue $\lambda$ we find that the following two relations hold true

$$\int_\Omega |\nabla u_+|^2\; dx = \lambda \int_\Omega h(x,u_+)u_+\; dx \tag{8}$$

and

$$\int_\Omega |\nabla u_-|^2\; dx = \lambda \int_\Omega u_-^2\; dx\,. \tag{9}$$

In this context, hypothesis (H1) and relations (3), (8) and (9) imply

$$\lambda_1 \int_\Omega u_+^2\; dx \le \int_\Omega |\nabla u_+|^2\; dx = \lambda \int_\Omega h(x,u_+)u_+\; dx \le \lambda \int_\Omega u_+^2\; dx$$

and

$$\lambda_1 \int_\Omega u_-^2\; dx \le \int_\Omega |\nabla u_-|^2\; dx = \lambda \int_\Omega u_-^2\; dx\,.$$

If $\lambda$ is an eigenvalue of problem (6) then $u \neq 0$ and thus, at least one of the functions $u_+$ and $u_-$ is not the zero function. Thus, the last two inequalities show that $\lambda$ is an eigenvalue of problem (6) only if $\lambda \ge \lambda_1$. □

**Lemma 2.** *$\lambda_1$ is an eigenvalue of problem (6). Moreover, the set of eigenvectors corresponding to $\lambda_1$ is a cone.*

*Proof.* Indeed, as we already pointed out, $\lambda_1$ is the lowest eigenvalue of problem (2), it is simple, i.e. all the associated eigenfunctions are merely multiples of each other (see, e.g. Gilbarg and Trudinger

[5]) and the corresponding eigenfunctions of $\lambda_1$ never change signs in $\Omega$. In other words, there exists $e_1 \in H_0^1(\Omega) \setminus \{0\}$, with $e_1(x) < 0$ for any $x \in \Omega$ such that

$$\int_\Omega \nabla e_1 \nabla v \, dx - \lambda_1 \int_\Omega e_1 v \, dx = 0 \,,$$

for any $v \in H_0^1(\Omega)$. Thus, we have $(e_1)_+ = 0$ and $(e_1)_- = e_1$ and we deduce that relation (7) holds true with $u = e_1 \in H_0^1(\Omega) \setminus \{0\}$ and $\lambda = \lambda_1$. In other words, $\lambda_1$ is an eigenvalue of problem (6) and undoubtedly, the set of its corresponding eigenvectors lies in a cone of $H_0^1(\Omega)$. The proof of Lemma 2 is complete. $\square$

**Lemma 3.** *$\lambda_1$ is isolated in the set of eigenvalues of problem (6).*

*Proof.* By Lemma 1 we know that in the interval $(0, \lambda_1)$ there is no eigenvalue of problem (6). On the other hand, hypothesis (H1) and relations (3) and (8) show that if $\lambda$ is an eigenvalue of problem (6) for which the positive part of its corresponding eigenfunction, that is $u_+$, is not identically zero then

$$\lambda_1 \int_\Omega u_+^2 \, dx \le \int_\Omega |\nabla u_+|^2 \, dx = \lambda \int_\Omega h(x, u_+) u_+ \, dx \le \lambda C \int_\Omega u_+^2 \, dx \,,$$

and thus, since $C \in (0, 1)$ we infer $\lambda \ge \frac{\lambda_1}{C} > \lambda_1$. We deduce that for any eigenvalue $\lambda \in (0, \lambda_1/C)$ of problem (6) we must have $u_+ = 0$. It follows that if $\lambda \in (0, \lambda_1/C)$ is an eigenvalue of problem (6) then it is actually an eigenvalue of problem (2) with the corresponding eigenfunction negative in $\Omega$. But, we already noticed that the set of eigenvalues of problem (2) is discrete and $\lambda_1 < \lambda_2$. In other words, taking $\delta = \min\{\lambda_1/C, \lambda_2\}$ we find that $\delta > \lambda_1$ and any $\lambda \in (\lambda_1, \delta)$ can not be an eigenvalue of problem (2) and, consequently, any $\lambda \in (\lambda_1, \delta)$ is not an eigenvalue of problem (6). We conclude that $\lambda_1$ is isolated in the set of eigenvalues of problem (6). The proof of Lemma 3 is complete. $\square$

Next, we show that there exists $\mu_1 > 0$ such that any $\lambda \in (\mu_1, \infty)$ is an eigenvalue of problem (6). With that end in view, we consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda h(x, u_+), & \text{in} \quad \Omega \\ u = 0, & \text{on} \quad \partial\Omega \,, \end{cases} \tag{10}$$

We say that $\lambda$ is an eigenvalue of problem (10) if there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla u \nabla v \, dx - \lambda \int_\Omega h(x, u_+) v \, dx = 0 \,,$$

for any $v \in H_0^1(\Omega)$.

We notice that if $\lambda$ is an eigenvalue for (10) with the corresponding eigenfunction $u$, then taking $v = u_-$ in the above relation we deduce that $u_- = 0$, and thus, we find $u \ge 0$. In other words, the eigenvalues of problem (10) possesses nonnegative corresponding eigenfunctions. Moreover, the above discussion show that an eigenvalue of problem (10) is an eigenvalue of problem (6).

For each $\lambda > 0$ we define the energy functional associated to problem (10) by $I_\lambda : H_0^1(\Omega) \to \mathbb{R}$,

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega H(x, u_+) \, dx \,,$$

where $H(x,t) = \int_0^t h(x,s) \, ds$. Standard arguments show that $I_\lambda \in C^1(H_0^1(\Omega), \mathbb{R})$ with the derivative given by

$$\langle I_\lambda'(u), v\rangle = \int_\Omega \nabla u \nabla v \, dx - \lambda \int_\Omega h(x, u_+) v \, dx \,,$$

for any $u$, $v \in H_0^1(\Omega)$. Thus, $\lambda > 0$ is an eigenvalue of problem (10) if and only if there exists a critical nontrivial point of functional $I_\lambda$.

**Lemma 4.** *The functional $I_\lambda$ is bounded from below and coercive.*

*Proof.* By hypothesis (H3) we deduce that

$$\lim_{t\to\infty} \frac{H(x,t)}{t^2} = 0, \quad \text{uniformly in } \Omega \,.$$

Then for a given $\lambda > 0$ there exists a positive constant $C_\lambda > 0$ such that

$$\lambda H(x,t) \le \frac{\lambda_1}{4} t^2 + C_\lambda, \quad \forall\, t \ge 0, \text{ a.e. } x \in \Omega \,,$$

where $\lambda_1$ is given by relation (3).

Thus, we find that for any $u \in H_0^1(\Omega)$ it holds true

$$I_\lambda(u) \ge \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{\lambda_1}{4}\int_\Omega u^2 \, dx - C_\lambda|\Omega| \ge \frac{1}{4}\|u\|^2 - C_\lambda|\Omega| \,,$$

where by $\|\cdot\|$ is denoted the norm on $H_0^1(\Omega)$, that is $\|u\| = (\int_\Omega |\nabla u|^2 \, dx)^{1/2}$. This shows that $I_\lambda$ is bounded from below and coercive. The proof of Lemma 4 is complete. $\square$

**Lemma 5.** *There exists $\lambda^\star > 0$ such that assuming that $\lambda \ge \lambda^\star$ we have* $\inf_{H_0^1(\Omega)} I_\lambda < 0$.

*Proof.* Hypothesis (H2) implies that there exists $t_0 > 0$ such that

$$H(x, t_0) > 0 \quad \text{a.e. } x \in \overline{\Omega} \,.$$

Let $\Omega_1 \subset \Omega$ be a compact subset, sufficiently large, and $u_0 \in C_0^1(\Omega) \subset H_0^1(\Omega)$ such that $u_0(x) = t_0$ for any $x \in \Omega_1$ and $0 \le u_0(x) \le t_0$ for any $x \in \Omega \setminus \Omega_1$.

Thus, by hypothesis (H1) we have

$$\begin{aligned} \int_\Omega H(x, u_0) \, dx \quad &\ge \quad \int_{\Omega_1} H(x, t_0) \, dx - \int_{\Omega\setminus\Omega_1} C u_0^2 \, dx \\ &\ge \quad \int_{\Omega_1} H(x, t_0) \, dx - C t_0^2 |\Omega \setminus \Omega_1| > 0 \,. \end{aligned}$$

We conclude that $I_\lambda(u_0) < 0$ for $\lambda > 0$ sufficiently large, and thus, $\inf_{H_0^1(\Omega)} I_\lambda < 0$. The proof of Lemma 5 is complete. $\square$

Lemmas 4 and 5 show that for any $\lambda > 0$ large enough, the functional $I_\lambda$ possesses a negative global minimum (see, [11, Theorem 1.2]), and thus, any $\lambda > 0$ large enough is an eigenvalue of problem (10) and consequently of problem (6). Combining that fact with the results of Lemmas 1, 2 and 3 we conclude that Theorem 1 holds true.

**Remark.** Finally, we notice that similar results as those given by Theorem 1 can be formulated for equations of type (6) but replacing the Laplace operator $\Delta u$ by the $p$-Laplace operator, that is $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, with $1 < p < \infty$. Certainly, in that case hypotheses (H1)-(H3) should be modified accordingly with the new situation. This statement is supported by the fact that the first eigenvalue of the $p$-Laplace operator on bounded domains satisfies similar properties as the one obtained in the case of the Laplace operator (see, e.g., [1]) combined with the remark that the results on problem (10) can be easily extended to the case of the $p$-Laplace operator.

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