Γ-convergence of power-law functionals with variable exponents

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Abstract. Γ-convergence results for power-law functionals with variable exponents are obtained. The main motivation comes from the study of (first-failure) dielectric breakdown. Some connections with the generalization of the ∞-Laplace equation to the variable exponent setting are also explored.

2000 Mathematics Subject Classification:

Key words: dielectric breakdown, Γ-convergence, Lebesgue-Sobolev spaces with variable exponent, power-law functionals

1 Introduction

During the last decade an intense research activity has focused on the characterization of the effective yield set in the context of polycrystal plasticity and on the related problem regarding the optimality of the classical inner and outer estimates (the so-called Sachs and Taylor bounds; see [25], [27]) for the effective yield set. We refer to Garroni & Kohn [14], Goldsztein [15], [16], and Kohn & Little [17] for a number of interesting results in this area, and for more details on the physical framework.

In the context of (first-failure) dielectric breakdown for composite materials made of two isotropic phases, a new approach to these problems has been proposed in Garroni *et al.* [13], where the authors introduce natural variational principles which provide a rigorous justification via Γ-convergence of the classical dielectric breakdown model as a limiting case of power-law models and, on the other hand, suggest a new nondegenerate variational principle in L^∞ which can be used to efficiently characterize the effective yield set. The results in [13], which concern power-law type functionals acting on gradients, have been recently extended by Bocea & Nesi [4] to allow for more general linear PDE constraints on the underlying fields in the framework of $\mathcal{A}$-quasiconvexity, leading, in particular, to variational characterizations of the yield (strength) set in the setting of electrical resistivity.

The original approach of Garroni, Nesi & Ponsiglione [13] is based on two Γ-convergence results (with respect to the strong topology of $L^1(Q)$), as $p \to \infty$, for the sequences of functionals $\{G_p\}$ and F_p, with $G_p, F_p : L^1(Q) \to [0, +\infty]$ defined by

$$G_p(u) := \begin{cases} \dfrac{1}{p} \displaystyle\int_Q |\lambda(x)\nabla u(x)|^p dx & \text{if } u \in W^{1,p}(Q) \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$F_p(u) := \begin{cases} \left(\displaystyle\int_Q |\lambda(x)\nabla u(x)|^p dx \right)^{\frac{1}{p}} & \text{if } u \in W^{1,p}(Q) \\ +\infty & \text{otherwise.} \end{cases}$$

Here, $Q = (0,1)^N$ is the unit cube in $\mathbb{R}^N$, and $\lambda \in L^\infty(Q)$ is such that $0 < a \leq \lambda(x) \leq b$ for a.e. $x \in Q$.

The main goal of this paper is to generalize the Γ-convergence results in Garroni *et al.* [13] to the case of functionals involving variable exponents. Precisely, the exponent p in the power-law regularization considered here is allowed to depend on the point $x \in \Omega$, where $\Omega \subset \mathbb{R}^N$ is an open set of finite Lebesgue measure $|\Omega| < +\infty$ with sufficiently smooth (e.g. Lipschitz) boundary. In light of the well-known properties of Γ-convergence, our results imply that various models of dielectric breakdown can in fact be justified as limiting cases of more flexible power-law models. The main difficulties that need to be overcome are due to the fact that the functionals involved are not homogeneous, and thus the classical methods cannot be used directly.

The meaning of "$p \to \infty$" in our setting is as follows: we consider a sequence $\{p_n\}$ of Lipschitz continuous functions $p_n : \overline{\Omega} \to (1, \infty)$, such that $p_n^- := \inf_{x\in\Omega} p_n(x) \to \infty$ as $n \to \infty$, and for which the Harnack-type inequality $\sup_{x\in\Omega} p_n(x) =: p_n^+ \leq \beta p_n^-$ holds for all $n \in \mathbb{N}$ sufficiently large, where $\beta > 1$ is a real constant.

In Theorems 1 and 2 (stated in Section 3 of the paper) we prove two Γ-convergence results with respect to the strong topology of $L^1(\Omega)$ for the natural generalizations of the functionals G_p and F_p to the variable exponent setting. These are given by

$$I_n(u) := \begin{cases} \displaystyle\int_\Omega \frac{1}{p_n(x)} |\lambda(x)\nabla u(x)|^{p_n(x)} dx & \text{if } u \in W^{1,p_n(\cdot)}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

and

$$J_n(u) := \begin{cases} |\lambda \nabla u|_{p_n(\cdot)} & \text{if } u \in W^{1,p_n(\cdot)}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

respectively, where $W^{1,p_n(\cdot)}(\Omega)$ stands for the appropriate variable exponent Sobolev space, and $|\cdot|_{p_n(\cdot)}$ is the Luxemburg norm in the space $L^{p_n(\cdot)}(\Omega)$.

We note that in the particular case $\lambda \equiv 1$ the Euler-Lagrange equation associated to the minimization of I_n in the space $W^{1,p_n(\cdot)}(\Omega)$ subject to appropriate boundary conditions is the $p_n(\cdot)$-Laplace equation

$$-\Delta_{p_n(x)} u(x) := -\mathrm{div}\left(|\nabla u(x)|^{p_n(x)-2} \nabla u(x) \right) = 0 \text{ in } \Omega. \tag{1.1}$$

Power laws with variable exponents and the corresponding PDEs involving this generalization of the p-Laplace operator appear, for example, in connection to the study of electrorheological fluids (see, e.g., Diening [9], Rajagopal & Ruzicka [23], or Ruzicka [24]). From a PDE point of view, the computation of the Γ-limits of I_n and J_n is motivated by a recent work of Manfredi, Rossi & Urbano [20], where the authors consider a sequence $\{p_n\} \subset C^1(\Omega)$ which converges uniformly to $+\infty$ in Ω, such that $p_n(x) > \alpha > N$ for all $x \in \Omega$, and with $\nabla \ln p_n \to \xi \in C(\Omega)$ uniformly in Ω, and show that weak solutions of (1.1) (which are precisely the minimizers of I_n in $W^{1,p_n(\cdot)}(\Omega)$) subject to appropriate boundary conditions converge uniformly, as $n \to \infty$, to a viscosity solution of

$$-\Delta_\infty u - |\nabla u|^2 \ln |\nabla u| \langle \xi, \nabla u \rangle = 0 \text{ in } \Omega. \tag{1.2}$$

We remark that in the constant exponent case, $p_n(x) = n$ for all $x \in \Omega$, we have $\xi \equiv 0$, and thus (1.2) reduces to the well-known ∞-Laplace equation

$$-\Delta_\infty u = 0 \text{ in } \Omega,$$

which has been introduced by Aronsson [1], [2] as the candidate for the Euler-Lagrange equation associated to a properly interpreted minimization problem for the functional $u \mapsto \|\nabla u\|_{L^\infty(\Omega)}$ in $W^{1,\infty}(\Omega)$. More recently, this has been justified using the notions of absolutely minimizing Lipschitz extensions (AMLE) and viscosity solutions of the associated PDE; we refer to [3] and references therein for more details on these issues.

It is interesting to note that when $\lambda \equiv 1$ the supremal functional $u \mapsto \|\nabla u\|_{L^\infty(Q)}$ coincides with the restriction to $W^{1,\infty}(Q)$ of the Γ-limit of the sequence F_p computed in Garroni *et al.* [13]. Thus, the supremal functional associated to the the ∞-Laplace equation arises as a Γ-limit of power-law functionals. It is natural to ask whether a supremal functional associated to (1.2) (which is the generalization of the ∞-Laplace equation to the variable exponent setting) can also be obtained via Γ-convergence of certain sequences of power-law functionals with variable exponents.

A first attempt to answer this question would be to consider sequences of functionals where J_n is modified so as to involve in some fashion the so-called modular $\rho_{p_n(\cdot)}(\lambda \nabla u)$ (see Section 2 for its definition). In this direction, under the additional requirement that $\lim\limits_{n\to\infty} \frac{p_n^+}{p_n^-} = 1$, we study (see Theorem 3) the Γ-convergence of the sequence of functionals $\{K_n\}$ defined by

$$K_n(u) = \begin{cases} \left(\displaystyle\int_\Omega |\lambda(x) \nabla u(x)|^{p_n(x)} \, dx \right)^{1/p_n^+} & \text{if } u \in W^{1,p_n(\cdot)}(\Omega) \text{ with } |\lambda \nabla u|_{p_n(\cdot)} \leq 1 \\ |\lambda \nabla u|_{p_n(\cdot)} & \text{if } u \in W^{1,p_n(\cdot)}(\Omega) \text{ with } |\lambda \nabla u|_{p_n(\cdot)} > 1 \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

However, although of independent interest in itself, the Γ-convergence result regarding the sequence $\{K_n\}$ does not, in fact, shed light on the answer to the question stated in the previous paragraph for a general sequence $\{p_n\}$ of variable exponents. There is, nevertheless, one particular choice of the sequence $\{p_n\}$ where the question can be answered in the affirmative. Precisely, we consider, for $x \in \Omega$ (an open, bounded domain in $\mathbb{R}^N$ with smooth boundary), and $n \in \mathbb{N}$, $p_n(x) := np(x)$, where $p : \overline{\Omega} \to (1, \infty)$ is a given function in $C^1(\Omega) \cap W^{1,\infty}(\Omega)$, and we study the Γ-convergence of functionals of the form

$$u \mapsto \left(\int_\Omega \frac{1}{np(x)} |\nabla u(x)|^{np(x)}\, dx \right)^{1/n}. \tag{1.3}$$

Variational integrals of this type have been considered by Zhikov in [28] (cf. Lindqvist & Lukkari [19]). It turns out (see Section 6 for the precise statement) that in this case the Γ-limit involves the supremal functional

$$u \mapsto \operatorname{esssup}_{x\in\Omega} \left(|\nabla u(x)|^{p(x)} \right),$$

which corresponds to the PDE

$$-\Delta_\infty u - |\nabla u|^2 \ln |\nabla u| \, \langle \nabla \ln p, \nabla u \rangle = 0 \text{ in } \Omega. \tag{1.4}$$

For more on this relationship we refer to Lindqvist & Lukkari [19]. We note that when $p_n(x) = np(x)$ one has $\nabla \ln p_n \to \xi = \nabla \ln p$ uniformly in Ω, and thus (1.2) reduces to (1.4). The existence and uniqueness of viscosity solutions for this equation are proven in [20] and [19], respectively.

The plan of the paper is as follows: In Section 2 we review some properties of the variable exponent Lebesgue-Sobolev spaces. In Section 3 we give the definition of Γ-convergence, and we state our Γ-convergence results for the functionals I_n, J_n, and K_n. The proofs of these results are left for the next two sections of the paper. Finally, in Section 6, we state and prove a Γ-convergence result for power-law functionals of the type (1.3).

2 Variable exponent Lebesgue-Sobolev spaces

In this section we provide a brief review of the basic properties of the variable exponent Lebesgue-Sobolev spaces. For more details we refer to the book by Musielak [22] and the papers by Edmunds *et al.* [10, 11, 12], Kovacik & Rákosník [18], Mihăilescu & Rădulescu [21], and Samko & Vakulov [26]. Let $\Omega \subset \mathbb{R}^N$ be an open set. Throughout this paper we denote by $|\Omega|$ the N-dimensional Lebesgue measure of the set Ω. For any Lipschitz continuous function $p : \overline{\Omega} \to (1, \infty)$ define

$$p^- := \inf_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \sup_{x\in\Omega} p(x).$$

It is usually assumed that $p^+ < +\infty$, since this condition is known to imply many desirable features for the associated variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$. This function space is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

$L^{p(\cdot)}(\Omega)$ is a Banach space when endowed with the so-called *Luxemburg norm*, defined by

$$|u|_{p(\cdot)} := \inf\left\{\mu > 0 \ : \ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

We note that the variable exponent Lebesgue space is a special case of an Orlicz-Musielak space. For constant functions p, $L^{p(\cdot)}(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$, endowed with the standard norm

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx\right)^{1/p}.$$

We recall that if $1 < p^- \le p^+ < +\infty$ the space $L^{p(\cdot)}(\Omega)$ is separable and reflexive. If $0 < |\Omega| < \infty$, and if p_1, p_2 are variable exponents so that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous and its norm does not exceed $|\Omega| + 1$.

We denote by $L^{p^{'}(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $1/p(x) + 1/p^{'}(x) = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p^{'}(\cdot)}(\Omega)$ the Hölder type inequality

$$\left|\int_\Omega uv \, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p^{'-}}\right) |u|_{p(\cdot)} |v|_{p^{'}(\cdot)} \tag{2.1}$$

holds.

A key role in manipulating the variable exponent Lebesgue and Sobolev (see below) spaces is played by the *modular* of the space $L^{p(\cdot)}(\Omega)$, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)} \, dx.$$

Lebesgue and Sobolev spaces with $p^+ = +\infty$ have been investigated in [10] and [18]. In this case one defines $\Omega_\infty := \{x \in \Omega;\ p(x) = +\infty\}$ and the modular is given by

$$\rho_{p(\cdot)}(u) := \int_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} \, dx + \operatorname{ess\,sup}_{x \in \Omega_\infty} |u(x)| \, .$$

In the particular case $\Omega_\infty = \Omega$ we recover the usual Lebesgue space $L^\infty(\Omega)$. If $u \in L^{p(\cdot)}(\Omega)$ then the following relations hold:

$$|u|_{p(\cdot)} > 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+}\,; \tag{2.2}$$

$$|u|_{p(\cdot)} < 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-}\,; \tag{2.3}$$

$$|u|_{p(\cdot)} = 1 \Leftrightarrow \rho_{p(\cdot)}(u) = 1\,. \tag{2.4}$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) :\ |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

On this space we may consider one of the following equivalent norms

$$\|u\|_{p(\cdot)} := |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)},$$

or

$$\|u\| := \inf\left\{\mu > 0;\ \int_\Omega \left(\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} + \left|\frac{u(x)}{\mu}\right|^{p(x)}\right)\, dx \le 1\right\},$$

where, in the definition of $\|u\|_{p(\cdot)}$, $|\nabla u|_{p(\cdot)}$ stands for the Luxemburg norm of $|\nabla u|$. We note that under reasonable assumptions on the function p the space $W^{1,p(\cdot)}(\Omega)$ is also a separable and reflexive Banach space.

3 Main results

We start by recalling the definition of Γ-convergence (introduced in [7], [8]) in metric spaces. The reader is referred to [6] for a comprehensive introduction to the subject.

Definition 1. *Let X be a metric space. A sequence $\{I_n\}$ of functionals $I_n : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is said to $\Gamma(X)$-converge to $I_\infty : X \to \overline{\mathbb{R}}$, and we write $\Gamma(X) - \lim_{n\to\infty} I_n = I_\infty$, if the following hold:*
(i) for every $u \in X$ and $\{u_n\} \subset X$ such that $u_n \to u$ in X, we have

$$I_\infty(u) \le \liminf_{n\to\infty} I_n(u_n);$$

(ii) for every $u \in X$ there exists a sequence $\{u_n\} \subset X$ (called a recovery sequence*) such that $u_n \to u$ in X and*

$$I_\infty(u) \ge \limsup_{n\to\infty} I_n(u_n).$$

Let $\Omega \subset \mathbb{R}^N$ be an open set of finite Lebesgue measure $|\Omega| < +\infty$, with sufficiently smooth boundary. To simplify the presentation we will assume in what follows that $|\Omega| = 1$. However, this additional assumption is only imposed so that unnecessary complications in the proofs can be avoided; our results still hold, with appropriate modifications, in the general case.

Consider a sequence $\{p_n\}$ of Lipschitz continuous functions $p_n : \overline{\Omega} \to (1, \infty)$, satisfying the conditions

$$p_n^- \to \infty \text{ as } n \to \infty, \tag{3.1}$$

and

$$\text{there exists a real constant } \beta > 1 \text{ such that } p_n^+ \le \beta p_n^- \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

We remark that, in particular, (3.1) and (3.2) imply that we have

$$p_n^{'\,-} \to 1 \text{ and } \left(p_n^+\right)^{\frac{1}{p_n^-}} \to 1 \text{ as } n \to \infty. \tag{3.3}$$

Let $\lambda \in L^\infty(\Omega)$ be such that $0 < a \le \lambda(x) \le b$, where a and b are two positive real numbers. For each $n \in \mathbb{N}$, consider the functionals $I_n, J_n : L^1(\Omega) \to [0, +\infty]$ defined by

$$I_n(u) = \begin{cases} \displaystyle\int_\Omega \frac{1}{p_n(x)} |\lambda(x)\nabla u(x)|^{p_n(x)}\, dx & \text{if } u \in W^{1,p_n(\cdot)}(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$J_n(u) = \begin{cases} |\lambda \nabla u|_{p_n(\cdot)} & \text{if } u \in W^{1,p_n(\cdot)}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Our main results are given by the following theorems:

Theorem 1. *Define* $I_\infty : L^1(\Omega) \to [0, +\infty]$ *by*

$$I_\infty(u) = \begin{cases} 0 & \textit{if } |\lambda(x)\nabla u(x)| \le 1 \textit{ a.e. } x \in \Omega \\ +\infty & \textit{otherwise.} \end{cases}$$

Then $\Gamma(L^1(\Omega)) - \lim\limits_{n\to\infty} I_n = I_\infty$.

Theorem 2. *Define* $J_\infty : L^1(\Omega) \to [0, +\infty]$ *by*

$$J_\infty(u) = \begin{cases} ||\lambda \nabla u||_{L^\infty(\Omega;\mathbb{R}^N)} & \textit{if } u \in W^{1,\infty}(\Omega) \\ +\infty & \textit{otherwise.} \end{cases}$$

Then $\Gamma(L^1(\Omega)) - \lim\limits_{n\to\infty} J_n = J_\infty$.

Remark 1. *As already mentioned in the Introduction, it was shown in [20] that the unique minimizers of the functional* I_n *subject to appropriate boundary conditions, which are the weak solutions of (1.1), converge uniformly, as* $n \to \infty$*, to a viscosity solution of (1.2). The results in [20] are obtained under hypotheses on the sequence* $\{p_n\}$ *which are similar to (albeit more stringent than) our assumptions (3.1) and (3.2).*

Next, we would like to point out yet another Γ-convergence result which can be established in the case where the sequence $\{p_n\}$ satisfies (3.1), together with a stronger condition than (3.2), namely

$$\lim_{n\to\infty} p_n^+/p_n^- = 1\,. \tag{3.4}$$

Let $\Omega = B(0,1) \subset \mathbb{R}^N$ (the ball of radius 1 centered at the origin), and let $p : \overline{\Omega} \to (1, \infty)$ be given by $p(x) = \sin(|x|) + 2$. It is easy to check that if the functions $p_n : \overline{B}(0,1) \to (1, \infty)$ are defined by $p_n(x) := np(\frac{x}{n})$, then the sequence $\{p_n\}$ satisfies (3.1) and (3.4).

Theorem 3. *Assume that the sequence* $\{p_n\}$ *satisfies (3.1) and (3.4), and consider the sequence* $\{K_n\}$ *of functionals* $K_n : L^1(\Omega) \to [0, +\infty]$ *defined by*

$$K_n(u) = \begin{cases} \left(\int_\Omega |\lambda(x)\nabla u(x)|^{p_n(x)}\, dx \right)^{1/p_n^+} & \textit{if } u \in W^{1,p_n(\cdot)}(\Omega) \textit{ with } |\lambda \nabla u|_{p_n(\cdot)} \le 1 \\ |\lambda \nabla u|_{p_n(\cdot)} & \textit{if } u \in W^{1,p_n(\cdot)}(\Omega) \textit{ with } |\lambda \nabla u|_{p_n(\cdot)} > 1 \\ +\infty & \textit{otherwise.} \end{cases}$$

Then, $\Gamma(L^1(\Omega)) - \lim\limits_{n\to\infty} K_n = J_\infty$*, where* J_∞ *is defined as in Theorem 2.*

4 Proof of Theorem 1

We start by verifying (ii) in Definition 1. If $I_\infty(u) = \infty$ the inequality is clear for any sequence $u_n \to u$ strongly in $L^1(\Omega)$. On the other hand, if $I_\infty(u) < +\infty$ we must have $I_\infty(u) = 0$ and, consequently, $|\lambda(x)\nabla u(x)| \leq 1$ for a.e. $x \in \Omega$. For each $n \in \mathbb{N}$, let $u_n := u$, and note that we have

$$\limsup_{n\to\infty} I_n(u_n) = \limsup_{n\to\infty} \int_\Omega \frac{1}{p_n(x)} |\lambda(x)\nabla u(x)|^{p_n(x)}\, dx \leq \limsup_{n\to\infty} \frac{|\Omega|}{p_n^-} = 0 = I_\infty(u),$$

where we have used hypothesis (3.1). Thus the constant sequence $\{u_n\} = \{u\}$ is a recovery sequence for the Γ-limit.

To prove (i) in Definition 1 we may assume, without loss of generality, that $u_n \in W^{1,p_n(\cdot)}(\Omega)$, and

$$\liminf_{n\to\infty} I_n(u_n) = \lim_{n\to\infty} I_n(u_n) < \infty\,. \tag{4.1}$$

Let $x \in \Omega$ be a Lebesgue point for $\lambda\nabla u \in L^1(\Omega)$. For any ball $B(x,r) \subset \Omega$ and $n \in \mathbb{N}$ sufficiently large we have, in view of (2.1),

$$\int_{B(x,r)} |\lambda(y)\nabla u_n(y)| dy \leq \left(\frac{1}{p_n^-} + \frac{1}{{p'_n}^-}\right) |\lambda\nabla u_n|_{p_n(\cdot)}\, |\chi_{B(x,r)}|_{p'_n(\cdot)}. \tag{4.2}$$

For sufficiently small $r > 0$ (so that $|B(x,r)| < 1$) we have

$$|\chi_{B(x,r)}|_{p'_n(\cdot)} = \inf\left\{\mu > 0\ :\ \int_{B(x,r)} \frac{1}{\mu^{p'_n(y)}}\, dy \leq 1\right\} \leq |B(x,r)|^{\frac{1}{{p'_n}^-}}. \tag{4.3}$$

Next, we claim that

$$|\lambda\nabla u_n|_{p_n(\cdot)} \leq \max\left\{1, \left(p_n^+ I_n(u_n)\right)^{\frac{1}{p_n^-}}\right\}. \tag{4.4}$$

Indeed, if $|\lambda\nabla u_n|_{p_n(\cdot)} > 1$, then (2.2) gives

$$|\lambda\nabla u_n|_{p_n(\cdot)}^{p_n^-} \leq \int_\Omega |\lambda(x)\nabla u_n(x)|^{p_n(x)}\, dx \leq p_n^+ \int_\Omega \frac{1}{p_n(x)} |\lambda(x)\nabla u_n(x)|^{p_n(x)}\, dx,$$

which implies that

$$|\lambda\nabla u_n|_{p_n(\cdot)} \leq \left(p_n^+ I_n(u_n)\right)^{\frac{1}{p_n^-}}.$$

Combining (4.2), (4.3), and (4.4), we obtain

$$\int_{B(x,r)} |\lambda(y)\nabla u_n(y)| dy \leq \left(\frac{1}{p_n^-} + \frac{1}{{p'_n}^-}\right) |B(x,r)|^{\frac{1}{{p'_n}^-}} \max\left\{1, \left(p_n^+ I_n(u_n)\right)^{\frac{1}{p_n^-}}\right\},$$

which, in view of (3.1), (3.3), and (4.1), implies that

$$\limsup_{n\to\infty} \int_{B(x,r)} |\lambda(y)\nabla u_n(y)| dy \leq |B(x,r)|. \tag{4.5}$$

For each $n \in \mathbb{N}$, consider the sets

$$\Omega_n^+ = \{x \in \Omega;\ |\lambda(x)\nabla u_n(x)| \geq 1\} \quad \text{and} \quad \Omega_n^- = \{x \in \Omega;\ |\lambda(x)\nabla u_n(x)| \leq 1\}.$$

Let $q \geq 1$ be an arbitrary real number. By (3.1) we have that $q < p_n^-$ for sufficiently large $n \in \mathbb{N}$. Using the classical Hölder's inequality we deduce that

$$\begin{aligned}
\int_\Omega |\lambda(x)\nabla u_n(x)|^q\,dx &\leq \left(\int_\Omega |\lambda(x)\nabla u_n(x)|^{p_n^-}\,dx\right)^{\frac{q}{p_n^-}} |\Omega|^{\frac{p_n^- - q}{p_n^-}} \\
&= \left(\int_{\Omega_n^-} |\lambda(x)\nabla u_n(x)|^{p_n^-}\,dx + \int_{\Omega_n^+} |\lambda(x)\nabla u_n(x)|^{p_n^-}\,dx\right)^{\frac{q}{p_n^-}} \\
&\leq \left(1 + \int_\Omega |\lambda(x)\nabla u_n(x)|^{p_n(x)}\,dx\right)^{\frac{q}{p_n^-}} \leq \left(1 + p_n^+ I_n(u_n)\right)^{\frac{q}{p_n^-}}.
\end{aligned}$$

Thus, the sequence $\{\nabla u_n\}$ is bounded in $L^q(\Omega;\mathbb{R}^N)$ for any $q \geq 1$. Since $u_n \to u$ in $L^1(\Omega)$ we deduce, in view of the Poincaré-Wirtinger inequality, that $\{u_n\}$ is bounded in $L^q(\Omega)$. It follows that $\{u_n\}$ is bounded in $W^{1,q}(\Omega)$, and thus we may extract a subsequence (not relabelled) such that $u_n \rightharpoonup u$ weakly in $W^{1,q}(\Omega)$. Using a well-known lower semicontinuity result we find

$$\int_{B(x,r)} |\lambda(y)\nabla u(y)|dy \leq \liminf_{n\to\infty} \int_{B(x,r)} |\lambda(y)\nabla u_n(y)|dy,$$

which implies, in view of (4.5), that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |\lambda(y)\nabla u(y)|dy \leq 1.$$

Since almost every $x \in \Omega$ is a Lebesgue point for $\lambda\nabla u$, passing to the limit $r \to 0^+$ in the above inequality yields $|\lambda(x)\nabla u(x)| \leq 1$ for a.e. $x \in \Omega$. It follows that $I_\infty(u) = 0$, and this implies that the inequality in (i) of Definition 1 holds. This concludes the proof of Theorem 1. $\square$

5 Proofs of Theorems 2 and 3

We start by establishing several auxiliary results which will be needed in the sequel. The following two lemmas generalize to the variable exponent setting the classical result which asserts that if $\Omega \subset \mathbb{R}^N$ has finite Lebesgue measure, and if $u \in L^\infty(\Omega)$, then

$$\lim_{q\to\infty} \|u\|_{L^q(\Omega)} = \|u\|_{L^\infty(\Omega)}. \tag{5.1}$$

Lemma 1. *Let $\{p_n\}$ be a sequence of Lipschitz continuous functions, $p_n : \overline{\Omega} \to (1,\infty)$, such that $p_n^+ \to \infty$ as $n \to \infty$. Then, if $u \in L^\infty(\Omega) \setminus \{0\}$, we have*

$$\lim_{n\to\infty} \left(\int_\Omega \left(\frac{|u(x)|}{\|u\|_{L^\infty(\Omega)}}\right)^{p_n(x)} dx\right)^{1/p_n^+} = 1. \tag{5.2}$$

Proof. Define $v \in L^\infty(\Omega)$ by

$$v(x) := \frac{u(x)}{\|u\|_{L^\infty(\Omega)}}.$$

We have $|v(x)| \le 1$ for a.e. $x \in \Omega$, and using the fact that $p_n(x) \le p_n^+$ for all $x \in \Omega$ and $n \in \mathbb{N}$ we deduce that

$$\int_\Omega |v(x)|^{p_n^+}\, dx \le \int_\Omega |v(x)|^{p_n(x)}\, dx \le |\Omega| = 1, \quad \forall\, n\,.$$

Thus, we have

$$\|v\|_{L^{p_n^+}(\Omega)} \le \left(\int_\Omega |v(x)|^{p_n(x)}\, dx\right)^{1/p_n^+} \le 1 \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ in the above inequality, and taking into account (5.1), we obtain that (5.2) holds. □

Remark 2. *In particular, (5.2) holds when our hypothesis (3.1) is satisfied. Also, note that (5.1) follows from (5.2); indeed, given an arbitrary sequence $\{q_n\}$ of real numbers such that $q_n \to \infty$, it suffices to consider $p_n(x) := q_n$ in (5.2).*

Lemma 2. *Let $u \in L^\infty(\Omega)$, and let $\{p_n\}$ be a sequence of Lipschitz continuous functions $p_n : \overline{\Omega} \to (1, \infty)$ such that (3.1) and (3.2) hold. Then*

$$\lim_{n\to\infty} |u|_{p_n(\cdot)} = \|u\|_{L^\infty(\Omega)}\,. \tag{5.3}$$

Proof. We may assume, without loss of generality, that $u \neq 0$. We will show that

$$\lim_{n\to\infty} \left| \frac{u}{\|u\|_{L^\infty(\Omega)}} \right|_{p_n(\cdot)} = 1, \tag{5.4}$$

which is equivalent to (5.3). First, note that since $|u(x)| \le \|u\|_{L^\infty(\Omega)}$ for a.e. $x \in \Omega$ we have

$$\int_\Omega \left| \frac{u(x)}{\|u\|_{L^\infty(\Omega)}} \right|^{p_n(x)} dx \le 1 \text{ for all } n \in \mathbb{N}.$$

The definition of the Luxemburg norm then gives

$$\left| \frac{u}{\|u\|_{L^\infty(\Omega)}} \right|_{p_n(\cdot)} \le 1 \text{ for all } n \in \mathbb{N}. \tag{5.5}$$

In view of (2.3) and (5.5) we obtain that

$$\left| \frac{u}{\|u\|_{L^\infty(\Omega)}} \right|_{p_n(\cdot)}^{p_n^-} \ge \int_\Omega \left| \frac{u(x)}{\|u\|_{L^\infty(\Omega)}} \right|^{p_n(x)} dx,$$

and hence

$$\left| \frac{u}{\|u\|_{L^\infty(\Omega)}} \right|_{p_n(\cdot)} \ge \left[\left(\int_\Omega \left| \frac{u(x)}{\|u\|_{L^\infty(\Omega)}} \right|^{p_n(x)} dx \right)^{1/p_n^+} \right]^{p_n^+/p_n^-} \quad \text{for all } n \in \mathbb{N}. \tag{5.6}$$

By Lemma 1, the sequence $\{a_n\}$ with $a_n := \left(\int_\Omega \left|\frac{u(x)}{\|u\|_{L^\infty(\Omega)}}\right|^{p_n(x)} dx\right)^{1/p_n^+}$ converges to 1 as $n \to \infty$. On the other hand, (3.2) implies that the sequence $\{p_n^+/p_n^-\}$ is bounded. Consequently,

$$\lim_{n\to\infty} a_n^{p_n^+/p_n^-} = \lim_{n\to\infty} \exp\left(\frac{p_n^+}{p_n^-}\ln(a_n)\right) = 1\,.$$

Passing to the limit as $n \to \infty$ in (5.6) gives

$$\limsup_{n\to\infty} \left|\frac{u}{\|u\|_{L^\infty(\Omega)}}\right|_{p_n(\cdot)} \geq 1\,.$$

Hence, taking into account (5.5), we deduce that (5.4) holds. □

Remark 3. *We point out that in the case when the additional assumption $|\Omega| = 1$ is removed (that is, we just assume that $|\Omega| < +\infty$) the conclusion of Lemma 2 (see (5.3)) should read*

$$\lim_{n\to\infty} \left|\frac{u}{|\Omega|^{1/p_n}}\right|_{p_n(\cdot)} = \|u\|_{L^\infty(\Omega)}\,. \tag{5.7}$$

Clearly, (5.7) implies (5.3). We also note that given an arbitrary sequence of real numbers $q_n \to \infty$, by taking $p_n(x) := q_n$ in (5.7) we again recover (5.1).

Lemma 3. *Let $\Omega \subset \mathbb{R}^N$ be an open domain with $|\Omega| < +\infty$, and let $h : \overline{\Omega} \to (1, \infty)$ be a given Lipschitz continuous function. Then*

$$\left||v|^s\right|_{\frac{h(\cdot)}{s}}^{1/s} = |v|_{h(\cdot)} \quad \text{for all } v \in L^{h(\cdot)}(\Omega) \text{ and } s \in (1, h^-). \tag{5.8}$$

Proof. We may assume that $v \neq 0$. Since $\left|\frac{v}{|v|_{h(\cdot)}}\right|_{h(\cdot)} = 1$, and taking into account (2.4), we obtain

$$\int_\Omega \left[\left(\frac{|v(x)|}{|v|_{h(\cdot)}}\right)^s\right]^{h(x)/s} dx = \int_\Omega \left(\frac{|v(x)|}{|v|_{h(\cdot)}}\right)^{h(x)} dx = 1\,.$$

Thus, invoking (2.4) again, we conclude that

$$\left|\left(\frac{|v|}{|v|_{h(\cdot)}}\right)^s\right|_{\frac{h(\cdot)}{s}} = 1,$$

which shows that (5.8) holds. □

We are now ready to prove Theorem 2.

We start by establishing the existence of a recovery subsequence for the Γ-limit. Let $u \in L^1(\Omega)$ be arbitrary. We only need to consider the case $J_\infty(u) < +\infty$. Thus, $u \in W^{1,\infty}(\Omega)$ and $J_\infty(u) = \|\lambda\nabla u\|_{L^\infty(\Omega;\mathbb{R}^N)}$. For $n \in \mathbb{N}$ define $u_n := u$. We have $u_n \in W^{1,p_n(\cdot)}(\Omega)$, and using Lemma 2 we obtain

$$\limsup_{n\to\infty} J_n(u_n) = \limsup_{n\to\infty} |\lambda\nabla u|_{p_n(\cdot)} = \|\lambda\nabla u\|_{L^\infty(\Omega;\mathbb{R}^N)}.$$

Therefore, the constant sequence $\{u_n\} = \{u\}$ is a recovery sequence for the Γ-limit.

It remains to show that for any $u \in L^1(\Omega)$ we have

$$J_\infty(u) \le \liminf_{n\to\infty} J_n(u_n), \tag{5.9}$$

whenever $\{u_n\} \subset L^1(\Omega)$ is such that $u_n \to u$ in $L^1(\Omega)$.

We may assume, without loss of generality, that $u_n \in W^{1,p_n(\cdot)}(\Omega)$, and

$$\liminf_{n\to\infty} J_n(u_n) = \lim_{n\to\infty} J_n(u_n) < \infty\,. \tag{5.10}$$

Let $q \ge 1$ be an arbitrary real number. By (3.1), $q < p_n^-$ for sufficiently large $n \in \mathbb{N}$. Using Hölder's inequality (2.1) we deduce that

$$\int_\Omega |\lambda(x)\nabla u_n(x)|^q\, dx \le \left(\frac{p_n^+ - q}{p_n^+} + \frac{q}{p_n^-}\right) |1|_{\frac{p_n(\cdot)}{p_n(\cdot)-q}}\, ||\lambda\nabla u_n|^q|_{\frac{p_n(\cdot)}{q}}\,.$$

Since $|\Omega| = 1$ we have

$$\int_\Omega 1^{\frac{p_n(x)}{p_n(x)-q}}\, dx = 1,$$

and thus, taking into account (2.4), $|1|_{\frac{p_n(\cdot)}{p_n(\cdot)-q}} = 1$. On the other hand, in view of (3.2) and Lemma 3, we obtain that

$$\begin{aligned}
\|\lambda\nabla u_n\|_{L^q(\Omega;\mathbb{R}^N)} &\le \left[1 + q\left(\frac{1}{p_n^-} - \frac{1}{p_n^+}\right)\right]^{1/q} ||\lambda\nabla u_n|^q|^{1/q}_{\frac{p_n(\cdot)}{q}} \\
&\le \left[1 + q(\beta-1)\frac{1}{p_n^+}\right]^{1/q} ||\lambda\nabla u_n|^q|^{1/q}_{\frac{p_n(\cdot)}{q}} \\
&\le \left[1 + q(\beta-1)\frac{1}{p_n^+}\right]^{1/q} |\lambda\nabla u_n|_{p_n(\cdot)} = \left[1 + q(\beta-1)\frac{1}{p_n^+}\right]^{1/q} J_n(u_n).
\end{aligned} \tag{5.11}$$

Hence, by (3.1) and (5.10), we obtain that the sequence $\{\nabla u_n\}$ is bounded in $L^q(\Omega;\mathbb{R}^N)$ for any $q \ge 1$, and since $u_n \to u$ in $L^1(\Omega)$ we deduce, after eventually extracting a subsequence (not relabelled), that $u_n \rightharpoonup u$ weakly in $W^{1,q}(\Omega)$. The weak lower semicontinuity of the norm implies that

$$\|\lambda\nabla u\|_{L^q(\Omega;\mathbb{R}^N)} \le \liminf_{n\to\infty} \|\lambda\nabla u_n\|_{L^q(\Omega;\mathbb{R}^N)}. \tag{5.12}$$

Passing to the limit as $n \to \infty$ in (5.11), we deduce that

$$\limsup_{n\to\infty} \|\lambda\nabla u_n\|_{L^q(\Omega;\mathbb{R}^N)} \le \limsup_{n\to\infty} \left\{ \left[1 + q(\beta-1)\frac{1}{p_n^+}\right]^{1/q} |\lambda\nabla u_n|_{p_n(\cdot)} \right\} = \lim_{n\to\infty} |\lambda\nabla u_n|_{p_n(\cdot)}\,, \tag{5.13}$$

where we have used (5.10). Combining (5.12) and (5.13) we find that for any $q > 1$ we have

$$\|\lambda\nabla u\|_{L^q(\Omega;\mathbb{R}^N)} \le \lim_{n\to\infty} |\lambda\nabla u_n|_{p_n(\cdot)} = \lim_{n\to\infty} J_n(u_n). \tag{5.14}$$

It can be shown, using a localization argument similar to the one used in the proof of Theorem 1, that $\nabla u \in L^\infty(\Omega;\mathbb{R}^N)$. Thus, letting $q \to \infty$ in (5.14), and taking into account (5.1), we deduce that

$$J_\infty(u) \le \lim_{n\to\infty} J_n(u_n) = \liminf_{n\to\infty} J_n(u_n).$$

Hence (5.9) holds, which concludes the proof of Theorem 2. $\square$

Before we can prove Theorem 3, we need to establish an auxiliary result which is a slight refinement of Lemma 1.

Lemma 4. *Let $\{p_n\}$ be a sequence of Lipschitz continuous functions, $p_n : \overline{\Omega} \to (1,\infty)$, such that (3.1) and (3.4) are satisfied, and let $u \in L^\infty(\Omega)$ be such that $\|u\|_{L^\infty(\Omega)} \le 1$. Then*

$$\lim_{n\to\infty} \left(\int_\Omega |u(x)|^{p_n(x)}\, dx \right)^{1/p_n^+} = \|u\|_{L^\infty(\Omega)}\,.$$

Proof. Since we have

$$|u(x)| \le \|u\|_{L^\infty(\Omega)} \le 1 \quad \text{for a.e. } x \in \Omega,$$

and using the fact that $p_n^- \le p_n(x) \le p_n^+$ for all $x \in \Omega$ and all $n \in \mathbb{N}$, we obtain

$$\left(\int_\Omega |u(x)|^{p_n^+}\, dx \right)^{1/p_n^+} \le \left(\int_\Omega |u(x)|^{p_n(x)}\, dx \right)^{1/p_n^+} \le \left[\left(\int_\Omega |u(x)|^{p_n^-}\, dx \right)^{1/p_n^-} \right]^{p_n^-/p_n^+}.$$

Taking into account (5.1) and (3.4), the conclusion of Lemma 4 follows by passing to the limit as $n \to \infty$ in the above estimates. $\square$

Proof of Theorem 3. The proof follows along the lines of the proof of Theorem 2. We will only point out the main differences.

Let $u \in L^1(\Omega)$ be arbitrary. To prove the existence of a recovery sequence, we only need to consider the nontrivial case: $J_\infty(u) < \infty$. Thus, $u \in W^{1,\infty}(\Omega)$ and $J_\infty(u) = \|\lambda\nabla u\|_{L^\infty(\Omega;\mathbb{R}^N)}$. Define $u_n \in W^{1,p_n(\cdot)}(\Omega)$ by $u_n := u$ for $n \in \mathbb{N}$. We have the following alternative: either $\|\lambda\nabla u\|_{L^\infty(\Omega;\mathbb{R}^N)} \le 1$ or $\|\lambda\nabla u\|_{L^\infty(\Omega;\mathbb{R}^N)} > 1$. In the first case, applying (5.5) to $\lambda|\nabla u| \in L^\infty(\Omega)$ gives $|\lambda\nabla u|_{p_n(\cdot)} \le 1$, which implies that

$$K_n(u_n) = K_n(u) = \left(\int_\Omega |\lambda(x)\nabla u(x)|^{p_n(x)}\, dx \right)^{1/p_n^+}.$$

On the other hand, if $\|\lambda\nabla u\|_{L^\infty(\Omega;\mathbb{R}^N)} > 1$, we have, by Lemma 2, that $|\lambda\nabla u|_{p_n(\cdot)} > 1$ for $n \in \mathbb{N}$ sufficiently large. Thus, in this case we have

$$K_n(u_n) = K_n(u) = |\lambda\nabla u|_{p_n(\cdot)}.$$

In view of Lemmas 2 and 4 we conclude that we have, overall,

$$\lim_{n\to\infty} K_n(u_n) = J_\infty(u),$$

and hence $\{u_n\} = \{u\}$ is again a recovery sequence for the Γ-limit.

It remains to prove that

$$J_\infty(u) \leq \liminf_{n\to\infty} K_n(u_n) \tag{5.15}$$

whenever $\{u_n\} \subset L^1(\Omega)$ and $u \in L^1(\Omega)$ are such that $u_n \to u$ in $L^1(\Omega)$. We may assume, without loss of generality that $u_n \in W^{1,p_n(\cdot)}(\Omega)$, and

$$\liminf_{n\to\infty} K_n(u_n) = \lim_{n\to\infty} K_n(u_n) < \infty.$$

Let $q \geq 1$ be an arbitrary real number. By (3.1), $q < p_n^- \leq p_n^+$ for sufficiently large $n \in \mathbb{N}$. For $n \in \mathbb{N}$ we have either $|\lambda \nabla u_n|_{p_n(\cdot)} \leq 1$ or $|\lambda \nabla u_n|_{p_n(\cdot)} > 1$. Revisiting the proof of Theorem 2 we recall that, in either case, (5.11) gives

$$\|\lambda \nabla u_n\|_{L^q(\Omega;\mathbb{R}^N)} \leq \left[1 + q(\beta - 1)\frac{1}{p_n^+}\right]^{1/q} |\lambda \nabla u_n|_{p_n(\cdot)}\,, \tag{5.16}$$

while when $|\lambda \nabla u_n|_{p_n(\cdot)} \leq 1$, we obtain, taking into account (2.3),

$$\|\lambda \nabla u_n\|_{L^q(\Omega;\mathbb{R}^N)} \quad \leq \quad \left[1 + q(\beta - 1)\frac{1}{p_n^+}\right]^{1/q} \left(\int_\Omega |\lambda(x)\nabla u_n(x)|^{p_n(x)}\, dx\right)^{1/p_n^+}. \tag{5.17}$$

Thus, by (5.16) and (5.17), we have that

$$\|\lambda \nabla u_n\|_{L^q(\Omega;\mathbb{R}^N)} \leq \left[1 + q(\beta - 1)\frac{1}{p_n^+}\right]^{1/q} K_n(u_n).$$

We deduce that the sequence $\{\nabla u_n\}$ is bounded in $L^q(\Omega;\mathbb{R}^N)$ for any $q \geq 1$. To conclude that (5.15) holds, one may now proceed along the lines of the last part of the proof of Theorem 2. □

6 A Γ-convergence result in the particular case $p_n(x) = np(x)$

Motivated by the discussion in the Introduction, in this section we state and prove a Γ-convergence result in the particular case where the exponents p_n have the form $p_n(x) := np(x)$, for $x \in \Omega$ and $n \in \mathbb{N}$. Here, Ω is an open, bounded domain in $\mathbb{R}^N$ with smooth boundary, and $p : \overline{\Omega} \to (1, \infty)$ is a given function of class $C^1(\Omega) \cap W^{1,\infty}(\Omega)$. We note that the sequence $\{np(\cdot)\}$ satisfies our hypotheses (3.1) and (3.2). However, the more stringent condition (3.4) assumed in Theorem 3 is not satisfied, except for the case where p is a constant.

Consider the sequence $\{S_n\}$ of functionals $S_n : L^1(\Omega) \to [0, +\infty]$ given by

$$S_n(u) := \begin{cases} \left(\displaystyle\int_\Omega \frac{1}{np(x)} |\nabla u(x)|^{np(x)}\, dx\right)^{1/n} & \text{if } u \in W^{1,p_n(\cdot)}(\Omega) \\ +\infty & \text{otherwise}\,, \end{cases}$$

Theorem 4. $\Gamma(L^1(\Omega)) - \lim_{n\to\infty} S_n = S_\infty$, *where* $S_\infty : L^1(\Omega) \to [0, +\infty]$ *is defined by*

$$S_\infty(u) := \begin{cases} || \, |\nabla u|^{p(\cdot)}||_{L^\infty(\Omega)} & \text{if } u \in W^{1,\infty}(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{6.1}$$

Proof. The existence of a (constant) recovery sequence for the Γ-limit follows along the same lines as in the previous proofs. The crucial fact in this case is that for $u \in W^{1,\infty}(\Omega)$ we have

$$\lim_{n\to\infty} \left(\int_\Omega \frac{1}{np(x)} |\nabla u(x)|^{np(x)} \, dx \right)^{1/n} = \lim_{n\to\infty} \left(\int_\Omega \left(|\nabla u(x)|^{p(x)} \right)^n dx \right)^{1/n} = || \, |\nabla u|^{p(\cdot)}||_{L^\infty(\Omega)}.$$

To complete the proof, we need to show that if $\{u_n\} \subset L^1(\Omega)$ and $u \in L^1(\Omega)$ are such that $u_n \to u$ in $L^1(\Omega)$, then

$$S_\infty(u) \le \liminf_{n\to\infty} S_n(u_n). \tag{6.2}$$

We may assume, without loss of generality, that $u_n \in W^{1,np(\cdot)}(\Omega)$ (which, in particular, implies that $|\nabla u_n|^{p(\cdot)} \in L^n(\Omega)$), and that

$$\liminf_{n\to\infty} S_n(u_n) = \lim_{n\to\infty} S_n(u_n) < \infty. \tag{6.3}$$

Let $q \ge 1$ be arbitrary. For $n \in \mathbb{N}$ sufficiently large, we have

$$\begin{aligned} \int_\Omega |\nabla u_n(x)|^{qp^+} dx &\le |\Omega| + \int_\Omega |\nabla u_n(x)|^{\frac{p(x)qp^+}{p^-}} dx \le |\Omega| + \left(\int_\Omega |\nabla u_n(x)|^{np(x)} dx \right)^{\frac{qp^+}{np^-}} |\Omega|^{1-\frac{qp^+}{np^-}} \\ &\le |\Omega| + \left(np^+\right)^{\frac{qp^+}{np^-}} S_n(u_n)^{\frac{qp^+}{p^-}} |\Omega|^{1-\frac{qp^+}{np^-}}, \end{aligned} \tag{6.4}$$

where we have used Hölder's inequality. Thus, $\{\nabla u_n\}$ is bounded in $L^{qp^+}(\Omega; \mathbb{R}^N)$. Since $u_n \to u$ in $L^1(\Omega)$ we deduce by Poincaré-Wirtinger's inequality that $\{u_n\}$ is bounded in $L^{qp^+}(\Omega; \mathbb{R}^N)$, and thus $\{u_n\}$ is bounded in $W^{1,qp^+}(\Omega)$. It follows that we can extract a subsequence (not relabelled) such that $u_n \rightharpoonup u$ weakly in $W^{1,qp^+}(\Omega)$. Since $p(x) \le p^+$ for any $x \in \Omega$, $W^{1,qp^+}(\Omega)$ is continuously embedded in $W^{1,qp(\cdot)}(\Omega)$, and we deduce that $u_n \rightharpoonup u$ weakly in $W^{1,qp(\cdot)}(\Omega)$. Then [21, Lemma 3.4] yields

$$\int_\Omega |\nabla u(x)|^{qp(x)} dx \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n(x)|^{qp(x)} dx. \tag{6.5}$$

An alternative argument for (6.5) is as follows: Let $f : \Omega \times \mathbb{R}^N \to [0, +\infty)$ be defined by $f(x, v) := |v|^{qp(x)}$. Note that f is continuous, it satisfies the growth condition $0 \le f(x, v) \le C(1+|v|^{qp^+}), \forall\, (x, v) \in \Omega \times \mathbb{R}^N$, and that $f(x, \cdot)$ is convex for all $x \in \Omega$. Since $u_n \rightharpoonup u$ weakly in $W^{1,qp^+}(\Omega)$, (6.5) now follows from well-known weak lower semicontinuity results for functionals of the form $I(u) := \int_\Omega f(x, \nabla u(x)) dx$ (see, e.g., [5]).

Applying again Hölder's inequality, we find

$$\int_\Omega |\nabla u_n|^{qp(x)} dx \le \left(\int_\Omega |\nabla u_n|^{np(x)} dx \right)^{\frac{q}{n}} |\Omega|^{1-\frac{q}{n}} \le (np^+)^{\frac{q}{n}} S_n(u_n)^q |\Omega|^{1-\frac{q}{n}}.$$

Thus, taking into account (6.3),

$$\limsup_{n\to\infty}\left(\int_\Omega |\nabla u_n|^{qp(x)}dx\right)^{\frac{1}{q}} \leq |\Omega|^{\frac{1}{q}} \liminf_{n\to\infty} S_n(u_n). \tag{6.6}$$

Finally, using the fact that

$$\left(\liminf_{n\to\infty}\int_\Omega |\nabla u_n|^{qp(x)}dx\right)^{\frac{1}{q}} \leq \limsup_{n\to\infty}\left(\int_\Omega |\nabla u_n|^{qp(x)}dx\right)^{\frac{1}{q}},$$

and in view of (6.5) and (6.6), we obtain that

$$\||\nabla u|^{p(\cdot)}\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{q}} \liminf_{n\to\infty} S_n(u_n).$$

Letting $q \to \infty$, we deduce that (6.2) holds. $\square$

Acknowledgements. The research of M. Bocea was partially supported by the U.S. National Science Foundation under Grant No. DMS-0806789. M. Mihăilescu has been partially supported by Grant CNCSIS 79/2007 "Degenerate and Singular Nonlinear Processes".

References

[1] Aronsson, G. 1967 Extensions of functions satisfying Lipschitz conditions. *Ark. Mat.* **6**, 551-561.

[2] Aronsson, G. 1968 On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$. *Ark. Mat.* **7**, 395-425.

[3] Aronsson, G., Crandall, M. G. & Juutinen, P. 2004 A tour of the theory of absolutely minimizing functions. *Bulletin (New Series) of the AMS* **41** No. 4, 439-505.

[4] Bocea, M. & Nesi, V. 2008 Γ-convergence of power-law functionals, variational principles in L^∞, and applications. *SIAM J. Math. Anal.* **39**, 1550-1576.

[5] Dacorogna, B. 1989 *Direct methods in the calculus of variations. Applied mathematical sciences,* vol. 78. Berlin, Germany: Springer.

[6] Dal Maso, G. 1993 *An introduction to* Γ*-convergence. Progress in nonlinear differential equations and their applications,* vol. 8. Boston, MA: Birkäuser.

[7] De Giorgi, E. 1975 Sulla convergenza di alcune succesioni di integrali del tipo dell'area. *Rend. Mat.* **8**, 277-294.

[8] De Giorgi, E. & Franzoni, T. 1975 Su un tipo di convergenza variazionale. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **58**, 842-850.

[9] Diening, L. 2002 *Theoretical and numerical results for electrorheological fluids*, Ph.D. thesis, University of Frieburg, Germany.

[10] Edmunds, D. E., Lang, J. & Nekvinda, A. 1999 On $L^{p(x)}$ norms. *Proc. Roy. Soc. London Ser. A* **455**, 219-225.

[11] Edmunds, D. E. & Rákosník, J. 1992 Density of smooth functions in $W^{k,p(x)}(\Omega)$. *Proc. Roy. Soc. London Ser. A* **437**, 229-236.

[12] Edmunds, D. E. & Rákosník, J. 2000 Sobolev embeddings with variable exponent. *Studia Math.* **143**, 267-293.

[13] Garroni, A., Nesi, V. & Ponsiglione, M. 2001 Dielectric breakdown: optimal bounds. *Proc. R. Soc. Lond. A* **457**, 2317-2335.

[14] Garroni, A. & Kohn, R. V. 2003 Some three-dimensional problems related to dielectric breakdown and polycrystal plasticity. *Proc. Royal Soc. London A* **459**, 2613-2625.

[15] Goldsztein, G. H. 2001 Rigid perfectly plastic two-dimensional polycrystals. *Proc. Royal Soc. London A* **457**, 2789-2798.

[16] Goldsztein, G. H. 2003 Two-dimensional rigid polycrystals whose grains have one ductile direction. *Proc. Royal Soc. London A* **459**, 1949-1968.

[17] Kohn, R. V. & Little, T. D. 1999 Some model problems of polycrystal plasticity with deficient basic crystals. *SIAM J. Appl. Math.* **59**, 172-197.

[18] Kováčik, O. & Rákosník, J. 1991 On spaces $L^{p(x)}$ and $W^{1,p(x)}$. *Czechoslovak Math. J.* **41**, 592-618.

[19] Lindqvist, P. & Lukkari, T. 2009 Preprint. A curious equation involving the ∞-Laplacian. *arXiv:0902.1771 - arxiv.org*

[20] Manfredi, J. J., Rossi, J. D. & Urbano, J. M. 2009 In press. Limits as $p(x) \to \infty$ of $p(x)$-harmonic functions. *Nonlinear Analysis, T.M.A.*

[21] Mihăilescu, M. & Rădulescu, V. 2006 A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids. *Proc. R. Soc. Lond. A* **462**, 2625-2641.

[22] Musielak, J. 1983 *Orlicz spaces and modular spaces. Lecture notes in mathematics,* vol. 1034, Berlin, Germany: Springer.

[23] Rajagopal, K. R. & Ruzicka, M. 2001 Mathematical modelling of electrorheological fluids. *Cont. Mech. Term.* **13**, 59-78.

[24] Ruzicka, M. 2002 *Electrorheological fluids: modeling and mathematical theory*, Berlin, Germany: Springer.

[25] Sachs, G. 1928 Zur Albeitung einer Fleissbedingun. *Z. Ver. Dtsch. Ing.* **72**, 734-736.

[26] Samko, S. & Vakulov, B. 2005 Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators. *J. Math. Anal. Appl.* **310**, 229-246.

[27] Taylor, G. 1938 Plastic strains in metals. *J. Inst. Metals* **62**, 307-324.

[28] Zhikov, V. 1986 Averaging of functionals of the calculus of variations and elasticity theory. *Izv. Akad. Nauk SSSR Ser. Mat.* **50**(4), 675-710, 877.