Linkedness and Path-Pairability in the Cartesian Product of Graphs

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Contents

List of Figures

Abstract

In this dissertation I summarize my work in the field of linkedness and path-pairability of graphs with primary focus on the inheritance of the mentioned properties in the Cartesian product of graphs. We obtain a general additive inheritance bound for linkedness. We determine the exact linkedness number of hypercubes, as well as affine and projective grids of arbitrary dimensions. Similar inheritance of the path-pairability property is investigated. We show that unlike in the case of linkedness, a multiplicative lower bound can be achieved for the inharitance of path-pairability. Further results regarding maximum degree and maximum diameter conditions of path-pairable graphs are presented. In all these topics I have published, accepted or submitted papers in various mathematical journals.

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1. Introduction

In this dissertation we investigate linkedness and path-pairability properties of graphs. The concept of linkedness naturally arose with and is strongly connected to the study of communicational networks. In this area I have the following 3 published, accepted or submitted papers: [20], [21], and [22]. The paper [22] is already published, [20] is accepted for publications, [21] is submitted and these 3 papers are not contained in any PhD dissertation different from the present thesis.

The classical formulation of linkedness type problems is the following: we represent a communicational network by an undirected graph without loops or multiple edges. Users of the network corresponding to certain vertices wish to communicate with each other. In order to guarantee secure and undisturbed communication we have to establish individual channels for each of the communicating parties. We may require various different properties of the channels to be assigned to the pairs, creating countless variants of the base problem and raising several interesting questions. In this dissertation, we focus on two main variants, the vertex-disjoint and the edge-disjoint path problems.

Definition 1. *An undirected graph G is k-linked if, for every ordered set of $2k$ vertices $S = (s_1, \dots, s_k)$ and $T = (t_1, \dots, t_k)$, there exist internally vertex-disjoint paths $P_1, \dots, P_k$ such that each P_i is an $s_i t_i$-path.*

Definition 2. *An undirected graph G is weakly-k-linked if, for every ordered set of $2k$ vertices $S = (s_1, \dots, s_k)$ and $T = (t_1, \dots, t_k)$, there exist edge-disjoint paths $P_1, \dots, P_k$ such that each P_i is an $s_i t_i$-path.*

We always assume for obvious reasons that s_i and t_i do not share the same vertex of the graph.

Both concepts are natural strengthenings of well known connectivity-related properties (see Corollary 7). The main field of study in the past few decades has been the connection between connectivity and linkedness properties of graphs. While it follows easily that linkedness requires sufficiently high connectivity, it was not clear if high enough connectivity can force a graph to be k-linked for arbitrary values of k. The question was answered in 1974 by Larman and Mani [17]. They proved the existence of a function $f(k) : \mathbb{Z} \to \mathbb{Z}$ with the property that every $f(k)$-connected graph is k-linked. The same result was proved independently by Jung [11] and both proofs are based on an earlier theorem of Mader [19].

Theorem 1 (Larman, Mani [17], Jung [11]). *If an undirected graph G is $2^{\binom{3k}{2}}$ connected, then it is k-linked.*

The above theorem provides an exponential upper bound on $f(k)$ in terms of k. Since then, several techniques concerning dense subgraphs and minors have been invented, and utilized with success to push down the upper bound. The first polynomial bound ($O(n\sqrt{\log n})$) was proved by Robertson and Seymour [27]. Bollobás and Thomasen [1] gave the first linear upper bound ($22k$). To date, the best known result ($10k$) is due to Thomas and Wollen [32]. Note that, since calculating linkedness of a graph is known to be an NP-complete problem (Karp, [12]), while calculating connectivity is clearly in P, one can hardly assume an if-and-only-if connection between linkedness and connectivity. Recent results [14] give evidence that further parameters (such as girth) play important roles in shaping that relation.

Meanwhile, important progress was made in the study of weakly linked graphs and their relation to edge-connectivity. Tutte's famous result on the existence of edge-disjoint spanning trees showed that $2k$-edge-connected graphs are weakly k-linked. Huck gave an almost sharp upper bound proving that $(k+2)$-edge-connected graphs are weakly k-linked. As weakly-k-linked graphs are necessarily k-edge-connected, the gap between the lower and upper bounds is almost completely closed. See a more detailed description in Chapter 3.

Another variant of linkedness called *path-pairability* was introduced by Csaba, Faudree, Gyárfás and Lehel [3]. The motivation is to establish edge-disjoint channels in a network given any pairing of its vertices. In other words, path-pairability is a variant of weak-linkedness where every vertex of the communicational network is a user, initiating a conversation with a given partner (conference calls involving more than two participants are not studied in this dissertation). Unlike the case of linkedness or weak-linkedness, path-pairability is not closely related to vertex or edge-connectivity. Connections with other graphs parameters such as maximum degree or diameter have been partially studied, leaving still plenty of room for further research.

This dissertation primarily focuses on the study of linkedness and path-pairability in the Cartesian products of graphs. Graph products have been extensively studied in the last century from various points of views, raising an abundance of intriquing problems. In graph theory, a central question concerning product graphs is the inheritance of the different graph parameters. The main results presented in the dissertation investigate the inheritance of linkendess and path-pairability in the Cartesian product of graphs.

We discuss the above mentioned topics in the following order.

- Section 1 will be an introduction.
- Section 2 will establish notation and terminology. It will also present well known connectivity-related results for later utilization.
- Section 3 will survey the development of research in the field of linkedness and highlight its important milestones. Related concepts such as orderedness or generalized linkedness will be parts of that discussion.
- Section 4 will give a detailed introduction to path-pairability. Substantial results of the dissertation on diameter bounds of path-pairable graphs are presented.
- Section 5 will introduce the Cartesian product of graphs and give insight to its basic structural properties. The section will present the main results of the dissertation, concerning inheritance of linkedness and path-pairability in the Cartesian product of graphs.
- Section 6 will discuss additional questions and remarks.

2. Notation, terminology, and folklore results

We set notation and terminology and recall some straightforward corollaries of Menger's Theorems about certain connectivity properties of graphs. The vertex and edge sets of an undirected graph G are denoted by $V(G)$ and $E(G)$. As usual, $d(x)$ denotes the degree of the vertex $x \in V(G)$ and $\Gamma(x)$ (or $\Gamma_G(x)$) denotes the set of neighbours of x. For a subset $H \subset V(G)$, the size of H is denoted by $|H|$, while $d(H)$ denotes the number of edges between H and its complement $G \backslash H$ (sometimes denoted by $G - H$). We also use the notation $v(G) = |V(G)|$ and $e(G) = |E(G)|$. We use $\delta(G), \Delta(G)$ and $\underline{d}(G)$ (also δ, Δ and $\underline{d}$, unless misleading) for the minimum degree, maximum degree and average degree of a graph G, respectively. Paths and cycles of n vertices are denoted by P_n and C_n. An xy path is a path joining x to y, while an $x_1, x_2, \ldots, x_n$ path is the concatenation of appropriate $x_i x_{i+1}$ paths. The diameter of a graph G is denoted by $d(G)$. This notation might be at first glance confusing for the reader. Note that $d(G)$ in the previous translation (number of edges between $V(G)$ and $\emptyset$) makes no sense. Subgraphs of a given graph G are not investigated in this dissertation, hence $d(H)$ for $H \subset G$ is unambiguous. Further notation related to the Cartesian product of graphs will be introduced in a later chapter.

We recall the definitions of the most common connectivity concepts and some well known results concerning them.

Definition 3. *A simple undirected graph G is k-connected (or k-vertex-connected), if the removal of any vertex set of size at most $k-1$ does not result in a disconnected graph or a graph of a single vertex. By definition, k-connected graphs consist of at least $k+1$ vertices and have minimum degree $\delta \geq k$.*

Definition 4. *A simple undirected graph G is k-edge-connected, if the removal of any edge set of size at most $k-1$ does not result in a disconnected graph. By definition, k-edge-connected graphs have minimum degree $\delta \geq k$.*

Theorem 2 (Menger)**.** *Let $G = (V, E)$ be a simple undirected graph and $A, B \subset V$.*

(1) *The minimum number of vertices separating A from B in G is equal to the maximum number of vertex-disjoint AB paths in G.*
(2) *The minimum number of edges separating A from B in G is equal to the maximum number of edge-disjoint AB paths in G.*

Corollary 5. *Let $G = (V, E)$ be a simple undirected graph.*

(1) *If G is k-connected, then for every pair of vertices $x, y \in V(G)$, there exist $P_1, \ldots, P_k$ internally disjoint xy paths joining the two vertices.*

(2) *If G is k-edge-connected, then for every pair of vertices $x, y \in V(G)$, there exist $P_1, \ldots, P_k$ edge-disjoint xy paths joining the two vertices.*

Corollary 6. *Let $G = (V, E)$ be a simple undirected graph.*

(1) *If G is k-connected, then for every $\{x_1, \ldots, x_k, y\} \in V(G)$, there exist $P_1, \ldots, P_k$ internally disjoint $x_i y$ paths joining y and the x_i's.*

(2) *If G is k-edge-connected, then for every $\{x_1, \ldots, x_k, y\} \in V(G)$, there exist $P_1, \ldots, P_k$ edge-disjoint $x_i y$ paths joining y and the x_i's.*

Corollary 7. *Let $G = (V, E)$ be a simple undirected graph.*

(1) *If G is k-connected, then for every $\{x_1, \ldots, x_k\} \in V(G)$ and $\{y_1, \ldots, y_k\} \in V(G)$, there exist $P_1, \ldots, P_k$ internally disjoint $x_i y_{\pi(i)}$ paths, joining every x_i to $y_{\pi(i)}$ for some permutation $\pi \in S_k$.*

(2) *If G is k-connected, then for every $\{x_1, \ldots, x_k\} \in V(G)$ and $\{y_1, \ldots, y_k\} \in V(G)$, there exist $P_1, \ldots, P_k$ edge-disjoint $x_i y_{\pi(i)}$ paths, joining every x_i to $y_{\pi(i)}$ for some permutation $\pi \in S_k$.*

We recall two further properties of k-connected graphs for later utilization.

Proposition 8. *Let $G = (V, E)$ be a simple undirected graph. If G is k-connected, then for every $\{x_1, \ldots, x_k\} \in V(G)$, there exist a cycle C containing all the x_i's.*

Proposition 9. *Let $G = (V, E)$ be a simple undirected graph. If G is k-connected, then for every $\{x, y_1, \ldots, y_{k-1}, z\} \in V(G)$, there exist a path P from x to z containing (in some order) all the $y_i' s$.*

3. Linkedness

One can easily construct k-linked graphs with arbitrary value of k. We are primarily interested in edge-minimal examples. Not surprisingly, the question of k-linkedness eventually boils down to and shows strong connection with connectivity.

We start with a technical remark. Linkedness of graphs is often defined by the following alternative way:

Definition 10. *An undirected graph G is k-linked if for every ordered set of $2k$* **pairwise disjoint** *vertices $S = (s_1, \ldots, s_k)$ and $T = (t_1, \ldots, t_k)$, there exist vertex-disjoint paths $P_1, \ldots, P_k$, such that each P_i is an $s_i t_i$-path.*

While this new definition might seem a weakening of the original concept, it can be proved easily that the two definitions are equivalent and describe the same graph property. We prove the equivalence and will use Definition 10 throughout this dissertation (unless stated otherwise).

Proposition 11. *Definition 1 and Definition 10 are equivalent.*

Proof. There is only one direction of the equivalence to prove, namely that Definition 10 implies Definition 1. Let us assume that G is k-linked in the weaker sense, and let $S = (s_1, \dots, s_k)$ and $T = (t_1, \dots, t_k)$ denote terminals placed on the vertices of G, such that the number of x, y terminals sharing the underlying vertex is exactly c. We prove by induction on c that G is k-linked in the stronger sense, if the graph has minimum degree $\delta \geq 2k-1$. Our statement is straighforward for $c = 0$. Now let us assume that $c \geq 1$, and let x be a vertex consisting of at least 2 terminals. As $\Gamma_G(x) \geq 2k-1$ and $(G-x)$ contains at most $2k-2$ terminals, there exist a vertex $y \in \Gamma_G(x)$ that contains no terminal. Let us relocate one of the terminals at x to y. The number of terminals sharing a common vertex in the new arrangement is definitely smaller than c, thus by induction hypothesis, the pairs can be joined by disjoint paths. Easy to see that none of these paths use edge $\overline{xy}$, hence our path system can be extended by that edge. This new system is a solution to the original assignment of the terminals. $\square$

It remains to show that $\delta(G) \geq 2k-1$ holds for a k-linked graph G. We, in fact, prove that k-linked graphs are $(2k-1)$-connected.

3.1. **Linkedness and connectivity.** Easy to see that k-linked graphs are $(2k-1)$-connected. Indeed, in a less than $(2k-1)$-connected graph G, place terminals $s_1, t_1, \dots, s_{k-1}, t_{k-1}$ in a vertex cut D of size at most $2k-2$. That makes it impossible to join another pair of terminals s_k and t_k, if they are located in different components of $G-D$. It is also fairly easy to construct k-linked graphs that are $(2k-1)$-connected, but not $2k$-connected: the complete bipartite graph $K_{2k-1,N}$ is a possible example for $N \geq 2k-1$. The main question of the last decades in the study of linkedness has been the exact relation in the converse direction. It was conjectured that sufficiently high connectivity implies linkedness. Observe first that in the special case, when the graph G contains a subgraph H isomorphic to K_{2k}, $2k$-connectivity is indeed sufficient to imply k-linkedness. Corollary 5 provides a useful tool to find for every terminal u a $P_{uu'}$ path with endpoint u' in H, such that none of these paths share a vertex. The joining of the (u_i, v_i) pairs of terminals can be finished by using the $u_i'v_i'$ edges of H. A similar technique can be utilized in the presence of a subgraph isomorphic to $K_{3k} - 2k \cdot K_2$ by building uu' paths the same way. Choose any $2k$ of the $3k$ vertices, stop the paths once they enter the subgraph at any vertex, and finish the linking via paths of length at most 2.

Unfortunately, the presence of large cliques cannot be guaranteed by means of sufficiently high connectivity. The situation, however, changes drastically, once we switch our focus from clicques to large complete topological subgraphs. As already mentioned in the introduction, Mader was the first to prove results in that direction.

Theorem 3 (Mader [19]). *If G is $2^{\binom{3k}{2}}$-connected, then it contains a K_{3k} topological subgraph.*

The graph G is said to contain the graph H as a *topological subgraph* or *subdivision*, if H can be obtained from some subgraph of G, by contracting paths to edges. One of the most known result involving topological subgraphs is Kuratov's theorem that gives a characterization of planar graphs. The theorem says that a graph is planar, if and only if it does not contain K_5 or $K_{3,3}$ as topological subgraphs. Larman and Mani (also, independently Jung) [11]) used Theorem 3 to prove that sufficiently high connectivity forces a graph to be k-linked.

Theorem 4 (Larman, Mani [17], Jung [11]). *If G is $2k$-connected and contains K_{3k} as a topological subgraph, then G is k-linked.*

Corollary 12. *If G is $2^{\binom{3k}{2}}$-connected, then G is k-linked.*

The main idea of the proof of Theorem 4 is based on the previously mentioned path building technique. We build for every terminal u a $P_{uu'}$ path ending at vertex u' in the topological subgraph. Appropriate choice of the paths guarantees that every one of them enters the topological subraphs at a path between two vertices. Moreover, the paths can be positioned in a way, such that the above presented joining can be achieved, just as in our earlier examples, where a subgraph $K_{3k} - 2k \cdot K_2$ was present. For the exact details of the proof we refer the reader to [17].

Theorem 4 verifies the existence of a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$, such that every $f(k)$-connected graph is k-linked. In fact, it follows from the theorem that $f(k) \leq 2^{\binom{3k}{2}}$. An important idea that led to the improvement of the upper bound on $f(k)$, was changing our focus from topological subgraphs to large minors. The undirected graph G is said to contain the graph H as a minor, if H can be obtained from some subgraph of G, by contracting connected subgraphs to vertices. Note that, if G contains H as a subdivision, it also contains H as a minor. The converse implication is not necessarily true. For example, the Petersen graph contains the graph K_5 as a minor, but it cannot contain a subdivision of K_5 as it has no vertex of degree 5 or more. Kuratowsky's theorem and non-planarity of the Petersen graph are often misinterpreted by mixing up minors with subdivisions (observe that the Petersen graph is not planar as it contains a subdivision of $K_{3,3}$).

Thomason [33] placed upper bounds on the the average degree of a graph, such that it guarantees the existence of a complete minor of a given size. Applying their results, Robertson and Seymour [27] bettered the bound of Jung, Larman and Mani.

Theorem 5 (Robertson and Seymour, [27]). *$f(k) \leq c \cdot k\sqrt{logk}$.*

Two years later, Bollobás and Thomasen gave the first linear upper bound on $f(k)$. Their proof is based on the existence of sufficiently dense, but not necessarily complete minors.

Theorem 6 (Bollobás, Thomasen [1]). *If G is $22k$-connected then it is k-linked.*

The upper bound was improved several times in the last decade by Kawarabayashi, Kostochka, and Yu [13] and by Thomas and Wollen [32].

Theorem 7 (Kawarabayashi, Kostochka, Yu [13]). *$f(k) \leq 16k$.*

Theorem 8 (Thomas, Wollen [32]). *$f(k) \leq 10k$.*

To date, this is best known upper bound for $f(k)$ in the general case.

Although in this dissertation we study finite graphs exclusively, we mention an intriquing result of Mader [18], which disproves the above discussed upper bounds for infinite graphs. Mader gave examples of infinite graphs with arbitrarily high (finite) connectivity, such that the mentioned graphs are not even 2-linked. On the other hand, Thomassen [34] proved that every uncountable $2k$-connected graph is k-linked.

3.2. **Special graph classes.**

3.2.1. *Graphs with large girth.* If girth conditions are placed on the graph, stronger relation can be proved between linkedness and connectivity. In 1991, Mader [19] proved the following theorem.

Theorem 9 (Mader [19])**.** *Every $2k$-connected graph with sufficiently large girth is k-linked. Furthermore, the condition on the connectivity is sharp. There exist $(2k-1)$-connected graphs with arbitrarily large girth that are not k-linked.*

In 2004, Kawarabayashi [14] gave an exact upper bound on the girth sufficient to imply k-linkedness for $2k$-connected graphs.

Theorem 10 (Kawarabayashi [14])**.** *Every $2k$-connected graph with girth at least 11 is k-linked, if k is not 4 or 5. If k is 4 or 5, girth 19 suffices.*

While Kawarabayashi's result may strengthen our belief that the ratio between linkedness and connectivity is roughly $\frac{1}{2}$ for most graphs, we mention that there exist "almost" $3k$-connected graphs that are not k-linked. Consider the graph G described as follows: take the complete graph on $3k-1$ vertices and delete a matching of size k, that is, $G = K_{3k-1} - k \cdot K_2$. It can be easily verified that G is $(3k-2)$-connected. Choose the k pairs of terminals in accordance with the deleted edges. As none of the terminals can be joined to its pair by a direct edge, every joining path must consist of at least three vertices, that is, the union of the presumed paths contains at least $3k$ vertices, which is clearly not possible.

A very fastidious reader might consider our previous contruction simply too small as our impossibility argument is based on the lack of space in the graph. It is indeed special in the sense that the targeted linkedness is comparable with the size of the graph, that is, $link(G) = O(v(G))$. For a more general example, take the complete $3t$-partite graph of equal class sizes of $2a$, $a, t \in \mathbb{Z}^+$. Our graph is clearly $(3t-1)a$-connected. Observe that it is at most $\frac{1}{3} \cdot (3t \cdot 2a)$-linked, as paths joining terminals that belong to the same class must include an external vertex (thus every joining path contains at least 3 vertices). The linkedness-connectivity ratio of the examined graph is $\frac{\frac{1}{3}\cdot(3t\cdot 2a)}{(3t-1)\cdot 2a} = \frac{1}{3-\frac{1}{t}}$ that tends to $\frac{1}{3}$ as $t \to \infty$.

This observation supports our belief that the linkedness-connectivity ratio behaves differently in case of small girth, where the best known general result is Theorem 8.

Apparently, the $\frac{1}{2}$ linkedness-connectivity ratio holds for certain graphs with small girth as well. Complete graphs, as well as complete bipartite graphs are illustrative examples of that kind. In Section 5, we prove that affine and projective grids of any dimension but 3 share this property. It would be interesting to see, what other conditions can guarantee similar ratio in the small girth case.

3.2.2. *2-Linked and 3-Linked Graphs.* For specified (small) values of k, even stronger connections have been explored. An early result of Jung [11] eliminates the gap presented in the previous section for $k = 2$.

Theorem 11 (Jung [11]). *If G is* 6*-connected, then it is 2-linked.*

In order to show that Jung's result is sharp, consider a family of 5-connected planar graphs with at least one non-triangular face (see Figure 3.2.2). Obviously, planar graphs with a face contaning 4 or more vertices cannot be 2-linked. Indeed, placing (u, v) and (u', v') pairs of terminals on a nontriangular face (we may assume without loss of generality that this particular face is the outer face of the graph) in the cyclic order $uu'vv'$, any P and P' paths joining the terminals will share a vertex. For non-planar graphs, milder conditions are sufficient.

Theorem 12 (Jung [11]). *If G is a* 4*-connected non-planar graph, then G is 2-linked.*

The classification of 2-linked graphs was completed in 1980 by Seymour [29], Shiloach [30] and Thomassen [34]. The full description of 3-connected but not 4-connected, 2-linked graphs, however, requires a rather lenghty and technical discussion that might be less interesting for the reader, hence we omit the details. Several result regarding 3-linked graphs were proved recently by Chen, Gould, Kawarabayashi, Pfender, and Wei and also by Thomas and Wollen. We list a few of these results.

Theorem 13 (Chen, Gould, Kawarabayashi, Pfender, Wei [2]). *Every 6-connected graph with $\delta(G) \geq 18$ is 3-linked.*

Theorem 14 (Thomas, Wollen [32]). *Every 6-connected graph on n vertices with $5n - 14$ edges is 3-linked.*

3.2.3. *Planar graphs.* It is a well known result that planar graphs are at most 5-connected, hence a planar graph cannot be more than 3-linked. In fact, it can be proved easily that no planar graph is 3-linked.

Proposition 13. *A planar graph G cannot be 3-linked.*

Proof. Assume on the contrary that there exists a 3-linked planar graph G. Certainly, G is 5-connected and triangulated. Consider an arbitrary embedding to the plane with outer face ABC. Label the third vertices of the unique inner triangles, corresponding to edges AB, BC and CA, by C', A' and B', respectively (see Figure 3.2.3). Obviously, A', B', C' are pairwise different vertices. Choosing the pairing AA', BB' and CC', the paths should join opposite vertices of a hexagon $AB'CA'BC'$. This is clearly not possible, hence the graph is not 3-linked. □

We have already mentioned another elementary observation, that is, if G is planar and at least 2-linked, then it is triangulated (if G is planar and has a non-triangle face then it cannot be 2-linked). We omit the full description of 2-linked planar graphs for the same reason we did not fully describe 2-linked graphs in general, but close up the discussion of 2-linked planar graphs with a relevant theorem of Goddard.

Theorem 15 (Goddard [7]). *If G is a 4-connected, triangulated planar graph, then it is 2-linked.*

Note that Goddard's original theorem states even more, namely that 4-connected, triangulated planar graph are 4-ordered, thus they are 2-linked. We discuss orderedness of graphs and their relationship to linkedness in a later chapter.

3.3. **Related graph properties and generalizations.**

3.3.1. *Weak-linkedness.* In several applications, one does not need to establish internally vertex disjoint communication channels. Vertices of a network, representing communication towers may receive and transmit several different data streams simultaneously and without data collision, as long as separate edges are allocated for the transmissions. In this model, we require a network to assign an individual channel for each of the communicating parties, such that the paths carrying the messages share no edges of the graph.

Definition 14. *An undirected graph G is weakly-k-linked if, for every ordered set of $2k$ vertices $S = (s_1, \dots, s_k)$ and $T = (t_1, \dots, t_k)$, there exist edge-disjoint paths $P_1, \dots, P_k$, such that each P_i is an $s_i t_i$-path.*

We wish to highlight that the two sets S and T may contain vertices multiple times. In the extrem case $x_1 = \cdots = x_k$ and $y_1 = \cdots = y_k$, we are requested to find k edge-disjoint paths between two vertices. It means that weakly k-linked graphs are k-edge-connected. This necessary lower bound is sharp, as one can easily construct k-edge-connected but not $(k+1)$-edge-connected, weakly-k-linked graphs. Just as in the case of linkedness, implication in the converse direction has been of great interest. Thomassen [34] conjectured that k-edge-connectivity implies weakly-k-linkedness for odd k. Note that if k is even, the previous bound is clearly insufficient: consider first $k = 2$ and $G = C_4$. Placing two pairs of terminals on opposite sides of the cycle, the pairs clearly cannot be linked without the repeated use of and edge. For $k \geq 2$, replace the edges of C_4 by $\frac{k}{2}$ parallel edges and label the vertices by A, B, C, D, in a cyclic order. Easy to see that, if we place $k-1$ pairs of terminals on A and C, the paths joining the pairs will necessarily use either all the edges incident to B or D. Choosing these vertices as the kth pair of terminals completes our counterexample.[1]

The famous theorem of Tutte, concerning the existence of k-edge disjoint spanning trees implies that $2k$-edge-connected graphs are weakly-k-linked.

Theorem 16 (Tutte)**.** *An undirected graph G contains k edge-disjoint spanning trees, if and only if, for every partition $\mathcal{F} = \{V_1, \ldots, V_t\}$ of the vertex set, $\sum_{i=1}^{t} \frac{d(V_i)}{2} \geq k(t-1)$ holds.*

Corollary 15. *If G is $2k$-edge-connected, then G is weakly k-linked.*

Proof. Given a partition $\mathcal{F} = \{V_1, \ldots, V_t\}$ of $V(G)$, every set has degree $2k$ or more, else the graph would not be $2k$-edge-connected. That means $\sum_{i=1}^{t} \frac{d(V_i)}{2} \geq \sum_{i=1}^{t} \frac{2k}{2} \geq k(t-1)$, hence G contains k edge-disjoint spanning trees. We can use each tree to join a given pair of terminals that guarantees disjointness. $\square$

The upper bound was bettered for certain values of k by Hirata [31] in 1984, and by Okamura [25] in 1990.

[1] As throughout this dissertation we concern ourselves with simple undirected graphs without multiple edges, it might be seemly to present a counterexample of that ilk at this current problem as well. It can be done fairly easily. Replace the four bunches of k parallel edges by for copies of K_{k+1}, and join A,B,C, and D to $\frac{k}{2}$ different vertices in each copy that replaced a bunch of parallel edges incident to that very vertex. We leave the verification of the example to the reader.

Theorem 17 (Hirata [31])**.** *Every* $(2k+1)$*-edge-connected graph is weakly-*$(k+2)$*-linked for* $k \geq 2$*.*

Theorem 18 (Okamura [25])**.** *For* $k \geq 2$*, every* $4k$*-edge-connected graph is weakly-*$3k$*-linked, and every* $(4k+2)$*-edge-connected graph is weakly* $(3k+2)$*-linked.*

One year later, the theorem of Huck closed the gap almost completely between the necessary and sufficient conditions. The theorem leaves an uncertainity factor of a constant ± 1, regarding the exact connection between weak-linkedness and edge-connectivity. To date, this is the best known general result.

Theorem 19 (Huck [9])**.** *If* k *is odd and* G *is* $(k+1)$*-edge-connected, then* G *is weakly* k*-linked. If* k *is even and* G *is* $(k+2)$*-edge-connected, then* G *is weakly* k*-linked.*

While one might think that the above constant error term could be eliminated with a reasonable effort invested, we wish to state that this kind of gap-closings are often cumbersome tasks. To illustrate our point, we recall the history of another problem of similar fashion. Haggkvist and Thomassen showed with an elegant and quick proof that, in a $(k+1)$-edge-connected graph, for every path-forest[2] of at most k edges, there exists a cycle containing the forest. Note that, in order to satisfy such a property, a graph certainly has to be at least k-edge-connected, but it was not clear if it has to be $(k+1)$-edge-connected, thus the lower and upper bounds failed to collide by an error term of 1. Several years later, Kawarabayashi proved that k-edge-connectivity is sufficient to guarantee that above property in most graphs. That particular proof is about 140 pages long and is much more complicated than the one that yielded the presented weaker bound.

3.3.2. *Orderedness.* We discuss a natural strengthening of Proposition 8 that claims the existence of cycles in a k-connected graph, containing a chosen set of vertices. Our motivation is the same as in case of linkedness. We would like to gain control of the (cyclic) order, in which the cycle encounters the vertices. We define the following concept:

Definition 16. *The graph* G *is* k*-ordered for* $k \geq 3$ *if, for every choice of* k *cyclically ordered vertices* $x_1, \ldots, x_k$*, there exists a cycle containing the vertices in the given cyclic order.*

[2] A path-forest is a forest, in which every connected component is a path.

Observe that k-ordered graphs are necessarily $(k-1)$-connected, othervise two appropriately chosen, consecutive terminals x and y of the cyclic order can be separated by the other terminals. Indeed, fill up a cut of size at most $k-3$ with terminals, such that non of them is next to x or y in the cyclic order. This rather straightforward lower bound is sharp in the sense that there exist $(k-1)$-connected, k-ordered graphs for arbitrary values of k (see [23]). However, $(k-1)$-connectivity is not sufficient to imply k-orderedness. Just as in the case of linkedness, the question regarding an upper bound on the connectivity that forces a graph G to be k-ordered, naturally arose. Instead of studying further that direct connection, we examine the relation between linkedness and orderedness. That will eventually imply lower and upper bounds in terms of connectivity and orderedness (see Theorem 20).

Proposition 17. *If G is k-linked, then it is k-ordered. If G is $2k$-ordered, then it is k-linked.*

Proof. Suppose that G is linked (here we use Definition 1 !), and we are given a cyclic order of a k-tuple of vertices $(x_1, \ldots, x_k)$. Choose another k-tuple $y_1, \ldots, y_k$, such that $y_i = x_{i+1}$ modulo k. Since G is linked, it contains k disjoint paths $P_1, \ldots, P_k$, such that P_i joins x_i and y_i. Easy to see that $\bigcup_{i=1}^{k} P_i$ is a cycle that contains $(x_1, \ldots, x_k)$ in the given cyclic order.

Now suppose that G is $2k$-ordered, and that we are given two ordered k-tuples of (pairwise disjoint) vertices $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$. Consider the cyclic order of the $2k$-tuple $(x_1, y_1, \ldots, x_k, y_k)$, and the cycle containing them in this given cyclic order. The appropriate segments of the cirle between vertices x_i and y_i provide the required paths. $\square$

It is believed that for most graphs the ratio between linkedness and orderedness is $\frac{1}{2}$.This assumption is not true in general. Firs of all, we mention that, while 6-connected graphs have been long known to be 2-linked, it is still an open question if they are 4-ordered.

Conjecture 18. *If G is 6-connected, then it is 4-ordered.*

Secondly, let us define G as the union of two copies of the complete graph K_n joined by a perfect matching. [3]

Proposition 19. *The Cartesian product of the complete graphs K_{2m} and K_2 is m-linked and is at most $\lceil \frac{4}{3}m \rceil$-ordered, for $m \geq 4$.*

Proof. The m-linkedness of the Cartesian product of K_m and K_2 is stated and proved in a later chapter, in Lemma 28. Assume now that our graph is $2t$-ordered and place $2t$ terminals $(x_1, \dots, x_{2t})$ to the ends of t edges of our perfect matching, such that x_i and x_{i+1} lie on different edges and are contained by different complete subgraphs. Easy to see that any path between x_i and x_{i+1} contains at least two internal vertices. It means that the non-labeled vertices can provide at most $\frac{4m-2t}{2} = 2m - t$ paths joining the labeled ones, that is, $2t \leq 2m - t$ and so $2t \geq \frac{4}{3}m$. $\square$

Based on [4], we summarize the best known bounds regarding orderedness and connectivity.

Theorem 20. *For $k \geq 3$, if a graph G is*

(1) *k-ordered $\Rightarrow$ $(k-1)$-connected, but k-ordered $\not\Rightarrow$ k-connected,*
(2) *k-linked $\Rightarrow$ k-ordered, but k-linked $\not\Rightarrow$ $(k+1)$-ordered,*
(3) *k-ordered $\Rightarrow$ $\lfloor \frac{k}{2} \rfloor$-linked, but k-ordered $\not\Rightarrow$ $(\lfloor \frac{k}{2} \rfloor + 1)$-linked,*
(4) *$10k$-connected $\Rightarrow$ k-ordered, but $(2k-4)$-connected $\not\Rightarrow$ k-ordered.*

3.3.3. *Generalized linkedness.* A common generalization of many of the discussed concepts is $(l_1, \dots, l_t)$-linkedness. A graph G is $(l_1, \dots, l_t)$-linked if, for any pairwise disjoint sets of vertices $L_1, \dots, L_t$ of sizes $l_1, \dots, l_t$, there exist t disjoint connected subgraphs $G_1, \dots, G_t$ of G, such that $L_i \subset V(G_i)$, $i = 1, \dots, t$. Easy to see that $l_1 = \dots = l_{t-1} = 1$, $l_t \geq 2$ defines t-connected graphs on at least $(l_t + t - 1)$ vertices, while $(l_1, \dots, l_t)$-linkedness for $l_1 = \dots = l_t = 2$ is an equivalent definition of t-linked graphs.

The $(l_1, \dots, l_t)$-linked graphs are certainly $min(l_i) + 1$ connected, and $\sum_{i=1}^{t} l_i$-linked graphs are $(l_1, \dots, l_t)$-linked, hence high enough connectivity implies the generalized

[3] The graph obtained this way is also called the Cartesian Product of the complete graphs K_m and K_2. Product graphs, especially Cartesian product of graphs will be discussed in more details in a later chapter.

linkedness as well. The exact bounds, just in case of classical linkedness, are far from being sharp. We present a handful of known results by Chen, Gould, Kawarabayashi, Pfender, Wei [2] and by Mori [24].

Theorem 21 (Chen Gould, Kawarabayashi, Pfender, Wei [2])**.** *Every 7-connected graph containing K_9 as a minor is (2, 5)-linked.*

Theorem 22 (Mori [24])**.** *Let G be a planar graph with at least six vertices. Then G is (3, 3)-linked, if and only if G is maximal and 4-connected.*

4. Path-pairability

The concept of k-path-pairability was introduced by Csaba, Faudree, Lehel and Gyárfás in the early '90-s. This concept is a natural weakening of weak-linkedness, by prohibiting the subsequent assignment of vertices as terminals. In other words, a path-pairable communicational network does not allow participants to initiate more than one call at a time.

Definition 20. *Given a fixed integer k, a graph G on at least $2k$ vertices is k-path-pairable if, for any pair of disjoint sets of vertices $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$ of G, there exist k edge-disjoint paths $P_1, P_2, \ldots, P_k$, such that P_i is a path from x_i to y_i, $1 \le i \le k$. The path-parability number of a graph G is the largest positive integer k, for which G is k-path-pairable, and it is denoted by* $\mathrm{pp}(G)$*. A k-path-pairable graph on $2k$ vertices is simply called path-pairable.*

Obviously, weakly-k-linked graphs are k-path-pairable, thus one can easily give plenty of examples for k-path-pairable graphs. It is much more intriguing that graphs that are not highly weakly-linked can have large path-pairability. In fact, while weakly k-linked graphs are known to be k-edge-connected, one can easily construct k-path-pairable graphs for arbitrary k that are not even 2-edge-connected. The most illustrative examples are the star graphs with an even number of vertices that are path-pairable and are clearly not even 2-edge-connected. Note also that, while the graph K_{2k} is the only k-linked or weakly-k-linked graph on $2k$ vertices, it has many path-pairable subgraphs (even beside the induced star-graphs) with reasonably fewer edges.

Small examples of path-pairable graphs can be easily constructed. The three dimensional cube Q_3 and the Petersen graph P are both path-pairable. The graph shown in Figure 4 is known to be the only path-pairable graph with maximal degree 3 on 12 or more vertices. There are also additional straightforward candidates for infinite path-pairable families as well. It can be proved easily that the n-partite graph $K_{a_1,\dots,a_n}$ ($\sum_{i=1}^{n} a_i$ is even) is path-pairable with the exception of the case $n = a_1 = 2$. We would like to describe k-path-pairable graphs and gain a deeper understanding of the necessary and sufficient graph conditions that yield k-path-pairablity. Just as in the case of linkedness, the question is still open and we are far away from a full description. In the next section, we list a few necessary and sufficient conditions without the wish of being exhaustive.

4.1. **Necessary and sufficient conditions.** Although k-path-pairable graphs may not be highly edge-connected, they do have to possess a structure that is similar to k-edge-connected graphs. As every subset S of at most k vertices may potentially host $|S|$ teminals belonging to different pairs, the condition $d(S) \geq |S|$ is necessary in order to channel the $|S|$ edge-disjoint paths between S and $V(G) - S$. We call the discussed condition the *cut-condition.*

Definition 21 (Cut-condition)**.** *A graph G satisfies the k-cut-condition if, for every $S \subset V(G)$ and $|S| \leq k$, $d(S) \geq |S|$ holds. A graph G on $2n$ vertices satisfies the cut-condition if, for every $S \subset V(G)$ and $|S| \leq n$, $d(S) \geq |S|$ holds.*

Satisfying the k-cut-condition is necessary, but not sufficient for a graph G to be k-path-pairable. Consider the disjoint union of the star graph $K_{1,k}$ and the complete graph K_N on $N \geq 2k$ vertices. Join each but the central vertex of degree one to an arbitrary vertex of K_N by an edge, such that different vertices of $K_{1,k}$ are joined to different vertices of K_N. The graph G obtained this way is clearly not k-path-pairable. To prove this, place $k+1$ terminals in different vertices of $K_{1,k}$, and place the pair of the terminal of the central vertex in K_N. Any path joining that very pair severs another terminal by devouring both edges of a degree two vertex. On the other hand, any $S \subset V(G)$ subset that contains a vertex of K_N certainly satisfies the cut-condition. For $S \subset K_{1,k}$, verification of the cut condition is also rather straightforward. There is an edge of the matching between K_n and any chosen non-central vertex. The terminal vertex, if chosen, has at least one neighbor that is not in S, hence $d(S) \geq |S|$.

Appropriate fine-tuning of the construction provides examples of graphs that are not path-pairable (here we speak of the parameter-free variant of path-pairability), while they satisfy the cut-condition. Take the disjoint union of $K_{1,k}$ and K_{k-1}, and join the two graphs by a matching of size $k-1$ that avoids the central vertex of the star graph. Join the remaining unique vertex of degree one to any vertex of K_{n-1}. Just as before, the arrangement of degree-two vertices, joined to the center of the star, prohibit the channeling of k edge-disjoint paths. We claim that our graph G satisfies the cut-condition for $k \geq 6$. Assume on the contrary that $S \subset V(G)$ of size at most k violates the condition. We proceed by a case-by-case analysis.

Case 1: If $K_{k-1} \subset S$, S must contain an additional vertex, that is, $|S| = k$. Easy to see that adding neither the center, nor any leaf of the star graph to the vertex set of K_{k-1} violates the condition.

Case 2: If $|S \cap K_{k-1}| = k-2$, then $d(S) \geq k-2$, due to the edges leaving S within K_{k-1}. Also, at least $k-4$ of them have a neighbor in $K_{1,k}$ not belonging to S. It yields $d(S) \geq (k-2)+(k-4) \geq k$, thus S cannot violate the cut-condition.

Case 3: If $1 \leq |S \cap K_{k-1}| \leq k-3$, then $d(S) \geq 2k-6 \geq k$, which follows easily, by considering the edges leaving S within K_{k-1}.

Case 4: If $S \subset K_{1,k}$, S must contain the center of the star, else it obviously holds the condition. Observe that each non-central vertex of $K_{1,k}$ has an edge leaving S toward K_{k-1}, and so does at least one edge of the star. It completes the proof.

There are not many, but only obvious known sufficient conditions that guarantee k-path-pairability. Theorem 19 states that $(k+2)$-edge-connected graphs are weakly-k-linked, hence they are k-path-pairable as well. It would be especially interesting to find properties that imply k-path-pairability but do not imply k-linkedness. For example, it is not known, if the upper bound on edge-connectivity forcing k-path-pairability could be improved. In general, sufficient conditions for k-path-pairability and path-pairability are subjects to further investigation.

4.2. **Maximum degree.** We have already concluded that neither high connectivity, nor high edge density is a necessary condition for path-pairability. Now we investigate similar questions regarding the maximum degree in a graphs.

For the fixed parameter variant was studied by Faudree, Gyárfás, and Lehel [5], who gave examples of k-path-pairable graphs for arbitrary values of k, with a constant maximum degree $\Delta = 3$.

Theorem 23 (Faudree, Gyárfás, Lehel [5]). *There exists a k-path-pairable graph G with $\Delta(G) = 3$ for every $k \in \mathbb{Z}$.*

Note that the example given in [5] has exponential size in terms of k. It means that, although the construction gives rise of graphs with arbitrarily high path-pairability, path-pairable graphs cannot be obtained by this method.

We start with the elementary observation that the star graphs are the only path-pairable tress, thus every path-pairable graph on n vertices that is not a star has at least n vertices. Faudree, Gyárfás, and Lehel [6] improved that rather obvious lower bound and proved the following result:

Theorem 24 (Faudree, Gyárfás, Lehel [6]). *If G is a path-pairable graph of order n, and $\Delta(G) \leq n-2$, then $e(G) \geq \frac{3}{2}n - \log_2 n - c$ for some constant c.*

The results presented in [6] suggest that decreasing the maximum degree in a path-pairable graph necessarily involves a higher edge density. We would like to see, how small the maximum degree of a (parameter-free) path-pairable graph can be, i.e. if similar theorem to Theorem 23 can be proved. Note that eliminating the large maximum degree in a network, even at the expense of higher edge density, is a reasonable goal from practical perspectives as well. Vertices of high degree might be heavily used and often overburdened with data transmission tasks, and are therefore prone to malfunction. It is a natural desire to design networks that balance the transmission tasks equally. That motivates the study of path-pairable graphs with small maximum degree.

The problem of finding the minimum of $\Delta(G)$ revealed an essential difference between fixed parameter path-pairability and the much stronger parameter-free path-pairability. In latter case, the maximum degree $\Delta(G)$ does have to increase together with the size of a path-pairable graph G, as proved by the same authors [6] as above.

Theorem 25 (Faudree, Gyárfás, Lehel [6]). *If G is a path-pairable graph on n vertices with maximum degree Δ then $n \leq 2\Delta^{\Delta}$.*

The proof of the theorem in [6] is based on the natural idea that joining $\frac{n}{2}$ pairs of vertices in a graph on n vertices requires lots of edges. This is especially true, if we manage to choose the pairs, such that they lie far away from each other. We briefly summarize that technique of choosing. Let us construct an auxiliary graph H with vertex set $V(H) = V(G)$, such that $x, y \in V(H)$ are joined by an edge, if $d(x, y) > t$. We claim that $\delta(H) \geq \frac{n}{2}$, thus, by the theorem of Dirac, it contains a Hamiltonian cycle. Indeed, the number of vertices at distance at most $\lfloor \log_\Delta \frac{n}{2} \rfloor$ from $x \in V(H)$ is less than $\frac{n}{2}$. The Hamiltonian cycle of H naturally provides a pairing of the vertices of G, such that the vertices of each pair lie at distance $t+1$ or more apart. In order to realize the edge-disjoint linking of the given pairing in G, one needs at least $\lfloor \log_\Delta \frac{n}{2} \rfloor$ edges for each path, that is, $\frac{n}{2} \cdot \lfloor \log_\Delta \frac{n}{2} \rfloor$ edges in total. The number of edges in G is at most $\frac{\Delta \cdot n}{2}$. The equailty obtained this way proves the required bound on n.

The theorem gives an approximate lower bound of $\frac{\log(n)}{\log\log(n)}$ on $\Delta(G)$. By contrast, to date the best known constructions have maximum degree $O(\sqrt{n})$ and are due to Kubicka, Kubicki and Lehel [15] and to Mészáros [21]. It is conjectured that the d-dimensional hypercube Q_d is path-pairable for odd values of d. We discuss this conjecture, as well as the mentioned constructions in details in the next chapter. It is also believed that the maximal degree in Q_{2k+1} is optimal, and that Faudree's result on the lower bound can (and yet has to) be improved.

We highlight another intriguing open problem that investigates the connection between maximum degree Δ and the relative path-pairability parameter, that is, path-pairability in terms of the graph size. We are especially interested in the following question:

Question 22. *If G_n is a family of $k(n)$-path-pairable graphs on n vertices, with a universal constant maximum degree Δ, how large can $k(n)$ possibly be?*

Theorem 23 implies that $log_2(n) \leq k(n)$ holds for infinitely many examples, while it can be concluded from the proof of Theorem 25 that $k(n) \leq c \cdot \frac{n}{log_2(n)}$, where c is an absolute constant. To date, these are the best known bounds, leaving plenty of room for improvements.

4.3. **Diameter.** Our sketched proof of Theorem 25 still has the following small technical incompletion: how do we know that there are vertices at distance $\lfloor \log_\Delta \frac{n}{2} \rfloor$ from each other? We prove a very short and straightforward lemma to remove that flaw.

Lemma 23. *If G is a connected graph on n vertices, with diameter d and maximal degree Δ, then $\Delta^{d+1} \geq n$.*

Proof. Take an arbitrary pair of vertices $x, y \in V(G)$, such that $d(x, y) = d$. As G is connected, a BFS-tree T with root x contains every vertex of G. The tree has depth d, hence it contains at most $1 + \Delta + (\Delta - 1)^2 + ... + (\Delta - 1)^d \leq \Delta^{d+1}$ vertices. $\square$

Lemma 23 immediately gives a general lower bound of $\sqrt[d]{n}$ on Δ that holds for any graph. Although the given bound is never sharp, it naturally raises the following question: how large (in terms of n) can be the diameter of a path-pairable graph? Observe that, while it is fairly easy to come up with k-path-pairable graphs of arbitrarily large diameter (high connectivity implies k-linkedness that implies k-path-pairability), the so far presented path-pairable graphs all have a bounded diameter (the highest presented path-pairable diameter is 3). It is therefore very natural to ask, if

(1) there exist path-pairable graphs with arbitrary large diameter, and
(2) in terms of n, how large can be the diameter of a path-pairable graph?

We answer both questions and state the first main theorem of the dissertation.

Theorem 26. *If G is a path-pairable graph on n vertices with diameter d, then $d \leq 6\sqrt{2} \cdot \sqrt{n}$ holds.*

Proof. Assume that G is a paith pairable graph on n vertices with diameter $d \geq 20$. Let $x, y \in V(G)$, such that $d(x, y) = d$. Define $S_i = \{z \in V(G) : d(x, z) = i\}$ for $i = 0, 1, 2, \ldots, d$ and $U_i = \bigcup_{j=0}^{i} S_j$. Set the notation $s_i = |S_i|$ and $u_i = |U_i|$. Note that there is no edge between any S_i and S_j $(i < j)$ classes unless they are consecutive, that is, $j = i + 1$. Also, observe that adding edges between consecutive classes changes neither the diameter nor the path-pairability property of the graph. Thus, we may assume without loss of generality that vertices belonging to consecutive S_i sets are joined by an edge. If S_d contains vertices in addition to y, move them to S_{d-1} by joining them to every vertex of S_{d-2} as well as to y, in case they were not adjacent. Note again that, while our operation may change the distribution of the vertices among the S_i classes, our newly obtained graph G' has the same diameter as G and is also path-pairable. In addition, $z \in S_i$ if and only if $d(y, z) = d - i$,

$0 \le i \le d$. We introduce the notation S'_i for S_{d-i} and divide our sets into three parts creating left, middle, and right segments A,B, and C, as follows:

$$A = \bigcup_{i=0}^{\lceil \frac{d}{3} \rceil} S_i \qquad B = \bigcup_{i=\lceil \frac{d}{3} \rceil+1}^{\lfloor \frac{2d}{3} \rfloor-1} S_i \qquad C = \bigcup_{i=\lfloor \frac{2d}{3} \rfloor}^{d} S_i$$

Our main goal is to give a lower estimate on the size of B. We first prove the following lemmas:

Lemma 24. *$s_{2k} + s_{2k+1} \ge k$, as long as $u_{2k+1} \le \frac{n}{2}$.*

Proof. We prove our statement by induction on k. Apparently, $s_0 + s_1 \ge 0$ and $s_2 + s_3 \ge 1$. Observe that the number of edges between S_{2k+2} and S_{2k+3} is at least $u_{2k+1} = \sum_{i=0}^{2k+1} s_i$. Indeed, placing u_{2k+1} terminals in U_{2k+2} and their pairs in $V(G) - U_{2k+2}$, there must be space for at least u_{2k+1} edge-disjoint paths passing from S_{2k+2} to S_{2k+3}. By induction hypothesis, $u_{2k+1} \ge \frac{1}{2}k(k+1)$ holds, while the number of edges between the two classes is at most $s_{2k+2}s_{2k+3} \le \left(\frac{s_{2k+2}+s_{2k+3}}{2}\right)^2$. It yields $\frac{1}{2}k(k+1) \le \left(\frac{s_{2k+2}+s_{2k+3}}{2}\right)^2$, that is, $s_{2k+2} + s_{2k+3} \ge \sqrt{2k(k+1)} \ge k+1$ if $k \ge 1$. $\square$

Lemma 25. *$|A|, |C| \ge min\left(\frac{n}{2}, \frac{d^2}{100}\right)$.*

Proof. Assume $|A| < \frac{n}{2}$. Using Lemma 24, we know that $|A| = u_{\lceil \frac{d}{3} \rceil} = s_0 + \cdots + s_{\lceil \frac{d}{3} \rceil} \ge 0 + 1 + \cdots + \left\lfloor \frac{\lceil \frac{d}{3} \rceil}{2} \right\rfloor \ge \frac{d^2}{100}$. By exchanging the role of x and y the same reasoning shows that $|A'| \ge \frac{d^2}{100}$ if $|A'| < \frac{n}{2}$, where $A' = \bigcup_{i=0}^{\lceil \frac{d}{3} \rceil} S'_i$. As $\lceil \frac{d}{3} \rceil + \lfloor \frac{2d}{3} \rfloor = d$, one can easily see that $C = A'$, which completes the proof. $\square$

If $\lceil \frac{d}{3} \rceil < t < \lfloor \frac{2}{3}d \rfloor$, the number of edges between S_t and S_{t+1} is at least $min\left(\frac{n}{2}, \frac{d^2}{100}\right)$. As seen before, placing $min\left(\frac{n}{2}, \frac{d^2}{100}\right)$ terminals in A and their pairs in C the set S_t has to be able to bridge $min\left(\frac{n}{2}, \frac{d^2}{100}\right)$ disjoint paths to S_{t+1}. The number of crossing edges between these two sets is at most $s_t \cdot s_{t+1}$. It means $\frac{s_t+s_{t+1}}{2} \ge \sqrt{s_t s_{t+1}} \ge min\left(\frac{\sqrt{n}}{\sqrt{2}}, \frac{d}{10}\right)$. If $d \ge 16$, then $\lfloor \frac{2}{3}d \rfloor - \lceil \frac{d}{3} \rceil - 2 \ge \frac{d}{3} - \frac{10}{3} \ge 2$ and B contains at least two different S_i sets. That gives us the requested lower bound on $|B|$:

$$|B| \ge \sum_{t=\lceil \frac{d}{3} \rceil+1}^{\lfloor \frac{2}{3}d \rfloor-2} \frac{s_t + s_{t+1}}{2} \ge (\frac{d}{3} - \frac{10}{3}) min(\frac{\sqrt{n}}{\sqrt{2}}, \frac{d}{10}) \ge \frac{d}{6} min(\frac{\sqrt{n}}{\sqrt{2}}, \frac{d}{10}) \tag{1}$$

As $|B| \leq n$, our equation proves that $d \leq 6\sqrt{2}\sqrt{n}$. $\square$

We show that our bound is optimal up to a constant factor, and present an infinite family of path-pairable graphs $\{G_n\}$, such that G_n has n vertices and diameter $O(\sqrt{n})$ for infinitely many values of n. Let $n = (2m) \cdot (4m+3)$ and define G as an equally blown up graph of the cycle C_{2m} of size n. That is, $V(G) = \{x_{i,j} : 0 \leq i \leq 2m-1, 0 \leq j \leq 4m+2\}$ and $x_{i,j}$ and $x_{i',j'}$ are connected if $i - i' = 1$ or $i - i' = -1$ (modulo $2m$). We use the notation $S_i = \{x_{i,j} \in V(G) : 0 \leq j \leq 4m+2\}$ and refer to the set as the ith class of G. Easy to see that G has diameter $m > \frac{1}{4}\sqrt{n}$ (in fact, $d(G) \approx \frac{1}{2\sqrt{2}}\sqrt{n}$). We mention that G also has maximum degree $O(\sqrt{n})$, the same order of magnitude as in [15], which is the best known result for path-pairable graphs with small degree.

Theorem 27. *The graph G_n defined above is path-pairable.*

Proof. Set an arbitrary pairing of the vertices of G. We accomplish the joining of the pairs in two phases. During the first phase, for each pair of terminals, we define a path that starts at one of the terminals and ends at some vertex in the class of its pair. If the ending vertex happens to be the actual pair of the terminal, we set this path as the joining path for the given pair, otherwise we continue with the second phase. If two terminals initially belong to the same class, then the pair simply skips the first phase of the joining. Direct our cycle C_{2m} and the blown-up graph G counterclockwise and label each pair x, y such that there exists a directed $x \to y$ path of length at most m. We start building the above mentioned path for pair (x, y) at vertex x. Fix m edge-disjoint matchings $M_1^i, \ldots, M_m^i$ of size $4m+3$ between every consecutive classes S_i and S_{i+1}. For a pair of terminals (x, y) lying in classes S_i and S_{i+d} (modulo $2m$) at distance d ($1 \leq d \leq m$), choose the edge of M_1^i being adjacent to x and label the other vertex adjacent to it by $p_1(x)$. In step j for $2 \leq j \leq d$ take the edge of M_j^{i+j-1} being adjacent to $p_{j-1}(x)$ and label its other end by $p_j(x)$. Apparently, y and $p_d(x)$ belong to the same class. Phase one ends by assigning an $P_{xy} : x - p_1(x) - \cdots - p_d(x)$ path to each (x, y) pair of terminals.

Observe that paths P and P' assigned to and starting at terminals x and x' of the same class do not contain a common vertex as they are given edges of the same matchings in every step. Now assume that edge $e = (x_{i,j}, x_{i+1,k})$ has been utilized by two paths P_1 and P_2. It means that $e \in M_t^i$ for some $1 \leq t \leq m$ and that P_1 and P_2 must have started in the same class. However, in order to share an edge they also have to share a vertex which contradicts our previous observation. It proves that phase one terminates without edge-collision.

In phase two we finish the joining. For the terminal y initially paired with x and for the endpoint $p_d(x)$ of path P_{xy} (both vertices lying in S_i for some i) consider the yet unused edges of the bipartite subgraph H_i spanned by S_i and S_{i+1}. As $d_{H_i}(y), d_{H_i}(p_d(x)) \geq 3m+3$, there exists at least $2m+3$ different vertices $z_1, z_2, \ldots, z_{2m+3} \in S_{i+1}$ such that $(x, z_k), (y, z_k) \in E(G), k = 1, 2, \ldots, 2m+3$. Observe that any vertex in S_i is an endpoint of at most $m+1$ paths defined in phase one, hence out of the $2m+3$ listed candidates at most $2m+2$ could have been assigned to another pair $(y', p_{d'}(x'))$. It means x and y can be joined by the path $x - p_1(x) - \cdots - p_d(x) - z_i - y$ with an appropriate choice of z_i from the above list. That completes the proof.

In phase two we finish the joining. Observe that each vertex of the graph hosts at most $m+1$ terminals after phase 1. For terminals x and y being stationed in S_i consider the yet unused edges of the bipartite subgraph H_i spanned by S_i and S_{i+1}. As $d_H(x), d_H(y) \geq 3m+3$ there exists $z \in S_{i+1}$ such that $(x, z), (y, z) \in E(G)$ hence x and y can be joined by that path of length 2. Together with the path generated during phase one for x joining can be completed. One can easily verify that no edge of H_i has been used multiple times during phase two. That completes the proof. $\square$

We mention that, by replacing the $\frac{n}{2}$ upper bound by a fixed parameter k in Lemma 24 and 25, a more general form of Theorem 26 can be proved.

Theorem 28. *If G is a k-path-pairable graph on n vertices with diameter d then $d \leq 6 \cdot \frac{n}{\sqrt{k}}$.*

The presented bound is sharp up to a constant factor: take an arbitrary long path P_N and consider its blown-up $\hat{P}_N$, where we substitute each vertex by a click K_t[4] and each edge by an appropriate complete bipartite graph. The presented graph has $n = t \cdot N$ vertices and diameter $N = \frac{n}{t}$. It can be proved also fairly easily that it is $k = t^2$-path-pairable.

Proposition 26. *The blown-up graph $\hat{P}_N$ described above is t^2-path-pairable for $N \geq t$.*

[4]Note that in the most common definition of blown-up graphs substitution of the vertices is done by empty graphs (just as we did previously) rather than cliques.

Proof. Given a distribution of t^2 pairs of vertices, we can carry out pairing by starting at one end of the path and greedily joining terminals to vertices of the consecutive class and finishing the joining of terminals within the classes. We label our classes by $C_1, \dots, C_N$, starting at the left end of the path. For a terminal u, we will assign several $u', u'', \dots$ pseudopairs in the consecutive classes until we finally pair one with the appropriate v pair. We start by pairing terminals that lie in the same class by direct edges of the cliques. From now on we may assume that, for every pair (u, v), one of the terminals is closer to the left end of the path, hence it will be encountered earlier in our left-to-right sweeping algorithm than its pair. Being at class C_i, the consecutive class C_{i+1} contains at most t terminals. If some of them have appropriate pseudopair in C_i, they can be joined by direct edges (here we are massively using that path-pairability prohibits repeated terminal assignment of a vertex). Then, the remaining terminals of C_i can be assigned a new pseudopair in C_{i+1}, maintaining the condition that a vertex $x \in C_{i+1}$ hosts at most t terminals and pseudopairs that have not been paired. Having visited at most t^2 terminals, this condition can be easily maintained using Hall's Theorem concerning matchings. Having reached $t^2 + a$ terminals, we must have encountered at least a pairs, that is, the number of still unmatched terminals is at most $t^2 - a$, thus our above reasoning works just as well as before.

□

5. The Cartesian product of graphs and parameter inheritance

Products of graphs were first defined in 1912 by Whitehead and Russell [28]. They were repeatedly rediscovered later, notably by Sabidussi [8] in 1960. Several different types of graph products (such as Cartesian product, strong product, lexicographical and zig-zag products) have been introduced and investigated in the last century; for a comprehensive survey on the evolution of graph products we refer the reader to [10]. In this dissertation we study Cartesian products of graphs. Occasional references of "product of graphs" is always meant to be translated in this chapter as Cartesian product of graphs (in Chapter 6, further graph products will be discussed briefly). The *Cartesian product* of graphs G and H is the graph $G\Box H$ with vertices $V(G\Box H) = V(G) \times V(H)$, and $(x, u)(y, v)$ is an edge if $x = y$ and $uv \in E(H)$ or $xy \in E(G)$ and $u = v$. The definition can be translated less formally as follows.

The Cartesian product of graphs G and H is the graph defined on the Descartes product of the vertex sets, such that every row of the product consists of a copy of G and every column is a copy of H with the order of vertices fixed for both G and H (see Figure 5 for a small illustrative example). The product of graphs $G_1, \ldots, G_t$ for $t \geq 3$ is defined recursively (observe that the Cartesian product is an associative operation). The graphs $G_1, \ldots, G_t$ are called factors of $G_1 \square \ldots \square G_t$.

Grid graphs are probably the most illustrative examples of product graphs obtained by the Cartesian product. One can easily to see that taking the Cartesian product of two paths P_a and P_b yields a two-dimensional grid graph of size $a \times b$. Higher dimensional grids can be obtained by taking the product of several paths $P_{a_1}, \ldots, P_{a_d}$ of appropriate lengths. In particular, multiplying a single edge $e = \overline{xy}$ with itself results in the d-dimensional hypercube Q_d. The symbol $\square$ for the multiplication in fact resembles the cycle C_4, also known as Q_2 as the Cartesian product of two edges. Note that the product graph $P_{a_1} \square \ldots \square P_{a_d}$ is sometimes referred to as a *d dimensional affine grid*, while the product of d cycles $C_{a_1} \square \ldots \square C_{a_d}$ is often called a *d dimensional projective grid*.

Both affine and projective grids have a interesting recursive structure; the Cartesian product of m-dimensional and n-dimensional grids is an $(m+n)$-dimensional one. That convenient property gives rise to many recursive or inductive proof techniques in the study of grid graphs. As a very straightforward textbook example, one can easily prove inductively that affine grids are two-colorable. If the affine grid G is one dimensional, there is nothing to prove. In the remaining case $G = G_0 \square G_1$ where G_0, G_1 are smaller dimensional grids, both of which have proper $0-1$ colorings by induction hypothesis. We color $(x, y) \in V(G)$ with the color $x \in V(G_0)$ plus the color of $y \in G_1$ in the above colorings modulo 2. One can easily verify that we have defined a proper two-coloring of G; we leave the verification to the reader.

Observe that our proof above also works in much more general settings as we have used very little of the grid's structural properties. We have in fact proved that the Cartesian product of two arbitrary two-colorable graphs (not necessarily grids) is two colorable. This is already a rather simple result in the very popular and intriguing field of research often referred to as *parameter inheritance*.

Apparently, parameter inheritance offers countless exciting and challenging problems. For any graph parameter $\mathcal{P}$ it is meaningful to ask if there exist any connection and correlation between $G\Box H$ and its factors G and H having $\mathcal{P}$. For example, one might ask if bipartition property of graphs is inherited from components to the Cartesian product. It turns out that the Cartesian product is a bipartite graph if and only if every component is bipartite. One direction follows simple by iteratively applying our previous proof for two-colorability (recall, a graph is two-colorable if and only if it is bipartite). For the converse direction, one can easily verify by simple parity argument that if $G\Box H$ contains an odd cycle, then the union of vertical or horizontal edges used in the odd cycle contain an odd cycle in G or H.

We mainly concern ourself with numeric parameters, that is, parameters that assign for every graph a numeric (typically non-negative integer) value. We can reformulate our previous example the following way: non-empty bipartite graphs are known to be the only graphs with chromatic number 2, hence we have just proved that if $\chi(G) = \chi(H) = 2$, then $\chi(G\Box H) = 2$. Now let us say that we would like to extend our result for larger chromatic numbers and would like to express $\chi(G\Box H)$ in terms of $\chi(G)$ and $\chi(H)$. Certainly, $\chi(G\Box H) \geq \chi(G), \chi(H)$ by simple monotonity argument ($G\Box H$ has subgraphs isomorpic to G and H) and it is an easy exercise to show that in fact $\chi(G\Box H) = max\big(\chi(G), \chi(H)\big)$. We list a few more elementary correspondences for later utilization.

$$\begin{aligned}
v(G\Box H) &= v(G)\cdot v(H)\\
e(G\Box H) &= v(G)\cdot e(H) + v(H)\cdot e(G)\\
d_{G\Box H}\big((x,y)\big) &= d_G(x) + d_H(y)\\
\delta(G\Box H) &= \delta(G) + \delta(H)\\
\Delta(G\Box H) &= \Delta(G) + \Delta(H)\\
\underline{d}(G\Box H) &= \underline{d}(G) + \underline{d}(H)\\
d(G\Box H) &= d(G) + d(H)
\end{aligned}$$

Apparently, some parameters yield very loose connection or no connection at all between the product graph and its components. For example, one can easily see that the product of two nonempty graphs contains at least one copy of C_4 and so product graphs never have large girth, regardless of the girths of their components. A minimal edge-cover of a complete graph K_n contains a single vertex while the product graph $K_n \Box K_m$ can be only edge-covered by at least $min(n, m)$ vertices. In other cases, parameters can be calculated fairly easily as the corresponding objects only occur within layers of the product. As an example, let c_3 denotes the number of triangles in the graphs, then obviously $c_3(G \Box H) = v(G)c_3(H) + v(H)c_3(G)$ holds. It is also easy to see that $\omega(G \Box H) = max(\omega(G), \omega(H)$.

The central subject of this dissertation to examine inheritance of linkedness, weak-linkedness, and path-pairability in the Cartesian product of graphs. None of the mentioned parameters turns out to be a simple function of the appropriate parameters in the factor graphs. Our primary goal is to obtain sharp lower (and if possible, upper) bounds on the parameter of product.

As we have seen, linkedness and weak linkedness are stronly connected to connectivity and edge-connectivity. Before turning to the study of inheritance of linkedness we present some of the relevant results about the latter properties in the Cartesian product for later utilization.

Theorem 29 (Spacapan [36])**.** $\kappa(G \Box H) = min\big(\delta(G) + \delta(H), \kappa(G) \cdot |V(H)|, \kappa(H) \cdot |V(G)|\big)$.

Theorem 30 (Xu,Yang [38])**.** *Let G_1 and G_2 be λ_1 and λ_2 edge-connected graphs, respectively. Then $\lambda(G_1 \Box G_2) = min(\delta_1 + \delta_2, \lambda_1 \cdot v_2, \lambda_2 \cdot v_1)$.*

Corollary 27. *$\kappa(G \Box H) \geq \kappa(G) + \kappa(H)$ and $\lambda(G \Box H) \geq \lambda(G) + \lambda(H)$.*

We close up our introductory section of product graphs with the definition of parameter monotonicity. We call a parameter $\mathcal{P}$ *monotone with respect to the Cartesian product* if $\mathcal{P}(G \Box H) \geq max\big(\mathcal{P}(G), \mathcal{P}(H)\big)$ for every G and H. We have seen that the chromatic number χ is a monotone parameter and so are all the so far listed parameters with the exception of girth. We have also showed in Chapter 3, Proposition 19 that orderedness is not a monotone parameter either. As special cases of our main theorems, we will prove in the next chapter that linkedness, weak-linkedness, and path-pairability are all monotone parameters.

5.1. **Main results.**

5.1.1. *Linkedness.* We study inheritance of linkedness in the Cartesian product and state the second main theorem of the present dissertation. By Theorem 8 and Corollary 27 it follows immediately that the product $G\Box H$ of an a-linked and a b-linked graphs is at least $\lfloor \frac{a+b}{10} \rfloor$-linked. We improve that result and give a sharp lower bound on the linkedness of the Cartesian product of general graphs.

Theorem 31. *If G is an a-linked graph with $|V(G)| \geq 8a$ and H is a b-linked graph with $|V(H)| \geq 8b$ then $G\Box H$ is $(a+b-1)$-linked.*

Observe first that although $G\Box H$ is at least $(2a-1)+(2b-1) = 2(a+b-1)$-connected, the presented result cannot be simply derived from Kawarabayashi's Theorem (Theorem 10) as product graphs generally have girth of at most 4. Note also that the presented bound is sharp: take the complete graphs K_N and add a vertex x by joining it to $2k-1$ different vertices of K_N (we assume $N \geq 2k-1$). The constructed graph G is $(2k-1)$-connected and k-linked, while $G\Box G$ is $(4k-2)$-connected, hence it is at most $(2k-1)$-linked.

We first settle the case when a or b is equal to 1. Note that being 1-linked is equivalent to connectivity.

Lemma 28. *Let the graph G be k-linked and let the graph H be connected. Then $G\Box H$ is k-linked as well.*

Proof. Let M denote the set of $2k$ (arbitrarily chosen and paired) terminals in $G\Box H$. Take a G-layer G_x $(x \in H)$ with terminals $u_1, \ldots, u_t$ $(1 \leq t \leq 2k)$. If $t = 2k$, use the condition that G_x is k-linked and find the necessary paths within the layer. Otherwise, let $D = \{u_1, \ldots, u_t\}$, $S = M - D$ and let T consist of $2k-t$ non-terminal vertices in G_x.

We use Lemma 7 for the graph $G\Box H$ which is $(2k-1)$-connected as G is k-linked and H is connected. Using Lemma 7 one can find $2k-t$ paths $P_1, \ldots, P_{2k-t}$ from S to T in $G\Box H - D$. For each P_i path let p_i denote its terminal endpoint in S and let p_i' denote the first vertex of P_i in G_x (the vertex where P_i first "enters" G_x). Truncate P to a $p_i - p_i'$ path. Using the condition that G_x is k-linked, one can find k paths $Q_1, \ldots, Q_k$ that join the $2k$ vertices of the set $D \cup \{p_i' : 1 \leq i \leq 2k-t\}$, with the obvious matching (p_i' is paired with the original pair of p_i). Note that the truncation is performed in order to get the above P_i and Q_i paths disjoint. Also, observe that the path system $Q_1, \ldots, Q_k$ extended by paths $P_1, \ldots, P_{2k-t}$ at the p_i' vertices is an appropriate path system for the initial matching. That completes the proof. $\square$

From now on, we may assume $a \geq b \geq 2$. We prove a more general form of Theorem 31:

Theorem 32. *If G is an a-linked graph with $|V(G)| \geq 8a$ and H is a $(2b-1)$-connected graph with $a \geq b$, then $G \Box H$ is $(a+b-1)$-linked.*

Proof of Theorem 32. Our main goal in the proof is to carry out one of the following tasks:

i) Join one terminal to its pair within a layer and proceed by induction in an appropriate subgraph.
ii) For every pair (x, y) find paths P_x, P_y with other endvertices x' and y', such that x' and y' share the same horizontal layer. Following that we will find a path Q joining x' and y' and join x and y by the concatenation $P_x Q P_y$.

For the latter task observe that, as the total number of terminals is $2a + 2b - 2$ and $a \geq b$, two approriate G-layers will be sufficient to contain and join all the x'-s and y'-s. The bottleneck of the idea is that all the P_x, P_y paths have to be disjoint. We also want to make sure that these paths enter only one of the above distinguished horizontal layers containing the x'-s and y'-s. We will use Lemma 7 to guarantee such conditions. We call a G-layer *crowded* if it contains more than $2a - 1$ terminals. Observe that crowded G-layers necessarily contain at least one pair of matching terminals.

If there exists a crowded G-layer G_x $(x \in H)$ in $G \Box H$, take a pair $u_1, v_1 \in G_x$. As $|\Gamma_H(x)| \geq 2b - 1$, there exists $y \in \Gamma_H(x)$ such that G_y contains no terminal. The appropriate neighbours of u_1 and v_1 in G_y can be joined by a path within G_y. We can join u_1 and v_1 by extending that path on both ends by the vertical edges from u_1 and from v_1 to G_y. For every remaining terminal u of G_x we find a vertical neighbour not belonging to G_y as follows (note that case i) and case ii) do not exclude each other).

i) Link u to its pair if they are adjacent by a vertical edge.
ii) If the terminal u has a vertical neighbour u' that is neither a terminal nor has it been previously assigned as a vertical neighbour to another terminal in G_x, choose u'.

iii) If neither of the previous cases applies, then H_u contains all terminals lying outside of G_x and its pair v lies in G_x. Switch (u_1, v_1) to (u, v) and start the procedure again with joining u and v. The second round terminates without encountering the same problem.

Define a new pairing of the remaining $a+b-2$ pairs of terminals by substituting every u by u'. Observe that G and $H-x-y$ are a-linked and $(b-1)$-linked and have at least $8a$ and $8b-2$ vertices, respectively. By inductional hypothesis, $G\square(H-x-y)$ is $(a+b-2)$-linked and so there exist $a+b-2$ paths joining the newly defined $a+b-2$ pairs. The extension of these paths by the appropriate $\overline{uu'}$ edges results in a path system that joins the original pairing.

Assume now that $G\square H$ contains no crowded G-layer. For a terminal u our first goal is to find a path with horizontal edges to a vertex $u' \in G_u$ such that $H_{u'}$ is devoid of terminals and endvertices of previously routed paths of the same kind. We carry out this task in several rounds, defining a u' vertex and a corresponding $u-u'$ path for every u terminal of a given G-layer within a round. As long as the number of terminals on layers being or having been processed does not exceed $2a-1$, Lemma 7 provides an easy way for the assignment. We will frequently use the following *truncation* operation during our proof. Assume we are given a path P of horizontal edges with a terminal end u and a non-terminal endvertex $\hat{u}$, whose $H_{\hat{u}}$ layer does not contain terminals or vertices of previously defined paths. Starting with u, we read the vertices of P in precedence order until we find the first vertex u' that has the same properties as $\hat{u}$. We stop and truncate P to an uu' path. Observe that the main importance of the truncation operation is gaining control of the length of the joining paths that is not automatically guaranteed by Menger's theorem. The above truncation of the paths makes sure that we can find at every step an appropriate canditate for the role of $\hat{u}$.

Consider all G-layers $G_1, \ldots, G_n$ containing $0 < s_1 \leq \cdots \leq s_n < 2a$ terminals. Choose $1 \leq t \leq n$ such that $\sum_{i=1}^{t-1} s_i \leq 2a-1$ and $\sum_{i=1}^{t} s_i > 2a-1$. We design our algorithm as follows:

i) In round 1, choose a set of s_1 vertices in G_1 whose corresponding H-layers do not contain any terminal. Use Menger's theorem to find s_1 disjoint paths between the terminals of G_1 and the newly chosen set. Truncate these paths and define the set D_1 as the set of the non-terminal endpoints of the truncated paths.

ii) In round i for $2 \le i \le t-1$, let T denote the set of terminals in G_i and let D_i be the projection of D_{i-1} to G_i. Choose a set S of s_i vertices in G_i whose corresponding H-layers do not contain any terminal or vertex of D_i. Easy to see that $|D_i| = \sum_{j=1}^{i} s_j \le 2a-1$, hence the conditions of Lemma 7 hold. Take s_i paths joining (in some order) S and T. Truncate the paths and update D_i by adding the set of the paths's non-terminal endpoints.

iii) In the remaining $n-t+1$ rounds ($t \le i \le n$), choose a set of s_i vertices in G_i whose corresponding H-layers do not contain any terminal. Use Menger's Theorem to find s_i disjoint paths joining (in some order) the terminals and the newly chosen vertices.

We refer to the previous phase as a *global horizontal shift*. Observe that each terminal u was given a non-terminal vertex $u' \in G_u$ and an uu' path $P_{uu'}$ of horizontal edges, such that:

A) $P_{uu'}$ does not intersect with other paths defined in the phase.

B) $H_{u'}$ consist of at most $n-t+1$ vertices belonging to other paths defined in the phase (at most one at each layer during the last $n-t+1$ steps).

Note that the condition $V(G) \ge 8a$ guarantees that every step of the horizontal shift can be carried out without running out of space; we have at most $4a$ terminals in the graph, each of which requires at most one new H layer during that phase. Our next goal is to carry out a *global vertical shift*. We take two G-layers that contain neither terminals nor vertices belonging to paths of the previous phase and call them G_α and G_β. For each u' of the previous phase we define a vertex u'' and a $u' - u''$ path in $H_{u'}$ such that:

i) $u'' \in G_\alpha$ or $u'' \in G_\beta$,

ii) if (u, v) are a pair, then u'' and v'' belong to the same G-layer,

iii) G_α and G_β both have at most a pairs of (u'', v'') vertices,

iv) the path $P_{u'u''}$ does not intersect other paths of the recent or the previous phase (with the exception of $P_{uu'}$). In addition, if $u'' \in G_\alpha$, then $P_{u'u''} \cap G_\beta = \emptyset$, if $u'' \in G_\beta$, then $P_{u'u''} \cap G_\alpha = \emptyset$.

Clearly, G_α and G_β will provide room for the final step of joining the terminals. As both layers are a-linked, all (u'', v'') pairs can be joined by disjoint paths. Our initial pair (u, v) will be joined by an $uu'u''v''v'v$ path. It remains to show that the $P_{u'u''}$ can be found with the above conditions. Distribute the (u, v) terminal pairs among G_α and G_β an arbitrary balanced way (the layers receive $\lfloor\frac{a+b-1}{2}\rfloor$ and $\lceil\frac{a+b-1}{2}\rceil$ terminal pairs). For given u' an u'' vertices we may assume without loss of generality that $u'' \in G_\alpha$. The underlying $H_{u'}$-layer is $(2b-1)$-connected. It contains at most $n-t+1$ vertices of horizontal paths and the projection of u' to G_β. If $n - t + 2 \leq 2b - 2$, we can find a $P_{u'u''}$ path that contains none of the listed vertices. In that case we have constructed all of the requested $P_{u'u''}$ paths, thus finished the proof. If $n - t + 2 > 2b - 2$, then $s_1 = \cdots = s_n = 1$ or $s_1 = \cdots = s_{n-1} = 1, s_n = 2$. These rather simple cases can be handled by very simple case-by-case analysis. Choose an empty H layer for every pair of terminals. As each G-layer is $(2a-1)$-connected, and there are $a+b-1$ pairs of terminals, we can set a path between a terminal u and the assigned u' endpoint within G_u without entering the other assigned H-layers. We join (u, v) by an $uu'v'v$ path. We leave the detailed analysis to the reader.

There are a few more cases to consider. If there is only one G-layer that is devoid of terminals, label it G_α and choose an arbitrary G_β layer with at most a terminals. As the average terminal load of a G layer is $\frac{2a+2b-2}{8b-1}$, such layer can be easily found. In that case, G_β skips phase one and we proceed to phase two, saving all the horizontal edges for the final joining.

If there is no available G-layer, we need a more elaborate work to proceed. Let G_x denote a G-layer with the fewest number of terminals. By simple averaging, one can easily show (just as we did above) that G_x contains at most a terminals. We, in fact, prove that x has a neighbour $y \in \Gamma_H(x)$ such that G_y contains at most a terminals. Recall that H is $(2b-1)$-connected, hence $|\Gamma_H(x)| \geq 2b - 1$. Assuming that every G-layer corresponding to a neighbour of x consists of at least $a + 1$ terminals would yield at least $(2b - 1)(a + 1) > (2a + 2b - 2)$ terminals in total that is not possible. Let $y \in \Gamma_H(x)$ be chosen as described above. We distinguish two cases:

(1) If G_x and G_y do not share a pair of terminals, that is, every pair has terminals in at most one of them, simply label $G_\alpha = G_x$, $G_\beta = G_y$ and use the shifting techniques without applying horizontal shift on these two layers. The two layers will collect all the terminals and perform the final joining.

(2) If there is a pair of terminals (u, v) such that $u \in G_x$ and $v \in G_y$, we use an inductional step and reduce the problem to a smaller graph. Apply horizontal shift on the terminals of G_x and G_y such that every terminal t gets a pseudopair, u' and v' share an H layer and no other pair of pseudopairs share their H layers. Recall that x and y are adjacent and so are u' and v'. Join the pseudopairs and so u and v by the union of the three paths. For a remaining pseudopair t' (that is neither equal to u' nor to v'), choose an arbitrary vertical edge with other endpoint t'' such that $t'' \notin V(G_x) \cup V(G_y)$. Apply induction on G and $H - x - y$ just as in the main proof. Note that the base of our induction is the case $b = 1$ which is covered in Lemma 28.

$\square$

We believe that the statement of Theorem 31 is true even without the indicated size conditions. Remember that we only assume the condition to guarantee enough room for the shifting techniques. Nevertheless, in the case when $\frac{\text{link}(G)}{v(G)} > \frac{1}{8}$ not only the shifting techniques fail to work but one has to deal with an aboundance of terminals congested on the layers. Linking of the terminals in that case is likely to lead a rather lenghty and tedious case-by-case analysis involving ad hoc solutions which we do not find particularly interesting and do not investigate in this dissertation.

One may wonder how the constant term (-1) comes into the picture in the bound of Theorem 31. That $\text{link}(G \square H) \geq \text{link}(G) + \text{link}(H)$ does not hold in general is especially interesting in light of Corollary 27 (i.e. $\kappa(G \square H) \geq \kappa(G) + \kappa(H)$ and $\lambda(G \square H) \geq \lambda(G) + \lambda(H)$). We have already verified that the sharpness of this rather unexpected bound is due to a natural connectivity constraint: there exist graphs G and H with $\text{link}(G) = a$, $\kappa(G) = 2a - 1$,$\text{link}(H) = b$, $\kappa(H) = 2b - 1$, and $\kappa(G \square H) = 2(a+b-1)$. As $(a+b)$-linked graphs are $\big(2(a+b)-1\big)$-connected, $G \square H$ is at most $(a+b-1)$-linked. That observation does explain our presented bound but it naturally raises additional questions. Let G be an a-linked, $2a$-connected graph and let H be a b-linked, $2b$-conneced graph. What can we say about $\text{link}(G \square H)$? The product graph $G \square H$ is $2(a + b)$-connected, hence the above reasoning no longer prohibits it to be $(a + b)$-linked. In fact, with appropriate re-tuning of the "double-shifting" technique one can prove that $\text{link}(G \square H)$ is $(a + b)$-linked under the above conditions (including the size constraints of Theorem 31). Based on the same shifting technique we present the following possible extension of Theorem 31:

Theorem 33. *If G is an a-linked, k-connected graph and H is a h-connected graph ($h \leq k$, G and H are sufficiently large), then $G\Box H$ is $\frac{a}{2a+1}(k+h)$-linked.*

Proof. Let us denote $L := \frac{a}{2a+1}(k+h)$. If there exists a crowded G-layer (containing at least $k+1$ terminals), find a matching pair of terminals (which exists by pigeonhole principal), join them, empty the layer as before, and proceed by induction. Otherwise, global shift horizontally, allocate $t := \lceil \frac{L}{a} \rceil$ empty G-layers $G_{\alpha_1}, \ldots, G_{\alpha_t}$, distribute the terminal pairs among them via vertical paths and reduce the problem to linking within horizontal layers. We do not elaborate on the proof but encourage the reader to copy the proof of Theorem 31 for unveiling the actual steps of the linking. □

Theorem 33 illustrates that the linkedness number of the Cartesian product of an a-linked and a b-linked graph may be reasonable greater than $a+b-1$. As the matter of fact, high linkedness of the product graph avails no lower bound on the linkedness of its components. The theorem of Bollobás and Thomason [1] together with the result of Spacapan [36] show that large minimum degree is sufficient to imply high linkedness while the component graphs are connected, but might not even be 2-connected. In other words, there exist a function f_δ such that $\delta(G)+\delta(H) \geq f_\delta(k)$ implies $G\Box H$ is k-linked. Using the improved bound presented in [32] we know that $f_\delta(k) \leq 10k$. Even better bounds might be obtained on f_δ by further investigation of the problem that we do not discuss here.

5.1.2. *Path-pairability.* Before actually turning to the investigation of path-pairability of product graphs, we make a quick detour to weak-linkedness. It is reasonable to believe that the shifting technique in the proof of Theorem 31 can be simply reshaped by using the appropriate edge-variant of Menger's Theorem and their discussed corollaries. This assumption turns out to be misleading. The main reason our earlier proof cannot be simply reworded is an essential difference between the connection of linkedness and connectivity and the connection of their edge disjoint variants. While k-linked graphs are necessarily $(2k-1)$-connected, there exist weakly-k-linked graphs that are not even $(k+1)$-edge-connected. The difference in the implied connectivity lower bounds restrains the use of Lemma 7 and only enables us to obtain a weaker result that we do not present. Nevertheless, almost sharp results for the inheritance of weak-linkedness can be easily derived from Huck's Theorem (Theorem 19).

Corollary 29. *If G is a weakly-a-linked graph and H is a weakly-b-linked graph, then $G\square H$ is weakly-$2 \cdot \lfloor \frac{a+b-1}{2} \rfloor$-linked.*

Now recall that k-path-pairability does not imply edge-connectivity that grows with k, hence inheritance in that case cannot be handled by simple utilization of Theorem 19. We invent new horizontal and vertical shifts to cope with the problem. Our new approach is heavily based on the assumption that terminals placed in the graphs do not share a vertex, thus our method is exclusively shaped for path-pairability.

Recall that the universal lower bounds we obtained for the inheritance of linkedness and weak-linkedness in the Cartesian product were linear in terms of the appropriate parameters of the factors, regardless of their sizes. Our current goal is to conclude that such correspondence does not hold for path-pairability. We in fact prove that, given sufficient space (but no other parameter constraint), the Cartesian product of an a-path-pairable graph and a b-path-pairable graph is $O(a \cdot b)$ path-pairable. First of all we verify that, just as linkedness and weak-linkedness, path-pairability is a monotone parameter.

Lemma 30. *If G is a k-path-pairable graph and H is a connected graph, then $G\square H$ is k-path-pairable.*

Proof. We prove the statement by induction on $|V(H)|$. The case $|V(H)| = 1$ is straightforward. Let T be a spanning tree of H, let $x \in V(T)$ be a leaf containing at most k terminals, and let $y \in \Gamma_T(x)$ be chosen arbitrarily. As G is k-path-pairable, the terminals of the layer G_x can be shifted to vertical layers whose intersections with G_y do not contain terminals (recall that $|V(G)| \geq 2k$). The terminals then can be moved to G_y and the joining can be finished in $G\square(H - x)$ by the induction hypothesis. If at any point all terminals are contained by a simple G-layer, we can directly use path-pairability of the layer to finish the joining. $\square$

From now on, we may assume without loss of generality that G and H are at least 2-path-pairable graphs. We state and prove two different theorems regarding inheritance of path-pairability. The first theorem is of the same ilk as Theorem 31 and Corollary 29, while the second one is meant to demonstrate the difference between the inheritance of path-pairability and linkedness.

Theorem 34. *If G is an a-path-pairable graph with $|V(G)| \geq 8a$ and H is a b-path-pairable graph with $|V(H)| \geq 8b$, then $G\square H$ is $(a+b)$-path-pairable.*

Proof. We may assume that $2 \leq b \leq a$. Let M denote the set of $2(a+b)$ (arbitrarily chosen and paired) terminals in $G\square H$. We first prove the theorem in the "base" case when no G-layer contains terminals belonging to $a+1$ or more pairs. The assumption in fact implies that no layer contains more than $2a$ terminals. We mimic the horizontal and vertical shifts presented in the proof of Theorem 31. Take a G-layer G_x ($x \in H$) with $t \leq 2a$ terminals $u_1, \dots, u_t$. Observe that if $t = a+s$ where $1 \leq s \leq a$, then G_x contains at least s pairs of terminals. For a terminal u of G_x that has no pair in G_x, we choose a pseudopair $u' \in G_x$, such that different terminals get different pseudopairs and $H_{u'}$ will contain no other terminal but it will contain the pseudopair of v, the terminal pair of u. Since $|V(G)| \geq 8a$, we can freely assign vertical layers for the pseudopairs of each pair of terminals. The initial terminals occupy at most $2(a+b) \leq 4a$ vertical layers, thus we have at least $4a$ additional empty layers to allocate while we only need $a+b$. In the first phase, we pair terminals on every G-layer with its pair or pseudopair. Having done this, pairing of the pseudopairs can be finished using vertical edges of the connected H-layers containing them.

Observe immediately that our presented technique wastes a lots of potential in pairing the pseudopairs. Using that every H-layer is b-path-pairable, $\frac{2a+2b}{2b} \leq a$ additional empty H-layers are sufficient to finish the pairing, hence the lower bounds on the graph sizes in the theorem can be improved to $|V(G)| \geq 5a$ and $|V(H)| \geq 5b$ in the discussed case.

Now we turn to the examination of the general case. As $4(a+1) > 2(a+b)$ at most 3 G-layers contain $(a+1)$ or more types of terminals. Our goal is to reduce our problem to the base case by redistributing the terminals among the G-layers. We achieve this goal by assigning a pseudopair for each terminal within its original H-layer. Observe that, as the solution of the base case contains a horizontal shift, the combination of the initial redistribution and the solution of the base case will use no vertical edge more than once. For the redistribution of the terminals, we follow a case-by-case analysis.

(1) Assume first that G_x is the only G-layer that contains $u_1, \dots, u_{a+t}$ terminals belonging to different pairs, where $1 \leq t \leq b$. Some of these terminals

may have their pair on the same horizontal layer, we will take that into consideration. Clearly, there are at most $(a+2b-t)$ terminals outside of G_x. We claim that one of the G-layers in the graph $G\square(H-x)$ contains at most $a-t$ terminals, else $G\square(H-x)$ would contain at least $(8b-1)(a-t+1) > a+2b-t$ terminals, clearly contradicting our previous observation. Take a G-layer G_y with the above property. We want to choose t of the terminals in G_x (if their pair is in G_x as well, then we choose both of them) and assign them pseudopairs in G_y, together with vertical paths joining the terminals to their pseudopairs. Note that we cannot assign a pseudopair to a vertex that already contains a terminal. The terminals initially placed in G_y prohibit the assignment of pseudopairs for at most $a-t$ of the terminals (singleton or paired) of G_x, that is, at least $(a+t)-(a-t)=2t$ of the terminals $u_1,\dots,u_{a+t}$ can get pseudopairs in G_y, while we only needed t. If any of the chosen t terminals has an initial pair in G_x, we move that terminal along as well (this in no longer prohibited due to our choice of the terminals). Note also that the total number of types of terminals and pseudopairs in G_y is at most $(a-t)+t=a$ after the redistributing step, as prescribed in the base case. We can now apply the solution of the base case on a new set of terminals, where pseudopairs take the place of their initial terminals.

(2) If two G-layers contain at least $a+1$ types of terminals, the remaining terminals occupy at most $2b-2$ G-layers, that is, there exists at least $6b$ G-layers that are free of terminals. We define for a terminal u a pseudopair u' in H_u, such that

   (a) $G_{u'}$ contains no terminal and contains at most a pseudopairs at the end of the procedure,

   (b) uu' pairs are joined within $H_u = H_{u'}$ by edge-disjoint horizontal paths.

   Indeed, to satisfy the first condition, observe that we have at most $2a+2b$ terminals that we distribute among $6b$ empty layers without any additional constraint (remember, here a terminal and its pair do not have to get pseudopairs assigned to the same G-layer), thus a balanced distribution can be chosen with at most $\lceil \frac{2a}{3b} \rceil \leq a$ terminals on a G-layer. The second condition can be guaranteed by 2-path-pairability, as every H is at least 2-path-pairable $(b \geq 2)$ and we assign at most 2 pseudopairs within an H-layer.

(3) The case with three overloaded layers (G_1, G_2, G_3) works similarly to the previous one. Observe first that in the examined case $3(a+1) \leq 2a+2b$, hence $b \geq \frac{a+3}{2} \geq 2,5$. By pigeon-hole principal, we have at least $6a+3b+2$ empty G-layers at our disposal, each of them expected to receive $\frac{2a+2b}{6a+3b+2} < 2 \leq a$ pseudopairs on average. Since $b \geq 3$ that completes our proof.

□

We have proved above a linear inheritance of path-pairability. For certain graphs it gives the right order of magnitude of inheritance (see Proposition 47). We now show how the presented lower bound changes dramatically once the product graph offers sufficient space for more joining paths. We in fact prove that, as long as both G and H have size $O((ab)^\alpha)$ for $\alpha \in [\frac{1}{2}, 1]$, they can pair terminals of equal order of magnitude.

Theorem 35. *If G is an a-path-pairable graph and H is a b-path-pairable graph and $v(G), v(H) \geq 4S$, $S < \frac{(a+1)(b+1)}{2}$, then $G\Box H$ is S-path-pairable.*

Corollary 31. *If G is an a-path-pairable graph and H is a b-path-pairable graph and $v(G), v(H) \geq 4 \cdot \frac{(a+1)\cdot(b+1)}{2} - 1$, then $G\Box H$ is $(\frac{(a+1)\cdot(b+1)}{2} - 1)$-path-pairable.*

Proof of Theorem 35. We use the same techniques as in the proof of Theorem 34. We may assume $2 \leq b \leq a$. If no G-layer contains more than a different types of terminals, we can join the pairs that share a G-layer and assign pseudopairs to the terminals having their pairs on a different layer. The pseudopairs can be chosen such that pseudopairs of a pair of terminals are located on the same H-layer and their H-layer contains no terminal or pseudopair of another terminal. We only need $v(G) \geq 3S$ to provide sufficient space that certainly holds. Since every H-layer is connected, pairing of the pseudopairs can be carried out within the H-layers.

If $G_{x_1}, \ldots, G_{x_t}$-layers contain more than a-types of terminals, observe first that $t \leq b$, else $G\Box H$ would consist of at least $(a+1)(b+1)$ terminals, contradicting $S < \frac{(a+1)(b+1)}{2}$. It means that in every vertical layer that contains a terminal u, we can assign a pseudopair u' and - using that H is b-path-pairable and so is every vertical layer in $G\Box H$- define edge disjoint uu' paths for every u. We can distribute the pseudopairs among the initially empty horizontal layers such that none of them contains more than a pseudopairs, using that $v(H) \geq 4S \geq 2S + \frac{2S}{a}$ (the number of horizontal layers necessary to carry out our assignment above). Joining of the pseudopairs can be carried out as described in the above base case. □

We show that Corollary 31 is sharp up to a constant factor. That is, the order of magnitude in the inheritance of path-pairability cannot be expanded more than indicated in Theorem 35 by simply providing an abudance of space in the product graph. To prove our claim, we first make the following observation: if $G_0 \subset G$ and $H_0 \subset H$ subsets violate the cut-condition, that is, $e(G_0) < |G_0|$ and $e(H_0) < |H_0|$, the product set $G_0 \Box H_0$ does not necessarily have the same condition. In order to generate violating product sets, stronger assumptions are needed:

Proposition 32. *Let G and H be graphs and $G_0 \subset G$, $H_0 \subset H$, such that $2 \cdot e(G_0) < |G_0|$ and $2 \cdot e(H_0) < |H_0|$. Then $e(G_0 \Box H_0) < |G_0 \Box H_0|$, that is, $G_0 \Box H_0$ violates the cut condition.*

Proof. Cearly $|G_0 \Box H_0| = |G_0| \cdot |H_0|$, while $e(G_0 \Box H_0) = |G_0| \cdot e(H_0) + |H_0| \cdot e(G_0) > \frac{|G_0| \cdot |H_0|}{2} + \frac{|G_0| \cdot |H_0|}{2} = |G_0| \cdot |H_0|$. □

Now consider the blown-up path $\hat{P_N}$ described in Chapter 4. Let G, H be copies of $\hat{P_N}$ with class sizes a, b and diameters greater than $4a^2 + 1$ and $4b^2 + 1$, respectively. Recall that G and H are a^2 and b^2 path-pairable graphs (see Proposition 26)

Moreover, let $G_0 \subset G$ and $H_0 \subset H$ be formed by $2a^2 + 1$ and $2b^2 + 1$ consecutive classes, starting at the left end of the blown-up paths. The sets G_0 and H_0 satisfy the conditions of Proposition 32, thus $G \Box H$ is not $(2a^2 + 1) \cdot (2b^2 + 1)$-path-pairable, regardless of the initial sizes of G and H. That justifies our claim.

5.2. **Results for grid graphs.** We study linkedness properties of the n-dimensional affine and projective grids, that is, the Cartesian product of n paths and n cycles. In particular, we determine the linkedness number of the n-dimensional hypercube. Similar questions regarding weak-linkedness and path-pairability of the above families will be investigated as well.

5.2.1. *Linkedness.* We have proved in Theorem 31 that if G is a sufficiently large k-linked graph and H is a sufficiently large 2-linked graph, then $G \Box H$ is $(k+1)$-linked. We now revisit the proof of Theorem 31 and show that even weaker conditions on H are sufficient to make $G \Box H$ $(k+1)$-linked. Our Lemma 33 below is the key ingredient in the examination of the linkedness of grid graphs.

Lemma 33. *If G is a k-linked graph with $k \geq 2$, $|G| \geq max(9, 4k)$ and H is a 2-connected graph, then $G\square H$ is $(k+1)$-linked.*

Corollary 34. *If G is a k-linked graph with $k \geq 2$, $|G| \geq max(9, 4k)$, then $G\square C_m$ is $(k+1)$-linked, where C_m denotes the cycle of length m.*

Proof. Assume we are given the pairing of $2k+2$ terminals in $G\square H$. We use the technique of the proof of Theorem 31 and follow a case-by-case analysis.

(1) If there exist a G-layer G_i with $3 \leq s_i \leq k$ elements, then no G-layer is crowded (no G-layer contains $2k$ or more terminals). Choose $G_\alpha = G_i$ and apply the horizontal and vertical shift techniques on the remaining $2k+2-s_i \leq 2k-1$ terminals. Observe that one can use Lemma 9 in every G-layer during the horizontal shift. In the vertical phase the H-layer is 2-connected and the path joining u' and u'' only has one vertex to avoid (corresponding to G_α or G_β).

(2) If there exist G-layers G_i and G_j such that $s_i = 1$, $s_j = 2$ or $s_i = s_j = 2$, choose $G_\alpha = G_i$ and $G_\alpha = G_j$. Solution for the previous case works here as well.

(3) If $s_1 = \cdots = s_{2k+2} = 1$, use the separate technique presented for small cases at the end of proof of Theorem 31. Fix an empty H-layer for every pair, place pseudopairs on the layers, join the terminals with their pseudopairs, then join the corresponding pseudopairs. Note that every G-layer is $(2k-1)$-connected and it joins at most one terminal to its pseudopair such that the path avoids k additional vertices. As $2k-1 > k$, such pairing can be carried out.

(4) If $s_1 = \cdots = s_{n-1} = 1$, $k+1 \leq s_n \leq 2k-1$, choose $\{G_\alpha, G_\beta\} = \{G_1, G_2\}$ and apply the shifting technique. Lemma 9 handles every G-layer just as in Case 1.

(5) If $s_1 = s_2 = 1$, $s_3 = 2k$, join a pair u_1, v_1 within G_2 using Lemma 9 and shift the remaining terminals vertically. If a terminal u_2 has no available neighbour, then $v_2 \in G_2$ and we can switch pair (u_1, v_1) to (u_2, v_2) and repeat the argument, just as in the crowded layer case of the proof of Theorem 31.

(6) If $s_1 = 2$, $s_2 = 2k$, similar technique works as in Case 5.

(7) If $s_n \geq 2k+1$, we have all terminals (or all but one) on the same G_x-layer for some $x \in H$. Let $y, z \in \Gamma_H(x)$. We can distribute the pairs of terminals

between G_y and G_z by using appropriate vertical $\overline{xy}$ and $\overline{xz}$ edges and join u', v' endpoints within the horizontal layer. If $s_n = 2k+1$, the missing terminal can be routed to the appropriate layer. We leave the details to the reader.

(8) If $s_1 = s_2 = k+1$, we may assume none of the layers contain a pair, otherwise we can proceed by matching one-one pair within the layers, allocating new terminal vertex u' instead of the original terminal u on the layer, shifting and using induction as previously. Let $G_1 = G_x$, $G_2 = G_y$ for some $x, y \in H$ and let $z \in \Gamma_H(x) - \{y\}$ (as H is 2-connected, such z must exist). Shift terminals horizontally within G_x if necessary in order to get for every terminal u a uu' path with endpoint u', such that $H_{u'}$ contains neither a terminal nor a vertex belonging to the shifting paths (in case there was no shift necessary, let $u' = u$). We pick a single terminal $u \in G_x$ and take a path $uu'u''$ where u'' denotes the projection of u' to G_y. We connect u'' with the pair of u in G_y using Lemma 9. For the remaining $2k$ pairs, we set a vertical paths for each terminal in G_x and G_y to G_z. For a terminal $w \in G_x$ there is no obstacle in H_w to find a path to its projection to G_z. If $w \in G_y$, we use the fact that H is 2-connected and that it contains at most one vertex of G_x we might have used previously. Having set the vertical paths, we join the projections in G_z.

□

Having proved our central lemma, we first inspect the d-dimensional hypercube Q_d. As Q_d is d-connected, the linkedness number of Q_d is at most $\lceil \frac{d}{2} \rceil$. Equality holds for $d = 1$ and 2. Q_3 is not 2-linked as being a planar graph with a non-triangle face. Q_4 is 2-linked, which statement can be proved by a rather short and easy case-by-case analysis. We prove that the above presented obvious upper bound is sharp for higher dimensions.

Proposition 35. *Let Q_d denote the n-dimensional hypercube. Then* $\mathrm{link}(Q_d) = \lceil \frac{d}{2} \rceil$ *if $d \neq 3$.*

Proof of Theorem 35. We use inductional reasoning in our proof. We first settle the smallest missing case.

Lemma 36. *The five dimensional hypercube Q_5 is 3-linked (but not 4-linked).*

Proof of Lemma 36. We distinguish two cases:

(1) Assume there exist terminals x_1, y_1 satisfying $d(x_1, y_1) \leq 4$ where $d(x, y)$ denotes the distance of the vertices x and y. In other words, our current assumption is that x_1 and y_1 are not "opposite" vertices of Q_5. Because of symmetries of the graph Q_5, we may assume without loss of generality that $x_1 = (0, 0, 0, 0, 0)$ and $y_1 = (\underline{v}, 0)$, $\underline{v} \in Q_4$. Also, let us denote $Q_5 = Q_4^0 \cup Q_4^1$, the decomposition of Q_5 into affine hyperplanes being isomorphic to Q_4 (with respect to the last coordinate). Certainly, $x_1, y_1 \in Q_4^0$ and we may assume that $Q_4^0 - x_2 - y_2 - x_3 - y_3$ is connected (otherwise switch to pair (x_2, y_2)). Join x_1 to y_1 in Q_4 by any path of length 4 encountering no other terminal. We want to join the remaining two pairs in Q_4^1. If a terminal $u \in \{x_2, y_2, x_3, y_3\}$ lies in Q_4^0, we define a crossing path that ends at $u' \in Q_4^1$. If the projection of u to Q_4^1 is not a terminal vertex (or if it happens to be the pair of u), we take that very edge as the required path. In every other case there is a $v \in \Gamma_{Q_4^0}(u)$ such that the projection of v to Q_4^1 is available, yielding an appropriate path of length 2.

(2) If $d(x_1, y_1) = d(x_2, y_2) = d(x_3, y_3) = 5$, there exist - up to isomorphism - 5 possible arrangements of the terminals. We leave the easy case-by-case analysis to the reader.

$\square$

As $Q_d = Q_{d-2} \square C_4$, Corollary 34 applies (for $d \geq 4$) and so the proof is complete. $\square$

Proposition 35 can be immediately generalized to d-dimensional affine grids with a minimal effort invested.

Proposition 37. *Let* $G = P_{m_1} \square \ldots \square P_{m_d}$. *Then*

i) $\operatorname{link}(G) = 1$ *if* $d = 3$, $m_1 = m_2 = 2$ *and*
ii) $\operatorname{link}(G) = \lceil \frac{d}{2} \rceil$ *if* $d \neq 3$, $m_i \geq 2$ *or* $d = 3$, $m_3 \geq m_2 \geq 3$.

Proof. The first statement is obvious as G is a non-triangulated planar graph. For $d \neq 3$, let Q_d be an induced subgraph of G containing terminals $x_1, \ldots, x_p$, $p \geq 1$. As $G - x_2 - \cdots - x_p$ is $(d-p)$-connected, the set of remaining terminals can be routed to Q_d and the joining can be performed. The case $d = 3$, $m_3 \geq m_2 \geq 2$ can be solved by the previous idea using the fact that $P_2 \square P_3 \square P_3$ is 2-linked. $\square$

We use Lemma 34 for the investigation of linkedness in projective grids and prove that the obvious upper bound of linkedness is sharp in this case as well.

Lemma 38. *For cycles of length m and n ($m, n \geq 3$)* $\text{link}(C_m \square C_n) = 2$.

Proof. It can be shown by a simple but rather lengthy case-by-case analysis that $C_3 \square C_3$, $C_3 \square C_4$ and $C_4 \square C_4$ are 2-linked. If $max(m, n) \geq 5$, one of the cycles can be shortened by substituting an empty layer with vertical / horizontal edges joining its neighbours and proceed by induction. □

Proposition 39. *For cycles of length $m_1, \ldots, m_d$ ($m_i \geq 3$, $d \geq 2$)* $\text{link}(C_{m_1} \square \ldots \square C_{m_d}) = d$.

Proof. It follows directly from Lemma 34 and Lemma 38. □

5.2.2. *Weak linkedness.* We turn to the examination of weak-linkedness of the above families. It follows from Theorem 19 that $d - 2 \leq \text{wlink}(P_{m_1} \square \ldots \square P_{m_d}) \leq d$ and $2d - 2 \leq C_{m_1} \square \ldots \square C_{m_d} \leq 2d$. We first prove the following lemma:

Lemma 40. *If G is weakly-k-linked and $|V(G)| \geq 2k$ for $k \geq 2$, then $G \square K_2$ is weakly-$(k+1)$-linked.*

Proof. Assume that an assignment of $2(k+1)$ terminals is given in $G \square K_2$. Let G_0 and G_1 denote the two G-layers in the product graph. Our general approach is a divide-and-conquer technique, in which we form two sets of the pairs and pair them within the two G-layers. We proceed by a case-by-case analysis.

(1) If G_0 contains all the terminals, pick any arbitrary (u, v) pair, assign u', v' pseudopairs in G_1, such that $\overline{uu'}$ and $\overline{vv'}$ are vertical edges. Join u' to v' in G_1, as well as join all the remaining pairs in G_0, using that it is weakly-k-linked. Similar reasoning works if all the terminals are contained in G_1.
(2) If G_0 contains at most k types of terminals and has at least one (u, v) pair among them, join all pairs within G_0. For the remaining terminals of G_0, assign pseudopairs belonging to different vertices. If all the terminals happen to lie on different vertices, there is nothing to be done. Our goal is to have all the yet unpaired terminals of G_0 on different vertices, so that they can be channeled to G_1. As G_0 contains at most k-types of terminals and has at least $2k$ vertices, the weakly-k-linkedness takes care of that sorting. Pairing of the remaining terminals can be carried out within G_1.

(3) If G_0 contains at most k types of terminals but has no pair among them, take a u terminal in G_0 and channel its pair to G_0, using the appropriate vertical edge. Now G_0 has at most $k+1$ terminals and at least $2k-1 \geq k+1$ vertices with yet unused vertical edges, thus the solution of the previous case can be applied.

(4) In the remaining case, both G-layers contain exactly one terminal of each pair. We choose $k+1$ appropriate vertical edges between the two G-layers and channel k-terminals in one direction and a single terminal in the other. As a result, we will be able to join k pairs within one G-layer and the remaining one in the other. To do this, take arbitrary terminal $u \in G_0$ and channel it to G_1, using the unique $\overline{uu'}$ vertical edge. Take a terminal $v \in G_1$, such that it is not the pair of u, neither is it placed on the other end of the $\overline{uu'}$ edge. As $k+1 \geq 3$, there must be such a terminal. Using that G_1 is weakly-k-linked, join the channeled image of u to its pair within G_1. Also, join all the remaining terminals of G_1 but v, that is, $k-1$ additional terminals, to pseudopairs. The pseudopairs shall be placed on pairwise different vertices, none of which corresponds to the vertex of v. As we have $2k-2$ choices for these $k-1$ vertices, such assignment can be carried out. The rest of the proof is rather obvious: we channel all the yet unpaired terminals to G_0 and finish the pairing.

□

Corollary 41. *The d-dimensional hypercube Q_d is weakly-d-linked if $d \geq 3$.*

Proof. Our previous lemma takes care of the main part of a potential inductional proof. Note that Q_2 is not weakly-2-linked. It can be showed by a rather lenghty and cumbersome case-by-case analysis that Q_3 is weakly-3-linked. We leave the verification of the statement to the reader. □

Weak-linkedness of high-dimensional affine grids can be derived easily from our corollary.

Proposition 42. *The d dimensional affine grid $G = P_{m_1} \square \ldots \square P_{m_d}$ is weakly-d-linked, if $d \geq 3$.*

Proof. We use induction on $d + V(G)$. Corollary 41 handles starting cases of the induction. Let $m_i \geq 3$ for some $1 \leq i \leq d$ and $x \in P_{m_i}$, such that x is an endpoint of the path, and the $(d-1)$-dimensional layer $G' = P_{m_1}\square \ldots \{x\} \ldots \square P_{m_d}$ contains at most d terminals. As G' is $(d-1)$-edge-connected, the terminals can all be sorted into different vertices of G' (one terminal stays put, the others get a set of pseudopairs assigned and we apply Lemma 9) and they can be channeled to $G - G'$ via edges corresponding to P_i. Pairing of the vertices then can be finished within $G - G'$. $\square$

It can be also easily proved that a two dimensional affine grid $P_a \square P_b$ is weakly-2-linked if both $a, b \geq 3$. We leave the verification of this problem to the reader.

We believe that a similar result holds for high dimensional projective grids as well. Our case-by-case analysis of the problem, however, has led too many subcases and we could not analyse it in its full depth. We state the expected result as a conjecture.

Conjecture 43. *The d dimensional projective grid $C_{m_1}\square \ldots \square C_{m_d}$ is weakly-$2d$-linked.*

5.2.3. *Path-pairability.* Path-pairability of hypercubes is one of the central conjectures in the study of the topic. The one-dimensional hypercube is path-pairable and the two-dimensional is not. It can be showed by a lengthy case-by-case analysis that Q_3 is path-pairable. It is conjectured that the same parity-pattern holds for higher dimesions as well. We do know that hypercubes of even dimension are not path-pairable.

Proposition 44 ([6])**.** *Q_n is not path-pairable for n even.*

Proof. Choose the pairing of the vertices $u, v \in \mathbb{F}_2^n$ such that $u + v = (1, 1, \ldots, 1, 1)$. This is certainly a pairing and the distance between the vertices in any of the 2^{n-1} pairs is exactly n. Assuming that edge-disjoint paths can be established between the pairs, they require $n \cdot 2^{n-1}$ edges, that is, the total number of edges in the hypercube. Exactly one path starts at each vertex $x \in \mathbb{F}_2^n$ while every other path entering x at some edge leaves it using another one. It means the paths together cover an odd number of edges joined to x, contradicting our previous observation. $\square$

The problem concerning path-pairability of the odd-dimensional hypercubes for $n \geq 5$ is open. Note that if the conjecture is true, it will provide an infinite family of path-pairable graphs, such that the product of any two of them ($Q_{2a+1}\square Q_{2b+1} =$

$Q_{2(a+b+1)}$) is not path-pairable. It will also give examples of path-pairable graphs of maximum degree $\Delta = O(\log_2(n))$.

Conjecture 45 ([3])**.** *The* $(2k+1)$*-dimensional hypercube* Q_{2k+1} *is path-pairable for all* $k \in \mathbb{N}$*.*

Path-pairability of high dimensional large grids is also an unearthed area in the research of path-pairability. We mention a few elementary observations for the projective case (the affine case can be dealt with similarly). As we have seen, path-pairability may rise high in the presence of sufficient space. High dimensional huge grids are just the perfect candidates for this purpose. It follows from the multiplicative inheritance described in Corollary 31 that sufficciently large d-dimensional projective grids are $O(2^d)$-path-pairable. We show that it is also upperly bounded by a function of d, regardless of the grid's size, by presenting a set $H \subset V(C_{m_1}\square \dots \square C_{m_d})$ that violates the cut-condition.

Proposition 46. *The* d*-dimensional projective grid* $G = C_{m_1}\square \dots \square C_{m_d}$ *is less than* $(2d)^{d-1} \cdot (2d+1)$*-path-pairable if* $m_i \geq (2d+1)$*.*

Proof. The statement is obvious if $|G| < (2d)^{d-1} \cdot (2d+1)$. Consider now the d-dimensional subgrid $G_0 = P_{2d}\square \dots \square P_{2d}\square P_{2d+1}$. Easy to see that G_0 violates the cut-condition as $V(G_0) = (2d)^{d-1}\cdot(2d+1) > 2\cdot((d-1)(2d)^{d-2}(2d+1)+(2d)^{d-1}) = d(G_0)$. It shows that G is less than $(2d)^{d-1} \cdot (2d+1)$-path-pairable. $\square$

In summary, we have proved that $c_1 \cdot 2^d \leq \mathrm{pp}(C_{m_1}\square \dots \square C_{m_d}) \leq (2d)^{d-1} \cdot (2d+1)$ for suffciently large d-dimensional grid. Needless to say, the presented gap has yet to be narrowed or even closed in order to gain a complete understanding on path-pairability of grids.

5.3. **Path-pairable products.** We have omitted the investigation of inheritance of linkedness for the case when the graph's linkedness number gets real close to its actual size. Our decision is a biased one but it is motivated by the fact that extremely high linkedness requires extremely high connectivity and edge density, thus our product graph will be much like the product of complete graphs with a handful of missing edges. As explained before, examination of linkedness is cumbersome even for the product of complete graphs, hence we do not investigate the extremal cases.

At path-pairability, however, we are facing an entirely different situation. We have seen that, in order to achieve high path-pairability, neither high connectivity nor high edge density is required. We also know that for fixed values of k, being k-linked, weakly-k-linked, and k-path-pairable are properties that are automatically inherited from graphs possessing them to their products with other graphs. Observe, on the other hand, that path-pairability is not inherited automatically, that is, the path-pairability of G does not imply path-pairability of $G\Box H$. As an example, the diameter bound of Theorem 26 shows that if $G\Box H$ is path-pairable, then $diam(G), diam(H) \leq 6\sqrt{2} \cdot \sqrt{v(G) \cdot v(H)}$. If $diam(H) > 2\sqrt{6}\sqrt{v(G) \cdot v(H)}$, say, $H = P_{3 \cdot v(G)}$, $G\Box H$ cannot be path-pairable.

Even if both G and H are path-pairable, path-pairability of $G\Box H$ cannot be guaranteed. We present an illustrative counterexample taking the product of star graphs.

Proposition 47. *The Cartesian product $K_{1,b}\Box K_{1,d}$ is at most $\lceil \frac{b+d}{2} \rceil$-path-pairable.*

Proof. The product graph has a unique vertex z_{a+b} of degree $a+b$. Let C and R denote the sets of vertices of degree two in an arbitrary column and an arbitrary row not containing z_{a+b}. Let x be an additional vertex of degree two and let y denote the intersection $C \cap R$. We place terminals in $C \cup R \cup \{x\}$ such that x and y form a pair and the unique vertices of degree $a+1$ and $b+1$ in the union (denoted by z_{a+1} and z_{b+1}) form another. Observe that paths that join the above two pairs both use either the edge between z_{a+1} and z_{a+b} or between z_{b+1} and z_{a+b}, hence the pairing cannot be achieved. $\Box$

We do not know if path-pairability of the components is necessary to create a path-pairable product. We believe it is not. Overall, our understanding of path-pairable products is very limited. Our current goal is to extend this knowledge by introducing new path-pairable product graphs.

Kubicka, Kubicki, and Lehel [15] investigated path-pairability of two dimensional complete grid graphs and proved that the Cartesian product $K_a\Box K_b$ of the complete graphs K_a and K_b is path-pairable. For $a = b$ the construction gives examples of path-pairable graphs with maximum degree $\Delta = 2a - 2 \approx 2\sqrt{n}$. We improve the upper bound on $\Delta(G)$ to $\sqrt{n}$ for sufficiently large balanced complete bipartite graphs. To date, this is the best known upper bound on the possible minimum of the maximal degree of a path-pairable graph.

Theorem 36. *The product graph $K_{m,m}\square K_{m,m}$ is path-pairable for even values of m if $m \geq 104$.*

Proof. Let us denote the two classes of the bipartite graph $K_{m,m}$ by A_1 and A_2. We introduce further notation for certain sets of the vertices in the product graph $G = K_{m,m}\square K_{m,m}$ as follows: $A_{11} = A_1\square A_1$, $A_{12} = A_1\square A_2$, $A_{21} = A_2\square A_1$, and $A_{22} = A_2\square A_2$. We will refer to these sets as *classes* of G. We also set a cyclic order of the four classes clockwise. References *next class* and *previous class* are translated in accordance with that given cyclic order. We label the m^2 elements of each class by an (u, v) pair where $u = 1, \ldots, m$ and $v = 1, \ldots, m$.

Given a pairing of the vertices, we carry out the joining of the terminals in three phases named: *swarming*, *line-up*, and *final match*. For a pair of terminals of G we first ship them to the same class (swarming), then send them forward to the same row/column of the next class (line-up). Finally, we join the paths by their newly established ends with a single vertex of the next class (final match). Note that during the phases terminals of different pairs might temporarily share vertices but will eventually get sorted to their partners at the end of the final match phase.

Swarming In this phase, we ship one terminal of each pair to the class of its partner. If a pair lies with both vertices within a class, they simply skip the swarming phase. A terminal (u, v), belonging to class A_{11} and heading to A_{12}, shall follow the path $(u, v) \to (u + 1, v)$, where $(u + 1, v)$ denotes the appropriate vertex of A_{12} and addition is calculated modulo m. Similarly, we ship (u, v) to A_{21} via the path $(u, v) \to (u, v + 1)$. Should (u, v) be shipped to A_{22}, we allocate it the path $(u, v) \to (u+1, v) \to (u+1, v+2)$ where $(u+1, v)$ belongs to A_{12} and $(u+1, v+2)$ belongs to A_{22}. Terminals belonging to other classes will be shipped by the same rules, increasing the appropriate coordinate by 1 at the first step, and increasing the other one by 2 in the second step, if applicable. Getting shipped via paths of length two is always carried out clockwise.

One can easily verify, that the above arrangment of paths assures that, if $m \geq 5$, no edge is being utilized twice during the swarming phase. We now choose the terminal to be shipped for each pair, such that at the end of the swarming phase, every class hosts exactly $\frac{m^2}{2}$ pairs. Starting with an arbitrary selection, we can assume without loss of generality, that A_{11} hosts the most pairs, and that at least one terminal $x \in A_{11}$ received its pair y from a class hosting less than $\frac{m^2}{2}$ pairs. Shipping x to the class of y instead balances the distribution of the pairs. Repetition of the previous step leads to an equal distribution.

We define G' with $V(G') = V(G)$, and a new edge set $E(G')$ by deleting those edges from $E(G)$ we have used in the swarming phase. Observe, that by the given shipping method, every vertex of G hosts at most 5 terminals and uses at most 8 of its edges, that is, the minimal degree of G' is at least $m - 8$. We continue the path building in G'.

Line-up We ship each pair of terminals to the next class, such that terminals shipped by a horizontal edge shall share the same column of the new class, while vertically shipped terminals will arrive in the same row. For every pair, there are at least $m - 16$ available columns/rows in the next class. Our intention is to pair up the pairs with the rows/columns, such that every one of them will contain $\frac{m}{2}$ pairs. We recall a straigthforward corollary of Hall's Matching Theorem.

Lemma 48. *A balanced bipartite graph $G = (A, B, E)$ on $n + n$ vertices with minimum degree at least $\frac{n}{2}$ contains a perfect matching.*

We define the following bipartite graph $G = (A, B, E)$ as follows: represent each pair of terminals hosted in A_{11} by a vertex in A, while each column of A_{12} is represented by $\frac{m}{2}$ independent vertices in B. Certainly, $|A| = |B| = \frac{m^2}{2}$. We connect two vertices of A and B by an edge, if both terminals of the corresponding pair have horizontal edges to the corresponding column of A_{12}. Easy to see, that the graph has minimum degree at least $\frac{m^2}{2} - 16m$, hence, by Lemma 48, it contains a perfect matching for $m \geq 64$.

Observe, that if two pairs of terminals sharing a vertex of a class C are distributed to the same vertical layer of the next class C', at least one of the terminals will not be able to get shipped there. We need to guarantee a matching between the pairs and the layers of C' without such a collision. Recall, that each vertex of C hosts at most 5 terminals, hence each pair of terminals has at most 8 additional pairs to collide with. Consider a perfect matching for which the number of above collisions is minimal. Let (x, y) and (x', y') be colliding pairs of terminals being sent to layer

L of C'. We may assume x and x' share the same vertex of C. We want to find a pair (u, v) sent to a layer $L' \neq L$ of E such that

i) (x, y) can be sent from C to L' (instead of L) during the line-up without causing further collision,
ii) (u, v) can be sent from C to L (instead of L') during the line-up without causing further collision.

The pair (x, y) can be initially sent to $m - 16$ layers of C', at most 8 of which might contain terminals that initially shared vertex with (x, y) in C. In order to avoid further collisions we exclude these layers, leaving us at least $m - 24$ choices of L'. We also want to exclude layers that alreay received terminals from the vertex of x or y, yielding at most 8 additional excluded layers, that is, at least $m - 32$ choices of L' and so $(m - 32) \cdot \frac{m}{2}$ choices for (u, v). We want to choose (u, v) such that it initially had not shared vertex in C with any terminal currently hosted in L and that u and v still can be moved (having withdrawn from L') from C to L (the corresponding edges have not been used yet). For the first constraint, recall that L contains $\frac{m}{2}$ pairs, every one of which shares vertex with at most 8 additional terminals. It means there exist at most $4m$ additional terminals that initially cannot be sent to L, because the appropriate edges had already been used during the first phase. Now assume that the appropriate edge that is supposed to channel u or v to L has already been used. It can either occur if another terminal was sent from that particular vertex of C to L during the line-up or if the edges were used during the swarming phase. The first conditions means that (u, v) collides with the other pair of terminals that was sent to L, hence (u, v) is one of the above listed $4m$ pairs. In the remaining case, the missing edge is one of those at most $8 \cdot \frac{n}{2} = 4m$ edges the whole layer L used up during the swarming. The mentioned edges have at most $4m$ endpoints in C and at most $5 \cdot 4m = 20m$ pairs of terminals correspoding to them. Overall, it means that if $(m - 32) \cdot \frac{m}{2} > 24m$ (that is, $m > 56$) , one can find an appropriate (u, v). By swapping the positions of (u, v) and (x, y), we reduce the number of collisions that contradicts our assumption.

We repeat the same procedure for the remaining three classes. It can be easily verified that no edge is used more than once. We define G'' by deleting the used edges from G'. We proceed in G'' to the final match.

Final match For a row/column filled with $\frac{m}{2}$ pairs of terminals, we assign every pair a vertex of the appropriate row/column of the next class that is adjacent to both terminals (see Figure 5.3). Note that during the first two phases, each vertex has used at most 13 of its edges. We use Lemma 48 to find the appropriate assignment. Let A form the set, in which every pair of terminals of a certain row/column is represented by a vertex. The set B is formed by any $\frac{m}{2}$ vertices of the appropriate column/row of the next class. We connect vertices by edges, if both terminals of the pair are adjacent to the appropriate vertex in the next class. Our bipartite graph has two classes of size $\frac{m}{2}$ and minimum degree $\frac{m}{2} - 26$. If $m \geq 104$, the required matching is provided by Lemma 48. That completes the proof. □

Corollary 49. *There exists a path-pairable graph G on n vertices with $\Delta(G) = \sqrt{n}$ for infinitely many values of n.*

6. Additional remarks and open questions

We have examined linkedness and path-pairability properties of the Cartesian-product of undirected graphs. In the final chapter we present open problems in a few closely related topics. These short summaries are not meant to analyze the presented questions in depth equal to our work in the previous chapters but are intended to attract the reader's attention to potential directions of further research.

6.1. **Path-pairable planar graphs.** There are countless additional parameters whose cross-examination with the path-pairability property might be of interest. We take a brief tour in the study of k-path-pairable and especially path-pairable planar graphs. Recall first that both linkedness and weak-linkedness numbers of planar graphs are upperly limited due to simple connectivity conditions. We proved in Proposition 13 that planar graphs cannot be 3-linked. Similarly, it can be proved that planar graphs are at most weakly-5-linked. We also know that path-pairability is not limited by any means of connectivity, hence theoretically planar graphs with arbitrarily high path-pairability number may exist. We are, in fact, interested in the most spectacular species of this family and look for infinite sets of path-pairable planar graphs. Observe that we have already come across such a family as star graphs $\{K_{1,2n-1}\}_{n\in\mathbb{N}}$ are both path-pairable and planar (and so is every planar graph on

n vertices that contains a vertex of degree $n-1$). While small examples of path-pairable planar graphs can be constructed rather easily (see Figure 6.1), it is not straightforward if there exist other edge-minimal infinite families.

We hereby present such a family. Observe first that the complete bipartite graph $K_{2,2m-2}$ only fails to be path-pairable for those pairing of the vertices where the two high degree vertices (let us call them P and Q) form a pair. Let us assume $m \geq 6$ and construct a graph G from $K_{2,2m-2}$ as follows: join the $2m-2$ degree two vertices to each other such that they form a path P_{2m-2} and denote their endpoints by P' and Q'. Remove $\overline{PP'}$ and $\overline{QQ'}$ edges and join P and Q by a new edge. Easy to see that G is planar and has maximum degree $\Delta(G) = 2m-2 = n-2$, hence it does not contain a subgraph isomorphic to $K_{1,n-1}$. Let $\mathcal{M}$ be a given pairing of $V(G)$. The vertices P, P', Q', Q are contained in at most 4 pairs. For a pair (u, v) containing none of the above vertices, we set a path of length two via P or Q, such that both of them are utilized for at least one such (u, v) pair ($n \geq 12$ assures there at at least two pairs of that ilk). Also, if any of the above four vertices has a pair outside that four-touple, there is either a unique edge that joins them or a unique path of length 2 via P or Q (that is the case when we pair P' or Q' with a non-labelled vertex). We set these joining paths as well and examine the remaining (at most 4) pairs in a case-by-case analysis:

(1) If P and Q form a pair but P' and Q' do not, use the direct edge PQ for joining them.
(2) If P' and Q' form a pair but P and Q do not, use path $P'QPQ'$ to join P' and Q'.
(3) If both (P, Q) and (P', Q') are paired, use $P'QPQ'$ to join P' and Q' as before. Let $S, T \in P_{2m-2}$ such that $\overline{PS}$ and $\overline{QT}$ are unused edges. Let P_{ST} denote the unique path joining S and T in P_{2m-2}. The path $PSP_{ST}TQ$ joins P and Q.
(4) If P and Q' form a pair, they can simply joined by a direct edge. The same hold for the pairing of P' and Q.
(5) If P and P' are paired but Q and Q' do not form a pair, use the path $P'QP$ to join P and P'. The same method works in case of the pairing of Q and Q'.

(6) If both PP' and QQ' form pairs, use paths $P'QP$ and $Q'PSP_{ST}TQ$ where S, T and P_{ST} are defined as previously. That completes the case-by-case analysis and thus the proof.

The above example is only intended to enrich our list of path-pairable planar graphs and strengthen our belief in their diversity. The above graphs are not extremal in any reasonable sense. They are not even edge-minimal as one can easily prove that the presented techniques work with a path of length of $\approx m$ within the larger class of $K_{2,2m-2}$. As long as maximum degree is concerned, our family yields an especially minor improvement. On the other hand, the solution technique of Theorem 25 yields a reasonably stronger lower bound on the maximum degree of the path-pairable planar graphs. As a planar graph on n vertices consists of at most $3n-6$ edges, every pairing of the vertices of a path-pairable planar graph must contain a pair at distance 5 or less. It implies that one can find a vertex whose constant (6) neighbourhood consists at least half of the vertices of the graph, thus $\Delta \geq c \cdot \sqrt[6]{n}$. Even better lower bound can be obtained by the so called "separator theorem" proved by Lipton and Tarjan [26]. It claims that the vertex set of every planar graph G can be partitioned into three classes $V(G) = A \cup B \cup C$ such that $|A|, |B| \leq \frac{2}{3}|V(G)|$, $|C| \leq 2\sqrt{2}|V(G)|$ and there are no edges between A and B. It clearly follows from the theorem that if G is a sufficiently large path-pairable planar graph on n vertices, then $\Delta(G) \geq c \cdot n$ for a constant c. This improved bound already meets in order of magnitude with the maximum degree of the studied path-pairable graphs. However, the presented classes all fail to raise an infinite planar graph family, thus the problem regarding the existence of arbitrary large path-pairable planar graphs with maximum degree $\Delta(G) \approx \sqrt{|V(G)|}$ is still open.

We also make a quick comment on the diameter of the path-pairable planar graphs. To date, there is no known path-pairable planar graph of diameter at least 4. Using the edge bound of planar graphs, one can appropriately modify the proof of Theorem 26 and show that the diameter of a path-pairable planar graph is upperly bounded by $c \cdot \log n$, where n denotes the number of vertices and c is a constant. Whether or not the presented bounds can be realized by actual path-pairable planar graphs, or even less, whether an infinite family of path-pairable planar graphs with unbounded diameter can be found, are still open questions that require additional research.

6.2. **Linkedness and path-pairability of directed graphs.** Linkedness, weak-linkedness, and path-pairability of directed graphs can be defined as follows:

Definition 50. *A directed graph $D = (V, A)$ is k-linked if, for every ordered set of vertices $X = (x_1, \dots, x_k)$ and $Y = (y_1, \dots, y_k)$ there exist internally-vertex-disjoint directed paths $P_1, \dots, P_k$ such that each P_i is a directed $x_i y_i$-path starting in x_i and ending in y_i.*

Definition 51. *A directed graph $D = (V, A)$ is weakly-k-linked if, for every ordered set of vertices $X = (x_1, \dots, x_k)$ and $Y = (y_1, \dots, y_k)$ there exist edge-disjoint directed paths $P_1, \dots, P_k$ such that each P_i is a directed $x_i y_i$-path starting in x_i and ending in y_i.*

Definition 52. *A directed graph $D = (V, A)$ is k-path-pairable if, for every ordered set of $2k$ pairwise different vertices $X = (x_1, \dots, x_k)$ and $Y = (y_1, \dots, y_k)$ there exist edge-disjoint directed paths $P_1, \dots, P_k$ such that each P_i is a directed $x_i y_i$-path starting in x_i and ending in y_i. A directed graph is path-pairable if it is k-path-pairable on $2k$ vertices for some k.*

While in the undirected case the complete graph on n vertices is clearly $\lfloor \frac{n}{2} \rfloor$-linked, the existence of k-linked directed graphs for arbitrary k is not imminent (we do not allow the existence of directed edges $\vec{uv}$ and $\vec{vu}$ at the same time). We give a construction of k-linked directed graphs as follows: let G be the complete 3-partite graph with vertex set $V(G) = A \cup B \cup C$ where $|A| = |B| = |C| = 3k$. We orient all edges $A \to B \to C \to A$, that is, set A has incoming edges from C and outgoing edges to B, while edges between B and C are oriented from B to C. We claim that, with that orientation, $\vec{G}$ is k-linked. Indeed, easy to see that, regardless of the choice of the terminal vertices $x_1, \dots, x_k$ and $y_1, \dots, y_k$, each of the three sets will contain at least k non-terminal vertices $z_1^A, \dots, z_k^A$, $z_1^B, \dots, z_k^B$ and $z_1^C, \dots, z_k^C$. The subgraph induced by $\{x_i, y_i, z_i^A, z_i^B, z_i^C\}$ contains a directed path from x_i to y_i and so linking can be completed.

Just as we learned in case of undirected graphs, directed k-linkedness implies strong $(2k-1)$-connectivity. While one would expect implication in the converse direction (an $f(k)$ function similar to the one in Section 2.1.), surprisingly enough, such connection does not exist in the directed case. Thomassen [35] gave an example of strongly k-connected directed graphs for arbitrary value of k that are not even 2-linked.

Theorem 37 (Thomassen [35]). *There exist strongly k-connected directed graphs which are not 2-linked for every $k \in \mathbb{Z}^+$.*

Sufficient conditions for a directed graph D to be k-linked have been widely investigated. We refer the reader to [16] for a detailed survey.

Weak-linkedness of directed graphs has been analyzed to its extent. The folklore theorem of Edmonds gives a full characterization of weakly-k-connected directed graphs.

Theorem 38 (Edmonds). *If $D = (V, A)$ is rooted k-edge-connected with respect of root r, then there exists k edge-disjoint directed spanning tree with root r.*

Corollary 53. *$D = (V, A)$ is weakly k-linked if and only if it is k-edge-connected.*

As k-path-pairability follows from k-linkedness, we only investigate the existence of path-pairable directed graphs. We call an undirected graph $G = (V, E)$ path-pairably-orientable if appropriate orientation of the edges in E results in a path-pairable directed graph $\overrightarrow{G} = (V, \overrightarrow{E})$.

Proposition 54. *The graph $K_{2m} - m \cdot K_2$ is path-pairably-orientable for $m \geq 3$.*

Proof. We order the vertices such that the edges of the deleted perfect matching form the main diagonals. We orient the remaining edges of the graph clockwise. Assume we are given a pairing of the vertices. For $m = 3$, there are four different constructions up to isomorphism. Figure 6.2 provides a possible realization of the linking for each of the cases (we leave the verification of the remaining cases to the reader).

We proceed by induction on m. Observe that by deleting two vertices that form a diagonal in the clockwise oriented $K_{2(m+1)} - (m + 1) \cdot K_2$, we obtain a smaller graph which is isomorphic to $K_{2m} - m \cdot K_2$. More than that, diameters remain diameters after the deletion and so the remaining graph is directed path-pairable by the inductional hypothesis. We denote the deleted vertices by x and y. Given a pairing of the vertices of $K_{2(m+1)} - (m + 1) \cdot K_2$, assume first that x and y form a pair. We may assume that the required path has to start at x and end at y. Linking can be carried out by simply choosing an arbitrary vertex z of the arc $\overrightarrow{xy}$. Linking of the remaing pairs goes by induction. Assume now that x and y have different pairs in the pairing. Let u and v be arbitrary vertices of the arcs $\overrightarrow{xy}$ and $\overrightarrow{yx}$, respectively. For every possible choice of u and v we define the cycle $uxvyu$ with the appropriate orientation. Easy to see that, regardless of the position of $x's$ and $y's$ pairs, both of them can be linked to x and y using an appropriate one of the given four-cycles. As we only used edges belonging to x or y, linking of the remaining pairs can be carried out by induction. $\square$

Cartesian product of directed graphs can be defined the same way we constructed Cartesian product of undirected graphs with the additional rule of keeping the original direction of every edge. The *Cartesian product* of graphs $\vec{G}$ and $\vec{H}$ is the graph $\vec{G}\Box\vec{H}$ with vertices $V(\vec{G}\Box\vec{H}) = V(\vec{G}) \times V(\vec{H})$, and $\overrightarrow{(x,u)(y,v)}$ is a directed edge if $x = y$ and $\vec{uv} \in E(\vec{H})$ or $\vec{xy} \in E(\vec{G})$ and $u = v$. Parameter inheritance of linkedness and path-pairability properties (as well as many further graph properties) are bound to offer several intriguing and challenging questions. Inheritance of linkedness is especially exciting as, unlike the undirected case, inheritance of strong connectivity yields no lower bound on directed linkedness (see Theorem 37).

6.3. **Other graph products.** There are several interesting additional variants of graph products (see Figure 6.3). We list a few of the most studied ones and briefly summarize some of the known results regarding inheritance of connectivity and linkedness in the listed products. Weak-linkedness can (as usual) approached through edge-connectivity, using Theorem 19. We do not discuss these problem, neither do we investigate path-pairability of the given products.

6.3.1. *Tensor product.* The tensor product of graphs G and H is the graph $G \times H$ with vertices $V(G\Box H) = V(G) \times V(H)$ and $(x,u)(y,v)$ is an edge in $G \times H$ if $xy \in E(G)$ and $xy \in E(H)$.

Observe that the tensor product of connected graphs is not necessarily connected. For example, the tensor product $P_a \times P_b$ consists of exactly two components. The observation holds for factors with higher connectivity as well. Define the graph G on the vertex set $V(G) = \{1, 2, \ldots, 2m\}$ for $m \in \mathbb{Z}^+$ and join $i, j \in V(G)$ by an edge if $i - j$ is odd. Easy to see that $G = K_{m,m}$, hence it is m-connected. Also, our labelling of the vertices clearly shows that (x, u) and (y, v) are connected in $G \times H$ if and only if $x+y+u+v$ is even. That splits the vertex set $V(G\times H)$ into two connectivity classeswith respect to the parity of the sum of the coordinates. It shows that neither connectivity nor linkedness is inherited automatically in case of the tensor product. Nevertheless, it would be interesting to know what other conditions are needed to obtain results that are similar to Theorem 31.

6.3.2. *Strong product.* The strong product of graphs G and H is the graph $G\boxtimes H$ with vertices $V(G\boxtimes H)=V(G)\times V(H)$ and $(x,u)(y,v)$ is an edge if $x=y$ and $uv\in E(H)$ or $xy\in E(G)$ and $u=v$ or $uv\in E(H)$ and $xy\in E(G)$. In other word, the edge-set of $G\boxtimes H$ is the union of the edge sets of $G\square H$ and $G\times H$.

Though most of the edges of $G\boxtimes H$ are inherited from $G\times H$, connectivity of the strong-product is guaranteed by the frame provided by $G\square H$. Inheritance of connectivity for strong products has been investigated by Špacapan [37].

Theorem 39 (Špacapan, [37])**.** *If G,H are connected graphs, then $\kappa(G\boxtimes H)\geq min\big(\kappa(G)(1+\delta(H)),\kappa(H)(1+\delta(G))\big)$.*

Together with Theorem 8, the current theorem implies that $\mathrm{link}(G\boxtimes H)\geq(\frac{2}{5}-\varepsilon)\cdot\mathrm{link}(G)\cdot\mathrm{link}(H)$ for every $\varepsilon>0$ if $\mathrm{link}(G)$ and $\mathrm{link}(H)$ are large enough. Also, it can be shown easily that $\mathrm{link}(G\boxtimes H)\leq 2\cdot\mathrm{link}(G)\cdot\mathrm{link}(H)$; a vertex joined by k edges to k different vertices of a complete graph K_n $(n\geq k+1)$ is a possible example where the inequality is sharp. The deeper understanding of the inheritance of linkedness in strong products requires further research.

6.3.3. *Lexicographical product.* The *lexicographical product* of graphs G and H is the graph $G\circ H$ with vertices $V(G\circ H)=V(G)\times V(H)$ and $(x,u)(y,v)$ is an edge if $uv\in E(H)$ or $xy\in E(G)$ and $u=v$. Lexicographical product is particularly interesting being the first (and only) non-commutative graph product of our list, that is, $G\circ H\neq H\circ G$ in general.

The theorem of Yang and Xu [39] provides the appropriate connectivity inheritance result.

Theorem 40 (Yang, Xu [39])**.** *If G,H are connected graphs, then $\kappa(G\circ H)=\kappa(G)\cdot v(H)$.*

Just as in the case of the strong product, there is still plenty of room for further exploration in this particular variant as well.

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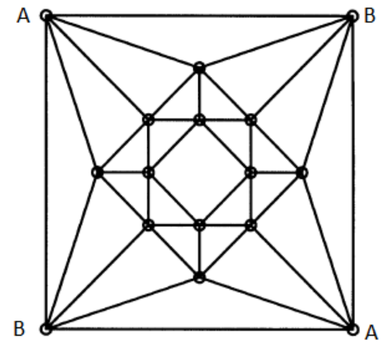

FIGURE 1. Example for a 5-connected, not 2-linked graph.

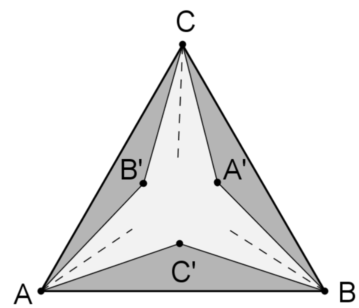

FIGURE 2. Planar graphs are not 3-linked.

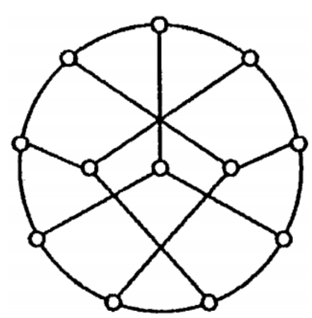

FIGURE 3. Path-pairable graph of order 12

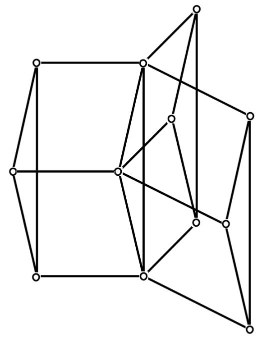

FIGURE 4. Cartesian product of a claw and a triangle

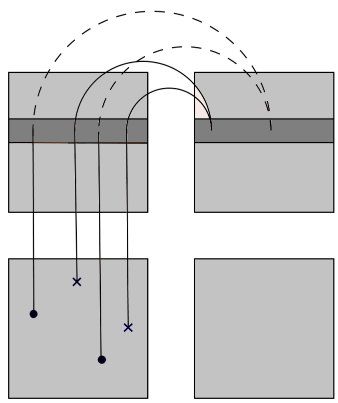

FIGURE 5. Line-up and final match phases.

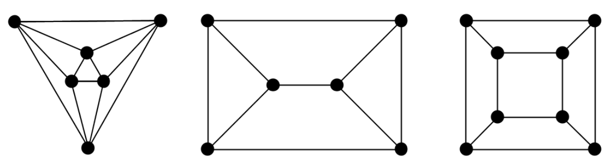

FIGURE 6. Small path-pairable planar graphs can be easily constructed.

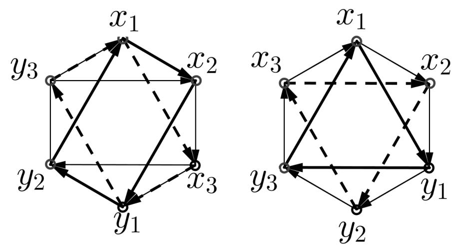

FIGURE 7. Linking via edge-disjoint directed cycles.

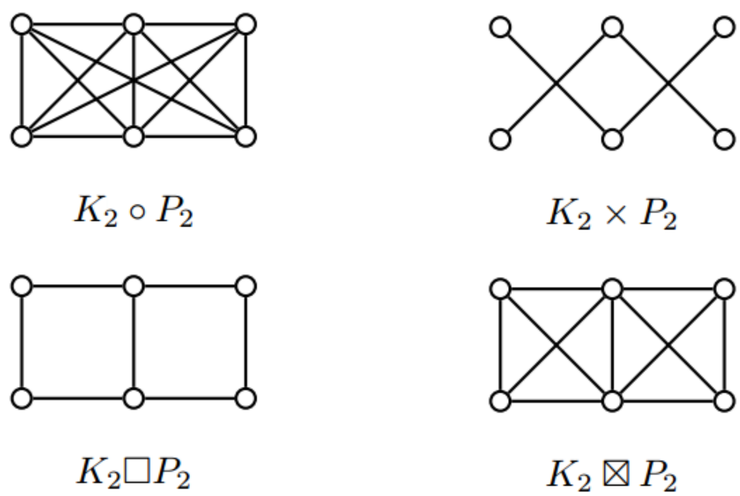

FIGURE 8. Four kinds of products of K_2 and P_2