ON CONSTRUCTIONS OF MATRIX BIALGEBRAS

submitted by

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DECLARATION

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ABSTRACT

The thesis consists of two parts. In the first part consisting of Chapter 2 and 3, matrix bialgebras, generalizations of the quantized coordinate ring of $n \times n$ matrices are considered. The defining parameter of the construction is an endomorphism of the tensor-square of a vector space. In the investigations this endomorphism is assumed to be either an idempotent or nilpotent of order two. In Theorem 2.2.2, 2.3.2 and 2.5.3 it is proved that the Yang-Baxter equation gives not only a sufficient condition – as it was known before – for certain regularity properties of matrix bialgebras, such as the Poincaré-Birkhoff-Witt basis property or the Koszul property, but it is also necessary, under some technical assumptions. The proofs are based on the methods of the representation theory of finite-dimensional algebras.

In the second part consisting of Chapter 4 and 5, the quantized coordinate rings of matrices, the general linear group and the special linear group are considered, together with the corresponding Poisson algebras called semiclassical limit Poisson algebras. In Theorem 4.1.1 and 5.1.1 it is proved that the subalgebra of cocommutative elements in the above mentioned algebras and Poisson algebras are maximal commutative, and maximal Poisson-commutative subalgebras respectively. The proofs are based on graded-filtered arguments.

Pötyinek.

És mindenkinek, aki szeret.

"(...) you'll begin challenging your own assumptions.
Your assumptions are your windows on the world.
Scrub them off every once in a while,
or the light won't come in."

— Alan Alda (62nd Commencement Address,
Connecticut College, New London, 1980)

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CONTENTS

PREFACE

The field of quantum groups and quantum algebra emerged at the intersection of ring theory, Lie theory and C^*-algebras in the 1980s from the works of V. G. Drinfeld [Dri], M. Jimbo [Ji], L. D. Faddeev et al. [FRT], Yu. I. Manin [Man], and S. L. Woronowicz [Wo].

Although the frameworks they applied were different in nature and diverged even further in the last four decades, one of the common guiding principles was to observe phenomena that are in parallel with the classical counterparts, such as the (Brauer-)Schur-Weyl duality (see [Hay] or Sec. 8.6 in [KS]), analogous representation theory (see Sec. 10.1 in [ChP]), existence of a Poincaré-Birkhoff-Witt basis (see I.6.8 in [BG]) or existence of a Haar state (see I.2. in [NT]).

In the thesis, we follow the track laid by Yu. I. Manin and M. Takeuchi, and investigate matrix bialgebras (see Def. 2.1.1) from the point of view of properties of quadratic graded algebras and their symptoms on the Hilbert series of the algebra. The terminology on these bialgebras is very diverse, they are also called quantum semigroups in [Man] (Ch. 7), matric bialgebras or conormalizer algebras in [Ta], or matrix-element bialgebras in [Su]. Moreover, matrix bialgebras are special cases of the FRT-construction in the sense of [Lu], and of the universal coacting bialgebra or coend-construction (see [EGNO] and Subsec. 2.1.1).

Main examples of matrix bialgebras include every FRT-bialgebra $\mathcal{M}(\hat{R})$, where $\hat{R}$ satisfies the Yang-Baxter equation (see [KS],[Hay]), in particular the quantized coordinate ring $\mathcal{O}_q(M_n)$ of $n \times n$ matrices for a non-zero scalar q. Further examples are the covering bialgebras of quantum SL_2 Hopf-algebras (see [DVL]) or the quantum orthogonal bialgebra $\widetilde{M}_q^+(n)$ (see [Ta]).

A well-investigated case is that of the Hecke-type FRT-bialgebras $\mathcal{M}(\hat{R})$ where $\hat{R}$ satisfies both the Yang-Baxter and the Hecke equations. These algebras are known to have several favorable properties under mild conditions (see [AA],[Hai1],[Su]). After introducing the conventions and definitions of the studied topics in **Chapter 1**, we give results in the reverse direction in **Chapter 2**. Namely a matrix bialgebra $\mathcal{M}(p)$ – associated to an element $p \in \mathrm{End}(V \otimes V)$ with minimal polynomial of degree two – cannot have the appropriate Hilbert-series implied by the above properties, without being a Hecke-type FRT-bialgebra.

One of these favorable properties is the existence of a Poincaré-Birkhoff-Witt basis. For a Hecke-type FRT-bialgebra $\mathcal{M}(\hat{R})$ this property holds, assuming q is not a third root of unity and the corresponding symmetric and exterior algebras have compatible PBW-bases (see Theorem 3 in [Su]).

In **Theorem 2.2.2** we show that a matrix bialgebra $\mathcal{M}(p)$ for an idempotent element p (with natural assumptions on certain dimensions) can have Hilbert-series $(1-t)^{-n^2}$ only if $1+bp$ satisfies the Yang-Baxter equation for some $b \neq 0, -1$ that is not a third root of unity. In **Theorem 2.3.2** we show a similar result for the case $p^2 = 0$. Note that if the minimal polynomial of p has degree two then we may assume that either $p^2 = p$ or $p^2 = 0$. The methods applied in the proofs are based on the representation theory of some serial and biserial algebras.

A weaker property of a quadratic graded algebra, compared to the existence of a PBW-basis, is the Koszul property. It was a question of Manin to characterize Koszul matrix bialgebras associated to orthogonal idempotent elements in $\mathrm{End}(V \otimes V)$ over $\mathbb{C}$ (see Section VI/6. and Problem IX/12. in [Man]). By Theorem 2.5 in [Hai1] it is known that a Hecke-type FRT-bialgebra is Koszul, assuming q is not a root of unity. In **Theorem 2.5.3** we give a more general, partial characterization of the Koszul property, in the presence of rank-related assumptions.

In both of the above cases, the idea in the background is to compare the representation theoretical decomposition of $V^{\otimes d}$ over the dual bialgebra of $\mathcal{M}(p)$ to its classical decomposition over the symmetric group (for $d = 3$ and 4, respectively). In **Section 2.4** we show how this comparison can help to give upper bounds on the coefficients of the Hilbert series of the bialgebras for arbitrary d.

In **Chapter 3** we discuss the 2-dimensional case and a motivating example. Originally our objective was to check a conjecture in [Ta] stating that the quantum orthogonal bialgebra $\widetilde{M}_q^+(n)$ has a Poincaré-Birkhoff-Witt basis (for $n = 3$ it is claimed to hold). The bialgebra is obtained by a modification of a (non-Hecke type) FRT-bialgebra so that it is a matrix bialgebra $\mathcal{M}(p)$ for an idempotent element p.

For $n = 3$ the algebra is defined by 9 generators and 36 quadratic relations, hence using a computer algebra system it is possible but not really enlightening to compute the first few terms of its Hilbert series. It turns out that $\widetilde{M}_q^+(n)$ does not have a Poincaré-Birkhoff-Witt basis even for $n = 3$ (see **Section 3.2**). The above theorems show that it is not a coincidence, but is equivalent to the fact that $\widetilde{M}_q^+(n)$ cannot be defined as Hecke-type FRT-bialgebra.

An alternative motivation is the recent activity in search of quantum $\mathbb{P}^n$ spaces (see [ZZ]), which are Artin-Schelter regular algebras of global dimension n with Hilbert series $(1-t)^{-n}$. Generalizing the fact that $\mathcal{O}_q(M_n)$ is a quantum projective space of dimension n^2, a possible source for further examples could be matrix bialgebras. The above results can be interpreted as no-go theorems in the direction that any matrix bialgebra with the appropriate Hilbert series must be a Hecke-type FRT-bialgebra.

In **Chapter 4** (based on [Me1]) we deal with the most classical matrix bialgebras and its variants, namely, the quantized coordinate rings $\mathcal{O}_q(M_n)$, $\mathcal{O}_q(SL_n)$ and $\mathcal{O}_q(GL_n)$ of $n \times n$ matrices, the special linear group and the general linear group, respectively (see [BG],[FRT],[PW]). In this case we assume that the base field is $\mathbb{C}$ and $q \in \mathbb{C}^\times$ is not a root of unity. These algebras are under active research, certain fundamental properties of them were described only recently (see for example [Ya1]).

In [DL1] M. Domokos and T. Lenagan determined generators for the subalgebra of cocommutative elements $\mathcal{O}_q(GL_n)^{\text{coc}}$ in $\mathcal{O}_q(GL_n)$ with q being not a root of unity. Their proof was based on the observation that these are exactly the invariants of some quantum analog of the conjugation action of GL_n on $\mathcal{O}(GL_n)$ which may be called modified adjoint coaction. It turned out that this ring of invariants is basically the same as in the classical setting, namely it is a polynomial ring generated by the quantum versions of the trace functions (see Subsec. 4.2.2).

The correspondence between $\mathcal{O}_q(GL_n)^{\text{coc}}$ and $\mathcal{O}(GL_n)^{\text{coc}}$ does not stop on the level of their algebra structure. In [AY] A. Aizenbud and O. Yacobi proved the quantum analog of Kostant's theorem stating that $\mathcal{O}_q(M_n)$ is a free module over the ring of invariants under the adjoint coaction of $\mathcal{O}_q(GL_n)$, provided that q is not a root of unity. Hence the description of $\mathcal{O}_q(GL_n)$ as a module over $\mathcal{O}_q(GL_n)^{\text{coc}}$ is available. The classical theorem of Kostant can be interpreted as the $q = 1$ case of this result.

In **Theorem 4.1.1** we show another strong relation: $\mathcal{O}_q(GL_n)^{\text{coc}}$ is a maximal commutative subalgebra in $\mathcal{O}_q(GL_n)$, and similarly for $\mathcal{O}_q(M_n)$ and $\mathcal{O}_q(SL_n)$. In fact we show a stronger statement, **Theorem 4.1.2**, stating that the centralizer of the cocommutative element σ_1 is $\mathcal{O}_q(GL_n)^{\text{coc}}$ and similarly for the other two cases.

The maximal commutative property of this subalgebra is a genuinely noncommutative aspect, it does not hold if $q = 1$ and neither if q is a root of unity. On first sight the phenomenon seems to have no commutative counterpart. In **Chapter 5** (based on [Me2]) we show that this is not the case. We consider the semiclassical limit Poisson algebras of the quantized coordinate rings (see Subsec. 5.2.5). These Poisson algebras received considerable attention recently, among other things because of the connection between the primitive ideals of the quantized coordinate ring $\mathcal{O}_q(SL_n)$ and the symplectic leaves of the Poisson manifold SL_n (see [Go],[HL2],[Ya2]). In **Theorem 5.1.1** we show that the Poisson-subalgebras of cocommutative elements in each Poisson-algebras form maximal Poisson-commutative subalgebras. The arguments are based on the ideas of Chapter 4.

1

PRELIMINARIES

1.1 CONVENTIONS

Throughout the thesis, we work over an algebraically closed field $\mathbb{k}$ of characteristic zero. The set $\mathbb{N}$ includes 0, and $\mathbb{N}^+$ does not. The Kronecker delta $\delta_{a,b}$ is one if $a = b$ and zero otherwise. Let us collect some linear algebraic notations and conventions we apply.

For a $\mathbb{k}$-vector space V, we denote its $\mathbb{k}$-*dimension* by $\dim V$, its *dual vector space* by $V^\vee$, and the $\mathbb{k}$-algebra of its *linear endomorphisms* by $\mathrm{End}(V)$. Direct sums, kernels, images and cokernels of maps are denoted as usual, suppressing the base field from the notation. By Vect_f we denote the category of finite-dimensional vector spaces over the field $\mathbb{k}$.

Tensor products $U \otimes V$ and $V^{\otimes d}$ (for vector spaces U, V and $d \in \mathbb{N}$) are also understood over $\mathbb{k}$ unless it is explicitly written. To simplify notations, if $\dim V < \infty$, the standard algebra identifications

$$\mathrm{End}(V)^{\otimes d} \xrightarrow{\cong} \mathrm{End}(V^{\otimes d}) \qquad (d \in \mathbb{N}) \tag{1.1}$$

$$a_1 \otimes \ldots \otimes a_d \mapsto (v_1 \otimes \ldots \otimes v_d \mapsto a_1(v_1) \otimes \ldots \otimes a_d(v_d))$$

are used without further mention. On the other hand, the *transpose (or dualization)* algebra anti-isomorphism $^\vee : \mathrm{End}(V) \to \mathrm{End}(V^\vee)$ and the vector space isomorphism $\phi : \mathrm{End}(V) \to \mathrm{End}(V)^\vee$ given by $\phi(a) = (b \mapsto \mathrm{Trace}(ab))$ will be explicit, since in the presence of a bilinear form on V, there may be non-equivalent identifications among these spaces. The rank of $a \in \mathrm{End}(V)$ is denoted by $\mathrm{rk}(a)$.

We will use the following standard indexing notations. For indexing a basis of a (finite-dimensional) vector space V, we typically use subscripts as $v_1, \ldots, v_n$, while for indexing the dual basis $f^1, \ldots, f^n \in V^\vee$ we use superscripts. Compatibly with the usual isomorphism $V \otimes V^\vee \to \mathrm{End}(V)$, $v \otimes f \mapsto \big(w \mapsto vf(w)\big)$ we denote by $e_i^j \in \mathrm{End}(V)$ the image of $v_i \otimes f^j$.

To be compatible also with the Einstein summation convention (though we will not omit summation signs), the coordinates of an endomorphism $\varphi \in \mathrm{End}(V)$ are denoted as $\varphi = \sum_{i,j} r_j^i e_i^j$ for some $r_j^i \in \mathbb{k}$. Hence if we write elements of V as column vectors, elements of $V^\vee$ as row vectors, and use matrix-vector multiplication from the left, then r^i_j appears in the i-th row and j-th column in the matrix $[\varphi]$.

By an algebra, we always mean a unital, associative $\Bbbk$-algebra. By $\Bbbk\langle H\rangle$ (resp. $\Bbbk[H]$) we denote the free (resp. free commutative) unital algebra over the alphabet H. Modules of $\Bbbk$-algebras are understood as unital left modules, unless it is stated otherwise explicitly. For an algebra A, the abelian category of left A-modules (resp. finite-dimensional left A-modules) is denoted by $A-\mathrm{Mod}$ (resp. $A-\mathrm{Mod}_f$). For an A-module M with structure map $\rho : A \to \mathrm{End}(M)$ and an element $a \in A$ we may write $\mathrm{Ker}_M(a)$ instead of $\mathrm{Ker}(\rho(a))$ (and similarly for Im). If $\varphi : A \to B$ is a morphism of algebras then the restriction functor $\mathrm{Res}_\varphi : B-\mathrm{Mod} \to A-\mathrm{Mod}$ is defined on an object $M \in B-\mathrm{Mod}$ as $a \cdot m = \varphi(a)m$ for each $m \in M$ and $a \in A$.

Similarly, by a $\Bbbk$-*coalgebra*, we mean a $\Bbbk$-vector space C endowed with a coassociative comultiplication Δ that is counital with respect to the counit ε. In details, the linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : A \to \Bbbk$ have to satisfy both $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$ and $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$. A right *comodule* of a $\Bbbk$-coalgebra C is defined as a vector space W with a linear map $\rho : W \to W \otimes C$ such that $(\rho \otimes \mathrm{id}_C) \circ \rho = (\mathrm{id}_W \otimes \Delta) \circ \rho$. Morphisms, kernel and quotients of these structures can be defined suitably (see Ch. 2 in [Rad]). The kernel of a morphism of coalgebras is called a *coideal*, that is, a subspace $I \subseteq C$ such that $\Delta(I) \subseteq I \otimes C + C \otimes I$.

As the combination of the notions of algebra and coalgebra, we define a $\Bbbk$-*bialgebra* as a $\Bbbk$-vector space A endowed with both a (unital, associative) $\Bbbk$-algebra structure, and a (counital, coassociative) $\Bbbk$-coalgebra structure, such that for all $a, b \in A$, $\Delta(ab) = \Delta(a)\Delta(b)$, $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$, $\Delta(1_A) = 1_{A\otimes A}$ and $\varepsilon(1_A) = 1_\Bbbk$. In short, Δ and ε are $\Bbbk$-algebra morphisms. Morphisms of bialgebras are defined as algebra morphisms that are coalgebra morphisms. Kernels of these are called *biideals*, ideals that are also coideals.

1.2 GRADED ALGEBRAS

For a monoid (i.e. unital semigroup) $\mathcal{S}$ and a $\Bbbk$-algebra A, A is $\mathcal{S}$-*graded* if there is a fixed decomposition $A = \bigoplus_{s\in\mathcal{S}} A_s$ for some subspaces $\{A_s \mid s \in \mathcal{S}\}$ such that $A_s \cdot A_t \subseteq A_{st}$ for all $s, t \in \mathcal{S}$. Similarly, for an $\mathcal{S}$-graded algebra A, an A-module M is $\mathcal{S}$-graded, if $A_s M_t \subseteq M_{st}$ for all $s, t \in \mathcal{S}$.

In the following, unless stated explicitly, by a *graded algebra* A we mean an $\mathbb{N}$-graded algebra, equivalently, a $\mathbb{Z}$-graded algebra with $A_i = 0$ for $i < 0$. Similarly, if A is a graded algebra, then *graded A-module* is a short-hand for $\mathbb{Z}$-graded A-module. The *Hilbert series* of a graded algebra is defined as $H(A,t) = \sum_{d=0}^{\infty}(\dim A_d)t^d$.

The most elementary graded algebra we use is the (unital) *tensor algebra*

$$\mathcal{T}(V) = \bigoplus_{d\in\mathbb{N}} V^{\otimes d}$$

with tensor product $w \cdot w' := w \otimes w'$ as the multiplication for $w, w' \in \mathcal{T}(V)$. It has a natural graded algebra structure given by the above direct sum decomposition. The tensor algebra is universal in the sense that for any algebra A, every linear map $V \to A$ extends to a unique algebra morphism $\mathcal{T}(V) \to A$.

Definition 1.2.1. A graded algebra A is called *quadratic* if the natural embedding $A_1 \hookrightarrow A$ extended to $p : \mathcal{T}(A_1) \to A$ is surjective and the two-sided ideal generated by $\mathrm{Ker}(p) \cap A_2$ is $\mathrm{Ker}(p)$.

Equivalently, a quadratic algebra is of the form $\mathcal{T}(V)/(\mathrm{Rel})$ for some vector space V and a subspace $\mathrm{Rel} \subseteq V^{\otimes 2}$, inheriting the grading from $\mathcal{T}(V)$. Define the *quadratic dual* $A^!$ of a quadratic algebra $A = \mathcal{T}(V)/(\mathrm{Rel})$ as

$$A^! := \mathcal{T}(V^\vee)/(\mathrm{Rel}^o)$$

where $\mathrm{Rel}^o = \{f \in V^\vee \mid f(r) = 0 \ (\forall r \in \mathrm{Rel})\}$.

One of the most studied subclasses of quadratic algebras is the class of Koszul algebras. An algebra A is called *Koszul* if and only if the natural embedding $A^! \hookrightarrow \mathrm{Ext}_A(\Bbbk, \Bbbk)$ is an isomorphism (for further details, see [PP]). By Corollary 2.2 in Ch. 2 of [PP], a necessary condition for an algebra A to be Koszul is that it is *numerically Koszul* i.e.

$$H(A^!, -t)H(A,t) = 1$$

where $H(A,t)$ is the Hilbert series of A. More explicitly, A is numerically Koszul if and only if

$$\sum_{k=0}^{d}(-1)^k \dim A^!_k \dim A_{d-k} = 0 \tag{1.2}$$

for all $d \geq 1$. Note that if A is quadratic, then for $1 \leq d \leq 3$, Eq. 1.2 always holds (see Sec. 2.4 in [PP]).

1.2.1 *PBW-basis*

Assume that $n := \dim V < \infty$ and fix a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ in V. In this subsection, we suppress the tensor signs when writing elements of $V^{\otimes d}$ for some $d \in \mathbb{N}$. The definitions agree with those used in [PP].

Define the *degree-lexicographic ordering* on the set of monomials

$$\mathrm{Mon}(\mathcal{B}) := \{v_{i_1} \dots v_{i_d} \in \mathcal{T}(V) \mid 1 \le i_1, \dots, i_d \le n,\ d \in \mathbb{N}\}$$

as $v_{i_1} \dots v_{i_d} <_{\mathrm{deglex}} v_{i_1} \dots v_{i_e}$ if and only if $d < e$ or $d = e$ and there is a $k \le d$ such that $(i_1, \dots, i_{k-1}) = (j_1, \dots, j_{k-1})$ and $i_k < j_k$. Note that $<_{\mathrm{deglex}}$ is a semigroup ordering, i.e. $m <_{\mathrm{deglex}} m'$ implies $smt <_{\mathrm{deglex}} sm't$ for all $m, m', s, t \in \mathrm{Mon}(\mathcal{B})$. Moreover, $<_{\mathrm{deglex}}$ is a well-ordering i.e. any subset of monomials has a minimal element.

With the purpose of defining the PBW-property of a quadratic algebra $A = \mathcal{T}(V)/I$, consider the set of quadratic monomials that cannot be written as a linear combination of smaller monomials modulo I:

$$S^{(2)} := \{v_i v_j \in \mathrm{Mon}(\mathcal{B}) \mid v_i v_j \notin \big(I_2 + \mathrm{Span}(v_k v_l \mid v_k v_l <_{\mathrm{deglex}} v_i v_j)\big)\}$$

and define the set of all monomials that cannot be written as a linear combination of smaller monomials using relations in I_2:

$$S := \{v_{i_1} \dots v_{i_d} \in \mathrm{Mon}(\mathcal{B}) \mid v_{i_j} v_{i_{j+1}} \in S^{(2)},\ \forall j \le d-1\}.$$

Then S is a $\Bbbk$-vector space generating system of A. Indeed, if $v_{i_1} \dots v_{i_d} \in \mathrm{Mon}(\mathcal{B}) \backslash S$ then it can be expressed modulo I, as a linear combination of monomials smaller with respect to $<_{\mathrm{deglex}}$, using that $<_{\mathrm{deglex}}$ is a semigroup ordering. As $<_{\mathrm{deglex}}$ is a well-ordering and $\mathcal{T}(V)$ is spanned by $\mathrm{Mon}(\mathcal{B})$, we obtain that S is indeed a $\Bbbk$-vector space generating system.

The quadratic algebra A is called a *PBW-algebra* (or said to *have a PBW-basis*), if S is independent, i.e. it is a $\Bbbk$-basis of A.

Following [Su], we say that A has a *polynomial* (resp. *exterior*) *ordering algorithm* with respect to $<_{\mathrm{deglex}}$ if $S^{(2)} \subseteq \{v_i v_j \mid i \le j\}$ (resp. $S^{(2)} \subseteq \{v_i v_j \mid i < j\}$). In this case, for all $d \in \mathbb{N}$, $\dim A_d$ is at most $\dim \mathrm{Sym}(V)_d = \binom{n+d-1}{d}$ (resp. $\dim \Lambda(V)_d = \binom{n}{d}$), since the generating system S consists of the ordered (resp. strictly ordered) monomials.

Assuming that A has a polynomial (resp. exterior) ordering algorithm, it is called a *polynomial* (resp. *exterior*) *PBW-algebra*, if it is a PBW-algebra, or equivalently, if $\dim A_d$ equals $\dim \mathrm{Sym}(V)_d$ (resp. $\dim \Lambda(V)_d$) for all $d \in \mathbb{N}$.

Remark 1.2.2. In the terminology of Gröbner bases (see [Mo]), an algebra $A = \mathcal{T}(V)/I$ is a PBW-algebra if and only if the reduced Gröbner basis of I with respect to $<_{\mathrm{deglex}}$ consists of quadratic elements.

Set the leading monomial $\mathrm{lm}(g) = m \in \mathrm{Mon}(\mathcal{B})$ if $g = cm + \sum_i c_i m_i$ for some $m_i \in \mathrm{Mon}(\mathcal{B})$, $c, c_i \in \Bbbk$ such that $c \ne 0$ and $m_i <_{\mathrm{deglex}} m$ for all i. A subset G of an ideal $I \lhd \mathcal{T}(V)$ is a (noncommutative) *Gröbner basis*, if

$$\mathrm{lm}(G) := \big(\mathrm{lm}(g) \mid g \in G\big) = \big(\mathrm{lm}(r) \mid r \in I\big) =: \mathrm{lm}(I)$$

A Gröbner basis G is called *reduced*, if for each $g \in G$, $c = 1$ in the above definition (i.e. g is monic), moreover,

$$g - \operatorname{lm}(g) \in \operatorname{Span}\big(m \in \operatorname{Mon}(\mathcal{B}) \mid m \in \operatorname{lm}(G)\big)$$

and $\operatorname{lm}(G)$ is an irredundant basis of $\operatorname{lm}(I)$.

Lemma 1.2.3. *A quadratic algebra $A = \mathcal{T}(V)/I$ has a polynomial ordering algorithm if and only if for each $i > j$*

$$v_i v_j \in I_2 + \operatorname{Span}(v_k v_l \mid v_k < v_i,\ v_k \leq v_l) \tag{1.3}$$

Similarly, A has an exterior ordering algorithm if and only if for each $i \geq j$

$$v_i v_j \in I_2 + \operatorname{Span}(v_k v_l \mid v_k < v_i,\ v_k < v_l) \tag{1.4}$$

Note that the right hand sides do not depend on j.

Proof. It is clear that if Eq. 1.3 (resp. 1.4) holds then $\{v_i v_j \mid i > j\} \subseteq \operatorname{Mon}(\mathcal{B}) \backslash S^{(2)}$ (resp. same with $i \geq j$) as $v_k < v_i$ implies $v_k v_l <_{\text{deglex}} v_i v_j$. Conversely, assume that A has a polynomial (resp. exterior) ordering algorithm. Then for all $i > j$ (resp. $i \geq j$)

$$I_2 + \operatorname{Span}(v_k v_l \mid v_k v_l <_{\text{deglex}} v_i v_j) =$$

$$= I_2 + \operatorname{Span}(v_k v_l \mid v_k v_l <_{\text{deglex}} v_i v_j,\ v_k \leq v_l)$$

(resp. the same with $v_k < v_l$). Indeed, each $v_k v_l$ such that $v_k v_l <_{\text{deglex}} v_i v_j$ can be written modulo I_2 as a sum of monomials in $S^{(2)} \subseteq \{v_i v_j \mid i \leq j\}$ (resp. $i < j$) that are also smaller than $v_i v_j$. Moreover, we claim that it is

$$= I_2 + \operatorname{Span}(v_k v_l \mid v_k < v_i,\ v_k \leq v_l)$$

(resp. the same with $v_k < v_l$) independently of j. Indeed, $\supseteq$ is clear. For the converse, assume that $v_k = v_i$, $v_l < v_j$ and $v_k \leq v_l$ (resp. $v_k < v_l$). Then $v_i = v_k \leq v_l < v_j < v_i$ by $i > j$ (resp. $v_i = v_k < v_l < v_j \leq v_i$ by $i \geq j$), but that is a contradiction. $\square$

From the Diamond Lemma (see Appendix I.11 in [BG] or Theorem 2.1 in Chapter 4 of [PP]) one may deduce the following well-known fact:

Fact 1.2.4. *A quadratic algebra $A = \mathcal{T}(V)/I$ with a polynomial (resp. exterior) ordering algorithm is a polynomial (resp. exterior) PBW-algebra if and only if $\dim(A_3)$ equals $\binom{n+2}{3}$ (resp. $\binom{n}{3}$), where $n = \dim V$.*

In the proof of Cor. 2.2.4, we will use the following lemma, that is implicit in the proof of Thm. 4.1 in Ch 4 of [PP].

Lemma 1.2.5. *Let $v_1, \ldots, v_n$ be a basis of V and assume that $A = \mathcal{T}(V)/I$ has an exterior ordering algorithm with respect to $<_{\text{deglex}}$. Then $A^!$ has a polynomial ordering algorithm with respect to the degree-lexicographic ordering corresponding to the reversely ordered basis $v_n, v_{n-1}, \ldots, v_1$.*

1.3 FINITE-DIMENSIONAL ALGEBRAS

In the following, by a *quiver* we mean a finite, oriented graph Q with vertex set $V(Q)$ and set of arrows (i.e. directed edges) $A(Q)$. A finite-dimensional $\Bbbk$-*representation* $\rho : Q \to \mathrm{Vect}_f$ of Q is a map that associates a finite-dimensional $\Bbbk$-vector space $\rho(v)$ to each $v \in V(Q)$, and a linear map $\rho(\alpha) : \rho(s_\alpha) \to \rho(t_\alpha)$ to each arrow $\alpha : s_\alpha \to t_\alpha$ in $A(Q)$. The category of finite-dimensional $\Bbbk$-representations of Q is denoted by $\mathrm{rep}(Q)$ (following the notation of [SS]).

It is well-known that there is an equivalence between $\Bbbk$-representations of Q, and (left) modules of the *path algebra* of Q:

$$\Bbbk Q := \Bbbk\langle v,\, \alpha \mid v \in V(Q),\, \alpha \in A(Q)\rangle \Big/ \Big(v^2 - v,\, 1 - \sum_{v \in V(Q)} v,\, \alpha v - \delta_{v,s_\alpha}\alpha,\, v\alpha - \delta_{v,t_\alpha}\alpha,\, \alpha_2\alpha_1 \;\Big|\; v \in V(Q),\, \alpha, \alpha_1, \alpha_2 \in A(Q),\, \alpha : s_\alpha \to t_\alpha,\, t_{\alpha_1} \neq s_{\alpha_2}\Big)$$

In other words, $\Bbbk Q$ is spanned by the directed paths in Q, using concatenation of paths as multiplication. Note that sometimes $\Bbbk Q$ is defined in the opposite way (with $v\alpha - \delta_{v,s_\alpha}\alpha$ and so on), in that case the equivalence holds for right modules. For further details about representations of quivers, one may consult [SS].

1.3.1 *Representations theory of the four subspace quiver*

Let $\mathbb{S}_4$ be the so-called *four subspace quiver*, i.e. the quiver with vertices labeled by $0, 1, 2, 3, 4$ and one arrow pointing to 0 from all other vertices. This is a Euclidean quiver i.e. the underlying graph is extended Dynkin of type $\tilde{\mathbb{D}}_4$. The indecomposable finite-dimensional $\Bbbk$-representations of $\mathbb{S}_4$ over an algebraically closed field were first described in [GP]. A complete description of these representations may be found in [MZ].

The *defect* $\partial(\rho)$ of a finite-dimensional $\Bbbk$-representation ρ of $\mathbb{S}_4$ is defined as

$$\partial(\rho) := -2\dim\rho(0) + \sum_{j=1}^{4} \dim\rho(j)$$

It is well-known that a finite-dimensional indecomposable representation ρ of $\mathbb{S}_4$ is regular (resp preinjective, resp. postprojective) if and only if $\partial(\rho) = 0$ (resp. > 0, resp. < 0), see Corollary 3.5 and 3.8 in Chapter XIII. of [SS]. In particular, if an indecomposable representation ρ has $\dim\rho(i) + \dim\rho(i+2) = \dim\rho(0)$ for $i = 1, 2$ then it is regular.

By Theorem XIII.3.13 in [SS], the category $\operatorname{add}\mathcal{R}(\mathbb{S}_4)$ of regular representations of $\mathbb{S}_4$ decomposes as

$$\operatorname{add}\mathcal{R}(\mathbb{S}_4) = \bigoplus_{\lambda \in \mathbb{P}^1} \operatorname{add}\mathcal{T}_\lambda \tag{1.5}$$

as an abelian category. A representative for the isomorphism class of each indecomposable module in $\mathcal{T}_\lambda$ can be given as follows (by the Appendix of [MZ]).

For $\lambda \notin \{0, 1, \infty\}$ and $m \in \mathbb{N}^+$ define the representation $\rho := R^{(\lambda)}[m]$ as $\rho(0) = \Bbbk^{2m}$ with a basis $e_1, \ldots, e_{2m}$ (and $e_0 := 0$ to simplify notation) together with the sequence of subspaces

$$\rho(1) = \operatorname{Span}(e_k \mid 1 \le k \le m)$$

$$\rho(2) = \operatorname{Span}(e_{m+k} \mid 1 \le k \le m)$$
$$\rho(3) = \operatorname{Span}(e_k + e_{m+k} \mid 1 \le k \le m)$$
$$\rho(4) = \operatorname{Span}(e_{k-1} + \frac{1}{1-\lambda} e_k + e_{m+k} \mid 1 \le k \le m)$$

and the arrows of $\mathbb{S}_4$ are mapped to the corresponding embeddings. The coefficient is defined as $\frac{1}{1-\lambda}$ instead of λ to keep compatibility with Subsec. 2.2.2.

For $\kappa \in \{0, 1, \infty\}$, $i = 1, 2$ and $m \in \mathbb{N}^+$, define $\rho := R_i^{(\kappa)}[2m]$ as follows. Let $\sigma \in \operatorname{Sym}(\{1, 2, 3, 4\})$ be the permutation given by the following table:

κ	0	0	1	1	∞	∞
i	1	2	1	2	1	2
σ	(23)	(14)	(34)	(12)	(12)(34)	id

Let $\rho(0) = \Bbbk^{2m}$ and

$$\rho\big(\sigma(1)\big) = \operatorname{Span}\big(e_k \mid 1 \le k \le m\big)$$

$$\rho\big(\sigma(2)\big) = \operatorname{Span}\big(e_{m+k} \mid 1 \le k \le m\big)$$
$$\rho\big(\sigma(3)\big) = \operatorname{Span}\big(e_k + e_{m+k} \mid 1 \le k \le m\big)$$
$$\rho\big(\sigma(4)\big) = \operatorname{Span}\big(e_{k-1} + e_{m+k} \mid 1 \le k \le m\big)$$

and the arrows are mapped to the corresponding embeddings. For $\kappa \in \{0, 1, \infty\}$, $i = 1, 2$ and $m \in \mathbb{N}^+$, define $R_{3-i}^{(\kappa)}[2m-1]$ as the quotient of $R_i^{(\kappa)}[2m]$ by the subrepresentation given by $\rho'(0) = \Bbbk e_{m+1}$ and $\rho'(j) = \rho(j) \cap \Bbbk e_{m+1}$ for $j = 1, 2, 3, 4$.

1.3.2 *Modules over biserial algebras*

In this subsection we discuss the module theory of special biserial algebra (based on [WW]), that we will apply in Section 2.2 and 2.3.

The notion of path-algebra can be generalized to include relations as follows. Let Q be a (finite) quiver, and consider the path-algebra $\Bbbk Q$. An ideal $I \lhd \Bbbk Q$ is called *admissible*, if there is an $m \geq 2$ such that $R^m \subseteq I \subseteq R^2$, where $R = (\alpha \mid \alpha \in A(Q))$. (If Q is acyclic and hence $\Bbbk Q$ is finite-dimensional, then R is the Jacobson-radical of $\Bbbk Q$.) A finite-dimensional algebra A is a *bound quiver algebra* if it is isomorphic to an algebra quotient of the path-algebra $\Bbbk Q$ of a finite quiver Q with an admissible ideal I. In the following, we suppress this (non-unique choice of) isomorphism and identify A with $\Bbbk Q/I$.

A well-understood subclass of bound quiver algebras is the class of *special biserial algebras* (see [WW]) defined as follows: for each $v \in V(Q)$, there are at most two arrows $\alpha_1, \alpha_2 \in A(Q)$ that touch v, moreover, for each $\alpha_1 \in A(Q)$ there is at most one arrow α_2 (resp. α_3) such that $\alpha_1\alpha_2 \notin I$ (resp. $\alpha_3\alpha_1 \notin I$). For our investigations, it is enough to restrict ourselves to *monomial* algebras, i.e. where I is generated by monomials of the arrows.

Following [WW], we may describe the isomorphism types of finite-dimensional indecomposable modules of a monomial special biserial algebra as follows. Let us call a quiver L a *walk-quiver*, if the underlying undirected graph of L is a path-graph i.e. non-empty connected graph without cycles and loops, with degrees at most two at each vertex.

For a given bound quiver algebra $\Bbbk Q/I$, let us define a *V-sequence* as a quiver-homomorphism (directed graph-homomorphism) $v : L \to Q$ where L is a walk-quiver, moreover,

- if $\bullet \overset{\beta_1}{\leftarrow} \bullet \cdots \bullet \overset{\beta_r}{\leftarrow} \bullet$ is a directed path in L, then $v(\beta_1)\dots v(\beta_r) \notin I$, and
- if $\beta_1 \neq \beta_2$ are distinct arrows in L such that either their sources or their targets agree, then $v(\beta_1) \neq v(\beta_2)$.

Similarly, let us call Z a *tour-quiver*, if Z is not a directed cycle, but the underlying undirected graph of Z a cycle-graph i.e. connected graph without loops on at least two vertices with all vertices of undirected degree two.

Let us define a *primitive V-sequence* as a quiver-homomorphism $u : Z \to Q$ where Z is a tour-quiver, moreover,

- if $\bullet \overset{\beta_1}{\leftarrow} \bullet \cdots \bullet \overset{\beta_r}{\leftarrow} \bullet$ is a directed path in Z, then $u(\beta_1)\dots u(\beta_r) \notin I$,
- if $\beta_1 \neq \beta_2$ are distinct arrows in Z such that either their sources or their targets agree, then $u(\beta_1) \neq u(\beta_2)$, and

- there is no quiver-automorphism $\sigma \neq \mathrm{id}$ of Z such that $u \circ \sigma = u$.

Given a quiver homomorphism $h : X \to Q$ and a representation ρ of X, we may induce a representation $F_h\rho$ of Q as follows:

$$(F_h\rho)(y) := \bigoplus_{h(x)=y} \rho(x) \qquad \big(y \in V(Q)\big)$$

$$(F_h\rho)(\alpha) := \bigoplus_{h(\beta)=\alpha} \rho(\beta) \qquad \big(\alpha \in A(Q)\big)$$

Note that F_h commutes with direct sum of representations. (In fact F_h can be extended to an additive functor.)

For a walk-quiver L there is a unique (up to isomorphism) faithful indecomposable representation $\mathbf{L}$ of L:

$$\begin{aligned} \mathbf{L}(x) &= \Bbbk \qquad (x \in V(L)) \\ \mathbf{L}(\beta) &= \mathrm{id}_{\Bbbk} \qquad (\beta \in A(L)) \end{aligned}$$

For a tour-quiver Z, $m \in \mathbb{N}^+$, $\lambda \in \Bbbk^\times$ and $\beta_0 \in A(Z)$ we may define a faithful indecomposable representation $\mathbf{Z}(m, \lambda, \beta_0)$ of Z as

$$\begin{aligned} \mathbf{Z}(m,\lambda,\beta_0)(x) &= \Bbbk^m \qquad (x \in V(Z)) \\ \mathbf{Z}(m,\lambda,\beta_0)(\beta) &= \mathrm{id}_{\Bbbk^m} \qquad (\beta \in A(Z)\setminus\{\beta_0\}) \\ \mathbf{Z}(m,\lambda,\beta_0)(\beta_0) &= J_m(\lambda) = \begin{bmatrix} \lambda & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & \lambda \end{bmatrix} \end{aligned}$$

where $J_m(\lambda)$ is the Jordan block of rank m with eigenvalue λ. One may observe that if $\beta_0, \beta_1 \in A(Z)$ then

$$\mathbf{Z}(m,\lambda,\beta_0) \cong \mathbf{Z}(m,\lambda^{\varepsilon(\beta_0,\beta_1)},\beta_1)$$

where $\varepsilon(\beta_0,\beta_1) = 1$ if β_0 and β_1 have the same orientation along the circle, and $\varepsilon(\beta_0,\beta_1) = -1$ if they have opposite orientations.

Given a V-sequence $v : L \to Q$ (resp. primitive V-sequence $u : Z \to Q$), we may define representations of Q as

$$\mathbf{M}(v) := F_v\mathbf{L} \qquad \mathbf{M}(u,m,\lambda,\beta_0) := F_u\mathbf{Z}(m,\lambda,\beta_0)$$

Then (suitable modification of) Prop. 2.3 in [WW] claims the following.

Proposition 1.3.1. *Let $\Bbbk Q/I$ be a monomial special biserial algebra. Then $\mathbf{M}(v)$ and $\mathbf{M}(u,n,\lambda,\beta_0)$ are all indecomposable representations of Q, annihilated by I.*

Conversely, any indecomposable representation of Q annihilated by I is isomorphic to one of the above representations.

Moreover, the list is irredundant in the sense, that no representation $\mathbf{M}(v)$ is isomorphic to $\mathbf{M}(u,n,\lambda,\beta_0)$; for some V-sequences $v : L \to Q$ and $v' : L' \to Q$, we have $\mathbf{M}(v) \cong \mathbf{M}(v')$ if and only if there is a quiver-isomorphism $\sigma : L \to L'$ such that $v' = v \circ \sigma$; and for any primitive V-sequences $u : Z \to Q$ and $u : Z' \to Q$, $\mathbf{M}(u,n,\lambda,\beta_0)$ is isomorphic to $\mathbf{M}(u',n',\lambda',\beta_0')$ if and only if $n = n'$, and there is a quiver-isomorphism $\sigma : Z \to Z'$ such that $u' = u \circ \sigma$ and $\lambda' = \lambda \cdot \varepsilon(\sigma(\beta_0),\beta_0')$.

Let L be a walk-graph. Let us call an induced directed subgraph $H \subseteq L$ a *source subgraph* (resp. *sink subgraph*), if there is no arrow from the complement of H to H (resp. from H to the complement). Such a subgraph is called connected, if the underlying undirected graph is connected.

Proposition 1.3.2. *Let L and L' be walk-quivers and $v : L \to Q$ and $v' : L' \to Q$ be V-sequences of the bound quiver algebra $\Bbbk Q/I$. Then*

$$\dim \operatorname{Hom}_{\Bbbk Q/I}\big(\mathbf{M}(v),\mathbf{M}(v')\big) = |\{f : H \to H' \text{ quiver-isomorphism} \mid H \subseteq L \text{ is a connected source subgraph}, H' \subseteq L' \text{ is a sink subgraph}, v' \circ f = v\}|$$

The proof of the claim is the same as of Lemma 4.2 in [WW].

1.3.3 *Norm-square on monoids*

In Section 2.4, we will need the following observations on the notion of norm-square defined on commutative monoids.

Recall that for a commutative monoid $\mathcal{S}$ (i.e. a commutative unital semigroup, which we will denote additively), a non-invertible element $a \in \mathcal{S}$ is called an *atom* if $a = b + c$ implies that b or c is invertible. A monoid is called *atomic*, if every non-invertible element can be written as a sum of finitely many atoms. The monoid is *factorial* if it is isomorphic to $\mathbb{N}[I] := \mathbb{N}^{\oplus I}$ for some set I.

Denote by $\mathcal{A}(\mathcal{S})$ (resp. $\mathcal{S}^\times$) the set of atoms (resp. invertible elements) in $\mathcal{S}$. Define the *norm-square* of an $s \in \mathcal{S}$ as

$$N_{\mathcal{S}}(s) := \sup\Big(\sum_{a\in\mathcal{A}(\mathcal{S})} k_a^2 \,\Big|\, \sum_{a\in\mathcal{A}(\mathcal{S})} k_a a = s\Big) \tag{1.6}$$

using the convention that $\sup(\emptyset) = -\infty$. Note that $s \mapsto N_{\mathcal{S}}(s)^{\frac{1}{2}}$ is not a norm in the sense that it satisfies neither subadditivity, nor absolute homogeneity, in general. On the other hand, for a factorial monoids, it agrees with the usual notion of norm-square, hence the name.

Lemma 1.3.3. *Let $\varphi : \mathcal{S} \to \mathcal{S}'$ be a homomorphism of atomic monoids such that $\varphi^{-1}\big((\mathcal{S}')^\times\big) \subseteq \mathcal{S}^\times$. Then for any $s \in \mathcal{S}$ one has $N_{\mathcal{S}}(s) \leq N_{\mathcal{S}'}\big(\varphi(s)\big)$.*

Moreover, if $\mathcal{S}$ and $\mathcal{S}'$ are factorial then $N_{\mathcal{S}}(s) = N_{\mathcal{S}'}\big(\varphi(s)\big)$ if and only if the restriction of φ to $\{a \in \mathcal{A}(\mathcal{S}) \mid \exists b \in \mathcal{S},\ s = a + b\}$ is injective into $\mathcal{A}(\mathcal{S}')$.

Proof. If s is invertible in $\mathcal{S}$ then both sides of the inequality are $-\infty$, hence we may assume this is not the case. As $\mathcal{S}$ is atomic, $s = \sum_{a \in A} k_a a$ for some finite subset $A \subseteq \mathcal{A}(\mathcal{S})$, $k_a \in \mathbb{N}^+$. As $\mathcal{S}'$ is also atomic, for any $a \in \mathcal{A}(\mathcal{S}')$ – using that $\varphi(a)$ is not invertible by the assumption, – we may decompose

$$\varphi(a) = \sum_{b \in B_a} l_{a,b} b$$

for some nonempty finite subsets $B_a \subseteq \mathcal{A}(\mathcal{S}')$ and $l_{a,b} \in \mathbb{N}^+$ for all $b \in B_a$. Then

$$\varphi(s) = \sum_{a \in A} k_a \sum_{b \in B_a} l_{a,b} b = \sum_{b \in \cup_a B_a} \Big(\sum_{a\colon B_a \ni b} k_a l_{a,b} \Big) b$$

Hence

$$N_{\mathcal{S}'}\big(\varphi(s)\big) \geq \sum_{b \in \cup_a B_a} \Big(\sum_{a\colon B_a \ni b} k_a l_{a,b} \Big)^2 \geq \sum_{b \in \cup_a B_a} \sum_{a\colon B_a \ni b} k_a^2 \geq \sum_{a \in A} k_a^2$$

by $l_{a,b} \geq 1$ and $B_a \neq \emptyset$.

The second and third inequalities are satisfied with equality if and only if $l_{a,b} = 1$ for all $b \in B_A$, $a \in A$, and $\{B_a \mid a \in A\}$ consists of disjoint one-element sets, i.e. φ injects A into $\mathcal{A}(\mathcal{S}')$. Hence the "only if" part of the equality statement holds using that $\mathcal{S}$ is atomic, even without the assumption that $\mathcal{S}$ and $\mathcal{S}'$ are factorial.

Conversely, assume that $s \in \mathcal{S} = \mathbb{N}[I]$, $\mathcal{S}' = \mathbb{N}[J]$ and that for all $i \in I_s := \{k \in I \mid s_k > 0\}$ we have $\varphi(e_i) = e_{j_i}$ for some $j_i \in J$. Then φ restricted to $\mathbb{N}[I_s] \subseteq \mathbb{N}[I]$ is induced by an injective map $\varphi' : I_s \to J$. In particular, φ restricted to $\mathbb{N}[I_s]$ is an isomorphism onto $\mathbb{N}[\varphi'(I_s)]$, and hence

$$N_{\mathcal{S}}(s) = N_{\mathbb{N}[I_s]}(s) = N_{\mathbb{N}[\varphi'(I_s)]}(\varphi(s)) = N_{\mathcal{S}'}(\varphi(s))$$

so the claim follows. □

1.3.4 *Endomorphisms*

The monoids we will encounter in Sec. 2.4 are typically monoids of modules in the following sense.

Let R be a $\mathbb{k}$-algebra, and denote by $\mathrm{Indec}_f(R)$ (resp. $\mathrm{Irr}_f(R)$) the set of isomorphism classes of finite-dimensional, indecomposable (resp. simple) R-modules. Then $\mathbb{N}[\mathrm{Indec}_f(R)]$ can be identified with the commutative monoid of the isomorphism classes of finite-dimensional R-modules under direct sum. Similarly, its submonoid $\mathbb{N}[\mathrm{Irr}_f(R)]$ is the monoid of semisimple finitely-dimensional R-modules.

Let us define the monoid homomorphism

$$\mathbb{N}[\mathrm{Indec}_f(R)] \to \mathbb{N}[\mathrm{Irr}_f(R)] \qquad M \mapsto M^{ss} := \bigoplus_{S \in \mathrm{Irr}_f(R)} S^{[M:S]}$$

where the *composition multiplicity* $[M : S]$ is defined as

$$[M : S] := \dim \mathrm{Hom}_R(Q_S, M) \tag{1.7}$$

for the projective cover Q_S of $S \in \mathrm{Irr}_f(R)$. In other words, M^{ss} is the semisimple R-module such that $[M : S] = [M^{ss} : S]$ for all $S \in \mathrm{Irr}_f(R)$. In fact $M \mapsto M^{ss}$ is additive on short exact sequences, but we will not need this property.

The goal of defining M^{ss} is to give a simple bound on $\dim \mathrm{End}_R(M)$ as follows.

Lemma 1.3.4. *For a finite-dimensional R-module M,*

$$\dim \mathrm{End}_R(M) \leq \dim \mathrm{End}_R(M^{ss})$$

with equality if and only if M is semisimple.

Proof. Let N be a maximal proper submodule of M and take $S := M/N$. By the left-exactness of the co- and contravariant Hom-functors,

$$\begin{aligned} \dim \mathrm{End}_R(M) &\leq \dim \mathrm{Hom}_R(S, M) + \dim \mathrm{Hom}_R(N, M) \leq \\ &\leq \sum_{X,Y \in \{N,S\}} \dim \mathrm{Hom}_R(X, Y) = \dim \mathrm{Hom}_R(S \oplus N, S \oplus N) \end{aligned}$$

Hence the inequality follows by induction on the length of M.

If M is semisimple then there is clearly an equality in the statement. Conversely, if M is not semisimple, take a maximal proper submodule N that contains the socle of M. Then $\mathrm{Hom}_R(S, M) = \mathrm{Hom}_R(S, N)$ hence we may repeat the previous inequality without the term $\dim \mathrm{Hom}_R(S, S) = 1$ on the right hand side. □

Note that the term on the right hand side of the lemma can be expressed as

$$\dim \mathrm{End}_R(M^{ss}) = \sum_{S \in \mathrm{Irr}(R)} [M : S]^2 = N_{\mathcal{S}}(M^{ss}) \tag{1.8}$$

where $\mathcal{S} = \mathbb{N}[\mathrm{Irr}_f(R)]$. Indeed, the first equality follows by the Schur lemma, as $\Bbbk$ is assumed to be algebraically closed and $\dim M$ is finite. The second equality is by the definition of $N_{\mathcal{S}}$.

For $M \in \mathcal{S} = \mathbb{N}[\mathrm{Indec}_f(R)]$, we have $N_{\mathcal{S}}(M) < \dim \mathrm{End}_R(M)$, if $M \in \mathcal{S}$ is not semisimple. In particular, Lemma 1.3.4 does not follow from Lemma 1.3.3.

Remark 1.3.5. In general, $M \mapsto (\dim \mathrm{End}_R(M))^{\frac{1}{2}}$ is not necessarily a norm on $\mathbb{R}[\mathrm{Indec}_f(R)]$, or equivalently, the bilinear extension of $(M, N) \mapsto \frac{1}{2}(\dim \mathrm{Hom}(M, N) + \dim \mathrm{Hom}(N, M))$ is not necessarily positive definite.

1.4 MONOIDAL CATEGORIES

The topic of Hopf algebras and quantum groups is closely connected to the theory of monoidal categories. Here, we list the definitions used in the next chapter, following the conventions of [EGNO].

In this work, a *category* $\mathcal{C}$ is always assumed to be *additive* and $\Bbbk$*-linear* i.e. $\mathcal{C}$ has a zero object $\mathbf{0}$, finite direct sums, and for every $x, y \in \mathcal{C}$, $\mathrm{Hom}(x, y)$ is endowed with a fixed $\Bbbk$-vector space structure such that composition of morphisms is $\Bbbk$-linear. Similarly, a *functor* F is assumed to be $\Bbbk$-linear, i.e. $F(f + g) = F(f) + F(g)$ and $F(cf) = cF(f)$ for any $f, g \in \mathrm{Hom}(x, y)$ and $c \in \Bbbk$. For further standard definitions, including kernel, cokernel, exact functor, length of an object, abelian category, and monoidal category, see [EGNO].

1.4.1 *Tensor bialgebra*

Let V be a finite-dimensional $\Bbbk$-vector space. Now we define a coalgebra structure on $\mathcal{T}(E)$ where $E := \mathrm{End}(V)$, that we will use in Def. 2.1.1. Recall that $\mathcal{T}(E)$ is a graded algebra by Sec. 1.2.

First, consider the comultiplication on E defined as

$$\Delta(a) := \tau_{(12)} \circ (a \otimes \mathrm{id}) = (\mathrm{id} \otimes a) \circ \tau_{(12)} \qquad (a \in E) \tag{1.9}$$

where $E^{\otimes 2} \cong \mathrm{End}(V^{\otimes 2})$ by Eq. 1.1, and $\tau_{(12)}(u \otimes v) = v \otimes u$ for $u, v \in V$. Note that E is coassociative as

$$((\Delta \otimes \mathrm{id}) \circ \Delta)(a) = (\mathrm{id} \otimes \tau_{(12)})(\mathrm{id} \otimes a \otimes \mathrm{id})(\tau_{(12)} \otimes \mathrm{id}) = ((\mathrm{id} \otimes \Delta) \circ \Delta)(a)$$

and counital with counit $\varepsilon(a) = \mathrm{Trace}(a)$.

More explicitly, using the notation of Sec. 1.1, let $v_1, \ldots, v_n \in V$ be a basis of V and hence $\{e_i^j \mid 1 \leq i, j \leq n\}$ is a basis in E, where $e_i^j(v_k) = \delta_{j,k} v_i$ for all i, j, k. The coproduct can be expressed as

$$\Delta(e_i^k) = \tau_{(12)} \circ \left(e_i^k \otimes \sum_{j=1}^{n} e_j^j\right) = \sum_{j=1}^{n} e_j^k \otimes e_i^j$$

Remark 1.4.1. Using the above definition of Δ, the map $\phi : \mathrm{End}(V) \to E^\vee$ given by $\phi(a) = (b \mapsto \mathrm{Trace}(ab))$ gives an algebra isomorphism between $\mathrm{End}(V)$ endowed with composition and $E^\vee$ with the multiplication $\Delta^\vee$.

Indeed, denoting the dual basis of $\{e_i^j \in E \mid i,j \leq n\}$ by $\{f_j^i \in E^\vee \mid i,j \leq n\}$ (note that the indexes are switched), we obtain $\phi(e_i^j) = f_i^j$ by $\phi(e_i^j)(e_j^i) = \mathrm{Trace}(e_i^j e_j^i) = 1$. On the other hand, $\Delta^\vee(f_k^j \otimes f_l^i) = \delta_{j,l} f_k^i$, analogously to $e_k^j \cdot e_l^i = \delta_{j,l} e_k^i$, hence ϕ is indeed an algebra isomorphism. Note that the other possible definition $\Delta^{\mathrm{op}}(a) = (a \otimes \mathrm{id}) \circ \tau_{(12)}$ would make ϕ an algebra anti-isomorphism.

The coalgebra structure of E can be extended to the tensor algebra $\mathcal{T}(E)$ with product $w \cdot w' := w \otimes w'$ such that $\mathcal{T}(E)$ is a bialgebra. Indeed, by the universality of $\mathcal{T}(E)$ as an algebra (see Sec. 1.2), the linear maps $\varepsilon : E \to \Bbbk$ and $\Delta : E \to E \otimes E \hookrightarrow \mathcal{T}(E) \otimes \mathcal{T}(E)$ extend to unique algebra morphisms $\mathcal{T}(E) \to \Bbbk$ and $\mathcal{T}(E) \to \mathcal{T}(E) \otimes \mathcal{T}(E)$, that are also denoted by ε and Δ by a slight abuse of notation. Since the equations $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$ and $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$ hold on E and all sides are algebra morphisms, they hold on the whole $\mathcal{T}(E)$ by the uniqueness part of the universality of $\mathcal{T}(E)$ as an algebra. Hence $\mathcal{T}(E)$ is indeed a bialgebra.

More explicitly, for $b = a_1 \cdot \ldots \cdot a_d \in E^{\otimes d}$ $(d \geq 2)$,

$$\Delta(b) = \Delta(a_1) \cdot \ldots \cdot \Delta(a_d) = \big(\tau_{(12)}(a_1 \otimes \mathrm{id})\big) \cdot \ldots \cdot \big(\tau_{(12)}(a_d \otimes \mathrm{id})\big) =$$
$$= \tau^{(d)}(a_1 \cdot \ldots \cdot a_d) \otimes (\mathrm{id} \cdot \ldots \cdot \mathrm{id}) = \tau^{(d)}(b \otimes \mathrm{id}_{V^{\otimes d}}) \qquad (1.10)$$

where $\tau^{(d)} \in E^{\otimes d} \otimes E^{\otimes d}$ is defined as $\tau^{(d)}(u \otimes v) = v \otimes u$ for $u, v \in V^{\otimes d}$. Similarly, we have $\Delta(b) = (\mathrm{id}_{V^{\otimes d}} \otimes b)\tau^{(d)}$.

The tensor bialgebra is universal among bialgebras on E (also called free bialgebra in [Rad] or free matric bialgebra in [Ta]), i.e. for any bialgebra B and coalgebra morphism $\varphi : E \to B$, there is a unique bialgebra morphism $\mathcal{T}(E) \to B$ extending φ. The proof is similar to the previous argument defining the bialgebra structure on $\mathcal{T}(E)$, see Theorem 5.3.1 in [Rad].

We will also need the (right) comodule structure of $V^{\otimes d}$ over the subcoalgebra $E^{\otimes d} \hookrightarrow \mathcal{T}(E)$. Let $w_1, \ldots, w_{n^d}$ be a basis of $V^{\otimes d}$, denote its dual basis by $g^1, \ldots, g^{n^d}$, and the corresponding matrix units $u \mapsto w_i g^j(u)$ by $b_i^j \in E^{\otimes d}$. The comodule structure on $V^{\otimes d}$ is defined as

$$\rho_{V^{\otimes d}} \,:\, w_i \longmapsto \sum_{j=1}^{n} w_j \otimes b_i^j \in V^{\otimes d} \otimes E^{\otimes d} \qquad (1.11)$$

for all $i = 1, \ldots, n^d$. In the case of $\rho_{V^{\otimes d}}$, the coordinate-free definition, $w \mapsto \tau_{(12)}(w \otimes \mathrm{id}_{E^{\otimes d}})$ where $\tau_{(12)}(w \otimes (u \mapsto vf(u))) = v \otimes (u \mapsto wf(u))$ would yield complicated notation, hence we omit its use.

Remark 1.4.2. If $\dim V = \infty$ then $\tau_{(12)} \notin \mathrm{End}(V)^{\otimes 2}$, hence Δ cannot be defined as above. Indeed, if $\tau_{(12)} = \sum_{i=1}^{m} \varphi_i \otimes \psi_i$ then

$$v \otimes u = \tau_{(12)}(u \otimes v) =$$

$$= \sum_{i=1}^{m} \varphi_i(u) \otimes \psi_i(v) \in \mathrm{Span}(\varphi_i(u) \mid i = 1, \dots, m) \otimes V$$

in particular for fixed u, we have $v \in \mathrm{Span}(\varphi_i(u) \mid i = 1, \dots, m)$ for all v.

1.4.2 *Ring categories*

In Subsec. 2.1.1 we will need the following notions.

Definition 1.4.3. A *ring category* $\mathcal{C}$ (Def. 4.2.3, [EGNO]) is an abelian category (in the sense of Def. 1.3.1, [EGNO]) with a monoidal structure (in the sense of Def. 2.1.1, [EGNO]) such that the tensor product is a $\Bbbk$-linear exact functor in both arguments, x has finite length in $\mathcal{C}$ for all $x \in \mathrm{Ob}(\mathcal{C})$, $\dim \mathrm{Hom}_{\mathcal{C}}(x, y) < \infty$ for all $x, y \in \mathrm{Ob}(\mathcal{C})$, and $\dim \mathrm{End}_{\mathcal{C}}(\mathbf{1}) = 1$ for the unit object $\mathbf{1}$.

An example is the category Vect_f of finite-dimensional $\Bbbk$-vector spaces equipped with the usual tensor product. In Subsec. 2.1.1, we will consider abelian monoidal subcategories $\mathcal{D}$ of Vect_f, which are automatically ring categories. Explicitly, an *abelian monoidal subcategory* $\mathcal{D}$ of Vect_f consists of a nonempty subclass $\mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathrm{Vect}_f)$ and $\Bbbk$-subspaces $\mathrm{Hom}_{\mathcal{D}}(x, y) \subseteq \mathrm{Hom}_{\mathrm{Vect}_f}(x, y)$ for all $x, y \in \mathrm{Ob}(\mathcal{D})$ such that $x \in \mathrm{Ob}(\mathcal{D}) \Rightarrow \mathrm{id}_x \in \mathcal{D}$ and morphisms are closed under taking compositions, finite direct sums, kernels, cokernels and tensor products. Equivalently, the embedding $\mathcal{D} \hookrightarrow \mathrm{Vect}_f$ is a ($\Bbbk$-linear) exact *monoidal functor* (in the sense of Def. 2.4.1, [EGNO]).

Given a bialgebra A, we may consider the ring category of fin. dim. (right) comodules $\mathrm{Comod}_f(A)$ (see Sec. 5.2 in [EGNO]), together with the forgetful functor $F_A : \mathrm{Comod}_f(A) \to \mathrm{Vect}_f$ mapping each comodule (resp. morphism) to the underlying vector space (resp. map). This way we obtain an assignment

$$\mathrm{Rep} : A \mapsto (\mathrm{Comod}_f(A), F_A)$$

that is in fact functorial i.e. a bialgebra homomorphism $\varphi : A \to A'$ induces a monoidal functor (named *corestriction*)

$$\mathrm{Cores}_{\varphi} : \mathrm{Comod}_f(A) \to \mathrm{Comod}_f(A')$$

in a composition-preserving way. Note that both F_A and Cores_{φ} are faithful and exact monoidal functors. (Calling Rep a functor would lead to size issues, as there is no class of monoidal categories with forgetful functors, using the Von Neumann–Bernays–Gödel foundation discussed in [AHS].)

By the next proposition, Rep has a left-adjoint in the following sense:

Proposition 1.4.4 (*Sec. 8, Prop. 4 in [JS]*)**.** *Let $\mathcal{C}$ be a ring category and $X : \mathcal{C} \to \mathrm{Vect}_f$ a monoidal functor. Then there is a bialgebra A with the following universal property: for any bialgebra A' and monoidal functor $X' : \mathcal{C} \to \mathrm{Comod}_f(A')$ satisfying $F_{A'} \circ X' = X$ there is a unique bialgebra homomorphism $\varphi : A \to A'$ such that $F_{A'} \circ \mathrm{Cores}_\varphi = F_A$.*

Definition 1.4.5. The *universal coacting bialgebra* $\mathrm{End}(X)^\vee$ of a functor $X : \mathcal{C} \to \mathrm{Vect}_f$ is defined as the bialgebra A given by Prop. 1.4.4.

The bialgebra $\mathrm{End}(X)^\vee$ (where the dual is understood as graded dual) is unique up to unique bialgebra isomorphism, since it is defined by a universal property. The construction of A is called the *coend construction* (see Sec 1.10 in [EGNO]). By Theorem 5.4.1 in [EGNO], if X is faithful and exact (such a functor is called a fiber functor) then $\mathcal{C}$ is equivalent to $\mathrm{Comod}_f(A)$.

The enhancement of the proposition for Hopf algebras and rigid monoidal categories is referred as Tannaka duality, see [JS], [Sch] or [Ul].

2

MATRIX BIALGEBRAS

In this chapter we define and investigate the matrix bialgebra $\mathcal{M}(p)$ corresponding to a map $p \in \mathrm{End}(V \otimes V)$.

First, in Section 2.1 we collect the definitions and basic facts about $\mathcal{M}(p)$ and the corresponding, dually defined Schur algebra $\mathrm{Sch}_d(p)$ and (generalized) Hecke algebra $\mathcal{H}_d(p)$. Then we characterize the existence of a PBW-basis (see Theorem 2.2.2 and Corollary 2.2.4), assuming that p is an idempotent and polynomial-type up to degree three (see Def. 2.2.1). In Section 2.3 we show an analogous theorem (Theorem 2.3.2) for the case when p is nilpotent of order two, an polynomial-type up to degree three in the sense of Def. 2.3.1.

In Section 2.4, we describe a method that can be used to bound the coefficients of the Hilbert series of matrix bialgebras (see Proposition 2.4.8). In Section 2.5, assuming that p is an orthogonal projection, we partially characterize the Koszul property of the bialgebra (see Theorem 2.5.3).

Throughout the chapter, $\Bbbk$ denotes an algebraically closed field, and V is a fixed $\Bbbk$-vector space of dimension $n < \infty$.

2.1 THE MATRIX BIALGEBRA $\mathcal{M}(p)$

Let $E := \mathrm{End}(V)$. The central object of our study is the following.

Definition 2.1.1. Let $p \in E^{\otimes 2} \cong \mathrm{End}(V \otimes V)$, and define the algebra

$$\mathcal{M}(p) := \mathcal{T}(E)/\left(a \circ p - p \circ a \mid a \in E^{\otimes 2}\right)$$

where $\mathcal{T}(E)$ denotes the tensor algebra of E (see Subsec. 1.2).

In particular, $\mathcal{M}(p)$ is a finitely generated quadratic algebra in the sense of Def. 1.2.1. Note that $\mathcal{M}(p) = \mathcal{M}(c\mathrm{id} + dp)$ by definition, for any $c, d \in \Bbbk$, $d \neq 0$. For an alternative definition of $\mathcal{M}(p)$ using the RTT-relations, see the end of Subsec. 2.1.1.

Recall from Subsec. 1.4.1 the bialgebra structure defined on $\mathcal{T}(E)$.

Proposition 2.1.2. *$\mathcal{M}(p)$ is a bialgebra quotient of $\mathcal{T}(E)$, i.e. the ideal $\left(a \circ p - p \circ a \mid a \in E^{\otimes 2}\right)$ is a biideal in $\mathcal{T}(E)$.*

By the proposition, we call $\mathcal{M}(p)$ the *matrix bialgebra* of p. As we will see in Subsec. 2.1.1, $\mathcal{M}(p)$ is isomorphic to the universal coacting bialgebra $\mathrm{End}(X)^\vee$ (see Def. 1.4.5), where X is the embedding of the smallest abelian monoidal subcategory $\mathcal{C}$ of Vect_f that contains p.

The standard example for $\mathcal{M}(p)$ is $p = \tau_{(12)}$ where $\tau_{(12)}(u \otimes v) = v \otimes u$ for all $u, v \in V$. Then

$$\mathcal{M}(\tau_{(12)}) \cong \mathcal{O}(M_n) \cong \Bbbk[x_{i,j} \mid 1 \leq i, j \leq n]$$

the coordinate ring of the $n \times n$ matrices over $\Bbbk$, endowed with the bialgebra structure $\Delta(x_{i,j}) = \sum_{k=1}^n x_{i,k} \otimes x_{k,j}$ for all $1 \leq i, j \leq n$. More generally, if p satisfies the Yang-Baxter equation (see Eq. 2.19), then $\mathcal{M}(p)$ is also called an FRT-bialgebra (see Section 9.1 in [KS]).

From another point of view, consider $p_{\mathrm{sym}} = \frac{1}{2}(\mathrm{id} - \tau_{(12)})$. Then we also have $\mathcal{M}(p_{\mathrm{sym}}) = \mathcal{M}(\tau_{(12)})$ by definition, i.e. the bialgebra $\mathcal{O}(M_n)$ can be constructed using an idempotent p. Similarly, if r satisfies the Hecke equation $(r + \mathrm{id})(r - q) = 0$ for some $q \in \Bbbk^\times$, then $p := \frac{1}{1+q}(q\mathrm{id} - r)$ is an idempotent and $\mathcal{M}(p) = \mathcal{M}(r)$.

If r satisfies both the Yang-Baxter equation and the Hecke equation, then the bialgebra $\mathcal{M}(r)$ have numerous regularity properties, assuming q is not a root of unity (see [Hai1], [Hai2], [Su] and Subsec. 2.2.3).

In Sec. 2.2 and 2.5 (resp. Sec. 2.3), we will investigate properties of the bialgebra $\mathcal{M}(p)$ arising from an idempotent (resp. nilpotent of order two) p, comparing these with the well-behaved case of $\mathcal{M}(r)$ where r is as in the previous paragraph. If the minimal polynomial of $p \in \mathrm{End}(V \otimes V)$ is of degree two, then we may always reduce to the above mentioned cases of $p^2 = p$ or $p^2 = 0$, as $\Bbbk$ is algebraically closed.

Proof of Proposition 2.1.2. As $E^{\otimes 2}$ is a sub-coalgebra of $\mathcal{T}(E)$, it is enough to prove that

$$\mathrm{Rel} := \left\{ a \circ p - p \circ a \mid a \in E^{\otimes 2} \right\}$$

is a coideal in $E^{\otimes 2}$. Indeed, if Rel is a coideal of $E^{\otimes 2}$ meaning that $\Delta(\mathrm{Rel}) \subseteq \mathrm{Rel} \otimes E^{\otimes 2} + E^{\otimes 2} \otimes \mathrm{Rel}$ then

$$\Delta(E^{\otimes d} \cdot \mathrm{Rel}) \subseteq (E^{\otimes d} \cdot \mathrm{Rel}) \otimes E^{\otimes (d+2)} +$$

$$+ E^{\otimes (d+2)} \otimes (E^{\otimes d} \cdot \mathrm{Rel}) \subseteq E^{\otimes (d+2)} \otimes E^{\otimes (d+2)}$$

for any $d > 2$. Similarly for $\Delta(E^{\otimes d} \cdot \mathrm{Rel})$, hence $I = (\mathrm{Rel})$ is indeed a biideal of $\mathcal{T}(E)$.

Let us denote composition by concatenation, and $\tau := \tau_{(12)}$. Then, by Eq. 1.10, for any $a \in E^{\otimes 2}$,

$$\Delta(ap - pa) = \tau(ap \otimes \mathrm{id}) - \tau(pa \otimes \mathrm{id}) =$$

$$
\begin{aligned}
&= \tau(ap \otimes \mathrm{id}) - \tau(a \otimes p) + \tau(a \otimes p) - \tau(pa \otimes \mathrm{id}) \\
&= \Big(\tau(a \otimes \mathrm{id})(p \otimes \mathrm{id}) - (p \otimes \mathrm{id})\tau(a \otimes \mathrm{id})\Big) \\
&+ \Big(\tau(a \otimes \mathrm{id})(\mathrm{id} \otimes p) - (\mathrm{id} \otimes p)\tau(a \otimes \mathrm{id})\Big) \\
&= \Big(\Delta(a)(p \otimes \mathrm{id}) - (p \otimes \mathrm{id})\Delta(a)\Big) \\
&+ \Big(\Delta(a)(\mathrm{id} \otimes p) - (\mathrm{id} \otimes p)\Delta(a)\Big)
\end{aligned}
$$

Denoting the tensor-components of $\Delta(a)$ as $\Delta(a) = \sum_i g_i \otimes h_i$, we obtain

$$
\Delta(ap - pa) = \sum_i (g_i p - p g_i) \otimes h_i + \sum_i g_i \otimes (h_i p - p h_i)
$$

that is an element of $\mathrm{Rel} \otimes E^{\otimes 2} + E^{\otimes 2} \otimes \mathrm{Rel}$. Hence Rel is indeed a coideal.
□

Remark 2.1.3. A proof of Prop. 2.1.2 using computations with coordinates can be found in Subsec. 9.1.1 of [KS].

2.1.1 *Universal property*

The subsequent chapters do not build directly on this subsection, its sole purpose is to connect the explicit definition of $\mathcal{M}(p)$ given in Def. 2.1.1 to the literature (see for example [Lu]) and the generalities discussed in Subsec. 1.4.2.

We define an abelian monoidal subcategory $\mathcal{C} \overset{X}{\hookrightarrow} \mathrm{Vect}_f$ containing V and p such that the following proposition holds.

Proposition 2.1.4. *For* $p \in \mathrm{End}(V \otimes V)$*,* $\mathcal{M}(p)$ *is isomorphic to the universal coacting bialgebra* $\mathrm{End}(X)^\vee$*.*

First, we define a subcategory $\widetilde{\mathcal{C}}$ in Vect_f that is *small* i.e. $\mathrm{Ob}(\widetilde{\mathcal{C}})$ and $\cup_{x,y \in \widetilde{\mathcal{C}}} \mathrm{Hom}(x, y)$ are sets instead of proper classes. Then we will define $\mathcal{C}$ as an intersection of abelian monoidal subcategories of $\widetilde{\mathcal{C}}$. The purpose of this technical step involving $\widetilde{\mathcal{C}}$ is twofold. First, a not necessarily small category may have too many subcategories (that do not form a set), hence considering their intersection may lead to foundational issues. Secondly, intersecting abelian monoidal subcategories containing different (but of course isomorphic) realizations of the tensor product (resp. direct sums, kernels, cokernels) of two objects or morphisms would not necessarily yield a monoidal (resp. abelian) subcategory.

Define $\widetilde{\mathcal{C}}$ as the full subcategory of Vect_f with objects as the quotient spaces of all subspaces of the form $\oplus_{i=1}^m V^{\otimes k_i}$ for $m \in \mathbb{N}$ and $k_1, \ldots, k_m \in$

$\mathbb{N}$ (cf. Birkhoff's HSP theorem), and with morphisms as all $\mathbb{k}$-linear maps between them (i.e. it is a full subcategory of Vect_f). Clearly, this is closed under compositions, finite direct sums, kernels and cokernels. Moreover, one can check that it is closed under tensor products as $\otimes$ is exact in both arguments. Therefore $\widetilde{\mathcal{C}}$ is an abelian monoidal subcategory of Vect_f.

Define $\mathcal{C}$ as minimal among the abelian monoidal subcategories of $\widetilde{\mathcal{C}}$ that contain V and p, moreover, the direct sums, kernels, cokernels and tensor product are realized in the following fixed way, instead of being defined only up to isomorphism. The kernel-object of a morphism $Q_1 \to Q_2$ where $Q_1 = U/U'$ for some

$$U' \subseteq U \subseteq \oplus_{i=1}^{m} V^{\otimes k_i}$$

$m \in \mathbb{N}$ and $k_1, \ldots, k_m \in \mathbb{N}$ is realized as the subspace $\mathrm{Ker}(U \to Q_2) \subseteq U$ modulo its intersection with U', and the corresponding kernel-morphism is the embedding. Similarly, we use the most standard choices for direct sums, cokernels and tensor products.

By the assumptions, $\mathcal{C}$ exists uniquely since the intersection (that makes sense as $\widetilde{\mathcal{C}}$ is small) of – not necessarily minimal – categories of the above mentioned form is again an abelian monoidal subcategory. By Subsec. 1.4.2, $\mathcal{C}$ is a ring category. Hence we may apply Prop. 1.4.4 to construct $\mathrm{End}(X)^{\vee}$ where $X : \mathcal{C} \hookrightarrow \mathrm{Vect}_f$ is the (faithful and exact, but not full) embedding functor.

Prop. 2.1.4 is proved in [Man] (Prop. VI/9.), [Ta] (Prop. 3.4/a) and Sudbery [Su] (Theorem 1), with slightly different formulations. For the sake of completeness, we include its proof.

Proof of Proposition 2.1.4. Let A' be a bialgebra and $X' : \mathcal{C} \to \mathrm{Comod}_f(A')$ such that $F_{A'}X' = X$. Then $X'(V)$ is a comodule over A' with underlying vector space $F_{A'}X'(V) = X(V) = V$. Recall from Subsec. 1.4.1, that $\mathcal{T}(E)$ is universal among bialgebras coacting on V, hence there is a unique bialgebra homomorphism $\psi : \mathcal{T}(E) \to A'$ such that $\mathrm{Cores}_{\psi}(V) = X'(V)$.

Let us denote composition by concatenation. We claim that it is enough to deduce that $ap - pa \in \mathrm{Ker}(\psi)$ for any $a \in E^{\otimes 2}$. Indeed, in that case ψ factorizes as $\mathcal{T}(E) \overset{\pi}{\twoheadrightarrow} \mathcal{M}(p) \overset{\varphi}{\to} A'$ giving the existence of a bialgebra morphism φ. For this φ, we have that $F_{A'}\mathrm{Cores}_{\varphi} = F_{\mathcal{M}(p)}$ holds on V and p. Using that the functors $F_{\mathcal{M}(p)}$, $F_{A'}$ and Cores_{φ} are monoidal and exact, the subcategory of $\mathcal{C}$ on which the above equality holds forms an abelian monoidal subcategory of Vect_f, hence it contains the whole abelian monoidal category "generated" by p and V, that is $\mathcal{C}$. The uniqueness of φ follows from the faithfulness of $F_{A'}$.

Now we prove that $ap - pa \in \mathrm{Ker}(\psi)$ for all $a \in E^{\otimes 2}$. Consider the coactions $\rho : V^{\otimes 2} \to V^{\otimes 2} \otimes \mathcal{T}(E)$ and $\rho' : V^{\otimes 2} \to V^{\otimes 2} \otimes A'$. We know that

$$(\mathrm{id}_{V^{\otimes 2}} \otimes \psi)\rho = \rho' \tag{2.1}$$

as ψ is a bialgebra morphism. Moreover, for any $w \in V^{\otimes 2}$

$$\rho'(pw) = (p \otimes \mathrm{id}_{\mathcal{T}(E)})\rho'(w) \tag{2.2}$$

since p is an A'-comodule homomorphism. Denote the basis of $V^{\otimes 2}$ by $w_1, \dots, w_{n^2}$, its dual basis by $g^1, \dots, g^{n^2}$, and the matrix units $u \mapsto w_i g^j(u)$ by $b_i^j \in E^{\otimes 2}$ $(1 \leq i, j \leq n^2)$. By Eq.1.11,

$$\rho(w_i) = \sum_{j=1}^{n} w_j \otimes b_i^j = \sum_{j=1}^{n} w_j \otimes w_i \otimes g^j \tag{2.3}$$

and similarly for ρ' . Then we may compute

$$\rho'(pw_i) \overset{2.1}{=} (\mathrm{id}_{V^{\otimes 2}} \otimes \psi)\rho(pw_i) \overset{2.3}{=} \sum_j (w_j \otimes \psi(pw_i \otimes g^j)) = \sum_j (w_j \otimes \psi(pb_i^j))$$

On the other hand,

$$\rho'(pw_i) \overset{2.2}{=} (p \otimes \mathrm{id}_{E^{\otimes 2}})\rho'(w_i) \overset{2.1}{=} (p \otimes \psi)\rho(w_i) \overset{2.3}{=} \sum_k (pw_k \otimes \psi(b_i^k))$$

Using the conventions of Sec. 1.1, we have $pw_k = \sum_j [p]_k^j w_j$ and $b_i^j p = \sum_k [p]_k^j b_i^k$. Hence,

$$= \sum_k \sum_j ([p]_k^j w_j \otimes \psi(b_i^k)) = \sum_j \Big(w_j \otimes \psi\big(\sum_k [p]_k^j b_i^k\big)\Big) = \sum_j (w_j \otimes \psi(b_i^j p))$$

As $w_1, \dots, w_{n^2}$ form a basis, we have $\psi(pb_i^j - b_i^j p) = 0$ for all $1 \leq i, j \leq n^2$, and the claim follows. □

The definitions given in Def. 2.1.1 and Prop. 2.1.4 have several equivalent formulations. We mention two that appear in [Su] and in [KS] (first investigated in [FRT].

In [Su] before Prop. 1. the following universal construction for quadratic coordinate algebras is given. Let $C_1, \dots, C_r$ be quadratic (in the sense of Def. 1.2.1) quotient algebras of the tensor algebra $\mathcal{T}(V^\vee)$. The *matrix-element bialgebra* $\mathcal{M}$ determined by $C_1, C_2, \dots, C_r$ is defined as the largest quotient algebra of $\mathcal{T}(E^\vee)$ such that there are algebra morphisms $\rho_i : C_i \to C_i \otimes \mathcal{M}$ completing the following commutative diagrams

$$\begin{array}{ccc} V^\vee & \longrightarrow & V^\vee \otimes E^\vee \\ \downarrow & & \downarrow \\ C_i & \xrightarrow[\rho_i]{} & C_i \otimes \mathcal{M} \end{array}$$

for $i = 1, 2, \dots, r$. Beyond the universal definition, there is an explicit construction for $\mathcal{M}$ in [Su]. Namely denote $S_i = \mathrm{Ker}(\mathcal{T}(V^\vee) \to C_i)$ and define the standard isomorphism

$$\tau_{(23)} : V^\vee \otimes V^\vee \otimes V \otimes V \to (E^\vee)^{\otimes 2}$$

via $f \otimes g \otimes v \otimes w \mapsto (x \mapsto f(xv)) \otimes (y \mapsto g(yw))$. For any vector space W and subspace $U \subseteq W$, let $U^o := \{f \in W^\vee \mid f(u) = 0, \ \forall u \in U\}$. Then, [Su] Theorem 1/a claims that

$$\mathcal{M} \cong \mathcal{T}(E^\vee)/\big(\tau_{(23)} \sum_{i=1}^{r} S_i \otimes S_i^o\big) \tag{2.4}$$

Moreover, in [Su] Theorem 1/c, it is shown that if $|\mathbb{k}| \geq r$ then $\mathcal{M} \cong \mathcal{M}(p)$ for a $p \in \mathrm{End}\big((V^\vee)^{\otimes 2}\big)$ that is a semisimple endomorphism with eigenspaces $S_1, \dots, S_r$. In our treatment, we hid the duals on the level of the definitions.

An alternative definition of $\mathcal{M}(p)$ is given in Sec. 9 of [KS]. Denote temporarily $\dim V$ by N. For a map $R \in \mathrm{End}(V \otimes V)$, let $\hat{R} := \tau_{(12)} \circ R$ and define $\mathcal{A}(R)$ as the quotient of $\mathbb{k}\langle u^i_j \mid 1 \leq i, j \leq N\rangle$ by the relations

$$\sum_{k,l} (\hat{R}^{i,j}_{k,l} u^k_m u^l_n - u^i_k u^j_l \hat{R}^{k,l}_{m,n}) \qquad (1 \leq i, j, m, n \leq N) \tag{2.5}$$

We claim that $\mathcal{A}(R) = \mathcal{M}(\hat{R})$. Indeed, we may write out the defining relation of $\mathcal{M}(\hat{R})$ in the basis $e^i_m \otimes e^j_n \in \mathrm{End}(V \otimes V) \cong E^{\otimes 2}$ $(1 \leq i, j, m, n \leq N)$, using $e^s_k \circ e^i_m = \delta_{s,m} e^i_k$ and the coefficients

$$\hat{R} = \sum_{s,t,k,l} \hat{R}^{k,l}_{s,t} e^s_k \otimes e^t_l$$

We obtain

$$\begin{aligned} &\hat{R} \circ (e^i_m \otimes e^j_n) - (e^i_m \otimes e^j_n) \circ \hat{R} = \\ &= \sum_{k,l} \hat{R}^{k,l}_{m,n} (e^i_k \otimes e^j_l) - (e^k_m \otimes e^l_n) \hat{R}^{i,j}_{k,l} \end{aligned}$$

that agrees with Eq. 2.5 (multiplied by -1). The relations in Eq. 2.5 are also called the *RTT-relations*.

2.1.2 *Dual, Schur bialgebra*

In the current setup, there is a natural analog of the Schur algebra and q-Schur algebra, which classically appear in the study of symmetric groups and Hecke algebras.

Definition 2.1.5. For $p \in \operatorname{End}(V \otimes V)$ and $d \in \mathbb{N}$, define

$$\operatorname{Sch}_d(p) := \mathcal{M}_d(p)^\vee$$

where $\mathcal{M}_d(p)$ is the homogeneous component of $\mathcal{M}(p)$ of degree d.

As $\mathcal{M}(p)_d$ is a finite-dimensional coalgebra (see Eq. 1.10), $\operatorname{Sch}_d(p)$ is a finite-dimensional algebra for all $d \in \mathbb{N}$ with multiplication $\Delta_d^\vee$, the dual of the comultiplication of $\mathcal{M}(p)$ in degree d. Together for all d, these form the graded-dual bialgebra

$$\operatorname{Sch}(p) := \bigoplus_{d \in \mathbb{N}} \operatorname{Sch}_d(p)$$

The coalgebra structure on $\operatorname{Sch}(p)$ is defined by the sum of the algebra morphisms on each $\operatorname{Sch}_d(p)$ as

$$\bigoplus_{i+j=d} m_{i,j}^\vee : \operatorname{Sch}_d(p) \to \bigoplus_{i+j=d} \operatorname{Sch}_i(p) \otimes \operatorname{Sch}_j(p)$$

that are well-defined by $\dim \operatorname{Sch}_i(p) < \infty$ for all i.

Definition 2.1.6. For $d \geq 2$ a fixed integer, let

$$p_{i,i+1} = \mathrm{id}_V^{\otimes(i-1)} \otimes p \otimes \mathrm{id}_V^{\otimes(d-1-i)} \qquad (i = 1, \dots, d-1) \tag{2.6}$$

and define $\mathcal{H}_d(p)$ as the (unital) subalgebra of $E^{\otimes d} := \operatorname{End}(V^{\otimes d})$ generated by $p_{1,2}, \dots, p_{d-1,d}$ and $\mathrm{id}_{V^{\otimes d}}$. By definition, $V^{\otimes d}$ is a (left) $\mathcal{H}_d(p)$-module.

For the $\Bbbk$-algebra $\operatorname{Sch}_d(p)$ ($d \geq 2$) one may obtain the following description, that is a suitable variation of Theorem 2.1 in [Hai1] or Prop. 4.2.5 in [Rae] (see also [RVdB]).

Proposition 2.1.7. *For any $d \geq 2$, $\operatorname{Sch}_d(p) \cong \operatorname{End}_{\mathcal{H}_d(p)}(V^{\otimes d})$ as a $\Bbbk$-algebra.*

Proof. First, for $d = 2$ the coalgebra surjection $E^{\otimes 2} \twoheadrightarrow \mathcal{M}_2(p)$ defines an algebra injection $g : \operatorname{Sch}_2(p) \hookrightarrow (E^{\otimes 2})^\vee$ where the multiplication on $(E^{\otimes 2})^\vee$ is $\Delta_2^\vee$. Let $\phi : E^{\otimes 2} \to (E^{\otimes 2})^\vee$ be the map $\phi(a) = (b \mapsto \operatorname{Trace}(ab))$, that is an algebra isomorphism by Remark 1.4.1. Then, for $\mathrm{Rel} := \big(a \circ p - p \circ a \mid a \in E^{\otimes 2}\big)$

$$\begin{aligned}\operatorname{Sch}_2(p) \cong \operatorname{Im}(\phi^{-1} \circ g) = \big\{b \in E^{\otimes 2} \mid \operatorname{Trace}(b \circ c) = 0,\ \forall c \in \mathrm{Rel}\big\} = \\ = \{b \in E^{\otimes 2} \mid \operatorname{Trace}(p \circ b \circ a) = \operatorname{Trace}(b \circ p \circ a),\ \forall a \in E^{\otimes 2}\} \\ = \{b \in E^{\otimes 2} \mid b \circ p = p \circ b\} = \operatorname{End}_{\mathcal{H}_2(p)}(V^{\otimes 2})\end{aligned}$$

For $d > 2$, we may use the same argument to deduce

$$\operatorname{Im}\big(\operatorname{Sch}_d(p) \hookrightarrow E^{\otimes d}\big) = \bigcap_{i=1}^{d-1} \{b \in E^{\otimes d} \mid b \circ p_{i,i+1} = p_{i,i+1} \circ b\}$$

The claim follows. □

2.2 IDEMPOTENT CASE

In the following, we assume that $p \in \mathrm{End}(V \otimes V)$ such that $p^2 = p$. The goal of the section is to prove Theorem 2.2.2 and as a consequence Corollary 2.2.4.

The motivation is the following: let $q \in \Bbbk^\times$ and assume that $r \in \mathrm{End}(V \otimes V)$ satisfies the Hecke equation $(r + \mathrm{id})(r - q) = 0$, and the *Yang-Baxter equation*, i.e.

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23}$$

for $r_{12} = r \otimes \mathrm{id}$ and $r_{23} = \mathrm{id} \otimes r$, suppressing the notation of composition. By Theorem 3 in [Su], $\mathcal{M}(r)$ has a PBW-basis under suitable assumptions on $A_{\mathrm{Ker}(r)}$ and $A_{\mathrm{Im}(r)}$ where $A_U = \mathcal{T}(V)/(U)$. Now we would like to prove the converse in the sense of Corollary 2.2.4 (cf. Fact 1.2.4).

Recall that $p_{12} := p \otimes \mathrm{id}$ and $p_{23} := \mathrm{id} \otimes p$ in $\mathrm{End}(V^{\otimes 3})$.

Definition 2.2.1. We say that an idempotent p is *polynomial-type up to degree three*, if p is of rank $\binom{n}{2}$, $\mathrm{id}_{V \otimes V \otimes V} - p_{12} - p_{23}$ is invertible, $p_{12} - p_{23}$ is not nilpotent, and

$$\dim(\mathrm{Ker}(p_{12}) \cap \mathrm{Ker}(p_{23})) = \binom{n+2}{3}$$
$$\dim(\mathrm{Im}(p_{12}) \cap \mathrm{Im}(p_{23})) = \binom{n}{3}$$

where $n = \dim V$.

Theorem 2.2.2. *Let $n \geq 2$ and $p \in \mathrm{End}(V \otimes V)$ be an idempotent that is polynomial-type up to degree three. Then*

$$\dim \mathcal{H}_3(p) \geq 6 - \delta_{n,2} \qquad \dim \mathcal{M}_3(p) \leq \binom{n^2+2}{3}$$

and the following are equivalent:

i) $\dim \mathcal{M}_3(p) = \binom{n^2+2}{3}$,

ii) $\dim \mathcal{H}_3(p) = 6 - \delta_{n,2}$,

iii) there are distinct scalars $a, b \in \Bbbk^\times$ such that $a + b$ and $a^2 + ab + b^2$ are nonzero, and $r = a + bp$ satisfies the Yang-Baxter equation.

Remark 2.2.3. An example for an idempotent that is polynomial-type up to degree three is $p = p_{\mathrm{sym}}$ (as it follows from Remark 2.2.5 and Lemma 2.2.12). In this case $\mathcal{T}(V)/(\mathrm{Im}(p)) \cong \Bbbk[x_1 \ldots, x_n]$, hence the name.

Note also that $\mathrm{id} - p_{12} - p_{23}$ being invertible (resp. $p_{12} - p_{23}$ being not nilpotent) is an open condition in the sense that the subset of such idempotents in $\mathrm{End}(V \otimes V)$ form a Zariski-open subset.

For any $U \subseteq V \otimes V$ define $A_U := \mathcal{T}(V)/(U)$. Recall from Subsec. 1.2.1 the definition of ordering algorithm, polynomial and exterior PBW-algebras.

Corollary 2.2.4. *Let $n \geq 2$ and $p \in \mathrm{End}(V \otimes V)$ an idempotent, polynomial-type up to degree three. Assume that there is an ordered basis $v_1, \ldots, v_n$ of V such that $A_{\mathrm{Im}(p)}$ (resp. $A_{\mathrm{Ker}(p)}$) is a polynomial (resp. exterior) PBW-algebra with respect to it.*

Then $\mathcal{M}(p)$ is a PBW-algebra if and only if there are distinct scalars $a, b \in \Bbbk^\times$ such that $a + b$ and $a^2 + ab + b^2$ are nonzero, and $r = a + bp$ satisfies the Yang-Baxter equation (Eq. 2.19).

Remark 2.2.5. $\mathrm{id} - p_{12} - p_{23}$ is invertible if and only if

$$\mathrm{Ker}(p_{12}) \cap \mathrm{Im}(p_{23}) = 0 = \mathrm{Ker}(p_{23}) \cap \mathrm{Im}(p_{12})$$

Indeed, for any $0 \neq v \in \mathrm{Ker}(\mathrm{id} - p_{12} - p_{23})$ we have $(\mathrm{id} - p_{12})v = p_{23}v$ and $(\mathrm{id} - p_{23})v = p_{12}v$, where either $(\mathrm{id} - p_{12})v$ or $p_{12}v$ is nonzero, hence is a nonzero element of $\mathrm{Ker}(p_{12}) \cap \mathrm{Im}(p_{23})$ or $\mathrm{Ker}(p_{23}) \cap \mathrm{Im}(p_{12})$.

Conversely, for $w \in \mathrm{Ker}(p_{12}) \cap \mathrm{Im}(p_{23})$ we have $(\mathrm{id} - p_{12} - p_{23})w = w - p_{23}w = 0$, and similarly for $w \in \mathrm{Ker}(p_{23}) \cap \mathrm{Im}(p_{12})$.

By Prop. 2.1.7, for any $d \geq 2$, the algebra $\mathrm{Sch}_d(p)$ (and hence the coalgebra $\mathcal{M}_d(p)$) can be understood using the $\mathcal{H}_d(p)$-module structure of $V^{\otimes d}$. In the case $p^2 = p$, $\mathcal{H}_d(p)$ is a quotient of the algebra

$$P_d := \Bbbk\langle x_1, \ldots, x_{d-1}\rangle / \big(x_i^2 - x_i \ (1 \leq i < d),\ x_i x_j - x_j x_i \ (|i - j| > 1)\big) \quad (2.7)$$

by mapping x_i to $p_{i,i+1}$ $(i = 1, \ldots, d-1)$. The proof of Theorem 2.2.2 is built on the fact that the representations of P_3 are understood. For $d \geq 4$, the method does not extend, as then P_d is of wild representation type. Indeed, $P_d/(x_i \mid i \geq 4) \cong P_4$, and P_4 is of wild representation type, see for example p. 214 in [OS].

2.2.1 *Excursion: relation to the four subspace quiver*

The modules of P_d can also be described using quiver representations as follows. For the terminology of quiver-representations, see Section 1.3.

Let $\mathbb{S}_r$ be the quiver of r subspaces, i.e. it is the directed graph on the vertices labeled by $0, 1, \ldots, r$ and one arrow pointing to 0 from all other vertices. An ordered choice $(U_1, \ldots, U_r)$ of r subspaces in a $\Bbbk$-vector space U_0 defines a $\Bbbk$-representation of $\mathbb{S}_r$: each vertex i is mapped to U_i $(i = 0, \ldots, r)$, and the corresponding arrows are mapped to the embeddings. Conversely, every representation that assigns injective linear maps to the arrows (equivalently, the representation has no simple injective direct summand) can be given this way.

Hence a P_d-module M with structure map $\varphi : P_d \to \mathrm{End}_{\Bbbk}(M)$ induces a representation ρ_d of $\mathbb{S}_{2d-2}$ defined by the subspaces

$$\mathrm{Im}\,\varphi(x_1), \ldots, \mathrm{Im}\,\varphi(x_{d-1}),\ \mathrm{Ker}\,\varphi(x_1), \ldots, \mathrm{Ker}\,\varphi(x_{d-1}) \tag{2.8}$$

Let us denote by $P_d\mathrm{-Mod}_f$ the abelian category of finite-dimensional P_d-modules. The above assignment extends to a fully faithful functor $F_d : P_d\mathrm{-Mod}_f \to \mathrm{rep}(\mathbb{S}_{2d-2})$, since the homomorphisms $\varphi : \rho \to \rho'$ of these $\Bbbk$-representations of $\mathbb{S}_r$ are in obvious bijection with linear maps $\rho(0) \to \rho'(0)$ that commutes with all x_i, i.e. that map the subspace $\rho(i)$ into $\rho'(i)$ for all $1 \leq i \leq n$.

It is well-known that the quiver $\mathbb{S}_{2d-2}$ is of tame representation type if and only if $d \leq 3$. For $d = 3$, the underlying unoriented graph of $\mathbb{S}_4$ is the four subspace quiver, which is extended Dynkin of type $\widetilde{\mathbb{D}}_4$, whose representations are completely described (see [GP] or Subsec. 1.3.1).

For simplicity, let us denote the generators of P_3 by x and y, i.e. $P_3 \cong \Bbbk\langle x, y\rangle/(x^2 - x, y^2 - y)$. For the definition of $\mathrm{add}\mathcal{R}(\mathbb{S}_4)$ and $\mathrm{add}\mathcal{T}_\lambda$, see Eq. 1.5.

Proposition 2.2.6. *The functor F_3 restricted to finite-dimensional P_3-modules defines an equivalence of $\Bbbk$-linear abelian categories with image $\bigoplus_{\lambda\in\Bbbk} \mathrm{add}\mathcal{T}_\lambda \subsetneq \mathrm{add}\mathcal{R}(\mathbb{S}_4)$.*

Proof. Let us denote F_3 by F. As F is fully faithful, it is enough to determine its image in $\mathrm{rep}(\mathbb{S}_4)$. First we show that for any finite-dimensional P_3-module M, $F(M)$ is a direct sum of regular indecomposable representations such that

$$\rho(i) \oplus \rho(i+2) = \rho(0) \qquad (i = 1, 2) \tag{2.9}$$

where $\rho(j)$ is considered as a subspace of $\rho(0)$ for all j. Indeed, for any submodule $N \leq M$, we may compute the defect (see Subsec. 1.3.1):

$$\partial(F(N)) = \sum_{u\in\{x,y\}} \Big(\dim \mathrm{Im}\, u|_N + \dim \mathrm{Ker}\, u|_N - \dim N\Big) = 0 \tag{2.10}$$

as N is closed under x and y. Hence $F(M)$ is a direct sum of regular indecomposable $\mathbb{S}_4$-representations, by Subsec. 1.3.1. Similarly, Eq. 2.9 follows from the definition of F. Conversely, to any regular indecomposable representation ρ satisfying 2.9, one can define the corresponding P_3-module where x (resp. y) acts via projection onto $\rho(3)$ (resp. $\rho(4)$) with kernel $\rho(1)$ (resp. $\rho(2)$).

Next we show that $\mathrm{Im}F$ is closed under extensions and taking regular submodules. Indeed, for P_3-modules N and Q, an extension ρ such that $\rho/F(N) \cong F(Q)$ is a direct sum of regular representations, as $\mathrm{add}\mathcal{R}(\mathbb{S}_4)$

is closed under extension. Moreover, $\rho(i) \cap \rho(i+2) = 0$ and $\dim \rho(i) + \dim \rho(i+2) = \dim \rho(0)$, as the same hold for both $F(N)$ and $F(Q)$ for $i = 1, 2$. By the previous paragraph, $\rho \cong F(M)$ for some P_3-module M. Similarly, if $\rho \leq F(M)$ is a regular subrepresentation, then we may apply the same argument, where $\rho(i) \cap \rho(i+2) = 0$ is automatic and $\dim \rho(i) + \dim \rho(i+2) \leq \dim \rho(0)$ holds with equality by the regularity assumption (see Eq. 2.10).

By the previous paragraph, it is enough to determine which simple regular representations are in ImF. For the notation on the representation of $\mathcal{S}_4$, see Subsec. 1.3.1. We show that ImF does not contain a representation isomorphic to $R_i^{(\infty)}[1]$ ($i = 1, 2$), but it does contain one isomorphic to $R_i^{(\kappa)}[1]$ ($\kappa = 0, 1$, $i = 1, 2$) and $R^{(\lambda)}[1]$ ($\lambda \in \Bbbk \backslash \{0, 1\}$). By Eq. 1.5, these are all the simple regular representations of $\mathcal{S}_4$.

For $\lambda \in \Bbbk \backslash \{0, 1\}$, a P_3-module representing the isomorphism class $R^{(\lambda)}[1]$ (denoted this way instead of $F^{-1}(R^{(\lambda)}[1])$) can be given by the homomorphism $P_3 \to \mathrm{End}(\Bbbk^2)$

$$x \mapsto \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad y \mapsto \begin{bmatrix} 0 & 0 \\ \lambda - 1 & 1 \end{bmatrix} \tag{2.11}$$

using the definition of $R^{(\lambda)}[1]$ given in Subsec. 1.3.1. Similarly, for $\kappa = 0, 1$ and $i = 1, 2$, the P_3-module $R_i^{(\kappa)}[1]$ can be defined by $x^{(\kappa,i)}, y^{(\kappa,i)} \mapsto 0$ where the variables $x^{(\kappa,i)}$ and $y^{(\kappa,i)}$ are given by the following table:

(κ, i)	$(0,1)$	$(0,2)$	$(1,1)$	$(1,2)$
$x^{(\kappa,i)}$	x	$1-x$	x	$1-x$
$y^{(\kappa,i)}$	y	$1-y$	$1-y$	y

(2.12)

Moreover, $\rho = R_i^{(\infty)}[1]$ ($i = 1, 2$) is not a P_3-module since for a P_3-module, we have $\mathrm{Ker}(u) \cap \mathrm{Im}(u) = 0$ for $i = 1, 2$ and $u \in \{x, y\}$, however $\rho(i) \cap \rho(i+2) \neq 0$. □

2.2.2 *Modules of P_3*

Recall that $P_3 \cong \Bbbk\langle x, y\rangle / (x^2 - x, y^2 - y)$. In this subsection, we describe P_3-modules without referring to the four subspace quiver $\mathcal{S}_4$ and the previous subsection. On one hand, this yields a more self-contained discussion, moreover in Subsec. 2.2.3 we will need a description of the annihilator of each module anyway, that are easier to compute this way.

By a slight abuse of notation, we will use the same notation for P_3-modules, that was used in the previous section for the representations of $\mathbb{S}_4$. In particular, we will give modules denoted as

$$\left\{R^{(\lambda)}[m],\ R_i^{(\kappa)}[m] \mid m \in \mathbb{N}^+,\ \lambda \in \Bbbk\backslash\{0,1\},\ \kappa = 0,1,\ i = 1,2\right\} \quad (2.13)$$

that constitute a complete list of isomorphism classes of indecomposable P_3-modules. By Prop. 2.2.6 this notation will not cause ambiguity.

Note the key fact that $(x-y)^2 = x + y - xy - yx$ is a central element in P_3, as $x(x-y)^2 = x - xyx = (x-y)^2 x$, and similarly for y. For a $\Bbbk$-algebra A, denote by $A-\mathrm{Mod}_f$ the abelian category of finite-dimensional A-modules.

Lemma 2.2.7. *Let A be a $\Bbbk$-algebra, and $z \in A$ a central element. Then*

$$A-\mathrm{Mod}_f \cong \bigoplus_{\lambda\in\Bbbk} \bigcup_{k=1}^{\infty} \left(A/(z-\lambda)^k\right)-\mathrm{Mod}_f$$

where the union is defined using the natural embeddings induced by

$$A/(z-\lambda)^l \twoheadrightarrow A/(z-\lambda)^k$$

for $l > k$. In particular, $\mathrm{Irr}_f(A) = \bigsqcup_{\lambda\in\Bbbk} \mathrm{Irr}_f(A/(c-\lambda))$.

Proof. Let M be a finite-dimensional A-module. The eigenspaces of z are A-submodules by centrality. Using that $\Bbbk$ is algebraically closed, there is a unique decomposition $M \cong \oplus_{\lambda\in\Bbbk} M_\lambda$ such that $(z-\lambda)^k$ annihilates M_λ for high enough k. Clearly, $\mathrm{Hom}(M_\lambda, M_{\lambda'}) = 0$ for $\lambda \neq \lambda' \in \Bbbk$, hence the decomposition gives a direct sum decomposition of the categories. □

By the lemma, instead of $P_3-\mathrm{Mod}_f$, it is enough to understand the modules of the algebras

$$A_{\lambda,k} := P_3/((x-y)^2 - \lambda)^k \quad (2.14)$$

These are finite-dimensional by the next remark. In the terminology of finite-dimensional hereditary algebras (see [SS]), the Auslander-Reiten quivers of $\cup_{k\in\mathbb{N}^+} A_{\lambda,k} - \mathrm{Mod}_f$ are tubes of rank one for $\lambda \neq 0,1$ and of rank two otherwise, by the following lemmas.

Remark 2.2.8. The following subset is a $\Bbbk$-basis of P_3:

$$\{1, (xy)^j x, (yx)^j y, (xy)^{j+1}, (yx)^{j+1} \mid j \in \mathbb{N}\}$$

Moreover, the coefficient of $(yx)^k$ in $((x-y)^2-\lambda)^k$ is $(-1)^k \neq 0$, hence $(yx)^k$ can be expressed in $A_{\lambda,k}$ as a linear combination of $(xy)^k$ and monomials of degree at most $k-1$. In particular, $\dim A_{\lambda,k} \leq 2k$.

As a side result of Lemma 2.2.9 and 2.2.10, we will obtain that in fact $\dim A_{\lambda,k} = 2k$ and hence $\{1, (xy)^j x, (yx)^j y, (xy)^{j+1}, (yx)^{j+1}\ 0 \leq j \leq k-1\}\backslash\{(yx)^k\}$ is a $\Bbbk$-basis of $A_{\lambda,k}$. (A fact that is also easy to see by the definition of $A_{\lambda,k}$.)

Lemma 2.2.9. *For all $k \in \mathbb{N}^+$ and $\lambda \in \Bbbk \backslash \{0,1\}$,*

$$A_{\lambda,k} \cong M_2[t]/(t^k) \sim \Bbbk[t]/(t^k)$$

where M_2 is the algebra of 2×2 matrices (over $\Bbbk$) and $\sim$ denotes Morita equivalence.

Proof. Consider the unital homomorphism $\varphi_{\lambda,k} : A_{\lambda,k} \to M_2[t]/(t^k)$ defined as

$$x \mapsto \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad y \mapsto \begin{bmatrix} 0 & 0 \\ t+\lambda-1 & 1 \end{bmatrix} \tag{2.15}$$

extended multiplicatively. This map is well-defined, because for $u \in \{x, y\}$, $\varphi_{\lambda,k}(u^2) = \varphi_{\lambda,k}(u)$, moreover,

$$\varphi_{\lambda,k}\big((x-y)^2\big) = \begin{bmatrix} 1 & -1 \\ 1-t-\lambda & -1 \end{bmatrix}^2 = \begin{bmatrix} t+\lambda & 0 \\ 0 & t+\lambda \end{bmatrix}$$

hence $\varphi_{\lambda,k}\big(((x-y)^2-\lambda)^k\big) = 0$. We may also verify that $\varphi_{\lambda,k}$ is surjective since $t \cdot [\mathrm{id}] \in \operatorname{Im}(\varphi_{\lambda,k})$ and

$$\varphi_{\lambda,k}\big(x(1-y)\big) = \begin{bmatrix} t+\lambda & 0 \\ 0 & 0 \end{bmatrix} \qquad \varphi_{\lambda,k}\big(y(1-x)\big) = \begin{bmatrix} 0 & 0 \\ 0 & t+\lambda \end{bmatrix}$$

where $t + \lambda \in \Bbbk[t]/(t^k)$ is invertible if $\lambda \neq 0$. Therefore if $\lambda - 1 \neq 0$ then every matrix unit is in $\operatorname{Im}(\varphi_{\lambda,k})$, hence $\varphi_{\lambda,k}$ is surjective. By $\dim A_{\lambda,k} \leq 2k$ (see Remark 2.2.8), $\varphi_{\lambda,k}$ is an isomorphism. □

Lemma 2.2.10. *For $k \in \mathbb{N}^+$ and $\lambda \in \{0,1\}$,*

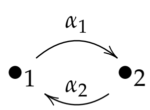
$$A_{\lambda,k} \cong \Bbbk Q/\big((\alpha_i\alpha_{3-i})^k \mid i = 1,2\big)$$

where

$$\Bbbk Q := \Bbbk\langle e, \alpha_1, \alpha_2\rangle/(e^2 - e,\ \alpha_1^2,\ \alpha_2^2,\ e\alpha_1 - \alpha_1,\ \alpha_2 e - \alpha_2,\ \alpha_1 e,\ e\alpha_2)$$

The notation $\Bbbk Q$ stands for the fact that it is the path algebra (see Subsec. 1.3.1) of the quiver Q with two vertices connected by one arrow in both directions:

Q :



(2.16)

Proof. First note that it is enough to prove the statement for $\lambda = 0$, since $x \mapsto 1 - x$, $y \mapsto y$ defines an $A_{0,k} \to A_{1,k}$ isomorphism. Indeed, $(1-x)^2 = (1-x)$, moreover in P_3,

$$(1-x-y)^2 = 1 - x - y + xy + yx = 1 - (x-y)^2 \tag{2.17}$$

so the above mentioned map is an $A_{0,k} \to A_{1,k}$ homomorphism, that is involutive, hence an isomorphism.

For $\lambda = 0$, let us define $\varphi_k : A_{0,k} \to \Bbbk Q/I$, where $I = \big((\alpha_i\alpha_{3-i})^k \mid i = 1,2\big)$, as

$$x \mapsto e, \qquad y \mapsto e + \alpha_1 + \alpha_2 + \sum_{i=0}^{k-2} C_i\Big((\alpha_2\alpha_1)^{i+1} - (\alpha_1\alpha_2)^{i+1}\Big)$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. Then φ_k is well-defined as $\varphi_k(x)^2 = \varphi_k(x)$ and

$$\varphi_k(y)^2 = \left(e + \alpha_1 + \alpha_2 + \sum_{i=0}^{k-2} C_i\Big((\alpha_2\alpha_1)^{i+1} - (\alpha_1\alpha_2)^{i+1}\Big)\right)^2 =$$

$$= e + \alpha_1 + \alpha_2 + \alpha_1\alpha_2 + \alpha_2\alpha_1 - 2\sum_{i=0}^{k-2} C_i(\alpha_1\alpha_2)^{i+1}$$

$$+ \sum_{i=0}^{k-2} C_i\Big(\alpha_1(\alpha_2\alpha_1)^{i+1} - \alpha_2(\alpha_1\alpha_2)^{i+1}\Big)$$

$$+ \sum_{i=0}^{k-2} C_i\Big((\alpha_2\alpha_1)^{i+1}\alpha_2 - (\alpha_1\alpha_2)^{i+1}\alpha_1\Big)$$

$$+ \sum_{i=0}^{k-2}\sum_{j=0}^{k-2} C_iC_j\big((\alpha_2\alpha_1)^{i+j+2} + (\alpha_1\alpha_2)^{i+j+2}\big)$$

$$= e + \alpha_1 + \alpha_2 + \alpha_2\alpha_1 + (1 - 2C_0)\alpha_1\alpha_2$$

$$+ \sum_{l=0}^{k-3}\Big(\sum_{i+j=l} C_iC_j - 2C_{l+1}\Big)(\alpha_1\alpha_2)^{l+2} + \sum_{l=0}^{k-3}\Big(\sum_{i+j=l} C_iC_j\Big)(\alpha_2\alpha_1)^{l+2}$$

$$= e + \alpha_1 + \alpha_2 + \alpha_2\alpha_1 - \alpha_1\alpha_2 + \sum_{i=1}^{k-2} C_i\Big((\alpha_2\alpha_1)^{i+1} - (\alpha_1\alpha_2)^{i+1}\Big) = \varphi_k(y)$$

where we used $C_0 = 1$ and the standard recursion formula $\sum_{i+j=n} C_iC_j = C_{n+1}$. Moreover, $\varphi_k(x-y) \in \operatorname{Rad}(\Bbbk Q/I)$ so $\varphi_k(x-y)^{2k} \subseteq \operatorname{Rad}(\Bbbk Q/I)^k = 0$. Therefore φ_k is indeed well-defined.

One can also prove recursively that φ_k is surjective. Indeed, for all $j = 1, \dots, k-1$

$$\varphi_k\big(x(x-y)^{2j+1}\big) \in (-1)(\alpha_1\alpha_2)^j\alpha_1 + \operatorname{Rad}(\Bbbk Q/I)^{2j+2}$$

and similarly for $x(x-y)^{2j}$, $(1-x)(x-y)^{2j+1}$ and $(1-x)(x-y)^{2j}$. As $\dim \Bbbk Q/I = 2k$ and $\dim A_{0,k} \le 2k$ by Remark 2.2.8, the claim follows. □

By Lemma 2.2.9 and 2.2.10, $A_{\lambda,k}$ is a uniserial ring for all $\lambda \in \Bbbk$, $k \in \mathbb{N}^+$. As uniserial rings are particular cases of monomial special biserial algebras, we may determine the indecomposable modules of $A_{\lambda,k}$ by Prop. 1.3.1.

For $\lambda \neq 0, 1$, $A_{\lambda,k}$ has a unique (up to isomorphism) indecomposable $2m$-dimensional module for each $1 \leq m \leq k$, let us denote it by $R^{(\lambda)}[m]$.

Similarly, for $\lambda \in \{0,1\}$, $i \in \{1,2\}$ and $1 \leq m \leq 2k$ we may define the indecomposable $A_{\lambda,k}$-module $R_i^{(\lambda)}[m]$ as follows. In the notation of Subsec. 1.3.2, consider the walk-quiver L_{m-1}:

$$\bullet \xleftarrow{\beta_1} \bullet \xleftarrow{\beta_2} \bullet \xleftarrow{\beta_3} \dots \xleftarrow{\beta_{m-1}} \bullet$$

Let Q be the quiver as in Eq. 2.16, and define a V-sequence $v : L_{m-1} \to Q$ by $\beta_j \mapsto \alpha_{1+\bar{j}}$ for all $j \leq m-1$ where $\bar{j} \in \{0,1\}$ is j modulo 2. Using the isomorphism given in 2.2.10, define the representation $R_i^{(\lambda)}[m] := \mathbf{M}(v)$ (see Subsec. 1.3.2).

More explicitly, $A_{\lambda,k}$ has two non-isomorphic simple modules. We denote by $R_1^{(\lambda)}[1]$ (resp. by $R_2^{(\lambda)}[1]$) the one with annihilator $(x, \lambda - y)$ (resp. $(1-x, 1-\lambda-y)$). Moreover, for all $1 \leq m \leq k$ and $i \in \{1,2\}$, $A_{\lambda,k}$ has a unique (up to isomorphism) indecomposable m-dimensional module with socle (i.e. minimal semisimple submodule) isomorphic to $R_i^{(\lambda)}[1]$, we denote it by $R_i^{(\lambda)}[m]$. Therefore, by Lemma 2.2.7, Eq. 2.13 is a complete list of finite-dimensional P_3-modules.

By Prop. 1.3.2, we may deduce the following combinatorial lemma.

Proposition 2.2.11. *For all* $\lambda \in \{0,1\}$, $i, i' \in \{1,2\}$ *and* $m, m' \in \mathbb{N}^+$,

$$\dim \operatorname{Hom}_{P_3}\left(R_i^{(\lambda)}[m], R_{i'}^{(\lambda)}[m']\right) = \\ = \begin{cases} \left\lfloor \frac{\min(m,m')+1}{2} \right\rfloor & \text{if } (m \leq m',\ i = i') \text{ or} \\ & \qquad (m > m',\ m+i \equiv m'+i' \bmod 2) \\ \left\lfloor \frac{\min(m,m')}{2} \right\rfloor & \text{otherwise} \end{cases}$$

In particular,

$$[R_i^{(\lambda)}[m] : R_{i'}^{(\lambda)}[1]] = \left\lfloor \frac{m + \delta_{i,i'}}{2} \right\rfloor$$

The statement can also be visually verified using Loewy diagrams. As a demonstration, denote $S(i) := R_i^{(0)}[1]$ $(i = 1, 2)$, and observe

$$R_1^{(0)}[5] : \begin{matrix} S(1) \\ S(2) \\ S(1) \\ S(2) \\ S(1) \end{matrix} \qquad R_2^{(0)}[4] : \begin{matrix} S(1) \\ S(2) \\ S(1) \\ S(2) \end{matrix} \qquad \dim \operatorname{Hom}_{P_3}(R_1^{(0)}[5], R_2^{(0)}[4]) = 2$$

In the proof of Theorem 2.2.2, we will need the following straightforward observations, connecting the polynomial-type assumption on p with the module theory of P_3.

Lemma 2.2.12. *Let M be a finite-dimensional P_3-module. Then $1 - x - y$ acts invertibly on M if and only if*

$$\operatorname{Hom}\big(R_i^{(1)}[m], M\big) = 0 \qquad (i = 1, 2,\ m \in \mathbb{N}^+)$$

Similarly, $x - y$ is nilpotent on M if and only if every direct summand of M is isomorphic to $R_i^{(0)}[m]$ for some $i = 1, 2$ and $m \in \mathbb{N}^+$. Moreover,

$$\dim\big(\operatorname{Ker}_M(x) \cap \operatorname{Ker}_M(y)\big) = \dim \operatorname{Hom}_{P_3}(R_1^{(0)}[1], M)$$

$$\dim\big(\operatorname{Im}_M(x) \cap \operatorname{Im}_M(y)\big) = \dim \operatorname{Hom}_{P_3}(R_2^{(0)}[1], M)$$

$$\operatorname{rk}_M(u) = \big[M : R_2^{(0)}[1]\big] + \big[M : R_i^{(1)}[1]\big] + \sum_{\lambda \in \Bbbk \setminus \{0,1\}} \big[M : R^{(\lambda)}[1]\big] \tag{2.18}$$

for $(u, i) \in \{(x, 2), (y, 1)\}$.

Proof. First, note that all of the above formulas are additive in M, hence it is enough to check for M indecomposable. By Lemma 2.2.7, an indecomposable P_3-module N is induced from an $A_{1,k}$-module for some $k \in \mathbb{N}^+$ if and only if the element $1 - (x - y)^2 = (1 - x - y)^2$ (see Eq. 2.17) acts non-invertibly on M. Hence the first claim follows by the definition of $R_i^{(1)}[m]$. Similarly, $x - y$ is nilpotent if and only if $P_3 \to \operatorname{End}_{\Bbbk}(M)$ factors through $P_3 \twoheadrightarrow A_{0,k}$ for some $k \in \mathbb{N}^+$.

If $0 \neq v \in \operatorname{Ker}_M(x) \cap \operatorname{Ker}_M(y)$ then $\Bbbk v$ is a P_3-submodule isomorphic to $R_1^{(0)}[1] \cong P_3/(x, y)$, and vice versa. Similarly, for $\operatorname{Im}_M(x) \cap \operatorname{Im}_M(y)$ and $R_2^{(0)}[1] = P_3/(1 - x, 1 - y)$.

Moreover, both sides of 2.18 are additive on short exact sequences of P_3-modules, hence it is enough to check the equation for simple P_3-modules. For these we have $\operatorname{rk}_{R^{(\lambda)}[1]}(x) = 1$ for $\lambda \in \Bbbk \setminus \{0, 1\}$ and $\operatorname{rk}_{R_i^{(\lambda)}[1]}(x) = \delta_{i,2}$ for $\lambda = 0, 1$ and $i = 1, 2$ (and similarly for y). □

2.2.3 *Yang-Baxter equation*

In this subsection we investigate for a given idempotent $p \in \operatorname{End}(V \otimes V)$, whether there is an element $r \in \operatorname{Span}(\mathrm{id}, p) = \mathcal{H}_2(p)$ that satisfies the Yang-Baxter equation

$$r_{12} r_{23} r_{12} = r_{23} r_{12} r_{23} \tag{2.19}$$

where $r_{12} = \mathrm{id} \otimes r$ and $r_{23} = \mathrm{id} \otimes r$.

Proposition 2.2.13. *Let M be a finite-dimensional P_3-module and $a, b \in \Bbbk^\times$ such that $a+b$ and a^2+ab+b^2 are nonzero.* Then

$$D_{a,b} := \big((a+bx)(a+by)(a+bx) \\ - (a+by)(a+bx)(a+by)\big) \in \operatorname{Ann}(M)$$

if and only if M is isomorphic to the direct sum of copies of

$$R_1^{(0)}[1],\ R_2^{(0)}[1],\ R^{(\lambda)}[1]$$

where $\lambda = \frac{1}{b^2}(a^2+ab+b^2)$.

If $a^2+ab+b^2 = 0$ then $D_{a,b} \in \operatorname{Ann}(M)$ if and only if M is isomorphic to the direct sum of copies of $R_i^{(0)}[m]$ for $i = 1,2$ and $1 \leq m \leq 3$.

Remark 2.2.14. The element $D_{a,b}$ annihilates $M = V^{\otimes 3}$ if and only if $r := a+bp$ satisfies the Yang-Baxter Equation. If we use the normalization $a = q$ and $b = -(1+q)$, then r satisfies $(r+\mathrm{id})(r-q) = 0$, hence $\mathcal{H}_3(p)$ is the Hecke algebra $\mathcal{H}_{q,3}$, and $\lambda = 1 - \frac{q}{(1+q)^2}$.

Proof. After simplification,

$$D_{a,b} = b^3(xyx - yxy) + ab(a+b)(x-y)$$

Since $\operatorname{Ann}(\oplus_i M_i) = \cap_i \operatorname{Ann}(M_i)$, we may assume that M is indecomposable.

By Remark 2.2.8, the images of x, y, xyx and yxy in $\operatorname{End}_\Bbbk(R^{(\lambda)}[m])$ are independent for $\lambda \in \Bbbk\backslash\{0,1\}$, $m > 1$ and similarly for $R_i^{(\lambda)}[m]$ for $\lambda \in \{0,1\}$, $i \in \{1,2\}$, $m > 3$.

By the relations

$$-x\big((x-y)^2 - \lambda\big) = xyx + (\lambda-1)x$$

$$-y\big((x-y)^2 - \lambda\big) = yxy + (\lambda-1)y$$

for $\lambda \in \Bbbk\backslash\{0,1\}$ we have $D_{a,b} \in \operatorname{Ann}\big(R^{(\lambda)}[1]\big)$ if and only if $\lambda - 1 = \frac{a(a+b)}{b^2}$ as we claimed.

Considering $R_i^{(\lambda)}[1]$ for $\lambda \in \{0,1\}$, one may observe that $D_{a,b}$ is not an element of the ideals

$$\operatorname{Ann}(R_1^{(1)}[1]) = (x, 1-y) \qquad \operatorname{Ann}(R_2^{(1)}[1]) = (1-x, y)$$

by $a+b \neq 0$. On the other hand, it is contained in

$$\operatorname{Ann}(R_1^{(0)}[1]) = (x, y) \qquad \operatorname{Ann}(R_2^{(0)}[1]) = (1-x, 1-y)$$

Moreover, $D_{a,b} \in \mathrm{Ann}(R_i^{(0)}[3])$ $(i = 1,2)$ if and only if $a^2 + ab + b^2 = 0$. Indeed,

$$\mathrm{Ann}(R_1^{(0)}[3]) = -(x-y)^2 \cdot (x,y) =$$
$$= (xyx - x, yxy - y) \ni xyx - yxy - (x-y)$$
$$\mathrm{Ann}(R_2^{(0)}[3]) = (x-y)^2 \cdot (1-x, 1-y) =$$
$$= (y - xy - yx + xyx, x - yx - xy + yxy) \ni xyx - yxy - (x-y)$$

Similarly, for $R_i^{(0)}[2]$ $(i = 1,2)$. Hence $D_{a,b} \in \mathrm{Ann}(R_i^{(0)}[m])$ $(i = 1,2,$ $1 \leq m \leq 3)$ if and only if $ab(a+b) = -b^3$. □

Remark 2.2.15. If $a + b = 0$ then $D_{a,b} = b^3(xyx - yxy)$ that is contained in $\mathrm{Ann}(R_i^{(\lambda)}[1])$ for $\lambda = 0,1$ and $i = 1,2$, but not contained in $\mathrm{Ann}(R_i^{(\lambda)}[m])$ if $m > 1$, neither in $\mathrm{Ann}(R^{(\lambda)}[m])$ for any $\lambda \in \Bbbk \backslash \{0,1\}$ and $m \in \mathbb{N}^+$.

2.2.4 *Proof of Theorem 2.2.2*

The proof of Theorem 2.2.2 is based on the following proposition.

Proposition 2.2.16. *Let $n \geq 2$ and M a P_3-module of dimension n^3 such that the action of $1 - x - y$ (resp. $x - y$) is invertible (resp. not nilpotent) on M. Moreover, assume that*

$$\dim\left(\mathrm{Ker}_M(x) \cap \mathrm{Ker}_M(y)\right) = \binom{n+2}{3}$$
$$\dim\left(\mathrm{Im}_M(x) \cap \mathrm{Im}_M(y)\right) = \binom{n}{3}$$
$$\mathrm{rk}_M(x) = \mathrm{rk}_M(y) = \binom{n}{2} n.$$

Then

$$\dim \mathrm{End}_{P_3}(M) \leq \binom{n^2+2}{3}$$

with equality if and only if there is a $\lambda \in \Bbbk \backslash \{0,1\}$ such that

$$M \cong \left(R_1^{(0)}[1]\right)^{n_{0,1}} \oplus \left(R_2^{(0)}[1]\right)^{n_{0,2}} \oplus \left(R^{(\lambda)}[1]\right)^{c} \tag{2.20}$$

where $n_{0,1} = \binom{n+2}{3}$, $n_{0,2} = \binom{n}{3}$ and $c = 2\binom{n+1}{3}$.

Remark 2.2.17. If M is assumed to be semisimple, then Prop. 2.2.16 is more straightforward. Also it is a consequence of Prop. 2.4.10. However, the general case cannot be reduced to the semisimple one.

The length of the proof may also be justified by the fact that if $x - y$ is nilpotent then the proposition does not hold. That is why $p_{12} - p_{23}$ is assumed to be not nilpotent in Def. 2.2.1, though we were unable to find an idempotent $p \in \mathrm{End}(V \otimes V)$ such that $V^{\otimes 3}$ is an explicit counterexample to the statement of Theorem 2.2.2 without the assumption above.

Indeed, by Lemma 2.2.12, $x - y$ is nilpotent if and only if M is an $A_{0,k}$-module for some k. Consider

$$M := (R_1^{(0)}[2])^{k_{1,2}} \oplus (R_1^{(0)}[3])^{k_{1,3}} \oplus (R_2^{(0)}[2])^{k_{2,2}} \oplus (R_2^{(0)}[3])^{k_{2,3}}$$

where $k_{1,2} = k_{2,2} = n$,

$$k_{1,3} = \binom{n+1}{3} + \binom{n}{2} = \frac{1}{6}(n-1)n(n+4)$$

$$k_{2,3} = \binom{n+1}{3} - \binom{n}{2} - n = \frac{1}{6}(n-4)n(n+1)$$

Then M satisfies every assumption of Prop. 2.2.16 except $x - y$ is not nilpotent on M, by Remark 2.2.5 and Lemma 2.2.12. Note also that by Prop. 2.2.13 there are $a, b \in \Bbbk^\times$ such that $a^2 + ab + b^2 = 0$ and $D_{a,b} \in \mathrm{Ann}(M)$.

On the other hand, by Prop. 2.2.11,

$$\dim \mathrm{End}_{P_3}(M) = k_{1,2}^2 + k_{2,2}^2 + 2\Big(k_{1,3}^2 + k_{2,3}^2 + k_{1,2}k_{1,3} + k_{2,2}k_{2,3}$$

$$+ k_{1,2}k_{2,2} + k_{1,2}k_{2,3} + k_{1,3}k_{2,2} + k_{1,3}k_{2,3}\Big) = \binom{n^2+2}{3} + n^2$$

The example shows that careful estimations are required in the proof.

In the proof of Prop. 2.2.16, we will need the following lemma.

Lemma 2.2.18. *Let $k, m, m' \in \mathbb{N}$ such that $m + 3 \leq m' \leq k$, and let N be a finite-dimensional $A_{0,k}$-module such that all of its indecomposable direct summands are isomorphic to $R_i^{(0)}[l]$ for some $i = 1, 2$ and $m \leq l \leq m'$. Then*

$$\dim \mathrm{End}_{A_{0,k}}(R_j^{(0)}[m] \oplus N \oplus R_{j'}^{(0)}[m']) <$$
$$< \dim \mathrm{End}_{A_{0,k}}(R_j^{(0)}[m+2] \oplus N \oplus R_{j'}^{(0)}[m'-2])$$

for any $j, j' \in \{1, 2\}$.

Using Loewy diagrams (see Subsec. 2.2.2), the lemma can be visualized as

$$\dim \mathrm{End}_{A_{0,k}} \begin{matrix} & S(2) \\ & S(1) \\ S(1) \oplus & S(2) \\ & S(1) \\ & S(2) \end{matrix} = 4 < 6 = \dim \mathrm{End}_{A_{0,k}} \begin{matrix} S(1) & S(2) \\ S(2) \oplus & S(1) \\ S(1) & S(2) \end{matrix}$$

for the modules $R_1^{(0)}[1] \oplus R_2^{(0)}[5]$ and $R_1^{(0)}[3] \oplus R_2^{(0)}[3]$.

Proof. For any $N_1, N_2 \in A_{0,k}-\mathrm{Mod}_f$ denote

$$\langle N_1, N_2 \rangle := \dim \mathrm{Hom}_{A_{0,k}}(N_1, N_2) + \dim \mathrm{Hom}_{A_{0,k}}(N_2, N_1)$$

By Prop. 2.2.11, for any $j, j' \in \{1, 2\}$ and $m, l \leq k$,

$$\dim \mathrm{Hom}_{A_{0,k}}\left(R_{j'}^{(0)}[l], R_j^{(0)}[m+2]\right) - \dim \mathrm{Hom}_{A_{0,k}}\left(R_{j'}^{(0)}[l], R_j^{(0)}[m]\right) = \\ = \begin{cases} 1 & \text{if } l \geq m+2 \text{ or } (l = m+1 \text{ and } j = j') \\ 0 & \text{otherwise} \end{cases}$$

Similarly, if we reverse the order of modules in both Hom, then the result is the same with $j \neq j'$ instead of $j = j'$. Consequently,

$$\left\langle R_j^{(0)}[m+2], R_i^{(0)}[l] \right\rangle - \left\langle R_j^{(0)}[m], R_i^{(0)}[l] \right\rangle = \begin{cases} 2 & \text{if } l \geq m+2 \\ 1 & \text{if } l = m+1 \\ 0 & \text{otherwise} \end{cases} \tag{2.21}$$

Therefore, using $\dim \mathrm{End}_{A_{0,k}}(R_j^{(0)}[m+2]) - \dim \mathrm{End}_{A_{0,k}}(R_j^{(0)}[m]) = 1$ by Prop 1.3.2, we may compute

$$\dim \mathrm{End}_{A_{0,k}}(R_j^{(0)}[m+2] \oplus N \oplus R_{j'}^{(0)}[m'-2]) \\ - \dim \mathrm{End}_{A_{0,k}}(R_j^{(0)}[m] \oplus N \oplus R_{j'}^{(0)}[m']) = \\ = \left\langle R_j^{(0)}[m+2], N \right\rangle + \left\langle R_{j'}^{(0)}[m'-2], N \right\rangle + \left\langle R_j^{(0)}[m+2], R_{j'}^{(0)}[m'-2] \right\rangle \\ - \left\langle R_j^{(0)}[m], N \right\rangle - \left\langle R_{j'}^{(0)}[m'], N \right\rangle - \left\langle R_j^{(0)}[m], R_{j'}^{(0)}[m'] \right\rangle \tag{2.22}$$

where

$$\left\langle R_j^{(0)}[m+2], R_{j'}^{(0)}[m'-2] \right\rangle - \left\langle R_j^{(0)}[m], R_{j'}^{(0)}[m'] \right\rangle = \\ = \left\langle R_j^{(0)}[m+2], R_{j'}^{(0)}[m'-2] \right\rangle - \left\langle R_j^{(0)}[m], R_{j'}^{(0)}[m'-2] \right\rangle \\ + \left\langle R_j^{(0)}[m], R_{j'}^{(0)}[m'-2] \right\rangle - \left\langle R_j^{(0)}[m], R_{j'}^{(0)}[m'] \right\rangle \geq 1 + 0$$

by Eq. 2.21, using $m' - 2 \geq m + 1$ and $m < m' - 1$. Hence the quantity in Eq. 2.22 is

$$1 + \sum_{i=1,2} \left(\sum_{l \geq m+2} 2\left[N :^{\oplus} R_i^{(0)}[l]\right] + \left[N :^{\oplus} R_i^{(0)}[m+1]\right] \right. \\ \left. - \sum_{l \geq m'} 2\left[N :^{\oplus} R_i^{(0)}[l]\right] - \left[N :^{\oplus} R_i^{(0)}[m'-1]\right] \right) > 0$$

that is positive by $m' - 1 \geq m + 1$, where $[N :^{\oplus} Q]$ denotes the multiplicity of Q as a direct summand of N. □

In the proof we will apply Prop. 2.2.11 several times, that describes the dimension of the space of endomorphisms of P_3-modules.

Proof of Proposition 2.2.16. The proof is done in four steps: First we decompose M as $M_0 \oplus M_{\text{cont}}$ and convert the assumptions into equations about P_3-module multiplicities (see Eq. 2.25, 2.26 and 2.27). In the second (resp. third) step, we apply Lemma 2.2.18 (resp. Lemma 1.3.4), to reduce to a case where the decomposition of M_0 (resp. M_{cont}) is simpler. Finally, we turn the problem into a numerical issue, that we solve in Lemma 2.2.20.

First step: The P_3-module M can be decomposed as

$$M = M_0 \oplus M_1 \oplus M_{\text{cont}} \tag{2.23}$$

by Lemma 2.2.7, 2.2.9 and 2.2.10, where the indecomposable direct summands of M_{cont} (resp. M_0, M_1) are isomorphic to $R^{(\lambda)}[m]$, $\lambda \in \Bbbk\backslash\{0,1\}$ and $m \in \mathbb{N}^+$ (resp. $R_i^{(0)}[m]$, $R_i^{(1)}[m]$ for some $i = 1, 2$, and $m \in \mathbb{N}^+$). In fact $M_1 = 0$ by Lemma 2.2.12 using that $1 - x - y$ is invertible. Hence

$$\dim \operatorname{End}_{P_3}(M) \overset{2.2.7}{=} \dim \operatorname{End}_{P_3}(M_0) + \dim \operatorname{End}_{P_3}(M_{\text{cont}}) \tag{2.24}$$

by Lemma 2.2.7, so we bound these two quantities separately.

By Lemma 2.2.12, $M_{\text{cont}} \neq 0$ as $x - y$ is not nilpotent, and

$$\dim \operatorname{Hom}_{P_3}(R_1^{(0)}[1], M_0) = \binom{n+2}{3} \tag{2.25}$$

$$\dim \operatorname{Hom}_{P_3}(R_2^{(0)}[1], M_0) = \binom{n}{3} \tag{2.26}$$

using the assumptions. Moreover, we claim that

$$[M_0 : R_1^{(0)}[1]] - n^2 = [M_0 : R_2^{(0)}[1]] \leq n\binom{n}{2} - 1 \tag{2.27}$$

Indeed, by Lemma 2.2.9 and 2.2.12,

$$2n\binom{n}{2} - 2[M_0 : R_2^{(0)}[1]] \overset{\mathrm{rk}_M(x)=\binom{n}{2}n}{=} 2\mathrm{rk}_M(x) - 2[M_0 : R_2^{(0)}[1]] =$$

$$\overset{2.2.12}{=} 2 \sum_{\lambda \in \Bbbk\backslash\{0,1\}} [M_{\text{cont}} : R^{(\lambda)}[1]] \overset{2.2.9}{=} \dim M_{\text{cont}} \overset{2.23}{=} \dim M - \dim M_0$$

$$\overset{2.2.11}{=} n^3 - [M_0 : R_1^{(0)}[1]] - [M_0 : R_2^{(0)}[1]]$$

where this value is ≥ 1 by $M_{\text{cont}} \neq 0$. Eq. 2.27 follows by rearrangement using $n^3 - 2n\binom{n}{2} = n^2$.

Second step: Our goal is to deduce Eq. 2.28 and as a consequence Eq. 2.34, using Lemma 2.2.18. First, we claim that we may assume that there is an $m_{\min}$ such that if $R_i^{(0)}[m]$ is a direct summand of M_0 for some $i = 1, 2$ then $m_{\min} \leq m \leq m_{\min} + 2$.

Indeed, if there are summands $R_j^{(0)}[m]$ and $R_{j'}^{(0)}[m']$ of M_0 such that $m' - m \geq 3$ then we may replace them with $R_j^{(0)}[m+2]$ and $R_{j'}^{(0)}[m'-2]$. By Prop. 2.2.11, we do not change the quantities in Eq. 2.25, 2.26 and 2.27, but strictly increase $\dim \operatorname{End}_{P_3}(M_0)$ by Lemma 2.2.18.

We show that $m_{\min} \leq 2$. Indeed, if $m_{\min} \geq 3$ then

$$n^3 = \dim M \geq \dim M_0 \geq$$

$$\overset{2.2.11\,\&\, m_{\min}\geq 3}{\geq} 3 \sum_{i=1,2} \dim \operatorname{Hom}_{P_3}(R_i^{(0)}[1], M_0)$$

$$\overset{2.25\&2.26}{\geq} 3\binom{n+2}{3} + 3\binom{n}{3} = n^3 + 2n$$

where $[N :^{\oplus} Q]$ denotes the multiplicity of Q as a direct summand of N. This is a contradiction. Consequently, we may assume that

$$M_0 \cong \bigoplus_{i=1,2} \bigoplus_{m=1}^{4} R_i^{(0)}[m]^{k_{i,m}} \tag{2.28}$$

for some $k_{i,m} \in \mathbb{N}$ ($i = 1, 2$, $m = 1, 2, 3, 4$). In fact either $k_{1,1} = k_{2,1} = 0$ or $k_{1,4} = k_{2,4} = 0$, but we do not use this.

Claim 2.2.19. In the above notations,

$$\sum_{m=1}^{4} k_{1,m} = \binom{n+2}{3} \qquad \sum_{m=1}^{4} k_{2,m} = \binom{n}{3} \tag{2.29}$$

$$k_{2,3} + k_{2,4} + k_{1,2} + k_{1,3} + 2k_{1,4} \leq 2\binom{n+1}{3} - 1 \tag{2.30}$$

$$n^2 = k_{1,1} + k_{1,3} - k_{2,1} - k_{2,3} \tag{2.31}$$

$$k_{1,2} + k_{1,4} = k_{2,2} + k_{2,4} \tag{2.32}$$

The equations are linearly dependent, but it is simpler to write out all.

Proof. The first two equations are clear by Eq. 2.25, 2.26 and Prop. 2.2.11. Moreover, still by Prop. 2.2.11

$$[M_0 : R_i^{(0)}[1]] = \sum_{j=1,2} \sum_{m=1}^{4} \left\lfloor \frac{m + \delta_{j,i}}{2} \right\rfloor k_{j,m} =$$

$$= (k_{i,1} + k_{i,2} + 2k_{i,3} + 2k_{i,4}) + (k_{3-i,2} + k_{3-i,3} + 2k_{3-i,4}) \tag{2.33}$$

In particular, by Eq. 2.27 and Eq. 2.29,

$$n\binom{n}{2} - 1 \overset{2.27}{\geq} [M_0 : R_2^{(0)}[1]] \overset{2.29}{=} \binom{n}{3} + (k_{2,3}+k_{2,4}) + (k_{1,2}+k_{1,3}+2k_{1,4})$$

Using $n\binom{n}{2} - \binom{n}{3} = 2\binom{n+1}{3}$, we obtain Inequality 2.30. Moreover, Eq. 2.31 and 2.32 may be deduced from Eq. 2.27, 2.29 and 2.33 as

$$\begin{aligned} n^2 &\overset{2.27}{=} [M_0 : R_1^{(0)}[1]] - [M_0 : R_2^{(0)}[1]] = \\ &\overset{2.33}{=} k_{1,1}+k_{1,3}-k_{2,1}-k_{2,3} \\ &\overset{2.29}{=} \binom{n+2}{3} - k_{1,2} - k_{1,4} - \binom{n}{3} + k_{2,2} + k_{2,4} \end{aligned}$$

Using $n^2 = \binom{n+2}{3} - \binom{n}{3}$, we obtain the claim. □

By Prop. 2.2.11, we have

$$\dim \operatorname{End}_{P_3} M_0 = v^T B v$$

where $v = (k_{1,1}, k_{1,2}, k_{1,3}, k_{1,4}, k_{2,1}, k_{2,2}, k_{2,3}, k_{2,4})$ and

$$B = \left(\left(\dim \operatorname{Hom}_{P_3}\left(R_i^{(\lambda)}[m], R_{i'}^{(\lambda)}[m']\right)\right)\right)_{(i,m),(i',m')} =$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 & 0 & 1 & 1 & 2 \end{bmatrix}$$

where (i,m) and (i',m') runs on $\{1,2\} \times \{1,2,3,4\}$ (ordered lexicographically). It is elementary to check, using Claim 2.2.19, that

$$\begin{aligned} 2v^T B v = v^T(B+B^T)v &= \sum_{s=1}^{4}\left(\sum_{i=1}^{2}\sum_{m=s}^{4} k_{i,m}\right)^2 + (k_{1,1}+k_{1,3}-k_{2,1}-k_{2,3})^2 = \\ &= \left(\binom{n+2}{3} + \binom{n}{3}\right)^2 + \left(k_{1,3}+k_{2,3}+2k_{1,2}+2k_{1,4}\right)^2 + \\ &\quad + \left(k_{1,3}+k_{2,3}+k_{1,4}+k_{2,4}\right)^2 + (k_{1,4}+k_{2,4})^2 + n^4 \end{aligned} \tag{2.34}$$

Third step: The goal in this step is to deduce that M_{cont} my be assumed to be semisimple, and hence Eq. 2.36 holds.

Indeed, replacing $R^{(\lambda)}[m]$ by $(R^{(\lambda)}[1])^m$ for all $\lambda \in \Bbbk\backslash\{0,1\}$ and $m \in \mathbb{N}^+$ does not change the quantities in Eq. 2.25, 2.26 and 2.27, but – by Lemma 1.3.4 – strictly increases $\dim \operatorname{End}_{P_3}(M_{\text{cont}})$ if M_{cont} was not semisimple yet. Explicitly, by $\dim R^{(\lambda)}[1] = 2$,

$$\dim \operatorname{End}_{P_3}(M_{\text{cont}}) \leq \sum_{\lambda \in \Bbbk} \left[M_{\text{cont}} : R^{(\lambda)}[1]\right]^2 \leq$$

$$\leq \left(\frac{1}{2}\dim(M_{\text{cont}})\right)^2 \overset{2.23}{=} \frac{1}{4}\Big(\dim M - \dim M_0\Big)^2 \tag{2.35}$$

with equality if and only if $M_{\text{cont}} \cong (R^{(\lambda)}[1])^{\frac{1}{2}\dim M_{\text{cont}}}$. By Eq. 2.28 and $\dim R_i^{(0)}[m] = m$, we have

$$= \frac{1}{4}\Big(n^3 - \sum_{m=1}^{4} mk_{1,m} - \sum_{m=1}^{4} mk_{2,m}\Big)^2$$

$$\overset{2.29}{=} \frac{1}{4}\left(4\binom{n+1}{3} - k_{1,2} - 2k_{1,3} - 3k_{1,4} - k_{2,2} - 2k_{2,3} - 3k_{2,4}\right)^2$$

$$\overset{2.32}{=} \left(2\binom{n+1}{3} - k_{2,3} - k_{2,4} - k_{1,2} - k_{1,3} - 2k_{1,4}\right)^2 \tag{2.36}$$

using $n^3 - \binom{n+2}{3} - \binom{n}{3} = 4\binom{n+1}{3}$.

Fourth step: We claim that for the proof of the proposition (i.e. for bounding the sum of the right hand sides of Eq. 2.34 and 2.36), it is enough to prove the following lemma.

Lemma 2.2.20. *Let $n \geq 2$, $c = 2\binom{n+1}{3}$ and consider the compact convex polytope*

$$K := \Big\{(x,y,z) \in \mathbb{R}^3 \mid x,y,z \geq 0,\ x+y+z \leq c-1,\ 2x+y \leq c+n\Big\}$$

If $n = 3$ then include $x \leq 1$ in the definition of K. Then the function $f : K \to \mathbb{R}$ defined as

$$f(x,y,z) = \frac{1}{2}\Big((2x+y)^2 + (y+z)^2 + z^2\Big) + (c-x-y-z)^2$$

attains its maximum value (which is c^2) if and only if $(x,y,z) = (0,0,0)$.

Indeed, we may apply the lemma for

$$(x,y,z) = \big(k_{1,2} + k_{1,4}, k_{1,3} + k_{2,3}, x_{1,4} + x_{2,4}\big).$$

The assumption $x + y + z \le c - 1$ holds by Eq. 2.30. Moreover, by Eq. 2.32 and 2.29,

$$2x + y = 2(k_{1,2} + k_{1,4}) + (k_{1,3} + k_{2,3}) =$$
$$\overset{2.32}{=} (k_{1,2} + k_{1,4}) + (k_{2,2} + k_{2,4}) + (k_{1,3} + k_{2,3})$$
$$\overset{2.29}{\le} \binom{n+2}{3} + \binom{n}{3} = 2\binom{n+1}{3} + n$$

For $n = 3$, $x = k_{1,2} + k_{1,4} = k_{2,2} + k_{2,4} \le \binom{n}{3} = 1$ by Eq. 2.29 and 2.32. Therefore, by Lemma 2.2.20, Eq. 2.34 and 2.36, we obtain

$$\dim \mathrm{End}_{P_3}(M) \overset{2.24}{=} \dim \mathrm{End}_{P_3} M_0 + \dim \mathrm{End}_{P_3}(M_{\mathrm{cont}}) \le$$
$$\overset{2.34\&2.36}{\le} \frac{1}{2}\left(\binom{n+2}{3} + \binom{n}{3}\right)^2 + \frac{n^4}{2} + f\big(k_{1,2} + k_{1,4}, k_{1,3} + k_{2,3}, x_{1,4} + x_{2,4}\big) \le$$
$$\overset{2.2.20}{\le} \frac{n^6 + 13n^4 + 4n^2}{18} + 4\binom{n+1}{3}^2 = \binom{n^2+2}{3}$$

with equality if and only if $k_{i,m} = 0$ for all $i = 1, 2$ and $m = 2, 3, 4$.

In the reduction steps we strictly increased $\dim \mathrm{End}_{P_3}(M)$ except when $m' - m$ was at most 2 and $M_{\mathrm{cont}} \cong (R^{(\lambda)}[1])^{\frac{1}{2}\dim M_{\mathrm{cont}}}$, hence the bound may be satisfied with equality if and only if Eq. 2.20 holds.

Proof of the Lemma 2.2.20. The gradient and Hessian of f are

$$\nabla f(x, y, z) = [4x + 2y,\ 2x + 2y + z,\ y + 2z] + 2\big(x + y + z - c\big)[1, 1, 1]$$

$$\nabla^2 f(x, y, z) = \begin{bmatrix} 6 & 4 & 2 \\ 4 & 4 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

where the second has leading principal minors 6, 8, and 10, hence is $\nabla^2 f$ is positive definite, and f is strictly convex. Consequently, as P is compact and convex, maximum points of f are among the extreme points of P.

First assume that $n \ge 4$. Then $\frac{c+n}{2} \le c - 1$ and the extreme points are

$$[0, 0, 0], \qquad [0, 0, c-1], \qquad [0,\ c-1,\ 0], \qquad \left[\frac{c+n}{2}, 0, 0\right]$$
$$\left[\frac{c+n}{2}, 0, \frac{c-n-2}{2}\right] \qquad \left[n+1, c-n-2, 0\right]$$

where $c - n - 2 \ge 0$ for all $n \ge 3$.

One can check that the values of f on these points are smaller than c^2, except the case of $[0, 0, 0]$. Indeed,

$$f(0, 0, c-1) = f(0, c-1, 0) = (c-1)^2 + 1^2 < c^2$$

for all n. Moreover,

$$f(n+1, c-n-2, 0) = \frac{1}{2}(c+n)^2 + \frac{1}{2}(c-n-2)^2 + 1^2 =$$

$$= c^2 - 2c + (n^2 + 2n + 3) < c^2$$

as $n^2 + 2n + 3 < 2c = 4\binom{n+1}{3} = \frac{2}{3}n(n^2-1)$ for $n \geq 4$. For the other two extreme points, we obtain

$$f\Big(\frac{c+n}{2}, 0, \frac{c-n-2}{2}\Big) = \frac{1}{2}(c+n)^2 + \Big(\frac{c-n-2}{2}\Big)^2 + 1^2$$

$$f\Big(\frac{c+n}{2}, 0, 0\Big) = \frac{1}{2}(c+n)^2 + \Big(\frac{c-n}{2}\Big)^2$$

where the first is clearly at most $f(n+1, c-n-2, 0) < c^2$. For the second one, we get

$$\frac{1}{2}(c+n)^2 + \Big(\frac{c-n}{2}\Big)^2 = c^2 + n^2 - \frac{1}{4}(c-n)^2 < c^2$$

by $c - n > 2n$ using $n \geq 4$, hence the claim follows for $n \geq 4$.

For $n = 3$, $c = 2\binom{4}{3} = 8$ and the equations defining K are $x, y, z \geq 0$, $x + y + z \leq 7$ and $x \leq 1$ (as we exceptionally assumed for $n = 3$) since $2x + y \leq 11$ is superfluous. The extreme points are

$$[0,0,0], \qquad [0,0,7], \qquad [0,7,0],$$

$$[1,0,0], \qquad [1,6,0], \qquad [1,0,6]$$

where the values of f are 64, 50, 50, 51, 57.5 and 39, respectively. For $n = 2$, we have $c = 2$ and K is defined by $x, y, z \geq 0$ and $x + y + z \leq 1$. The values of f on the four extreme points are 4, 3, 2 and 2, hence the claim follows. □

The Lemma completes the proof of Prop. 2.2.16. □

Remark 2.2.21. For $n = 3$, without the assumption $x \leq 1$, K would have an extreme point $[4,3,0]$ with $f(4,3,0) = 66 > 64 = c^2$. On the other hand, including the assumption $x \leq \binom{n}{3}$ for all n would yield higher number of extreme points for K, resulting an even longer, superfluous computation.

Note also that if we replace the assumption $x + y + z \leq c - 1$ with $x + y + z \leq c$ in the definition of K, then Lemma 2.2.20 does not hold. Indeed, for $x = n$, $y = c - n$ and $z = 0$ we have $f(x, y, z) = c^2 + n^2$ (cf. Remark 2.2.17).

Proof of Theorem 2.2.2. By Prop. 2.1.7, 2.2.13 and 2.2.16, the bound on the dimension of $\mathcal{M}_3(p)$ holds and both i) and iii) are equivalent to the fact that there is a single $\lambda \in \Bbbk\backslash\{0,1\}$ such that Eq. 2.20 holds, using the fact that we know $\left[V^{\otimes 3} : R_1^{(0)}[1]\right]$ and $\left[V^{\otimes 3} : R_2^{(0)}[1]\right]$ by the assumption that p is polynomial-type up to degree three (see Eq. 2.25 and 2.26).

On the other hand, if $n \geq 3$, then both $\binom{n+2}{3}$ and $\binom{n}{3}$ are positive, hence $R_1^{(0)}[1]$ and $R_2^{(0)}[1]$ are P_3-direct summands of $V^{\otimes 3}$ i.e. they give two non-isomorphic simple $\mathcal{H}_3(p)$-modules of dimension 1. Moreover, as $p_{12} - p_{23}$ is not nilpotent there exists a $\lambda \in \Bbbk\backslash\{0,1\}$ such that $R^{(\lambda)}[1]$ defines an $\mathcal{H}_3(p)$-module. Hence $\dim \mathcal{H}_3(p) \geq 1 + 1 + 2^2 = 6$, with equality if and only if $V^{\otimes 3}$ is the direct sum of copies of these three simple modules. For $n = 2$, $\binom{n}{3} = 0$ so the same result holds with 5 instead of 6. Consequently, ii) is also equivalent to the other assertions. □

2.2.5 *PBW-basis*

In this subsection we may characterize the existence of a PBW-basis in $\mathcal{M}(p)$ in the sense of Subsec. 1.2.1.

Let $S := \operatorname{Ker}(p)$ and $T := \operatorname{Im}(p)$, and identify $E = \operatorname{End}(V)$ with $V \otimes V^{\vee}$. We will suppress the tensor signs between elements of V and $V^{\vee}$. The proof of Cor. 2.2.4 follows directly from Theorem 2.2.2 and the following proposition, that is implicit in Theorem 3, [Su].

Proposition 2.2.22. *Let $p \in E^{\otimes 2}$ be an idempotent, $v_1, \dots, v_n$ an ordered basis of V and $f_1, \dots, f_n$ its dual basis. Assume that A_T (resp. A_S) has a polynomial (resp. exterior) ordering algorithm with respect to the given ordered basis.*

Then $\mathcal{M}(p)$ has a polynomial ordering algorithm with respect to the basis

$$\{v_i f_k \in E \mid 1 \leq i, k \leq n\}$$

where $v_i f_l > v_j f_k$ if and only if $i > j$ or $i = j$ and $l < k$.

Proof. By Eq. 2.4,

$$\mathcal{M}(p) \cong \mathcal{T}(E)/(\tau_{(23)}(SS^o + TT^o))$$

Using the ordering on E defined in the statement, for all $1 \leq a, b \leq n$, let

$$K^{\leq}_{(a,b)} := \operatorname{Span}(v_i v_j f_l f_k \in E^{\otimes 2} \mid v_i f_l < v_a f_b,\ v_i f_l \leq v_j f_k)$$

By Lemma 1.2.3 the claim is equivalent to

$$v_a v_c f_b f_d \in SS^o + TT^o + K^{\leq}_{(a,b)} \tag{2.37}$$

for all $1 \leq a, b, c, d \leq n$ such that $v_a f_b > v_c f_d$ i.e. $a > c$ or $a = c$ and $b < d$.

For fixed $1 \le a, b \le n$, define the following spaces

$$K_a^< = \mathrm{Span}(v_i v_j \mid i < a,\ i < j)$$
$$K_a^= = \mathrm{Span}(v_i v_i \mid i < a)$$
$$L_b^\le = \mathrm{Span}(f_l f_k \mid l > b,\ l \ge k)$$

We use the last definition also for $b = 0$.

As A_T (resp. A_S) has a polynomial (resp. exterior) ordering algorithm, and by Lemma 1.2.3, we have

$$v_a v_c \in T + K_a^< + K_a^= \qquad (n \ge a > c \ge 1) \tag{2.38}$$

$$v_a v_c \in S + K_a^< \qquad (n \ge a \ge c \ge 1) \tag{2.39}$$

Moreover, by Lemma 1.2.5, $A_S^! \cong \mathcal{T}(V^\vee)/(S^o)$ has a polynomial ordering algorithm for the dual ordering on $f_1, \dots, f_n$. Hence

$$f_b f_d \in S^o + L_b^\le \qquad (1 \le b < d \le n) \tag{2.40}$$

On the other hand, by the definition of the ordering on E,

$$K_{(a,b)}^\le = \mathrm{Span}(v_i v_j f_l f_k \mid i < a,\ i < j) + \mathrm{Span}(v_i v_i f_l f_k \mid i < a,\ l \ge k)$$
$$+ \mathrm{Span}(v_a v_j f_l f_k \mid l > b,\ a = i < j) + \mathrm{Span}(v_a v_a f_l f_k \mid l > b,\ l \ge k) =$$
$$= K_a^< V^\vee V^\vee + K_a^= L_0^\le + \sum_{j>a} v_a v_j \sum_{l>b} f_l V^\vee + v_a v_a L_b^\le$$

So, by Eq. 2.37, it is enough to verify

$$v_a v_c f_b f_d \in SS^o + TT^o + K_a^< V^\vee V^\vee + K_a^= L_0^\le + v_a v_a L_b^\le \tag{2.41}$$

for all $a > c$ or $a = c$ and $b < d$.

First assume that $a = c$ and $b < d$. Then, by Eq. 2.39 and 2.40, we have

$$v_a v_a f_b f_d \overset{2.40}{\in} v_a v_a (S^o + L_b^\le) \subseteq$$
$$\overset{2.39}{\subseteq} (S + K_a^<) S^o + v_a v_a L_b^\le$$
$$\subseteq SS^o + K_a^< V^\vee V^\vee + v_a v_a L_b^\le \tag{2.42}$$

Hence Eq. 2.41 holds.

Now assume that $a > c$. Then, by $S^o + T^o = V^\vee V^\vee$, Eq. 2.39 and 2.38, we have

$$v_a v_c f_b f_d \in v_a v_c (S^o + T^o) \subseteq$$
$$\subseteq (S + K_a^<) S^o + (T + K_a^< + K_a^=) T^o$$

$$\subseteq SS^o + TT^o + K_a^< V^\vee V^\vee + K_a^= V^\vee V^\vee$$

where

$$K_a^= V^\vee V^\vee = K_a^= L_0^\leq + K_a^= \sum_{k<l} f_k f_l$$

by the definition of $L_0^\leq$. To rearrange the last summand, we apply the previous paragraph: for all $j < a$ and $k < l$,

$$v_j v_j f_k f_l \overset{2.42}{\in} SS^o + K_j^< V^\vee V^\vee + v_j v_j L_k^\leq \subseteq$$

$$\subseteq SS^o + K_a^< V^\vee V^\vee + K_a^= L_0^\leq$$

Therefore Eq. 2.41 indeed holds in all cases. □

Proof of Corollary 2.2.4. By Prop. 2.2.22, $\mathcal{M}(p)$ has a polynomial ordering algorithm. Hence, by Fact 1.2.4, it is a PBW-algebra if and only if $\dim \mathcal{M}_3(p) = \binom{n^2+2}{3}$. Therefore the claim follows by Theorem 2.2.2. □

2.3 NILPOTENT CASE

In the following, we assume that $p \in \mathrm{End}(V \otimes V)$ such that p is nilpotent of order 2, i.e. $p^2 = 0$. Our goal is to prove Theorem 2.3.2 by reducing it to Theorem 2.2.2.

Matrix bialgebras arising from such a nilpotent element include the Hecke-type FRT-bialgebras for $q = -1$. In particular, the bialgebra $O^i(M_2)$ corresponding to the isolated quantum group investigated in [Skr].

Recall that we denote $p_{12} = p \otimes \mathrm{id}$ and $p_{23} = \mathrm{id} \otimes p$ in $\mathrm{End}(V^{\otimes 3})$, and $n := \dim V$.

Definition 2.3.1. Let us call a nilpotent (of order two) $p \in \mathrm{End}(V \otimes V)$ *polynomial-type up to degree three*, if p is of rank $\binom{n}{2}$, $p_{12} + p_{23}$ is not nilpotent, and

$$\dim(\mathrm{Ker}(p_{12}) \cap \mathrm{Ker}(p_{23})) = \binom{n+2}{3} \tag{2.43}$$

$$\dim(\mathrm{Im}(p_{12}) \cap \mathrm{Im}(p_{23})) = \binom{n}{3} \tag{2.44}$$

$$\mathrm{Ker}(p_{12}) \cap \mathrm{Im}(p_{23}) \subseteq \mathrm{Im}(p_{12}) \tag{2.45}$$

$$\mathrm{Im}(p_{12}) \cap \mathrm{Ker}(p_{23}) \subseteq \mathrm{Im}(p_{23}) \tag{2.46}$$

where $n = \dim V$. Note that the definition differs from Def. 2.2.1.

Theorem 2.3.2. *Let* $\dim V \geq 2$ *and* $p \in \mathrm{End}(V \otimes V)$ *such that* $p^2 = 0$ *that is polynomial-type up to degree three. Then the statements of Theorem 2.2.2 hold for* p*, with characterization* $iii)$ *replaced by*

iii)′ there are distinct scalars $\lambda, \mu \in \Bbbk^\times$ *such that*

$$p_{12}p_{23}p_{12} - \lambda p_{12} - \mu p_{23}p_{12}p_{23} + \mu\lambda p_{23}$$

annihilates $V^{\otimes 3}$.

Remark 2.3.3. Assume that there are $a, b \in \Bbbk^\times$ such that $a + bp$ satisfies the Yang-Baxter equation (see Eq. 2.19). Assume also that the Hilbert series of $A_{\mathrm{Ker}(p)}$ (resp. $A_{\mathrm{Im}(p)}$) agrees with $(1-t)^{-n}$ (resp. $(1+t)^n$) in degrees 1, 2 and 3, where $A_U = \mathcal{T}(V)/U$ for $U \subseteq V \otimes V$. In Remark 2.3.10 we prove that in this case p is polynomial-type up to degree three and $iii)'$ holds with $\mu = 1$, assuming that $p_{12} + p_{23}$ is not nilpotent. The latter extra assumption holds for $O^i(M_2)$ (see [Skr]).

By Prop. 2.1.7, for all $d \geq 2$ we have $\mathrm{Sch}_d(p) \cong \mathrm{End}_{\mathcal{H}_d(p)}(V^{\otimes d})$. As $\mathcal{H}_d(p)$ is the quotient of

$$P'_d := \Bbbk\langle y_1, \ldots, y_{d-1}\rangle / \big(y_i^2 \ (1 \leq i < d),\ y_i y_j - y_j y_i \ (|i-j| > 1)\big)$$

similarly to idempotent case, we may also write $\mathrm{End}_{P'_3}$ instead of $\mathrm{End}_{\mathcal{H}_d(p)}$. For $d = 3$, we describe $P'_3{-}\mathrm{Mod}_f$ in the next subsection.

2.3.1 *Representations of* P'_3

In this subsection, we describe the indecomposable modules of $P'_3 = \Bbbk\langle x', y'\rangle / \big((x')^2, (y')^2\big)$ in the form of Corollary 2.3.6.

Note that $(x'+y')^2 = x'y' + y'x'$ is central in P'_3. Hence, by Lemma 2.2.7, it is enough to understand the modules of $A'_{\lambda,k} := P'_3/\big((x'y'+y'x'-\lambda)^k\big)$.

Lemma 2.3.4. *For all* $k \in \mathbb{N}^+$ *and* $\lambda \in \Bbbk^\times$

$$A'_{\lambda,k} \cong M_2[t]/(t^k) \sim \Bbbk[t]/(t^k)$$

Proof. The unital homomorphism $A'_{\lambda,k} \to M_2[t]/(t^k)$ given by

$$x' \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad y' \mapsto \begin{bmatrix} 0 & 0 \\ t+\lambda & 0 \end{bmatrix} \tag{2.47}$$

is clearly a homomorphism. If $\lambda \neq 0$ then it is also invertible. □

For all $\lambda \neq 0$, $m \in \mathbb{N}^+$, let us denote by $\mathbf{T}^{(\lambda)}[m]$ the unique $2m$-dimensional indecomposable module of P'_3 induced from $A'_{\lambda,k}$ (for any $k \geq m > 0$).

Lemma 2.3.5. *Let* $B'_k := P'_3/(x',y')^k$. *Then*

$$\bigcup_{k=1}^{\infty} (A'_{0,k}\text{−Mod}_f) \cong \bigcup_{k=1}^{\infty} (B'_k\text{−Mod}_f)$$

Proof. We claim that

$$(x',y')^{k+1} \subseteq (x'y'+y'x')^k \subseteq (x',y')^k \tag{2.48}$$

for all $k \in \mathbb{N}^+$. Indeed, the second containment is clear. Moreover, for k odd,

$$(x'y'+y'x')^k = x'y'x'\dots y'x' + y'x'y'\dots x'y'$$

hence $(x'y'+y'x')^k x' = y'x'y'\dots x'y'x'$ and $(x'y'+y'x')^k x' = y'x'y'\dots x'y'x'$. Similarly, for k even, hence Eq. 2.48 holds.

By the previous paragraph, there are natural surjections

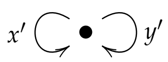
$$B'_{k+1} \twoheadrightarrow A'_{0,k} \twoheadrightarrow B'_k$$

for all $k \in \mathbb{N}^+$. The claim follows. □

The motivation for the second lemma is the following. For all $k \geq 2$,

$$B'_k \cong \Bbbk Q/I^k$$

is a monomial special biserial algebra (see Subsec. 1.3.2), with quiver Q that has a single vertex and two loops (x' and y'):



and $I^k = (x',y')^k$.

By Prop. 1.3.1, we may describe the indecomposable modules in $\mathcal{C} := \bigcup_{k=1}^{\infty}(B'_k\text{−Mod}_f)$. For the terminology, see Subsec. 1.3.2.

Fix $k \geq 2$. By definition, a V-sequence for $\Bbbk Q/I^k$ is a walk-quiver L with arrow-labeling $v : A(L) \to \{x',y'\}$ such that L contains no directed path with k arrows, moreover, the labeling v is alternating, i.e. if distinct arrows $\beta_1, \beta_2 \in A(L)$ have a common end-point (in the undirected sense) then $v(\beta_1) \neq v(\beta_2)$.

Similarly, a primitive V-sequence for $\Bbbk Q/I^k$ is a tour-quiver Z with arrow-labeling $u : A(Z) \to \{x',y'\}$ such that Z contains no directed path of k arrows, moreover, u is alternating, and there is no quiver-automorphism $\sigma \neq \mathrm{id}$ of Z such that $u \circ \sigma = u$. Consequently, by Lemma 2.2.7:

Corollary 2.3.6. *The isomorphism types of indecomposable modules of* P'_3 *can be listed as follows:*

- $\mathbf{T}^{(\lambda)}[m]$ *for some* $\lambda \in \Bbbk^\times$ *and* $m \in \mathbb{N}^+$,

- $\mathbf{M}(v)$ *where L is a walk-quiver and $v : A(L) \to \{x', y'\}$ is an alternating labeling,*
- $\mathbf{M}(u, m, \lambda, \beta_0)$ *where $m \in \mathbb{N}^+$, $\lambda \in \Bbbk^\times$, Z is a tour-quiver, $\beta_0 \in A(Z)$, $u : A(Z) \to \{x', y'\}$ is an alternating labeling, and there is no quiver-automorphism $\sigma \neq \mathrm{id}$ of Z such that $u \circ \sigma = u$.*

The redundancies in the list are as in Prop. 1.3.1.

In the next subsection, we will need the following notions and properties about the above mentioned modules. For a quiver C that is either a walk-quiver or a tour-quiver let us define the number of *sinks* and *complete sinks* as

$$\mathrm{sk}(C) := \{c \in V(C) \mid \nexists \beta \in A(C),\ \beta : c \to d\}$$
$$\mathrm{csk}(C) := \{c \in \mathrm{sk}(C) \mid \deg(c) = 2\}$$

where $\deg(c)$ stands for the graph-theoretic undirected degree of the vertex c.

Let $L_{k,l}$ be the walk-quiver that is the join of a path with $k-1$ edges, and a path with $l-1$ edges, at their sinks, as in the figure.

$$L_{k,l} : \quad \bullet \xrightarrow{\beta_{k-1}} \cdots \xrightarrow{\beta_2} \bullet \xrightarrow{\beta_1} \bullet \xleftarrow{\gamma_1} \bullet \xleftarrow{\gamma_2} \cdots \xleftarrow{\gamma_{l-1}} \bullet$$

Then let $\mathbf{V}[k, l] := \mathbf{M}(v)$ where $v : A(L_{k,l}) \to \{x', y'\}$ is the labeling on $L_{k,l}$ such that $v(\beta_1) = x'$ (this determines the labeling completely). Note that $\dim \mathbf{V}[k, l] = k + l - 1$.

Lemma 2.3.7. *For $k, k', l, l' \in \mathbb{N}^+$,*

$$\dim \mathrm{Hom}_{P_3'}(\mathbf{V}[k, l], \mathbf{V}[k', l']) = \delta_{k>k'}\delta_{l>l'} - 1 +$$
$$+ \dim \mathrm{Hom}_{P_3}(R_1^{(0)}[k] \oplus R_2^{(0)}[l], R_1^{(0)}[k'] \oplus R_2^{(0)}[l'])$$

By Prop. 1.3.2, the lemma may be verified by a combinatorial case checking. Moreover, analogously to Lemma 2.2.12, the following holds.

Lemma 2.3.8. *Let C be either a walk-quiver or a tour-quiver, and $v : A(C) \to \{x', y'\}$ alternating labeling of C such that there is no quiver-automorphism $\sigma \neq \mathrm{id}$ of C satisfying $u \circ \sigma = u$. Then*

$$\dim\left(\mathrm{Ker}_M(x') \cap \mathrm{Ker}_M(y')\right) = m \cdot sk(C)$$
$$\dim\left(\mathrm{Im}_M(x') \cap \mathrm{Im}_M(y')\right) = m \cdot csk(C)$$
$$\mathrm{rk}_M z = m \cdot \{\beta \in A(C) \mid v(\beta) = z\} \qquad (z \in \{x', y'\})$$

for $M = \mathbf{M}(v)$ and $m = 1$ if C is a walk-quiver, and $M = \mathbf{M}(v, m, \lambda, \beta_0)$ if C is a tour-quiver ($\lambda \in \Bbbk^\times$, $m \in \mathbb{N}^+$ and $\beta_0 \in A(C)$ arbitrary).

Similarly to Subsec. 2.2.3, condition $iii)'$ in Theorem 2.3.2 determines the isomorphism types of indecomposable summands of $V^{\otimes 3}$ as follows.

For each $\mu \in \mathbb{k}^\times$ let $\mathbf{U}^{(\mu)} := \mathbf{M}(u, 1, \mu, \beta_0)$ where Z is the Kronecker quiver (i.e. two vertices with two parallel arrows pointing from one vertex to the other), $u : A(Z) \to \{x', y'\}$ is an alternating labeling such that $u(\beta_0) = x'$. Moreover, let $\mathbf{V} := \mathbf{V}[1, 1]$, and $\mathbf{T}^{(\lambda)} := \mathbf{T}^{(\lambda)}[1]$ for any $\lambda \in \mathbb{k}^\times$.

Proposition 2.3.9. *Let M be a finite-dimensional P_3'-module and $\lambda, \mu \in \mathbb{k}^\times$. Then*

$$x'y'x' - \lambda x' - \mu y'x'y' + \mu\lambda y' \in \mathrm{Ann}(M) \tag{2.49}$$

if and only if M is isomorphic to the direct sum of copies of the modules $\mathbf{V}$, $\mathbf{U}^{(\mu)}$, and $\mathbf{T}^{(\lambda)}$.

Proof. We may assume that M is indecomposable. If $M \cong \mathbf{M}(v)$ for some alternating $v : A(L) \to \{x', y'\}$, then we prove that $M \cong \mathbf{V}$. Indeed, let $w \in V(L)$ and assume that $\beta \in A(L)$ has source w (for the notation, see Subsec. 1.3.2). If $v(\beta) = x'$ then $x'\mathbf{L}(w)$ is independent of the span of $x'y'x'\mathbf{L}(w)$, $y'x'y'\mathbf{L}(w)$ and $y'\mathbf{L}(w)$, hence 2.49 cannot hold. Similarly, if $v(\beta) = y'$.

Assume that M is of the form $\mathbf{M}(u, m, \mu, \beta_0)$. Then there is $w \in V(Z)$ such that $\beta_1, \beta_2 \in A(Z)$ have source w, $u(\beta_1) = x'$ and $u(\beta_2) = y'$. If x' and y' have distinct targets then $x'\mathbf{L}(w)$ is independent of the span of the other spaces, hence 2.49 cannot hold. Therefore, Z is the Kronecker-quiver, and we may assume that $u(\beta_0) = x'$. We still need to prove that $x' - \mu y' \in \mathrm{Ann}(M)$ (in particular, $m = 1$). Since $x'y'x'$ and $y'x'y'$ annihilate M, this follows.

Assume that $M \cong \mathbf{T}^{(\nu)}[m]$ for some $\nu \in \mathbb{k}^\times$ and $m \in \mathbb{N}^+$. Then by Eq. 2.47, the matrix of $x'y'x' - \lambda x' - \mu y'x'y' + \mu\lambda y'$ in the appropriate basis over $\mathbb{k}[t]/(t^m)$ is

$$\begin{bmatrix} 0 & (t+\nu) - \lambda \\ -\mu(t+\nu)^2 + \mu\lambda(t+\nu) & 0 \end{bmatrix}$$

that is zero only if $m = 1$ and $\nu = \lambda$. The claim follows. ∎

Remark 2.3.10. The relation of the Yang-Baxter equation to Prop. 2.2.13 and the polynomial-type assumption is the following.

By elementary computation, there are $a, b \in \mathbb{k}^\times$ such that

$$(a + bx')(a + by')(a + bx') - (a + by')(a + bx')(a + by') \in \mathrm{Ann}(M)$$

if and only if Eq. 2.49 holds with $\mu = 1$. In that case, $a = -\lambda$ (and b arbitrary). Indeed, the expression above simplifies as

$$b^3(x'y'x' - y'x'y') + ab^2(x' - y')$$

Now, assume the conditions given in Remark 2.3.3. We prove the statement given there. Clearly, $\mathrm{rk}(p) = \binom{n}{2}$, Eq. 2.43 and 2.44 hold, by the assumption on $A_{\mathrm{Ker}(p)}$ and $A_{\mathrm{Im}(p)}$. The previous paragraph implies that $iii)'$ in Theorem 2.3.2 is true for p with $\mu = 1$. Hence we only have to show Eq. 2.45 and 2.46.

Note that it is enough to show the above equations for the indecomposable P_3'-module summands of $V^{\otimes 3}$. By Prop. 2.3.9, these are among $\mathbf{V}$, $\mathbf{U}^{(\mu)}$ and $\mathbf{T}^{(\lambda)}$. In the case of $\mathbf{V}$, $\mathrm{Im}(x') = \mathrm{Im}(y') = 0$. For $\mathbf{U}^{(\mu)}$, the subspaces $\mathrm{Ker}(x')$, $\mathrm{Ker}(y')$, $\mathrm{Im}(x')$ and $\mathrm{Im}(y')$ all agree with the unique one-dimensional submodule of $\mathbf{U}^{(\mu)}$. While for $\mathbf{T}^{(\lambda)}$, $\mathrm{Ker}(x') \cap \mathrm{Im}(y')$ and $\mathrm{Im}(x') \cap \mathrm{Ker}(y')$ are both zero. The claim of Remark 2.3.3 follows.

2.3.2 *Proof of Theorem 2.3.2*

Using Prop. 2.2.16, we may deduce its analog for the $p^2 = 0$ case. This is the main step in the proof of Theorem 2.3.2.

Proposition 2.3.11. *Let $n := \dim V \geq 2$ and $p \in \mathrm{End}(V \otimes V)$ such that $p^2 = 0$ and p polynomial-type up to degree three in the sense of Def. 2.3.1. Then*

$$\dim \mathcal{M}_3(p) \leq \binom{n^2+2}{3} \tag{2.50}$$

with equality if and only if there are $\lambda, \mu \in \Bbbk^\times$ such that

$$V^{\otimes 3} \cong \mathbf{V}^{n^2} \oplus \left(\mathbf{U}^{(\mu)}\right)^{n_{0,2}} \oplus \left(\mathbf{T}^{(\lambda)}\right)^{c} \tag{2.51}$$

where $n_{0,2} = \binom{n}{3}$ and $c = 2\binom{n+1}{3}$.

Recall that for a ring R, $\mathbb{N}[\mathrm{Indec}_f(R)]$ denotes the monoid of isomorphism classes of finite-dimensional left R-modules (with direct sum). Let $P_3 := \Bbbk\langle x, y\rangle / (x^2 - x, y^2 - y)$ as in Sec. 2.2. Consider the following submonoid of $\mathbb{N}[\mathrm{Indec}_f(P_3)]$.

$$\mathcal{S} = \langle \mathbf{V}, \mathbf{T}^{(\lambda)}[m], \mathbf{M}(u, m, \mu, \beta_0) \mid \lambda, \mu \in \Bbbk^\times,\ m \in \mathbb{N}^+,\ u, \beta_0 \text{ as above}\rangle \tag{2.52}$$

Fix $\lambda_0 \in \Bbbk \backslash \{0, 1\}$, and let us define an additive map

$$h : \mathcal{S} \to \mathbb{N}[\mathrm{Indec}_f(P_3)]$$
$$h(\mathbf{V}) := R_1^{(0)}[1]$$
$$h\big(\mathbf{T}^{(\lambda)}[m]\big) := R^{(\lambda_0)}[m]$$

for all $\lambda \in \Bbbk^\times$ and $m \in \mathbb{N}^+$. Moreover, if $M \cong \mathbf{M}(u, m, \mu, \beta_0)$ for some choice of $u : A(Z) \to \{x', y'\}$, m, μ and β_0, then – by the definition of $\mathbf{M}(u, m, \mu, \beta_0)$ – we have

$$M/\mathrm{rad} M \cong \mathbf{V}^{\mathrm{sk}(Z)\cdot m} \tag{2.53}$$

$$\mathrm{rad} M \cong \bigoplus_{s \in \mathrm{sk}(Z)} \mathbf{V}[k_s, l_s]^m \tag{2.54}$$

for some $k_s, l_s \in \mathbb{N}^+$ ($s \in \mathrm{sk}(Z)$), where $\mathrm{rad}M$ is the (Jacobson) radical of the module M. Hence we may define

$$h(M) := \bigoplus_{s \in \mathrm{sk}(Z)} \left(R_1^{(0)}[k_s]^m \oplus R_2^{(0)}[l_s]^m \right)$$

The following lemma describing the properties of h – especially Eq. 2.56 – reduces Prop. 2.3.11 to 2.2.16.

Lemma 2.3.12. *For all $M, N \in \mathcal{S}$,*

$$\dim h(M) = \dim M \tag{2.55}$$

$$\dim \mathrm{Hom}_{P_3}(h(M), h(N)) \geq \dim \mathrm{Hom}_{P_3'}(M, N) \tag{2.56}$$

with strict inequality if $\mathbf{T}^{(\lambda)}[1] \leq M$ and $\mathbf{T}^{(\lambda')}[1] \leq N$ for $\lambda \neq \lambda' \in \Bbbk^\times$. Moreover,

$$\mathrm{rk}_{h(M)} x = \mathrm{rk}_M x' \qquad \mathrm{rk}_{h(M)} y = \mathrm{rk}_M y' \tag{2.57}$$

$$\dim \mathrm{Hom}_{P_3}\left(R^{(\lambda_0)}[m], h(M)\right) = \sum_{\lambda \in \Bbbk^\times} \dim \mathrm{Hom}_{P_3'}\left(\mathbf{T}^{(\lambda)}[m], M\right) \tag{2.58}$$

$$\dim \mathrm{Hom}_{P_3}\left(R_i^{(0)}[1], h(M)\right) = \dim\left(\mathrm{Im}_M(x') \cap \mathrm{Im}_M(y')\right) + \delta_{i,1}[M : \mathbf{V}] \tag{2.59}$$

$$\dim \mathrm{Hom}_{P_3}\left(R_i^{(1)}[1], h(M)\right) = 0 \tag{2.60}$$

for any $\lambda \in \Bbbk^\times$ and $i = 1, 2$.

The proof of the lemma follows from the definition of h and Lemma 2.3.7 and 2.3.8.

Proof of Proposition 2.3.11. By Corollary 2.3.6, $V^{\otimes 3}$ can be decomposed as $V^{\otimes 3} \cong M_{0,\mathrm{walk}} \oplus M_{0,\mathrm{tour}} \oplus M_{\mathrm{cont}}$ where every indecomposable summand of $M_{0,\mathrm{walk}}$ (resp. $M_{0,\mathrm{tour}}$, M_{cont}) is isomorphic to a module of the form $\mathbf{M}(v)$ (resp. $\mathbf{M}(u, m, \mu, \beta_0)$, $\mathbf{T}^{(\lambda)}[m]$). We claim that $M_{0,\mathrm{walk}} \cong \mathbf{V}^{n^2}$.

First observe that $M_{0,\mathrm{walk}}$ is a direct sum of n^2 indecomposable P_3'-modules. Indeed, for an indecomposable P_3'-module M, we have

$$\dim M - \mathrm{rk}_M(x') - \mathrm{rk}_M(y') = \begin{cases} 1 & M \text{ is of the form } \mathbf{M}(v) \\ 0 & \text{otherwise} \end{cases}$$

by Lemma 2.3.8 for $\mathbf{M}(v)$ and $\mathbf{M}(u, m, \mu, \beta_0)$, and by the definition for $\mathbf{T}^{(\lambda)}[m]$. The left hand side is additive in M, hence the number of independent indecomposable direct summands of $M_{0,\mathrm{walk}}$ is

$$\dim V^{\otimes 3} - \mathrm{rk}(p_{12}) - \mathrm{rk}(p_{23}) = n^3 - 2n \binom{n}{2} = n^2 \tag{2.61}$$

Now we show that each summand of $M_{0,\text{walk}}$ is isomorphic to $\mathbf{V}$. As p is polynomial-type in the sense of Def. 2.3.1, for each indecomposable summand M of $V^{\otimes 3}$, we have

$$\mathrm{Ker}_M(x') \cap \mathrm{Im}_M(y') \subseteq \mathrm{Im}_M(x')$$

$$\mathrm{Im}_M(x') \cap \mathrm{Ker}_M(y') \subseteq \mathrm{Im}_M(y').$$

If $M \cong \mathbf{M}[v]$ for some $v : A(L) \to \{x', y'\}$ and walk-quiver L, this is possible only if either L is the one-point quiver L_1 (with no edges), or $\mathrm{sk}(L) = \mathrm{csk}(L)$, i.e. L has no vertex with in-degree 1 and out-degree 0. Indeed, if $\beta : d \to c$ is the only arrow touching c and $v(\beta) = y'$, then the one-dimensional subspace $\mathbf{L}(c) \subseteq M$ (see Subsec. 1.3.2) will give $\mathrm{Ker}_M(x') \cap \mathrm{Im}_M(y') \not\subseteq \mathrm{Im}_M(x')$. Similarly, for $v(\beta) = x'$.

We have to prove that for every indecomposable summand $M \cong \mathbf{M}[v]$ of $M_{0,\text{walk}}$, the case of$\mathrm{sk}(L) = \mathrm{csk}(L)$ is impossible. For this purpose, observe that for an indecomposable P_3'-module M, we have

$$\begin{aligned}\dim\left(\mathrm{Ker}_M(x') \cap \mathrm{Ker}_M(y')\right) - \dim\left(\mathrm{Im}_M(x') \cap \mathrm{Im}_M(y')\right) = \\ = \begin{cases} \mathrm{sk}(L) - \mathrm{csk}(L) & M \text{ is of the form } \mathbf{M}(v) \\ 0 & \text{otherwise} \end{cases}\end{aligned} \tag{2.62}$$

by Lemma 2.3.8 for $\mathbf{M}(v)$ and $\mathbf{M}(u, m, \mu, \beta_0)$, and by the definition for $\mathbf{T}^{(\lambda)}[m]$. Note that $\mathrm{sk}(L) - \mathrm{csk}(L) \in \{0, 1, 2\}$. The left hand side of Eq. 2.62 is additive in M. We may compute

$$\begin{aligned}\dim\left(\mathrm{Ker}(p_{12}) \cap \mathrm{Ker}(p_{23})\right) - \dim\left(\mathrm{Im}(p_{12}) \cap \mathrm{Im}_M(p_{23})\right) = \\ = \binom{n+2}{3} - \binom{n}{3} = n^2\end{aligned} \tag{2.63}$$

Together with Eq. 2.62 and 2.63, this shows that $\mathbf{V}^{n^2}$ is a summand of $V^{\otimes 3}$. Hence, by Eq. 2.61, we got $M_{0,\text{walk}} \cong \mathbf{V}^{n^2}$, as claimed.

By the previous paragraphs, (the isomorphism type of) $V^{\otimes 3}$ is in the submonoid $\mathcal{S}$ (see Eq. 2.52). In particular, we may apply h on $V^{\otimes 3}$.

It is enough to show that the P_3-module $h(V^{\otimes 3})$ satisfies the assumptions of Prop. 2.2.16. Indeed, then Eq. 2.56 implies the stated inequality (Eq. 2.50), moreover, in the case of equality, $V^{\otimes 3}$ cannot have submodules isomorphic to $\mathbf{T}^{(\lambda)}[1]$ and $\mathbf{T}^{(\lambda')}[1]$ for $\lambda \neq \lambda' \in \Bbbk^\times$, and $h(V^{\otimes 3})$ is of the form given by Eq. 2.20, by Prop. 2.2.16. That is possible only if $M_{\text{cont}} = \mathbf{T}^{(\lambda)}[1]^c$ for some $\lambda \in \Bbbk^\times$ and $\mathrm{rad}\,\mathrm{rad}\, M_{0,\text{tour}} = 0$, by the definition of h.

We need to prove that $\operatorname{rad}\operatorname{rad} M_{0,\text{tour}} = 0$ implies that $M_{0,\text{tour}}$ is a power of $U^{(\mu)}$ for some $\mu \in \Bbbk^\times$. Observe that if $\operatorname{rad}\operatorname{rad} \mathbf{M}(u, m, \mu, \beta_0) = 0$ then the arrows of Z are oriented alternatingly (i.e. there is no directed path of two arrows in Z). As $\mathbf{M}(u, m, \mu, \beta_0)$ is indecomposable only if Z has no quiver-automorphism quiver-automorphism $\sigma \neq \mathrm{id}$ such that $u \circ \sigma = u$, we get that Z must be the Kronecker quiver (i.e. two vertices with two parallel arrows pointing from one vertex to the other). That is, $M_{0,\text{tour}} \cong (\mathbf{U}^{(\mu)})^{n_{0,2}}$ for some $\mu \in \Bbbk^\times$. Conversely, if $V^{\otimes 3}$ is as in Eq. 2.51 then the inequality holds with equality by elementary calculation.

Now we check the assumptions of Prop. 2.2.16 using Lemma 2.3.12. Clearly, $\dim h(V^{\otimes 3}) = n^3$ Eq. 2.55, $\operatorname{rk}_{h(V^{\otimes 3})}(x) = \operatorname{rk}_{h(V^{\otimes 3})}(y) = \binom{n}{2}$ by Eq. 2.57 and $1 - x - y$ is invertible on $h(V^{\otimes 3})$ by Eq. 2.60 and Lemma 2.2.12. Moreover, $x' + y'$ is not nilpotent on $V^{\otimes 3}$ if and only if $M_{\text{cont}} \neq 0$, equivalently $R^{(\lambda)}[1]$ is a submodule of $h(V^{\otimes 3})$, by Eq. 2.58. By Lemma 2.2.12 it is also equivalent to $x - y$ being not nilpotent on $h(V^{\otimes 3})$. Similarly, the conditions on the dimensions of $\operatorname{Ker}(x) \cap \operatorname{Ker}(y)$ and $\operatorname{Im}(x) \cap \operatorname{Im}(y)$ follow by Eq. 2.59 and Lemma 2.3.8, using that $M_{0,\text{walk}} \cong \mathbf{V}^{n^2}$. □

Proof of Theorem 2.3.2. It follows from Prop. 2.3.9 and 2.3.11 the same way as Theorem 2.2.2 did from Prop. 2.2.16 and 2.2.13. □

2.4 UPPER BOUND

In this section, our goal is to give an upper bound on $\dim \mathcal{M}_d(p)$ for some $p \in \operatorname{End}(V \otimes V)$, in the form of Prop. 2.4.8 and Cor. 2.4.9.

For $d = 3$, in Section 2.2 we used that the algebra generated by two idempotents is of tame representation type, hence $\dim \operatorname{End}_{P_3}(V^{\otimes 3})$ (i.e. $\dim \mathcal{M}_3(p)$) can be computed. As for $d \geq 4$ the analogous algebra P_d is wild, we investigate an alternative method, with necessarily weaker consequences.

The goal is to bound $\dim \operatorname{End}_{\mathcal{H}_d(p)}(V^{\otimes d})$ by applying Lemma 1.3.3 on monoid homomorphism with domain $\mathbb{N}[\operatorname{Irr}_f(\mathcal{H}_d(p))]$. In Subsec. 2.4.4 we discuss the (already described) case of $d = 3$, while in Subsec. 2.5.1, we investigate the case of $d = 4$. Note that in this section (except Subsec. 2.4.4) p is not assumed to be idempotent.

2.4.1 *Ordered Multiplicities*

Fix $d \in \mathbb{N}^+$ and denote by OPart_d the set of ordered partitions of size d, where an *ordered partition α of size d* is a sequence of positive integers $(\alpha_1, \ldots, \alpha_r)$ for some $r \in \mathbb{N}^+$ such that $\sum_{i=1}^r \alpha_i = d$ (but $\alpha_i \geq \alpha_{i+1}$ is not assumed). The length r is called the *height* of α and is denoted by $\mathrm{ht}(\alpha)$. Note that $|\mathrm{OPart}_d| = 2^{d-1}$ as

$$\alpha \mapsto \left\{ \sum_{i=1}^{k} \alpha_i \;\middle|\; k = 1, \ldots, d-1 \right\} \tag{2.64}$$

is a bijection onto $\mathcal{P}(\{1, 2, \ldots, d-1\})$.

Let $p \in \mathrm{End}(V \otimes V)$. For $\alpha \in \mathrm{OPart}_d$, consider the natural algebra injection

$$\mathcal{H}_\alpha(p) := \mathcal{H}_{\alpha_1}(p) \otimes \cdots \otimes \mathcal{H}_{\alpha_{\mathrm{ht}(\alpha)}}(p) \xrightarrow{\iota_\alpha} \mathcal{H}_d(p) \subseteq \mathrm{End}(V^{\otimes d})$$

where $\mathcal{H}_d(p)$ is defined in Def. 2.1.6. Consider the ideal

$$I_\alpha := \left(p_{j,j+1} \;\middle|\; 1 \le j \le d-1,\ j \ne \sum_{i=1}^{k} \alpha_i \ \left(\forall k \le \mathrm{ht}(\alpha)\right) \right) \lhd \mathcal{H}_\alpha(p)$$

that is of codimension at most one. Define Triv_α as the (isomorphism class of the) $\mathcal{H}_\alpha(p)$-module $\mathcal{H}_\alpha(p)/I_\alpha$.

Recall from Eq. 1.7 the definition of composition multiplicity $[M : S]$ of a module S as a factor of M. For an arbitrary finite-dimensional $\mathcal{H}_d(p)$-module M, let us define the *ordered multiplicities* corresponding to ordered partitions $\alpha \in \mathrm{OPart}_d$ as

$$\mathrm{OMult}^p_\alpha(M) := \left[\mathrm{Res}_{\iota_\alpha} M : \mathrm{Triv}_\alpha\right]$$

where $\mathrm{Res}_{\iota_\alpha} M$ is M understood as an $\mathcal{H}_\alpha(p)$-module, instead of an $\mathcal{H}_d(p)$-module. If $\mathrm{Triv}_\alpha = 0$ i.e. $I_\alpha = \mathcal{H}_\alpha(p)$ then let $\mathrm{OMult}^p_\alpha(M) := 0$. If d is understood from the context, we denote the resulting homomorphism as

$$\mathrm{OMult}^p : \mathbb{N}\left[\mathrm{Indec}_f(\mathcal{H}_d(p))\right] \to \mathbb{N}[\mathrm{OPart}_d]$$

The map is determined by its values on $\mathrm{Irr}_f(\mathcal{H}_d(p))$ by the following lemma.

Lemma 2.4.1. *For a finite-dimensional $\mathcal{H}_d(p)$-module M,*

$$\mathrm{OMult}^p(M^{ss}) = \mathrm{OMult}^p(M)$$

Proof. As Res_{ι_μ} is exact, and $N \mapsto [N : \mathrm{Triv}_\mu]$ is additive on short exact sequences, $N \mapsto [\mathrm{Res}_{\iota_\mu} N : \mathrm{Triv}_\mu]$ is also additive on short exact sequences. Hence $\left[\mathrm{Res}_{\iota_\mu}(M^{ss}) : \mathrm{Triv}_\mu\right] = \left[\mathrm{Res}_{\iota_\mu}(M) : \mathrm{Triv}_\mu\right]$. The claim follows. □

Recall from Eq. 1.6 the notion of norm-square $N_{\mathcal{S}}(s)$ for a commutative monoid $\mathcal{S}$ and $s \in \mathcal{S}$.

Lemma 2.4.2. *Let $\mathcal{S}_d(p)$ be the image of* OMult^p *in* $\mathbb{N}[\mathrm{OPart}_d]$. *Then*

$$\dim \mathrm{Sch}_d(p) \leq N_{\mathcal{S}_d(p)}(\mathrm{OMult}^p(V^{\otimes d}))$$

If $\mathcal{S}_d(p)$ is factorial then equality is attained if and only if $\mathcal{H}_d(p)$ is semisimple and OMult^p *is an isomorphism.*

Proof. By Prop. 2.1.7, Lemma 2.4.1 and Subsec. 1.3.4, we obtain

$$\dim \mathrm{Sch}_d(p) \overset{2.1.7}{=} \dim \mathrm{End}_{\mathcal{H}_d(p)}(V^{\otimes d}) \overset{1.3.4}{\leq} N_{\mathbb{N}\left[\mathrm{Irr}(\mathcal{H}_d(p))\right]}((V^{\otimes d})^{ss}) \leq$$

$$\overset{1.3.3}{\leq} N_{\mathcal{S}_d(p)}\left(\mathrm{OMult}^p((V^{\otimes d})^{ss})\right) \overset{2.4.1}{=} N_{\mathcal{S}_d(p)}(\mathrm{OMult}^p(V^{\otimes d}))$$

where the first inequality is an equality if and only if $\mathcal{H}_d(p)$ is semisimple. Moreover, $V^{\otimes d}$ is a faithful $\mathcal{H}_d(p)$-module, so if $\mathcal{S}_d(p)$ is factorial then any atom $a \in \mathcal{A}(\mathbb{N}\left[\mathrm{Irr}(\mathcal{H}_d(p))\right]) = \mathrm{Irr}(\mathcal{H}_d(p))$ divides $\mathrm{OMult}^p(V^{\otimes d})$. Hence the second inequality holds with equality if and only if OMult^p is an isomorphism, by Lemma 1.3.3. □

Consequently, the problem of finding an upper bound on $\dim \mathcal{M}_d(p)$ can be split into determining $\mathrm{OMult}^p(V^{\otimes d})$ and $\mathcal{S}_d(p)$.

2.4.2 *The case of symmetric groups*

Let $d \geq 2$ be a fixed integer, and denote the symmetric group on d elements by $\mathfrak{S}_d$. First we investigate the prime example: when $\mathcal{H}_d(p)$ is the image of the group algebra $\Bbbk\mathfrak{S}_d$.

For an ordered partition $\alpha \in \mathrm{OPart}_d$, denote by $\widetilde{\alpha} \in \mathrm{OPart}_d$ the ordered partition of height $\mathrm{ht}(\alpha)$ such that $\widetilde{\alpha}_i = \alpha_{\sigma(i)}$ for some permutation σ of $\{1, \dots, \mathrm{ht}(\alpha)\}$ and $\widetilde{\alpha}_i \geq \widetilde{\alpha}_{i+1}$ for all $i = 1, \dots, \mathrm{ht}(\alpha) - 1$. The subset $\{\widetilde{\alpha} \mid \alpha \in \mathrm{OPart}_d\}$ in OPart_d is denoted by Part_d as its elements are the classical (unordered) partitions.

Let us recall the *dominance order*, that endows Part_d with a lattice structure (see Def 3.2 in [Ja]). For $\nu, \mu \in \mathrm{Part}_d$, ν *dominates* μ (denoted as $\nu \geq \mu$) if and only if

$$\sum_{i=1}^{k} \nu_i \geq \sum_{i=1}^{k} \mu_i$$

for all $k = 1, \dots, \mathrm{ht}(\nu)$.

For $\nu \in \mathrm{Part}_d$, define $f^\nu \in \mathbb{N}[\mathrm{OPart}_d]$ as

$$f^\nu_\alpha := K_{\nu,\widetilde{\alpha}} \qquad (\alpha \in \mathrm{OPart}_d)$$

where $K_{\nu,\mu}$ is the *Kostka number* of $(\nu, \mu) \in \mathrm{Part}_d^2$ i.e. the number of semistandard Young tableaux of shape ν and weight μ (see p. 101 in [Mac]).

Remark 2.4.3. Recall that $K_{\nu,\nu} = 1$ and $K_{\nu,\mu} \neq 0$ if and only if $\nu \geq \mu$ in the dominance order.

Consider $p_{\text{sym}} = \frac{1}{2}(\text{id}_{V^{\otimes 2}} - \tau_{(12)}) \in \text{End}(V \otimes V)$ where $\tau_{(12)}$ is the flip $\tau_{(12)}(u \otimes v) = v \otimes u$ $(u, v \in V)$. Then

$$\mathcal{H}_d(p_{\text{sym}}) = \overline{\Bbbk\mathfrak{S}_d} \subseteq \text{End}(V^{\otimes d}) \tag{2.65}$$

where $\overline{\Bbbk\mathfrak{S}_d}$ denotes the image in $\text{End}(V^{\otimes d})$ of the group algebra $\Bbbk\mathfrak{S}_d$. Consequently, $\text{Sch}_d(p_{\text{sym}})$ is the ordinary Schur algebra $S(n, d)$ for $n = \dim V$ (hence the notation Sch), by Schur-Weyl duality. On the other hand, we have $\mathcal{M}(p_{\text{sym}}) \cong \oplus_d \text{End}_{\Bbbk\mathfrak{S}_d}(V^{\otimes d}) = \mathcal{O}(M_n)$.

It is classical that the simple modules of $\mathcal{H}_d(p_{\text{sym}})$ (resp. $\Bbbk\mathfrak{S}_d$) are parametrized by $\{\nu \in \text{Part}_d \mid \text{ht}(\nu) \leq \dim V\}$ (resp. Part_d), where the simple module S^ν is the Specht-module corresponding to the partition $\nu \in \text{Part}_d$ (for definition, see [Mac]).

Lemma 2.4.4. *For all $\nu \in \text{Part}_d$ such that $\text{ht}(\nu) \leq \dim V$, we have*

$$\text{OMult}^{p_{\text{sym}}}(S^\nu) = f^\nu$$

In particular, $\text{OMult}^{p_{\text{sym}}}$ *is injective.*

Proof. For the proof, note that for $\alpha \in \text{OPart}_d$, the simple module Triv_α of $\mathcal{H}_\alpha(p_{\text{sym}}) = \overline{\Bbbk\mathfrak{S}_\alpha} \subseteq \text{End}(V^{\otimes d})$ is isomorphic to the (module given by the) trivial representation of $\mathfrak{S}_\alpha := \mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_{\text{ht}(\alpha)}}$.

As $\Bbbk\mathfrak{S}_d$ is semisimple by Maschke's theorem, for any $\alpha \in \text{OPart}_d$,

$$\text{OMult}_\alpha^{p_{\text{Sym}}}(S^\nu) \overset{\text{def}}{=} [\text{Res}_{\iota_\alpha} S^\nu : \text{Triv}_\alpha] =$$

$$\overset{\text{Maschke}}{=} \text{Hom}_{\Bbbk\mathfrak{S}_\alpha}(\text{Triv}_\alpha, \text{Res}_{\iota_\alpha} S^\nu)$$

$$\overset{\text{Frob.}}{=} \text{Hom}_{\Bbbk\mathfrak{S}_d}(\text{Ind}_{\mathfrak{S}_\alpha}^{\mathfrak{S}_d} \text{Triv}_\alpha, S^\nu) \overset{\text{Young}}{=} K_{\nu,\widetilde{\alpha}}$$

using Frobenius reciprocity, and Young's rule (see Theorem 14.1 in [Ja]). Therefore $\text{OMult}^{p_{\text{sym}}}(S^\nu) = f^\nu$. Since $K_{\nu,\nu} = 1$ and $K_{\nu,\widetilde{\alpha}} \neq 0$ if and only if $\nu \geq \widetilde{\alpha}$, the matrix $(K_{\nu,\tilde{\alpha}})$ is unitriangular (using an enumeration of partitions, that is a refinement of the dominant order) so the set of its row-vectors $\{f^\nu \mid \nu \in \text{Part}_d\}$ is independent. □

Denote by $K^{-1}_{\mu,\nu}$ the inverse Kostka number corresponding to the partitions $\nu, \mu \in \text{Part}_d$, using the indexing convention such that $\sum_\mu K_{\nu,\mu} K^{-1}_{\mu,\xi} = \delta_{\nu,\xi}$.

Corollary 2.4.5. *The submonoid generated by* $\{f^\nu \mid \nu \in \mathrm{Part}_d,\ \mathrm{ht}(\nu) \leq \dim V\}$ *is the set of* $u \in \mathbb{N}[\mathrm{OPart}_d]$ *such that*

$$u_\alpha = u_{\widetilde{\alpha}} \qquad (\forall \alpha \in \mathrm{OPart}_d)$$
$$\sum_{\mu \in \mathrm{Part}_d} u_\mu K^{-1}_{\mu,\nu} \geq 0,\ (\mathrm{ht}(\nu) \leq \dim V)$$
$$\sum_{\mu \in \mathrm{Part}_d} u_\mu K^{-1}_{\mu,\nu} = 0,\ (\mathrm{ht}(\nu) > \dim V)$$

where ν *runs on* Part_d.

Proof. The first set of equations holds by definition. Consider the basis $\mathcal{B} = \{e^\nu \mid \nu \in \mathrm{Part}_d\}$ in $P := \{u \in \mathbb{Z}[\mathrm{OPart}_d] \mid u_\alpha = u_{\widetilde{\alpha}}\}$ defined as $e^\nu_\alpha = \delta_{\nu,\widetilde{\alpha}}$. The matrix whose rows are the coordinate-vectors of f^ν ($\nu \in \mathrm{Part}_d$) with respect to the basis $\mathcal{B}$ is the Kostka matrix $(K_{\nu,\mu})$. It has determinant 1. Hence the vectors form a $\mathbb{Z}$-module basis of $P \cong \mathbb{Z}[\mathrm{Part}_d]$. Consequently, the submonoid $\langle f^\nu \mid \nu \in \mathrm{Part}_d \rangle$ can be described as

$$\Big\{u \in P \mid \sum_{\mu \in \mathrm{Part}_d} u_\mu K^{-1}_{\mu,\nu} \geq 0,\ \forall \nu \in \mathrm{Part}_d\Big\}$$

To obtain all defining inequalities for the submonoid $\langle f^\nu \mid \nu \in \mathrm{Part}_d,\ \mathrm{ht}(\nu) \leq \dim V\rangle$ in $\mathbb{N}[\mathrm{OPart}_d]$, we have to further require $\sum_{\mu \in \mathrm{Part}_d} v_\mu K^{-1}_{\mu,\nu} = 0$ for all $\nu \in \mathrm{Part}_d$ such that $\mathrm{ht}(\nu) > \dim V$. □

Remark 2.4.6. We could also define p_{sym} as $\frac{1}{2}(\mathrm{id}_{V^{\otimes 2}} + \tau_{(12)})$, but in that case 1-dimensional $\mathcal{H}_d(p_{\mathrm{sym}})$-module $\mathrm{Triv}_{(d)}$ would not give the trivial representation of $\mathfrak{S}_d$.

2.4.3 *Subsymmetric case*

Let $d \geq 2$ still be a fixed integer, and $p \in \mathrm{End}(V \otimes V)$.

Definition 2.4.7. Let us call $\mathcal{H}_d(p)$ *subsymmetric* if

$$\mathrm{Im}\big(\mathrm{OMult}^p\big) \subseteq \langle f^\nu \mid \nu \in \mathrm{Part}_d \rangle_{\mathbb{N}[\mathrm{OPart}_d]}$$

where $\langle . \rangle$ stands for the submonoid generated.

In fact it is not a property of the algebra $\mathcal{H}_d(p)$, but a property of its generating subset $p_{1,2}, \ldots, p_{d-1,d}$. By Lemma 2.4.1, $\mathrm{Im}\big(\mathrm{OMult}^p\big) = \mathrm{Im}\big(\mathrm{OMult}^p|_{\mathrm{Irr}_f(\mathcal{H}_d(p))}\big)$.

Proposition 2.4.8. *If* $\mathcal{H}_d(p)$ *is subsymmetric, then*

$$\dim \mathrm{Sch}_d(p) \leq \sum_{\mu \in \mathrm{Part}_d} \Big(\sum_{\nu \in \mathrm{Part}_d} \mathrm{OMult}^p_\nu(V^{\otimes d}) K^{-1}_{\nu,\mu} \Big)^2$$

with equality if and only if $\mathcal{H}_d(p)$ *is semisimple and* OMult^p *injects* $\mathrm{Irr}(\mathcal{H}_d(p))$ *into* $\{f^\nu \mid \nu \in \mathrm{Part}_d\}$.

Proof. By Lemma 1.3.3 applied to $\mathcal{S} := \mathcal{S}_d(p)$ and $\mathcal{S}' := \langle f^\nu \mid \nu \in \mathrm{Part}_d \rangle$ we obtain

$$\dim \mathrm{Sch}_d(p) \overset{2.4.2}{\leq} N_{\mathcal{S}_d(p)}(\mathrm{OMult}^p(V^{\otimes d})) \overset{1.3.3}{\leq} N_{\mathcal{S}'}(\mathrm{OMult}^p(V^{\otimes d}))$$

Then, by Lemma 2.4.4, $\mathcal{S}' \cong \mathbb{N}[\mathrm{Irr}(\mathcal{H}_d(p_{\mathrm{Sym}}))]$ is factorial with $\mathcal{A}(\mathcal{S}') \cong \{f^\nu \mid \nu \in \mathrm{Part}_d,\ \mathrm{ht}(\nu) \leq \dim V\}$. Hence we have the following unique decomposition into atoms:

$$\mathrm{OMult}^p(V^{\otimes d}) = \sum_{\nu \in \mathrm{Part}_d} \mathrm{OMult}^p_\nu(V^{\otimes d}) e^\nu =$$

$$= \sum_{\mu \in \mathrm{Part}_d} \Big(\sum_{\nu \in \mathrm{Part}_d} \mathrm{OMult}^p_\nu(V^{\otimes d}) K^{-1}_{\nu,\mu} \Big) f^\mu$$

where $(e^\nu)_\mu = \delta_{\nu,\mu}$ for $\nu, \mu \in \mathrm{Part}_d$. The desired inequality follows.

By Lemma 1.3.3 applied to $\mathrm{OMult}^p : \mathbb{N}[\mathrm{Irr}(\mathcal{H}_d(p))] \to \mathcal{S}'$, the inequalities are all satisfied with equality if and only if OMult^p injects $\mathrm{Irr}(\mathcal{H}_d(p))$ into $\mathcal{A}(\mathcal{S}')$. $\square$

Corollary 2.4.9. *Assume that $\mathcal{H}_d(p)$ is subsymmetric, and*

$$\mathrm{OMult}^p_\mu(V^{\otimes d}) = \prod_{i=1}^{\mathrm{ht}(\mu)} \binom{n + \mu_i - 1}{\mu_i}$$

for all $\mu \in \mathrm{Part}_d$. Then

$$\dim \mathrm{Sch}_d(p) \leq \binom{n^2 + d - 1}{d}$$

with equality if and only if $\mathcal{H}_d(p)$ is semisimple and OMult^p is an isomorphism onto $\langle f^\nu \mid \nu \in \mathrm{Part}_d,\ \mathrm{ht}(\nu) \leq n \rangle$. In particular, if the assumptions hold for all $d \geq d_0$ for some $d_0 \in \mathbb{N}$ then $\mathrm{GKdim}(\mathcal{M}(p)) \leq n^2$.

Proof. First note that $\mathrm{Sch}_d(p_{\mathrm{sym}})$ is the ordinary Schur algebra $S(n,d)$ by Schur-Weyl duality. Hence $p = p_{\mathrm{sym}}$ satisfies the assumption and the stated inequality holds with equality. Moreover, Prop. 2.4.8 also holds with equality by Lemma 2.4.4.

For arbitrary p satisfying the assumptions, the bound in Prop. 2.4.8 depends only on $\mathrm{OMult}^p_\mu(V^{\otimes d})$. Since these numbers agree with the case of p_{sym}, we obtain that

$$\dim \mathrm{Sch}_d(p) \leq \dim S(n,d) = \binom{n^2 + d - 1}{d}$$

The characterization of the equality case follows by the equality case in Prop. 2.4.8. $\square$

Without further assumptions, the subsymmetric property of $\mathcal{H}_d(p)$ is complicated to check, as it depends on all of its irreducible representations. In the next subsection, we give a characterization of it in the case when p is an idempotent and $d = 3$. For the $d = 4$ case, see Prop. 2.5.5.

2.4.4 *Subsymmetric in degree three*

Proposition 2.4.10. *$\mathcal{H}_3(p)$ is subsymmetric if and only if $\mathrm{id} - p_{12} - p_{23}$ is invertible.*

Proof. By the definition of subsymmetric (Def. 2.4.7), we have to check that for any simple $\mathcal{H}_3(p)$-module M,

$$\mathrm{OMult}^p(M) \in \langle f^{(3)}, f^{(2,1)}, f^{(1,1,1)} \rangle \subseteq \mathbb{N}[\mathrm{OPart}_d] \cong \mathbb{N}^4$$

where the vectors $f^{(3)}, f^{(2,1)}, f^{(1,1,1)}$ in the standard basis $e^{(3)}, e^{(2,1)}, e^{(1,2)}, e^{(1,1,1)}$ of $\mathbb{Z}[\mathrm{OPart}_3]$ are the rows of

$$((K_{\nu\tilde{\alpha}}))_{\nu,\alpha \in \mathrm{Part}_d \times \mathrm{OPart}_d} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As $P_3 \twoheadrightarrow \mathcal{H}_3(p)$, it is enough to investigate the simple modules of P_3 i.e. of $A_{\lambda,1}$ ($\lambda \in \Bbbk$) by Lemma 2.2.7. For $\lambda \neq 0, 1$, $R^{(\lambda)}[1]$ is given in 2.15 by taking $t = 0$, hence $\mathrm{OMult}^p(R^{(\lambda)}[1]) = (0, 1, 1, 2)$. Moreover, by their definition,

$$\mathrm{OMult}^p(R_i^{(\lambda)}[1]) = \begin{cases} (1,1,1,1) & \text{if } \lambda = 0,\ i = 1 \\ (0,0,0,1) & \text{if } \lambda = 0,\ i = 2 \\ (0,1,0,1) & \text{if } \lambda = 1,\ i = 1 \\ (0,0,1,1) & \text{if } \lambda = 1,\ i = 2 \end{cases}$$

Therefore $\mathcal{H}_3(p)$ is subsymmetric if and only if the P_3-module $R_i^{(1)}[1]$ ($i = 1, 2$) does not define an $\mathcal{H}_3(p)$-module. By Lemma 2.2.12, it is equivalent to $1 - x - y$ acting invertibly on any $\mathcal{H}_3(p)$-module M, i.e. $\mathrm{id} - p_{12} - p_{23} \in \mathcal{H}_3(p)$ being invertible. □

As a consequence, assuming that $\mathrm{id} - p_{12} - p_{23}$ is invertible, we may apply Prop. 2.4.8 to bound $\dim \mathcal{M}(p)$:

$$\dim \mathrm{End}_{\mathcal{H}_3(p)}(V^{\otimes 3}) \leq a^2 + (b-a)^2 + (n^3 - a - 2b)^3$$

where $a = \mathrm{OMult}^p_{(3)}(V^{\otimes 3})$, $b = \mathrm{OMult}^p_{(2,1)}(V^{\otimes 3}) = n \cdot \dim \mathrm{Ker}(p)$ and $n = \dim V$. For the bound, note that the Kostka and inverse Kostka matrices for $d = 3$ are

$$K = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad K^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

The problem is that a is non-trivial to determine if $V^{\otimes 3}$ is not semisimple.

2.5 ORTHOGONAL PROJECTION CASE

In this section, we investigate matrix bialgebras in the special case, when $\Bbbk = \mathbb{C}$, V is a finite-dimensional complex Hilbert space and $p \in \mathrm{End}(V \otimes V)$ is an orthogonal projection. Then the construction yields cosemisimple bialgebras that were first considered by Manin (see Section VI/6 in [Man]). The main goal of the next subsection is to prove Theorem 2.5.2 and 2.5.3 below.

Theorem 2.5.2 is an application of Cor. 2.4.9 for $d = 4$, showing that the characterization of subsymmetric property for low degrees is possible but increases in complexity (see Prop. 2.4.10 and 2.5.5). While, Theorem 2.5.3 is motivated by Theorem 2.5 in [Hai1] stating that $\mathcal{M}(r)$ is Koszul, if r satisfies the Yang-Baxter equation and the Hecke equation $(r + \mathrm{id})(r - q) = 0$, assuming that q is not a root of unity.

Recall that $p_{12} = p \otimes \mathrm{id}_V \otimes \mathrm{id}_V$ and similarly for p_{23} and p_{34}.

Definition 2.5.1. An orthogonal projection $p \in \mathrm{End}(V \otimes V)$ is called *polynomial-type up to degree four*, if it is polynomial-type up to degree three (see Def. 2.2.1), and for $n = \dim V$

$$\dim \Big(\bigcap_{i=1}^{3} \mathrm{Ker}(p_{i,i+1}) \Big) = \binom{n+3}{4} \qquad \dim \Big(\bigcap_{i=1}^{3} \mathrm{Im}(p_{i,i+1}) \Big) = \binom{n}{4}$$

Theorem 2.5.2. *Assume that $p \in \mathrm{End}(V \otimes V)$ is polynomial-type up to degree four,*

$$\mathrm{rk}(p_{12}(\mathrm{id} - p_{23})p_{34}) = \mathrm{rk}((\mathrm{id} - p_{23})p_{34})$$
$$\mathrm{rk}(p_{34}(\mathrm{id} - p_{23})p_{12}) = \mathrm{rk}((\mathrm{id} - p_{23})p_{12})$$

and $\{\mathrm{Ker}(p_{i,i+1}) \mid i = 1,2,3\}$ generates a distributive lattice of subspaces. Then

$$\dim \mathcal{M}_4(p) \le \binom{n^2+3}{4}$$

with equality if and only if OMult^p is an isomorphism onto $\mathrm{Im}(\mathrm{OMult}^{p_{\mathrm{sym}}})$ for $d = 4$.

Theorem 2.5.3. *Assume that* $p \in \mathrm{End}(V \otimes V)$ *is an orthogonal projection, polynomial-type up to degree four and the following conditions hold:*

$$\begin{aligned}\dim\Big(\mathrm{Ker}(p_{23}(p_{12}-p_{34}))\cap\mathrm{Im}(p_{23}(\mathrm{id}-p_{12}))\\ \cap\mathrm{Ker}(p_{12}p_{34})\Big) &= \frac{3n+6}{4}\binom{n+1}{3}\\ \dim\Big(\mathrm{Ker}((\mathrm{id}-p_{23})(p_{12}-p_{34}))\cap\mathrm{Im}((\mathrm{id}-p_{23})p_{12})\\ \cap\mathrm{Ker}((\mathrm{id}-p_{12})(\mathrm{id}-p_{34}))\Big) &= \frac{3n-6}{4}\binom{n+1}{3}\\ \dim\mathrm{Ker}((p_{12}-p_{34})p_{23}) &= 4n\binom{n+1}{3}\end{aligned} \tag{2.66}$$

Then $\mathcal{M}(p)$ *is Koszul only if the equivalent conditions of Theorem 2.2.2 hold.*

The dimension conditions in Eq. 2.66 are satisfied for p_{sym} (see Remark 2.5.16). Unfortunately, these are not open conditions in the sense that in the space $\mathrm{Gr}_{\binom{n}{2}}(V\otimes V)$ of orthogonal projections on $V\otimes V$ of rank $\binom{n}{2}$, the elements that satisfy Eq. 2.66 do not form a Zariski-open subset. This is in contrast with Remark 2.2.3.

Recall from Subsec. 2.1.2 the definitions of $\mathcal{H}_d(p)$ and $\mathrm{Sch}_d(p)$ that are subalgebras of $E^{\otimes d} = \mathrm{End}(V^{\otimes d})$. As p is self-adjoint, they are closed under taking adjoints, and hence for each $\mathcal{H}_d(p)$-submodule (resp. $\mathrm{Sch}_d(p)$-submodule) $M \subseteq V^{\otimes d}$, $M^{\perp}$ is a submodule direct complement. In particular, $\mathcal{H}_d(p)$ and $\mathrm{Sch}_d(p)$ are semisimple subalgebras of $E^{\otimes d}$, and they are each other's centralizer, by the double centralizer theorem for semisimple algebras.

The fact that $\mathcal{H}_3(p)$ is semisimple is equivalent to $V^{\otimes 3}$ being a semisimple P_3-module. Let us denote the simple P_3-modules as $R_i^{(\kappa)}$, $R^{(\lambda)}$ for $\kappa = 0,1$, $i = 1,2$, $\lambda \in \Bbbk\backslash\{0,1\}$ (see Subsec. 2.2.2). Now if $x_1, x_2 \in P_3$ act as self-adjoint operators on $R^{(\lambda)}$ then $\lambda \in (0,1) \subseteq \mathbb{R}$. Indeed, by Eq. 2.11, the operators are self-adjoint with respect to the Hermitian sesquilinear form with matrix

$$\begin{bmatrix} 1-\lambda & \lambda-1 \\ \lambda-1 & 1 \end{bmatrix}$$

if $\lambda \in \mathbb{R}$, that is positive definite if and only if $0 < \lambda < 1$. The modules $R_i^{(\kappa)}$ are one-dimensional and x_1 and x_2 act via real scalars, so they are automatically self-adjoint.

Assuming $\mathrm{id} - p_{12} - p_{23}$ is invertible,

$$V^{\otimes 3} \cong (R_1^{(0)})^{n_{0,1}} \oplus (R_2^{(0)})^{n_{0,2}} \oplus \bigoplus_{\lambda\in\Bbbk\backslash\{0,1\}} (R^{(\lambda)})^{n_\lambda} \tag{2.67}$$

for some $n_{0,1}, n_{0,2}, n_\lambda \in \mathbb{N}$ ($\lambda \in (0,1)$), by Lemma 2.2.12. Moreover, if p is polynomial-type up to degree three, then $n_{0,1} = \binom{n+2}{3}$, $n_{0,2} = \binom{n}{3}$ and hence $\sum_\lambda n_\lambda = 2\binom{n+1}{3}$, by Lemma 2.2.12.

Remark 2.5.4. For the arguments in this section, it is enough to assume that $\Bbbk$ is an arbitrary algebraically closed field of characteristic zero, and V is finite-dimensional vector space endowed with a σ-sesquilinear non-degenerate Hermitian form $\beta : V \otimes V \to \Bbbk$ for some involutive automorphism σ of $\Bbbk$, such that for all $d \geq 1$, the induced (σ-sesquilinear, non-degenerate, Hermitian) form $\beta^{\otimes d}$ on $V^{\otimes d}$ is anisotropic i.e. $\beta^{\otimes d}(v,v) = 0$ implies $v = 0$.

Then the idempotent $p \in \mathrm{End}(V \otimes V)$ is required to be self-adjoint with respect to $\beta^{\otimes 2}$ i.e. $\beta^{\otimes 2}(pu, v) = \beta^{\otimes 2}(u, pv)$ for all $u, v \in V \otimes V$. For a simpler discussion, we restrict ourselves to the main special case, when $\Bbbk = \mathbb{C}$ and σ is the complex conjugation (hence the fixed field $\Bbbk^\sigma$ specializes to $\mathbb{R}$).

2.5.1 *Subsymmetric in degree four*

Theorem 2.5.2 is a direct consequence of Cor. 2.4.9 and the following proposition.

Proposition 2.5.5. *Let $p \in \mathrm{End}(V \otimes V)$ an orthogonal projection such that* $\mathrm{id} - p_{12} - p_{23}$ *is invertible,*

$$\mathrm{rk}(p_{12}(\mathrm{id} - p_{23})p_{34}) = \mathrm{rk}((\mathrm{id} - p_{23})p_{34}) \tag{2.68}$$

$$\mathrm{rk}(p_{34}(\mathrm{id} - p_{23})p_{12}) = \mathrm{rk}((\mathrm{id} - p_{23})p_{12}) \tag{2.69}$$

and $\{\mathrm{Ker}(p_{i,i+1}) \mid i = 1,2,3\}$ generates a distributive lattice of subspaces. Then $\mathcal{H}_4(p)$ is subsymmetric.

Remark 2.5.6. Though the notation $\mathrm{id} - p_{12} - p_{23}$ is ambiguous as it can be an element of $\mathrm{End}(V^{\otimes d})$ for any $d \geq 3$, its invertibility is still well-defined. Indeed, the element $s := \mathrm{id} - p_{12} - p_{23} \in \mathrm{End}(V^{\otimes 3})$ is invertible if and only if $\mathrm{id}_{V^{\otimes(k-1)}} \otimes s \otimes \mathrm{id}_{V^{\otimes(d-k-1)}}$ is invertible in $\mathrm{End}(V^{\otimes d})$ for any (hence all) $1 \leq k \leq d-1$.

Recall from Subsec. 2.4.3 that $\mathcal{H}_4(p)$ is subsymmetric if and only if for every simple $\mathcal{H}_4(p)$-module M,

$$\mathrm{OMult}^p(M) \subseteq \langle f^\nu \mid \nu \in \mathrm{Part}_4 \rangle_{\mathbb{N}[\mathrm{OPart}_4]}$$

where $((f^{\nu}_{\alpha})) = ((K_{\nu,\tilde{\alpha}})) \in \mathbb{Z}^{\mathrm{Part}_4 \times \mathrm{OPart}_4}$ is

$$\begin{array}{c} \\ (4) \\ (3{,}1) \\ (2{,}2) \\ (2{,}1{,}1) \\ (1{,}1{,}1{,}1) \end{array} \begin{array}{c} \begin{array}{cccccccc} (4) & (3{,}1) & (1{,}3) & (2{,}2) & (2{,}1{,}1) & (1{,}2{,}1) & (1{,}1{,}2) & (1{,}1{,}1{,}1) \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

and $K_{\nu,\mu}$ is the Kostka number corresponding to the partitions (ν, μ). Let M be a fixed simple $\mathcal{H}_4(p)$-module and denote by $y_i \in \mathrm{End}_{\Bbbk}(M)$ the action of $p_{i,i+1}$ on M. Consider

$$U_i := \mathrm{Ker}(y_i) \qquad (i = 1, 2, 3)$$

$$d_\alpha := \mathrm{OMult}^p_\alpha(M) = \dim\left(\bigcap\{U_j \mid j \neq \sum_{i=1}^{k} \alpha_i \ (\forall k \leq \mathrm{ht}(\alpha))\}\right)$$

for any $\alpha \in \mathrm{OPart}_4$.

By Cor. 2.4.5, the subsymmetric condition can be formulated as

$$\dim U_1 = \dim U_2 = \dim U_3 \tag{2.70}$$

$$\dim(U_1 \cap U_2) = \dim(U_2 \cap U_3) \tag{2.71}$$

and that

$$\sum_{\mu \in \mathrm{Part}_4} d_\mu K^{-1}_{\mu,\nu} \geq 0 \qquad (\nu \in \mathrm{Part}_4) \tag{2.72}$$

By [ELW], we may replace K^{-1} in the last inequality. Namely, define $K' \in \mathbb{Z}^{\mathrm{OPart}_4 \times \mathrm{Part}_4}$ as

$$K' = \begin{array}{c} \\ (4) \\ (3{,}1) \\ (1{,}3) \\ (2{,}2) \\ (2{,}1{,}1) \\ (1{,}2{,}1) \\ (1{,}1{,}2) \\ (1{,}1{,}1{,}1) \end{array} \begin{array}{c} \begin{array}{ccccc} (4) & (3{,}1) & (2{,}2) & (2{,}1{,}1) & (1{,}1{,}1{,}1) \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

We claim that, Eq. 2.72 is equivalent to

$$\sum_{\alpha \in \mathrm{OPart}_4} d_\alpha K'_{\alpha,\nu} \geq 0 \qquad (\nu \in \mathrm{Part}_4) \tag{2.73}$$

assuming Eq. 2.70 and 2.71 hold. (The replacement of K^{-1} with K' is not necessary, but it makes the proof of Prop. 2.5.5 more straightforward.)

Indeed, by Eq. 2.70 and 2.71,

$$(d_\alpha)_{\alpha\in\mathrm{OPart}_4} \in P := \{u \in \mathbb{Z}[\mathrm{OPart}_4] \mid u_\alpha = u_{\widetilde{\alpha}}\}$$

Moreover, P is freely generated by $\{f^\nu \mid \nu \in \mathrm{Part}_4\}$, as a $\mathbb{Z}$-module, by Lemma 2.4.4. On the other hand, by Proposition 5 in [ELW],

$$\sum_{\alpha\in\mathrm{OPart}_4} K_{\nu,\widetilde{\alpha}} K'_{\alpha,\mu} = \delta_{\nu,\mu} \qquad (\forall \nu,\mu \in \mathrm{Part}_4) \tag{2.74}$$

(for the notation $\widetilde{\alpha}$, see Subsec. 2.4.2). Hence, by $f^\nu_\alpha = K_{\nu,\widetilde{\alpha}}$ for all $\alpha \in \mathrm{OPart}_4$, we have

$$\sum_{\alpha\in\mathrm{OPart}_4} f^\nu_\alpha K'_{\alpha,\mu} = \delta_{\nu,\mu} = \sum_{\xi\in\mathrm{Part}_4} f^\nu_\xi K^{-1}_{\xi,\mu}$$

by the definition of K^{-1} and Eq. 2.74. Therefore both define the same affine monoid in P.

Now we turn to the proof of Prop. 2.5.5, that is naturally split into the following two lemmas.

Lemma 2.5.7. *If* $\mathrm{id} - p_{12} - p_{23}$ *is invertible, and Eq. 2.68 and 2.69 hold then Eq. 2.70 and 2.71 are satisfied.*

Proof. For all $i, j = 1,2,3$, $|i-j| = 1$, we verify that $1 - y_i$ maps U_i into U_j injectively, in particular $\dim U_i \geq \dim U_j$.

Indeed, since $V^{\otimes 4}$ is a semisimple faithful $\mathcal{H}_4(p)$-module, M is isomorphic to a direct summand of $V^{\otimes 4}$. Hence $1 - y_1 - y_2$ is invertible on M using the assumption on $\mathrm{id} - p_{12} - p_{23}$. Similarly, $1 - y_3 - y_4$ is invertible by Remark 2.5.6. Therefore

$$\mathrm{Ker}\big((1-y_i)|_{U_j}\big) = \mathrm{Im}(y_i) \cap \mathrm{Ker}(y_j) = 0$$

by Remark 2.2.5, so $\dim U_i \geq \dim U_j$ follows.

To obtain Eq. 2.71, we use the same argument as in the previous paragraph. First note that

$$M = (U_i \cap U_{i+1}) \oplus (\mathrm{Im}(y_i) + \mathrm{Im}(y_{i+1}))$$

as a vector space ($i = 1,2$), by $\mathrm{Im}(y_i)^\perp = \mathrm{Ker}(y_i)$ (since y_i is self-adjoint). Consider M as a P_3-module via $\gamma : P_3 \to \mathrm{End}_{\Bbbk}(M)$, such that $\gamma(x_i)$ is the projection onto $\mathrm{Im}(y_i) + \mathrm{Im}(y_{i+1})$ with kernel $U_i \cap U_{i+1}$.

Then for $\dim(U_1 \cap U_2) \geq \dim(U_2 \cap U_3)$ it is enough to verify that $\gamma(1 - x_1)$ maps $U_2 \cap U_3$ into $U_1 \cap U_2$ injectively. Hence we need

$$0 = \mathrm{Ker}\big(\gamma(1-x_1)|_{U_2\cap U_3}\big) = \mathrm{Im}\big(\gamma(x_1)\big) \cap \mathrm{Ker}\big(\gamma(x_2)\big) =$$

$$= (\mathrm{Im}(y_1) + \mathrm{Im}(y_2)) \cap \mathrm{Ker}(y_2) \cap \mathrm{Ker}(y_3)$$

$$= \mathrm{Im}\big((1-y_2)y_1\big) \cap \mathrm{Ker}(y_3)$$

That is, $\mathrm{rk}(y_3(1-y_2)y_1) = \mathrm{rk}((1-y_2)y_1)$. Similarly, $\gamma(1-x_2)|_{U_1\cap U_2}$ is injective if and only if $\mathrm{rk}(y_1(1-y_2)y_3) = \mathrm{rk}((1-y_2)y_3)$. As M is a summand of $V^{\otimes 4}$ and $\mathrm{rk}(y_{4-j}(1-y_2)y_j) \leq \mathrm{rk}((1-y_2)y_j)$ $(j=1,3)$ holds unconditionally, the rank conditions follow from Eq. 2.69. □

Lemma 2.5.8. *Assuming Eq. 2.70, 2.71 and that $\{U_1, U_2, U_3\}$ generates a distributive lattice of subspaces, the second condition in Eq. 2.73 holds.*

Proof. In details, Eq. 2.73 can be expanded as

$$0 \leq d_{(4)} \leq d_{(3,1)} \leq d_{(2,2)}$$

$$d_{(3,1)} + d_{(2,2)} - d_{(4)} \leq d_{(2,1,1)} \tag{2.75}$$

$$(d_{(2,1,1)} + d_{(1,2,1)} + d_{(1,1,2)}) - (d_{(3,1)} + d_{(2,2)} + d_{(1,3)}) + d_{(4)} \leq d_{(1,1,1,1)} \tag{2.76}$$

Clearly, we have

$$0 \leq \dim\Big(\bigcap_{i=1,2,3} U_i\Big) = d_{(4)} \leq d_{(3,1)} = \dim\left(U_1 \cap U_2\right)$$

Moreover, we may prove that $1-y_3$ maps $U_1 \cap U_2$ into $U_1 \cap U_3$ injectively, in particular $d_{(3,1)} \leq d_{(2,2)}$. Indeed, by $y_1y_3 = y_3y_1$, we have

$$(1-y_3)(U_1 \cap U_2) \subseteq U_1 \cap U_3$$

On the other hand,

$$\mathrm{Ker}\big((1-y_3)|_{U_1\cap U_2}\big) = U_1 \cap (U_2 \cap \mathrm{Im}(y_3)) = 0$$

using that $1-y_2-y_3$ is invertible (see Remark 2.5.6).

Inequality 2.75 is clear as

$$d_{(3,1)} + d_{(2,2)} - d_{(4)} =$$

$$= \dim(U_1 \cap U_2) + \dim(U_1 \cap U_3)$$

$$- \dim(U_1 \cap U_2 \cap U_3)$$

$$\leq \dim U_1 = d_{(3,1)}$$

Finally, Inequality 2.76 can be expanded as

$$\sum_{i=1,2,3} \dim U_i - \sum_{j=1,2} \dim(U_j \cap U_{j+1}) + \dim(U_1 \cap U_2 \cap U_3) \leq \dim M.$$

This follows from the assumption on $\{U_1, U_2, U_3\}$, by elementary properties of distributive subspace lattices (see Proposition 7.2 in [PP]). □

Proof of Proposition 2.5.5. By Lemma 2.5.7, 2.5.8, it is enough to see that $\{U_1, U_2, U_3\}$ generates a distributive lattice of subspaces. That follows from the fact that distributivity is preserved for direct summands of P_4-modules, moreover, $V^{\otimes 4}$ is a faithful semisimple $\mathcal{H}_4(p)$-module, hence M is a direct summand of $V^{\otimes 4}$ where $\{\mathrm{Ker}(p_{i,i+1}) \mid i = 1,2,3\}$ generates a distributive lattice by assumption. □

2.5.2 *Koszul property*

Recall (from Section 1.2) that if a quadratic algebra A is Koszul, then it is numerically Koszul, i.e. Eq. 1.2 holds for the Hilbert series of A and its quadratic dual $A^!$ for all $d \geq 1$. Since the equation always holds for $1 \leq d \leq 3$, in this section we investigate it for $d = 4$ in the case of $A = \mathcal{M}(p)$.

We may express the quadratic dual $\mathcal{M}(p)^!$ in terms of P_d-representations (for definition, see Eq. 2.7) as follows. Let $f_d : P_d \to P_d$ the algebra automorphism defined as $x_i \mapsto 1 - x_i$ for all $i = 1, \ldots, d-1$. Recall from Sec. 1.1 that for any P_d-module M, $\mathrm{Res}_{f_d} M$ is defined as the vector space M such that $x \in P_d$ acts as $x \cdot m := f_d(x)m$ for any $m \in M$.

Lemma 2.5.9. $\left(\mathcal{M}(p)^!_d\right)^\vee \cong \mathrm{Hom}_{P_d}(V^{\otimes d}, \mathrm{Res}_{f_d} V^{\otimes d})$ *as a vector space.*

Proof. For a quadratic algebra $A := \mathcal{T}(U)/(\mathrm{Rel})$ where $\mathrm{Rel} \subseteq U \otimes U$, $A^!$ is defined as $\mathcal{T}(V^\vee)/(\mathrm{Rel}^o)$. Hence for all $d \geq 1$

$$(A^!_d)^\vee = \bigcap_{i=1}^{d-1} V^{\otimes(i-1)} \otimes \mathrm{Rel} \otimes V^{\otimes(d-i-1)}$$

as a subspace of $V^{\otimes d}$. (For the notations, see Subsec. 1.2.1.) In particular, for $A = \mathcal{M}(p)$ we obtain

$$\left(\mathcal{M}(p)^!_d\right)^\vee = \bigcap_{i=1}^{d-1} \left\{ a \circ p_{i,i+1} - p_{i,i+1} \circ a \;\middle|\; a \in E^{\otimes d} \right\} \tag{2.77}$$

as a subspace of $E^{\otimes d}$, for all $d \geq 2$.

Let us suppress the composition signs. By $p_{i,i+1}^2 = p_{i,i+1}$, we may conclude that for all $i \leq d-1$,

$$\left\{ a p_{i,i+1} - p_{i,i+1} a \;\middle|\; a \in E^{\otimes d} \right\} = \left\{ b \in E^{\otimes d} \mid b p_{i,i+1} + p_{i,i+1} b = b \right\} \tag{2.78}$$

Indeed, for any $a \in E^{\otimes d}$, it is straightforward to check $b = a p_{i,i+1} - p_{i,i+1} a$ is an element of the right hand side of Eq. 2.78. Conversely, given b as above, take $a = b p_{i,i+1} - p_{i,i+1} b$. Then

$$a p_{i,i+1} - p_{i,i+1} a =$$

$$= (bp_{i,i+1} - p_{i,i+1}b)p_{i,i+1} - p_{i,i+1}(bp_{i,i+1} - p_{i,i+1}b)$$

$$= bp_{i,i+1} + p_{i,i+1}b - 2p_{i,i+1}bp_{i,i+1} = b$$

as $p_{i,i+1}bp_{i,i+1} = b(1 - p_{i,i+1})p_{i,i+1} = 0$. Hence Eq. 2.78 holds.

On the other hand,

$$\mathrm{Hom}_{P_d}(V^{\otimes d}, \mathrm{Res}_{f_d} V^{\otimes d}) = \{b \in E^{\otimes d} \mid bp_{i,i+1} = f_d(p_{i,i+1})b \ (i = 1, \dots, d-1)\}$$

where $f_d(p_{i,i+1}) = 1 - p_{i,i+1}$ for all i. The claim follows. □

Assuming that p is polynomial-type up to degree four (see Def. 2.5.1), the numerically Koszul assumption for $d = 4$ can be spelled out as follows.

Proposition 2.5.10. *Let $p \in \mathrm{End}(V \otimes V)$ be an orthogonal projection that is polynomial-type up to degree four, and consider the P_4-module*

$$N := V^{\otimes 4} \Big/ \Big(\bigcap_{i=1}^{3} \mathrm{Im}(x_i) + \bigcap_{i=1}^{3} \mathrm{Ker}(x_i) \Big)$$

Then,

$$\sum_{k=0}^{4} (-1)^k \dim \mathcal{M}(p)^!_k \dim \mathcal{M}(p)_{4-k} =$$

$$= \dim \mathrm{End}_{P_4}(N) + \dim \mathrm{Hom}_{P_4}(N, \mathrm{Res}_{f_4} N)$$

$$+ \frac{21}{4} n^2 \binom{n+1}{3}^2 - 2n^2 \sum_{\lambda \in (0,1)} n_\lambda^2 \tag{2.79}$$

where n_λ is defined by Eq. 2.67.

Theorem 2.5.3 can only be deduced from the proposition in the special case $n = 2$ (see Example 2.5.11). For $n \geq 3$, we need Lemma 2.5.12, 2.5.14, and the assumptions in Eq. 2.66 of the theorem.

Proof. By the definitions, we have $\dim \mathcal{M}(p)_1 = \dim \mathcal{M}(p)^!_1 = \dim E = n^2$ and denoting $r := \mathrm{rk}(p)$

$$\dim \mathcal{M}(p)_2 = \dim\{a \in E^{\otimes 2} \mid ap = pa\} = r^2 + (n^2 - r)^2$$

$$\dim \mathcal{M}(p)^!_2 = \dim\{a \in E^{\otimes 2} \mid ap + pa = a\} = 2r(n^2 - r)$$

Moreover, as $\mathrm{Res}_{f_3} R_i^{(0)} \cong R_{3-i}^{(0)}$ for $i = 1, 2$, and $\mathrm{Res}_{f_3} R^{(\lambda)} \cong R^{(\lambda)}$ for all λ, Prop. 2.5.9 implies

$$\dim \mathcal{M}(p)_3 = n_{0,1}^2 + n_{0,2}^2 + \sum_\lambda n_\lambda^2$$

$$\dim \mathcal{M}(p)^!_3 = 2n_{0,1}n_{0,2} + \sum_\lambda n_\lambda^2$$

Therefore

$$\begin{aligned}\sum_{k=0}^{4}(-1)^k \dim \mathcal{M}(p)^!_k \dim \mathcal{M}(p)_{4-k} =\\ = \dim \mathcal{M}(p)_4 - n^2\Big(n_{0,1}^2 + n_{0,2}^2 + \sum_\lambda n_\lambda^2\Big)\\ +2(n^2-r)r\big((n^2-r)^2+r^2\big)\\ -\Big(2n_{0,1}n_{0,2} + \sum_\lambda n_\lambda^2\Big)n^2 + \dim \mathcal{M}(p)^!_4\\ = \dim \mathcal{M}(p)_4 + \dim \mathcal{M}(p)^!_4 - 2n^2\sum_\lambda n_\lambda^2\\ + 2(n^2-r)r\big((n^2-r)^2+r^2\big) - n^2(n_{0,1}+n_{0,2})^2\end{aligned} \tag{2.80}$$

The subspaces $\cap_{i=1}^3 \mathrm{Ker}(x_i)$ and $\cap_{i=1}^3 \mathrm{Im}(x_i)$ are direct summands of the P_4-module $V^{\otimes 4}$, isomorphic to powers of the one-dimensional modules $\mathrm{Triv}_{(4)}$ and $\mathrm{Res}_{f_4}\mathrm{Triv}_{(4)}$. As p is polynomial-type up to degree four, these are of dimension $\binom{n+3}{4}$ and $\binom{n}{4}$, respectively. Hence, by Prop. 2.1.7 and Lemma 2.5.9, we obtain

$$\dim \mathcal{M}(p)_4 = \binom{n+3}{4}^2 + \binom{n}{4}^2 + \dim \mathrm{End}_{P_4}(N)$$

$$\dim \mathcal{M}(p)^!_4 = 2\binom{n+3}{4}\binom{n}{4} + \dim \mathrm{Hom}_{P_4}(N, \mathrm{Res}_{f_4}N)$$

By the polynomial-type assumption, we have $r = \binom{n}{2}$, $n_{0,1} = \binom{n+2}{3}$, and $n_{0,2} = \binom{n}{3}$. Hence

$$\begin{aligned}2.80 = \dim \mathrm{End}_{P_4}(N) + \dim \mathrm{Hom}_{P_4}(N, \mathrm{Res}_{f_4}N) - 2n^2\sum_\lambda n_\lambda^2\\ +2\binom{n+1}{2}\binom{n}{2}\Big(\binom{n+1}{2}^2 + \binom{n}{2}^2\Big) - n^2\Big(\binom{n+2}{3}+\binom{n}{3}\Big)^2\\ +\Big(\binom{n+3}{4}+\binom{n}{4}\Big)^2\end{aligned}$$

where the sum in the last two rows equal to $\frac{21}{4}n^2\binom{n+1}{3}^2$. The claim follows. □

Example 2.5.11. For $\dim V = 2$ and p polynomial-type up to degree four, Prop. 2.5.10 implies that $\mathcal{M}(p)$ can be Koszul only if the equivalent conditions of Theorem 2.2.2 hold. Indeed, as $\sum_\lambda n_\lambda = 2\binom{n+1}{3} = 2$, if $n_\lambda \neq 2$ for all λ, then $\sum_\lambda n_\lambda^2 \leq 2$, hence

$$2.79 = \frac{21}{4} n^2 \binom{n+1}{3}^2 - 2n^2 \sum_\lambda n_\lambda^2 \geq 21 - 16 > 0$$

For $\dim V = 3$ and p polynomial-type up to degree three, the same does not hold. Then $\sum_\lambda n_\lambda = 2\binom{n+1}{3} = 8$ and

$$2.79 = \frac{21}{4} n^2 \binom{n+1}{3}^2 - 2n^2 \sum_\lambda n_\lambda^2 = 756 - 18 \sum_\lambda n_\lambda^2$$

that is negative, if $n_\lambda = 7$ and $n_\mu = 1$ for some $\lambda, \mu \in (0,1)$, $\lambda \neq \mu$.

2.5.3 *Some P_4-modules*

For the proof of Theorem 2.5.3, let us define the following three P_4-modules, mimicking the corresponding irreducible $\Bbbk\mathfrak{S}_4$-modules.

For $\lambda \in (0,1)$, let $R^{(\lambda)}_{(2,2)} := \mathrm{Res}_g R^{(\lambda)}$ where the algebra morphism $g : P_4 \to P_3$ is defined as $g(x_3) = x_1$ and $g(x_i) = x_i$ for $i = 1, 2$, i.e. the maps can be given as

$$[x_1] = [x_3] = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad [x_2] = \begin{bmatrix} 0 & 0 \\ \lambda - 1 & 1 \end{bmatrix} \tag{2.81}$$

Moreover, for $\lambda, \mu \in (0,1)$, define $R^{(\lambda,\mu)}_{(3,1)}$ via the projection matrices

$$[x_1] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [x_3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[x_2] = \begin{bmatrix} 0 & 0 & 0 \\ \lambda - 1 & 1 & \mu - 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.82}$$

that are self-adjoint with respect to the unique (up to scalar multiple) Hermitian sesquilinear form with matrix

$$\begin{bmatrix} -(\lambda - 1) & \lambda - 1 & 0 \\ \lambda - 1 & 1 & \mu - 1 \\ 0 & \mu - 1 & -(\mu - 1) \end{bmatrix}$$

The form is positive definite if and only if $\lambda + \mu > 1$.

Finally, let

$$R^{(\lambda,\mu)}_{(2,1,1)} := \mathrm{Res}_{f_4} R^{(\lambda,\mu)}_{(3,1)}$$

If $\lambda = \mu$, then we simply write $R^{(\lambda)}_{(3,1)}$ instead of $R^{(\lambda,\lambda)}_{(3,1)}$ (and similarly for $R^{(\lambda,\lambda)}_{(2,1,1)}$).

Note also – using $g \circ f_4 = f_3 \circ g$ and $\mathrm{Res}_{f_3} R^{(\lambda)} \cong R^{(\lambda)}$ – that

$$\mathrm{Res}_{f_4} R^{(\lambda)}_{(2,2)} \cong \mathrm{Res}_{f_4}\mathrm{Res}_g R^{(\lambda)} \cong \mathrm{Res}_g \mathrm{Res}_{f_3} R^{(\lambda)} \cong R^{(\lambda)}_{(2,2)}$$

Recall the definition of ι_α from Subsec. 2.4.1. By the above definitions,

$$\mathrm{Res}_{\iota_{(4-i,i)}} R^{(\lambda)}_{(3,1)} \cong R^{(\lambda)} \oplus R^{(0)}_2 \qquad (i = 1, 3)$$

$$\mathrm{Res}_{\iota_{(4-i,i)}} R^{(\lambda)}_{(2,1,1)} \cong R^{(\lambda)} \oplus R^{(0)}_1 \qquad (i = 1, 3)$$

As a consequence, $R^{(\lambda)}_{(3,1)}$, $R^{(\lambda)}_{(2,2)}$ and $R^{(\lambda)}_{(2,1,1)}$ are simple modules.

Lemma 2.5.12. *Let N be a semisimple P_4-module, and denote*

$$U := \mathrm{Ker}\big(x_2(x_1 - x_3)\big) \cap \mathrm{Im}\big(x_2(1 - x_1)\big) \cap \mathrm{Ker}(x_1 x_3)$$

Then for any $\lambda \in (0, 1)$

$$[N : R^{(\lambda)}_{(3,1)}] \geq \dim U - \dim \mathrm{Ker}(x_1)$$

$$+ [\mathrm{Res}_{\iota_{(3,1)}} N : R^{(\lambda)}] + [\mathrm{Res}_{\iota_{(3,1)}} N : R^{(0)}_1] \tag{2.83}$$

Proof. First note that if

$$0 \neq u \in U \cap \mathrm{Ker}\big((x_1 - x_2)^2 - \lambda\big)$$

then

$$M := \mathrm{Span}(u, x_1 u, x_3 u) \cong R^{(\lambda)}_{(3,1)}$$

Indeed, define a map $R^{(\lambda)}_{(3,1)} \to M$ as $e_2 \mapsto -u$ and $e_i \mapsto x_i u$ for $i = 1, 3$.

We claim that it is a P_4-homomorphism (hence an isomorphism as $R^{(\lambda)}_{(3,1)}$ is simple and $u \neq 0$). The matrices of x_1 and x_3 are the same as 2.82, by $u \in \mathrm{Ker}(x_1 x_3)$ and $x_i^2 = x_i$ ($i = 1, 3$). Moreover, to compute the matrix of x_2, we may use $u \in \mathrm{Im}(x_2)$, $x_2 u = u$, and

$$u \in \mathrm{Ker}\big((x_1 - x_2)^2 - \lambda\big) \cap \mathrm{Ker}\big(x_2(x_1 - x_3)\big)$$

So we obtain,

$$x_2x_3u = x_2x_1u = x_2u - (x_2 - x_2x_1x_2)u =$$

$$= u - (x_1 - x_2)^2 x_2 u = -(\lambda - 1)u$$

Hence M is indeed isomorphic to $R^{(\lambda)}_{(3,1)}$.

Now we give a lower bound on $[N : R^{(\lambda)}_{(3,1)}]$, that is the same as $\dim\left(U \cap \operatorname{Ker}\left((x_1 - x_2)^2 - \lambda\right)\right)$, by the previous paragraph and $\dim\left(U \cap R^{(\lambda)}_{(3,1)}\right) = 1$. Therefore

$$\begin{aligned}[N : R^{(\lambda)}_{(3,1)}] &= \dim\left(U \cap \operatorname{Ker}\left((x_1 - x_2)^2 - \lambda\right)\right) \geq \\ &\geq \dim\left(\operatorname{Ker}\left((x_1 - x_2)^2 - \lambda\right) \cap \operatorname{Im}\left(x_2(1 - x_1)\right)\right) \\ &\quad + \dim(U) - \dim \operatorname{Im}\left(x_2(1 - x_1)\right) \end{aligned} \tag{2.84}$$

where

$$\dim\left(\operatorname{Ker}\left((x_1 - x_2)^2 - \lambda\right) \cap \operatorname{Im}\left(x_2(1 - x_1)\right)\right) = [\operatorname{Res}_{\iota_{(3,1)}} N : R^{(\lambda)}]$$

since for any $\lambda \in (0,1)$, $\operatorname{Ker}\left((x_1 - x_2)^2 - \lambda\right) \cong \left(R^{(\lambda)}\right)^d$ as a module over $\Bbbk\langle x_1, x_2\rangle \cong P_3$ by Lemma 2.2.9, and $x_2(1 - x_1)R^{(\lambda)} = R^{(\lambda)} \cap \operatorname{Im}(x_2)$ is one-dimensional. Moreover, for the third term in Eq. 2.84

$$\dim \operatorname{Im}\left(x_2(1 - x_1)\right) = \dim \operatorname{Im}(1 - x_1) - \dim\left(\operatorname{Ker}(x_2) \cap \operatorname{Im}(1 - x_1)\right) =$$

$$= \dim \operatorname{Ker}(x_1) - [\operatorname{Res}_{\iota_{(3,1)}} N : R^{(0)}_1]$$

The claim follows. □

Corollary 2.5.13. *By applying Eq. 2.83 on* $\operatorname{Res}_g N$*, we obtain*

$$[N : R^{(\lambda)}_{(2,1,1)}] \geq \dim U' - \dim \operatorname{Im}(x_1)$$

$$+[\operatorname{Res}_{\iota_{(3,1)}} N : R^{(\lambda)}] + [\operatorname{Res}_{\iota_{(3,1)}} N : R^{(0)}_2]$$

where U' *is*

$$\operatorname{Ker}\left((1 - x_2)(x_1 - x_3)\right) \cap \operatorname{Im}\left((1 - x_2)x_1\right) \cap \operatorname{Ker}\left((1 - x_1)(1 - x_3)\right)$$

Lemma 2.5.14. *Let* N *be a semisimple* P_4*-module. Then, for any* $\lambda \in (0,1)$

$$[N : R^{(\lambda)}_{(2,2)}] \geq \dim \operatorname{Ker}\left((x_1 - x_3)x_2\right) + [\operatorname{Res}_{\iota_{(3,1)}} N : R^{(\lambda)}] - \dim N$$

Proof. Let

$$U_\lambda := \mathrm{Ker}(x_1 - x_3) \cap \mathrm{Im}(x_2) \cap \mathrm{Ker}\big((x_1 - x_2)^2 - \lambda\big)$$

First note that if $0 \neq u \in U_\lambda$ then $M := \mathrm{Span}(u, x_1 u) \cong R^{(\lambda)}_{(2,2)}$. Indeed, the map $R^{(\lambda)}_{(2,2)} \to M$ defined as $e_2 \mapsto u$, $e_i \mapsto x_1 u$ $(i = 1,3)$ gives a nonzero P_4-homomorphism from the simple module $R^{(\lambda)}_{(2,2)}$, by Eq. 2.81.

The lower bound on $[N : R^{(\lambda)}_{(2,2)}]$ follows by

$$[N : R^{(\lambda)}_{(2,2)}] = \dim U_\lambda \geq \dim\left(\mathrm{Ker}(x_1 - x_3) \cap \mathrm{Im}(x_2)\right) +$$

$$+ \dim\Big(\mathrm{Ker}\big((x_1 - x_2)^2 - \lambda\big) \cap \mathrm{Im}(x_2)\Big) - \mathrm{rk}(x_2)$$

where $\dim\left(\mathrm{Ker}(x_1 - x_3) \cap \mathrm{Im}(x_2)\right) + \dim \mathrm{Ker}(x_2) = \dim \mathrm{Ker}\big((x_1 - x_3)x_2\big)$ and $\dim\left(\mathrm{Ker}\big((x_1 - x_2)^2 - \lambda\big) \cap \mathrm{Im}(x_2)\right) = [\mathrm{Res}_{\iota_{(3,1)}} N : R^{(\lambda)}]$, hence the claim follows. □

2.5.4 *Proof of Theorem 2.5.3*

Now, we may conclude Theorem 2.5.3 by Prop. 2.5.10, Lemma 2.5.12 and 2.5.14.

Proof of Theorem 2.5.3. It is enough to show that

$$\dim \mathrm{End}_{P_4}(N) + \dim \mathrm{Hom}_{P_4}(N, \mathrm{Res}_{f_4} N) \geq$$

$$\geq 2n^2 \sum_{\mu \in (0,1)} n_\mu^2 - \frac{21}{4} n^2 \binom{n+1}{3}^2 \tag{2.85}$$

with equality only if there is a $\lambda \in (0,1)$ such that $n_\lambda = \sum_\mu n_\mu = 2\binom{n+1}{3}$. Indeed, if $\mathcal{M}(p)$ is Koszul then by Prop. 2.5.10, the above inequality must hold with equality. If we prove that the case of equality implies $n_\lambda = 2\binom{n+1}{3}$ for some λ, then by Prop. 2.2.16 and Theorem 2.2.2, the statement of the theorem holds.

Now we apply the lemmas to obtain 2.85. Fix $\lambda \in (0,1)$ such that n_λ is maximal among n_μ's. As $R^{(\lambda)}_{(3,1)}$, $R^{(\lambda)}_{(2,2)}$ and $R^{(\lambda)}_{(2,1,1)}$ are non-isomorphic simple P_4-modules, and $\mathrm{Res}_{f_4} R^{(\lambda)}_{(3,1)} \cong R^{(\lambda)}_{(2,1,1)}$ while $\mathrm{Res}_{f_4} R^{(\lambda)}_{(2,2)} \cong R^{(\lambda)}_{(2,2)}$, we have

$$\dim \mathrm{End}_{P_4}(N) \geq [N : R^{(\lambda)}_{(3,1)}]^2 + [N : R^{(\lambda)}_{(2,1,1)}]^2 + [N : R^{(\lambda)}_{(2,2)}]^2 \tag{2.86}$$

$$\dim \mathrm{Hom}_{P_4}(N, \mathrm{Res}_{f_4} N) \geq [N : R^{(\lambda)}_{(3,1)}] \cdot [N : R^{(\lambda)}_{(2,1,1)}] + [N : R^{(\lambda)}_{(2,2)}]^2 \tag{2.87}$$

Let $c = 2\binom{n+1}{3}$. By Lemma 2.5.12, Cor. 2.5.13 and the assumptions Eq. 2.66,

$$[N : R^{(\lambda)}_{(3,1)}] + [N : R^{(\lambda)}_{(2,1,1)}] \geq \tag{2.88}$$

$$\geq \max\left(0, \frac{3n+6}{8}c + [\mathrm{Res}_{\iota_{(3,1)}} N : R^{(\lambda)}] + [\mathrm{Res}_{\iota_{(3,1)}} N : R^{(0)}_1] - \dim \mathrm{Ker}(x_1)\right)$$

$$+ \max\left(0, \frac{3n-6}{8}c + [\mathrm{Res}_{\iota_{(3,1)}} N : R^{(\lambda)}] + [\mathrm{Res}_{\iota_{(3,1)}} N : R^{(0)}_2] - \dim \mathrm{Im}(x_1)\right)$$

By the definition of N, we have

$$\dim \mathrm{Ker}(x_1) = n^2\binom{n+1}{2} - \binom{n+1}{4} = \frac{11n+6}{8}c$$

$$\dim N = n^4 - \binom{n+3}{4} - \binom{n}{4} = \frac{22n}{8}c$$

$$[\mathrm{Res}_{\iota_{(3,1)}} N : R^{(\lambda)}] = [\mathrm{Res}_{\iota_{(3,1)}} V^{\otimes 4} : R^{(\lambda)}] = n[V^{\otimes 3} : R^{(\lambda)}] = n \cdot n_\lambda$$

$$[\mathrm{Res}_{\iota_{(3,1)}} N : R^{(0)}_1] = n\binom{n+2}{3} - \binom{n+3}{4} = \frac{3n+6}{8}c$$

$$[\mathrm{Res}_{\iota_{(3,1)}} N : R^{(0)}_2] = n\binom{n}{3} - \binom{n}{4} = \frac{3n-6}{8}c$$

Therefore, by Eq. 2.88,

$$[N : R^{(\lambda)}_{(3,1)}] + [N : R^{(\lambda)}_{(2,1,1)}] \geq$$

$$\geq \max\left(0, n \cdot n_\lambda - \frac{5n-6}{8}c\right) + \max\left(0, n \cdot n_\lambda - \frac{5n+6}{8}c\right) \tag{2.89}$$

Similarly, by Lemma 2.5.14,

$$[N : R^{(\lambda)}_{(2,2)}] \geq \dim \mathrm{Ker}((x_1 - x_3)x_2) + [\mathrm{Res}_{\iota_{(3,1)}} N : R^{(\lambda)}] - \dim N =$$

$$= 2nc + n \cdot n_\lambda - \frac{11n}{4}c = n \cdot n_\lambda - \frac{3n}{4}c \tag{2.90}$$

Together Eq. 2.86, 2.87, 2.89 and 2.90 imply that

$$\dim \mathrm{End}_{P_4}(N) + \dim \mathrm{Hom}_{P_4}(N, \mathrm{Res}_{f_4} N) \geq$$

$$\geq \left(\max\left(0, n \cdot n_\lambda - \frac{5n-6}{8}c\right) + \max\left(0, n \cdot n_\lambda - \frac{5n+6}{8}c\right)\right)^2$$

$$+ 2\max\left(0, n \cdot n_\lambda - \frac{3n}{4}c\right)^2 \tag{2.91}$$

Hence it is enough to prove the following claim:

Claim 2.5.15. Let $n \geq 3$ integer, I a finite set, and for each $\mu \in I$, let $\alpha_\mu \in \mathbb{R}_{\geq 0}$ such that $\sum_\mu \alpha_\mu = 1$. Fix $\lambda \in I$ such that α_λ is maximal. Then

$$\frac{1}{2}\left(\max\left(0, \alpha_\lambda - \frac{5-\frac{6}{n}}{8}\right) + \max\left(0, \alpha_\lambda - \frac{5+\frac{6}{n}}{8}\right)\right)^2 + \max\left(0, \alpha_\lambda - \frac{3}{4}\right)^2 + \frac{21}{32} \geq \sum_{\mu \in I} \alpha_\mu^2 \tag{2.92}$$

with equality if and only if $\alpha_\lambda = 1$.

Indeed, by the claim applied on $\alpha_\lambda = \frac{n_\lambda}{c}$, we obtain Ineq. 2.85 (using Ineq. 2.91), with an equality if and only if $n_\lambda = c$.

Proof. First assume that $\alpha_\lambda \geq \max\left(\frac{3}{4}, \frac{5+\frac{6}{n}}{8}\right)$. Then the left hand side is

$$\frac{1}{2}\left(2\alpha_\lambda - \frac{5}{4}\right)^2 + \left(\alpha_\lambda - \frac{3}{4}\right)^2 + \frac{21}{32} = 3\alpha_\lambda^2 - 4\alpha_\lambda + 2$$

while the right hand side

$$\sum_{\mu \in I} \alpha_\mu^2 \leq \alpha_\lambda^2 + (1-\alpha_\lambda)^2 = 2\alpha_\lambda^2 - 2\alpha_\lambda + 1 \tag{2.93}$$

Their difference is $\alpha_\lambda^2 - 2\alpha_\lambda + 1 = (\alpha_\lambda - 1)^2 \geq 0$ hence the claim follows in this case.

Next, assume that $\frac{3}{4} > \alpha_\lambda > \frac{1}{4}$. Then

$$\sum_{\mu \in I} \alpha_\mu^2 \leq \alpha_\lambda^2 + (1-\alpha_\lambda)^2 \leq \left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{5}{8} < \frac{21}{32}$$

Now assume that $\alpha_\lambda \leq \frac{1}{4}$ and define the function

$$f : \left\{ y \in [0, \alpha_\lambda]^k \mid \sum_{i=1}^k y_i = 1 \right\} \to \mathbb{R}$$

as $y \mapsto \sum_{i=1}^k y_i^2$ where $k = |I|$. Then, as f is strictly convex, it attains its maximum on the extreme points of the compact and convex domain, i.e. on a point of the form

$$(\alpha_\lambda, \alpha_\lambda, \ldots, \alpha_\lambda, 1 - \left\lfloor \frac{1}{\alpha_\lambda} \right\rfloor \alpha_\lambda, 0, 0, \ldots, 0)$$

Hence

$$f(y) \leq \left\lfloor \frac{1}{\alpha_\lambda} \right\rfloor \alpha_\lambda^2 + \left(1 - \left\lfloor \frac{1}{\alpha_\lambda} \right\rfloor \alpha_\lambda\right)^2 \leq \alpha_\lambda + \alpha_\lambda^2 \leq \frac{5}{16} < \frac{21}{32}$$

The case we left out is when $\frac{5+\frac{6}{n}}{8} > \alpha_\lambda \geq \frac{3}{4}$ and hence $n \leq 5$. Then, by $\frac{3}{4} > \frac{5-\frac{6}{n}}{8}$, the difference of the two sides of Eq. 2.92 – using Eq. 2.93 – is at least

$$\frac{1}{2}\Big(\alpha_\lambda - \frac{5-\frac{6}{n}}{8}\Big)^2 + \Big(\alpha_\lambda - \frac{3}{4}\Big)^2 + \frac{21}{32} - \Big(2\alpha_\lambda^2 - 2\alpha_\lambda + 1\Big) =$$
$$= -\frac{1}{2}\alpha_\lambda^2 + \frac{\frac{6}{n}-1}{8}\alpha_\lambda + \frac{1}{2}\Big(\frac{5-\frac{6}{n}}{8}\Big)^2 + \frac{7}{32}$$

Let $\beta = \frac{\frac{6}{n}-1}{8}$ and note that $\beta \leq \frac{3}{4} \leq \alpha_\lambda < \frac{5+\frac{6}{n}}{8} = \beta + \frac{3}{4}$. Hence $0 \leq \alpha_\lambda - \beta < \frac{3}{4}$ and so

$$= -\frac{1}{2}\alpha_\lambda^2 + \beta\alpha_\lambda + \frac{1}{2}\Big(\frac{1}{2} - \beta\Big)^2 + \frac{7}{32}$$
$$= \beta^2 - \frac{1}{2}\beta + \frac{11}{32} - \frac{1}{2}(\alpha_\lambda - \beta)^2 >$$
$$> \beta^2 - \frac{1}{2}\beta + \frac{2}{32} = \Big(\beta - \frac{1}{4}\Big)^2 \geq 0$$

The claim follows. □

The proof of Theorem 2.5.3 is completed by Claim 2.5.15. □

Remark 2.5.16. The assumptions of Theorem 2.5.3 are satisfied by p_{sym}. Indeed, p_{sym} is polynomial-type up to degree three as we mentioned in Remark 2.2.3. Moreover, $V^{\otimes 4}$ decomposes as a $\mathcal{H}_4(p)$-module (equivalently, $\Bbbk\mathfrak{S}_4$-module) as follows:

$$V^{\otimes 4} \cong R_{(4)}^{k_4} \oplus \big(R_{(3,1)}^{(\lambda)}\big)^{k_{3,1}} \oplus \big(R_{(2,2)}^{(\lambda)}\big)^{k_{2,2}}$$
$$\oplus \big(R_{(2,1,1)}^{(\lambda)}\big)^{k_{2,1,1}} \oplus \big(R_{(1,1,1,1)}\big)^{k_{1,1,1,1}}$$

where $R_{(4)} := \mathrm{Triv}_{(4)}$, $R_{(1,1,1,1)}$ is the sign representation, $\lambda = \frac{3}{4}$, and the exponents are

$$k_4 = \binom{n+3}{4} \qquad k_{1,1,1,1} = \binom{n}{4} \qquad k_{2,2} = \frac{n}{2}\binom{n+1}{3}$$
$$k_{3,1} = \frac{3n+6}{4}\binom{n+1}{3} \qquad k_{2,1,1} = \frac{3n-6}{4}\binom{n+1}{3}$$

As it is well-known (see for example, Thm. 8.1.16 in [JK]) the number k_μ can be computed as the number of semi-standard Young tableaux over $\{1, 2, \ldots, n\}$ of shape μ (and arbitrary type).

By the proofs of Lemma 2.5.12 and 2.5.14, we obtain the first two equations of 2.66. Moreover, by the definition of $R_{(3,1)}^{(\lambda)}$, $R_{(2,2)}^{(\lambda)}$ and $R_{(2,1,1)}^{(\lambda)}$,

$$\dim \mathrm{Ker}\big((x_1 - x_3)x_2\big) = 2(k_{3,1} + k_{2,2} + k_{2,1,1}) = 4n\binom{n+1}{3}$$

hence p_{sym} indeed satisfies every assumption of Theorem 2.5.3.

3

EXAMPLES

The main example of matrix bialgebras is $\mathcal{O}_q(M_n)$, the quantized coordinate ring of $n \times n$ matrices. Instead of this well-studied algebra – that we discuss in the next chapter – first we investigate the matrix bialgebras with $\dim V = 2$ in Section 3.1, and the quantum orthogonal bialgebra of Takeuchi in Section 3.2. In Section 3.3, we show that the twist of a matrix bialgebra is still a matrix bialgebra.

3.1 DIMENSION TWO

Let V be a vector space of dimension two and β a symmetric non-degenerate $\Bbbk$-bilinear form on V. Our goal is to determine basic properties of the matrix bialgebra $\mathcal{M}(p)$ discussed in Section 2.2, for every β-symmetric idempotent $p \in E^{\otimes 2}$ of rank one. For a related classification of quantum deformations of GL_2, see [Skr].

These idempotents correspond to the points of

$$P = \left\{[w] \in \mathbb{P}(V \otimes V) \mid \beta^{\otimes 2}(w, w) \neq 0\right\}$$

Indeed, $\Bbbk w \oplus \Bbbk w^{\perp} = V \otimes V$ for any $[w] \in P$, hence the orthogonal projection p_w onto $\Bbbk w$ gives a symmetric idempotent. Conversely, taking $\mathrm{Im}(p) \in P$ for an idempotent p is clearly an inverse of the previous construction.

The orthogonal group $O(V, \beta)$ acts on P and the isomorphism type of $\mathcal{M}(p)$ (as a bialgebra) is invariant under this action. So, first we determine the $O(V, \beta)$-orbits of P. As $\Bbbk$ is algebraically closed, there is a basis $e_1, e_2 \in V$ such that

$$[\beta] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

In this basis, we have

$$O(V, \beta) = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \begin{bmatrix} 0 & a^{-1} \\ a & 0 \end{bmatrix} \;\middle|\; a \in \Bbbk^{\times} \right\} \cong \Bbbk^{\times} \rtimes (\mathbb{Z}/2)$$

Using the ordered basis $e_{1,1}$, $e_{1,2}$, $e_{2,1}$, $e_{2,2}$ in $V \otimes V$ where $e_{i,j} = e_i \otimes e_j$ $(i, j = 1, 2)$, we have

$$P = \{[a_1, a_2, a_3, a_4] \in \mathbb{P}^3 \mid a_1a_4 + a_2a_3 \neq 0\}$$

By elementary observations, we may decompose P into the following orbits:

$$O_1 = \{[1, 0, 0, a_4] \mid a_4 \neq 0\}$$

$$O_2(c) = \{[0, 1, c, 0],\ [0, c, 1, 0]\} \qquad (c \in \mathbb{k}^\times)$$

$$O_3(c) = \{[0, 1, c, a_4],\ [a_4, c, 1, 0] \mid a_4 \neq 0\} \qquad (c \in \mathbb{k}^\times)$$

$$O_4(c) = \{[a_1, 0, 1, a_4],\ [a_1, 1, 0, a_4] \mid a_1a_4 = c\} \qquad (c \in \mathbb{k}^\times)$$

$$\begin{aligned} O_5(b, c) = \{[a_1, a_2, a_3, a_4] \mid\ & a_1, a_2, a_3, a_4 \neq 0, \\ & (a_2 - ca_3)(a_2 - c^{-1}a_3) = 0 = a_1a_4 - ba_2a_3\} \\ & (b, c \in \mathbb{k}^\times,\ b \neq -1) \end{aligned}$$

where $O_2(c) = O_2(c^{-1})$, and $O_5(b, c) = O_5(b, c^{-1})$ for any $c \in \mathbb{k}^\times$.

To investigate $\mathcal{M}(p)$ for each p, we consider the following representatives from each orbit:

$$[1, 0, 0, 1] \qquad [0, 1, c, 0] \qquad [0, 1, c, 1] \qquad [1, 0, 1, c] \qquad [1, 1, c, bc]$$

where $b, c \neq 0$, $b \neq -1$.

By the aid of the computer algebra software (we used Magma), we may determine some properties of $\mathcal{M}(p)$ for each w mentioned above. It turns out that $\mathrm{Im}(p_{12}) \cap \mathrm{Im}(p_{23}) = 0$ is satisfied in all examples, except $w = [1, 1, 1, 1]$; $\mathrm{id} - p_{12} - p_{23}$ is invertible in all examples, except for $w = [1, 1, c, c]$; and $p_{12} - p_{23}$ is always non-nilpotent. Moreover, $\mathcal{H}_3(p)$ is non-semisimple for

$$w \in \{[0, 1, -1, 1],\ [1, 0, 1, b],\ [1, 1, -b, -b^2],\ [1, 1, -b, 1] \mid b \in \mathbb{k} \backslash \{0, -1\}\}$$

Recall that for $U \subseteq V \otimes V$ we defined $A_U := \mathcal{T}(V)/U$.

Example 3.1.1. For $w = [1, 0, 0, 1]$, the bialgebra $\mathcal{M}(p)$ satisfies Theorem 2.2.2 with equality and can be defined as

$$\mathbb{k}\langle a, b, c, d\rangle / (a^2 - d^2, b^2 - c^2, ab + cd, ac + bd, db + ca, dc + ba)$$

Moreover, $A_{\mathrm{Im}(p)} = \mathbb{k}\langle x, y\rangle / (x^2 + y^2) \cong \mathbb{k}\langle x, y\rangle / (xy + yx)$.

Example 3.1.2. The second example, where $w = [0,1,c,0]$, is not $\mathcal{O}_q(M_2)$ but the twist of $\mathcal{O}(M_2)$ (see Subsec. 3.3). Indeed, in our case $\mathrm{Ker}(p) = (e_1^2, e_2^2, e_1e_2 - ce_2e_1)$ and the eigenvalues of $(p_{12} - p_{23})^2$ are $\{0, \frac{3}{4}\}$, whereas for $\mathcal{O}_q(M_2)$ we would need $\mathrm{Ker}(p) = (e_1^2, e_2^2, e_1e_2 - c^{-1}e_2e_1)$ and the eigenvalues are $\lambda = 1 - q(1+q)^{-2}$ (see Remark 2.2.14).

The isomorphism type of $A_{\mathrm{Im}(p)} \cong \Bbbk\langle x,y\rangle/(xy + cyx)$ is a quantum polynomial ring for $c \neq -1$ and the commutative polynomial ring for $c = -1$.

Example 3.1.3. For $w = [0,1,-1,1]$, $\mathcal{M}(p)$ is the quantum semigroup of Jordanian matrices, discussed in Sec. 6/2. of [DVL], or [DR]. In this case, $A_{\mathrm{Im}(p)} \cong \Bbbk\langle x,y\rangle/(xy - yx + y^2)$, the Jordan plane. As one can check by computer, $p_{12}p_{23}p_{12}$ is not diagonalizable, hence $V^{\otimes 3} \cong (R_1^{(1)}[1])^4 \oplus R^{(\frac{3}{4})}[2]$, in particular $\mathcal{H}_3(p)$ is not semisimple.

Example 3.1.4. For $w = [1,0,1,c]$, we obtain

$$P_1(t) = \mathrm{CharPol}\big((p_{12} - p_{23})^2\big) = t^4\Big(t^2 + (\frac{1}{c} - \frac{3}{2})t + \frac{9}{16} - \frac{1}{c}\Big)^2$$

The discriminant of the last factor is (constant multiple of) $c+1$. Hence for $c \neq -1$ we have $V^{\otimes 3} \cong (R_1^{(1)}[1])^4 \oplus R^{(\lambda)}[1] \oplus R^{(\mu)}[1]$ for some $\lambda \neq \mu$ and so $\mathcal{H}_3(p_w)$ is semisimple. For $c = -1$, the roots are $\{0, \frac{5}{4}\}$, however – as one can check by computer – $p_{12}p_{23}p_{12}$ is not diagonalizable.

In this case, $A_{\mathrm{Im}(p)} \cong \Bbbk\langle x,y\rangle/(xy - qyx)$ for $q = \frac{1-2c\pm\sqrt{1-4c}}{2c}$, in particular for $c = -1$ it is $q = -\frac{3\pm\sqrt{5}}{2}$.

Example 3.1.5. For $w = [1,1,c,bc]$ and $b \neq -1$, we obtain that

$$P_2(t) = \mathrm{CharPol}\big((p_{12} - p_{23})^2\big) =$$

$$= \frac{t^4}{c(b+1)^4}\Big(c(b+1)^4t^2 - (b+1)^2\Big(\frac{3}{2}b^2c - bc^2 + 3bc - b + \frac{3}{2}c\Big)t$$

$$+\frac{9}{16}b^4c - b^3c^2 + \frac{7}{4}b^3c - b^3 - 2b^2c^2 + \frac{27}{8}b^2c - 2b^2 - bc^2 + \frac{7}{4}bc - b + \frac{9}{16}c\Big)^2$$

with discriminant (constant multiple of)

$$c^{-2}(b+1)^{-4}b(c+1)^2(b+c)(bc+1)$$

For $c = -1$, the roots of P_2 are 0 and $\frac{3}{4} + \frac{b}{(1+b)^2}$ and it can be checked that $\mathcal{M}(p) \cong \mathcal{O}_q(M_2)$ for $1 - \frac{q}{(1+q)^2} = \frac{3}{4} + \frac{b}{(1+b)^2}$. In particular, $\mathcal{H}_3(p)$ is semisimple. For $c = -b^{\pm 1}$, the roots of P_2 are the same, but $\mathcal{H}_3(p_w)$ can be checked to be non-semisimple for all $b, c \in \Bbbk^\times$, $b \neq -1$.

The isomorphism type of $A_{\mathrm{Im}(p)}$ (for $b \neq -1$) is $\Bbbk\langle x,y\rangle/(xy - yx + y^2)$ assuming $(1+c)^2 = 4bc$ and $c \neq 1$, otherwise it is $\Bbbk\langle x,y\rangle/(xy - qyx)$ for some $q \in \Bbbk^\times$.

Example 3.1.6. For $w = [1, 1, c, c]$, we have

$$\dim \operatorname{Ker}(p_{12}) \cap \operatorname{Im}(p_{23}) = 1 + \delta_{c,-1}$$

$$\dim \operatorname{Im}(p_{12}) \cap \operatorname{Im}(p_{23}) = \delta_{c,1}$$

Consequently, for $c \neq \pm 1$,

$$V^{\otimes 3} \cong (R_1^{(1)}[1])^4 \oplus R_1^{(0)}[1] \oplus R_2^{(0)}[1] \oplus R^{(\lambda)}[1]$$

where $\lambda = -\frac{1}{4}(c - c^{-1})$. In the case $c = -1$, we have

$$V^{\otimes 3} \cong (R_1^{(1)}[1])^4 \oplus (R_1^{(0)}[1])^2 \oplus (R_2^{(0)}[1])^2$$

while for $c = +1$,

$$V^{\otimes 3} \cong (R_1^{(1)}[1])^5 \oplus R_2^{(1)}[1] \oplus R_1^{(0)}[1] \oplus R_2^{(0)}[1]$$

hence $\dim \operatorname{End}_{\mathcal{H}_3(p)}(V^{\otimes 3})$ is $4^2 + 2 \cdot 2^2 = 24$ and $5^2 + 3 \cdot 1 = 28$ in the last two cases, respectively. In particular, the statement of Theorem 2.2.2 does not hold for $c = \pm 1$.

The isomorphism type of $A_{\operatorname{Im}(p)}$ is $\Bbbk\langle x, y\rangle / (x^2)$ if $c = 1$ and $\Bbbk\langle x, y\rangle / (xy)$ otherwise.

3.2 QUANTUM ORTHOGONAL MATRICES

In this section, we discuss the example of quantum orthogonal matrices introduced by Takeuchi in Sec. 4 of [Ta].

Let $V = \Bbbk^N$ with basis $x_1, \ldots, x_N$, and (following the notation of the literature) fix $q^{\frac{1}{2}} \in \Bbbk^\times$ that is not a root of unity. Define

$$n := \lfloor \frac{N}{2} \rfloor \qquad i' := N + 1 - i \qquad (i = 1, \ldots, N)$$

Consider the space $S \subseteq V \otimes V$ spanned by

$$x_i^2 \qquad (i \neq i')$$

$$x_i x_j + q^{-1} x_j x_i \qquad (i < j \neq i')$$

$$x_{i'} x_i + x_i x_{i'} - (q - q^{-1}) \sum_{k=1}^{i-1} q^{k-i+1} x_k x_{k'} \qquad (i < i')$$

$$x_{\frac{N+1}{2}}^2 - (q - 1) \sum_{k=1}^{n} q^{k-\frac{N}{2}} x_k x_{k'} \qquad (\text{if } 2 \nmid N)$$

Moreover, take the direct complement T spanned by

$$x_i x_j - q x_j x_i \qquad (i < j \neq i')$$

$$x_{i'}x_i - x_i x_{i'} - q^{-i}D - (q - q^{-1}) \sum_{k=i+1}^{n} q^{k-i-1} x_k x_{k'} \qquad (i < i')$$

where

$$D = (1 - q^{-1}) q^{\frac{N}{2}} x^2_{\frac{N+1}{2}}$$

if $2 \nmid N$ and $D = 0$ otherwise. In the notations of [BG], $\mathcal{T}(V)/(S) = \Lambda(O_q^N)$ and $\mathcal{T}(V)/(T) = \mathcal{O}(O_q^N)$.

In [Ta], the author considered the bialgebra $\mathcal{M}(p)$ – using the notation $\widetilde{M}_q^+(N)$ – where p is the projection onto T with kernel S. (Note that the roles of q and q^{-1} are reversed in the article.) He conjectured that these bialgebras have a PBW-basis for all $N \in \mathbb{N}^+$. By Prop. 2.2.16, a necessary condition for the existence of a PBW-basis of $\mathcal{M}(p)$ is that $(p_{12} - p_{23})^2$ has only one non-zero eigenvalue. Now we show that for $N = 3$ (resp. $N = 4$), this condition does not hold. In fact we can also check that it is not Koszul using Theorem 2.5.3.

3.2.1 *Dimension three*

First, let $N = 3$. The generators of S are

$$x_1^2,\ x_3^2,\ x_1x_2 + q^{-1}x_2x_1,\ x_2x_3 + q^{-1}x_3x_2,$$

$$x_3x_1 + x_1x_3,\ x_2^2 - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})x_1x_3$$

while for T, we have

$$x_1x_2 - qx_2x_1,\ x_2x_3 - qx_3x_2,\ x_3x_1 - x_1x_3 - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})x_2^2$$

Order the standard basis $e_{i,j} \in V \otimes V$ $(i = 1, 2, 3)$ as

$$e_{1,1},\ e_{3,3},\ e_{1,2}, e_{2,1},\ e_{2,3}, e_{3,2},\ e_{1,3}, e_{2,2}, e_{3,1}$$

We claim that the matrix of p in this basis can be expressed as the block-matrix

$$[p] = A_1^{\oplus 2} \oplus A_2^{\oplus 2} \oplus A_3$$

where

$$A_1 = [0] \qquad A_2 = \frac{1}{q + q^{-1}} \begin{bmatrix} q^{-1} & -1 \\ -1 & q \end{bmatrix}$$

$$A_3 = \frac{1}{q + q^{-1}} \begin{bmatrix} 1 & q^{\frac{1}{2}} - q^{-\frac{1}{2}} & -1 \\ q^{\frac{1}{2}} - q^{-\frac{1}{2}} & (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 & -(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \\ -1 & -(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) & 1 \end{bmatrix}$$

It is straightforward to check that using the above matrix, $ps = 0$ and $pt = t$ for all $s \in S$ and $t \in T$, hence it is indeed the projection onto T with kernel S.

As it can be computed (by software),

$$\det(\mathrm{id} - p_{12} - p_{23}) = -\frac{q^{10}(q^2-q+1)^6}{(q^2+1)^{16}} = -\frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^6}{(q+q^{-1})^{16}}$$

that is nonzero, if $q \neq 1$. Moreover, the eigenvalues of $(p_{12} - p_{23})^2$ are

$$0, \qquad 1 - \frac{1}{(q+q^{-1})^2}, \qquad \frac{2(q+q^{-1})-1}{(q+q^{-1})^2}$$

with multiplicities 11, 10 and 6, respectively. The last two are distinct if$q + q^{-1} \neq \frac{1}{2}$ i.e. $q \neq \frac{1}{4}(1 \pm i\sqrt{15})$.

Since $\Lambda(O_q^N) = \mathcal{T}(V)/\mathrm{Ker}(p)$ and $\mathcal{O}(O_q^N) = \mathcal{T}(V)/\mathrm{Im}(p)$ are PBW-algebras (see [KS]), the further conditions of p being polynomial-type up to degree three are satisfied. Hence Prop. 2.2.16 shows that $\dim \widetilde{M}_q^+(3) < \binom{3^2+2}{3} = 165$. In fact

$$\dim \widetilde{M}_q^+(3) = 10^2 + 1^2 + \binom{10}{2}^2 + \binom{6}{2}^2 = 135$$

Moreover, still by the PBW-property of $\Lambda(O_q^N)$ and $\mathcal{O}(O_q^N)$, p is polynomial-type up to degree four in the sense of Def. 2.5.1. By

$$\frac{21}{4}3^2\binom{3+1}{3}^2 - 2\cdot 3^2(5^2+3^2) = 36\cdot(21-17) > 0$$

and Prop. 2.5.10, the bialgebra $\widetilde{M}_q^+(3)$ is not Koszul.

3.2.2 *Dimension four*

Let $N = 4$. The generators of S are

$$x_1^2,\ x_2^2,\ x_3^2,\ x_4^2,\ x_4x_1 + x_1x_4,\ x_3x_2 + x_2x_3 - (q-q^{-1})x_1x_4,$$

$$x_1x_2 + q^{-1}x_2x_1,\ x_1x_3 + q^{-1}x_3x_1,\ x_2x_4 + q^{-1}x_4x_2,\ x_3x_4 + q^{-1}x_4x_3$$

while for T,

$$x_1x_2 - qx_2x_1,\ x_1x_3 - qx_3x_1,\ x_2x_4 - qx_4x_2,\ x_3x_4 - qx_4x_3,$$

$$x_4x_1 - x_1x_4 - (q-q^{-1})x_2x_3,\ x_3x_2 - x_2x_3$$

Order the standard basis $e_{i,j} \in V \otimes V$ $(i,j = 1,2,3,4)$ as

$$e_{1,1},\ e_{2,2},\ e_{3,3},\ e_{4,4},$$

$$e_{1,2},\ e_{2,1},\ e_{1,3}, e_{3,1},\ e_{2,4}, e_{4,2},\ e_{3,4}, e_{4,3},$$

$$e_{1,4}, e_{2,3}, e_{3,2}, e_{4,1}$$

We claim that the matrix of p in this basis can be expressed as the block-matrix

$$[p] = A_1^{\oplus 4} \oplus A_2^{\oplus 4} \oplus A_4$$

where A_1 and A_2 as before, and

$$A_4 = \frac{1}{(q+q^{-1})^2} \begin{bmatrix} 2 & q-q^{-1} & q-q^{-1} & -2 \\ q-q^{-1} & q^2+q^{-2} & -2 & -(q-q^{-1}) \\ q-q^{-1} & -2 & q^2+q^{-2} & -(q-q^{-1}) \\ -2 & -(q-q^{-1}) & -(q-q^{-1}) & 2 \end{bmatrix}$$

It is again straightforward to check that this is the matrix of the projection onto T with kernel S.

As it can be computed (by software),

$$\det(\mathrm{id} - p_{12} - p_{23}) = \frac{q^{32}(q^4+1)^8}{(q^2+1)^{48}} = \frac{(q^2+q^{-2})^8}{(q+q^{-1})^{48}}$$

that is nonzero if q is not a primitive 8-th root of unity. Moreover, the eigenvalues of $(p_{12} - p_{23})^2$ are

$$0, \qquad 1 - \frac{1}{(q+q^{-1})^2}, \qquad \frac{4(q+q^{-1})^2 - 4}{(q+q^{-1})^4}$$

$$x^4 - 5x^2 + 4 = (x^2-4)(x^2-1) = 0$$

with multiplicities 24, 32, and 8. The last two are distinct, if $(q+q^{-1})^2 \notin \{1, 4\}$, equivalently, if q is not a sixth root of unity.

By the same argument as before, p is polynomial-type up to degree four, hence Prop. 2.2.16 shows that $\dim \widetilde{M}_q^+(4) < \binom{4^2+2}{3} = 816$. In fact

$$\dim \widetilde{M}_q^+(4) = 20^2 + 4^2 + \binom{32}{2}^2 + \binom{8}{2}^2 = 688$$

However, using Prop. 2.5.10, we cannot decide whether $\widetilde{M}_q^+(4)$ is Koszul, since

$$\frac{21}{4} 4^2 \binom{4+1}{3}^2 - 2 \cdot 4^2 (16^2 + 4^2) = 16 \cdot (525 - 544) < 0$$

Hence in this case further investigations are necessary.

3.2.3 *Higher dimension*

For $N > 4$, the general form of p is complicated enough that computing eigenvalues of $(p_{12} - p_{23})^2$ remains a non-trivial task. Indeed, for $y \in \mathbb{Q}$ denote $a_y := q^y + q^{-y}$ and for $1 \leq u \leq N$ define $\bar{u} := \min(u, u')$. A tedious computation shows the following.

Proposition 3.2.1. *For $N \in \mathbb{N}^+$ the matrix $[p]$ of the projection onto T with kernel S in the standard basis $\{e_{i,j} \in V \otimes V \mid i,j = 1,\dots,N\}$ has the following coordinates:*

For $1 \leq k,l,m,n \leq N$ and $l \neq k'$, $n \neq m'$, and $\{k,l\} = \{m,n\}$,

$$[p]^{k,l}_{m,n} = (q+q^{-1})^{-1}\mathrm{sign}(k-l)\mathrm{sign}(m-n)q^{\frac{1}{2}(\mathrm{sign}(k-l)+\mathrm{sign}(m-n))}$$

For $1 \leq k,m \leq N$ such that $\bar{k} < \overline{m}$

$$[p]^{k,k'}_{m,m'} = \frac{(q-q^{-1})q^{m-\frac{N}{2}-\chi_{(m>n)}}}{q+q^{-1}} \cdot$$

$$\cdot\left(q^{k-\frac{N}{2}-\chi_{(k>n)}}\left(\frac{q^{\frac{N}{2}-1}}{a_{\frac{N}{2}-1}} - \frac{q^{\frac{N}{2}-\bar{k}}}{a_{\frac{N}{2}-\bar{k}}}\right) + \mathrm{sign}\Big(\frac{N+1}{2} - k\Big)a^{-1}_{\frac{N}{2}-\bar{k}}\right)$$

For $\overline{m} < \bar{k}$, we have $[p]^{k,k'}_{m,m'} = [p]^{m,m'}_{k,k'}$ as p is symmetric.
For $1 \leq k,m \leq N$ and $\bar{k} = \overline{m} \neq \frac{N+1}{2}$ we have

$$[p]^{k,k'}_{m,m'} = \frac{1}{q+q^{-1}}\left((q-q^{-1})q^{k+m-N-\chi_{(k>n)}-\chi_{(m>n)}}\left(\frac{q^{\frac{N}{2}-1}}{a_{\frac{N}{2}-1}} - \frac{q^{\frac{N}{2}-\bar{k}}}{a_{\frac{N}{2}-\bar{k}}}\right)\right.$$

$$\left.+\mathrm{sign}\Big(\frac{N+1}{2} - k\Big)\mathrm{sign}\Big(\frac{N+1}{2} - m\Big)\frac{a_{\frac{N}{2}-\bar{k}-1}}{a_{\frac{N}{2}-\bar{k}}}\right)$$

While if N is odd then

$$[p]^{\frac{N+1}{2},\frac{N+1}{2}}_{\frac{N+1}{2},\frac{N+1}{2}} = \frac{q-q^{-1}}{q+q^{-1}}\left(\frac{q^{\frac{N}{2}-1}}{a_{\frac{N}{2}-1}} - \frac{q^{-\frac{1}{2}}}{a_{-\frac{1}{2}}}\right)$$

All other coordinates of p are zero.

3.3 TWISTING

In the following, we show that Zhang-twist of a matrix bialgebra $\mathcal{M}(p)$ is isomorphic to another bialgebra of the form $\mathcal{M}(p')$ for some $p' \in E^{\otimes 2}$. This section is not used elsewhere in the thesis.

Let M be a monoid and A an M-graded $\Bbbk$-algebra. In [Z], the *Zhang twist* of A is defined as follows. A set $\{\tau_s \mid s \in M\}$ of M-graded $\Bbbk$-algebra automorphisms of A is called a *twisting system*, if

$$\mathrm{Im}(\tau_s \circ \tau_t - \tau_{st}) \subseteq \mathrm{RAnn}(A_t) \qquad (\forall s, t \in M)$$

Given a twisting system, one defines the Zhang twist A^τ as the vector space A endowed with the new multiplication $a *_\tau b := a \cdot \tau_s(b)$ for any $a \in A_s$ and $b \in A_t$, $s, t \in M$.

Now we apply a special case of this construction for our current setup. Let V be a finite-dimensional vector space, M a graded monoid i.e. a monoid M with a fixed monoid homomorphism $\deg : M \to \mathbb{N}$. We denote $M_d := \{m \in M \mid \deg(m) = d\}$. Assume that there is an M-grading on the tensor algebra $\mathcal{T}(V)$ such that

$$V^{\otimes d} = \bigoplus_{s \in M_d} \mathcal{T}(V)_s \qquad (d \in \mathbb{N})$$

Let $p \in E^{\otimes 2}$ an M-homogeneous idempotent on $V^{\otimes 2}$. Then $A_S = \mathcal{T}(V)/(S)$ and $A_T = \mathcal{T}(V)/(T)$ are also M-graded algebras, where $S = \mathrm{Ker}(p)$, $T = \mathrm{Im}(p)$.

Consider the monoid

$$M \times_{\deg} M := \{(m, m') \in M \times M \mid \deg(m) = \deg(m')\}$$

that is also a graded monoid. Then $\mathcal{T}\big(\mathrm{End}(V)\big)$ is graded by $M \times_{\deg} M$ as follows:

$$\mathcal{T}\big(\mathrm{End}(V)\big)_{(m,m')} := \mathrm{Hom}\big(\mathcal{T}(V)_m, \mathcal{T}(V)_{m'}\big) \subseteq \mathrm{End}(V)^{\otimes \deg(m)}$$

for any $(m, m') \in M \times_{\deg} M$. Since $\mathrm{Span}(a \circ p - p \circ a \mid a \in E^{\otimes 2})$ is an M-homogeneous subspace, the corresponding universal bialgebra $\mathcal{M}(p)$ is also $M \times_{\deg} M$-graded, by its definition.

Consider a monoid homomorphism $\gamma : M \to \mathrm{Aut}(V, M, p)$ where

$$\mathrm{Aut}(V, M, p) := \big\{g \in GL(V) \mid f \text{ is } M\text{-homogeneous, } (g \otimes g)p = p(g \otimes g)\big\}$$

Then $\gamma_m \in GL(V)$ ($m \in M$) induces an M-homogeneous automorphism of A_S and A_T so it uniquely defines a twisting system of both M-graded algebra. Given such a γ, that we will call a *compatible twisting system* (of A_S and A_T), we may define

$$\gamma^{-1} \otimes \gamma^\vee : M \times_{\deg} M \to GL\big(\mathcal{T}\big(\mathrm{End}(V)\big)\big)$$

$$(\gamma^{-1} \otimes \gamma^\vee)_{(m,m')}(f) = (\gamma_m^{-1})^{\otimes d} \circ f \circ (\gamma_{m'})^{\otimes d} \qquad \big(f \in \mathrm{End}(V^{\otimes d})\big)$$

where $d = \deg(m)$. As $(\gamma_m \otimes \gamma_m)p = p(\gamma_m \otimes \gamma_m)$ for all $m \in M$ by definition, $(\gamma^{-1} \otimes \gamma^\vee)_{(m,m')}$ fixes $pE^{\otimes 2}(1-p)$ and $(1-p)E^{\otimes 2}p$ hence it induces an automorphism of $\mathcal{M}(p)$.

We may assume that M is generated in degree one since $\mathcal{T}(V)$ is generated in degree one.

Lemma 3.3.1. *Let M be a graded monoid, $p \in E^{\otimes 2}$ an M-homogeneous idempotent, and γ a compatible twisting system. Then $\mathcal{M}(p)^{\gamma^{-1}\otimes\gamma^\vee} \cong \mathcal{M}(q_\gamma^{-1} p q_\gamma)$ as an algebra, where*

$$q_\gamma := \sum_{m \in M_1} \left(P_{V_m} \otimes \gamma_m\right) \in GL(V \otimes V)$$

and P_{V_m} is the projection to the direct summand $V_m \subseteq V$ ($m \in M_1$).

Remark 3.3.2. Given a bilinear form on V, if q_γ is not orthogonal then $q_\gamma^{-1} p q_\gamma$ is not necessarily self-adjoint, hence not an orthogonal projection.

Proof. First, note that the algebra $A_R = T(V)/(R)$ twisted with a compatible twisting system is isomorphic to A_{R^γ} where $R^\gamma := q_\gamma^{-1}(R)$ for any $R \subseteq V \otimes V$. Let us apply this for $\mathcal{M}(p)$, where

$$\mathrm{End}(E^{\otimes 2}) \ni q_{\gamma^{-1}\otimes\gamma^\vee} = \sum_{m,m' \in M_1} \left((P_{V_m} \otimes P_{V_{m'}}^\vee) \otimes (\gamma_m^{-1} \otimes \gamma_{m'}^\vee)\right) =$$

$$= \Big(\sum_{m \in M_1} (P_{V_m} \otimes \gamma_m)\Big)^{-1} \otimes \sum_{m' \in M_1} (P_{V_{m'}} \otimes \gamma_{m'})^\vee$$

$$= q_{\gamma^{-1}} \otimes q_\gamma^\vee \in \mathrm{End}(V^{\otimes 2} \otimes (V^{\otimes 2})^\vee)$$

using the identification $E^{\otimes 2} \cong V^{\otimes 2} \otimes (V^{\otimes 2})^\vee$ and that q_γ is invertible. Therefore the defining ideal of $\mathcal{M}(p)^{\gamma^{-1}\otimes\gamma^\vee}$ is

$$q_{\gamma^{-1}\otimes\gamma^\vee}\left(pE^{\otimes 2}(1-p) + (1-p)E^{\otimes 2}p\right) =$$

$$= q_{\gamma^{-1}} pE^{\otimes 2}(1-p) q_\gamma + q_{\gamma^{-1}}(1-p)E^{\otimes 2} p q_\gamma$$

that is the defining ideal of $\mathcal{M}(q_\gamma^{-1} p q_\gamma)$. ☐

Proposition 3.3.3. *Let γ be a compatible twisting system γ. Then $\mathcal{M}(p) \cong \mathcal{M}(q_\gamma^{-1} p q_\gamma)$ as graded coalgebras.*

Proof. The coalgebras in the statement are direct sums of finite-dimensional coalgebras, hence they are isomorphic if and only if $\mathcal{M}(p)_d^\vee \cong \mathcal{M}(q_\gamma^{-1} p q_\gamma)_d^\vee$ for all $d \in \mathbb{N}$.

Let $\widetilde{p} := q_\gamma^{-1} p q_\gamma$ and consider the subalgebras $\mathcal{H}_d(p)$ and $\mathcal{H}_d(\widetilde{p})$ of $E^{\otimes d}$ generated by $p_{1,2}, \dots, p_{d-1,d}$ and $\widetilde{p}_{1,2}, \dots, \widetilde{p}_{d-1,d}$, respectively. It is enough to prove that there is an invertible $r_d \in E^{\otimes d}$ such that $r_d^{-1} p_{k,k+1} r_d = \widetilde{p}_{k,k+1}$

for $k = 1, \ldots, d-1$. Indeed, in that case $e \mapsto r_d^{-1} e r_d$ defines an algebra isomorphism $\mathrm{End}_{\mathcal{H}_d(p)}(V^{\otimes d}) \to \mathrm{End}_{\mathcal{H}_d(\widetilde{p})}(V^{\otimes d})$ and by Prop. 2.1.7, $\mathcal{M}(p)_d^{\vee} \cong \mathrm{End}_{\mathcal{H}_d(p)}(V^{\otimes d})$ and similarly for $\widetilde{p}$.

For any $m_1, \ldots, m_{d-1} \in M_1$ define

$$r_d|_{V \otimes V_{m_1} \otimes V_{m_2} \otimes \cdots \otimes V_{m_{d-1}}} = \mathrm{id} \otimes \gamma_{m_1} \otimes (\gamma_{m_1}\gamma_{m_2}) \otimes \cdots \otimes (\gamma_{m_1} \ldots \gamma_{m_{d-1}})$$

Since γ_m is invertible and M-homogeneous for all $m \in M_1$, r_d is also invertible. Then

$$(r_d^{-1} p_{k,k+1} r_d)|_{V_{m_1} \otimes \cdots \otimes V_{m_{d-1}} \otimes V} =$$
$$= \mathrm{id}^{\otimes(k-1)} \otimes \Big(\big((\gamma_{m_1} \ldots \gamma_{m_{k-1}} \otimes \gamma_{m_1} \ldots \gamma_{m_k})^{-1} \circ p \circ$$
$$\circ (\gamma_{m_1} \ldots \gamma_{m_{k-1}} \otimes \gamma_{m_1} \ldots \gamma_{m_k}) \big) \Big|_{(V_{m_k} \otimes V_{m_{k+1}})} \Big) \otimes \mathrm{id}^{\otimes(d-k-1)}$$
$$= \mathrm{id}^{\otimes(k-1)} \otimes \Big(\big((\mathrm{id} \otimes \gamma_m)^{-1} p (\mathrm{id} \otimes \gamma_m) \big) \Big|_{(V_{m_k} \otimes V_{m_{k+1}})} \Big) \otimes \mathrm{id}^{\otimes(d-k-1)}$$

by $p(\gamma_m \otimes \gamma_m) = (\gamma_m \otimes \gamma_m)p$ for any $m \in M$. Therefore, by the definition of q_γ, we obtain that it is indeed $\widetilde{p}_{k,k+1}$ on every summand of $V^{\otimes d}$. □

Example 3.3.4. The multiparameter quantum matrices $\mathcal{O}_{\lambda,q}(M_n)$ (introduced in [ASchT]) is an example of twisting $\mathcal{O}_\lambda(M_n)$ ($\lambda \in \Bbbk^\times$), where the monoid $M = \mathbb{N}^n$, the grading of $\Bbbk^n = \mathrm{Span}(v_1, \ldots, v_n)$ is given by $(\Bbbk^n)_{e_i} = \Bbbk v_i$ and the twisting system is $\gamma_{e_i}(x_j) = q_{ji} x_j$ if $i < j$ and x_j if $j \geq i$.

4

QUANTIZED COORDINATE RINGS OF M_n, GL_n AND SL_n

4.1 MAIN RESULTS OF THE CHAPTER

In this chapter we investigate the well-studied quantizations of coordinate rings of the space of $n \times n$ matrices M_n, the general linear group GL_n and the special linear group SL_n. The algebras in the chapter are considered over the field $\mathbb{C}$.

In [DL1] it is proved (see Theorem 6.1) that the subalgebra of cocommutative elements in the quantized coordinate ring $\mathcal{O}_q(GL_n)$ form a commutative subalgebra, assuming q is not a root of unity. We prove the following related result, based on [Me1].

Theorem 4.1.1. *For $n \in \mathbb{N}^+$ and $q \in \mathbb{C}^\times$ not a root of unity, the subalgebra of cocommutative elements is a maximal commutative $\mathbb{C}$-subalgebra in $\mathcal{O}_q(M_n)$, $\mathcal{O}_q(GL_n)$ and $\mathcal{O}_q(SL_n)$.*

The proof of commutativity in [DL1] is constructive in the sense that a free generating system is determined in the form of sums of (principal) quantum minors, denoted by σ_i, $(i = 1, \dots, n)$ (see Sec. 4.2 for definition). It means that for Theorem 4.1.1, it is enough to prove that the intersection of the centralizers of $\sigma_1, \dots, \sigma_n$ is not larger than their generated subalgebra. In fact the following (stronger) statement holds:

Theorem 4.1.2. *For $n \in \mathbb{N}^+$ and $q \in \mathbb{C}^\times$ not a root of unity, the centralizer of $\sigma_1 = x_{11} + \dots + x_{nn}$ in $\mathcal{O}_q(M_n)$ (resp. $\sigma_1 \in \mathcal{O}_q(GL_n)$ and $\overline{\sigma}_1 \in \mathcal{O}_q(SL_n)$) as a unital $\mathbb{C}$-subalgebra is generated by*

- *$\sigma_1, \dots, \sigma_{n-1}, \sigma_n$ in the case of $\mathcal{O}_q(M_n)$,*
- *$\sigma_1, \dots \sigma_{n-1}, \sigma_n, \sigma_n^{-1}$ in the case of $\mathcal{O}_q(GL_n)$, and*
- *$\overline{\sigma}_1, \dots \overline{\sigma}_{n-1}$ in the case of $\mathcal{O}_q(SL_n)$.*

It is important to note that, while the theorems in [DL1] are quantum analogs of theorems established in the commutative case and they are also true if q is a root of unity (see [AZ]). Theorem 4.1.2, however has no direct commutative counterpart (only Poisson-algebraic, see Chapter 5). Also it fails if q is a root of unity since then the algebras have large center.

There is a connection between the subalgebra of cocommutative elements in $\mathcal{O}_q(M_n)$ and the center of the so-called reflection equation algebra. See [JW] for recent new results on the latter and a thorough overview of related literature.

First, in Section 4.3 we prove Proposition 4.3.1 that states that it is enough to prove Theorem 4.1.2 for any of the three algebras $\mathcal{O}_q(M_n)$, $\mathcal{O}_q(GL_n)$ or $\mathcal{O}_q(SL_n)$. Then, in Section 4.4 we discuss the computation for the case $n = 2$, that is the first step of the induction we use to prove Theorem 4.1.2 in Section 4.5.

4.2 DEFINITIONS

4.2.1 *Quantized coordinate rings*

Assume that $n \in \mathbb{N}^+$ and $q \in \mathbb{C}^\times$ is not a root of unity. Define the *quantized coordinate ring* of $n \times n$ matrices $\mathcal{O}_q(M_n)$ as a matrix bialgebra $\mathcal{M}(p)$ for $p \in \mathrm{End}(\mathbb{k}^n \otimes \mathbb{k}^n)$ defined as

$$p(e_i \otimes e_j) = \begin{cases} \frac{1}{q+q^{-1}}(q^{\mathrm{sign}(j-i)} e_i \otimes e_j + e_j \otimes e_i) & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

where $e_1, \dots, e_n$ is the standard basis of $\mathbb{k}^n$, and $\mathrm{sign}(u)$ is the sign function. More explicitly, $\mathcal{O}_q(M_n)$ is the unital $\mathbb{C}$-algebra generated by the n^2 generators $x_{i,j}$ for $1 \leq i, j \leq n$ that are subject to the following relations:

$$x_{i,j}x_{k,l} = \begin{cases} x_{k,l}x_{i,j} + (q - q^{-1})x_{i,l}x_{k,j} & \text{if } i < k \text{ and } j < l \\ q x_{k,l}x_{i,j} & \text{if } (i = k \text{ and } j < l) \text{ or } (j = l \text{ and } i < k) \\ x_{k,l}x_{i,j} & \text{otherwise} \end{cases}$$

for all $1 \leq i, j, k, l \leq n$. It turns out to be a Noetherian domain. (For a detailed exposition, see [BG].) As a matrix bialgebra $\mathcal{O}_q(M_n)$ is endowed with a coalgebra structure given in Prop. 2.1.2.

Similarly, we may define the quantized coordinate rings of GL_n and SL_n using the *quantum determinant*

$$\det_q := \sum_{s \in \mathfrak{S}_n} (-q)^{\ell(s)} x_{1,s(1)} x_{2,s(2)} \cdots x_{n,s(n)}$$

where $\ell(s)$ is the length of s in $\mathfrak{S}_n$ considered as a Coxeter group.

The quantum determinant generates the center of $\mathcal{O}_q(M_n)$. Also it is a *group-like* element, i.e. $\Delta(\det_q) = \det_q \otimes \det_q$. Hence – analogously to the classical case – one defines

$$\mathcal{O}_q(SL_n) := \mathcal{O}_q(M_n)/(\det_q - 1)$$
$$\mathcal{O}_q(GL_n) := \mathcal{O}_q(M_n)\big[\det_q^{-1}\big]$$

where we may invert $\det_q$ in the sense of Ore's theorem, as it is central (hence normal), and not a zero-divisor. The comultiplication and counit on $\mathcal{O}_q(M_n)$ induce bialgebra structures on the latter algebras as well. In particular, $\mathcal{O}_q(M_n)$ is a subbialgebra of $\mathcal{O}_q(GL_n)$. In the case of $\mathcal{O}_q(SL_n)$ and $\mathcal{O}_q(GL_n)$ we can define antipodes that turn them into Hopf algebras.

Recall that $\mathcal{O}_q(M_n)$ is an $\mathbb{N}$-graded algebra by its definition. Consequently, we may define the function $\deg : \mathcal{O}_q(M_n) \to \mathbb{N}$ as the maximum of the degrees of nonzero homogeneous components of an element.

Moreover, though $\det_q - 1$ is not homogeneous with respect to the total degree, it is homogeneous modulo n so the quotient algebra $\mathcal{O}_q(SL_n)$ becomes a $\mathbb{Z}/n\mathbb{Z}$-graded algebra.

4.2.2 *Quantum minors*

An element $a \in A$ is called *cocommutative* if $\Delta(a) = (\tau \circ \Delta)(a)$ where $\tau : A \otimes A \to A \otimes A$ is the flip $\tau(a \otimes b) = b \otimes a$. Hence we may consider A^{coc}, the subset of cocommutative elements in A. If A is a bialgebra, then A^{coc} is necessarily a subalgebra of A.

For $A = \mathcal{O}_q(M_n)$ the quantum determinant $\det_q$ is cocommutative since it is group-like. Moreover, by generalizing $\det_q$, an explicit description of $\mathcal{O}_q(M_n)^{\mathrm{coc}}$ is given in [DL1].

For $I, J \subseteq \{1, \dots, n\}$, $I = (i_1, \dots, i_t)$ and $J = (j_1, \dots, j_t)$ let us define the *quantum minor* corresponding to I and J as

$$[I \mid J] := \sum_{s \in \mathfrak{S}_t} (-q)^{\ell(s)} x_{i_1, j_{s(1)}} \dots x_{i_t, j_{s(t)}} \in \mathcal{O}_q(M_n).$$

Equivalently, $[I \mid J]$ is the quantum determinant of the subalgebra in $\mathcal{O}_q(M_n)$ generated by $\{x_{i,j}\}_{i \in I, j \in J}$, that is defined using the obvious identification between the mentioned subalgebra and $\mathcal{O}_q(M_t)$.

One may compute

$$\Delta\big([I \mid J]\big) = \sum_{|K|=t} [I \mid K] \otimes [K \mid J].$$

Hence we obtain cocommutative elements by taking

$$\sigma_i = \sum_{|I|=i} [I \mid I] \in \mathcal{O}_q(M_n)$$

for all $1 \leq i \leq n$. For $i = n$ it is $\det_q$. In the case of $i = 1$, it is $\sigma_1 = x_{1,1} + x_{2,2} + \cdots + x_{n,n}$.

We will use the notation σ_i and $\overline{\sigma}_i$ for the induced elements

$$\overline{\sigma}_i = \sigma_i + (\det_q - 1) \in \mathcal{O}_q(SL_n)$$

$$\sigma_i \in \mathcal{O}_q(M_n) \leq \mathcal{O}_q(GL_n)$$

Given an algebra isomorphism $A \to \mathcal{O}_q(M_t)$ for some t, we will write $\sigma_i(A)$ for the image of σ_i in A.

Theorem 6.1 in [DL1] states that

$$\mathcal{O}_q(M_n)^{\mathrm{coc}} = \mathbb{C}\langle \sigma_1, \ldots, \sigma_n \rangle \tag{4.1}$$

where $\mathbb{C}\langle H \rangle$ stands for the $\mathbb{C}$-subalgebra generated by a subset H. They also state that the algebra is freely generated as a commutative algebra, i.e. it is isomorphic to $\mathbb{C}[t_1, \ldots, t_{n-1}]$. Similarly,

$$\mathcal{O}_q(GL_n)^{\mathrm{coc}} = \mathbb{C}\langle \sigma_1, \ldots, \sigma_n, \sigma_n^{-1} \rangle \tag{4.2}$$

that is isomorphic $\mathbb{C}[t_1, \ldots, t_n, t_n^{-1}]$. The case of SL_n is proved in [DL2]:

$$\mathcal{O}_q(SL_n)^{\mathrm{coc}} = \mathbb{C}\langle \overline{\sigma}_1, \ldots, \overline{\sigma}_{n-1} \rangle \tag{4.3}$$

that is isomorphic to $\mathbb{C}[t_1, \ldots, t_{n-1}]$.

4.2.3 *PBW-basis in the quantized coordinate ring of matrices*

Several properties of $\mathcal{O}_q(M_n)$ can be deduced by the observation that it is an iterated Ore extension. It means that there exists a finite sequence of $\mathbb{C}$-algebras $R_0, R_1, \ldots, R_{n^2}$ such that $R_0 = \mathbb{C}$ and $R_{i+1} = R_i[y_i; \tau_i, \delta_i]$, the skew polynomial ring in y_i for an appropriate automorphism $\tau_i \in \mathrm{Aut}(R_i)$ and a derivation $\delta_i \in \mathrm{Der}(R_i)$.

An iterated Ore extension has a Poincaré-Birkhoff-Witt basis (in the sense of Subsec. 1.2.1). In the case of as $\mathcal{O}_q(M_n)$, the ordering can be chosen to be the lexicographic ordering on $\{x_{i,j} \mid 1 \leq i, j \leq n\}$ (see [BG]).

4.2.4 *Associated graded ring*

For a filtered ring $(R, \{\mathcal{F}^d\}_{d \in \mathbb{N}})$ i.e. where $\{\mathcal{F}^d\}_{d \in \mathbb{N}}$ is an ascending chain of subspaces in R such that $R = \cup_{d \in \mathbb{N}} \mathcal{F}^d$ and $\mathcal{F}^d \cdot \mathcal{F}^e \subseteq \mathcal{F}^{d+e}$ for all $d, e \in \mathbb{N}$, we define its associated graded ring

$$\mathrm{gr}(R) := \bigoplus_{d \in \mathbb{N}} \mathcal{F}^d / \mathcal{F}^{d-1}$$

where we use the notation $\mathcal{F}^{-1} = \{0\}$. The multiplication of $\mathrm{gr}(R)$ is defined in the usual way:

$$\mathcal{F}^d/\mathcal{F}^{d-1} \times \mathcal{F}^e/\mathcal{F}^{e-1} \to \mathcal{F}^{d+e}/\mathcal{F}^{d+e-1}$$
$$\left(x + \mathcal{F}^{d-1}, y + \mathcal{F}^{e-1}\right) \mapsto xy + \mathcal{F}^{d+e-1}$$

Clearly, it is a graded algebra by definition. In fact $\mathrm{gr}(.)$ can be made into a functor defined as follows: for a morphism of filtered algebras $f : \left(R, \{\mathcal{F}^d\}_{d\in\mathbb{N}}\right) \to \left(S, \{\mathcal{G}^d\}_{d\in\mathbb{N}}\right)$ (i.e. when $f(\mathcal{F}^d) \subseteq \mathcal{G}^d$) we define

$$\mathrm{gr}(f) : \mathrm{gr}(R) \to \mathrm{gr}(S)$$
$$\left(x_d + \mathcal{F}^{d-1}\right)_{d\in\mathbb{N}} \mapsto \left(f(x_d) + \mathcal{G}^{d-1}\right)_{d\in\mathbb{N}}$$

One can check that it is indeed well-defined and preserves composition. A basic property of $\mathrm{gr}(.)$ is that if we have a map $f : R \to S$ such that $f(\mathcal{F}^d) = \mathcal{G}^d$ then the $\mathrm{gr}(f)$ is also surjective.

4.3 EQUIVALENCE OF THE STATEMENTS

As it is mentioned in the introduction, Theorem 4.1.1 follows directly from Theorem 4.1.2. Indeed, since σ_i's are commuting generators in the subalgebra of cocommutative elements in $\mathcal{O}_q(M_n)$, $\mathcal{O}_q(GL_n)$ and $\mathcal{O}_q(SL_n)$ (see Eq. 4.1, 4.2 and 4.3), any commutative subalgebra containing the subalgebra of cocommutative elements is contained in the centralizer of σ_1.

Moreover, the following proposition shows that it is enough to prove Theorem 4.1.2 in the case of $\mathcal{O}_q(M_n)$.

Proposition 4.3.1. *Assume that $n \in \mathbb{N}^+$ and $q \in \mathbb{C}^\times$ is not a root of unity. The following are equivalent:*

i) The centralizer of $\sigma_1 \in \mathcal{O}_q(M_n)$ is generated by $\sigma_1, \ldots, \sigma_{n-1}, \sigma_n$.

ii) The centralizer of $\sigma_1 \in \mathcal{O}_q(GL_n)$ is generated by $\sigma_1, \ldots, \sigma_{n-1}, \sigma_n, \sigma_n^{-1}$.

iii) The centralizer of $\overline{\sigma}_1 \in \mathcal{O}_q(SL_n)$ is generated by $\overline{\sigma}_1, \ldots, \overline{\sigma}_{n-1}$.

For the proof, we need the following straightforward lemma:

Lemma 4.3.2. *Let $R = \oplus_{i\geq 0} R_i$ be an $\mathbb{N}$-graded algebra and $r \in R_k$ (for some $k > 0$) a central element that is not a zero-divisor. Then for all $d \in \mathbb{N}$, $(r-1) \cap R_d = 0$.*

Proof. As $r - 1$ is central, $0 \neq x \in (r-1)$ means that $x = y \cdot (r-1)$ for some $y \in R$. Let $y = \sum_{i=j}^{j'} y_i \in \oplus_i R_i$ be the homogeneous decomposition of y where $y_j, y_{j'} \neq 0$. Then the highest (resp. lowest) degree nonzero homogeneous component of $y \cdot (r-1)$ is $y_{j'} r$ (resp. $-y_j$), which is of degree $j' + k$ (resp. j) since r is not a zero-divisor. By $j \leq j' < j' + k$, x cannot be homogeneous. □

Proof of Proposition 4.3.1. Assume that $i)$ holds and let $h \in \mathcal{O}_q(GL_n)$ that commutes with σ_1. By the definition of $\mathcal{O}_q(GL_n)$, there exists a $k \in \mathbb{N}$ such that $h \cdot \det_q^k \in \mathcal{O}_q(M_n) \leq \mathcal{O}_q(GL_n)$ which also commutes with σ_1 since $\det_q$ is central. Therefore, by $i)$, we have $h \cdot \det_q^k \in \mathbb{C}\langle \sigma_1, \dots, \sigma_{n-1}, \sigma_n \rangle$ hence

$$h = h \cdot \det_q^k \cdot \det_q^{-k} \in \mathbb{C}\langle \sigma_1, \dots, \sigma_{n-1}, \sigma_n, \sigma_n^{-1} \rangle$$

and so $ii)$ follows.

Conversely, assume $ii)$ and take an $h \in \mathcal{O}_q(M_n) \subseteq \mathcal{O}_q(GL_n)$ that commutes with σ_1. By the assumption, $h \in \mathbb{C}\langle \sigma_1, \dots, \sigma_{n-1}, \sigma_n, \sigma_n^{-1} \rangle$ hence it is cocommutative in $\mathcal{O}_q(GL_n)$. Since $\mathcal{O}_q(M_n)$ is a subbialgebra of $\mathcal{O}_q(GL_n)$, h is cocommutative in $\mathcal{O}_q(M_n)$ as well, hence by Eq. 4.1 $i)$ follows.

Now we prove $i) \Leftrightarrow iii)$: First assume $i)$ and let $\overline{h} \in \mathcal{O}_q(SL_n)$ that commutes with $\overline{\sigma}_1$. Since $\mathcal{O}_q(SL_n)$ is $\mathbb{Z}/n\mathbb{Z}$-graded and $\overline{\sigma}_1$ is homogeneous with respect to this grading, its centralizer is generated by homogeneous elements. So we may assume that $\overline{h}$ is homogeneous as well.

Let $k = \deg(\overline{h}) \in \{0, 1, \dots, n-1\}$. Take an $h \in \mathcal{O}_q(M_n)$ that represents $\overline{h} \in \mathcal{O}_q(SL_n) = \mathcal{O}_q(M_n)/(\det_q - 1)$. Let $h = \sum_{j=0}^{d} h_{jn+k}$ be the $\mathbb{N}$-homogeneous decomposition of h where h_{jn+k} is homogeneous of degree $jn + k$ for all $j \in \mathbb{N}$. (As $\overline{h}$ is $\mathbb{Z}/n\mathbb{Z}$-homogeneous we may assume that h has nonzero homogeneous components only in degrees $\equiv k$ modulo n.) Let

$$h' := \sum_{j=0}^{d} h_{jn+k} \cdot \det_q^{d-j} \in \mathcal{O}_q(M_n)_{dn+k}$$

which is a homogeneous element of degree $dn + k$ representing $\overline{h}$ in $\mathcal{O}_q(M_n)$. Therefore $\sigma_1 h' - h'\sigma_1 \in (\det_q - 1) \cap \mathcal{O}_q(M_n)_{dn+k+1}$ because $\overline{\sigma}_1\overline{h} - \overline{h}\overline{\sigma}_1 = 0 \in \mathcal{O}_q(SL_n)$ and σ_1 is homogeneous of degree 1. By Lemma 4.3.2,

$$(\det_q - 1) \cap \mathcal{O}_q(M_n)_{dn+k+1} = 0$$

hence $\sigma_1 h' = h'\sigma_1$. By $i)$ we obtain $h' \in \mathbb{C}\langle \sigma_1, \dots, \sigma_n \rangle$ therefore $\overline{h} \in \mathbb{C}\langle \overline{\sigma_1}, \dots, \overline{\sigma_{n-1}} \rangle$ as we claimed.

Conversely, assume $iii)$ and let $h \in \mathcal{O}_q(M_n)$ such that $\sigma_1 h = h\sigma_1$. Since σ_1 is $\mathbb{N}$-homogeneous, its centralizer is also generated by homogeneous elements so we may assume that h is homogeneous. Then the image $\overline{h}$ of h in $\mathcal{O}_q(SL_n)$ is also homogeneous with respect to the $\mathbb{Z}/n\mathbb{Z}$-grading of $\mathcal{O}_q(SL_n)$.

By the assumption, $\overline{h}$ commutes with $\overline{\sigma}_1$ hence $\overline{h} \in \mathbb{C}\langle \overline{\sigma}_1, \dots, \overline{\sigma}_{n-1} \rangle$ by $iii)$. This decomposition of $\overline{h}$ can be lifted to $\mathcal{O}_q(M_n)$ giving an element $s \in \mathbb{C}\langle \sigma_1, \dots, \sigma_{n-1} \rangle$ such that $h - s \in (\det_q - 1)$. As $\overline{h}$ was $\mathbb{Z}/n\mathbb{Z}$-homogeneous, s can also be chosen to be $\mathbb{Z}/n\mathbb{Z}$-homogeneous since the generators $\overline{\sigma}_1, \dots, \overline{\sigma}_{n-1}$ are homogeneous as well.

Now we modify h and s so that they become $\mathbb{Z}$-homogeneous elements of the same degree. Let Let $k = \deg(\overline{h}) \in \{0, 1, \ldots, n-1\}$ and $d = \frac{1}{n}\big(\max(\deg h, \deg s) - k\big)$ and take $s = \sum_{j=0}^{d} s_{jn+k}$ the homogeneous decomposition of s. If $\deg(s) > \deg(h)$ then let

$$h' = h \cdot \det_q^{\frac{1}{n}\big(\deg(s)-\deg(h)\big)}$$

so now $d = \deg(s) = \deg(h')$. (The exponent is an integer since $\deg(h) \equiv \deg(s)$ modulo n.) Otherwise, let $h' = h$.

The same way as in the proof of $i) \Rightarrow iii)$, we can modify s as follows. Let

$$s' := \sum_{j=0}^{d} s_{jn+k} \cdot \det_q^{d-j}$$

Then $s' \in \mathbb{C}\langle \sigma_1, \ldots, \sigma_n \rangle$, it is $\mathbb{N}$-homogeneous of degree $nd + k$, and $s - s' \in (\det_q - 1)$. So

$$h' - s' = (h' - h) + (h - s) + (s - s') \in (\det_q - 1) \cap \mathcal{O}_q(M_n)_{nd+k}$$

that is zero by Lemma 4.3.2. Hence $h' \in \mathbb{C}\langle \sigma_1, \ldots, \sigma_n \rangle$ which gives $h \in \mathbb{C}\langle \sigma_1, \ldots, \sigma_n, \sigma_n^{-1} \rangle$. However,

$$\mathbb{C}\langle \sigma_1, \ldots, \sigma_n, \sigma_n^{-1} \rangle \cap \mathcal{O}_q(M_n) = \mathbb{C}\langle \sigma_1, \ldots, \sigma_n \rangle$$

by Eq. 4.1 and 4.2. The claim follows. □

4.4 CASE OF $\mathcal{O}_q(SL_2)$

In this section we prove Theorem 4.1.2 for $\mathcal{O}_q(SL_2)$, that is the base step of the induction that we use in the proof of Theorem 4.1.2.

In fact in the induction step we will show the statement for $\mathcal{O}_q(M_n)$ and not for $\mathcal{O}_q(SL_n)$. In the light of Prop. 4.3.1 these are equivalent. The reason why we use SL_2 in this part and not M_2 is that $\mathcal{O}_q(SL_2)$ has fewer elements (in the sense of Gelfand-Kirillov dimension) so the computations are shorter.

Proposition 4.4.1. *Assume that $q \in \mathbb{C}^\times$ is not a root of unity. The centralizer of $\overline{\sigma}_1 \in \mathcal{O}_q(SL_2)$ is $\mathbb{C}\langle \sigma_1 \rangle$.*

For the generators of $\mathcal{O}_q(SL_2)$ we will use the notations

$$a := \overline{x}_{1,1} \qquad b := \overline{x}_{1,2}$$

$$c := \overline{x}_{2,1} \qquad d := \overline{x}_{2,2}$$

where $\overline{x}_{i,j} = x_{i,j} + (\det_q - 1)$. In particular, $\overline{\sigma}_1 = a + d$.

By Theorem I.7.16. in [BG] we have a basis of $\mathcal{O}_q(SL_2)$ consisting of the following elements:

$$a^i b^k c^l,\ b^k c^l d^j,\ b^k c^l \qquad (i,j \in \mathbb{N}^+,\ k,l \in \mathbb{N})$$

We will use the $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathcal{O}_q(SL_2)$ defined as

$$\deg(a^i b^k c^l) = i \bmod 2$$

$$\deg(b^k c^l d^j) = j \bmod 2$$

Note, that it is not the $\mathbb{Z}/2\mathbb{Z}$-grading that $\mathcal{O}_q(SL_2)$ inherits from the $\mathbb{Z}$-grading of $\mathcal{O}_q(M_2)$. That would give $i+k+l$ and $k+l+j$ modulo 2, respectively. Still, this is a grading in the sense of graded algebras.

Proof. First, let us compute the adjoint action of $\overline{\sigma}_1 = a + d$ on the basis elements.

$$\begin{aligned}(a+d)\cdot a^i b^k c^l &= a^{i+1}b^k c^l + (1+q^{-1}bc)a^{i-1}b^k c^l = \\ &= a^{i+1}(b^k c^l) + a^{i-1}(b^k c^l + q^{-2(i-1)-1}b^{k+1}c^{l+1})\end{aligned}$$

Similarly,

$$\begin{aligned}a^i b^k c^l \cdot (a+d) &= q^{-(k+l)}a^{i+1}b^k c^l + q^{k+l}a^{i-1}(1+qbc)b^k c^l = \\ &= a^{i+1}(q^{-(k+l)}b^k c^l) + a^{i-1}(q^{k+l}b^k c^l + q^{k+l+1}b^{k+1}c^{l+1}).\end{aligned}$$

Hence for the commutator we get

$$\begin{aligned}\left[(a+d), a^i b^k c^l\right] &= a^{i+1}\big((1-q^{-(k+l)})b^k c^l\big) \\ &+ a^{i-1}\big((1-q^{k+l})b^k c^l \\ &+ (q^{-2(i-1)-1} - q^{k+l+1})b^{k+1}c^{l+1}\big).\end{aligned} \tag{4.4}$$

By the same computation on $b^k c^l d^j$ and $b^k c^l$, we may conclude that

$$\begin{aligned}\left[(a+d), b^k c^l d^j\right] &= \big((q^{-(k+l)} - 1)b^k c^l\big)d^{j+1} \\ &+ \big((q^{k+l} - 1)b^k c^l \\ &+ (q^{k+l+1} - q^{-2(j-1)-1})b^{k+1}c^{l+1}\big)d^{j-1} \\ \left[(a+d), b^k c^l\right] &= a(1-q^{-(k+l)})b^k c^l + (q^{-(k+l)} - 1)b^k c^l d.\end{aligned}$$

Generally, for a polynomial $p \in \mathbb{C}[t_1, t_2]$ and $i \geq 1$:

$$\begin{aligned}\left[(a+d), a^i p(b,c)\right] &= a^{i+1}\sum_m \big((1-q^{-m})p_m(b,c)\big) \\ &+ a^{i-1}\Big(\sum_m (1-q^m)p_m(b,c) \\ &+ (q^{-2(i-1)-1} - q^{m+1})bc\,p_m(b,c)\Big)\end{aligned} \tag{4.5}$$

where p_m is the m-th homogeneous component of p with respect to the $\mathbb{N}$-valued total degree on $\mathbb{C}[t_1, t_2] \cong \mathbb{C}\langle b, c\rangle$. The analogous computations for $p(b,c)d^j$ ($j \geq 1$) and $p(b,c)$ give

$$\begin{aligned} \left[(a+d), p(b,c)d^j\right] &= \sum_m \left((q^{-m}-1)p_m(b,c)\right)d^{j+1} \quad (4.6) \\ &+ \sum_m \left((q^m - 1)p_m(b,c)\right. \\ &+ \left.(q^{m+1} - q^{-2(j-1)-1})bcp_m(b,c)\right)d^{j-1} \\ \left[(a+d), p(b,c)\right] &= a\sum_m \left((1-q^{-m})p_m(b,c)\right) \quad (4.7) \\ &+ \sum_m \left((q^{-m}-1)p_m(b,c)\right)d \end{aligned}$$

To prove the statement, it is enough to show that in each subspace $\sum_{i=0}^{\alpha} a^i \cdot \mathbb{C}\langle b,c\rangle + \sum_{j=0}^{\alpha} \mathbb{C}\langle b,c\rangle \cdot d^i \subseteq \mathcal{O}_q(SL_2(\mathbb{C}))$ the space of elements that centralize $\overline{\sigma}_1$ has dimension $\alpha+1$. Indeed, $\dim \sum_{i=0}^{\alpha} \mathbb{C}\overline{\sigma}_1^i = \alpha+1$ by $\mathbb{C}\langle\overline{\sigma}_1\rangle \cong \mathbb{C}[t]$, and its elements centralize $\overline{\sigma}_1$ so then it is the whole centralizer.

Assume that the nonzero element g commutes with $\overline{\sigma}_1$. Express g in the above mentioned basis as

$$g = \sum_{i=1}^{\alpha} a^i r_i + \sum_{j=1}^{\beta} s_j d^j + u$$

where r_i, s_j and u are elements of $\mathbb{C}\langle b,c\rangle$, and α and β are the highest powers of a and d appearing in the decomposition (i.e. $r_\alpha \neq 0$ and $s_\beta \neq 0$). We will also write r_0 or s_0 for u, if it makes a formula simpler. Since $\overline{\sigma}_1$ is a homogeneous element with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading, we may assume that g is also homogeneous.

The proof is split into two cases: if g has degree $0 \in \mathbb{Z}/2\mathbb{Z}$ (hence α is even) then we will prove that the constant terms of the $\frac{\alpha}{2}+1$ polynomials $r_\alpha, r_{\alpha-2}, \ldots, r_2, u \in \mathbb{C}[b,c]$ determine g uniquely, and similarly, if $g \in \mathbb{Z}/2\mathbb{Z}$ has degree 1 (hence α is odd) then the constant terms of the $\frac{\alpha+1}{2}$ polynomials $r_\alpha, r_{\alpha-2}, \ldots, r_1 \in \mathbb{C}[b,c]$ also determine g uniquely. This is enough, since in the even case we obtain $\left(\frac{\alpha}{2}+1\right) + \frac{(\alpha-1)+1}{2} = \alpha+1$ for the dimension of the $\overline{\sigma}_1$-centralizing elements as the sum of dimensions of homogeneous $\overline{\sigma}_1$-centralizing elements in even and odd degrees. Similarly, if α is odd, it is $\frac{\alpha+1}{2} + \frac{\alpha-1}{2} = \alpha+1$ so it is indeed enough to prove the above claim.

First we prove that $r_\alpha \in \mathbb{C}\cdot 1$ in both cases. If $\alpha = 0$ then $r_\alpha = u$ so the $a^i b^k c^l$ terms in $[a+d, g]$ (decomposed in the monomial basis) are the same as the $a^i b^k c^l$ terms in $[a+d, u]$ by Eq. 4.5, 4.6 and 4.7. However, by

Eq. 4.7, these terms would be nonzero if $u \notin \mathbb{C}$. Now assume that $\alpha \geq 1$ and define the subspace

$$\mathcal{A}^d := \mathrm{Span}_{\mathbb{C}}(a^i b^k c^l,\ b^k c^l d^j,\ b^k c^l \mid i \leq d,\ k,l \in \mathbb{N})$$

for any $d \in \mathbb{N}$. Recall the fact that $\overline{\sigma}_1 \mathcal{A}^{\alpha-1}$, $\mathcal{A}^{\alpha-1}\overline{\sigma}_1$, $d\mathcal{A}^\alpha$ and $\mathcal{A}^\alpha d$ are all contained in $\mathcal{A}^\alpha$, using the defining relations. Hence

$$\begin{aligned} &\overline{\sigma}_1 g - g\overline{\sigma}_1 + \mathcal{A}^\alpha \subseteq \\ &\subseteq \overline{\sigma}_1(a^\alpha r_\alpha + \mathcal{A}^{\alpha-1}) - (a^\alpha r_\alpha + \mathcal{A}^{\alpha-1})\overline{\sigma}_1 + \mathcal{A}^\alpha \\ &= a \cdot a^\alpha r_\alpha - a^\alpha r_\alpha \cdot a + \mathcal{A}^\alpha. \end{aligned}$$

Using the decomposition $r_\alpha = \sum \lambda_{k,l} b^k c^l$ we have

$$a^\alpha r_\alpha \cdot a = \sum \lambda_{k,l} q^{-k-l} a^{\alpha+1} b^k c^l.$$

Since the elements $a^{\alpha+1}b^k c^l$ are independent even modulo $\mathcal{A}^\alpha$ by Subsec. 4.2.3, $a^\alpha r_\alpha \cdot a$ may agree with $a^{\alpha+1} r_\alpha$ modulo $\mathcal{A}^\alpha$ only if $\lambda_{k,l} = 0$ for all $(k,l) \neq (0,0)$. Therefore $r_\alpha \in \mathbb{C} \cdot 1$.

Now we prove that for all $1 \leq i \leq \alpha - 1$, r_{i+1} and the constant term of r_{i-1} determines $r_{i-1} \in \mathbb{C}[b,c]$. Indeed, by Eq. 4.5 we have

$$\begin{aligned} 0 = \mathrm{Coeff}_{a^i}([(a+d),g]) &= \sum_m \left((1-q^{-m})r_{i-1,m}\right) \\ &+ \sum_m (1-q^m) r_{i+1,m} \\ &+ \sum_m (q^{-2(i-1)-1} - q^{m+1}) bc r_{i+1,m} \end{aligned} \tag{4.8}$$

where and $r_{i,m}$ is the m-th homogeneous term of $r_i \in \mathbb{C}[b,c]$ and Coeff_{a^i} stands for the element in $\mathbb{C}[b,c]$ such that $a^i \cdot \mathrm{Coeff}_{a^i}(x)$ is a summand of x when it is decomposed in the monomial basis. The degree k part of the right hand side is

$$\begin{cases} (1-q^{-k})r_{i-1,k} + (1-q^k)r_{i+1,k} + (q^{1-2i} - q^{k-1})bc r_{i+1,k-2} & \text{if } k \geq 2 \\ (1-q^{-1})r_{i-1,1} + (1-q^1)r_{i+1,1} & \text{if } k = 1 \end{cases}$$

for all $1 \leq i \leq \alpha - 1$. Hence r_{i+1} determines r_{i-1} (using that q is not a root of unity) except for the constant term $r_{i-1,0}$ which has zero coefficient in Eq. 4.8 for all k.

We prove that $\deg s_{j+1} \leq \deg s_{j-1} - 2$ for all $j \geq 1$ where deg stands for the total degree of $\mathbb{C}[b,c]$. Analogously to Eq. 4.8, one can deduce the following by Eq. 4.6:

$$\begin{aligned} 0 = \mathrm{Coeff}_{d^j}\left([(a+d),g]\right) &= \sum_m \left((q^{-m}-1)s_{j-1,m}\right) + \\ &+ \sum_m (q^m-1)s_{j+1,m} \\ &+ \sum_m (q^{m+1} - q^{-2(j-1)-1})bcs_{j+1,m} \end{aligned}$$

The degree k part of the right hand side is

$$\begin{aligned} &(q^{-k}-1)s_{j-1,k} + (q^k-1)s_{j+1,k} + (q^{k-1}-q^{1-2j})bcs_{j+1,k-2} && \text{if } k \geq 2 \\ &(q^{-1}-1)s_{j-1,1} + (q^1-1)s_{j+1,1} && \text{if } k = 1 \end{aligned} \tag{4.9}$$

for all $1 \leq j \leq \beta - 1$. Note that $q^{k-1} - q^{1-2j} = 0$ can never happen for $k \geq 2$. If $s_{j+1} = 0$ then the statement is empty. If $s_{j+1} \neq 0$ then for $k = 2 + \deg s_{j+1} \geq 2$, we have $s_{j+1,k} = 0$ but $s_{j+1,k-2} = s_{j+1,\deg s_{j+1}} \neq 0$ hence Eq. 4.9 gives $s_{j-1,k} \neq 0$. So $\deg s_{j+1} \leq \deg s_{j-1} - 2$.

Now assume that α is even. By the previous paragraphs, the scalars r_α, $r_{\alpha-2,0}, \dots, r_{2,0}$ and u_0 determine all the polynomials $r_\alpha, r_{\alpha-2}, r_{\alpha-4}, \dots, r_2$ and u. We prove that they also determine the s_j's. Starting from $u = s_0$ one can obtain s_{j+1} by s_{j-1}. Indeed, since $\deg s_{j+1} \leq \deg s_{j-1} - 2$, if $\deg s_{j-1} \leq 1$ then $s_{j+1} = 0$, and similarly, for $k = \deg s_{j-1} \geq 2$ we have $s_{j+1,k-1} = 0$ and Eq. 4.9 gives

$$(q^{-k}-1)s_{j-1,k} = -(q^{k-1}-q^{1-2j})bc \cdot s_{j+1,k-2}.$$

Then, recursively for k, if $s_{j-1,k}$ and $s_{j+1,k}$ are given, by Eq. 4.9 they determine $s_{j+1,k-2}$, using that q is not a root of unity.

If α is odd, then by Eq. 4.5 one can obtain the following for the summand of $[(a+d),g]$ that does not contain a and d when decomposed in the given basis:

$$\begin{aligned} 0 = \mathrm{Coeff}_1\left([(a+d),g]\right) &= \sum_m \Big((1-q^m)r_{1,m} + (q-q^{m+1})bc \cdot r_{1,m} + \\ &+ (q^m-1)s_{1,m} + (q^{m+1}-q)bc \cdot s_{1,m}\Big) \end{aligned}$$

The homogeneous components of degree k are

$$\begin{aligned} &(1-q^k)r_{1,k} + (q-q^{k-1})bc \cdot r_{i+1,k-2} \\ &\qquad + (q^k-1)s_{1,k} + (q^{k+1}-q)bc \cdot s_{1,k-2} && \text{if } k \geq 2 \\ &(1-q)r_{1,1} + (q-1)s_{1,1} && \text{if } k = 1 \end{aligned} \tag{4.10}$$

Hence $r_\alpha, r_{\alpha-2,0}, \ldots, r_{1,0}$ determine not only r_i for $1 \leq i \leq \alpha$ but also s_1 by Eq. 4.10 applied for $k = \deg s_1 + 2$ and the same recursive argument as in the even case. Then, similarly, s_{j+1} is unique by s_{j-1} for all $2 \leq j \leq \beta - 1$ and the statement follows. □

4.5 PROOF OF THEOREM 4.1.2

In [DL1], to verify that the subalgebra of cocommutative elements in $A_n := \mathcal{O}_q(M_n)$ is generated by the σ_i's, the authors proved that the natural surjection

$$\eta : \mathcal{O}_q(M_n) \to \mathbb{C}[t_1, \ldots, t_n] \qquad x_{i,j} \mapsto \delta_{i,j} t_i$$

restricted to the subalgebra of cocommutative elements $\mathcal{O}_q(M_n)^{\mathrm{coc}}$ is an isomorphism and its image is the subalgebra of symmetric polynomials $D_n^{\mathfrak{S}_n}$ where $D_n := \mathbb{C}[t_1, \ldots, t_n]$. We use the same plan to prove that it is also the centralizer of $\sigma_1 \in \mathcal{O}_q(M_n)$. The idea is to translate the problem for associated graded rings where we may apply induction.

For this purpose, we will need the following intermediate quotient algebra between A_n and D_n:

$$B_{2,n} := A_n / (x_{1,j}, x_{i,1} \mid 2 \leq i, j \leq n)$$

Let us denote the corresponding natural surjection by $\varphi : A_n \to B_{2,n}$. Since $\operatorname{Ker} \eta \subseteq \operatorname{Ker} \varphi$ by their definition, η can be factored through φ. So our setup is:

$$C(\sigma_1) \subseteq \quad A_n \overset{\varphi}{\twoheadrightarrow} B_{2,n} \overset{\delta}{\twoheadrightarrow} D_n \tag{4.11}$$

where $\eta = \delta \circ \varphi$ and $C(\sigma_1)$ denotes the centralizer of σ_1 in A_n. The structure of $B_{2,n}$ is quite simple: $B_{2,n} \cong A_{n-1}[t]$ by the map $x_{i,j} \mapsto x_{i-1,j-1}$ for $i, j \geq 2$ and $x_{1,1} \mapsto t$. One can check that it is indeed an isomorphism since $x_{1,1}$ commutes with the elements of $\mathbb{C}\langle x_{1,1}, x_{i,j} \mid i, j \geq 2\rangle$ modulo $\operatorname{Ker} \varphi$.

These algebras are $\mathbb{N}$-graded algebras using the total degree of A_n, but we can also endow them by a filtration that is not the corresponding filtration of the grading.

Namely, for each $d \in \mathbb{N}$ let $\mathcal{A}^d$ be the subspace of A_n that is generated by the monomials in which $x_{1,1}$ appears at most d times, i.e. it is spanned by the ordered monomials of the form $x_{1,1}^i m$ where $i \leq d$ and m is an ordered monomial in the variables $x_{i,j}$, $(i, j) \neq (1, 1)$.

One can check that this is indeed a filtration: they are linear subspaces such that $\cup_d \mathcal{A}^d = A_n$ and $\mathcal{A}^d \cdot \mathcal{A}^e \subseteq \mathcal{A}^{d \cdot e}$ for all $d, e \in \mathbb{N}$. As $C(\sigma_1)$ is a subalgebra of A_n, we get an induced filtration $\mathcal{C}^d = \mathcal{A}^d \cap C(\sigma_1)$ $(d \in \mathbb{N})$ on $C(\sigma_1)$, and similarly, an induced filtration $\mathcal{B}^d := \varphi(\mathcal{A}^d)$ $(d \in \mathbb{N})$ on $B_{2,n}$ and $\mathcal{D}^d := \delta \circ \varphi(\mathcal{A}^d)$ $(d \in \mathbb{N})$ on D_n.

Proof of Theorem 4.1.2. We prove the statement by induction on n. The statement is verified for $\mathcal{O}_q(SL_2)$ in Sec. 4.4 so by Prop. 4.3.1 the case $n = 2$ is proved. Now assume that $n \geq 3$. We shall prove that

- $(\delta \circ \varphi)|_{C(\sigma_1)} : C(\sigma_1) \to D_n$ is injective, and
- the image $(\delta \circ \varphi)(C(\sigma_1))$ is in $D_n^{\mathfrak{S}_n}$.

These imply that the restriction of $\delta \circ \varphi$ to $C(\sigma_1)$ yields an isomorphism onto $D_n^{\mathfrak{S}_n}$, since by [DL1], $C(\sigma_1) \ni \sigma_i$ for all $i = 1, \ldots, n$ and $\delta \circ \varphi$ restricts to an isomorphism between $\mathbb{C}\langle \sigma_1, \ldots, \sigma_n \rangle$ and $D_n^{\mathfrak{S}_n}$.

First part, step 1: First we show that it is enough to prove that $\mathrm{gr}(\delta \circ \varphi)$ restricted to $\mathrm{gr}(C(\sigma_1))$ is injective to get the injectivity of $\delta \circ \varphi$ on $C(\sigma_1)$. Apply $\mathrm{gr}(.)$ to the filtered algebras in our setup presented in Diagram 4.11. It gives

$$\mathrm{gr}(C(\sigma_1)) \subseteq \mathrm{gr}(A_n) \xrightarrow{\mathrm{gr}(\varphi)} \mathrm{gr}(B_{2,n}) \xrightarrow{\mathrm{gr}(\delta)} \mathrm{gr}(D_n) \tag{4.12}$$

The surjectivity of the maps follow by $\varphi(\mathcal{A}^d) = \mathcal{B}^d$ and $\delta(\mathcal{B}^d) = \mathcal{D}^d$. Assuming that $\mathrm{gr}(\delta \circ \varphi)$ restricted to $\mathrm{gr}(C(\sigma_1))$ is injective, we get the injectivity of $(\delta \circ \varphi)|_{\mathcal{C}^0}$, moreover, we can apply an induction on d using the 5-lemma in the following commutative diagram of vector spaces for all $d \geq 1$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{C}^{d-1} & \longrightarrow & \mathcal{C}^d & \longrightarrow & \mathcal{C}^d/\mathcal{C}^{d-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta\circ\varphi|_{\mathcal{C}^{d-1}}} & & \downarrow{\scriptstyle \delta\circ\varphi|_{\mathcal{C}^d}} & & \downarrow{\scriptstyle \mathrm{gr}(\delta\circ\varphi|_{C(\sigma_1)})_d} & & \\
0 & \longrightarrow & \mathcal{D}^{d-1} & \longrightarrow & \mathcal{D}^d & \longrightarrow & \mathcal{D}^d/\mathcal{D}^{d-1} & \longrightarrow & 0
\end{array}$$

where the rows are exact by definition and $\mathrm{gr}(\delta \circ \varphi|_{C(\sigma_1)})_d$ and $\delta \circ \varphi\ |_{\mathcal{C}^{d-1}}$ are injective by the assumption and the induction hypothesis. Therefore $\delta \circ \varphi$ is injective on $\cup_d \mathcal{C}^d = C(\sigma_1)$.

Notice that $B_{2,n}$ and D_n are not only filtered by the $\varphi(x_{1,1})$ and t_1 degrees but they are also graded as $B_{2,n} \cong A_{n-1}[t]$ and $D_n \cong D_{n-1}[t]$ by $t_1 \mapsto t$ and $t_i \mapsto t_{i-1} \in D_{n-1}$ $(i \geq 2)$. Hence we will use the natural identifications of graded algebras $B_{2,n} \cong \mathrm{gr}(B_{2,n})$ and $\mathrm{gr}(D_n) \cong D_n$ (and so $\mathrm{gr}(\delta)$ is just δ).

Step 2: We prove that the image of the map $\mathrm{gr}(\varphi)$ restricted to $\mathrm{gr}(C(\sigma_1))$ is in $C(\varphi(\sigma_1)) \subseteq B_{2,n}$. (For a diagram about the maps involved in the proof of the first part, see Diagram 4.13.) Here, $C(\varphi(\sigma_1))$ is a graded subalgebra of $B_{2,n}$ since $\varphi(\sigma_1)$ is a sum of a central element $\varphi(x_{1,1})$ and of the elements $\varphi(x_{2,2}), \ldots, \varphi(x_{n,n})$ (that are homogeneous of degree zero) so $C(\varphi(\sigma_1)) = C(\varphi(x_{2,2} + \cdots + x_{n,n}))$ is homogeneous. The proof of this step is clear: For an $h \in \mathcal{C}^d \subseteq \mathcal{A}^d$ we have $0 = \varphi([\sigma_1, h]) = [\varphi(\sigma_1), \varphi(h)]$, hence $\mathrm{gr}(\varphi)(h + \mathcal{C}^{d-1}) \in C(\varphi(\sigma_1))$.

Step 3: We prove the injectivity of δ restricted to $C(\varphi(\sigma_1))$ by the induction. First note that $C(\varphi(\sigma_1)) \cong C_{A_{n-1}}(\sigma_1)[t]$ using the isomorphism $B_{2,n} \cong A_{n-1}[t]$. Then, by the induction hypothesis,

$$C_{A_{n-1}}(\sigma_1) = \mathbb{C}\langle \sigma_1(A_{n-1}), \dots, \sigma_{n-1}(A_{n-1})\rangle$$

Therefore $C(\varphi(\sigma_1)) = \mathbb{C}\langle \sigma_1(B_{2,n}), \dots, \sigma_{n-1}(B_{2,n}), \varphi(x_{1,1})\rangle$ where $\sigma_i(B_{2,n})$ is defined as the image of $\sigma_i(A_{n-1})$ under the above mentioned isomorphism. For these elements, we have $\delta(\sigma_i(B_{2,n})) = s_i(t_2, \dots, t_n)$ where $s_i(t_2, \dots, t_n)$ is the i-th elementary symmetric polynomial in the variables $t_2, \dots, t_n$. Hence δ restricted to $C(\varphi(\sigma_1))$ is indeed injective by the fundamental theorem of symmetric polynomials.

Now it is enough to prove the injectivity of $\mathrm{gr}(\varphi)$ restricted to $\mathrm{gr}(C(\sigma_1))$ to get the injectivity of $\delta \circ \varphi$ by Step 1 and 2.

Step 4: We find a subalgebra of $\mathrm{gr}(A_n)$ containing $\mathrm{gr}(C(\sigma_1))$ that has simple explicit description via the monomial basis. For $\mathrm{ad}\sigma_1 : A_n \to A_n$, $h \mapsto [\sigma_1, h]$, we have $C(\sigma_1) = \mathrm{Ker}(\mathrm{ad}\sigma_1)$ by definition. Although $\mathrm{ad}\sigma_1$ is not a morphism of algebras but a derivation of degree one, we can still take

$$\mathrm{Ker}(\mathrm{gr}(\mathrm{ad}\sigma_1)) := \{(h_d)_{d\in\mathbb{N}} \in \mathrm{gr}(A_n) \mid \sigma_1 h_d - h_d\sigma_1 + \mathcal{A}^d = 0 \in \mathcal{A}^{d+1}/\mathcal{A}^d\}$$

where $\mathrm{gr}(\mathrm{ad}\sigma_1)$ is understood as a map of graded vector spaces. Then, we can extend Diagram 4.12 as:

$$\begin{array}{ccccc} \mathrm{gr}(A_n) & \xrightarrow{\mathrm{gr}(\varphi)} & B_{2,n} & \xrightarrow{\delta} & D_n \\ \cup & & \cup & & \\ \mathrm{Ker}(\mathrm{gr}(\mathrm{ad}\sigma_1)) & & C(\varphi(\sigma_1)) & & \\ \cup & \nearrow & & & \\ \mathrm{gr}(C(\sigma_1)) & & & & \end{array} \tag{4.13}$$

Naturally, $\mathrm{gr}(C(\sigma_1)) \subseteq \mathrm{Ker}(\mathrm{gr}(\mathrm{ad}\sigma_1))$ since $\sigma_1 h_d - h_d\sigma_1 = 0 \in A_n$ implies $\sigma_1 h_d - h_d \sigma_1 \in \mathcal{A}^d$.

We give an explicit description of $\mathrm{Ker}(\mathrm{gr}(\mathrm{ad}\sigma_1))$. Observe that

$$\mathrm{gr}(A_n) \cong \bigoplus_{d\in\mathbb{N}} y^d \mathbb{C}\langle x_{i,j} \mid (i,j) \neq (1,1)\rangle$$

where y, the image of $x_{1,1}$, commutes with every $x_{i,j}$ for $2 \leq i,j \leq n$ and q-commutes with $x_{1,j}$ and $x_{i,1}$ for all $i,j \geq 2$. Indeed, by the monomial basis of A_n (see Sec. 4.2) we get the direct sum decomposition, moreover, the only defining relations involving $x_{1,1}$ are $x_{1,1}x_{1,j} = qx_{1,j}x_{1,1}$,

$x_{1,1}x_{i,1} = qx_{i,1}x_{1,1}$ and $x_{1,1}x_{i,j} = x_{i,j}x_{1,1} + (q - q^{-1})x_{i,1}x_{1,j}$ that reduce to q-commutativity of y and commutativity of y with the appropriate elements. The argument also gives that the image of the monomial basis of A_n is a monomial basis in $\mathrm{gr}(A_n)$.

In particular, we get that

$$\mathrm{Ker}\big(\mathrm{gr}(\mathrm{ad}\sigma_1)\big) \cong \bigoplus_{d \in \mathbb{N}} y^d \mathbb{C}\langle x_{i,j} \mid 2 \leq i, j \leq n \rangle$$

by the same isomorphism. Indeed, for an element $x_{1,1}^d m \in \mathcal{A}^d$ where m is an ordered monomial in the variables $x_{i,j}$ ($(i,j) \neq (1,1)$), we have

$$\mathrm{gr}(\mathrm{ad}\sigma_1)(x_{1,1}^d m + \mathcal{A}^{d-1}) = x_{1,1} \cdot x_{1,1}^d m - x_{1,1}^d m \cdot x_{1,1} + \mathcal{A}^d$$

since $x_{i,i} \cdot \mathcal{A}^d \subseteq \mathcal{A}^d$ and $\mathcal{A}^d \cdot x_{i,i} \subseteq \mathcal{A}^d$ for all $i \geq 2$. Then, by the above mentioned q-commutativity relations, we obtain $(1 - q^{-c(m)})x_{1,1}^{d+1} m + \mathcal{A}^d$ where $c(m)$ stands for the sum of exponents of the $x_{1,j}$'s and $x_{i,1}$'s ($2 \leq i, j \leq n$) appearing in m. The result is a monomial basis element in $\mathcal{A}^{d+1}/\mathcal{A}^d \subseteq \mathrm{gr}(A_n)$. For different monomials $x_{1,1}^d m$ and $x_{1,1}^{d'} m'$ we get different monomials $x_{1,1}^{d+1} m$ and $x_{1,1}^{d'+1} m'$ so $\mathrm{gr}(\mathrm{ad}\sigma_1)$ is diagonal in the monomial basis of $\mathrm{gr}(A_n)$ with the scalars $(1 - q^{-c(m)})$. Hence its kernel is

$$\{x_{11}^d m + \mathcal{A}^{d-1} \mid d \in \mathbb{N},\ c(m) = 0\}$$

since q is not a root of unity, as we stated. Therefore we get $\mathrm{Ker}\big(\mathrm{gr}(\mathrm{ad}\sigma_1)\big) \cong A_{n-1}[t]$ using $y \mapsto t$ and $x_{i,j} \mapsto x_{i-1,j-1}$ since y commutes with every $x_{i,j}$ for $2 \leq i, j \leq n$.

Now the injectivity part of the theorem follows: the isomorphisms $B_{2,n} \cong A_{n-1}[t]$ and $\mathrm{Ker}\big(\mathrm{gr}(\mathrm{ad}\sigma_1)\big) \cong A_{n-1}[t]$ established in step 4 are compatible, meaning that $\mathrm{gr}(\varphi)$ composed with them on the appropriate sides is $\mathrm{id}_{A_{n-1}[t]}$. In particular, $\mathrm{gr}(\varphi)$ restricted to $\mathrm{gr}\big(C(\sigma_1)\big) \subseteq \mathrm{Ker}\big(\mathrm{gr}(\mathrm{ad}\sigma_1)\big)$ is injective. By step 3, δ restricted to $C\big(\varphi(\sigma_1)\big)$ is also injective, so the composition $\delta \circ \mathrm{gr}(\varphi) = \mathrm{gr}(\delta \circ \varphi)$ is injective as well, using step 2. By step 1, this means that $\delta \circ \varphi$ is injective.

Second part: To prove $\eta\big(C(\sigma_1)\big) \subseteq D_n^{\mathfrak{S}_n}$, consider the following commutative diagram:

$$\begin{array}{ccccc}
A_n & \xrightarrow{\ \varphi\ } & B_{2,n} & \xrightarrow{\ \delta\ } & D_n \\
\cup & & \cup & & \cup \\
C(\sigma_1) & \longrightarrow & C\big(\varphi(\sigma_1)\big) & \longrightarrow & D_n^{\mathfrak{S}_{n-1}}
\end{array}$$

where $\mathfrak{S}_{n-1}$ acts on D_n by permuting $t_2, \ldots, t_n$. The diagram implicitly states that $\varphi\big(C(\sigma_1)\big) \subseteq C\big(\varphi(\sigma_1)\big)$ (which is clear) and that $\delta\big(C\big(\varphi(\sigma_1)\big)\big) \subseteq D_n^{\mathfrak{S}_{n-1}}$. The latter follows by the induction hypothesis for $n-1$: it gives that $C(\varphi(\sigma_1)) = \mathbb{C}\langle\sigma_1(B_{2,n}),\ldots,\sigma_{n-1}(B_{2,n}),\varphi(x_{1,1})\rangle$ by $B_{2,n} \cong A_{n-1}[t]$ and since $\delta(\varphi(x_{1,1})) = t_1$ and $\delta(\sigma_i(B_{2,n})) = s_i(t_2,\ldots,t_n)$, the i-th elementary symmetric polynomial in the variables $t_2,\ldots,t_n$, we obtain that $(\delta\circ\varphi)(C(\sigma_1))$ is symmetric in $t_2,\ldots,t_n$.

To prove symmetry in $t_1,\ldots,t_{n-1}$ too, consider the isomorphism $\gamma : \mathcal{O}_q(M_n) \cong \mathcal{O}_{q^{-1}}(M_n)$ given by $x_{i,j} \leftrightarrow x'_{n+1-i,n+1-j}$ where $x'_{i,j}$ denotes the variables in $\mathcal{O}_{q^{-1}}(M_n)$. This is indeed an isomorphism: interpreted in the free algebra it maps the defining relations of $\mathcal{O}_q(M_n)$ to the defining relations of $\mathcal{O}_{q^{-1}}(M_n)$. It also maps $\sigma_1 \in \mathcal{O}_q(M_n)$ into the σ_1 of $\mathcal{O}_{q^{-1}}(M_n)$ denoted by σ'_1. Moreover, $\overline{\gamma}\circ\eta = \eta'\circ\gamma$ where $\overline{\gamma} : D_n \to D_n$, $t_i \mapsto t_{n+1-i}$ $(i = 1,\ldots,n)$ and $\eta' : \mathcal{O}_{q^{-1}}(M_n) \to \mathbb{C}[t_1,\ldots,t_n]$, $x'_{i,j} \mapsto t_i\delta_{i,j}$ is the $\eta(= \delta\circ\varphi)$ of $\mathcal{O}_{q^{-1}}(M_n)$.

Hence $(\overline{\gamma}\circ\eta)(C(\sigma_1)) = \eta'(C(\sigma'_1))$ as $C(\sigma'_1)$ is symmetric under $\overline{\gamma}$. Applying the previous argument on $\mathcal{O}_{q^{-1}}(M_n)$ gives that $\eta'(C(\sigma'_1)) \subseteq D_n^{\mathfrak{S}_{n-1}}$ where $\mathfrak{S}_{n-1}$ still acts by permuting $t_2,\ldots,t_n$. Hence $\eta(C(\sigma_1))$ is symmetric in $t_1,\ldots,t_{n-1}$ too. So $\eta(C(\sigma_1))$ is symmetric in all the variables $t_1,\ldots,t_n$ by $n \geq 3$, as we claimed. □

Remark 4.5.1. In fact the proof of the injectivity of $\eta = \delta\circ\varphi$ is valid in the case $n = 2$ too, but the symmetry argument used to prove $\eta(C(\sigma_1)) \subseteq \mathbb{C}[t_1,\ldots,t_n]^{\mathfrak{S}_n}$ does not give anything if $n = 2$. That is why we had to start the induction at $n = 2$ instead of $n = 1$.

Remark 4.5.2. As it is discussed in [DL1], the set of cocommutative elements in $\mathcal{O}_q(GL_n)$ is the ring of invariants under the right coaction

$$\alpha : \mathcal{O}_q(GL_n) \to \mathcal{O}_q(GL_n)\otimes\mathcal{O}_q(GL_n)$$
$$a \mapsto \sum a_{(2)}\otimes a_{(3)}S(a_{(1)})$$

where we use Sweedler's notation. Although this coaction does not agree with the right adjoint coaction

$$a \mapsto \sum a_{(2)}\otimes S(a_{(1)})a_{(3)}$$

of the Hopf algebra $\mathcal{O}_q(GL_n)$ (that is also mentioned in the referred article) but they differ only by the automorphism S^2. Hence, by Theorem 4.1.1, the invariants of the right adjoint coaction also form a maximal commutative subalgebra.

We get other maximal commutative subalgebras by applying automorphisms of the algebras $\mathcal{O}_q(GL_n)$, $\mathcal{O}_q(M_n)$ or $\mathcal{O}_q(SL_n)$, though they do not have many automorphisms: it is proved in [Ya1] establishing a conjecture stated in [LL] that the automorphism group of $\mathcal{O}_q(M_n)$ is generated by the transpose operation on the variables and a torus that acts by rescaling the variables $x_{i,j} \mapsto c_i d_j x_{i,j}$ $(c_i, d_j \in \mathbb{C}^\times)$.

5

SEMICLASSICAL LIMIT POISSON ALGEBRAS

5.1 MAIN RESULTS OF THE CHAPTER

In this chapter, we prove the Poisson algebraic version of Theorem 4.1.1 and 4.1.2 for the corresponding semiclassical limit Poisson-algebra structures on $\mathcal{O}(M_n)$, $\mathcal{O}(GL_n)$ and $\mathcal{O}(SL_n)$.

Let $n \in \mathbb{N}^+$ and consider $\mathcal{O}(M_n)$, the algebra of polynomial functions on the affine space of $n \times n$ matrices M_n over the base field $\mathbb{C}$. In Subsection 5.2.5 and 5.2.6 we define a Poisson-algebra structure on $\mathcal{O}(M_n)$, and elements $c_k \in \mathcal{O}(M_n)$ $(1 \leq k \leq n)$ that are (scalar multiples of) the coefficients of the characteristic polynomial. Our goal is to prove the following result, based on [Me2].

Theorem 5.1.1. *For $n \in \mathbb{N}^+$ the subalgebra $\mathbb{C}[c_1, \ldots, c_n]$ (resp. $\mathbb{C}[c_1, \ldots, c_n, c_n^{-1}]$ and $\mathbb{C}[\overline{c}_1, \ldots, \overline{c}_{n-1}]$) is maximal Poisson-commutative in $\mathcal{O}(M_n)$ (resp. $\mathcal{O}(GL_n)$ and $\mathcal{O}(SL_n)$) with respect to the semiclassical limit Poisson structure.*

It is easy to deduce from [DL1] or [DL2] that $\{c_i, c_j\} = 0$ $(1 \leq i, j \leq n)$ in $\mathcal{O}(M_n)$ (see Proposition 5.5.1 below). Therefore Theorem 5.1.1 is a direct consequence of the following statement:

Theorem 5.1.2. *For $n \geq 1$ the Poisson-centralizer of c_1 in $\mathcal{O}(M_n)$ (resp. c_1 in $\mathcal{O}(GL_n)$, $\overline{c}_1$ in $\mathcal{O}(SL_n)$) equipped with the semiclassical limit Poisson bracket is generated as a subalgebra by*

- *$c_1, \ldots, c_n$ in the case of $\mathcal{O}(M_n)$,*
- *$c_1, \ldots, c_n, c_n^{-1}$ in the case of $\mathcal{O}(GL_n)$ and*
- *$\overline{c}_1, \ldots \overline{c}_{n-1}$ in the case of $\mathcal{O}(SL_n)$.*

The applied method is similar to that of Chapter 4, namely we modify the Poisson bracket by taking the associated graded filtered Poisson-algebras, and use induction.

It is well-known that the coefficient functions $c_1, \ldots, c_n \in \mathcal{O}(M_n)$ of the characteristic polynomial generate the subalgebra $\mathcal{O}(M_n)^{GL_n}$ of GL_n-invariants with respect to the adjoint action. This implies that the subalgebra coincides with the Poisson center of the coordinate ring $\mathcal{O}(M_n)$ endowed with the Kirillov-Kostant-Souriau (KKS) Poisson bracket. Hence Theorem 5.1.1 for $\mathcal{O}(M_n)$ can be interpreted as an interesting interplay between the KKS and the semiclassical limit Poisson structure. Namely, while the subalgebra $\mathcal{O}(M_n)^{GL_n}$ is contained in every maximal Poisson-commutative subalgebra with respect to the former Poisson bracket, it is contained in only one maximal Poisson-commutative subalgebra (itself) with respect to the latter Poisson bracket.

A Poisson-commutative subalgebra is also called an involutive (or Hamiltonian) system, while a maximal one is called a complete involutive system (see Sec. 5.2 or [Va]). Such a system is integrable if the (Krull) dimension of the subalgebra generated by the system is sufficiently large. In our case, the subalgebra generated by the elements $c_1, \ldots, c_{n-1}$ is not integrable, as its dimension is $n-1$ (resp. n for GL_n) instead of the required $\binom{n+1}{2} - 1$ (resp. $\binom{n+1}{2}$ for GL_n), see Remark 5.5.3.

The chapter is organized similarly to the previous chapter: first we prove that the three statements in Theorem 5.1.2 are equivalent (Proposition 5.3.1), then we prove Theorem 5.1.2 for $n = 2$ (Proposition 5.4.1) as a starting case of an induction presented in Sec. 5.5 that completes the proof of the theorem. In the chapter, every algebra is understood over the field $\mathbb{C}$.

5.2 DEFINITIONS

5.2.1 *Poisson algebras*

First we collect the basic notions about Poisson algebras we need here. For further details see [Va].

A (commutative) *Poisson algebra* $(A, \{.,.\})$ is a unital commutative associative algebra A together with a bilinear operation $\{.,.\} : A \times A \to A$ called the Poisson bracket such that it is antisymmetric, satisfies the Jacobi identity, and for any $a \in A$, $\{a, .\} : A \to A$ is a derivation. For Poisson algebras A and B, the map $\varphi : A \to B$ is a morphism of Poisson algebras if it is both an algebra homomorphism and a Lie-homomorphism.

There is a natural notion of *Poisson subalgebra* (i.e. a subalgebra that is also a Lie-subalgebra), *Poisson ideal* (i.e an ideal that is also a Lie-ideal) and quotient Poisson algebra (as the quotient Lie-algebra inherits the bracket). The *Poisson centralizer* $C(a)$ of an element $a \in A$ is defined as $\{b \in A \mid \{a, b\} = 0\}$. Clearly, it is a Poisson subalgebra. The *Poisson* *center* (or Casimir subalgebra) of A is $Z(A) := \{a \in A \mid C(a) = A\}$. Analogously, $a \in A$ is called *Poisson-central* if $C(a) = A$. One says that a subalgebra $C \leq A$ is *Poisson-commutative* (or involutive) if $\{c, d\} = 0$ for all $c, d \in C$ and it is *maximal Poisson-commutative* (or maximal involutive) if there is no Poisson-commutative subalgebra in A that strictly contains C.

Let A be a reduced, finitely generated commutative Poisson algebra. The rank $\mathrm{Rk}\{.,.\}$ of the Poisson structure $\{.,.\}$ is defined by the rank of the matrix $\left(\{g_i, g_j\}\right)_{i,j} \in A^{N\times N}$ for a generating system $g_1, \ldots, g_N \in A$. (One can prove that it is independent of the chosen generating system.) In the terminology of [Va], a maximal Poisson-commutative subalgebra C is called *Liouville integrable* if

$$\dim C = \dim A - \frac{1}{2}\mathrm{Rk}\{.,.\}$$

The inequality $\leq$ holds for any Poisson-commutative subalgebra (Proposition II.3.4 in [Va]), hence integrability is a maximality condition on the size of C that does not necessarily hold for every complete involutive system.

5.2.2 *Filtered Poisson algebras*

Definition 5.2.1. A *filtered Poisson algebra* is a Poisson algebra together with an ascending chain of subspaces $\{\mathcal{F}^d\}_{d\in\mathbb{N}}$ in A such that

- $A = \cup_{d\in\mathbb{N}}\mathcal{F}^d$,
- $\mathcal{F}^d \cdot \mathcal{F}^e \subseteq \mathcal{F}^{d+e}$ for all $d, e \in \mathbb{N}$, and
- $\{\mathcal{F}^d, \mathcal{F}^e\} \subseteq \mathcal{F}^{d+e}$ for all $d, e \in \mathbb{N}$.

Together with the filtration preserving morphisms of Poisson algebras, they form a category.

For a filtered Poisson algebra A, we may define its *associated graded Poisson algebra* $\mathrm{gr}A$ as

$$\mathrm{gr}(A) := \bigoplus_{d\in\mathbb{N}} \mathcal{F}^d / \mathcal{F}^{d-1}$$

where we used the simplifying notation $\mathcal{F}^{-1} = \{0\}$. The multiplication of $\mathrm{gr}(A)$ is defined the usual way:

$$\mathcal{F}^d/\mathcal{F}^{d-1} \times \mathcal{F}^e/\mathcal{F}^{e-1} \to \mathcal{F}^{d+e}/\mathcal{F}^{d+e-1}$$

$$\left(x + \mathcal{F}^{d-1}, y + \mathcal{F}^{e-1}\right) \mapsto xy + \mathcal{F}^{d+e-1}$$

Analogously, the Poisson structure of $\mathrm{gr}(A)$ is defined by $\big(x+\mathcal{F}^{d-1}, y+\mathcal{F}^{e-1}\big) \mapsto \{x,y\}+\mathcal{F}^{d+e-1}$. One can check that this way $\mathrm{gr}(A)$ is a Poisson algebra.

Let $(S,+)$ be an abelian monoid. (We will only use this definition for $S=\mathbb{N}$ and $S=\mathbb{Z}/n\mathbb{Z}$ for some $n\in\mathbb{N}$.) An *S-graded Poisson algebra R* is a Poisson algebra together with a fixed grading

$$R=\oplus_{d\in S}R_d$$

such that R is both a graded algebra (i.e. $R_d\cdot R_e\subseteq R_{d+e}$ for all $d,e\in S$) and a graded Lie algebra (i.e. $\{R_d,R_e\}\subseteq R_{d+e}$ for all $d,e\in S$) with respect to the given grading.

The above construction $A\mapsto \mathrm{gr}(A)$ yields an $\mathbb{N}$-graded Poisson algebra. In fact $\mathrm{gr}(.)$ can be turned into a functor: for a morphism of filtered Poisson algebras $f:\big(A,\{\mathcal{F}^d\}_{d\in\mathbb{N}}\big)\to\big(B,\{\mathcal{G}^d\}_{d\in\mathbb{N}}\big)$ we define

$$\mathrm{gr}(f):\mathrm{gr}(A)\to\mathrm{gr}(B) \qquad \big(x_d+\mathcal{F}^{d-1}\big)_{d\in\mathbb{N}}\mapsto\big(f(x_d)+\mathcal{G}^{d-1}\big)_{d\in\mathbb{N}}$$

One can check that it is indeed well-defined and preserves composition.

Remark 5.2.2. Given an $\mathbb{N}$-graded Poisson algebra $R=\oplus_{d\in\mathbb{N}}R_d$, one has a natural way to associate a filtered Poisson algebra to it. Namely, let $\mathcal{F}^d:=\oplus_{k\le d}R_k$. In this case, the associated graded Poisson algebra $\mathrm{gr}R$ of $\big(R,\{\mathcal{F}^d\}_{d\in\mathbb{N}}\big)$ is isomorphic to R.

5.2.3 *The Kirillov-Kostant-Souriau bracket*

A classical example of a Poisson algebra is given by the Kirillov-Kostant-Souriau (KKS) bracket on $\mathcal{O}(\mathfrak{g}^*)$, the coordinate ring of the dual of a finite-dimensional (real or complex) Lie algebra $\big(\mathfrak{g},[.,.]\big)$ (see [ChP] Example 1.1.3, or [We] Section 3).

It is defined as follows: a function $f\in\mathcal{O}(\mathfrak{g}^*)$ at a point $v\in\mathfrak{g}^*$ has a differential $\mathrm{d}f_v\in T_v^*\mathfrak{g}^*$ where we can canonically identify the spaces $T_v^*\mathfrak{g}^*\cong T_0^*\mathfrak{g}^*\cong\mathfrak{g}^{**}\cong\mathfrak{g}$. Hence we may define the Poisson bracket on $\mathcal{O}(\mathfrak{g}^*)$ as

$$\{f,g\}(v):=[\mathrm{d}f_v,\mathrm{d}g_v](v)$$

for all $f,g\in\mathcal{O}(\mathfrak{g}^*)$ and $v\in\mathfrak{g}^*$. It is clear that it is a Lie-bracket but it can be checked that the Leibniz-identity is also satisfied. For $\mathfrak{g}=\mathfrak{gl}_n$, it gives a Poisson bracket on $\mathcal{O}(M_n)$.

Alternatively, one can define this Poisson structure via semiclassical limits.

5.2.4 *Semiclassical limits*

Let $A = \cup_{d\in\mathbb{Z}}\mathcal{A}^d$ be a $\mathbb{Z}$-filtered algebra such that its associated graded algebra $\mathrm{gr}(A) := \oplus_{d\in\mathbb{Z}}\mathcal{A}^d/\mathcal{A}^{d-1}$ is commutative. The Rees ring of A is defined as

$$\mathrm{Rees}(A) := \bigoplus_{d\in\mathbb{Z}} \mathcal{A}^d h^d \subseteq A[h, h^{-1}]$$

Using the obvious multiplication, it is a $\mathbb{Z}$-graded algebra. The semiclassical limit of A is the Poisson algebra $\mathrm{Rees}(A)/h\mathrm{Rees}(A)$ together with the bracket

$$\{a + h\mathcal{A}^m, b + h\mathcal{A}^n\} := \frac{1}{h}[a,b] + \mathcal{A}^{n+m-2} \in \mathcal{A}^{n+m-1}/\mathcal{A}^{n+m-2}$$

for all $a + h\mathcal{A}^m \in \mathcal{A}^{m+1}/h\mathcal{A}^m$, $b + h\mathcal{A}^n \in \mathcal{A}^{n+1}/h\mathcal{A}^n$. The definition is valid as the underlying algebra of $\mathrm{Rees}(A)/h\mathrm{Rees}(A)$ is $\mathrm{gr}(A)$ that is assumed to be commutative, hence $[a,b] \in h\mathcal{A}^{m+n-1}$.

The Poisson algebra $\mathcal{O}(\mathfrak{g}^*)$ with the KKS bracket can be obtained as the semiclassical limit of the universal enveloping algebra $U\mathfrak{g}$, see [Go], Example 2.6.

5.2.5 *Semiclassical limits of quantized coordinate rings*

The semiclassical limits of $\mathcal{O}_q(SL_n)$ can be obtained via slight modification of process of Subsec. 5.2.4 (see [Go], Example 2.2).

Consider the $\Bbbk[t]$-algebra $R := \mathcal{O}_t(M_n)$ that is defined by the same formulas as $\mathcal{O}_q(M_n)$ (see Subsec. 4.2.1), but using the indeterminate t instead of $q \in \Bbbk^\times$. Then $\mathcal{O}_t(M_n)$ can be endowed with a $\mathbb{Z}$-filtration by defining $\mathcal{F}^n$ to be the span of monomials that are the product of at most n variables. However, instead of defining a Poisson structure on $\mathrm{Rees}(R)/h\mathrm{Rees}(R)$ with respect to this filtration, consider the algebra $R/(t-1)R$ that is isomorphic to $\mathcal{O}(M_n)$ as an algebra.

The *semiclassical limit Poisson bracket* is defined as

$$\{\bar{a}, \bar{b}\} := \frac{1}{t-1}(ab - ba) + (t-1)R \in R/(t-1)R$$

for any two representing elements $a, b \in R$ for $\bar{a}, \bar{b} \in R/(t-1)R$. One can check that it is a well-defined Poisson bracket.

Explicitly, the *Poisson structure* of $\mathcal{O}(M_n)$ defined above is

$$\{x_{i,j}, x_{k,l}\} = \begin{cases} 2x_{i,l}x_{k,j} & \text{if } i < k \text{ and } j < l \\ x_{i,j}x_{k,l} & \text{if } (i = k \text{ and } j < l) \text{ or } (j = l \text{ and } i < k) \\ 0 & \text{otherwise} \end{cases}$$

extended according to the Leibniz-rule (see [Go]).

It is a quadratic Poisson structure in the sense of [Va], Definition II.2.6. The semiclassical limit for GL_n and SL_n is defined analogously using $\mathcal{O}_q(GL_n)$ and $\mathcal{O}_q(SL_n)$ or by localization (resp. by taking quotient) at the Poisson central element det (resp. det -1) in $\mathcal{O}(M_n)$.

5.2.6 *Coefficients of the characteristic polynomial*

Consider the characteristic polynomial function $M_n \to \mathbb{C}[x]$, $A \mapsto \det(xI - A)$. Let us define the elements $c_0, c_1, \ldots, c_n \in \mathcal{O}(M_n)$ as

$$\det(xI - A) = \sum_{i=0}^{n} (-1)^i c_i x^{n-i}$$

In particular, $c_0 = 1$, $c_1 = \mathrm{tr}$ and $c_n = \det$. Their images in $\mathcal{O}(SL_n) \cong \mathcal{O}(M_n)/(\det - 1)$ are denoted by $\bar{c}_1, \ldots, \bar{c}_{n-1}$. If ambiguity may arise, we will write $c_i(A)$ for the element corresponding to c_i for an algebra A with a fixed isomorphism $A \cong \mathcal{O}(M_k)$ for some k.

The coefficient functions $c_1, \ldots, c_n$ can also be expressed via matrix minors as follows: For $I, J \subseteq \{1, \ldots, n\}$, $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$ define

$$[I \mid J] := \sum_{s \in \mathfrak{S}_k} \mathrm{sgn}(s) x_{i_1, j_{s(1)}} \ldots x_{i_k, j_{s(k)}}$$

i.e. it is the determinant of the subalgebra generated by $\{x_{i,j}\}_{i \in I, j \in J}$ that can be identified with $\mathcal{O}(M_k)$. Then

$$c_i = \sum_{|I|=i} [I \mid I] \in \mathcal{O}(M_n)$$

for all $1 \leq i \leq n$. It is well-known that $c_1, \ldots, c_n$ generate the same subalgebra of $\mathcal{O}(M_n)$ as the trace functions $A \mapsto \mathrm{Trace}(A^k)$, it is the subalgebra $\mathcal{O}(M_n)^{GL_n}$ of GL_n-invariants with respect to the adjoint action.

5.3 EQUIVALENCE OF THE STATEMENTS

Consider $\mathcal{O}(M_n)$ endowed with the semiclassical limits Poisson bracket. As it is discussed in the Introduction, Theorem 5.1.1 follows directly from Theorem 5.1.2 and Prop. 5.5.1.

The following proposition shows that it is enough to prove Theorem 5.1.2 for the case of $\mathcal{O}(M_n)$.

Proposition 5.3.1. *For any $n \in \mathbb{N}^+$ the following are equivalent:*

i) The Poisson-centralizer of $c_1 \in \mathcal{O}(M_n)$ is generated by $c_1, \ldots, c_n$.

ii) The Poisson-centralizer of $c_1 \in \mathcal{O}(GL_n)$ *is generated by* $c_1, \ldots, c_n, c_n^{-1}$.

iii) The Poisson-centralizer of $\bar{c}_1 \in \mathcal{O}(SL_n)$ *is generated by* $\bar{c}_1, \ldots, \bar{c}_{n-1}$.

Proof. The first and second statements are equivalent as det is a Poisson-central element, so we have $\{c_1, h \cdot \det^k\} = \{c_1, h\} \cdot \det^k$ for any $h \in \mathcal{O}(GL_n)$ and $k \in \mathbb{Z}$. Hence

$$\mathcal{O}(GL_n) \supseteq C(c_1) = \big(\mathcal{O}(M_n) \cap C(c_1)\big)[\det^{-1}]$$

proving $i) \iff ii)$.

$i) \iff iii)$: First, assume $i)$ and let $\bar{h} \in \mathcal{O}(SL_n)$ such that $\{\bar{c}_1, \bar{h}\} = 0$. Since $\mathcal{O}(SL_n)$ is $\mathbb{Z}/n\mathbb{Z}$-graded (inherited from the $\mathbb{N}$-grading of $\mathcal{O}(M_n)$) and $\bar{c}_1$ is homogeneous with respect to this grading, its Poisson-centralizer is generated by $\mathbb{Z}/n\mathbb{Z}$-homogeneous elements, so we may assume that $\bar{h}$ is $\mathbb{Z}/n\mathbb{Z}$-homogeneous.

Let $k = \deg(\bar{h}) \in \mathbb{Z}/n\mathbb{Z}$. Let $h \in \mathcal{O}(M_n)$ be a lift of $\bar{h} \in \mathcal{O}(SL_n)$ and consider the $\mathbb{N}$-homogeneous decomposition $h = \sum_{j=0}^{d} h_{jn+k}$ of h, where h_{jn+k} is homogeneous of degree $jn + k$ for all $j \in \mathbb{N}$. Define

$$h' := \sum_{j=0}^{d} h_{jn+k} \det^{d-j} \in \mathcal{O}(M_n)_{dn+k}$$

that is a homogeneous element of degree $dn + k$ representing $\bar{h} \in \mathcal{O}(SL_n)$ in $\mathcal{O}(M_n)$. Then $\{c_1, h'\} \in (\det -1) \cap \mathcal{O}(M_n)_{dn+k+1}$ since $\{\bar{c}_1, \overline{h'}\} = \{\bar{c}_1, \bar{h}\} = 0$, c_1 is homogeneous of degree 1 and the Poisson-structure is graded. Clearly, $(\det -1) \cap \mathcal{O}(M_n)_{dn+k+1} = 0$ hence $\{c_1, h'\} = 0$. Applying $i)$ gives $h' \in \mathbb{C}[c_1, \ldots, c_n]$ so $\bar{h} \in \mathbb{C}[\bar{c}_1, \ldots, \bar{c}_{n-1}]$ as we claimed.

Conversely, assume $iii)$ and let $h \in \mathcal{O}(M_n)$ such that $\{c_1, h\} = 0$. Since c_1 is $\mathbb{N}$-homogeneous, we may assume that h is also $\mathbb{N}$-homogeneous and so the image $\bar{h} \in \mathcal{O}(SL_n)$ of h is $\mathbb{Z}/n\mathbb{Z}$-homogeneous. By the assumption, $\bar{h} = p(\bar{c}_1, \ldots, \bar{c}_{n-1})$ for some $p \in \mathbb{C}[t_1, \ldots, t_{n-1}]$. Endow $\mathbb{C}[t_1, \ldots, t_n]$ with the $\mathbb{N}$-grading $\deg(t_i) = i$. As $\bar{h}$ is $\mathbb{Z}/n\mathbb{Z}$-homogeneous, we may choose $p \in \mathbb{C}[t_1, \ldots, t_{n-1}]$ so that its homogeneous components are all of degree $dn + \deg(\bar{h}) \in \mathbb{N}$ with respect to the above grading for some $d \in \mathbb{N}$.

By $h - p(c_1, \ldots, c_{n-1}) \in (\det -1)$ and the assumptions on degrees, we may choose a polynomial $q \in \mathbb{C}[t_1, \ldots, t_n]$ that is homogeneous with respect to the above grading and $q(t_1, \ldots, t_{n-1}, 1) = p$. Let $h' := h \cdot \det^r$ where $r := \frac{1}{n}(\deg q - \deg h) \in \mathbb{Z}$ so $\deg(h') = \deg(q) \in \mathbb{N}$. Then

$$h' - q(c_1, \ldots, c_n) \in (\det -1) \cap \mathcal{O}(M_n)_{\deg q} = 0$$

hence $h' \in \mathbb{C}[c_1, \ldots, c_n]$ and $h \in \mathbb{C}[c_1, \ldots, c_n, c_n^{-1}]$. This is enough as $\mathbb{C}[c_1, \ldots, c_n, c_n^{-1}] \cap \mathcal{O}(M_n) = \mathbb{C}[c_1, \ldots, c_n]$ by the definitions. □

5.4 CASE OF $\mathcal{O}(SL_2)$

In this section we prove Theorem 5.1.2 for $\mathcal{O}(SL_2)$ that is the first step of the induction in the proof of the general case.

We denote by a, b, c, d the generators $\overline{x}_{1,1}, \overline{x}_{1,2}, \overline{x}_{2,1}, \overline{x}_{2,2} \in \mathcal{O}(SL_2)$ and $\mathrm{tr} := \overline{c}_1 = a + d$.

Proposition 5.4.1. *The centralizer of* $\mathrm{tr} \in \mathcal{O}(SL_2)$ *is* $\mathbb{C}[\mathrm{tr}]$.

By $ad - bc = 1$ we have a monomial basis of $\mathcal{O}(SL_2)$ consisting of

$$a^i b^k c^l,\ b^k c^l d^j,\ b^k c^l \qquad (i, j \in \mathbb{N}^+,\ k, l \in \mathbb{N})$$

The Poisson bracket on the generators is the following:

$$\{a,b\} = ab \qquad \{a,c\} = ac \qquad \{a,d\} = 2bc$$

$$\{b,c\} = 0 \qquad \{b,d\} = bd \qquad \{c,d\} = cd$$

The action of $\{\mathrm{tr}, .\}$ on the basis elements can be written as

$$\{(a+d), a^i b^k c^l\} =$$

$$\begin{aligned} =&(k+l)a^{i+1}b^k c^l - 2ia^{i-1}b^{k+1}c^{l+1} - (k+l)a^i b^k c^l d \\ =&(k+l)a^{i+1}b^k c^l - (2i+k+l)a^{i-1}b^{k+1}c^{l+1} - (k+l)a^{i-1}b^k c^l \end{aligned}$$

By the same computation on $b^k c^l$ and $b^k c^l d^j$ one obtains

$$\begin{aligned} \{(a+d), b^k c^l\} =&(k+l)ab^k c^l - (k+l)b^k c^l d \\ \{(a+d), b^k c^l d^j\} =&(k+l+2j)b^{k+1}c^{l+1}d^{j-1} \\ &+(k+l)b^k c^l d^{j-1} - (k+l)b^k c^l d^{j+1} \end{aligned}$$

Hence for a polynomial $p \in \mathbb{C}[t_1, t_2]$ and $i \geq 1$:

$$\begin{aligned} \{(a+d), a^i p(b,c)\} &= a^{i+1}\sum_m m \cdot p_m(b,c) \\ &- a^{i-1}\sum_m \big((2i+m)bc + m\big) p_m(b,c) \end{aligned} \tag{5.1}$$

where p_m is the m-th homogeneous component of p. The analogous computations for $p(b,c)d^j$ ($j \geq 1$) and $p(b,c)$ give

$$\begin{aligned} \{(a+d), p(b,c)d^j\} &= -d^{j+1}\sum_m m \cdot p_m(b,c) \\ &+ d^{j-1}\sum_m \big((m+2j)bc + m\big) p_m(b,c) \end{aligned} \tag{5.2}$$

$$\{(a+d), p(b,c)\} = (a-d)\sum_m m \cdot p_m(b,c) \tag{5.3}$$

Proof of Proposition 5.4.1. Assume that $0 \neq g \in C(\mathrm{tr})$ and write it as

$$g = \sum_{i=1}^{\alpha} a^i r_i + \sum_{j=1}^{\beta} s_j d^j + u$$

where r_i, s_j and u are elements of $\mathbb{C}[b, c]$, and α and β are the highest powers of a and d appearing in the decomposition.

We prove that $r_\alpha \in \mathbb{C} \cdot 1$. If $\alpha = 0$ then $r_\alpha = u$ so the $a^i b^k c^l$ terms $(i > 0)$ in $\{a + d, g\}$ are the same as the $a^i b^k c^l$ terms in $\{a + d, u\}$ by Eq. 5.1, 5.2 and 5.3. However, by Eq. 5.3, these terms are nonzero if $u \notin \mathbb{C}$ and that is a contradiction. Assume that $\alpha \geq 1$ and for a fixed $k \in \mathbb{N}$ define the subspace

$$\mathcal{A}^k := \sum_{l \leq k} a^l \mathbb{C}[b, c, d] \subseteq \mathcal{O}(SL_2)$$

By $\{\mathrm{tr}, \mathcal{A}^{\alpha-1}\} \subseteq \mathcal{A}^\alpha$ we have

$$\mathcal{A}^\alpha = \{\mathrm{tr}, g\} + \mathcal{A}^\alpha = \{\mathrm{tr}, a^\alpha r_\alpha + \mathcal{A}^{\alpha-1}\} + \mathcal{A}^\alpha =$$

$$= a^\alpha \{\mathrm{tr}, r_\alpha\} + \alpha a^{\alpha-1} bc r_\alpha + \mathcal{A}^\alpha = a^\alpha \{\mathrm{tr}, r_\alpha\} + \mathcal{A}^\alpha$$

By Eq. 5.3 it is possible only if $\{\mathrm{tr}, r_\alpha\} = 0$ so $r_\alpha \in \mathbb{C}[b, c] \cap C(\mathrm{tr}) = \mathbb{C} \cdot 1$.

If $\alpha > 0$ we may simplify g by subtracting polynomials of tr from it. Indeed, by $r_\alpha \in \mathbb{C}^\times$ we have $g - r_\alpha \mathrm{tr}^\alpha \in \mathcal{A}^{\alpha-1} \cap C(\mathrm{tr})$ so we can replace g by $g - r_\alpha \mathrm{tr}^\alpha$. Hence we may assume that $\alpha = 0$. Then, again, $r_\alpha = u \in \mathbb{C} \cdot 1 \subseteq C(\mathrm{tr})$ so we may also assume that $u = 0$.

If g is nonzero after the simplification, we get a contradiction. Indeed, let $p(b, c) d^\gamma$ be the summand of g with the smallest $\gamma \in \mathbb{N}$. By the above simplifications, $\gamma \geq 1$. Then the coefficient of $d^{\gamma-1}$ in $\{\mathrm{tr}, g\}$ is the same as the coefficient of $d^{\gamma-1}$ in

$$\{\mathrm{tr}, p(b, c) d^\gamma\} = \{\mathrm{tr}, p(b, c)\} d^\gamma + 2\gamma bc p(b, c) d^{\gamma-1}$$

so it is $2\gamma bc p(b, c) d^{\gamma-1}$ that is nonzero if $p(b, c) \neq 0$ and $\gamma \geq 1$. That is a contradiction. □

5.5 PROOF OF THEOREM 5.1.2

Let $n \geq 2$ and let us denote $A_n := \mathcal{O}(M_n)$.

Proposition 5.5.1. *$\mathbb{C}[\sigma_1, \ldots, \sigma_n] \leq A_n$ is a Poisson-commutative subalgebra.*

Proof. Consider the principal quantum minor sums

$$\sigma_i = \sum_{|I|=i} \sum_{s \in \mathfrak{S}_i} t^{-\ell(s)} x_{i_1, i_{s(1)}} \ldots x_{i_t, i_{s(t)}} \in \mathcal{O}_t(M_n)$$

When A_n is viewed as the semiclassical limit $R/(t-1)R$ where $R = \mathcal{O}_t(M_n)$ (see Subsec. 5.2.5), one can see that σ_i represents $c_i \in R/(t-1)R \cong \mathcal{O}(M_n)$. In [DL1], it is proved that $\sigma_i\sigma_j = \sigma_j\sigma_i$ in $\mathcal{O}_q(M_n)$ if q is not a root of unity, in particular, if q is transcendental.

Since the algebra $\mathcal{O}_q(M_n)$ is defined over $\mathbb{Z}[q,q^{-1}]$, the elements $\sigma_1,\dots,\sigma_n$ (that are defined over $\mathbb{Z}[q,q^{-1}]$) commute in $\mathcal{O}_q(M_n(\mathbb{Z})) \leq \mathcal{O}_q(M_n(\mathbb{C}))$ as well. Hence $\sigma_1,\dots,\sigma_n$ also commute after extension of scalars, i.e. in the ring $\mathcal{O}_q(M_n(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathcal{O}_t(M_n(\mathbb{C}))$. Consequently, in $A_n \cong R/(t-1)R$ the subalgebra $\mathbb{C}[c_1,\dots,c_n]$ is a Poisson-commutative subalgebra, by the definition of semiclassical limit. □

By Prop. 5.5.1, $\mathbb{C}[c_1,\dots,c_n]$ is in the Poisson-centralizer $C(c_1)$. To prove the converse, Theorem 5.1.2, we need further notations. Consider the Poisson ideal

$$I := (x_{1,j},\, x_{i,1} \mid 2 \leq i,j \leq n) \lhd A_n$$

We will denote its quotient Poisson algebra by $B_{2,n} := A_n/I$ and the natural surjection by $\varphi : A_n \to B_{2,n}$. Note that $B_{2,n} \cong A_{n-1}[t]$ as Poisson algebras by $x_{i,j} + I \mapsto x_{i-1,j-1}$ ($2 \leq i,j \leq n$) and $x_{1,1} \mapsto t$ where the bracket of $A_{n-1}[t]$ is the trivial extension of the bracket of A_{n-1} by $\{t,a\} = 0$ for all $a \in A_{n-1}[t]$.

Furthermore, D_n will stand for $\mathbb{C}[t_1,\dots,t_n]$ endowed with the zero Poisson bracket. Define the map $\delta : B_{2,n} \to D_n$ as $x_{i,j} + I \mapsto \delta_{i,j}t_i$ that is morphism of Poisson algebras by $\{x_{i,i}, x_{j,j}\} \in I$. Note that $(\delta \circ \varphi)(c_i) = s_i$, the elementary symmetric polynomial in $t_1,\dots,t_n$. In particular, $\delta \circ \varphi$ restricted to $\mathbb{C}[c_1,\dots,c_n]$ is an isomorphism onto the symmetric polynomials in $t_1,\dots,t_n$ by the fundamental theorem of symmetric polynomials. In the proof of Theorem 5.1.2 we verify the same property for $C(\sigma_1)$.

Although the algebras A_n, $B_{2,n}$ and D_n are $\mathbb{N}$-graded Poisson algebras (see Subsec. 5.2.2) using the total degree of A_n and the induced gradings on the quotients, we will instead consider them as filtered Poisson algebras where the filtration is not the one that corresponds to this grading. For each $d \in \mathbb{N}$, let us define

$$\mathcal{A}^d = \{a \in A_n \mid \deg_{x_{1,1}}(a) \leq d\}$$

This is indeed a filtration on A_n. Note that the grading $\deg_{x_{1,1}}$ is incompatible with the bracket by $\{x_{1,1}, x_{2,2}\} = x_{1,2}x_{2,1}$. The algebras $B_{2,n}$, D_n and $C(c_1)$ inherit a filtered Poisson algebra structure as they are Poisson sub- and quotient algebras of A_n so we may take $\mathcal{B}^d := \varphi(\mathcal{A}^d)$, $\mathcal{D}^d := (\delta \circ \varphi)(\mathcal{A}^d)$ and $\mathcal{C}^d = \mathcal{A}^d \cap C(c_1)$. This way the natural surjections φ and δ and the embedding $C(c_1) \hookrightarrow A_n$ are maps of filtered Poisson algebras.

In the proof of Theorem 5.1.2 we use the associated graded Poisson algebras of $B_{2,n}$, D_n and $C(c_1)$ (see Subsec. 5.2.2). First we describe the structure of these. The filtrations on $B_{2,n}$ and D_n are induced by the $x_{1,1}$- and t_1-degrees, hence we have $\mathrm{gr}B_{2,n} \cong B_{2,n}$ and $\mathrm{gr}D_n \cong D_n$ as graded Poisson algebras (and $\mathrm{gr}\delta = \delta$), so we identify them in the following.

The underlying graded algebra of $\mathrm{gr}A_n$ is isomorphic to A_n using the $x_{1,1}$-degree but the Poisson bracket is different: it is the same on the generators $x_{i,j}$ and $x_{k,l}$ for $(i,j) \neq (1,1) \neq (k,l)$ but

$$\begin{aligned} \{x_{1,1}, x_{i,j}\}_{\mathrm{gr}} &= 0 \qquad (2 \le i,j \le n) \\ \{x_{1,1}, x_{1,j}\}_{\mathrm{gr}} &= x_{1,1}x_{1,j} \quad (2 \le j \le n) \\ \{x_{1,1}, x_{i,1}\}_{\mathrm{gr}} &= x_{1,1}x_{i,1} \quad (2 \le i \le n) \end{aligned}$$

where $\{.,.\}_{\mathrm{gr}}$ stands for the Poisson bracket of $\mathrm{gr}A_n$. Consequently, as maps we have $\mathrm{gr}\varphi = \varphi$, we still have $\{c_i, c_j\}_{\mathrm{gr}} = 0$ for all i, j, and the underlying algebra of $\mathrm{gr}C(c_1)$ can be identified with $C(c_1)$.

Note that $C(c_1)$ is defined by the original Poisson structure $\{.,.\}$ of A_n and not by $\{.,.\}_{\mathrm{gr}}$, even if it will be considered as a Poisson subalgebra of $\mathrm{gr}A_n$. The reason of this slightly ambiguous notation is that we will also introduce $C^{\mathrm{gr}}(x_{1,1}) \subseteq \mathrm{gr}A_n$ as the centralizer of $x_{1,1}$ with respect to $\{.,.\}_{\mathrm{gr}}$.

Our associated graded setup can be summarized as follows:

$$C(c_1) \quad \subseteq \quad \mathrm{gr}A_n \xrightarrow{\varphi} B_{2,n} \xrightarrow{\delta} D_n$$

Proof of Theorem 5.1.2. We prove the statement by induction on n. The statement is verified for $\mathcal{O}(SL_2)$ in Sec. 5.4 so, by Prop. 5.3.1, the case $n = 2$ is proved. Assume that $n \geq 3$. We shall prove that

- $(\delta \circ \varphi)|_{C(c_1)} : C(c_1) \to D_n$ is injective, and
- the image $(\delta \circ \varphi)\big(C(c_1)\big)$ is in $D_n^{\mathfrak{S}_n}$.

These imply that the restriction of $\delta \circ \varphi$ to $C(c_1)$ is an isomorphism onto $D_n^{\mathfrak{S}_n}$ since $C(c_1) \ni c_i$ for $i = 1, \ldots, n$ (see Subsec. 5.2.6) and $\delta \circ \varphi$ restricted to $\mathbb{C}[c_1, \ldots, c_n]$ is surjective onto $D_n^{\mathfrak{S}_n}$. The statement of the theorem follows.

To prove that $\delta \circ \varphi$ is injective on $C(c_1)$ it is enough to prove that δ is injective on $C\big(\varphi(c_1)\big)$ and that φ is injective on $C(c_1)$. Indeed, as φ is a Poisson map we have $\varphi\big(C(c_1)\big) \subseteq C\big(\varphi(c_1)\big)$.

First we prove δ is injective on $C\big(\varphi(c_1)\big)$. By $B_{2,n} \cong A_{n-1}[t]$ where t is Poisson-central, we have

$$B_{2,n} \supseteq C\big(\varphi(c_1)\big) \cong C_{A_{n-1}[t]}\big(t + c_1(A_{n-1})\big) = C_{A_{n-1}}\big(c_1(A_{n-1})\big)[t] \subseteq A_{n-1}[t]$$

By the induction hypothesis

$$C_{A_{n-1}}\big(c_1(A_{n-1})\big) = \mathbb{C}\big[c_1(A_{n-1}), \ldots, c_{n-1}(A_{n-1})\big]$$

Therefore δ restricted to $C\big(\varphi(c_1)\big)$ is an isomorphism onto the subalgebra $\mathbb{C}[s_1,\dots,s_{n-1}][t_1]\subseteq D_n$ where s_i is the symmetric polynomial in the variables $t_2,\dots,t_n$. In particular, δ is injective on $C\big(\varphi(c_1)\big)$.

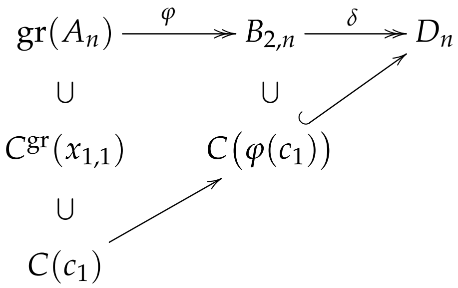
To verify the injectivity of φ on $C(c_1)$, define

$$C^{\mathrm{gr}}(x_{1,1}) := \{a \in \mathrm{gr}A_n \mid \{x_{1,1}, a\}_{\mathrm{gr}} = 0\}$$

The subalgebra $C(c_1)$ is contained in $C^{\mathrm{gr}}(x_{1,1})$ since for a homogeneous element a of degree d, we have

$$\mathcal{A}^{d+1}/\mathcal{A}^d \ni \{x_{1,1}, a\}_{\mathrm{gr}} + \mathcal{A}^d = \{x_{1,1} + \mathcal{A}^0, a + \mathcal{A}^{d-1}\} + \mathcal{A}^d = \{c_1, a\} + \mathcal{A}^d$$

hence $\{c_1, a\} = 0$ implies $\{x_{1,1}, a\}_{\mathrm{gr}} = 0 \in \mathrm{gr}A_n$. Our setup can be visualized on the following diagram:



Now it is enough to prove that φ restricted to $C^{\mathrm{gr}}(x_{1,1})$ is injective.

We can give an explicit description of $C^{\mathrm{gr}}(x_{1,1})$ in the following form:

$$C^{\mathrm{gr}}(x_{1,1}) = \mathbb{C}[x_{1,1}, x_{i,j} \mid 2 \le i,j \le n] \le \mathrm{gr}A_n$$

Indeed,

$$\{x_{1,1}, x_{i,j}\}_{\mathrm{gr}} = \begin{cases} x_{1,1}x_{i,j} & \text{if } j \neq i = 1 \text{ or } i \neq j = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore the map $\mathrm{ad}_{\mathrm{gr}}x_{1,1} : a \mapsto \{x_{1,1}, a\}_{\mathrm{gr}}$ acts on a monomial $m \in \mathrm{gr}A_n$ as $\{x_{1,1}, m\}_{\mathrm{gr}} = c(m)\cdot x_{1,1}m$ where $c(m)$ is the sum of the exponents of the $x_{1,j}$'s and $x_{i,1}$'s ($2 \le i,j \le n$) in m. Hence $\mathrm{ad}_{\mathrm{gr}}x_{1,1}$ maps the monomial basis of $\mathrm{gr}A_n$ injectively into itself. In particular,

$$C^{\mathrm{gr}}(x_{1,1}) = \mathrm{Ker}\big(\mathrm{ad}_{\mathrm{gr}}x_{1,1}\big) = \{a \in \mathrm{gr}A_n \mid c(m) = 0\} \cong A_{n-1}[t]$$

using the isomorphism $x_{1,1} \mapsto t$ and $x_{i,j} \mapsto x_{i-1,j-1}$.

The injectivity part of the theorem follows: φ is injective on $C^{\mathrm{gr}}(x_{1,1})$ (in fact it is an isomorphism onto $B_{2,n}$), and φ maps $C(c_1)$ into $C\big(\varphi(c_1)\big)$ on which δ is also injective.

To prove $(\delta\circ\varphi)\big(C(c_1)\big) \subseteq D_n^{\mathfrak{S}_n}$, first note that in the above we have proved that

$$(\delta\circ\varphi)\big(C(c_1)\big) \subseteq \delta\big(C\big(\varphi(c_1)\big)\big) \subseteq D_n^{\mathfrak{S}_{n-1}}$$

where $\mathfrak{S}_{n-1}$ acts on D_n by permuting $t_2, \dots, t_n$. Consider the automorphism γ of A_n given by the reflection to the off-diagonal:

$$\gamma(x_{i,j}) = x_{n+1-i,n+1-j}$$

It is not a Poisson map but a Poisson antimap (using the terminology of [ChP]), i.e. $\gamma(\{a,b\}) = -\{\gamma(a), \gamma(b)\}$. It maps c_1 into itself and consequently $C(c_1)$ into itself. For the analogous involution $\overline{\gamma} : D_n \to D_n$, $t_i \mapsto t_{n+1-i}$ $(i = 1, \dots, n)$ we have $(\delta \circ \varphi) \circ \gamma = \overline{\gamma} \circ (\delta \circ \varphi)$. Hence

$$(\delta \circ \varphi)\big(C(c_1)\big) = (\delta \circ \varphi \circ \gamma)\big(C(c_1)\big) = (\overline{\gamma} \circ \delta \circ \varphi)\big(C(c_1)\big) \subseteq \overline{\gamma}(D_n^{\mathfrak{S}_{n-1}})$$

proving the symmetry of $(\delta \circ \varphi)\big(C(c_1)\big)$ in $t_1, \dots, t_{n-1}$, so it is symmetric in all the variables by $n \geq 3$. □

Remark 5.5.2. In the case of the KKS Poisson structure (see Subsec. 5.2.3), every maximal Poisson-commutative subalgebra contains the Poisson center $\mathbb{C}[c_1, \dots, c_n]$, see [We]. (For an example of such a maximal Poisson-commutative subalgebra, see [KW].) This is in contrast with Theorem 5.1.1 in the sense that for the semiclassical Poisson structure, there is a single maximal Poisson-commutative subalgebra of $\mathcal{O}(M_n)$ containing $\mathbb{C}[c_1, \dots, c_n]$.

Remark 5.5.3. We prove that $\mathbb{C}[\overline{c}_1, \dots \overline{c}_{n-1}]$ is not an integrable complete involutive system (see Subsec. 5.2.1). First observe that the rank of the semiclassical Poisson bracket of $\mathcal{O}(SL_n)$ is $n(n-1)$.

Indeed, by Subsec. 5.2.1, the rank is the maximal dimension of the symplectic leaves in SL_n. The symplectic leaves in SL_n are classified in [HL1], Theorem A.2.1, based on the work of Lu, Weinstein and Semenov-Tian-Shansky [LW], [Sem]. The dimension of a symplectic leaf is determined by an associated element of $W \times W$ where $W = \mathfrak{S}_n$ is the Weyl group of SL_n. According to Proposition A.2.2, if $(w_+, w_-) \in W \times W$ then the dimension of the corresponding leaves is

$$\begin{aligned}\ell(w_+) + \ell(w_-) + \min\{m \in \mathbb{N} \mid w_+ w_-^{-1} = r_1 \cdot \dots \cdot r_m, \\ r_i \text{ is a transposition for all } i\}\end{aligned} \tag{5.4}$$

where $\ell(.)$ is the length function of the Weyl group that – in the case of SL_n – is the number of inversions in a permutation. By the definition of inversion using elementary transpositions, the above quantity is bounded by

$$\ell(w_+) + \ell(w_-) + \ell(w_+ w_-^{-1})$$

The maximum of the latter is $n(n-1)$ since $\ell(w_+) = \binom{n}{2} - \ell(w_+ t)$ where $t = (n \dots 1)$ stands for the longest element of $\mathfrak{S}_n$. Therefore

$$\ell(w_+) + \ell(w_-) + \ell(w_+ w_-^{-1}) =$$

$$= n(n-1) - \ell(w_+t) - \ell(w_-t) + \ell\big((w_+t)(w_-t)^{-1}\big) \leq n(n-1)$$

because $\ell(gh) \leq \ell(g) + \ell(h) = \ell(g) + \ell(h^{-1})$ for all $g, h \in \mathfrak{S}_n$. This maximum is attained on $w_+ = w_- = t$, even for the original quantity in Equation 5.4. Hence $\mathrm{Rk}\{.,.\} = n(n-1)$ for SL_n and $\mathrm{Rk}\{.,.\} = n(n-1)+1$ for M_n and GL_n. However, a complete integrable system should have dimension

$$\dim SL_n - \frac{1}{2}\mathrm{Rk}\{.,.\} = n^2 - 1 - \binom{n}{2} = \binom{n+1}{2} - 1$$

So it does not equal to $\dim \mathbb{C}[\bar{c}_1, \ldots \bar{c}_{n-1}] = n-1$ if $n > 1$. Similarly, the system is non-integrable for M_n and GL_n.

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