COLORINGS, PERFECT SETS AND GAMES ON GENERALIZED BAIRE SPACES

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Contents

CHAPTER 1

Introduction

Let κ be an uncountable regular cardinal such that $\kappa^{<\kappa} = \kappa$ holds. The *generalized Baire space for* κ, or the κ-Baire space for short, is the set ${}^{\kappa}\kappa$ of functions $f : \kappa \to \kappa$ equipped with the *bounded topology*, i.e., the topology given by the basic open sets

$$N_s = \{x \in {}^{\kappa}\kappa : s \subseteq x\}$$

associated to functions $s : \alpha \to \kappa$ for ordinals $\alpha < \kappa$. The *generalized Cantor space* ${}^{\kappa}2$ is defined analogously.

A systematic study of the descriptive set theory of these spaces was initiated by Alan Mekler and Jouko Väänänen [34,51], and extended by many prominent researchers. One of the main motivations behind these investigations is model theoretic. Models with domain κ can be coded, in a natural way, as elements of the κ-Cantor space, and some model theoretic properties can be reformulated as topological or descripitve set theoretic properties of this space. In particular, the study of these spaces provides a framework for the classification of uncountable models, which is one of the central themes in model theory. See, for example, [34, 52, 9] for more on these connections.

Descreptive set theory on generalized Baire spaces can look very different from the classical case. Whether the uncountable versions of classical theorems hold or not often depends on which additional axioms are assumed besides the usual ZFC axioms of set theory. Thus, these questions are closely related not just to model theory, but to other areas of set theory, such as infinitary combinatorics and large cardinal axioms, and can lead to a better understanding of the influence of such axioms.

One main theme of this thesis is the investigation, for generalized Baire spaces ${}^\kappa\kappa$, of the uncountable analogues of perfect set theorems and classical dichotomy theorems concerning colorings (or equivalently, graphs and hypergraphs) on the lower levels of the κ-Borel hierarchy.

In the uncountable setting, the failure of these dichotomies is consistent with ZFC in many cases. Consider, for example, the simplest such dichotomy, the κ-perfect set property for closed subsets of the κ-Baire space. This is the statement that any closed set $X \subseteq {}^\kappa\kappa$ of cardinality at least κ^+ contains a κ-perfect subset. (The concept of κ-perfectness [51] is a natural analogue of the concept of perfectness for subsets of Polish spaces). The existence of κ-Kurepa trees, or more generally, weak κ-Kurepa trees, witnesses the failure of the κ-perfect set property for closed subsets [34, 9]. Therefore this dichotomy fails if $V = L$ holds [9]. Furthermore, by an argument of Robert Solovay [19], this simplest dichotomy implies that κ^+ is an inaccessible cardinal in Gödel's universe L. Thus, all of the dichotomies studied in this work also imply the inacessibility of κ^+ in L.

Conversely, after Lévy-collapsing an inaccessible cardinal $\lambda > \kappa$ to κ^+, the κ-perfect set property holds for all closed subses, and in fact, for all subsets of ${}^\kappa\kappa$ definable from a κ-sequence of ordinals [39].

There are, in fact, a few different generalizations of perfectness for the κ-Baire space in literature. In classical descriptive set theory, all these notions correspond to equivalent definitions of perfectness for the Baire space. However, they are no longer equivalent in the uncountable setting.

Perfectness (and also scatterdness) was first generalized for subsets of the κ-Baire space by Jouko Väänänen in [51], where the concept of γ-perfectness (and γ-scatteredness) for infinite ordinals $\gamma \leq \kappa$ and subsets X of the κ-Baire space was defined based on a game of length γ played on X. A stronger notion of κ-perfectness is also widely used: a subset of the κ-Baire space is κ-perfect in this stronger sense iff it can be obtained as the set of κ-branches of a $<\kappa$-closed subtree T of ${}^{<\kappa}\kappa$ in which the set of splitting nodes is cofinal. Thus, this concept corresponds to a (strong) notion of κ-perfectness for subtrees T of ${}^{<\kappa}\kappa$. Concepts of γ-perfectness and γ-scatteredness (where $\omega \leq \gamma \leq \kappa$) for subtrees T of ${}^{<\kappa}\kappa$ which correspond more closely to Väänänen's notions can be defined based on versions of cut-and-choose games played on the trees T. (In the $\gamma < \kappa$ case, these games and notions were introduced in [11].) Specifically, it will not be hard to see that a subset X of the κ-Baire space is κ-perfect (in Väänänen's sense) iff it can be obtained as the set of κ-branches of a tree which is κ-perfect in this latter, weaker sense.

Although these notions are not equivalent, they are often interchangeable. For example, they lead to equivalent definitions of the κ-perfect set property, and are also equivalent with respect to most of the dichotomies studied in this work.

In the first part of Chapter 2, we detail connections between these notions of perfectness, scatteredness and the games underlying these definitions. Our observations lead to equivalent characterizations of the κ-perfect set property for closed subsets of the κ-Baire space in terms of the games considered here.

In particular, we show that Väänänen's generalized Cantor-Bendixson theorem [51] is in fact equivalent to the κ-perfect set property for closed subsets of the κ-Baire space. The consistency of this Cantor-Bendixson theorem was originally obtained in [51] relative to the existence of a measurable cardinal above κ. In [11], this statement is shown to hold after Lévy-collapsing an inaccessible cardinal to κ^+.

In Chapter 2, we also consider notions of density in itself for subsets of the κ-Baire space which are given by the different notions of perfectness studied here. We show that the statement

> "every subset of the κ-Baire space of cardinality at least κ^+ has a κ-dense in itself subset"

follows from a hypothesis which is consistent assuming the consistency of the existence of a weakly compact cardinal above κ. Previously, the above statement was known to be consistent relative to the existence of a measurable cardinal above κ [51, Theorem 1].

In Chapter 3, we consider a dichotomy for κ-perfect homogeneous subsets of *open colorings* on subsets of the κ-Baire space.

Given a set $X \subseteq {}^{\kappa}\kappa$, a *binary coloring* on X is any subset R of $[X]^2$. Such a coloring R may be identified in a natural way with a symmetric irreflexive binary relation R' on X; R is an *open coloring* iff R' is an open subset of $X \times X$. A partition $[X]^2 = R_0 \cup R_1$ is open iff R_0 is an open coloring on X. For a subset X of the κ-Baire space, we let $\mathrm{OCA}^*_\kappa(X)$ denote the following statement.

> $\mathrm{OCA}^*_\kappa(X)$: for every open partition $[X]^2 = R_0 \cup R_1$, either X is a union of κ many R_1-homogeneous sets, or there exists a κ-perfect R_0-homogeneous set.

The property $\mathrm{OCA}^*(X) = \mathrm{OCA}^*_\omega(X)$ for subsets X of the Baire space was studied in [7], and, in particular, was shown to hold for analytic sets.

We obtain the consistency of the κ-version of this result from an inaccessible cardinal above κ. More precisely, we prove that after Lévy-collapsing an inaccessible cardinal $\lambda > \kappa$ to κ^+, $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ holds; that is, $\mathrm{OCA}^*_\kappa(X)$ holds for all κ-analytic subsets of the κ-Baire space. Thus, $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ is equiconsistent with the existence of an inaccessible cardinal above κ.

We show that for an arbitrary subset X of the κ-Baire space, $\mathrm{OCA}^*_\kappa(X)$ is equivalent to the determinacy, for all open colorings $R \subseteq [X]^2$, of a cut and choose game associated to R.

We also investigate analogues, for open colorings, of the games generalizing perfectness considered in Chapter 2. We give some equivalent reformulations of $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ in terms of these games. For example, we prove that $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ is equivalent to the natural analogue, for open colorings, of Väänänen's generalized Cantor-Bendixson theorem.

In [51], Jouko Väänänen gave a generalization of the Cantor-Bendixson hierarchy for subsets of the κ-Baire space. This is done by considering modified versions, associated to trees without κ-branches, of the perfect set game defined in [51]. Thus, in the uncountable setting, trees without κ-branches play a role analogous to that of ordinals in the classical setting. In this approach, ordinals correspond to well-founded trees; specifically, the α^{th} level of the Cantor-Bendixson hierarchy corresponds to the game associated to the canonical well-founded tree of rank α.

Similar methods are used in e.g. [16, 18] to study transfinite Ehrenfeucht-Fraïssé games, and in [36] to study the analogue of inductive definitions, in general, for non well-founded trees.

In Chapter 2, we discuss how the Cantor-Bendixson hierarchy can be generalized for subtrees T of ${}^{<\kappa}\kappa$, by considering analogous modifications of the games generalizing perfectness for trees T and using adaptations of the approach in [51].

In Chapter 3, we consider analogues, for open colorings, of the games used to generalize the Cantor-Bendixson hierarhcy for subsets of the κ-Baire space and for subtrees of ${}^{<\kappa}\kappa$. These games allow trees without κ-branches to generalize different ranks associated to open colorings, leading to different generalized hierarchies. We prove comparison theorems for these games which show how the levels of the corresponding generalized hierarchies are related to each other. For example, in the specific case of the trivial coloring, these comparison theorems imply the following. Let X be a closed subset of the κ-Baire space, and let T be the tree of initial segments of X. Then the levels of the generalized Cantor-Bendixson hierarchies for X are always contained in (the set of κ-branches of) the levels of the generalized Cantor-Bendixson hierarhies for T.

In Chapter 4, we consider dichotomies for independent subsets with respect to given (families of) finitary $\mathbf{\Sigma}^0_2(\kappa)$ relations on subsets X of the κ-Baire space. Naturally, these can be reformulated as dichotomies for homogeneous subsets of given (families of) $\mathbf{\Pi}^0_2(\kappa)$ colorings on X.

In the first part of the chapter, we consider the κ-Silver dichotomy for $\mathbf{\Sigma}^0_2(\kappa)$ equivalence E relations on $\mathbf{\Sigma}^1_1(\kappa)$ subsets X of the κ-Baire space (i.e., the statement that if such an equivalence relation E has at least κ^+ many equivalence classes, then E has κ-perfectly many equivalence classess).

Recently, a considerable effort has been made to investigate set theoretical conditions implying (the consistency of) the satisfaction or the failure of the κ-Silver dichotomy for κ-Borel equivalence relations on the κ-Baire space. For example, the κ-Silver dichotomy fails for $\mathbf{\Delta}^1_1(\kappa)$ equivalence relations [8], and $V = L$ implies the failure of the κ-Silver dichotomy for κ-Borel equivalence relations in a strong sense [9, 10]. In the other direction, the κ-Silver dichotomy for κ-Borel equivalence relations is consistent relative to the consistency of $0^\#$ [8].

We show that after Lévy-collapsing an inaccessible cardinal $\lambda > \kappa$ to κ^+, the κ-Silver dichotomy holds for $\mathbf{\Sigma}^0_2(\kappa)$ equivalence relations on $\mathbf{\Sigma}^1_1(\kappa)$ subsets of the κ-Baire space. Thus, this statement is equiconsistent with the existence of an inaccessible cardinal above κ.

In the remainder of the chapter, we consider dichotomies for families $\mathcal{R}$ of at most κ many $\mathbf{\Sigma}^0_2(\kappa)$ relations (of arbitrary finite arity) on subsets of the κ-Baire space.

Our starting point is the following "perfect set property" for independent subsets with respect to such families of relations on κ-analytic subsets.

> $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$: if $\mathcal{R}$ is a collection of κ many finitary $\mathbf{\Sigma}^0_2(\kappa)$ relations on a κ-analytic set $X \subseteq {}^\kappa\kappa$ and X has an $\mathcal{R}$-independent subset of cardinality κ^+, then X has a κ-perfect $\mathcal{R}$-independent subset.

By a joint result of Jouko Väänänen and the author [46], $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ is consistent relative to the existence of a measurable cardinal above κ.

In the classical case, the countable version $\mathrm{PIF}_\omega(\boldsymbol{\Sigma}^1_1)$ of this dichotomy holds by a result of Martin Doležal and Wieslaw Kubiś [5]. (See also [24, 42] where specific cases of these results are shown.) In fact, they obtain $\mathrm{PIF}_\omega(\boldsymbol{\Sigma}^1_1)$ as a corollary of the following statement (which is also shown in [5]):

> *if $\mathcal{R}$ is a countable family of finitary $\boldsymbol{\Sigma}^0_2$ relations on a Polish space X and X has an $\mathcal{R}$-independent subset of Cantor-Bendixson rank $\geq \gamma$ for every countable ordinal γ, then X has a perfect $\mathcal{R}$-independent subset.*

We show what may be viewed as a κ-version of the above statement holds assuming only either $\diamondsuit_\kappa$ or the inaccessibility of κ. In fact, it is enough to assume a slightly weaker version $\mathcal{DI}_\kappa$ than $\diamondsuit_\kappa$ which also holds whenever κ is inaccessible (this principle will be defined in Chapter 4). In more detail, $\mathcal{DI}_\kappa$ implies, roughly, that

> *if $\mathcal{R}$ is a family of κ many finitary $\boldsymbol{\Sigma}^0_2(\kappa)$ relations on a closed set $X \subseteq {}^\kappa\kappa$ and X has $\mathcal{R}$-independent subsets "on all levels of the generalized Cantor-Bendixson hierarchy for player* **II***", then X has a κ-perfect $\mathcal{R}$-independent subset.*

As a corollary of our arguments, we obtain stronger versions of the main result of [46]. In particular, our results imply that the consistency strength of $\mathrm{PIF}_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$ is at most that of the existence of a weakly compact cardinal above κ.

In the last part of the Chapter 4, we show that a model theoretic dichotomy, motivated by the spectrum problem, is a special case of $\mathrm{PIF}_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$. The contents of this section are (essentially) the same as the contents of [46, Section 3].

1.1 Preliminaries and notation

The notation and terminology we use is mostly standard; see e.g. [20]. The Greek letters $\alpha, \beta, \gamma, \delta, \eta, \xi$ usually denote ordinals, and Ord denotes the class of all ordinals. We denote by Succ the class of successor ordinals, and Lim denotes the class of limit ordinals. Given ordinals $\alpha < \beta$, we use the notation $[\alpha, \beta) = \{\gamma < \beta : \alpha \leq \gamma\}$ and $(\alpha, \beta) = \{\gamma < \beta : \alpha < \gamma\}$, etc.

The Greek letters $\lambda, \kappa, \mu, \nu$ denote cardinals. In subsequent chapters, κ typically denotes a cardinal such that $\kappa^{<\kappa} = \kappa$.

For a set X, we let $\mathcal{P}(X)$ denote the powerset of X. If μ is a cardinal, then $[X]^\mu$ denotes the set of subsets of X which are of cardinality μ, and $[X]^{<\mu}$ denotes the set of subsets of X of cardinality $< \mu$. For $\gamma \in \mathrm{Ord}$, we also use the following notation:

$$[X]^\gamma_{\neq} = \{\langle x_i : i < \gamma\rangle \in {}^\gamma X : x_i \neq x_j \text{ for all } i < j < \gamma\},$$
$$[X]^{<\gamma}_{\neq} = \bigcup_{\beta<\gamma} [X]^\beta_{\neq} = \{\langle x_i : i < \beta\rangle \in {}^\beta X : \beta < \gamma \text{ and } x_i \neq x_j \text{ for all } i < j < \beta\}.$$

Given a set X, we let id_X denote the identity function on X. If f is a function $Y \subseteq \mathrm{dom}(f)$ and $Z \subseteq \mathrm{ran}(f)$, then $f[Y]$ denotes the pointwise image of Y under f, and $f^{-1}[Z]$ denotes the preimage of Z. For an ordinal γ, we let ${}^\gamma X$ denote the set of functions f with $\mathrm{dom}(f) = \gamma$ and $\mathrm{ran}(f) \subseteq X$. We let ${}^{<\gamma}X = \bigcup_{\alpha<\gamma} {}^\alpha X$. If μ, λ are cardinals, then $\lambda^{<\mu}$ denotes the cardinality of ${}^{<\mu}\lambda$. If Y is any set, $1 \leq n < \omega$ and $X \subseteq {}^{n+1}Y$, then we let $\mathbf{p}X$ denote the projection of X onto the first n coordinates, i.e.,

$$\mathbf{p}X = \{(x_0, \ldots, x_{n-1}) \in {}^nY : \text{ there exists } y \in Y \text{ such that } (x_0, \ldots, x_{n-1}, y) \in X\}.$$

If X is any set, then $Sym(X)$ denotes the permutation group of X, and we use $Inj(X)$ to denote the monoid of all injective functions from X into X. Let $n < \omega$. An n-ary relation R on X is *symmetric* iff $(x_0, \ldots, x_{n-1}) \in R$ implies $(x_{\rho(0)}, \ldots, x_{\rho(n-1)}) \in R$ for all permutations $\rho \in \mathrm{Sym}(n)$.

We say that an n-ary relation R on X is *irreflexive* iff $R \subseteq [X]^n_{\neq}$, that is, iff for all $(x_0, \ldots, x_{n-1}) \in R$ we have $x_i \neq x_j$ for all $i < j < n$.

We say that an n-ary relation R on X is *reflexive* iff its complement ${}^nX - R$ is irreflexive, that is, iff for all $(x_0, \ldots, x_{n-1}) \in {}^nX$ such that $x_i = x_j$ for some $i < j < n$ we have $(x_0, \ldots, x_{n-1}) \in R$.

Given a partial order $\mathbb{P} = \langle P, \leq_\mathbb{P}\rangle$ We confuse P with $\mathbb{P}$ when $\leq_\mathbb{P}$ is clear from the context. We also write $\perp$ and $\leq$ instead of $\perp_\mathbb{P}$ and $\leq_\mathbb{P}$ in this case. Let $p \in P$. We denote by $\mathrm{pred}_\mathbb{P}(p)$ the set of predecessors of p, i.e., $\mathrm{pred}_\mathbb{P}(p) = \{s \in P : s <_\mathbb{P} p\}$. We let $\mathrm{succ}_\mathbb{P}(p)$ denote the set of successors of p, i.e., $\mathrm{succ}_\mathbb{P}(p) = \{s \in P : s >_\mathbb{P} p\}$. Lastly, we denote by $\mathbb{P}{\restriction_p}$ the set of all $q \in P$ which are comparable with p. Thus, $\mathbb{P}{\restriction_p} = \mathrm{pred}_\mathbb{P}(p) \cup \{p\} \cup \mathrm{succ}_\mathbb{P}(p)$.

Definition 1.1. Let $\mathbb{P} = \langle P, \leq_\mathbb{P}\rangle$ be a partial order, let $Q \subseteq P$ and let κ be a cardinal.

(1) $\mathbb{P}$ is *$<\kappa$-closed* iff every decreasing sequence of length $< \kappa$ has a lower bound.

(2) Q is a *$<\kappa$-closed subset* of $\mathbb{P}$ iff the partial order $\langle Q, \leq_{\mathbb{P}} \restriction Q \times Q\rangle$ is $<\kappa$-closed, i.e., iff every $\leq_{\mathbb{P}}$-decreasing sequence of length $< \kappa$ of elements of Q has a lower bound in Q.

(3) Q is a *dense subset* of $\mathbb{P}$ iff for every $p \in P$, there exists $q \in Q$ such that $q \leq_{\mathbb{P}} p$.

Definition 1.2. Given an infinite cardinal κ and a partial order $\mathbb{P}$, we let $\mathrm{G}_\kappa(\mathbb{P})$ denote the following game of length κ. Two players **I** and **II** take turns building a decreasing sequence $\langle r_\alpha : \alpha < \kappa\rangle$ of elements of $\mathbb{P}$. Player **II** plays in all even rounds $2\alpha < \kappa$ (including limit rounds and round 0), and player **I** plays in all odd rounds $2\alpha + 1 < \kappa$. Player **II** wins a run of the game if she can play legally in all rounds $2\alpha < \kappa$.

Typically, given a run $\langle r_\alpha : \alpha < \kappa\rangle$ we denote the sequence of moves of player **II** by $\langle p_\alpha = r_{2\cdot\alpha} : \alpha < \kappa\rangle$, and we denote the sequence of moves of player **I** by $\langle q_\alpha = r_{2\cdot\alpha+1} : \alpha < \kappa\rangle$.

A partial order $\mathbb{P}$ is *$<\kappa$-strategically closed* iff player **II** has a winning strategy in $\mathrm{G}_\kappa(\mathbb{P})$.

1.1.1 The κ-Baire space

In this subsection, we assume κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$ holds.

Definition 1.3. The *generalized Baire space for κ*, or the κ-Baire space for short, is the set ${}^\kappa\kappa$ of functions $f : \kappa \to \kappa$ equipped with the *bounded topology*, i.e., the topology given by the basic open sets

$$N_s = \{x \in {}^\kappa\kappa : s \subseteq x\}$$

associated to functions $s : \alpha \to \kappa$ for ordinals $\alpha < \kappa$.

The bounded topology on the set ${}^\kappa 2$ of functions $f : \kappa \to 2$ is defined analogously. The *generalized Cantor space* ${}^\kappa 2$, or κ-Cantor space, is the set ${}^\kappa 2$ equipped with the bounded topology.

Unless otherwise mentioned, we assume that ${}^\kappa\kappa$ and ${}^\kappa 2$ are equipped with the bounded topology, throughout this work. (Thus, we also use ${}^\kappa\kappa$ and ${}^\kappa 2$ to denote the generalized Baire and Cantor spaces for κ.)

If $2 \leq n < \omega$, then the set ${}^n({}^\kappa\kappa)$ is equipped with the product topology (given by the bounded topology on ${}^\kappa\kappa$), and subsets $X \subseteq {}^n({}^\kappa\kappa)$ are equipped with the subspace topology.

Observe that the space ${}^n({}^\kappa\kappa)$ is homeomorphic to ${}^\kappa\kappa$ (by an argument analogous to the proof in the classical case).

Notation. We denote by $\mathcal{C}_\kappa$ the collection of closed subsets of the κ-Baire space.

Given a subset X of ${}^\kappa\kappa$, we let $\overline{X}$ denote the closure of X, and we let $\mathrm{Int}(X)$ denote its interior.

If $X \subseteq Y \subseteq {}^\kappa\kappa$, then we let $\overline{X}^Y$ and $\mathrm{Int}_Y(X)$ denote the closure and interior of X relative to Y.

The hypothesis $\kappa^{<\kappa} = \kappa$ is usually assumed when working with the κ-Baire and κ-Cantor space, because it implies that these spaces have some nice properties.

Fact 1.4 (see [9])**.** If $\kappa^{<\kappa} = \kappa$ is assumed, then the following hold for the κ-Baire space and the κ-Cantor space.

(1) The standard bases of both spaces are of size κ and consist of clopen sets.
(2) There exists a dense subset of size κ.
(3) The intersection (resp. union) of $< \kappa$ many open (closed) sets is open (closed).
(4) The κ-Baire category theorem holds; that is, the intersection of κ many open dense sets is dense.

Definition 1.5. Given a topological space X, the collection of κ-*Borel* subsets of X is the smallest set which contains the open subsets of X and is closed under complementation and taking unions and intersections of at most κ many sets.

Specifically, we will be interested in the second level of the κ-Borel hierarchy.

Definition 1.6. Let X be a topological space, and let $Y \subseteq X$.

(1) Y is a $\mathbf{\Sigma}^0_2(\kappa)$ subset of X iff it is the union of at most κ many closed subsets of X.
(2) Y is a $\mathbf{\Pi}^0_2(\kappa)$ subset of X iff it is the intersection of at most κ many open subsets of X.

Definition 1.7 ([34])**.** Let $1 \leq n < \omega$, and let $X \subseteq {}^n({}^\kappa\kappa)$.

(1) X is a $\mathbf{\Sigma}^1_1(\kappa)$, or κ-analytic, set iff X is the projection $\mathbf{p}Y$ of a closed subset $Y \subseteq {}^{n+1}({}^\kappa\kappa)$.
(2) X is a $\mathbf{\Pi}^1_1(\kappa)$ set iff its complement ${}^n({}^\kappa\kappa) - X$ is a $\mathbf{\Sigma}^1_1(\kappa)$ set.
(3) X is a $\mathbf{\Delta}^1_1(\kappa)$ set iff it is both a $\mathbf{\Sigma}^1_1(\kappa)$ set and a $\mathbf{\Pi}^1_1(\kappa)$ set.

Fact 1.8 ([9])**.** A subset $X \subseteq {}^n({}^\kappa\kappa)$. is $\mathbf{\Sigma}^1_1(\kappa)$ set if and only if X is continuous image of a closed subset of the κ-Baire space.

Fact 1.9. All κ-Borel subsets of ${}^n({}^\kappa\kappa)$ are $\mathbf{\Delta}_1^1(\kappa)$ sets [34]. However, there exists a $\mathbf{\Delta}_1^1(\kappa)$ subset of ${}^\kappa\kappa$ which is not a κ-Borel set [9].

We remark that an even stronger concept of Borel sets (that of Borel* sets) was also introduced for the κ-Baire space in [34] using a game theoretic definition.

Definition 1.10. Let X be a topological space.

(1) A subset C of X is *κ-compact* iff any open cover of C has a subcover of size $< \kappa$.

(2) A subset C of X is a *K_κ subset* iff it can be written as the union of at most κ many κ-compact subsets.

Definition 1.11. Suppose X is a topological space. We say that $R \subseteq {}^nX$ is an *open (n-ary) relation on* X iff R is an open subset of the product space ${}^nX$.

The concept of *closed relations, $\mathbf{\Pi}_2^0(\kappa)$ relations, $\mathbf{\Sigma}_2^0(\kappa)$ relations, κ-Borel relations*, etc., can be defined analagously.

Definition 1.12. Given a set X and $1 \leq n < \omega$, an *(n-ary) coloring* on X is an arbitrary subset R of $[X]^n$.

An n-ary coloring R can be identified, in a natural way, with a symmetric irreflexive relation $R' \subseteq [X]^n_{\neq}$, i.e., with

$$R' = \{(x_0, \dots, x_{n-1}) \in {}^nX : \{x_0, \dots, x_{n-1}\} \in R\}.$$

Suppose X is a topological space.

(1) We say that R is an *open coloring* on X iff R' is an open relation on X.

(2) We say that R is a *closed coloring* on X iff R' is a relatively closed subset of $[X]^n_{\neq}$ (or equivalently, iff $[X]^n - R$ is an open coloring on X).

(3) The concept of $\mathbf{\Pi}_2^0(\kappa)$ colorings, $\mathbf{\Sigma}_2^0(\kappa)$ colorings, etc., can be defined analogously to the concept of open colorings: that is, a coloring R on X is $\mathbf{\Pi}_2^0(\kappa)$ (resp. $\mathbf{\Sigma}_2^0(\kappa)$, etc.) iff R' is a $\mathbf{\Pi}_2^0(\kappa)$ (resp. $\mathbf{\Sigma}_2^0(\kappa)$, etc.) relation on X.

We say a partition $[X]^n = R_0 \cup R_1$ is *open (resp. closed, etc.)* iff R_0 is an open (closed, etc.) coloring on X.

Definition 1.13. Suppose X is an arbitrary set and $Y \subseteq X$.

(1) Given an n-ary coloring R_0 on X, we say Y is *R_0-homogeneous* iff $[Y]^n \subseteq R_0$.

(2) Let R be an n-ary relation on X. We say Y is *R-homogeneous* iff $[Y]^n_{\neq} \subseteq R$, i.e., iff for all pairwise different $y_0, \ldots, y_{n-1} \in Y$ we have $(y_0, \ldots, y_{n-1}) \in R$.

We say $Y \subseteq X$ is *R-independent* iff Y is $({}^nX - R)$-homogeneous, or equivalently, iff for all pairwise different $y_0, \ldots, y_{n-1} \in Y$ we have $(y_0, \ldots, y_{n-1}) \notin R$.

(3) If $\mathcal{R}$ is a family of finitary relations on X, then Y is defined to be *$\mathcal{R}$-independent* iff Y is R-independent for each $R \in \mathcal{R}$.

(4) If $\mathcal{R} = \langle R_\alpha : \alpha < \gamma \rangle$ is a sequence of finitary relations on X, then Y is defined to be $\mathcal{R}$-independent iff Y is independent w.r.t. $\{R_\alpha : \alpha < \kappa\}$.

Colorings $R_0 \subseteq [X]^n$ can be identified with partitions $[X]^n = R_0 \cup R_1$. In later chapters of this work (and especially in Chapter 3), we will also identify partitions

$$[X]^n = R_0 \cup R_1$$

with the symmetric *reflexive* n-ary relation R_1' on ${}^nX$ defined by R_1, i.e., with

$$R_1' = \{(x_0, \ldots, x_{n-1}) \in {}^nX : \{x_0, \ldots x_{n-1}\} \in R_1 \text{ or } x_i = x_j \text{ for some } i < j < n\}.$$

Thus, an open (resp. $\mathbf{\Pi}^0_2(\kappa)$, etc.) n-ary coloring R_0 on X will be identified with the closed (resp. $\mathbf{\Sigma}^0_2(\kappa)$, etc.) symmetric reflexive n-ary relation R_1' defined by its complement. Note that homogeneous subsets of open colorings correspond to independent subsets of closed relations (etc.) under this identification.

1.1.2 Trees

A *tree* is a partially ordered set $\langle T, \leq_T \rangle$ such that the set of predecessors of any element $t \in T$ is well-ordered by $\leq_T$, and T has a unique minimal element, called the *root* of T. We confuse the tree $\langle T, \leq_T \rangle$ with its domain T whenever $\leq_T$ is clear from the context. We also write $\leq$ and $\perp$ instead of $\leq_T$ and $\perp_T$ in this case. We use T, S, $U, \ldots$ and $\boldsymbol{t}$, $\boldsymbol{s}$, $\boldsymbol{u}, \ldots$ to denote trees.

If T is a tree, then its elements $t \in T$ are also called nodes. It $t \in T$, then $\mathrm{ht}_T(t)$ denotes the height of t, i.e., the the order type of $\mathrm{pred}_t(T)$. The α^{th} level of T consists of the nodes $t \in T$ of height α. The height $\mathrm{ht}(T)$ of the tree T is the minimal α such that the α^{th} level of T is empty. Thus, $\mathrm{ht}(T) = \sup\{\mathrm{ht}_T(t) + 1 : t \in T\}$.

A *subtree* of T is a subset $T' \subseteq T$ with the induced order which is downwards closed, i.e. if $t' \in T'$ and $t \in T$ and $t \leq_T t'$, then $t \in T$.

A *branch* of a tree T is a maximal chain of T, (i.e., a maximal linearly ordered subset of T). We let $Branch(T)$ denote the set of all branches of T. The length of a branch b is the order type of b. An α-branch is a branch of length α.

We let $\mathcal{T}_\alpha$ denotes the class of trees $\boldsymbol{t}$ such that every branch of $\boldsymbol{t}$ has length $<\alpha$. We denote by $\mathcal{T}_{\lambda,\alpha}$ the class of trees $T \in \mathcal{T}_\alpha$ of size $\leq\lambda$.

Trees in $\mathcal{T}_\omega$ are also called *well-founded trees*. Well-founded trees correspond, in a natural way, to ordinals (see Example 1.18 below).

Definition 1.14. Let T and T' be arbitrary trees. We write

$$T \leq T'$$

if there is an order-preserving map $f : T \to T'$, i.e., a map f such that

$$s <_T t \text{ implies } f(s) <_{T'} f(t) \text{ for all } s, t \in T.$$

We write $T \equiv T'$ if $T \leq T'$ and $T' \leq T$.

Note that $\leq$ is a partial ordering on the class of all trees. When restricted to $\mathcal{T}_\kappa$, it can be viewed as a substitute for the ordering of ordinals. Specifically, the restriction of $\leq$ to well-founded trees is equivalent to the ordering of the ordinals. The partial ordering $\leq$ on $\mathcal{T}_\kappa$ can be quite complicated in the case of uncountable cardinals κ; see for example [18, 34, 52, 53].

The σ-operation on trees, defined below, is originally due to Kurepa [27]. It can be seen as a generalization of the successor operation on ordinals.

Definition 1.15. For a tree T, let σT denote the tree of all ascending sequences of elements of T ordered by end extension. That is, σT consists of sequences $\langle t_\beta : \beta < \alpha\rangle$ such that α is an ordinal and $t_\gamma <_T t_\beta$ for all $\gamma < \beta < \alpha$. The ordering is defined as follows:

$$\langle s_\beta : \beta < \delta\rangle \leq_{\sigma T} \langle t_\beta : \beta < \alpha\rangle \qquad \text{iff } \delta \leq \alpha \text{ and } s_\beta = t_\beta \text{ for all } \beta < \delta.$$

Lemma 1.16 (Kurepa [27])**.** *If T is a tree, then*

$$T < \sigma T,$$

i.e., $T \leq \sigma T$ and $\sigma T \not\leq T$.

See for example [53, Lemma 9.55] for a proof. With the σ-operation, one can define a stronger ordering of trees: for trees T and T', let $T \ll T'$ iff $T \leq \sigma T'$. Note that $T \ll T'$ implies $T < T'$ by the above lemma, and that $\ll$ is well-founded [18].

We remark that there is an equivalent characterization of the partial orders $T \leq T'$ and $T \ll T'$ using a comparison game between trees [18]; see also [53, p. 256].

Fact 1.17. Let ξ be a limit ordinal, and let κ, λ be cardinals. Then

(1) $\mathcal{T}_\xi$ is closed under the σ-operation.
(2) $\mathcal{T}_{\lambda,\kappa}$ is closed under σ if and only if $\lambda^{<\kappa} = \lambda$. If ξ is not a cardinal, then $\mathcal{T}_{\lambda,\xi}$ is closed under σ if and only if $\lambda^{|\xi|} = \lambda$.

Suppose T is a well-founded tree. The *rank* $\mathrm{rk}_T(t)$ of nodes $t \in T$ is defined by recursion as follows: $\mathrm{rk}_T(t) = \sup\{\mathrm{rk}_T(s) : t <_T s\}$. The *rank* $\mathrm{rk}(T)$ of a well-founded tree T is the rank of its root.

There is a canonical way of associating, to any ordinal α, a well-founded tree of rank α. This is described in the example below.

Example 1.18. For any ordinal α, let $\boldsymbol{b}_\alpha$ denote the tree of descending sequences of elements of α, ordered by end extension. That is, $\boldsymbol{b}_\alpha$ consists of sequences of the form

$$\langle \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \rangle \qquad \text{such that} \qquad \alpha > \alpha_0 > \alpha_1 > \ldots > \alpha_{n-1}.$$

The ordering is defined as follows:

$$\langle \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \rangle \leq_{\boldsymbol{b}_\alpha} \langle \beta_0, \beta_1, \ldots, \beta_{m-1} \rangle \qquad \text{iff } n \leq m \text{ and } \alpha_i = \beta_i \text{ for all } i < n.$$

The root of $\boldsymbol{b}_\alpha$ is the empty sequence.

The tree $\boldsymbol{b}_\alpha$ is well-founded and has rank α. Moreover, if T is a well-founded tree of rank α, then $T \equiv \boldsymbol{b}_\alpha$.

Notice that $\boldsymbol{b}_\alpha \leq \boldsymbol{b}_\beta$ if and only if $\alpha \leq \beta$. We also have $\sigma\boldsymbol{b}_\alpha = \boldsymbol{b}_{\alpha+1}$ (and therefore also $\boldsymbol{b}_\alpha \ll \boldsymbol{b}_\beta$ if and only if $\alpha \leq \beta$).

There is another natural way to associate a tree to an ordinal α: consider the tree which consists of a single branch of length α. We will also denote this tree with the symbol α.

There is a natural supremum and an infimum for sets of trees with respect to $\leq$. These can be defined as follows.

Definition 1.19. If $\{T_i : i \in I\}$ is a family of trees, then let

$$\bigoplus_{i \in I} T_i$$

denote the tree which consists of a union of disjoint copies of the trees T_i ($i \in I$), identified at the root.

It is easy to see that $\bigoplus_{i \in I} T_i$ is the supremum of $\{T_i : i \in I\}$ with respect to $\leq$, in the following sense: for any tree T, we have that $T \geq \bigoplus_{i \in I} T_i$ if and only if $T \geq T_i$ for all $i \in I$.

Example 1.20. The *κ-fan* is the tree

$$\boldsymbol{f}_\kappa = \bigoplus_{\alpha < \kappa} \alpha.$$

That is, $\boldsymbol{f}_\kappa$ consists of branches of all lengths $<\kappa$ joined at the root. The set of its nodes is

$$\left\{a^\alpha_\beta : \beta < \alpha < \kappa\right\},$$

where $a^\alpha_0 = 0$ and $a^\alpha_\beta = (\alpha, \beta)$ for all $0 < \beta < \alpha < \kappa$. The ordering is defined as follows:

$$a^\alpha_\beta \leq a^\gamma_\delta$$

iff either we have $\alpha = \gamma$ and $\beta \leq \delta$ or we have $\beta = \delta = 0$.

When κ is clear from the context, we will write $\boldsymbol{f}$ instead of $\boldsymbol{f}_\kappa$.

The following operation, which was introduced by by Todorčević [47], gives the infimum of a family of trees with respect to $\leq$.

Definition 1.21. For a family $\{T_i : i \in I\}$ of trees, we define the tree $\bigotimes_{i \in I} T_i$ as follows:

$$\bigotimes_{i \in I} T_i = \{\langle t_i : i \in I\rangle \in \Pi_{i \in I} T_i : \ \mathrm{ht}_{T_i}(t_i) = \mathrm{ht}_{T_j}(t_j) \ \text{ for all } i, j \in I\},$$

$$\langle t_i : i \in I\rangle \leq \langle u_i : i \in I\rangle \qquad \text{iff } t_i \leq u_i \text{ for all } i \in I.$$

It is not hard to show that $\bigotimes_{i \in I} T_i$ is in fact the infimum of $\{T_i : i \in I\}$, i.e., that if T is any tree, then $T \leq \bigotimes_{i \in I} T_i$ if and only if $T \leq T_i$ for all $i \in I$. (See e.g. [18, Lemma 2.5] for a short proof.)

The infimum of two trees S and T is denoted by $S \otimes T$, and their supremum is denoted by $S \oplus T$.

We will also need the following "arithmetic" operations on trees. These operations generalize, in a sense, the addition and multiplication of ordinals.

Definition 1.22. For arbitrary trees S and T, we let

$$S + T$$

be the tree obtained from S by adding a copy of T at the end of each branch of S.

More precisely, the domain of $S + T$ consists of the nodes of S and nodes of the form (b, t) where $b \in \mathit{Branch}(S)$ and $t \in T$. The ordering is as follows: for all $s, s' \in S$, $b, b' \in \mathit{Branch}(S)$ and $t, t' \in T$ we write

$$(b, t) \leq (b', t') \quad \text{iff} \quad b = b' \text{ and } t \leq_T t',$$

we write $s \leq (b, t)$ iff $s \in b$ and we write $s \leq s'$ iff $s \leq_S s'$.

Note that $\boldsymbol{b}_\alpha + \boldsymbol{b}_\beta \equiv \boldsymbol{b}_{\beta+\alpha}$ holds for any ordinals α and β.

Definition 1.23. If S and T are arbitrary trees, then the tree

$$S \cdot T$$

is obtained from T by replacing every node $t \in T$ with a copy of S.

More precisely, the domain of $S \cdot T$ is

$$\{(g, s, t) : s \in S,\ t \in T, \text{ and } g : \mathrm{pred}_T(t) \to \mathit{Branch}(S)\}.$$

The order is defined as follows:

$$(g, t, s) \leq (g', t', s')$$

iff we have $t \leq_T t'$, $g = g' \restriction \mathrm{pred}_T(t)$ and either we have $t = t'$ and $s \leq_S s'$ or we have $t <_T t'$ and $s \in g(t')$.

For example, $\boldsymbol{b}_\alpha \cdot \boldsymbol{b}_\beta \equiv \boldsymbol{b}_{\alpha\cdot\beta}$ holds for any ordinals α and β. As another example, observe that $T \cdot n \equiv T + T \cdot (n-1)$ for all $1 < n < \omega$ and $T \cdot \omega \equiv \bigoplus_{n<\omega} T \cdot n$.

Fact 1.24. Let κ be a regular cardinal. Then $\mathcal{T}_\kappa$ is closed under all the operations $+$, $\cdot$, $\bigoplus$ and $\bigotimes$.

Definition 1.25. A tree T is *reflexive* iff for every $t \in T$ we have $T \leq \{s \in T : t \leq_T s\}$.

Specifically, if $\gamma \in \mathrm{Ord}$, then the tree denoted by γ (which consists of one branch of length γ) is reflexive if and only if γ is an *indecomposable ordinal*, i.e., iff $\alpha + \gamma = \gamma$ for all $\alpha < \gamma$.

Fact 1.26 (from [15, 17]; see also p. 8 of [52])**.** If κ, λ are uncountable cardinals and $S \in \mathcal{T}_{\lambda,\kappa}$, then there exists a reflexive tree $T \in \mathcal{T}_{\lambda,\kappa}$ such that $S \leq T$.

For more on the structure of trees and its role in infinitary logic and the descriptive set theory of the κ-Baire space, see for example [6, 18, 34, 50, 52, 53].

Trees and closed finitary relations on the κ-Baire space

We write ${}^{<\kappa}\kappa$ to denote the tree $\langle {}^{<\kappa}\kappa, \subseteq \rangle$, as well as its underlying domain. A subtree of ${}^{<\kappa}\kappa$ is (by definition) a downwards closed subset T of ${}^{<\kappa}\kappa$ ordered by $\subseteq$. Given a subtree T of ${}^{<\kappa}\kappa$, we also use T to denote its domain.

If T is a subtree of ${}^{<\kappa}\kappa$ and $t \in T$, then we write $\mathrm{ht}(t) = \mathrm{ht}_T(t) = \mathrm{dom}(t)$.

Notation. If T is a subtree of ${}^{<\kappa}\kappa$ and $1 < n < \omega$, then we let $T^{\otimes n}$ denote the $\otimes$ of n disjoint copies of T. That is,

$$T^{\otimes n} = \{(t_0, \ldots, t_{n-1}) \in {}^nT : \mathrm{ht}(t_0) = \cdots \mathrm{ht}(t_{n-1})\},$$

and is ordered as follows:

$$(t_0, \ldots, t_{n-1}) \leq (s_0, \ldots, s_{n-1}) \quad \text{iff} \quad t_i \subseteq s_i \text{ for all } i < n.$$

We also write $T \otimes T$ when $n = 2$, and we let $T^{\otimes 1} = T$.

By definition, a subtree of $T^{\otimes n}$ is a downwards closed subset $S \subseteq {}^nT$ such that

$$\mathrm{ht}(s_0) = \ldots = \mathrm{ht}(s_{n-1}) \quad \text{for all} \quad (s_0, \ldots, s_{n-1}) \in S,$$

equipped with the induced ordering. We will also confuse such trees with their domains.

If S is a subtree of $({}^{<\kappa}\kappa)^{\otimes n}$ and $s = (s_0, \ldots, s_{n-1}) \in S$, then we use the notation $\mathrm{ht}(s) = \mathrm{ht}_S(s) = \mathrm{dom}(s_0)$.

Definition 1.27. Given a subtree S is a subtree of $({}^{<\kappa}\kappa)^{\otimes n}$ where $1 \leq n < \omega$, we let

$$[S] = \{(x_0, \ldots, x_{n-1}) \in {}^n({}^\kappa\kappa) : (x_0{\restriction}\alpha, \ldots x_{n-1}{\restriction}\alpha) \in S \text{ for all } \alpha \in \kappa\}.$$

We identify $[S]$ with the set of κ-branches of S.

Observe that $[S]$ is a closed subset of ${}^n({}^\kappa\kappa)$. Specifically, if T is a subtree of ${}^{<\kappa}\kappa$, then $[T]$ is a closed subset of ${}^\kappa\kappa$, and if S is a subtree of $T^{\otimes n}$, then $[S]$ is a closed n-ary relation on $[T]$ (i.e., $[S]$ is a closed subset of ${}^n[T]$).

Conversely, suppose $R \subseteq {}^n({}^\kappa\kappa)$, where $1 \leq n < \omega$. We let

$$T_R = \{(x_0 \restriction \alpha, \dots, x_{n-1} \restriction \alpha) : (x_0, \dots, x_n) \in R \text{ and } \alpha < \kappa\}.$$

Then we have $[T_R] = \overline{R}$. Thus, R is a closed subset of ${}^n({}^\kappa\kappa)$ if and only if $R = [S]$ for a subtree S of $({}^{<\kappa}\kappa)^{\otimes n}$.

Definition 1.28. Let $1 \leq n < \omega$. We say that a subtree S of $({}^{<\kappa}\kappa)^{\otimes n}$ is *pruned* iff for every $s \in S$, there exists a κ-branch $x \in [S]$ which extends s (that is, if $s = (s_0, \dots, s_{n-1})$ and $x = (x_0, \dots, x_{n-1})$, then $s_i \subseteq x_i$ for all $i < n$).

Clearly, T_R is a pruned tree for all $R \subseteq {}^n({}^\kappa\kappa)$.

Suppose T is a subtree of ${}^{<\kappa}\kappa$ and $t \in T$. We let

$$[t]_T = \{t\} \cup \mathrm{succ}_T(t) = \{u \in T : t \subseteq u\}.$$

In the case $T = {}^{<\kappa}\kappa$, we omit it, i.e., we just write

$$[t] = \{u \in {}^{<\kappa}\kappa : t \subseteq u\}.$$

We use $T_{\restriction t}$ to denote the set of nodes $u \in T$ which are comparable with t.

Definition 1.29. Suppose that T is a subtree of ${}^{<\kappa}\kappa$ and $t \in T$.

(1) t is a *splitting node* of T iff t has at least two direct successors in T.
(2) t is a *cofinally splitting node* of T iff

$$\text{for all } \delta < \kappa \text{ there exists a splitting node } u \in [t]_T \text{ such that } \mathrm{ht}(u) \geq \delta,$$

or equivalently, iff for all $\delta < \kappa$ there exist $u_0, u_1 \in [t]_T$ such that $\mathrm{ht}(u_0), \mathrm{ht}(u_1) > \delta$, $u_0 \restriction \delta = u_1 \restriction \delta$ and $u_0 \perp u_1$.

Definition 1.30. Given a subtree T of ${}^{<\kappa}\kappa$, we say that T is a *$<\kappa$-closed tree* iff the partial order $\langle T, \supseteq \rangle$ is $<\kappa$-closed, i.e., iff every increasing sequence in T of length $< \kappa$ has an upper bound in T.

CHAPTER 2

Perfect sets and games

In the first part of the chapter, we consider different generalizations of the notions of perfectness and of scatteredness for the κ-Baire space ${}^{\kappa}\kappa$ associated to an uncountable cardinal $\kappa = \kappa^{<\kappa}$. The concept of γ-perfectness for infinite ordinals $\gamma \leq \kappa$ and subsets X of the κ-Baire space was first introduced by Jouko Väänenen, based on a game of length γ played on X [51]. A stronger notion of κ-perfectness for subsets of ${}^{\kappa}\kappa$ is also widely used and corresponds to a notion of κ-perfectness for subtrees of ${}^{<\kappa}\kappa$ (see Definition 2.1). Concepts of γ-perfectness and γ-scatteredness (where $\omega \leq \gamma \leq \kappa$) for subtrees T of ${}^{<\kappa}\kappa$ which correspond more closely to Väänänen's notions can be defined based on versions of cut-and-choose games played on the trees T. (In the $\gamma < \kappa$ case, these games and notions were introduced in [11].) In the classical descriptive set theory, all these definitions lead to equivalent notions of perfectness and scatteredness for the Baire space. However, this is no longer the case in the uncountable setting.

In Section 2.1, we detail connections between these notions of perfectness, scatteredness and the underlying games. Our observations lead to equivalent characterizations of the κ-perfect set property for closed subsets of the κ-Baire space in terms of the games considered here.

In particular, we show that Jouko Väänänen's generalized Cantor-Bendixson theorem [51] is in fact equivalent to the κ-perfect set property for closed subsets of the κ-Baire space. The consistency of this Cantor-Bendixson theorem was originally obtained in [51] relative to the existence of a measurable cardinal above κ. In [11], this statement is shown to hold after Lévy-collapsing an inaccessible cardinal to κ^+.

In Section 2.2, we discuss how the Cantor-Bendixson hierarchy can be generalized for subtrees T of ${}^{<\kappa}\kappa$. This is done by considering modified versions, associated to trees without κ-branches, of the games studied in Section 2.1. Thus, in the uncountable setting, trees without κ-branches play a role analogous to that of ordinals in the classical setting. The methods in Section 2.1 are similar to those used in [51], where the Cantor-Bendixson hierarchy was generalized for subsets of the κ-Baire space. In Chapter 3, we will obtain results, in a more general case, about how the levels of these different generalized Cantor-Bendixson hierarchies compare to each other.

In the last part of the chapter, we study notions of density in itself for the κ-Baire space which correspond to the notions of perfectness considered in the previous sections. We show that the statement

> "every subset of the κ-Baire space of cardinality at least κ^+ has a κ-dense in itself subset"

follows from a hypothesis which is consistent assuming the consistency of the existence of a weakly compact cardinal above κ. Previously, the above statement was known to be consistent relative to the existence of a measurable cardinal above κ [51, Theorem 1].

Many of the proofs in this chapter are based on simple observations or are modifications of known arguments. Nevertheless, the author feels that when combined, they may shed light on interesting connections between the concepts studied here.

We assume that κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$ throughout the chapter, unless otherwise mentioned.

2.1 Perfect and scattered subsets of the κ-Baire space

The notion of perfectness was originally generalized for subsets of the κ-Baire space by Jouko Väänänen [51], based on games of length $\omega \leq \gamma \leq \kappa$ (see Definitions 2.2 and 2.3 below). There is also another widely used definition of κ-perfectness which leads to a slightly stronger notion (see e.g. [39, 23, 8, 31]; see also Definition 2.1 and Example 2.6). The two notions of κ-perfectness are often interchangeable. For instance, they lead to equivalent definitions of the κ-perfect set property or of the κ-Silver dichotomy. More generally, the two concepts are interchangeable in questions dealing with the existence of κ-perfect independent sets with respect to families of finitary relations on the κ-Baire space. (See Proposition 2.5 and Corollary 2.10 below.)

In this work, we use the definition of κ-perfectness (and of γ-perfectness when $\omega \leq \gamma \leq \kappa$) given in [51]. In order to avoid ambiguity, we use the phrase "strongly κ-perfect" for the stronger notion, the definition of which is given right below.

Recall that a subtree T of ${}^{<\kappa}\kappa$ is *$<\kappa$-closed* if every increasing sequence in T of length $< \kappa$ has an upper bound in T. A node $t \in T$ is a *splitting node* of T if t has at least two direct successors in T.

Definition 2.1. Suppose κ is an infinite cardinal with $\kappa = \kappa^{<\kappa}$.

(1) We say that a subtree T of ${}^{<\kappa}\kappa$ is a *strongly κ-perfect tree* if it is $<\kappa$-closed and its set of splitting nodes is cofinal (i.e. every node of T is extended by a splitting node of T).

(2) We say a subset X of ${}^{\kappa}\kappa$ is a *strongly κ-perfect set* if $X = [T]$ for a strongly κ-perfect tree T.

Thus, X is strongly κ-perfect iff X is closed and T_X is a κ-perfect tree (where $T_X = \{x \restriction \alpha : x \in X, \alpha < \kappa\}$ is the tree of initial segments of elements of X.) Clearly, strong ω-perfectness is equivalent to perfectness for subtrees of ${}^{<\omega}\omega$ and for subsets of the Baire space ${}^{\omega}\omega$. (Recall e.g. from [21] that a subtree $T \subseteq {}^{<\omega}\omega$ is defined to be a perfect tree iff its set of splitting nodes is cofinal, and that a subset X of the Baire space ${}^{\omega}\omega$ is perfect if and only if $X = [T]$ for a perfect tree T.)

2.1.1 Väänänen's perfect set game

We now turn to the notion of γ-perfectness, for sets $X \subseteq {}^{\kappa}\kappa$ and ordinals $\omega \leq \gamma \leq \kappa$, as it was defined by Jouko Väänänen in [51]. This notion is based on the following game.

Definition 2.2 (from [51])**.** Suppose $X \subseteq {}^{\kappa}\kappa$ and $\gamma \leq \kappa$. The game $\mathcal{V}_\gamma(X)$, of length γ, is played as follows.

I		δ_0		δ_1	$\dots$		δ_α	$\dots$
II	x_0		x_1		$\dots$	x_α		$\dots$

In each round, player **II** first chooses an element $x_\alpha \in X$. Then, player **I** chooses an ordinal $\delta_\alpha < \kappa$ (and thus chooses a basic open neighborhood of x_α).

Player **I** has to choose δ_α so that $\delta_\beta < \delta_\alpha$ for all $\beta < \alpha$, and player **II** has to choose x_α in such a way that for all $\beta < \alpha$,

$$x_\beta \restriction \delta_\beta = x_\alpha \restriction \delta_\beta \text{ and } x_\alpha \neq x_\beta.$$

Player **II** wins this run of the game if she can play legally in all rounds $\alpha < \gamma$; otherwise player **I** wins.

For an arbitrary $x \in {}^\kappa\kappa$, the game $\mathcal{V}_\gamma(X, x)$ is defined just like $\mathcal{V}_\gamma(X)$, except player **II** has to start the game with $x_0 = x$ (and thus $x_0 \notin X$ is allowed).

We note that the definition of $\mathcal{V}_\gamma(X, x)$ given here is slightly different from but equivalent to the one in [51].

Definition 2.3 (from [51])**.** Let $\omega \leq \gamma \leq \kappa$ and $X \subseteq {}^\kappa\kappa$. The γ-*kernel* of X is defined to be

$$\mathrm{Ker}_\gamma(X) = \{x \in {}^\kappa\kappa : \text{ player } \mathbf{II} \text{ has a winning strategy in } \mathcal{V}_\gamma(X, x)\}.$$

A nonempty set X is γ-*perfect* iff $X = \mathrm{Ker}_\gamma(X)$.

Let $X \subseteq {}^\kappa\kappa$. Notice that $\mathrm{Ker}_\gamma(X)$ is closed and is a subset of $\overline{X}$. Thus, X is a γ-perfect set iff X is closed and player **II** has a winning strategy in $\mathcal{V}_\gamma(X, x)$ for all $x \in X$. The set $\mathrm{Ker}_\gamma(X)$ contains all γ-perfect subsets of X. In the $\gamma = \omega$ case, X is ω-perfect if and only if X is a perfect set in the original sense (i.e., iff X is closed and has no isolated points).

By the Gale-Stewart theorem, $\mathcal{V}_\omega(X, x)$ (and $\mathcal{V}_\omega(X)$) is determined for all $X \subseteq {}^\kappa\kappa$ and $x \in X$. However, this may not remain true for $\mathcal{V}_\gamma(X, x)$ if $\gamma > \omega$. See [51, p. 189 and Theorem 2] for counterexamples; see also [11, Section 1.5].

It is not hard to see that $\mathrm{Ker}_\kappa(X)$ is a κ-perfect set, and, more generally $\mathrm{Ker}_\gamma(X)$ is a γ-perfect set whenever γ is an indecomposable ordinal (i.e. $\alpha + \gamma = \gamma$ for all $\alpha < \gamma$). However, this is not necessarily the case for ordinals of the form $\gamma + 1$, where γ is indecomposable, as the next example shows.

Example 2.4. For all infinite $\delta \leq \kappa$, let

$$Z_\delta = \{z \in {}^\kappa 2 : \text{the order type of } \{\alpha : z(\alpha) = 0\} \text{ is } < \delta\}.$$

If γ is indecomposable, then $\mathrm{Ker}_{\gamma+1}(Z_{\gamma+1}) = Z_\gamma$, and therefore $\mathrm{Ker}_{\gamma+1}(Z_{\gamma+1})$ is not $(\gamma+1)$-perfect (it is, however, γ-perfect).

We remark that on [51, p. 189], this example is used to show that γ-perfectness implies $(\gamma+1)$-perfectness if and only if γ is a decomposable ordinal.

Notice that a strongly κ-perfect set is also a κ-perfect set. More generally, a closed set $X \subseteq {}^\kappa\kappa$ that is a union of strongly κ-perfect sets is κ-perfect. By Proposition 2.5 below, the converse also holds. We note that, in essence, this connection between κ-perfectness and strong κ-perfectness was observed already in [51] (see the proofs of Proposition 1 and Lemma 1 therein). A different formulation of item (1) below can also be found in [11] (see Proposition 1.2.12 therein). See also [39, Lemma 2.5].

Proposition 2.5 (essentially [51], [11])**.** *Let X be a closed subset of the κ-Baire space.*

(1)
$$\mathrm{Ker}_\kappa(X) = \bigcup\{Z \subseteq X : \ Z \textit{ is a strongly } \kappa\textit{-perfect set}\}.$$

(2) X is a κ-perfect set if and only if there exists a collection $\{X_i : i \in I\}$ of strongly κ-perfect sets such that $X = \bigcup_{i\in I} X_i$.

In particular, a κ-perfect set has cardinality 2^κ. Proposition 2.5 will follow from Proposition 2.69 below (see Corollary 2.70). We give a sketch of the proof below.

Proof (sketch). Item (2) follows immediately from item (1). To see item (1), suppose $Z \subseteq X$ is strongly κ-perfect and suppose $x \in Z$. Let $T = T_Z$. It is straightforward to construct a winning strategy τ for player **II** in $\mathcal{V}_\kappa(Z, x)$, using the fact that the tree T is strongly κ-perfect. Player **II** uses the fact that the set of splitting nodes of T is cofinal to define her moves in successor rounds of the game, and the $<\kappa$-closure of T to define her moves in limit rounds of the game. Then τ is also a winning strategy for player **II** in $\mathcal{V}_\kappa(X, x)$, and so $x \in \mathrm{Ker}_\kappa(X)$.

Conversely, suppose $x \in X \cap \mathrm{Ker}_\kappa(X)$. Let τ be a winning strategy for player **II** in $\mathcal{V}_\kappa(X, x)$. A strongly κ-perfect tree $T \subseteq T_X$ can be constructed by having player **II** use τ repeatedly in response to different partial plays of player **I**. The nodes of T will be initial segments of moves of player **II**. (For details on the construction, see the proof of Proposition 2.69.) The set $Z = [T]$ will be a strongly κ-perfect subset of X with $x \in Z$. □

The following example witnesses that the two notions of κ-perfectness do not coincide if κ is uncountable. It is a straightforward generalization from the $\kappa = \omega_1$ case of an exmaple of Taneli Huuskonen's.

Example 2.6 (Huuskonen, [51])**.** For a cardinal $\omega \leq \mu < \kappa$, let

$$X_\mu = \{x \in {}^\kappa 3 : |\{\alpha < \kappa : x(\alpha) = 2\}| < \mu\}.$$

Then X_μ is a κ-perfect set which is not strongly κ-perfect.

Let X be a subset of the κ-Baire space. By Proposition 2.5, X has a κ-perfect subset if and only if X has a strongly κ-perfect subset. Below are some further equivalent formulations of this requirement which will be utilized in this work.

Definition 2.7. An map $e : {}^{<\kappa}2 \to {}^{<\kappa}\kappa$ is a *perfect embedding* iff for all $t, u \in {}^{<\kappa}2$, we have

(i) $t \subseteq u$ implies $e(t) \subseteq e(u)$ and
(ii) $e(t^\frown 0) \perp e(t^\frown 1)$.

The perfect embedding e is *continuous* iff $e(t) = \bigcup\{e(t\restriction\alpha) : \alpha < \mathrm{ht}(t)\}$ for all $t \in {}^{<\kappa}\kappa$ such that $\mathrm{ht}(t) \in \mathrm{Lim}$.

Notice that if e is perfect embedding, then e is injective and $t \perp u$ implies $e(t) \perp e(u)$ for all $t, u \in {}^{<\kappa}2$. The following observation also clearly holds.

Claim 2.8. *Suppose $e : {}^{<\kappa}2 \to T$ is a perfect embedding into a subtree T of ${}^{<\kappa}\kappa$. Then*

$$T_e = \{t \in {}^{<\kappa}\kappa : t \subseteq e(u) \text{ for some } u \in {}^{<\kappa}2\}$$

is a strongly κ-perfect subtree of T. Conversely, if T has a strongly κ-perfect subtree, then there exists a continuous perfect embedding $e : {}^{<\kappa}2 \to T$.

Proposition 2.5, Claim 2.8 and [31, Lemma 2.9] and [8, Proposition 2] yield the following reformulations of a set containing a κ-perfect subset. We will often use the equivalence of these statements in our later arguments. In particular, we will typically use statements (3) and (4) below (and statements (1) and (2) in Corollary 2.10) interchangeably.

Lemma 2.9. *The following statements are equivalent for any subset X of ${}^\kappa\kappa$.*

(1) There exists a continous perfect embedding e such that $[T_e] \subseteq X$.
(2) There exists a perfect embedding e such that $[T_e] \subseteq X$.
(3) X contains a κ-perfect subset.
(4) X contains a strongly κ-perfect subset.
(5) There exists a continuous injection $\iota : {}^\kappa 2 \to X$.
(6) There exists a Borel injection $\iota : {}^\kappa 2 \to X$.

Proof. The first four statements are equivalent by Proposition 2.5 and Claim 2.8. It is clear that they imply item (5), and that item (5) implies item (6). For the implications (5)⇒(1) and (6)⇒(1), we refer the reader to the proofs of [31, Lemma 2.9] and [8, Proposition 2]. □

The equivalence of items (3)-(6) above imply that the existence of κ-perfect independent subsets w.r.t. families of finitary relations can be reformulated as follows.

Corollary 2.10. *Suppose $X \subseteq {}^\kappa\kappa$ and $\mathcal{R}$ is a family of finitary relations on X. Then the following are equivalent.*

(1) X contains a κ-perfect $\mathcal{R}$-independent subset.
(2) X contains a strongly κ-perfect $\mathcal{R}$-independent subset.
(3) There exists a continuous injection $\iota : {}^\kappa 2 \to X$ such that $\operatorname{ran}(\iota)$ is $\mathcal{R}$-independent.
(4) There exists a Borel injection $\iota : {}^\kappa 2 \to X$ such that $\operatorname{ran}(\iota)$ is $\mathcal{R}$-independent.

Recall that a topological space X is *scattered* iff every subset $Y \subseteq X$ contains an isolated point. The game $\mathcal{V}_\gamma(X, x)$ can also be used to generalize the concept of scatteredness for subsets X of the κ-Baire space.

Definition 2.11 (from [51])**.** Suppose $X \subseteq {}^\kappa\kappa$ and $\omega \leq \gamma \leq \kappa$. The γ-scattered part of X is defined to be

$$\operatorname{Sc}_\gamma(X) = \{x \in X : \text{ player } \mathbf{I} \text{ has a winning strategy in } \mathcal{V}_\gamma(X, x)\}.$$

The set X is *γ-scattered* iff $X = \operatorname{Sc}_\gamma(X)$.

Thus, X is γ-scattered if and only if player **I** wins $\mathcal{V}_\gamma(X)$. Observe that $\operatorname{Sc}_\gamma(X)$ is a relatively open and scattered subset of X. The set Z_δ defined in Example 2.4 is $\delta + 1$-scattered but δ-perfect [51].

Proposition 2.12 (Proposition 3 in [51]). *Let $X \subseteq {}^\kappa\kappa$. If $|\overline{X}| \leq \kappa$, then X is κ-scattered.*

Proof. Suppose $\overline{X} = \{y_\alpha : \alpha < \kappa\}$ and let $x \in X$. The strategy of player **I** in $\mathcal{V}_\kappa(X, x)$ is to choose δ_α in each round α in such a way that $x_\alpha \restriction \delta_\alpha \neq y_\alpha \restriction \delta_\alpha$ holds if $x_\alpha \neq y_\alpha$, and $x_\alpha \restriction \delta_\alpha \neq y_{\alpha-1} \restriction \delta_\alpha$ also holds if $\alpha \in \text{Succ}$ and $x_{\alpha-1} = y_{\alpha-1}$. Suppose that player **II** wins a run of $\mathcal{V}_\kappa(X, x)$ where player **I** uses this strategy (i.e., suppose she can play legally in all rounds). Let $x \in {}^\kappa\kappa$ be the function determined by this run, i.e., $x = \bigcup_{\alpha<\kappa} x_\alpha \restriction \delta_\alpha$. Then $x \in \overline{X}$ although $x \neq y_\alpha$ for all $\alpha < \kappa$, which is a contradiction. □

By [51, Theorem 3], it is consistent with GCH that the converse of Proposition 2.12 does not hold in the $\kappa = \omega_1$ case (assuming the consistency of ZFC).

The converse of Proposition 2.12 is implied by the κ-perfect set property for closed subsets of the κ-Baire space and is therefore consistent relative to the existence of an inaccessible cardinal $\lambda > \kappa$ (see Subsection 2.1.2 below). The consistency of the converse (in the $\kappa = \omega_1$ case, relative to the existence of a measurable cardinal) was first obtained in [51]; it follows from Theorem 4 therein.

The following example shows that it is not enough to assume that $|X| \leq \kappa$ in Proposition 2.12.

Example 2.13. Let

$$Y_0 = \{y \in {}^{<\kappa}2 : \text{there exists } \alpha < \kappa \text{ such that } y(\alpha) = 0 \text{ whenever } \alpha < \beta < \kappa\}.$$

Clearly, $|Y_0| = \kappa$ and $Y_0 \subseteq \text{Ker}_\kappa(Y_0)$.

We refer the reader to [51, p. 189 and 192] for an example of a closed set $X \subseteq {}^{\omega_1}\omega_1$ of cardinality $|X| = 2^\omega$ such that $\text{Sc}_\gamma(X) = \emptyset$ for all $\gamma < \omega_1$.

2.1.2 The κ-perfect set property and Väänänen's Cantor-Bendixson theorem

We say that the *κ-perfect set property* holds for a subset X of the κ-Baire space if either $|X| \leq \kappa$ or X has a κ-perfect subset. We let $\text{PSP}_\kappa(X)$ denote the statement that the κ-perfect set property holds for X. (Note that the notions of κ-perfect sets and strongly κ-perfect sets are interchangeable in this definition.)

For a collection Γ of subsets of the κ-Baire space, $\text{PSP}_\kappa(\Gamma)$ denotes the statement that $\text{PSP}_\kappa(X)$ holds for all $X \in \Gamma$.

Recall that $\mathcal{C}_\kappa$ denotes the collection of closed subsets of the κ-Baire space.

Remark 2.14. The statement $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ is equiconsistent with the existence of an inaccessible cardinal $\lambda > \kappa$, by results in [9,19,39] recalled below (and so is the statement that $\mathrm{PSP}_\kappa(X)$ holds for all sets $X \subseteq {}^\kappa\kappa$ definable from a κ-sequence of ordinals).

This fact is also of interest for the purposes of this thesis because $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ is a special case of many of the dichotomies studied here. In the sequel, we may sometimes use this fact without explicitly referring to these results or this remark.

A subtree T of ${}^{<\kappa}\kappa$ is defined to be a *weak κ-Kurepa tree* if $\mathrm{ht}(T) = \kappa$, $|[T]| > \kappa$ and the α^{th} level $T \cap {}^\alpha\kappa$ of T is of size $\leq |\alpha|$ for stationarily many $\alpha < \kappa$. If T is a weak κ-Kurepa tree, then the κ-perfect set property fails for $[T]$; see [9, Section 4.2] or [30, Section 7]. And so, the existence of weak κ-Kurepa trees implies that the κ-perfect set property cannot hold for all closed subsets of the κ-Baire space. Specifically, $V = L$ implies that the $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ fails for all uncountable regular κ, by [8, Lemma 4]. We note that the idea of using Kurepa trees to obtain counterexamples to the $\aleph_1$-perfect set property had already appeared in [51] and [34].

Thus, $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ implies that there are no κ-Kurepa trees, and therefore also implies that κ^+ is an inaccessible cardinal in L by a result of Robert Solovay; see [19, Sections 3 and 4].

Conversely, by a result of Philipp Schlicht, the κ-perfect set property holds for all subsets of the κ-Baire space which are definable from a κ-sequence of ordinals after Lévy-collapsing an inaccessible $\lambda > \kappa$ to κ^+ [39]. (In the case of $\mathrm{PSP}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$, this result already follows from a simpler argument also due to Philipp Schlicht; see [30, Proposition 9.9]. It is also a special case of our Theorem 3.14 below.)

In [51], Jouko Väänänen obtained the consistency (relative to the existence of a measurable cardinal above κ), of the following generalized Cantor-Bendixson theorem for closed subsets of the κ-Baire space:

every set $X \in \mathcal{C}_\kappa$ can be written as a disjoint union

$$X = \mathrm{Ker}_\kappa(X) \cup \mathrm{Sc}_\kappa(X)\,, \quad \text{where} \quad |\mathrm{Sc}_\kappa(X)\,| \leq \kappa. \tag{2.1}$$

This property may also be seen as a strong form of the determinacy of the games $\mathcal{V}_\kappa(X, x)$ for closed sets $X \in \mathcal{C}_\kappa$ and $x \in X$.

A straightforward generalization from the $\kappa = \omega_1$ case of [51, Theorem 4] shows that the set theoretical hypothesis $\mathrm{I}^-(\kappa)$ implies the Cantor-Bendixson theorem (2.1). The hypothesis $\mathrm{I}^-(\kappa)$ is equiconsistent with the existence of a measurable cardinal above κ. (See Definition 2.74 for the definition of $\mathrm{I}^-(\kappa)$, and see also the remarks following it.)

By a result of Geoff Galgon's, (2.1) holds already after Lévy-collapsing an inaccessible cardinal $\lambda > \kappa$ to κ^+ [11, Proposition 1.4.4].

Motivated by these results, we show in Proposition 2.16 below that the Cantor-Bendixson theorem (2.1) is in fact equivalent to $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$. While the proof is based on a few simple observations, it may be interesting to note that (2.1) follows already from $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ and does not need other combinatorial properties of $\mathrm{I}^-(\kappa)$ or of the Lévy-collapse.

The notion of κ-condensation points, defined below, will be useful in the proof of the equivalence of the properties in Proposition 2.16. Its relation to $\mathrm{Ker}_\kappa(X)$ and $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ noted in Proposition 2.16 may also be interesting in its own right.

Definition 2.15. If $X \subseteq {}^\kappa\kappa$ and $x \in X$, then x is a *κ-condensation point of X* iff

$$|X \cap N_{x\restriction\alpha}| > \kappa \quad \text{for all } \alpha < \kappa.$$

We let $CP_\kappa(X)$ denote the set of κ-condensation points of X.

Proposition 2.16. *The following statements are equivalent.*

(1) $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ *holds.*

(2) If $X \in \mathcal{C}_\kappa$*, then* $\mathrm{Ker}_\kappa(X) = CP_\kappa(X)$*, i.e.,* $\mathrm{Ker}_\kappa(X)$ *is the set of* κ*-condensation points of* X*.*

(3) Every $X \in \mathcal{C}_\kappa$ *can be written as a disjoint union*

$$X = \mathrm{Ker}_\kappa(X) \cup \mathrm{Sc}_\kappa(X)\,, \quad \textit{where} \quad |\mathrm{Sc}_\kappa(X)| \leq \kappa. \tag{2.1}$$

Proposition 2.16 implies that the statements (2) and (3) are also equiconsistent with the existence of an inaccessible cardinal above κ.

The proof of Proposition 2.16 is based on the following observation (which holds whether or not $\mathrm{PSP}_\kappa(X)$ is assumed).

Claim 2.17. *If X is a closed subset of ${}^\kappa\kappa$, then*

$$\mathrm{Ker}_\kappa(X) \subseteq CP_\kappa(X)\,; \qquad X - CP_\kappa(X) \subseteq \mathrm{Sc}_\kappa(X)\,; \qquad |X - CP_\kappa(X)| \leq \kappa.$$

Proof. First, suppose $x \in \mathrm{Ker}_\kappa(X)$. If $\delta < \kappa$, then $x \in \mathrm{Ker}_\kappa(X \cap N_{x\restriction\delta})$ and therefore $|X \cap N_{x\restriction\delta}| = 2^\kappa$ by Proposition 2.5. Therefore $x \in CP_\kappa(X)$.

If $x \in X$ is not a condensation point of X, then there exists an $\alpha(x) < \kappa$ such that $|X \cap N_{x\restriction\alpha(x)}| \leq \kappa$. This implies, by Proposition 2.12, that $x \in \mathrm{Sc}_\kappa\big(X \cap N_{x\restriction\alpha(x)}\big)$ and therefore $x \in \mathrm{Sc}_\kappa(X)$. This also implies the last statement of the claim because there are at most $\kappa^{<\kappa} = \kappa$ possibilities for $x\restriction\alpha(x)$. □

Proof of Proposition 2.16. Clearly, the generalized Cantor-Bendixson theorem (2.1) implies that $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ holds.

Now, assume $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$, and let $x \in CP_\kappa(X)$. For all $\delta < \kappa$, the set $X \cap N_{x\restriction\delta}$ is closed and has cardinality $> \kappa$, and therefore contains a κ-perfect subset X_δ, by $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ (or more specifically, by $\mathrm{PSP}_\kappa(X \cap N_{x\restriction\delta})$). Player **II** has the following winning strategy in $\mathcal{V}_\kappa(X, x)$: if the first move of player **I** is $\delta_0 < \kappa$, then player **II** uses her winning strategy in $\mathcal{V}_\kappa(X_{\delta_0})$ to define her moves in rounds $\alpha \geq 1$ of $\mathcal{V}_\kappa(X, x)$. Thus, by Claim 2.17, $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ implies that $\mathrm{Ker}_\kappa(X) = CP_\kappa(X)$ for all $X \in \mathcal{C}_\kappa$.

Lastly, suppose $X \in \mathcal{C}_\kappa$ and $\mathrm{Ker}_\kappa(X) = CP_\kappa(X)$. Then, by the fact that $\mathrm{Ker}_\kappa(X)$ and $\mathrm{Sc}_\kappa(X)$ are disjoint and by Claim 2.17, we also have $\mathrm{Sc}_\kappa(X) = X - CP_\kappa(X)$. Thus,

$$X = \mathrm{Ker}_\kappa(X) \cup \mathrm{Sc}_\kappa(X) \quad \text{and} \quad |\mathrm{Sc}_\kappa(X)| = |X - CP_\kappa(X)| \leq \kappa.$$

This shows that item (2) implies the generalized Cantor-Bendixson theorem (2.1). □

Remark 2.18. The argument in the proof of Proposition 2.16 also shows that the following statements are equivalent for any closed set $X \subseteq {}^\kappa\kappa$.

(1) $\mathrm{PSP}_\kappa(X \cap N_s)$ holds for all $s \in {}^{<\kappa}\kappa$.

(2) $\mathrm{Ker}_\kappa(X) = CP_\kappa(X)$, or in other words, $\mathrm{Ker}_\kappa(X)$ is the set of κ-condensation points of X.

(3) $X = \mathrm{Ker}_\kappa(X) \cup \mathrm{Sc}_\kappa(X)$, and $|\mathrm{Sc}_\kappa(X)| \leq \kappa$.

2.1.3 A cut-and-choose game

The κ-perfect set property can be characterized by the following cut-and-choose game $\mathcal{G}^*_\kappa(X)$ of length κ. The game $\mathcal{G}^*_\kappa(X)$ is the straightforward generalization of the perfect set game for subsets of reals; see e.g. [21, Section 21.A]). For subsets X of the κ-Cantor space ${}^\kappa 2$, the game $\mathcal{G}^*_\kappa(X)$ is equivalent to the κ-perfect set game $\mathcal{G}^*_{\kappa 2}(X)$ studied in [23]. (However, we have reversed the role of the players for technical reasons.) The equivalence of the two games can be shown using a straightforward modification of the argument in the countable case (see e.g. [21, Exercise 21.3]), but it also follows from [23, Lemma 7.2.2] and Proposition 2.20 right below.

Definition 2.19. For a subset X of the κ-Baire space, the game $\mathcal{G}^*_\kappa(X)$ of length κ is played as follows.

I		i_0		i_1	$\dots$		i_α	$\dots$
II	u^0_0, u^1_0		u^0_1, u^1_1		$\dots$	u^0_α, u^1_α		$\dots$

Player **II** starts each round by playing $u^0_\alpha, u^1_\alpha \in {}^{<\kappa}\kappa$ (and thus chooses basic open subsets of the κ-Baire space). Player **I** then plays $i_\alpha \in 2$ (and thus chooses between u^0_α and u^1_α). Player **II** has to satisfy the requirement that

$$u^0_\alpha, u^1_\alpha \supseteq u^{i_\beta}_\beta$$

for all $\beta < \alpha$. We also require that

$$\begin{aligned} u^0_\alpha \perp u^1_\alpha \quad & \text{if } \alpha \in \mathrm{Succ}, \text{ and} \\ u^0_\alpha = u^1_\alpha \quad & \text{if } \alpha \in \mathrm{Lim} \cup \{0\}. \end{aligned}$$

The nodes $u^{i_\alpha}_\alpha$ produced during a given run define an element $x = \bigcup_{\alpha<\kappa} u^{i_\alpha}_\alpha$ of the κ-Baire space. Player **II** wins the run if $x \in X$.

For a node $t \in {}^{<\kappa}\kappa$, the game $\mathcal{G}^*_\kappa(X,t)$ is defined just as $\mathcal{G}^*_\kappa(X)$, except player **II** has to start the game with $u^0_0 = u^1_0 = t$.

Note that we would obtain an equivalent game if we required $u^0_\alpha \perp u^1_\alpha$ instead at limit rounds and round 0. In the case of $\mathcal{G}^*_\kappa(X)$, the requirement $u^0_\alpha = u^1_\alpha$ in these rounds is a technical detail which will be convenient later.

The next proposition shows that for all sets $X \subseteq {}^\kappa\kappa$, the κ-perfect set property $\mathrm{PSP}_\kappa(X)$ is equivalent to the determinacy of $\mathcal{G}^*_\kappa(X)$. Its analogue for the game $\mathcal{G}^*_{^\kappa 2}(X)$ and sets $X \subseteq {}^\kappa 2$ appears in [23, Lemma 2.2.2].

Proposition 2.20 (essentially Lemma 7.2.2 of [23] for the game $\mathcal{G}^*_\kappa(X)$)**.** *Let X be a subset of the κ-Baire space.*

(1) Player **I** *has a winning strategy in $\mathcal{G}^*_\kappa(X)$ iff $|X| \leq \kappa$.*
(2) Player **II** *has a winning strategy in $\mathcal{G}^*_\kappa(X)$ iff X contains a κ-perfect subset.*

Thus, $\mathrm{PSP}_\kappa(X)$ *holds if and only if $\mathcal{G}^*_\kappa(X)$ is determined.*

Proposition 2.20 is a special case of Proposition 3.20 below (and is also stated as Corollary 3.23). We sketch the proofs of item (2) and the easier direction of item (1).

Proof (sketch). Item (2) is implied by the following observation. A winning strategy for player **II** in $\mathcal{G}^*_\kappa(X)$ determines, in a natural way, a perfect embedding $e : {}^{<\kappa}2 \to {}^{<\kappa}\kappa$ such that $[T_e] \subseteq X$. Conversely, a perfect embedding e with $[T_e] \subseteq X$ determines a winning strategy for player **II** in $\mathcal{G}^*_\kappa(X)$. (See Remark 2.6 for a more detailed formulation of this observation.)

Now, suppose that $X = \{x_\alpha : \alpha < \kappa\}$. Player **I** can play in successor rounds $\alpha + 1$ of $\mathcal{G}^*_\kappa(X)$ in a way that guarantees the following: if x is the element of ${}^\kappa\kappa$ produced during a given run, then $x \neq x_\alpha$ for all $\alpha < \kappa$. (More specifically, player **I** can choose $i_{\alpha+1} < 2$ so that $x_\alpha \not\supseteq u^{i_\alpha}_{\alpha+1}$ by the rule $u^0_{\alpha+1} \perp u^1_{\alpha+1}$.) Thus, $x \notin X$ holds whenever x is obtained from a run where player **I** uses this strategy. The converse direction can be shown using a special case of the argument in the proof of Proposition 3.20. This special case of the proof is also analogous to of the argument in [23, Lemma 7.2.2]. □

Remark 2.21. Notice that for any $X \subseteq {}^\kappa\kappa$ and $t \in {}^{<\kappa}\kappa$, the games $\mathcal{G}^*_\kappa(X, t)$ and $\mathcal{G}^*_\kappa(X \cap N_t)$ are equivalent. Thus, by Proposition 2.20,

(1) Player **I** has a winning strategy in $\mathcal{G}^*_\kappa(X, t)$ iff $|X \cap N_t| \leq \kappa$.
(2) Player **II** has a winning strategy in $\mathcal{G}^*_\kappa(X, t)$ iff $X \cap N_t$ contains a κ-perfect subset.

2.1.4 Perfect and scattered trees

In this subsection, we consider notions of κ-perfectness and κ-scatteredness for subtrees T of ${}^{<\kappa}\kappa$ which are given by certain cut-and-choose games played on such trees T. At the end of the subsection, we summarize some equivalent reformulations of the κ-perfect set property for closed subsets of the κ-Baire space which are obtained in this and previous subsections (see Corollary 2.40).

The notions of κ-perfect and κ-scattered trees T are defined with the help of (a reformulation of) the game $\mathcal{G}^*_\kappa([T])$. This leads to a slightly weaker notion of κ-perfectness for trees than the usual one (i.e., strong κ-perfectness; see Corollary 2.27 and Example 2.28). With this weaker notion, the following holds for all subsets $X \subseteq {}^\kappa\kappa$ (see Corollary 2.29):

$$X \text{ is } \kappa\text{-perfect if and only if } X = [T] \text{ for a } \kappa\text{-perfect tree } T.$$

In the case of ordinals $\omega \leq \gamma \leq \kappa$, notions of γ-perfectness and γ-scatteredness for subtrees T of ${}^{<\kappa}\kappa$ and infinite ordinals $\gamma \leq \kappa$ were introduced by Geoff Galgon [11] based on a strong of cut-and-choose game of length γ played on T (see Definitions 2.32 and 2.33).

When $\gamma = \kappa$, two notions of κ-perfectness and κ-scatteredness given by these games are, in fact, equivalent, by Proposition 2.35.

Let T be a subtree of ${}^{<\kappa}\kappa$. Because $[T]$ is a closed set, player **II** wins a given run of $\mathcal{G}^*_\kappa([T])$ if and only if she can play nodes $u^0_\alpha, u^1_\alpha \in T$ legally in all rounds $\alpha < \kappa$. Thus, $\mathcal{G}^*_\kappa([T])$ can be reformulated as the following game $\mathcal{G}^*_\kappa(T)$. (See Proposition 2.24 and its proof.) This reformulation allows for versions of length $\gamma < \kappa$ of the game to also be defined.

Definition 2.22. Let T be a subtree of ${}^{<\kappa}\kappa$, and let $\gamma \leq \kappa$. The game $\mathcal{G}^*_\gamma(T)$ has γ rounds and is played as follows.

I		i_0		i_1	$\dots$		i_α	$\dots$
II	u^0_0, u^1_0		u^0_1, u^1_1		$\dots$	u^0_α, u^1_α		$\dots$

In each round, player **II** first plays nodes $u^0_\alpha, u^1_\alpha \in T$ (and thus chooses basic open subsets of $[T]$). Player **I** then plays $i_\alpha \in 2$ (and thus chooses between u^0_α and u^1_α). Player **II** has to play so that

$$u^0_\alpha, u^1_\alpha \supseteq u^{i_\beta}_\beta$$

for all $\beta < \alpha$. We also require that

$$\begin{aligned} u^0_\alpha &\perp u^1_\alpha \quad \text{if } \alpha \in \mathrm{Succ}, \text{ and} \\ u^0_\alpha &= u^1_\alpha \quad \text{if } \alpha \in \mathrm{Lim} \cup \{0\}. \end{aligned}$$

Player **II** wins a run of the game if she can play legally in all rounds $\alpha < \gamma$; otherwise player **I** wins.

For a node $t \in T$, the game $\mathcal{G}^*_\gamma(T, t)$ is defined just as $\mathcal{G}^*_\gamma(T)$, except player **II** has to start the game with $u^0_0 = u^1_0 = t$.

Definition 2.23. Let T be a subtree of ${}^{<\kappa}\kappa$, and suppose $\omega \le \gamma \le \kappa$.

$$\mathrm{Ker}^*_\gamma(T) = \{t \in T : \text{ player } \mathbf{II} \text{ has a winning strategy in } \mathcal{G}^*_\gamma(T, t)\}.$$
$$\mathrm{Sc}^*_\gamma(T) = \{t \in T : \text{ player } \mathbf{I} \text{ has a winning strategy in } \mathcal{G}^*_\gamma(T, t)\}.$$

We say that nonempty tree T is *κ-perfect* iff $T = \mathrm{Ker}^*_\kappa(T)$. A tree T is *κ-scattered* iff $T = \mathrm{Sc}^*_\kappa(T)$.

Observe that $\mathrm{Ker}^*_\kappa(T)$ is a κ-perfect subtree of T which contains all κ-perfect subtrees of T. In the $\kappa = \omega$ case, a subtree T of ${}^{<\omega}\omega$ is ω-perfect if and only if T is a perfect tree in the classical sense (i.e., iff its set of splitting nodes is cofinal).

Note that if $s \in \mathrm{Sc}^*_\kappa(T)$ and $s \subseteq t \in T$, then $t \in \mathrm{Sc}^*_\kappa(T)$. We let $N(\mathrm{Sc}^*_\kappa(T))$ denote the relatively open subset of $[T]$ determined by $\mathrm{Sc}^*_\kappa(T)$ in the natural way:

$$\begin{aligned} N(\mathrm{Sc}^*_\kappa(T)) &= \bigcup\{N_s : s \in \mathrm{Sc}^*_\kappa(T)\} \cap [T] \\ &= \{x \in [T] : \text{ there exists } s \in \mathrm{Sc}^*_\kappa(T) \text{ such that } s \subseteq x\}. \end{aligned}$$

The game $\mathcal{G}^*_\omega(T)$ is determined by the Gale-Stewart theorem. For $\omega < \gamma \le \kappa$ however, there is no reason why $\mathcal{G}^*_\gamma(T)$ should be determined. By the next proposition and by Proposition 2.20, $\mathcal{G}^*_\kappa(T)$ is determined if and only $\mathrm{PSP}_\kappa([T])$ holds (see Corollary 2.26 below).

Proposition 2.24. *Let T be a subtree of ${}^{<\kappa}\kappa$. Then $\mathcal{G}^*_\kappa([T])$ is equivalent to $\mathcal{G}^*_\kappa(T)$, and $\mathcal{G}^*_\kappa([T], t)$ is equivalent to $\mathcal{G}^*_\kappa(T, t)$ for any node $t \in T$.*

Proof. Observe that, because $[T]$ is a closed set, player **II** wins a given (legal) run $r = \langle (u^0_\alpha, u^1_\alpha), i_\alpha : \alpha < \kappa\rangle$ of $\mathcal{G}^*_\kappa([T])$ if and only if $u^0_\alpha, u^1_\alpha \in T$ for all $\alpha < \kappa$. Thus, a given sequence $r = \langle (u^0_\alpha, u^1_\alpha), i_\alpha : \alpha < \kappa\rangle$ is a run of $\mathcal{G}^*_\kappa([T])$ where player **II** (resp. player **I**) wins iff r is a run of $\mathcal{G}^*_\kappa(T)$ where player **II** (resp. player **I**) wins. □

Recall that $CP_\kappa(X)$ denotes the set of κ-condensation points of a set $X \subseteq {}^\kappa\kappa$.

Corollary 2.25. *Suppose T is a subtree of ${}^{<\kappa}\kappa$.*

(1) $\mathrm{Sc}^*_\kappa(T) = \{t \in T : |[T] \cap N_t| \le \kappa\}$;

(2) $N(\mathrm{Sc}^*_\kappa(T)) = [T] - CP_\kappa([T])$.

(3)
$$\mathrm{Ker}^*_\kappa(T) = \{t \in T : [T] \cap N_t \textit{ has a (strongly) } \kappa\textit{-perfect subset}\} \tag{2.2}$$
$$= \{t \in T : T_{\restriction t} \textit{ contains a strongly } \kappa\textit{-perfect subtree}\}. \tag{2.3}$$

(4) $[\mathrm{Ker}^*_\kappa(T)] \subseteq CP_\kappa([T])$.

Note that by item (2) and Claim 2.17, $|N(\mathrm{Sc}^*_\kappa(T))| \leq \kappa$.

Proof. Item (1) and the equality (2.2) in item (3) follows from Proposition 2.24 and Proposition 2.20. These clearly imply items (2) and (3).

The equality of the sets in (2.2) and (2.3) follows from the observation that $[T'] \subseteq [T]$ implies $T' \subseteq T$ whenever T' is a pruned tree.

We note that it is simple prove that $\mathrm{Ker}^*_\kappa(T)$ is equal to the set in (2.3) directly, using the following observation: if $t \in T$, then winning strategies for player **II** in $\mathcal{G}^*_\kappa(T, t)$ correspond, in a natural way, to perfect embeddings $e : {}^{<\kappa}2 \to T$ such that $e(\emptyset) = t$ (see Remark 2.30 below). □

Corollary 2.26. *The following statements are equivalent.*

(1) $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ *(i.e., the κ-perfect set property holds for all closed subsets of ${}^\kappa\kappa$).*

(2) For all sutrees T of ${}^\kappa\kappa$,
$$T = \mathrm{Ker}^*_\kappa(T) \cup \mathrm{Sc}^*_\kappa(T),$$
or equivalently, $\mathcal{G}^*_\kappa(T, t)$ is determined for all $t \in T$.

Note that a strongly κ-perfect tree is also κ-perfect. More generally, a tree which is the union of strongly κ-perfect trees is also κ-perfect. The converse of this statement also holds, by the corollary below.

Corollary 2.27. *Suppose T is a subtree of ${}^{<\kappa}\kappa$.*

(1)
$$\mathrm{Ker}^*_\kappa(T) = \bigcup\{T' \subseteq T : \ T' \textit{ is a strongly } \kappa\textit{-perfect subtree of } T\}.$$

(2) T is a κ-perfect tree if and only if there exists a collection $\{T_\alpha : \alpha < \kappa\}$ of strongly κ-perfect subtrees of T such that
$$T = \bigcup_{\alpha<\kappa} T_\alpha.$$

Proof. The first statement is a reformulation of item (3) of Corollary 2.25. The second statement follows from and the first one and the assumption $\kappa^{<\kappa} = \kappa$. □

The following example shows that a κ-perfect tree may not be strongly κ-perfect.

Example 2.28 (Huuskonen, [51])**.** This is a reformulation of Example 2.6. For a cardinal $\omega \leq \mu < \kappa$, let

$$T_\mu = \{t \in {}^{<\kappa}3 : |\{\alpha < \kappa : t(\alpha) = 2\}| < \mu\}.$$

Then T_μ is a κ-perfect tree that is not strongly κ-perfect. Note that $[T_\mu] = X_\mu$ (where X_μ is defined as in Example 2.6).

Proposition 2.5 and Corollary 2.27 imply the following.

Corollary 2.29. *For any subtree T of ${}^{<\kappa}\kappa$,*

$$\mathrm{Ker}_\kappa([T]) = [\mathrm{Ker}^*_\kappa(T)]\,.$$

Thus, T is a κ-perfect tree if and only if $[T]$ is a κ-perfect set.

Thus, with the weaker notion of κ-perfectness for trees considered in this subsection, we have that

a set $X \subseteq {}^\kappa\kappa$ is a κ-perfect set if and only if $X = [T]$ for a κ-perfect tree T.

Proof. By Proposition 2.5 and Corollary 2.27,

$$\begin{aligned}\mathrm{Ker}_\kappa([T]) &= \bigcup\{[T'] : T' \text{ is a strongly } \kappa\text{-perfect subtree of } T\} && (2.4)\\ &= \left[\bigcup\{T' : T' \text{ is a strongly } \kappa\text{-perfect subtree of } T\}\right] = [\mathrm{Ker}^*_\kappa(T)]\,. && (2.5)\end{aligned}$$

To see the equality of the sets in (2.4) and (2.5), suppose $x \in [\mathrm{Ker}^*_\kappa(T)]$. Then by (2.5), there exits, for all $\alpha < \kappa$, a strongly κ-perfect tree T_α such that $x\restriction\alpha \in T_\alpha$ and all the nodes of T_α are comparable with $x\restriction\alpha$. The tree $T' = \bigcup_{\alpha<\kappa} T_\alpha$ is a strongly κ-perfect subtree of T such that $x \in [T']$. The other direction is clear. □

By Corollary 2.29, $\mathrm{Ker}_\kappa\big(\overline{X}\big) = [\mathrm{Ker}^*_\kappa(T_X)]$ holds for all subsets X of ${}^\kappa\kappa$. Let W be a set which consists of isolated branches splitting off from each node of ${}^{<\kappa}2$; for instance, let

$$W = \{x \in {}^\kappa 3 : |\alpha < \kappa : x(\alpha) \in 2\}| < \kappa\}.$$

Then W is discrete and therefore κ-scattered. However, $\overline{W} = {}^\kappa 2 \cup W$, implying that $[\mathrm{Ker}^*_\kappa(T_W)] = \mathrm{Ker}_\kappa\big(\overline{W}\big) = {}^\kappa 2$.

Remark 2.30. Let $\omega \leq \gamma \leq \kappa$ and let T be a subtree of ${}^{<\kappa}\kappa$. Winning strategies for player **II** in $\mathcal{G}^*_\gamma(T)$ correspond, in a natural way, to embeddings $e : {}^{<\gamma}2 \to 2$ such that

$$t \subseteq s \text{ implies } e(t) \subseteq e(s) \text{ and } e(s^\frown 0) \perp e(s^\frown 1) \text{ for all } t, s \in {}^{<\gamma}2. \tag{2.6}$$

Winning strategies for player **II** in $\mathcal{G}^*_\gamma(T, t)$ correspond to embeddings $e : {}^{<\gamma}2 \to T$ with

$$e(\emptyset) = t$$

such that (2.6) holds. In particular, winning strategies for player **II** in $\mathcal{G}^*_\kappa(T, t)$ correspond to such perfect embeddings $e : {}^{<\kappa}2 \to T$ with $e(\emptyset) = t$.

We give a more detailed description of this correspondence because it is used in many of our arguments. If τ is a winning strategy for player **II** in $\mathcal{G}^*_\gamma(T)$, then a map $e_\tau : {}^{<\gamma}2 \to T$ satisfying (2.6) can be defined in the following way. Suppose $\alpha < \gamma$ and $s \in {}^{\alpha}2$. Let

$$(u^0_\alpha, u^1_\alpha) = \tau\left(\langle i^s_\beta : \beta < \alpha\rangle\right) \quad \text{where} \quad \begin{array}{ll} i^s_{\beta+1} = s(\beta) & \text{whenever } \beta + 1 < \alpha \text{ and} \\ i^s_\xi = 0 & \text{for all } \xi \in \mathrm{Lim} \cap \alpha. \end{array}$$

Let

$$e_\tau(s) = u^{s(\alpha-1)}_\alpha \quad \text{if } \alpha \in \mathrm{Succ}, \quad \text{and let} \quad e_\tau(s) = u^0_\alpha \quad \text{if } \alpha \in \mathrm{Lim}.$$

If $e : {}^{<\gamma}2 \to T$ satisfies (2.6), then one can define, in a similar way, a winning strategy $\tau(e)$ for player **II** in $\mathcal{G}^*_\gamma(T)$ such that $e_{\tau(e)} = e$.

The example below shows that when $\gamma < \kappa$, the game $\mathcal{G}^*_\gamma(T)$ would not lead to a satisfactory definition of γ-perfectness.

Example 2.31. For $\omega \leq \gamma < \kappa$, let

$$S_\gamma = \{t \in {}^{<\kappa}2 : t(\alpha) = 0 \text{ for all } \alpha \geq \gamma\}.$$

Then $[S_\gamma]$ is a discrete set, but $S_\gamma = \mathrm{Ker}^*_\gamma(S_\gamma)$.

A notion of γ-perfectness and γ-scatteredness of subtrees T of ${}^{<\kappa}\kappa$ was introduced by Geoff Galgon based on a game $\mathcal{G}_\gamma(T)$ [11]. The game $\mathcal{G}_\gamma(T)$ is played like $\mathcal{G}^*_\gamma(T)$, except player **I** also plays ordinals $\delta_\alpha < \kappa$ at the end of each round, in ascending order. The nodes $u^0_{\alpha+1}, u^1_{\alpha+1} \in T$ that player **II** plays in rounds $\alpha + 1$ have to agree on the first δ_α coordinates, i.e., $u^0_{\alpha+1} \restriction \delta_\alpha = u^1_{\alpha+1} \restriction \delta_\alpha$.

Example 2.31 shows that this modification is indeed necessarry for the game to lead to a reasonable notion of γ-perfectness for ordinals $\gamma < \kappa$. When $\gamma = \kappa$, the two games are equivalent, and therefore lead to the same notion of κ-perfectness and κ-scatteredness (see Proposition 2.35 below).

Definition 2.32. (from [11]) Let T be a subtree of ${}^{<\kappa}\kappa$, and let $\gamma \leq \kappa$. The game $\mathcal{G}_\gamma(T)$ has γ rounds and is played as follows.

I		i_0, δ_0		i_1, δ_1	$\dots$		i_α, δ_α	$\dots$
II	u_0^0, u_0^1		u_1^0, u_1^1		$\dots$	u_α^0, u_α^1		$\dots$

In each round, player **II** first plays nodes $u_\alpha^0, u_\alpha^1 \in T$ (and thus chooses basic open subsets of $[T]$). Player **I** then plays ordinals $i_\alpha < 2$ and $\delta_\alpha < \kappa$.

Player **I** has to play δ_α in such a way that $\delta_\alpha > \delta_\beta$ for all $\beta < \alpha$, and Player **II** has to play so that

$$u_\alpha^0, u_\alpha^1 \supseteq u_\beta^{i_\beta}$$

for all $\beta < \alpha$. In successor rounds $\alpha = \alpha' + 1$, player **II** also has to make sure that

$$u_{\alpha'+1}^0 \perp u_{\alpha'+1}^1 \quad \text{and} \quad u_{\alpha'+1}^0 \restriction \delta_{\alpha'} = u_{\alpha'+1}^1 \restriction \delta_{\alpha'}.$$

In rounds $\alpha \in \mathrm{Lim} \cup \{0\}$, she has to play so that $u_\alpha^0 = u_\alpha^1$. Player **II** wins a run of the game if she can play legally in all rounds $\alpha < \gamma$; otherwise player **I** wins.

For a node $t \in T$, the game $\mathcal{G}_\gamma(T, t)$ is defined just as $\mathcal{G}_\gamma(T)$, except player **II** has to start the game with $u_0^0 = u_0^1 = t$.

Definition 2.33 (from [11]). Let T be a subtree of ${}^{<\kappa}\kappa$ and let $\omega \leq \gamma \leq \kappa$. The γ-*kernel* of T is defined to be

$$\mathrm{Ker}_\gamma(T) = \{t \in T : \text{ player } \mathbf{II} \text{ has a winning strategy in } \mathcal{G}_\gamma(T, t)\}.$$

The γ-*scattered part* of T is defined to be

$$\mathrm{Sc}_\gamma(T) = \{t \in T : \text{ player } \mathbf{I} \text{ has a winning strategy in } \mathcal{G}_\gamma(T, t)\}.$$

A nonempty tree T is a γ-*perfect tree* iff $T = \mathrm{Ker}_\gamma(T)$. A tree T is a γ-*scattered tree* iff $T = \mathrm{Sc}_\gamma(T)$.

Note that T is a γ-scattered tree if and only if player **I** wins $\mathcal{G}_\gamma(T)$, and if and only if $\emptyset \in \mathrm{Sc}_\gamma(T)$. We denote by $N(\mathrm{Sc}_\gamma(T))$ the relatively open subset of $[T]$ determined by $\mathrm{Sc}_\gamma(T)$, i.e.,

$$\begin{aligned} N(\mathrm{Sc}_\kappa(T)) &= \bigcup\{N_s : s \in \mathrm{Sc}_\kappa(T)\} \cap [T] \\ &= \{x \in [T] : \text{ there exists } s \in \mathrm{Sc}_\kappa(T) \text{ such that } s \subseteq x\}. \end{aligned}$$

Note that $\mathrm{Ker}_\gamma(T)$ is a subtree of T which contains all γ-perfect subtrees of T. If γ is an indecomposable ordinal, then $\mathrm{Ker}_\gamma(T)$ is a γ-perfect tree. The next example shows that this may not hold for decomposable ordinals.

Example 2.34 (from [51])**.** This is a reformulation of Example 2.4. For any infinite ordinal $\delta \leq \kappa$, let

$$U_\delta = \{u \in {}^{<\kappa}2 : \text{the order type of } \{\alpha : u(\alpha) = 0\} \text{ is } < \delta\}.$$

If γ is indecomposable, then the tree U_γ is γ-perfect but not $(\gamma+1)$-perfect. Therefore $\mathrm{Ker}_{\gamma+1}(U_{\gamma+1}) = U_\gamma$ is not $(\gamma+1)$-perfect.

Note that, if $\omega \leq \delta < \kappa$, then $[U_\delta] = Z_\delta$ (where Z_δ is the set defined in Example 2.4).

The determinacy of the games $\mathcal{G}_\gamma(T,t)$ was investigated in [11]; see Section 1.5 therein. We note that in Geoff Galgon's original definition of these games in [11], the requirement $u^0_\alpha \perp u^1_\alpha$ is made at limit rounds α as well (instead of $u^0_\alpha = u^1_\alpha$, as in Definition 2.32). Thus, the game $G(T,t,\gamma+1)$ from [11] is equivalent to the game $\mathcal{G}_{\gamma+2}(T,t)$ used here. When γ is a limit ordinal, the two games $G(T,t,\gamma)$ (from [11]) and $\mathcal{G}_\gamma(T,t)$ are equivalent.

We require $u^0_\alpha = u^1_\alpha$ in limit rounds α because with this definition, the games $\mathcal{G}_\gamma(T)$ and $\mathcal{V}_\gamma(T)$ can be compared: $\mathcal{G}_\gamma(T)$ is always easier for player **II** to win and harder for player **I** to win than $\mathcal{V}_\gamma([T])$; see Proposition 2.53. (This is also the reason behind the analogous rule in the definitions of $\mathcal{G}^*_\gamma(T)$ and $\mathcal{G}^*_\kappa(T)$.)

It is clear from the definitions that $\mathcal{G}^*_\gamma(T,t)$ is easier for player **II** to win and harder for player **I** to win than $\mathcal{G}_\gamma(T,t)$. Example 2.31 shows that when $\gamma < \kappa$, the converse of this statement does not hold, i.e., the two games may not be equivalent. The tree S_γ defined in that example is such that

$$\mathrm{Ker}^*_\gamma(S_\gamma) = S_\gamma = \mathrm{Sc}_\gamma(S_\gamma)\,.$$

In fact, the set $[S_\gamma]$ of its κ-branches is discrete.

We note that if μ is a regular cardinal and $\kappa = \mu^+$, then $\mathcal{G}_{\mu+1}(T)$ and $\mathcal{G}^*_{\mu+1}(T)$ are equivalent for player **I** whenever the set of splitting nodes of T is cofinal and each level of T has size $\leq \mu$ [11, Corollary 1.5.9]. (We remark that [11, Corollary 1.5.9] actually states this for the games $G(T, \emptyset, \mu+1)$ considered there. However, observe that if the set of splitting nodes of T is cofinal and $\gamma < \kappa$, then $\mathcal{G}_{\mu+1}(T)$ is equivalent to $\mathcal{G}_{\mu+2}(T)$ and is therefore also equivalent to $G(T, \emptyset, \mu+1)$.)

If $\gamma = \kappa$, the two games $\mathcal{G}^*_\kappa(T,t)$ and $\mathcal{G}_\kappa(T,t)$ are equivalent (for both players) by the proposition below, and therefore lead to the same notion of κ-perfectness and κ-scatteredness for trees.

Proposition 2.35. *Let T be a subtree of ${}^{<\kappa}\kappa$. Then the games $\mathcal{G}^*_\kappa(T,t)$ and $\mathcal{G}_\kappa(T,t)$ are equivalent for all $t \in T$. That is,*

$$\mathrm{Ker}^*_\kappa(T) = \mathrm{Ker}_\kappa(T) \qquad \textit{and} \qquad \mathrm{Sc}^*_\kappa(T) = \mathrm{Sc}_\kappa(T)\,.$$

Thus, T is κ-perfect in the sense of Definition 2.23 if and only if T is κ-perfect in the sense of Definition 2.33, and the analogous statement holds for κ-scatteredness.

We note that this proposition also follows from Corollaries 2.25 and 2.27, and [11, Propositions 1.2.13, 1.2.14 and 1.5.18]. The proof below shows the equivalence of the two games directly. The idea behind it will also be used in a later argument proving a stronger statement (see the Proposition 2.60; see also Remark 2.36. The proof will be given, in a slightly more general form, as the proof of Proposition 3.30).

Proof. We describe the proof of the two directions that do not follow immediately from the definitions. The moves in $\mathcal{G}_\kappa = \mathcal{G}_\kappa(T,t)$ will be denoted by u^0_α, u^1_α, i_α and δ_α, as usual. The moves in $\mathcal{G}^*_\kappa = \mathcal{G}^*_\kappa(T,t)$. will be denoted by v^0_α, v^1_α and i^*_α.

The idea, in both cases, is that while the players play the first $\alpha+1$ rounds of $\mathcal{G}_\kappa$, they play the first $\delta_\alpha+1$ rounds of $\mathcal{G}^*_\kappa$. If

$$(v^0_{\delta_\alpha+1}, v^1_{\delta_\alpha+1})$$

is a legal move for player **II** in round $\delta_\alpha+1$ of $\mathcal{G}^*_\kappa$, then

$$v^0_{\delta_\alpha+1}\restriction\delta_\alpha = v^1_{\delta_\alpha+1}\restriction\delta_\alpha \qquad \text{and} \qquad v^0_{\delta_\alpha+1} \perp v^1_{\delta_\alpha+1}. \tag{2.7}$$

(Note that the first equation also holds in round δ_α instead of $\delta_\alpha + 1$, but the second statement may not hold if $\delta_\alpha \in \mathrm{Lim}$.) Thus, player **I** is able to play in $\mathcal{G}^*_\kappa$ in such a way that the moves

$$(u^0_{\alpha+1}, u^1_{\alpha+1}) = (v^0_{\delta_\alpha+1}, v^1_{\delta_\alpha+1})$$

will also be a legal moves for player **II** in rounds $\alpha + 1$ of $\mathcal{G}_\kappa$. For limit rounds α, the players play the first $\eta_\alpha = \sup\{\delta_\beta + 1 : \beta < \alpha\}$ many rounds of $\mathcal{G}^*_\kappa$ while they play the first α rounds in $\mathcal{G}_\kappa$.

In more detail, suppose τ is a winning strategy for player **II** in $\mathcal{G}^*_\kappa$. Let $\alpha < \kappa$, and suppose player **I** has played $\langle i_\beta, \delta_\beta : \beta < \alpha\rangle$ in $\mathcal{G}_\kappa$ so far. Let

$$\eta_\beta = \sup\{\delta_{\beta'} + 1 : \beta' < \beta\}$$

for all $\beta \leq \alpha$ (note that in successor rounds $\beta = \beta' + 1$, we have $\eta_\beta = \delta_{\beta'} + 1$). The strategy of player **II** in round α of $\mathcal{G}_\kappa$ is to play

$$(u^0_\alpha, u^1_\alpha) = (v^0_{\eta_\alpha}, v^1_{\eta_\alpha}),$$

where the moves $v^0_{\eta_\alpha}$ and $v^1_{\eta_\alpha}$ are obtained from a partial run of $\mathcal{G}^*_\kappa$ where player **II** uses τ and player **I** plays

$$\begin{aligned} i^*_{\eta_\beta} &= i_\beta && \text{for all } \beta < \alpha\text{, and} \\ i^*_\eta &= 0 && \text{for all } \eta < \eta_\alpha \text{ such that } \eta \neq \eta_\beta \text{ for any } \beta < \alpha. \end{aligned}$$

Note that $\eta_\alpha < \kappa$ for all $\alpha < \kappa$ (here, we use that $\delta_\beta < \kappa$ for all $\beta < \kappa$ and that κ is regular). Therefore player **II** can indeed define (u^0_α, u^1_α) in each round $\alpha < \kappa$ of $\mathcal{G}_\kappa$ in the way described above. This move is legal because (2.7) holds whenever $\alpha \in \mathrm{Succ}$, and

$$u^i_\alpha = v^i_{\eta_\alpha} \supseteq v^{i^*_{\eta_\beta}}_{\eta_\beta} = u^{i_\beta}_\beta \tag{2.8}$$

holds for all $\beta < \alpha$ and $i < 2$. Thus, the strategy just defined is a winning strategy for player **II** in $\mathcal{G}_\kappa$.

Using the same idea, we now describe a winning strategy for player **I** in $\mathcal{G}^*_\kappa$ assuming he has a winning strategy ρ in $\mathcal{G}_\kappa$. Suppose $\eta < \kappa$, and suppose that player **II** has played $\langle (v^0_\epsilon, v^1_\epsilon) : \epsilon \leq \eta\rangle$ in $\mathcal{G}^*_\kappa$ so far. Using ρ, player **I** can define ordinals $\alpha < \kappa$ and $\langle \eta_\beta < \kappa : \beta \leq \alpha\rangle$ and a partial run

$$\big\langle (u^0_\beta, u^1_\beta), i_\beta, \delta_\beta : \beta < \alpha \big\rangle$$

of $\mathcal{G}$ such that the following hold. Player **I** defines the moves i_β and δ_β according to ρ, we have $\eta_\beta = \sup\{\delta_{\beta'} + 1 : \beta' < \beta\}$ for all $\beta \leq \alpha$, and α is the ordinal such that

$$\eta_\beta \leq \eta < \eta_\alpha \text{ for all } \beta < \alpha.$$

Lastly, $(u^0_\beta, u^1_\beta) = (v^0_{\eta_\beta}, v^1_{\eta_\beta})$ for all $\beta < \alpha$. Note that α is a successor ordinal by the continuity of the function $\alpha + 1 \to \kappa$; $\beta \mapsto \eta_\beta$. Thus, we have

$$\eta_{\alpha-1} \leq \eta < \eta_\alpha.$$

The strategy of player **I** in round η of $\mathcal{G}^*_\kappa$ is to play

$$\begin{aligned} i^*_\eta &= i_{\eta_{\alpha-1}} && \text{if } \eta = \eta_{\alpha-1}, \text{ and} \\ i^*_\eta &= 0 && \text{if } \eta > \eta_{\alpha-1}. \end{aligned}$$

The moves $(u^0_\beta, u^1_\beta) = (v^0_{\eta_\beta}, v^1_{\eta_\beta})$ are legal for player **II** in rounds $\beta < \kappa$ of $\mathcal{G}_\kappa$ as long as they are legal moves in rounds η_β of $\mathcal{G}^*_\kappa$. (This is true by (2.7) and because (2.8) holds by the choice of the i^*_η's.) Thus, if player **II** were able to win a run of $\mathcal{G}^*_\kappa$ where player **I** uses this strategy, then she would be able to win a run of $\mathcal{G}_\kappa$ where player **I** uses ρ, contradicting the assumption that ρ is a winning strategy. □

Remark 2.36. As Example 2.31 shows, if $\gamma < \kappa$, then there exists a tree S_γ such that player **I** wins $\mathcal{G}_\gamma(T)$ (equivalently, he wins $\mathcal{G}_\gamma(T, t)$ for all $t \in T$), but player **II** wins $\mathcal{G}^*_\gamma(T, t)$ for all $t \in S_\gamma$.

However, there is a "modified version of $\mathcal{G}^*_\gamma(T, t)$" which is easier for player **I** to win and harder for player **II** to win than $\mathcal{G}_\gamma(T, t)$. The idea is that in this "modified game", player **I** gets to decide γ times how many additional rounds ξ_α of $\mathcal{G}^*_\kappa(T, t)$ the players should play. That is, player **I** first chooses an ordinal $\xi_0 < \kappa$, and then the players play ξ_0 rounds of $\mathcal{G}^*_\kappa(T, t)$. Next, player **I** chooses $\xi_1 < \kappa$, and the players play ξ_1 more rounds of $\mathcal{G}^*_\kappa(T, t)$ (continuing from the position they were in after the first ξ_0 rounds). In general, for each $\alpha < \gamma$, player **I** first chooses an ordinal $\xi_\alpha < \kappa$, and then the two players play ξ_α more rounds of $\mathcal{G}^*_\kappa(T, t)$. Thus, the players play $\xi = \Sigma_{\alpha<\gamma}\xi_\alpha$ rounds of $\mathcal{G}^*_\kappa(T, t)$ altogether. Player **II** wins a run of this modified game if she can play legally in all ξ many rounds of $\mathcal{G}^*_\kappa(T, t)$.

This remark will be made precise with the help of the games defined in the next section, in Proposition 2.60.

In the corollary below, we summarize the connections between the κ-kernels and the κ-scattered parts of a given subtree T of ${}^{<\kappa}\kappa$ and its set $[T]$ of κ-branches. These connections follow from Corollaries 2.25 and 2.29, Claim 2.17., and Proposition 2.35.

Recall that $CP_\kappa(X)$ denotes the set of κ-condensation points of a subset X of the κ-Baire space.

Corollary 2.37. *Suppose T is a subtree of ${}^{<\kappa}\kappa$.*

$$\mathrm{Ker}_\kappa([T]) = [\mathrm{Ker}_\kappa(T)] = [\mathrm{Ker}^*_\kappa(T)] \subseteq CP_\kappa([T])\,; \tag{2.9}$$

$$\mathrm{Sc}_\kappa([T]) \supseteq N(\mathrm{Sc}_\kappa(T)) = N(\mathrm{Sc}^*_\kappa(T)) = [T] - CP_\kappa([T])\,. \tag{2.10}$$

Furthermore, the following statements are equivalent.

(1) $\mathrm{PSP}_\kappa([T]\cap N_t)$ *holds for all* $t \in T$.
(2) $\mathrm{Ker}_\kappa([T]) = CP_\kappa([T])$.
(3) Equality holds everywhere in (2.9) *and* (2.10). *That is,*

$$\mathrm{Ker}_\kappa([T]) = [\mathrm{Ker}_\kappa(T)] = [\mathrm{Ker}^*_\kappa(T)] = CP_\kappa([T])\,;$$

$$\mathrm{Sc}_\kappa([T]) = N(\mathrm{Sc}_\kappa(T)) = N(\mathrm{Sc}^*_\kappa(T)) = [T] - CP_\kappa([T])\,.$$

Specifically, given a subtree T of ${}^{<\kappa}\kappa$,

$$\mathrm{Ker}_\kappa([T]) = [\mathrm{Ker}_\kappa(T)]$$

always holds. If $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$, then $\mathrm{Sc}_\kappa([T]) = N(\mathrm{Sc}_\kappa(T))$ also holds.

Question 2.38. For which cardinals $\kappa = \kappa^{<\kappa}$, which infinite ordinals $\gamma < \kappa$ and which families $\mathcal{T}$ of subtrees of ${}^{<\kappa}\kappa$ do either of the following statements hold or consistently hold:

(1) $\mathrm{Ker}_\gamma([T]) = [\mathrm{Ker}_\gamma(T)]$ for all trees $T \in \mathcal{T}$;
(2) $\mathrm{Sc}_\gamma([T]) = N(\mathrm{Sc}_\gamma(T))$ for all subtrees $T \in \mathcal{T}$?

We note that by Proposition 2.53 below,

$$\mathrm{Ker}_\gamma([T]) \subseteq [\mathrm{Ker}_\gamma(T)] \qquad \text{and} \qquad \mathrm{Sc}_\gamma([T]) \supseteq N(\mathrm{Sc}_\gamma(T))$$

hold for all cardinals $\kappa = \kappa^{<\kappa}$, all infinite $\gamma < \kappa$ and all subtrees T of ${}^{<\kappa}\kappa$.

Conjecture 2.39. *Suppose κ is a weakly compact cardinal. Then* $\mathrm{Ker}_\gamma([T]) = [\mathrm{Ker}_\gamma(T)]$ *holds for all limit ordinals $\gamma \in \kappa \cap \mathrm{Lim}$ and all subtrees T of ${}^{<\kappa}\kappa$. Furthermore,*

$$[\mathrm{Ker}_{\gamma+1}(T)] \subseteq \mathrm{Ker}_\gamma([T])$$

holds for all ordinals $\gamma < \kappa$ and all subtrees T of ${}^{<\kappa}\kappa$.

We note that by [11, Corollary 1.1.60], statements (1) and (2) in the question above hold for $\gamma = \omega$ whenever κ is a weakly compact cardinal.

See Question 2.54 and Conjecture 2.55 for a more general version of the above question and conjecture.

In the corollary below, we summarize some equivalent reformulations of $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ (i.e., the κ-perfect set property for closed subsets of ${}^\kappa\kappa$) which are given in or implied by Propositions 2.16, 2.35 and Corollary 2.37. It may be interesting to note that the decomposition theorems for closed sets $X \subseteq {}^\kappa\kappa$ and for subtrees T of ${}^{<\kappa}\kappa$ given in items (2) and (3) below are in fact equivalent with $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$.

Corollary 2.40. *The following statements are equivalent.*

(1) $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ *holds.*

(2) Every closed subset X of ${}^\kappa\kappa$ can be written as a disjoint union

$$X = \mathrm{Ker}_\kappa(X) \cup \mathrm{Sc}_\kappa(X)\,, \quad \textit{where} \quad |\mathrm{Sc}_\kappa(X)| \leq \kappa.$$

(3) Every subtree T of ${}^{<\kappa}\kappa$ can be written as a disjoint union

$$T = \mathrm{Ker}_\kappa(T) \cup \mathrm{Sc}_\kappa(T)\,.$$

That is, $\mathcal{G}_\kappa(T,t)$ is determined for every subtree T of ${}^{<\kappa}\kappa$ and every $t \in T$.

(4) If T is a subtree of ${}^{<\kappa}\kappa$, then

$$\mathrm{Ker}_\kappa([T]) = [\mathrm{Ker}_\kappa(T)] = [\mathrm{Ker}^*_\kappa(T)] = CP_\kappa([T])\,;$$

$$\mathrm{Sc}_\kappa([T]) = N(\mathrm{Sc}_\kappa(T)) = N(\mathrm{Sc}^*_\kappa(T)) = [T] - CP_\kappa([T])\,.$$

Note again that the games $\mathcal{G}_\kappa(T,t)$ and $\mathcal{G}^*_\kappa(T,t)$ are equivalent, and therefore their role is interchageable in item (3).

By the above corallary, each of the statements (2)-(4) is also equiconsistent with the existence of an inaccessible cardinal above κ (see Remark 2.14).

2.2 Generalizing the Cantor-Bendixson hierarchy via games

Recall that $\mathcal{T}_\kappa$ denotes the class of trees without branches of length $\geq\kappa$. We begin this section by recalling from [51] how trees $\boldsymbol{t} \in \mathcal{T}_\kappa$ can be used to generalize the Cantor-Bendixson hierarchy for subsets X of the κ-Baire space. This is done via modified versions $\mathcal{V}_{\boldsymbol{t}}(X)$ of the games $\mathcal{V}_\kappa(X)$ associated to trees $\boldsymbol{t} \in \mathcal{T}_\kappa$. In this approach, ordinals correspond to well-founded trees; specifically, the α^{th} level of the Cantor-Bendixson hierarchy for a set X corresponds to the game $\mathcal{V}_{\boldsymbol{b}_\alpha}(X)$ (where $\boldsymbol{b}_\alpha$ is the canonical well-founded tree of rank α).

In the second part of the section, we consider analogous modifications of $\mathcal{G}_{\boldsymbol{t}}(T)$ and $\mathcal{G}^*_{\boldsymbol{t}}(T)$ of the games $\mathcal{G}_\kappa(T)$ and $\mathcal{G}^*_\kappa(T)$ for trees $\boldsymbol{t}$ of height $\leq \kappa$. We describe how these games can be used to generalize the Cantor-Bendixson hierarchy for subtrees T of ${}^{<\kappa}\kappa$.

We also mention some of our results from Section 3.2 about how the levels of the generalized Cantor-Bendixson hierarchies discussed in this section compare to each other. (These will be proven in a slightly more general form in Section 3.2).

We remark that the methods described in this section are adaptations of methods used in e.g. [16, 18] to study transfinite Ehrenfeucht-Fraïssé games; see also [52, 53]. Similar methods are also used in [36] to study the analogue of inductive definitions, in general, for non well-founded trees.

2.2.1 The Cantor-Bendixson hierarchy for subsets of the κ-Baire space

Let $\boldsymbol{t}$ be a tree of height $\leq \kappa$ and suppose $X \subseteq {}^\kappa\kappa$. The game $\mathcal{V}_{\boldsymbol{t}}(X)$ (defined right below) is like $\mathcal{V}_\kappa(X)$, except that at the beginning of each round α, player **I** also plays a node $t_\alpha \in \boldsymbol{t}$, in such a way that the t_α's form an ascending chain in $\boldsymbol{t}$. The tree $\boldsymbol{t}$ acts like a clock for player **I**: in order to win a run, he must make sure player **II** cannot move legally before he runs out of nodes in $\boldsymbol{t}$. That is, player **II** wins a run if and only if she can continue playing legally as long as player **I** can move up the tree $\boldsymbol{t}$.

Definition 2.41 (from [51])**.** Let $\boldsymbol{t} \in \mathcal{T}_{\kappa+1}$, and let $X \subseteq {}^\kappa\kappa$. The game $\mathcal{V}_{\boldsymbol{t}}(X)$ is played as follows.

I	t_0		δ_0	t_1		δ_1	$\dots$	t_α		δ_α	$\dots$
II		x_0			x_1		$\dots$		x_α		$\dots$

At the beginning of each round, player **I** plays an element $t_\alpha \in \boldsymbol{t}$ such that $t_\beta < t_\alpha$

for all $\beta < \alpha$. Next, player **II** chooses an element $x_\alpha \in X$. Lastly, player **I** chooses an ordinal $\delta_\alpha < \kappa$.

Player **I** has to choose δ_α so that $\delta_\beta < \delta_\alpha$ for all $\beta < \alpha$, and player **II** has to choose x_α in such a way that for all $\beta < \alpha$,

$$x_\alpha \restriction \delta_\beta = x_\beta \restriction \delta_\beta \text{ and } x_\alpha \neq x_\beta.$$

The first player who cannot play legally loses the run, and the other player wins. (In other words, if player **I** cannot play t_α legally, then he loses this run and player **II** wins. If player **II** cannot play x_α legally, then she loses this run and player **I** wins.)

For an arbitrary $x \in {}^\kappa\kappa$, the game $\mathcal{V}_{\boldsymbol{t}}(X, x)$ is defined just like $\mathcal{V}_{\boldsymbol{t}}(X)$, except player **II** has to start the game with $x_0 = x$ (and thus $x_0 \notin X$ is allowed).

Notice that if $\boldsymbol{t}$ is the tree which consists of a single branch of length γ, then $\mathcal{V}_{\boldsymbol{t}}(X)$ is equivalent to $\mathcal{V}_\gamma(X)$. (Recall that this tree $\boldsymbol{t}$ is also denoted by γ.) If $\boldsymbol{t}$ and $\boldsymbol{u}$ are trees such that $\boldsymbol{t} \leq \boldsymbol{u}$ (i.e., there exists an order preserving map $f : \boldsymbol{t} \to \boldsymbol{u}$), then $\mathcal{V}_{\boldsymbol{t}}(X)$ is easier is for player **I** to win and harder for player **II** to win than $\mathcal{V}_{\boldsymbol{u}}(X)$.

By the Gale-Stewart theorem, $\mathcal{V}_{\boldsymbol{t}}(X)$ is determined whenever $\boldsymbol{t}$ has height $\leq \omega$. (This may not be the case when $\mathrm{ht}(\boldsymbol{t}) > \omega$, however.)

Definition 2.42 (from [51])**.** For any subset $X \subseteq {}^\kappa\kappa$ and any tree $\boldsymbol{t} \in \mathcal{T}_{\kappa+1}$, we let

$$\mathrm{Ker}_{\boldsymbol{t}}(X) = \{x \in {}^\kappa\kappa : \text{ player } \mathbf{II} \text{ has a winning strategy in } \mathcal{V}_{\boldsymbol{t}}(X, x)\};$$
$$\mathrm{Sc}_{\boldsymbol{t}}(X) = \{x \in X : \text{ player } \mathbf{I} \text{ has a winning strategy in } \mathcal{V}_{\boldsymbol{t}}(X, x)\}.$$

A nonempty set X is $\boldsymbol{t}$-*perfect* iff $X = \mathrm{Ker}_{\boldsymbol{t}}(X)$. A set X is $\boldsymbol{t}$-*scattered* iff $X = \mathrm{Sc}_{\boldsymbol{t}}(X)$.

The set $\mathrm{Sc}_\kappa(X)$ is a relatively open and $\boldsymbol{t}$-scattered subset of X. If $X \subseteq {}^\kappa\kappa$, then $\mathrm{Ker}_{\boldsymbol{t}}(X)$ is a closed subset of ${}^\kappa\kappa$, and therefore a $\boldsymbol{t}$-perfect set is always closed.

Observe that the set $\mathrm{Ker}_{\boldsymbol{t}}(X)$ is $\boldsymbol{t}$-perfect if the tree $\boldsymbol{t}$ is reflexive, (i.e., iff for every $t \in \boldsymbol{t}$, T can be mapped in an order preserving way into the set $\{s \in T : t \leq_T s\}$). Example 2.4 shows that $\mathrm{Ker}_{\boldsymbol{t}}(X)$ may not be $\boldsymbol{t}$-perfect if $\boldsymbol{t}$ is not reflexive (note that the tree which consists of one branch of length γ is reflexive iff γ is an indecomposable ordinal).

If X is a topological space (specifically, if X is a subset of ${}^{\kappa}\kappa$), the α^{th} *Cantor-Bendixson derivative* of X ($\alpha \in \text{Ord}$) is defined, using recursion, as follows:

$$X^{(0)} = X,$$
$$X^{(\alpha+1)} = \{x \in X^{(\alpha)} : x \text{ is a limit point of } X^{(\alpha)}\},$$
$$X^{(\xi)} = \bigcap_{\alpha<\xi} X^{(\alpha)} \quad \text{if } \xi \in \text{Lim}.$$

The *Cantor-Bendixson rank* $\text{rk}_{CB}(\text{X})$ of X is the unique ordinal α with $X^{(\alpha)} = X^{(\alpha+1)}$.

If X is a closed subset of ${}^{\kappa}\kappa$, then the (ω-)perfect kernel of X can be expressed as

$$\text{Ker}_{\omega}(X) = \bigcap \{X^{(\alpha)} : \alpha < \text{rk}_{CB}(\text{X})\}.$$

If $X \subseteq {}^{\kappa}\kappa$ is arbitrary, then $\bigcap \{X^{(\alpha)} : \alpha < \text{rk}_{CB}(\text{X})\} = X \cap \text{Ker}_{\omega}(X)$ and is equal to the largest dense in itself subset of X. The (ω-)scattered part of X can be expressed as $\text{Sc}_{\omega}(X) = \bigcup \{X - X^{(\alpha)} : \alpha < \text{rk}_{CB}(\text{X})\}$.

Recall from Example 1.18 that $\boldsymbol{b}_{\alpha}$ is the tree of descending sequences of elements of α, ordered by end extension.

Claim 2.43 (from [51])**.** *If $X \subseteq {}^{\kappa}\kappa$, then*

$$X^{(\alpha)} = \text{Ker}_{\boldsymbol{b}_{\alpha}}(X) \cap X = X - \text{Sc}_{\boldsymbol{b}_{\alpha}}(X)$$

for all ordinals α. Therefore

$$\text{Ker}_{\omega}(X) = \bigcap \{\text{Ker}_{\boldsymbol{t}}(X) : \boldsymbol{t} \in \mathcal{T}_{\omega}\}, \qquad \textit{and} \qquad \text{Sc}_{\omega}(X) = \bigcup \{\text{Sc}_{\boldsymbol{t}}(X) : \boldsymbol{t} \in \mathcal{T}_{\omega}\}.$$

Proof. The first statement can be seen easily by induction on α. The second statement follows from the first one by the fact that if $\boldsymbol{t} \in \mathcal{T}_{\omega}$, then there exists an ordinal α such that $\boldsymbol{t} \leq \boldsymbol{b}_{\alpha}$ (as mentioned in Example 1.18). In the case that X is not closed, we also use the observation that $\text{Ker}_{\boldsymbol{t}}(X) = \overline{\text{Ker}_{\boldsymbol{t}}(X) \cap X}$ holds for any tree $\boldsymbol{t}$ (see Claim 2.66 below). □

The following result of Jouko Väänänen's is the uncountable analogue of Claim 2.43.

Theorem 2.44 (Theorem 5 in [51])**.** *If X is a subset of the κ-Baire space, then*

$$\text{Ker}_{\kappa}(X) = \bigcap \{\text{Ker}_{\boldsymbol{t}}(X) : \boldsymbol{t} \in \mathcal{T}_{\kappa}\}, \qquad \textit{and} \qquad \text{Sc}_{\kappa}(X) = \bigcup \{\text{Sc}_{\boldsymbol{t}}(X) : \boldsymbol{t} \in \mathcal{T}_{\kappa}\}.$$

Thus, the κ-perfect kernel $\mathrm{Ker}_\kappa(X)$ of a closed subset X of the κ-Baire space can be obtained as the intersection of the *levels*

$$\mathrm{Ker}_{\boldsymbol{t}}(X) \qquad (\boldsymbol{t} \in \mathcal{T}_\kappa)$$

of a "generalized Cantor-Bendixson hierarchy" for player **II**, associated to X.

For arbitrary subsets $X \subseteq {}^\kappa\kappa$, the largest κ-dense in itself subset $\mathrm{Ker}_\kappa(X) \cap X$ of X (see Section 2.3) can be obtained as the intersection of the levels

$$\mathrm{Ker}_{\boldsymbol{t}}(X) \cap X \qquad (\boldsymbol{t} \in \mathcal{T}_\kappa).$$

The analogous statement holds for player **I** as well (by Theorem 2.44): $X - \mathrm{Sc}_{\boldsymbol{t}}(X)$ is the intersection of the levels

$$X - \mathrm{Sc}_{\boldsymbol{t}}(X) \qquad (\boldsymbol{t} \in \mathcal{T}_\kappa)$$

of a "generalized Cantor-Bendixson hierarchy" for player **I**, associated to X.

As noted in [51], it is possible to prove analogous representation theorems for arbitrary trees $\boldsymbol{t}$ (instead of κ); see [36] for similar results.

2.2.2 Cantor-Bendixson hierarchies for subtrees of ${}^{<\kappa}\kappa$

Suppose $\boldsymbol{t}$ is a tree of height $\leq\kappa$ and T is a subtree of ${}^{<\kappa}\kappa$. The following game $\mathcal{G}_{\boldsymbol{t}}(T)$ is like $\mathcal{G}_\kappa(T)$, except player **I** also plays nodes (in ascending order) from the "clock-tree" $\boldsymbol{t}$. To distinguish between the two different roles trees can play when considering the games $\mathcal{G}_{\boldsymbol{t}}(T)$, we always use S, T, $U\dots$ to denote subtrees of ${}^{<\kappa}\kappa$, and we denote the "clock-trees" by $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{u}\dots$.

Definition 2.45. Let T be a subtree of ${}^{<\kappa}\kappa$, and let $\boldsymbol{t} \in \mathcal{T}_{\kappa+1}$. The game $\mathcal{G}_{\boldsymbol{t}}(T)$ is played as follows.

I	t_0		i_0, δ_0	t_1		i_1, δ_1	$\dots$	t_α		i_α, δ_α	$\dots$
II		u^0_0, u^1_0			u^0_1, u^1_1		$\dots$		u^0_α, u^1_α		$\dots$

At the beginning of each round, player **I** plays a node $t_\alpha \in \boldsymbol{t}$ such that $t_\beta < t_\alpha$ holds for all $\beta < \alpha$. Next, player **II** plays $u^0_\alpha, u^1_\alpha \in T$, and lastly, player **I** plays ordinals $i_\alpha < 2$ and $\delta_\alpha < \kappa$.

Player **I** has to play δ_α in such a way that $\delta_\alpha > \delta_\beta$ for all $\beta < \alpha$, and player **II** has to play so that

$$u^0_\alpha, u^1_\alpha \supseteq u^{i_\beta}_\beta$$

for all $\beta < \alpha$. In successor rounds $\alpha = \alpha' + 1$, player **II** also has to make sure that

$$u^0_{\alpha'+1} \perp u^1_{\alpha'+1} \quad \text{and} \quad u^0_{\alpha'+1}{\restriction}\delta_{\alpha'} = u^1_{\alpha'+1}{\restriction}\delta_{\alpha'}.$$

In rounds $\alpha \in \mathrm{Lim} \cup \{0\}$, she has to play so that $u^0_\alpha = u^1_\alpha$. The first player who cannot play legally loses the round, and the other player wins.

For a node $t \in T$, the game $\mathcal{G}_{\boldsymbol{t}}(T, t)$ is defined just like $\mathcal{G}_{\boldsymbol{t}}(T)$ except player **II** has to start the game with $u^0_0 = u^1_0 = t$.

Definition 2.46. Suppose T is a subtree of ${}^{<\kappa}\kappa$ and $\boldsymbol{t} \in \mathcal{T}_{\kappa+1}$. We let

$$\mathrm{Ker}_{\boldsymbol{t}}(T) = \{t \in T : \text{ player } \mathbf{II} \text{ has a winning strategy in } \mathcal{G}_{\boldsymbol{t}}(T,t)\};$$
$$\mathrm{Sc}_{\boldsymbol{t}}(T) = \{t \in T : \text{ player } \mathbf{I} \text{ has a winning strategy in } \mathcal{G}_{\boldsymbol{t}}(T,t)\}.$$

We say that a nonempty tree T is $\boldsymbol{t}$*-perfect* iff $T = \mathrm{Ker}_{\boldsymbol{t}}(T)$. A tree T is $\boldsymbol{t}$*-scattered* iff $T = \mathrm{Sc}_{\boldsymbol{t}}(T)$.

Observe that $\mathrm{Ker}_{\boldsymbol{t}}(T)$ is a subtree of T, and if $s \in \mathrm{Sc}_{\boldsymbol{t}}(T)$ and $t \in T$ extends s, then $t \in \mathrm{Sc}_{\boldsymbol{t}}(T)$. Thus, T is $\boldsymbol{t}$-scattered if and only if $\emptyset \in \mathrm{Sc}_{\boldsymbol{t}}(T)$ and if and only if player **I** wins $\mathcal{G}_{\boldsymbol{t}}(T)$. Again, $\mathrm{Ker}_{\boldsymbol{t}}(T)$ is a $\boldsymbol{t}$-perfect subtree if $\boldsymbol{t}$ is reflexive, but this may fail for non-reflexive trees $\boldsymbol{t}$ (by e.g. Example 2.34).

By the proposition below, the κ-perfect kernel $\mathrm{Ker}_\kappa(T)$ of a subtree T of ${}^{<\kappa}\kappa$ can be represented as the intersection of the levels of a "generalized Cantor-Bendixson hierarchy" for player **II** and T. We show the analogous representation theorem for player **I** as well, and in the case of limit ordinals $\xi \leq \kappa$.

Proposition 2.47. *If T is a subtree of ${}^{<\kappa}\kappa$ and ξ is a limit ordinal, then*

$$\mathrm{Ker}_\xi(T) = \bigcap \{\mathrm{Ker}_{\boldsymbol{t}}(T) : \boldsymbol{t} \in \mathcal{T}_\xi\}, \qquad \textit{and} \qquad \mathrm{Sc}_\xi(T) = \bigcup \{\mathrm{Sc}_{\boldsymbol{t}}(T) : \boldsymbol{t} \in \mathcal{T}_\xi\}.$$

Note that the second equation is equivalent to the following claim:

$$T - \mathrm{Sc}_\xi(T) = \bigcap \{T - \mathrm{Sc}_{\boldsymbol{t}}(T) : \boldsymbol{t} \in \mathcal{T}_\xi\}.$$

The proof below is analogous to the proof of [51, Theorem 5], and is similar to arguments in e.g. [18, 36, 53]. We give a detailed proof because some later arguments will be modifications of this one.

Proof. We prove the direction in each of the above equalities that is not immediately clear.

First, suppose that $s \in T - \mathrm{Ker}_\xi(T)$. We need to find a tree $\boldsymbol{u}' \in \mathcal{T}_\xi$ such that player **II** does not win $\mathcal{G}_{\boldsymbol{u}'}(T, s)$. Let $\boldsymbol{u}$ be the tree which consists of pairs $(\gamma + 1, \tau)$ such that $\gamma < \xi$ and τ is a winning strategy for player **II** in $\mathcal{G}_{\gamma+1}(T, s)$. The tree $\boldsymbol{u}$ is ordered by end-extension; that is

$$(\gamma + 1, \tau) \leq (\gamma' + 1, \tau')$$

iff $\gamma \leq \gamma'$ and τ agrees with τ' in the first γ rounds of $\mathcal{G}_{\gamma+1}(T, s)$. Observe that $\boldsymbol{u} \in \mathcal{T}_\xi$; indeed, a ξ-branch of $\boldsymbol{u}$ would determine a winning strategy for player **II** in $\mathcal{G}_\xi(T, s)$.

Claim 2.48. *Suppose $\boldsymbol{t}$ is a tree. Then player* **II** *wins $\mathcal{G}_{\boldsymbol{t}}(T, s)$ if and only if $\boldsymbol{t} \leq \boldsymbol{u}$.*

Proof of Claim 2.48. Suppose τ is a winning strategy for player **II** in $\mathcal{G}_{\boldsymbol{t}}(T, s)$. Then τ determines an order preserving map $f : \boldsymbol{t} \to \boldsymbol{u}$; $t \mapsto (\gamma_t + 1, \tau_t)$ as follows. If $t \in \boldsymbol{t}$, then let γ_t be the order type of $\mathrm{pred}_{\boldsymbol{t}}(t)$, and let τ_t be the strategy for player **II** in $\mathcal{G}_{\gamma_t+1}(T, s)$ which is obtained, roughly, by restricting τ to $\mathrm{pred}_{\boldsymbol{t}}(t) \cup \{t\}$. That is, if $\langle t_\beta : \beta \leq \alpha\rangle$ is the sequence of elements of $\mathrm{pred}_{\boldsymbol{t}}(t) \cup \{t\}$ in ascending order, then let

$$\tau_t\big(\langle \delta_\beta, i_\beta : \beta < \alpha\rangle\big) = \tau\big(\langle t_\beta, \delta_\beta, i_\beta : \beta < \alpha\rangle^\frown\langle t_\alpha\rangle\big)$$

for all legal partial plays $\langle \delta_\beta, i_\beta : \beta < \alpha\rangle$ of player **I** in $\mathcal{G}_{\gamma_t+1}(T, s)$. Clearly, τ_t is a winning strategy for **II** in $\mathcal{G}_{\gamma_t+1}(T, s)$, and the map f is order preserving.

To see the other direction, it is enough to define a winning strategy τ for player **II** in $\mathcal{G}_{\boldsymbol{u}}(T, s)$. Suppose $p = \langle u_\beta, \delta_\beta, i_\beta : \beta < \alpha\rangle^\frown\langle u_\alpha\rangle$ is a legal partial play of player **I** in $\mathcal{G}_{\boldsymbol{t}}(T, s)$, and that $t_\alpha = \langle \gamma_{\alpha+1}, \tau_\alpha\rangle$. Then let

$$\tau(p) = \tau_\alpha(\langle \delta_\beta, i_\beta : \beta < \alpha\rangle).$$

Note that $\tau(p)$ is well defined because $\alpha \leq \gamma_\alpha$. It is clear that, with this definition, τ is a winning strategy for **II** in $\mathcal{G}_{\boldsymbol{t}}(T, s)$. This completes the proof of Claim 2.48.

Consider the tree $\boldsymbol{u}' = \sigma\boldsymbol{u}$ (the tree of ascending chains in $\boldsymbol{u}$; see Definition 1.15). Then we have $\boldsymbol{u} < \boldsymbol{u}'$ and $\boldsymbol{u}' \in \mathcal{T}_\xi$ (by Lemma 1.16 and Fact 1.17). Therefore, by Claim 2.48, the tree $\boldsymbol{u}'$ is as required.

Now, suppose ρ is a winning strategy for player **I** in $\mathcal{G}_\xi(T, t)$. Let $\boldsymbol{s}_\rho$ be the tree which consists of legal partial plays $\langle u_\beta : \beta \leq \alpha\rangle$ of player **II** in $\mathcal{G}_\xi(T, t)$ against the strategy ρ. (That is, $\boldsymbol{s}_\rho$ consists of those partial plays of successor length of **II** against ρ where she has not lost yet.) The tree $\boldsymbol{s}_\rho$ is ordered by end extension (i.e., $\langle u_\beta : \beta \leq \alpha\rangle \leq \langle u'_\beta : \beta \leq \alpha'\rangle$ if and only if $\alpha \leq \alpha'$ and $u_\beta = u'_\beta$ for all $\beta \leq \alpha$).

Because ρ is a winning strategy for player **I**, $\boldsymbol{s}_\rho$ does not have any branches of length ξ. Indeed, such a branch would define a run of $\mathcal{G}_\xi(T, t)$ in which player **I** uses ρ, but player **II** wins. Thus, $\boldsymbol{s} = \sigma \boldsymbol{s}_\rho$ is also a tree in $\mathcal{T}_\xi$. It is therefore enough to show the following.

Claim 2.49. *Player* **I** *has a winning strategy in* $\mathcal{G}_{\boldsymbol{s}}(T, t)$.

Proof of Claim 2.49. Player **I** obtains a winning strategy in $\mathcal{G}_{\boldsymbol{s}}(T, t)$ by copying the partial plays of player **II** into $\boldsymbol{s}'$ and defining the rest of his moves δ_β, i_β using ρ.

In more detail, suppose that player **II** has played $\langle u_\beta : \beta < \alpha\rangle$ in $\mathcal{G}_{\boldsymbol{s}}(T, t)$ so far. Then $p_\beta = \langle u_{\beta'} : \beta' \leq \beta\rangle \in \boldsymbol{s}_\rho$, and therefore player **I** can play

$$t_\alpha = \langle p_\beta : \beta < \alpha\rangle \in \boldsymbol{s}.$$

If $\alpha \in \mathrm{Succ}$, then player **I** also lets $\langle i_{\alpha-1}, \delta_{\alpha-1}\rangle = \rho(\langle u_\beta : \beta < \alpha\rangle$. □

The games $\mathcal{G}_{\boldsymbol{t}}(T)$ for well-founded trees $\boldsymbol{t} \in \mathcal{T}_\omega$ lead to the notion of Cantor-Bendixson derivatives for subtrees T of ${}^{<\kappa}\kappa$ given in Definition 2.50 below. We note that Definition 2.50 was motivated by, but is different from, the notion of Cantor-Bendixson derivatives (for subtrees of ${}^{\kappa}\kappa$) given in [11, Definition 1.1.45]. We chose this definition precisely because it implies that the α^{th} derivative $T^{(\alpha)}$ corresponds to the game $\mathcal{G}_{\boldsymbol{b}_\alpha}(T)$ (in the sense of Claim 2.51 below).

We also note that the verbatim analogue of the definition of Cantor-Bendixson derivatives of subtrees of ${}^{\omega}\omega$ (found e.g. in [21, Exercise 6.15]) will not give a satisfactory notion of Cantor-Bendixson derivatives; see Remark 2.59 below.

A node t of a subtree T of ${}^{<\kappa}\kappa$ is a *cofinally splitting* node of T iff for all $\delta < \kappa$ there exists a splitting node $s \in T$ such that $s \supseteq t$ and $\mathrm{ht}(s) \geq \delta$.

Definition 2.50. Suppose T is a subtree of ${}^{\kappa}\kappa$. The α^{th} *Cantor-Bendixson derivative* of T ($\alpha \in \mathrm{Ord}$) is defined, using recursion, as follows:

$$\begin{aligned}
T^{(0)} &= T,\\
T^{(\alpha+1)} &= \big\{t \in T^{(\alpha)} : t \text{ is a cofinally splitting node of } T^{(\alpha)}\big\},\\
T^{(\xi)} &= \bigcap_{\alpha<\xi} T^{(\alpha)} \quad \text{if } \xi \in \mathrm{Lim}.
\end{aligned}$$

The *Cantor-Bendixson rank* $\mathrm{rk}_{CB}(\mathrm{T})$ of T is the unique ordinal α with $T^{(\alpha)} = T^{(\alpha+1)}$.

It is easy to show, by induction on α, that the following statement holds.

Claim 2.51. *Suppose T is a subtree of ${}^{<\kappa}\kappa$. For all ordinals α, we have*

$$T^{(\alpha)} = \mathrm{Ker}_{\boldsymbol{b}_\alpha}(T) = T - \mathrm{Sc}_{\boldsymbol{b}_\alpha}(T).$$

Corollary 2.52. *Suppose T is a subtree of ${}^{\kappa}\kappa$. We have*

$$\mathrm{Ker}_\omega(T) = \bigcap \left\{\mathrm{Ker}_{\boldsymbol{t}}(T) : \boldsymbol{t} \in \mathcal{T}_\omega\right\} = \bigcap \left\{T^{(\alpha)} : \alpha < \mathrm{rk}_{CB}(\mathrm{T})\right\};$$
$$\mathrm{Sc}_\omega(T) = \bigcup\{\mathrm{Sc}_{\boldsymbol{t}}(T) : \boldsymbol{t} \in \mathcal{T}_\omega\} = \bigcup \left\{T - T^{(\alpha)} : \alpha < \mathrm{rk}_{CB}(\mathrm{T})\right\}.$$

Proof. This statement follows from Proposition 2.47, Claim 2.51, and Definition 2.50 (and the fact that if $\boldsymbol{t} \in \mathcal{T}_\omega$, then $\boldsymbol{t} \leq \boldsymbol{b}_\alpha$ for some ordinal α). □

Proposition 2.53 below gives the compares the levels $\mathrm{Ker}_{\boldsymbol{t}}(T)$ and $T - \mathrm{Sc}_{\boldsymbol{t}}(T)$ of the generalized Cantor-Bendixson hierarchies for a subtree T of ${}^{<\kappa}\kappa$ and the levels of the hierarchies associated to its set $[T]$ of κ-branches.

Proposition 2.53. *Suppose T is a subtree of ${}^{<\kappa}\kappa$ and $\boldsymbol{t} \in \mathcal{T}_{\kappa+1}$. If $x \in [T]$ and $t \subsetneq x$, then $\mathcal{G}_{\boldsymbol{t}}(T, t)$ is easier for player* **II** *to win and harder for player* **I** *to win than $\mathcal{V}_{\boldsymbol{t}}([T], x)$. That is,*

$$\mathrm{Ker}_{\boldsymbol{t}}([T]) \subseteq [\mathrm{Ker}_{\boldsymbol{t}}(T)] \qquad \textit{and} \qquad \mathrm{Sc}_{\boldsymbol{t}}([T]) \supseteq N(\mathrm{Sc}_{\boldsymbol{t}}(T))\,.$$

Note that the second statement holds iff $[T] - \mathrm{Sc}_{\boldsymbol{t}}([T]) \subseteq [T - \mathrm{Sc}_{\boldsymbol{t}}(T)]$.

The proposition implies that if $[T]$ if a $\boldsymbol{t}$-perfect set and T is a pruned, then T is a $\boldsymbol{t}$-perfect tree. If T is a $\boldsymbol{t}$-scattered tree, then $[T]$ is a $\boldsymbol{t}$-scattered set.

We will prove a more general version of Proposition 2.53 in Subsection 3.2.3.

Question 2.54. For which cardinals $\kappa = \kappa^{<\kappa}$, which trees $\boldsymbol{t} \in \mathcal{T}_\kappa$ and which families $\mathcal{T}$ of subtrees of ${}^{<\kappa}\kappa$ do either of the following statements hold or consistently hold:

(1) $\mathrm{Ker}_{\boldsymbol{t}}([T]) = [\mathrm{Ker}_{\boldsymbol{t}}(T)]$ for all trees $T \in \mathcal{T}$;
(2) $\mathrm{Sc}_{\boldsymbol{t}}([T]) = N(\mathrm{Sc}_{\boldsymbol{t}}(T))$ for all subtrees $T \in \mathcal{T}$?

Conjecture 2.55. *Suppose κ is a weakly compact cardinal. If every branch of the tree $\boldsymbol{t} \in \mathcal{T}_\kappa$ is of limit length, then $\mathrm{Ker}_{\boldsymbol{t}}([T]) = [\mathrm{Ker}_{\boldsymbol{t}}(T)]$ holds for all subtrees T of ${}^{<\kappa}\kappa$.*

This question and conjecture are the analagoues of Question 2.38 and Conjecture 2.39 for trees $\boldsymbol{t} \in \mathcal{T}_\kappa$.

We now consider the game $\mathcal{G}^*_{\boldsymbol{t}}(T)$ which is like $\mathcal{G}^*_\kappa(T)$ except player **I** also plays nodes in a "clock-tree" $\boldsymbol{t}$ of height $\leq \kappa$. The games $\mathcal{G}^*_{\boldsymbol{t}}(T,t)$ may be used to give a different possible generalization of the Cantor-Bendixson hierarchy for subtrees T of ${}^{<\kappa}\kappa$ (though not in the straightforward way; see Remarks 2.59 and 2.64).

Definition 2.56. Suppose T is a subtree of ${}^{<\kappa}\kappa$ and $\boldsymbol{t} \in \mathcal{T}_{\kappa+1}$. The game $\mathcal{G}^*_{\boldsymbol{t}}(T)$ is played as follows.

I	t_0		i_0	t_1		i_1	$\dots$	t_α		i_α	$\dots$
II		u_0^0, u_0^1			u_1^0, u_1^1		$\dots$		u_α^0, u_α^1		$\dots$

At the beginning of each round, player **I** plays a node $t_\alpha \in \boldsymbol{t}$ such that $t_\beta < t_\alpha$ holds for all $\beta < \alpha$. Next, player **II** plays $u_\alpha^0, u_\alpha^1 \in T$. Lastly, player **I** chooses between u_α^0 and u_α^1 by playing $i_\alpha < 2$.

Player **II** has to play in such a way that

$$u_\alpha^0, u_\alpha^1 \supseteq u_\beta^{i_\beta}$$

for all $\beta < \alpha$. She also has to make sure that

$$\begin{aligned} u_\alpha^0 \perp u_\alpha^1 \quad & \text{if } \alpha \in \mathrm{Succ}, \text{ and} \\ u_\alpha^0 = u_\alpha^1 \quad & \text{if } \alpha \in \mathrm{Lim} \cup \{0\}. \end{aligned}$$

The first player who cannot play legally loses the round, and the other player wins.

For a node $t \in T$, the game $\mathcal{G}^*_{\boldsymbol{t}}(T,t)$ is defined just like $\mathcal{G}^*_{\boldsymbol{t}}(T)$ except player **II** has to start the game with $u_0^0 = u_0^1 = t$.

Definition 2.57. Suppose T is a subtree of ${}^{<\kappa}\kappa$ and $\boldsymbol{t} \in \mathcal{T}_{\kappa+1}$. We let

$$\begin{aligned} \mathrm{Ker}^*_{\boldsymbol{t}}(T) &= \{t \in T : \text{ player } \mathbf{II} \text{ has a winning strategy in } \mathcal{G}^*_{\boldsymbol{t}}(T,t)\}; \\ \mathrm{Sc}^*_{\boldsymbol{t}}(T) &= \{t \in T : \text{ player } \mathbf{I} \text{ has a winning strategy in } \mathcal{G}^*_{\boldsymbol{t}}(T,t)\}. \end{aligned}$$

It can be seen immediately from the definitions that $\mathcal{G}^*_{\boldsymbol{t}}(T)$ is harder for player **I** to win and easier for player **II** to win than $\mathcal{G}_{\boldsymbol{t}}(T)$. More precisely, we have the following.

Claim 2.58. *If T is a subtree of ${}^{<\kappa}\kappa$ and $\boldsymbol{t} \in \mathcal{T}_{\kappa+1}$, then*

$$\mathrm{Sc}^*_{\boldsymbol{t}}(T) \subseteq \mathrm{Sc}_{\boldsymbol{t}}(T) \qquad \text{and} \qquad \mathrm{Ker}^*_{\boldsymbol{t}}(T) \supseteq \mathrm{Ker}_{\boldsymbol{t}}(T)\,.$$

*In other words, let $t \in \boldsymbol{t}$. If player **I** wins $\mathcal{G}^*_{\boldsymbol{t}}(T,t)$, then he wins $\mathcal{G}_{\boldsymbol{t}}(T,t)$. If player **II** wins $\mathcal{G}_{\boldsymbol{t}}(T,t)$, then she wins $\mathcal{G}^*_{\boldsymbol{t}}(T,t)$.*

Example 2.31 shows that the games $\mathcal{G}_{\boldsymbol{t}}(T,t)$ and $\mathcal{G}^*_{\boldsymbol{t}}(T,t)$ are not necessarily equivalent, even when $\boldsymbol{t} = \gamma < \kappa$.

Remark 2.59. Example 2.31 also implies that the games $\mathcal{G}^*_{\boldsymbol{b}_\alpha}(T)$ do not lead to a satisfactory generalization of Cantor-Bendixson derivatives for subtrees T of ${}^{<\kappa}\kappa$.

Given a subtree T of ${}^{<\kappa}\kappa$ we define, recursively,

$$\begin{aligned}
T^{(w,0)} &= T,\\
T^{(w,\alpha+1)} &= \big\{t \in T^{(w,\alpha)} : t \text{ is a splitting node of } T^{(w,\alpha)}\big\},\\
T^{(w,\xi)} &= \bigcap_{\alpha<\xi} T^{(w,\alpha)} \quad \text{if } \xi \in \mathrm{Lim}.
\end{aligned}$$

It is easy to see by induction that

$$T^{(w,\alpha)} = \mathrm{Ker}^*_{\boldsymbol{b}_\alpha}(T) = T - \mathrm{Sc}^*_{\boldsymbol{b}_\alpha}(T).$$

holds for all ordinals α. Thus, if S_ω is the tree defined in Example 2.31, then

$$S_\omega = \mathrm{Ker}^*_\omega(S_\omega) = \mathrm{Ker}^*_{\boldsymbol{b}_\alpha}(S_\omega) = S_\omega^{(w,\alpha)}$$

holds for all ordinals α, even though $[S_\omega]$ is a discrete set. This shows that the above derivatives are not a satisfactory generalization of Cantor-Bendixson derivatives.

We note that the definition of the derivatives $T^{(w,\alpha)}$ is also the verbatim analogue of the definition, in the $\kappa = \omega$ case, of Cantor-Bendixson derivatives for subtrees of ${}^{\omega}\omega$ (found e.g. in [21, Exercise 6.15]).

In Remark 2.36, we commented that there is a "modified version of $\mathcal{G}^*_\gamma(T,t)$" which is easier for player **I** to win and harder for player **II** to win than $\mathcal{G}_\gamma(T,t)$. We now make (a more general version of) the above statement precise.

Let $\boldsymbol{f}$ denote the κ-*fan*, i.e., the tree which consists of branches of all lengths $<\kappa$ joined at the root. For any tree $\boldsymbol{t}$, we let $\boldsymbol{f} \cdot \boldsymbol{t}$ be the tree which is obtained from $\boldsymbol{t}$ by replacing each node $t \in \boldsymbol{t}$ with a copy of $\boldsymbol{f}$. (See Example 1.20 and Definition 1.23 for the precise definitions of $\boldsymbol{f} = \boldsymbol{f}_\kappa$ and of $\boldsymbol{f} \cdot \boldsymbol{t}$.) Note that if $\boldsymbol{t} \in \mathcal{T}_\kappa$, then $\boldsymbol{f} \cdot \boldsymbol{t} \in \mathcal{T}_\kappa$, by Fact 1.24.

Proposition 2.60. *Suppose T is a subtree of ${}^{<\kappa}\kappa$. It $\boldsymbol{t}$ is a tree of height $\leq \kappa$, then*

$$\mathrm{Sc}_{\boldsymbol{t}}(T) \subseteq \mathrm{Sc}^*_{\boldsymbol{f}\cdot\boldsymbol{t}}(T) \qquad \text{and} \qquad \mathrm{Ker}_{\boldsymbol{t}}(T) \supseteq \mathrm{Ker}^*_{\boldsymbol{f}\cdot\boldsymbol{t}}(T)\,.$$

In other words, $\mathcal{G}_{\boldsymbol{t}}(T,t)$ is harder for player **I** *to win and easier for player* **II** *to win than $\mathcal{G}^*_{\boldsymbol{f}_\kappa\cdot\boldsymbol{t}}(T,t)$ for all $t \in T$.*

This proposition is a special case of Proposition 3.30, which will be proven in Section 3.2. The idea of the proof is similar to the one behind the proof of Proposition 2.35 (the $\boldsymbol{t} = \kappa$ case). The proof also uses the following observation. During one round of $\mathcal{G}_{\boldsymbol{t}}(T,t)$ where player **I** plays $t \in \boldsymbol{t}$, he can play an arbitrary number $\xi < \kappa$ of rounds in $\mathcal{G}^*_{\boldsymbol{f}\cdot\boldsymbol{t}}(T,t)$ by playing the nodes in a branch (of length ξ) of the copy of $\boldsymbol{f}$ which replaces $\boldsymbol{t}$. Thus, player **I** can play as many rounds as needed in a run of $\mathcal{G}^*_{\boldsymbol{f}\cdot\boldsymbol{t}}(T,t)$ while the first α rounds of a run of $\mathcal{G}_{\boldsymbol{t}}(T,t)$ are being played. This also shows that $\mathcal{G}^*_{\boldsymbol{f}\cdot\boldsymbol{t}}(T,t)$ is equivalent to the "modified game" described in Remark 2.36 when $\boldsymbol{t} = \gamma$, and that a similar statement holds for trees $\boldsymbol{t}$ in general.

Corollary 2.61. *Suppose $\boldsymbol{t} \in \mathcal{T}_{\kappa+1}$ and $\boldsymbol{t} \equiv \boldsymbol{f}\cdot\boldsymbol{t}$. If T is any subtree of ${}^{<\kappa}\kappa$, then the games $\mathcal{G}^*_{\boldsymbol{t}}(T,t)$ and $\mathcal{G}_{\boldsymbol{t}}(T,t)$ are equivalent for all $t \in T$, i.e.,*

$$\mathrm{Sc}^*_{\boldsymbol{t}}(T) = \mathrm{Sc}_{\boldsymbol{t}}(T) \qquad \text{and} \qquad \mathrm{Ker}^*_{\boldsymbol{t}}(T) = \mathrm{Ker}_{\boldsymbol{t}}(T)\,.$$

Specifically, the games $\mathcal{G}^*_\kappa(T,t)$ and $\mathcal{G}_\kappa(T,t)$ are equivalent for all $t \in T$.

Example 2.62. We give an example of a tree $\boldsymbol{t} \in \mathcal{T}_\kappa$ such that $\boldsymbol{t} \equiv \boldsymbol{f}\cdot\boldsymbol{t}$. We write $\boldsymbol{f}^1 = \boldsymbol{f}$ and let $\boldsymbol{f}^n = \boldsymbol{f}^{n-1}\cdot\boldsymbol{f}$ if $1 < n < \omega$. Let

$$\boldsymbol{f}^\omega = \bigoplus_{n<\omega} \boldsymbol{f}^n,$$

and let $\boldsymbol{t} = \boldsymbol{f}^\omega$. It is clear that $\boldsymbol{f}\cdot\boldsymbol{t} \equiv \boldsymbol{t}$, and we have $\boldsymbol{t} \in \mathcal{T}_\kappa$ due to Fact 1.24.

The next corollary follows from Proposition 2.60 and the fact that $\boldsymbol{t} \in \mathcal{T}_\kappa$ implies $\boldsymbol{f}\cdot\boldsymbol{t} \in \mathcal{T}_\kappa$.

Corollary 2.63. *If T is a subtree of ${}^{<\kappa}\kappa$, then*

$$\begin{aligned}\mathrm{Ker}^*_\kappa(T) &= \bigcap\left\{\mathrm{Ker}^*_{\boldsymbol{f}\cdot\boldsymbol{t}}(T) : \boldsymbol{t} \in \mathcal{T}_\kappa\right\} = \bigcap\left\{\mathrm{Ker}^*_{\boldsymbol{t}}(T) : \boldsymbol{t} \in \mathcal{T}_\kappa\right\},\\ \mathrm{Sc}^*_\kappa(T) &= \bigcup\left\{\mathrm{Sc}^*_{\boldsymbol{f}\cdot\boldsymbol{t}}(T) : \boldsymbol{t} \in \mathcal{T}_\kappa\right\} = \bigcup\left\{\mathrm{Sc}^*_{\boldsymbol{t}}(T) : \boldsymbol{t} \in \mathcal{T}_\kappa\right\}.\end{aligned}$$

Remark 2.64. The games $\mathcal{G}^*_{\boldsymbol{f}\cdot\boldsymbol{t}}(T,t)$ may be used to give a different possible generalization of the Cantor-Bendixson hierarchy for subtrees T of ${}^{<\kappa}\kappa$. Proposition 2.60 shows that the alternate hierarchies (for both players **I** and **II**) are stronger than the original ones, in the sense that

$$\mathrm{Ker}^*_{\boldsymbol{f}\cdot\boldsymbol{t}}(T) \subseteq \mathrm{Ker}_{\boldsymbol{t}}(T) \qquad \text{and} \qquad T - \mathrm{Sc}^*_{\boldsymbol{f}\cdot\boldsymbol{t}}(T) \subseteq T - \mathrm{Sc}_{\boldsymbol{t}}(T)$$

holds at all levels of the hierarchies (i.e., for all $\boldsymbol{t} \in \mathcal{T}_\kappa$).

These games may also be used to give an alternate, stronger, notion of $\boldsymbol{t}$-perfectness for subtrees T of ${}^{<\kappa}\kappa$, and an alternate, weaker, notion of $\boldsymbol{t}$-scatteredness.

In the well-founded case, these modified games lead to the following Cantor-Bendixson derivatives for subtrees T of ${}^{<\kappa}\kappa$. Given a subtree T of ${}^{<\kappa}\kappa$, let

$$T^* = \bigcap_{\xi<\kappa} \mathrm{Ker}^*_\xi(T)\,.$$

That is, $t \in T^*$ iff player **II** wins $\mathcal{G}^*_\xi(T,t)$ for all $\xi < \kappa$, and iff (by Remark 2.30) for all $\xi < \kappa$, there exists an embedding $e : {}^{<\xi}2 \to T$ such that

$$t \subseteq s \text{ implies } e(t) \subseteq e(s) \text{ and } e(s^\frown 0) \perp e(s^\frown 1) \text{ for all } t,s \in {}^{<\xi}2.$$

For a subtree T of ${}^{<\kappa}\kappa$, we define, recursively,

$$T^{(s,0)} = T, \qquad T^{(s,\alpha+1)} = \left(T^{(s,\alpha)}\right)^*, \qquad T^{(s,\alpha)} = \bigcap_{\beta<\alpha} T^{(s,\beta)} \quad \text{if } \alpha \in \mathrm{Lim}.$$

Then the following statements hold:

$$T^{(s,\alpha)} = \mathrm{Ker}^*_{\boldsymbol{f}\cdot\boldsymbol{b}_\alpha}(T) = T - \mathrm{Sc}^*_{\boldsymbol{f}\cdot\boldsymbol{b}_\alpha}(T) \quad \text{for all } \alpha < \kappa;$$

$$\mathrm{Ker}^*_{\boldsymbol{f}\cdot\omega}(T) = T - \mathrm{Sc}^*_{\boldsymbol{f}\cdot\omega}(T) = \bigcap \left\{T^{(s,\alpha)} : \alpha \in \mathrm{Ord}\right\}.$$

(This can be shown by e.g. modifying the proof of Proposition 3.30, i.e., the slightly more general version of Proposition 2.60.)

2.3 Density in itself for the κ-Baire space

In this section, we consider notions of density in itself for the κ-Baire space which correspond to the notions of perfectness considered in the previous sections. We show that the statement "every subset of ${}^\kappa\kappa$ of cardinality $\geq\kappa^+$ has a κ-dense in itself subset"

follows from a hypothesis $\mathrm{I}^{\mathsf{w}}(\kappa)$ which is consistent assuming the consistency of the existence of a weakly compact cardinal $\lambda > \kappa$. Previously, (an equivalent formulation of) this statement was known to follow from a hypothesis $\mathrm{I}^-(\kappa)$ which is equiconsistent with the existence of a measurable cardinal $\lambda > \kappa$, by a result of Jouko Väänänen's [51, Theorem 1]. The hypothesis $\mathrm{I}^{\mathsf{w}}(\kappa)$ is a weaker version of $\mathrm{I}^-(\kappa)$.

The notions of strong κ-perfectness and $\boldsymbol{t}$-perfectness, for trees $\boldsymbol{t}$ of height $\leq \kappa$, lead to the following possible generalizations of density in itself for subsets of the κ-Baire space.

Definition 2.65. Let $X \subseteq {}^\kappa\kappa$ and let $\boldsymbol{t}$ be a tree of height $\leq \kappa$.

(1) We say X is *strongly κ-dense in itself* if $\overline{X}$ is a strongly κ-perfect set.
(2) We say X is *$\boldsymbol{t}$-dense in itself* if $\overline{X}$ is a $\boldsymbol{t}$-perfect set.

Specifically, if $\omega \leq \gamma \leq \kappa$, then X is *γ-dense in itself* iff $\overline{X}$ is γ-perfect. Clearly, a subset X of ${}^\kappa\kappa$ is ω-dense in itself if and only if it is dense in itself (in the original sense, i.e., iff X contains no isolated points). The set Y_0 defined in Example 2.13 is κ-dense in itself and is of cardinality κ.

The notions of κ-density in itself and strong κ-density in itself are often interchangeable; see Proposition 2.69 and Corollary 2.72.

The following observation is immediate from the definition of $\mathrm{Ker}_{\boldsymbol{t}}(X)$. As a corollary, we obtain an equivalent definition of $\boldsymbol{t}$-density in itself.

Claim 2.66. *Suppose $\boldsymbol{t} \in \mathcal{T}_{\kappa+1}$ and every branch of $\boldsymbol{t}$ is infinite. If $X \subseteq {}^\kappa\kappa$, then*

(1) every $x \in \mathrm{Ker}_{\boldsymbol{t}}(X)$ is a limit point of $X \cap \mathrm{Ker}_{\boldsymbol{t}}(X)$,
(2) and therefore

$$\mathrm{Ker}_{\boldsymbol{t}}(X) = \overline{X \cap \mathrm{Ker}_{\boldsymbol{t}}(X)}.$$

Corollary 2.67. *Suppose $\boldsymbol{t} \in \mathcal{T}_{\kappa+1}$ and every branch of $\boldsymbol{t}$ is infinite. A subset $X \subseteq {}^\kappa\kappa$ is $\boldsymbol{t}$-dense in itself if and only if*

$$X \subseteq \mathrm{Ker}_{\boldsymbol{t}}(X)$$

i.e., iff player **II** *has a winning strategy in $\mathcal{V}_{\boldsymbol{t}}(X, x)$ for all $x \in X$.*

Proof. If $X \subseteq \mathrm{Ker}_{\boldsymbol{t}}(X)$, then $\overline{X} = \mathrm{Ker}_{\boldsymbol{t}}(X)$ by Claim 2.66 and therefore X is $\boldsymbol{t}$-dense in itself. To see the other direction, suppose $\overline{X}$ is $\boldsymbol{t}$-perfect. Let $x \in X$, and let τ be a winning strategy for player **II** in $\mathcal{V}_{\boldsymbol{t}}(\overline{X}, x)$. Using τ and the density of X in the set $\overline{X} = \mathrm{Ker}_{\boldsymbol{t}}(\overline{X})$, it is easy to define a winning strategy for player **II** in $\mathcal{V}_{\boldsymbol{t}}(X, x)$. □

Remark 2.68. Let $\gamma \leq \kappa$ be an indecomposable ordinal, and let $X \subseteq {}^{\kappa}\kappa$. Then $\mathrm{Ker}_{\gamma}(X)$ is γ-perfect, and therefore $X \cap \mathrm{Ker}_{\gamma}(X)$ is γ-dense in itself by Claim 2.66. By Corollary 2.67, $X \cap \mathrm{Ker}_{\gamma}(X)$ is the largest γ-dense in itself subset of X. (Note that, specifically, these observations hold for $\gamma = \kappa$). However, this may not hold for decomposable ordinals γ, as Example 2.4 shows.

More generally, suppose $\boldsymbol{t}$ is a reflexive tree of height $\leq \kappa$ (see Definition 1.25). Then $X \cap \mathrm{Ker}_{\boldsymbol{t}}(X)$ is $\boldsymbol{t}$-dense in itself and is the largest $\boldsymbol{t}$-dense in itself subset of X. Thus,

$$X \text{ has a } \boldsymbol{t}\text{-dense in itself subset} \qquad \text{iff} \qquad X \cap \mathrm{Ker}_{\boldsymbol{t}}(X) \neq \emptyset$$

and therefore if and only if player **II** wins $\mathcal{V}_{\boldsymbol{t}}(X)$.

Recall that by Example 2.6, the notions of κ-perfectness and strong κ-perfectness are not equivalent, and therefore neither are the two corresponding notions of κ-density in itself. However, the following connection holds between the two notions.

Proposition 2.69. *Let X be a subset of the κ-Baire space.*

(1)
$$X \cap \mathrm{Ker}_{\kappa}(X) = \bigcup\{Y \subseteq X : Y \textit{ is strongly } \kappa\textit{-dense in itself}\}.$$

(2) X is κ-dense in itself if and only if there exists a collection $\{X_i : i \in I\}$ of strongly κ-dense in itself sets such that $X = \bigcup_{i \in I} X_i$.

We prove this proposition in detail, because some of the proofs in later parts of the thesis will be similar to the argument presented here. The construction in the proof (of the strongly κ-perfect tree T) is a modification of the construction in the proof of [39, Lemma 2.5]. The idea behind it is, in essence, the same as in the proof of [51, Proposition 1].

Proof of Proposition 2.69. Item (2) follows immediately from item (1). To see item (1), first observe that a set $Y \subseteq {}^{\kappa}\kappa$ is strongly κ-dense in itself if and only it the tree T_Y (of initial segments of elements of Y) is a strongly κ-perfect tree. (This is because $\overline{Y} = [T_Y]$.)

Suppose $Y \subseteq X$ is strongly κ-dense in itself and $x \in Y$. Then it is straightforward to construct a winning strategy τ for player **II** in $\mathcal{V}_\kappa(Y, x)$, using the fact that T_Y is a strongly κ-perfect tree. Player **II** uses the fact that the set of splitting nodes of T_Y is cofinal to define her moves in successor rounds of the game, and the $<\kappa$-closure of T_Y to define her moves in limit rounds of the game. Clearly, τ is also a winning strategy for player **II** in $\mathcal{V}_\kappa(X, x)$, and so $x \in \mathrm{Ker}_\kappa(X)$.

Conversely, suppose $x \in X \cap \mathrm{Ker}_\kappa(X)$. Let τ be a winning strategy for player **II** in $\mathcal{V}_\kappa(X, x)$. We define a strongly κ-perfect tree T by having player **II** use τ repeatedly in response to different partial plays of player **I**. The nodes of T will be initial segments of moves of player **II**. We also make sure that all the moves of player **II** obtained during the construction of T end up being κ-branches of T. Thus, the set Y of all such moves of **II** will be a κ-dense in itself set with $x \in Y$.

In more detail, we construct $\langle u_s, x_s, \delta_s : s \in {}^{<\kappa}2\rangle$ such that $u_s \in {}^{<\kappa}\kappa$, $x_s \in X$, and $\delta_s < \kappa$, and the following items hold for all $s, r \in {}^{<\kappa}2$. (An explanation of the last item will be given right below).

(i) $u_s = x_s \restriction \delta_s$;
(ii) if $r \subseteq s$, then $u_r \subseteq u_s$;
(iii) $u_{r^\frown 0} \perp u_{r^\frown 1}$.
(iv) $x_s = \tau\big(\langle \delta_{s\restriction\beta} : \beta < \mathrm{ht}(s), s(\beta) = 1\rangle\big)$.

By item (iv), x_s is obtained from a partial run

$$\langle x_{s\restriction\beta}, \delta_{s\restriction\beta} : \beta < \mathrm{ht}(s), s(\beta) = 1\rangle^\frown\langle x_s\rangle$$

of $\mathcal{V}_\kappa(X, x)$ where player **II** uses the strategy τ. These partial runs split exactly for those $s \in {}^{<\kappa}2$ such that $s = r^\frown 1$ for some $r \in {}^{<\kappa}2$ or s is the union of nodes of the form $r^\frown 1$. If $s = r^\frown 1$, then the partial run for r is extended by player **I** playing δ_r, and player **II** choosing x_s using τ. Whenever s is the union of nodes of the form $r^\frown 1$ (and therefore $\mathrm{ht}(s) \in \mathrm{Lim}$), the partial run is extended by player **II** choosing x_s using τ.

If $s = r^\frown 0$, then the partial run for s does not extend the partial run for r, and so $x_s = x_r$. More generally,

$$\text{if } s \supseteq r \text{ and } s(\alpha) = 0 \text{ for all } \alpha \in [\mathrm{ht}(r), \mathrm{ht}(s)), \text{ then } x_s = x_r. \tag{2.11}$$

To see that this recursive construction can indeed be done, it is enough to check the following: if $\alpha < \kappa$ and $\langle u_r, x_r, \delta_r : r \in {}^{<\alpha}2\rangle$ have been constructed and $\langle x_s : s \in {}^{\alpha}2\rangle$ are as in item (iv), then

(a) $x_{r^\frown 0} \neq x_{r^\frown 1}$

(b) if $r \subsetneq s$, then $u_r \subsetneq x_s$

for all $s \in {}^{\alpha}2$ and $r \in {}^{<\alpha}2$. Item (a) follows immediately by the above observations. If $s(\beta) = 0$ for all $\beta \in [\mathrm{ht}(r), \alpha)$, then (b) holds by ((2.11)). Otherwise, let $\beta \geq \mathrm{ht}(r)$ be such that $s(\beta) = 1$. Then by (iv) and induction,

$$x_s \supseteq x_{s\restriction\beta} \restriction \delta_{s\restriction\beta} = u_{s\restriction\beta} \supseteq u_r$$

By (a) and (b), δ_s and thus u_s can be chosen for all $s \in {}^{\alpha}2$ in such a way that items (i) to (iii) hold.

Once the recursive construction is complete, we let

$$T = \{u_s\restriction\alpha : s \in {}^{<\kappa}2, \alpha < \kappa\} \quad \text{and} \quad Y = \{x_s : s \in {}^{<\kappa}2\}.$$

Then $Y \subseteq X$ and $x \in Y$, and T is a strongly κ-perfect tree by items (ii) and (iii). Notice that by (2.11), $x_s \in [T]$ for all $s \in {}^{<\kappa}2$, and therefore $Y \subseteq [T]$. Conversely, $T \subseteq T_Y$ by item (i). Thus, $T_Y = T$, showing that the set Y is strongly κ-dense in itself. □

Corollary 2.70. *For any subset $X \subseteq {}^{\kappa}\kappa$,*

$$\begin{aligned}\mathrm{Ker}_\kappa(X) &= \bigcup\{\overline{Y} : Y \subseteq X \text{ and } Y \text{ is strongly } \kappa\text{-dense in itself}\}\\ &= \bigcup\{\overline{Y} : Y \subseteq X \text{ and } \overline{Y} \text{ is a strongly } \kappa\text{-perfect set}\}.\end{aligned}$$

Proof. This follows from Proposition 2.69, Claim 2.66 and the observation that the set on the right hand side of the above equation is a closed set. (If x is in the closure of the set on the right hand side, then there exists, for all $\alpha < \kappa$, a strongly κ-dense in itself set $Y_\alpha \subseteq X \cap N_{x\restriction\alpha}$. Let $Y = \bigcup_{\alpha<\kappa} Y_\alpha$. Then Y is a strongly κ-dense in itself subset of X, and $x \in \overline{Y}$.) □

The following statement is the same as Proposition 2.5. As mentioned there, this was in essence observed already in [51]. A different formulation of item (1) can be found in [11].

Corollary 2.71 (essentially [51], [11])**.** *Let X be a subset of the κ-Baire space.*

(1) If X is closed, then

$$\mathrm{Ker}_\kappa(X) = \bigcup\{Z \subseteq X : Z \text{ is a strongly } \kappa\text{-perfect set}\}.$$

(2) X is a κ-perfect set iff X is closed and there exists a collection $\{X_i : i \in I\}$ of strongly κ-perfect sets such that $X = \bigcup_{i \in I} X_i$.

By Proposition 2.69 and Remark 2.68, we have the following equivalent characterizations of a set containing a κ-dense in itself subset.

Corollary 2.72. *If $X \subseteq {}^\kappa\kappa$, then the following statements are equivalent.*

(1) X contains a κ-strongly dense in itself subset.
(2) X contains a κ-dense in itself subset.
(3) $X \cap \mathrm{Ker}_\kappa(X) \neq \emptyset$.
*(4) Player **II** wins $\mathcal{V}_\kappa(X)$.*

In the remainder of this section, we consider a "κ-dense in itself subset property" for arbitrary subsets X of the κ-Baire space.

Definition 2.73. We let DISP_κ denote the following statement.

DISP_κ: every subset X of ${}^\kappa\kappa$ of cardinality $\geq\kappa^+$ has a κ-dense in itself subset.

Notice that DISP_κ implies that the κ-perfect set property holds for closed subsets of the κ-Baire space. Therefore the consistency strength of DISP_κ is at least that of the existence of an inaccessible cardinal $\lambda > \kappa$. A straightforward generalization (from the $\kappa = \omega_1$ case) of [51, Theorem 1] shows that that DISP_κ is implied by a hypothesis $\mathrm{I}^-(\kappa)$ (defined below) which is equiconsistent with the existence of a measurable cardinal $\lambda > \kappa$.

We note that by a result of Philipp Schlicht [39], after Lévy-collapsing an inaccessible $\lambda > \kappa$ to κ^+, every subset of ${}^\kappa\kappa$ which is definable from a κ-sequence of ordinals has a κ-perfect (and therefore κ-dense in itself) subset. Thus, the statement that $\mathrm{PSP}_\kappa(X)$ holds for all sets $X \subseteq {}^\kappa\kappa$ is consistent with DC_κ relative to the existence of an inaccessible above κ [39], and therefore so is DISP_κ.

We show in this section that DISP_κ is implied by a weaker version $\mathrm{I}^{\mathsf{w}}(\kappa)$ of $\mathrm{I}^-(\kappa)$, which is consistent (with ZFC) assuming the consistency of the existence of a weakly compact cardinal $\lambda > \kappa$. (See Definition 2.75 and Theorem 2.76 below.) Therefore the consistency strength of DISP_κ lies between the existence of an inaccessible cardinal above κ and the existence of a weakly compact cardinal above κ.

Definition 2.74 (from [46])**.** We let $\mathrm{I}^-(\kappa)$ denote the following hypothesis.

> $\mathrm{I}^-(\kappa)$: there exists a κ^+-complete normal ideal $\mathcal{I}$ on κ^+ such that the partial order $\langle \mathcal{I}^+, \subseteq \rangle$ contains a dense $<\kappa$-closed subset.

This hypothesis is the modification of the hypothesis $\mathrm{I}(\kappa)$, introduced in [14], which is appropriate for limit cardinals κ. (We note that $\mathrm{I}(\kappa)$ is the same statement as $\mathrm{I}^-(\kappa^+)$.) See also [12, 33, 45] and [51] where the specific case of $\mathrm{I}(\omega)$ is considered.

If κ is a regular cardinal and $\lambda > \kappa$ is measurable, then Lévy-collapsing λ to κ^+ yields a model of ZFC in which $\mathrm{I}^-(\kappa)$ holds. The corresponding statement for $\mathrm{I}(\kappa)$ is an unpublished result of Richard Laver. The proofs can be reconstructed from the $\mathrm{I}(\omega)$ case, which is shown in [12].

We define a weaker version $\mathrm{I}^{\mathsf{w}}(\kappa)$ of the hypothesis $\mathrm{I}^-(\kappa)$ which already holds after Lévy-collapsing a weakly compact cardinal $\lambda > \kappa$ to κ^+

By a λ*-model*, we mean a transitive model M of ZFC^- (ZFC without the power set axiom) such that $|M| = \lambda$, $\lambda \in M$ and ${}^{<\lambda}M \subseteq M$. We denote by $H(\lambda)$ the collection of sets which are heredetarily of cardinality $< \lambda$.

Definition 2.75. We denote by $\mathrm{I}^{\mathsf{w}}(\kappa)$ the following set theoretical hypothesis:

> $\mathrm{I}^{\mathsf{w}}(\kappa)$: for every $X \in H(\kappa^{++})$, there exists a κ^+-model M with $X \in M$, and there exists a κ^+-complete ideal $\mathcal{I}$ on κ^+ such that $\mathcal{I} \subseteq M$ and $\langle \mathcal{I}^+ \cap M, \subseteq \rangle$ contains a dense $<\kappa$-closed subset.

Theorem 2.76 ([40]). *Suppose κ is a regular uncountable cardinal, $\lambda > \kappa$ is weakly compact, and G is $\mathrm{Col}(\kappa, <\lambda)$-generic. Then $\mathrm{I}^{\mathsf{w}}(\kappa)$ holds in $V[G]$.*

This theorem is a joint result of Philipp Schlicht and the author; an article containing the proof is in preparation [40].

Claim 2.77. *Let κ be a cardinal. If $\mathrm{I}^{\mathsf{w}}(\kappa)$ holds, then $\kappa^{<\kappa} = \kappa$.*

Proof. This argument is a straightforward analogue of the proof of the fact that $\mathrm{I}(\omega)$ implies CH which is found on [45, p. 1413].

Suppose, seeking a contradiction, that $\mathrm{I}^{\mathsf{w}}(\kappa)$ holds and $\kappa^{<\kappa} > \kappa$. Then there exists $\alpha < \kappa$ and $A \subseteq {}^{\alpha}\kappa$ such that $|A| = \kappa^+$. By $\mathrm{I}^{\mathsf{w}}(\kappa)$, we can fix a κ^+-model M such that an enumeration of A is in M, and we can fix a κ^+-complete ideal $\mathcal{I} \subseteq M \cap \mathcal{P}(A)$ and a dense $<\kappa$-closed subset K of $\mathcal{I}^+ \cap M$.

It is easy to construct a continuous increasing sequence $\langle t_\beta \in {}^{<\kappa}\kappa : \beta < \alpha\rangle$ and a decreasing sequence $\langle A_\beta \in K : \beta < \alpha\rangle$ such that

$$A_\beta \subseteq N_{t_\beta} \text{ for all } \beta < \alpha.$$

(We use the κ^+-completeness of $\mathcal{I}$ and the density of K at successor stages of the construction and the fact that K is $<\kappa$-closed at limit stages.)

Then $\bigcap_{\beta<\alpha} A_\alpha$ is in $\mathcal{I}^+$ and has at most one element, contradiction. □

We now show that $\mathrm{I}^{\mathsf{w}}(\kappa)$ implies the property DISP_κ. Notice that $\langle \mathcal{I}^+ \cap M, \subseteq\rangle$ contains a dense $<\kappa$-closed subset if and only if player **II** has a winning *tactic* τ in the game $\mathrm{G}_\kappa(\langle \mathcal{I}^+ \cap M, \subseteq\rangle)$ in the sense that in each round $\alpha < \kappa$ in the game, the move $\tau\big(\langle Y_\beta : \beta < \alpha\rangle\big)$ of player **II** depends only on the intersection $\bigcap_{\beta<\alpha} Y_\beta$ of the moves Y_β of player **I** so far. In particular, this statement implies that $\langle \mathcal{I}^+ \cap M, \subseteq\rangle$ is a $<\kappa$-strategically closed partial order (see Definition 1.2.)

Proposition 2.78. *Suppose that* $\mathrm{I}^{\mathsf{w}}(\kappa)$ *holds. Then* DISP_κ *holds, i.e., every set* $X \subseteq {}^\kappa\kappa$ *of cardinality* $\geq\kappa^+$ *has a* κ*-dense in itself subset.*

Proof. This argument is a straightforward strenghening of the proof of [51, Theorem 1].

Suppose $\mathrm{I}^{\mathsf{w}}(\kappa)$ holds. Let X be a subset of the κ-Baire space of cardinality $\geq\kappa^+$; we may assume that $|X| = \kappa^+$. Let M be a κ^+-model such that an enumeration of X is in M, and let $\mathcal{I} \subseteq M$ be a κ^+-complete ideal on X such that $\langle \mathcal{I}^+ \cap M, \subseteq\rangle$ is $<\kappa$-strategically closed. (As remarked in the paragraph right above this proposition, the existence of such M and $\mathcal{I}$ is implied by $\mathrm{I}^{\mathsf{w}}(\kappa)$.)

Using a winning strategy τ for player **II** in $\mathrm{G}_\kappa(\mathcal{I}^+ \cap M) = \mathrm{G}_\kappa(\langle \mathcal{I}^+ \cap M, \subseteq\rangle)$, we define a winning strategy for player **II** in $\mathcal{V}_\kappa(X)$, as described below. This latter winning strategy can then be used to obtain a κ-dense in itself subset of X; see Corollary 2.72.

The following concept is needed to describe the winning strategy of player **II**. Given $Y \subseteq X$ and $x \in Y$, we say that x is an $\mathcal{I}$*-point* of Y iff all basic open neighborhoods U of x satisfy $Y \cap U \in \mathcal{I}^+$.

Claim 2.79. *Every* $Y \in \mathcal{I}^+$ *has an* $\mathcal{I}$*-point.*

Proof of Claim 2.79. Assume that $Y \subseteq X$ has no $\mathcal{I}$-points. Then, for all $x \in Y$, we can choose a basic open neighborhood U_x of x such that $Y \cap U_x \in \mathcal{I}$. Thus, since there are $\kappa^{<\kappa} = \kappa$ basic open sets, Y is a union of $\leq\kappa$ many elements of $\mathcal{I}$. Therefore $Y \in \mathcal{I}$

by the κ^+-completeness of $\mathcal{I}$.

Let $\langle X_\alpha : \alpha < \kappa \rangle$ denote the sequence of moves of player **II** in $\mathrm{G}_\kappa(\mathcal{I}^+ \cap M)$ and let $\langle Y_\alpha : \alpha < \kappa \rangle$ denote the sequence of moves of player **I**.

Player **II** has the following winning strategy in $\mathcal{V}_\kappa(X)$. She lets x_0 be any $\mathcal{I}$-point of $X_0 = \tau(\emptyset) \subseteq X$. If the next move of player **I** in $\mathcal{V}_\kappa(X)$ is $\delta_0 < \kappa$, then in $\mathrm{G}_\kappa(\mathcal{I}^+ \cap M)$ let player **I** play $Y_0 = N_{x_0 \restriction \delta_0} \cap X_0$. Notice that for all $Z \in M$ and for all basic open sets N_s (where $s \in {}^{<\kappa}\kappa$), we have $N_s \cap Z = \{y \in Z : s \subseteq y\} \in M$. Specifically, $X_1 \in M$ and therefore $Y_1 \in M$. Since x_0 is an $\mathcal{I}$-point of X_1, this implies $Y_1 \in \mathcal{I}^+ \cap M$. Then player **II** can let $X_1 = \tau\big(\langle Y_1 \rangle\big)$ and let x_1 be any $\mathcal{I}$-point of X_1.

In general, player **II** obtains her sequence $\langle x_\alpha : \alpha < \kappa \rangle$ of moves in $\mathcal{V}_\kappa(X)$ by simultaneously playing a run $\langle X_\beta, Y_\beta : \beta < \kappa \rangle$ of $\mathrm{G}_\kappa(\mathcal{I}^+ \cap M)$ using her winning strategy τ, where the moves Y_β of player **I** are determined by his moves in $\mathcal{V}_\kappa(X)$. More specifically, if $\alpha < \kappa$ and player **I** has played $\langle \delta_\beta : \beta < \alpha \rangle$ so far in $\mathcal{V}_\kappa(X)$, then let $Y_\beta = N_{x_\beta \restriction \delta_\beta} \cap X_\beta$ for all $\beta < \alpha$, and let $X_\alpha = \tau\big(\langle Y_\beta : \beta < \alpha \rangle\big)$. Player **II** then chooses an arbitrary $\mathcal{I}$-point x_α of X_α as her α^{th} move in $\mathcal{V}_\kappa(X)$. □

By the previous proposition and Theorem 2.76, we have the following.

Corollary 2.80. *Suppose κ is a regular uncountable cardinal, $\lambda > \kappa$ is weakly compact, and G is $\mathrm{Col}(\kappa, <\lambda)$-generic. Then DISP_κ holds in $V[G]$.*

Thus, the consistency strength of DISP_κ lies between cardinal the existence of an inaccessible above κ and a weakly compact cardinal above κ.

Question 2.81. What is the consistency strength of DISP_κ?

Observe that the statement DISP_κ implies the following statement:

$$\begin{aligned}&\text{every subset } X \subseteq {}^\kappa\kappa \text{ can be written as a union}\\ &\qquad \text{of the } \kappa\text{-dense in itself set } X \cap \mathrm{Ker}_\kappa(X) \text{ and a set of cardinality } \leq \kappa. \end{aligned} \tag{2.12}$$

However, there is no reason why DISP_κ (or (2.12)) should imply the determinacy, for all $X \subseteq {}^\kappa\kappa$ and all $x \in X$, of the games $\mathcal{V}_\kappa(X, x)$.

Question 2.82. Is it consistent that the games $\mathcal{V}_\kappa(X, x)$ are determined for *all* subsets X of the κ-Baire space and all $x \in X$?

Question 2.83. Is the following Cantor-Bendixson theorem for *all* subsets of the κ-Baire space consistent?

Every subset $X \subseteq {}^{\kappa}\kappa$ can be written as a disjoint union

$$X = (X \cap \mathrm{Ker}_\kappa(X)) \cup \mathrm{Sc}_\kappa(X)\,, \quad \text{where} \quad |\mathrm{Sc}_\kappa(X)\,| \leq \kappa. \tag{2.13}$$

The statement (2.13) can also be viewed as a strong form of the statement DISP_κ, or as a strong form of the determincacy of the games $\mathcal{V}_\kappa(X, x)$ for all subsets $X \subseteq {}^{\kappa}\kappa$ and all $x \in X$.

CHAPTER 3

Open colorings on generalized Baire spaces

In the first part of this chapter, we look at an uncountable analogue $\mathrm{OCA}_\kappa(X)$ of the Open Coloring Axiom for subsets X of the κ-Baire space. We investigate more closely a natural variant $\mathrm{OCA}^*_\kappa(X)$, concerning the existence of κ-perfect homogeneous sets (the definitions of both $\mathrm{OCA}_\kappa(X)$ and $\mathrm{OCA}^*_\kappa(X)$ are found at the beginning of Section 3.1.) The first main result of this chapter, Theorem 3.14, states that after Lévy-collapsing an inaccessible $\lambda > \kappa$ to κ^+, $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ holds; that is, $\mathrm{OCA}^*_\kappa(X)$ holds for all κ-analytic subsets X of ${}^\kappa\kappa$ (and therefore so does $\mathrm{OCA}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$). Thus, $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ is equiconsistent with the existence of an inaccessible cardinal $\lambda > \kappa$.

In the second part of this chapter, we study analogues for open colorings of the games considered in Chapter 2. We first show that for arbitrary $X \subseteq {}^\kappa\kappa$, $\mathrm{OCA}^*_\kappa(X)$ is equivalent to the determinacy, for all open colorings $R_0 \subseteq [X]^2$, of a cut and choose game associated to R_0 (see Proposition 3.20.)

We then study games played on subsets $X \subseteq {}^\kappa\kappa$ or subtrees of ${}^{<\kappa}\kappa$ which allow trees without κ-branches to generalize different ranks associated to open colorings. We prove comparison theorems which show how the levels of the different generalized hierarchies given by these games compare to each other (see Subsection 3.2.3). At the end of this section, we investigate the behavior of our games of length κ. In particular, we show that $\mathrm{OCA}^*_\kappa(X)$ holds for closed subsets of the κ-Baire space if and only if the natural analogue, for open colorings, of Jouko Väänänen's generalized Cantor-Bendixson theorem holds (see Corollary 3.56).

We assume κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$ throughout the chapter, unless otherwise mentioned.

An open coloring $R_0 \subseteq [X]^2$ on a set $X \subseteq {}^\kappa\kappa$ can be identified with the open partition $[X]^2 = R_0 \cup R_1$. The coloring R_0 can also be identified with the closed symmetric binary relation R_1' on X defined by its complement, i.e., with

$$R_1' = \{(x, y) \in X^2 : \{x, y\} \notin R_0 \text{ or } x = y\}.$$

Thus, the questions studied in this chapter can be reformulated, equivalently, in terms of either binary open colorings on X, open partitions of $[X]^2$, or closed binary relations on X (for subsets X of the κ-Baire space). We will use these formulations interchangeably throughout the chapter.

3.1 A dichotomy for open colorings on subsets of the κ-Baire space

Definition 3.1. Suppose κ is a cardinal such that $\kappa^{<\kappa} = \kappa$ and $X \subseteq {}^\kappa\kappa$. We let $\mathrm{OCA}_\kappa(X)$ denote the following statement.

> $\mathrm{OCA}_\kappa(X)$: for every open partition $[X]^2 = R_0 \cup R_1$, either X is a union of κ many R_1-homogeneous sets, or there exists an R_0-homogeneous set of cardinality κ^+.

If Γ is a collection of subsets of ${}^\kappa\kappa$, then $\mathrm{OCA}_\kappa(\Gamma)$ denotes the statement that $\mathrm{OCA}_\kappa(X)$ holds for all $X \in \Gamma$.

Definition 3.2. Suppose κ is a cardinal such that $\kappa^{<\kappa} = \kappa$ and $X \subseteq {}^\kappa\kappa$. We let $\mathrm{OCA}^*_\kappa(X)$ denote the following statement.

> $\mathrm{OCA}^*_\kappa(X)$: for every open partition $[X]^2 = R_0 \cup R_1$, either X is a union of κ many R_1-homogeneous sets, or there exists a κ-perfect R_0-homogeneous set.

If Γ is a collection of subsets of ${}^\kappa\kappa$, then $\mathrm{OCA}^*_\kappa(\Gamma)$ denotes the statement that $\mathrm{OCA}^*_\kappa(X)$ holds for all $X \in \Gamma$.

Thus, $\mathrm{OCA}^*_\kappa(X)$ is the variant of $\mathrm{OCA}_\kappa(X)$ where, instead of an R_0-homogeneous subset of size κ^+, one looks for a κ-perfect R_0-homogeneous subset. In particular, $\mathrm{OCA}_\kappa(X)$ is implied by $\mathrm{OCA}^*_\kappa(X)$.

The Open Coloring Axiom (OCA) was introduced by Todorčević [48]. It states that $\mathrm{OCA}(X) = \mathrm{OCA}_\omega(X)$ holds for all subsets X of the Baire space ${}^\omega\omega$. (A weaker but symmetric version of the Open Coloring Axiom was introduced in [1]). Since its introduction, the Open Coloring Axiom and its influence on the structure of the real line has become an important area of investigation; see for example [48, 49, 7, 54]. The property $\mathrm{OCA}^*(X) = \mathrm{OCA}^*_\omega(X)$ for subsets X of the Baire space was studied in e.g. [7] and [49, Chapter 10].

In this section, we study $\mathrm{OCA}^*_\kappa(X)$ and $\mathrm{OCA}_\kappa(X)$ for subsets X of the κ-Baire space (for uncountable cardinals $\kappa = \kappa^{<\kappa}$). The main result of this section, Theorem 3.14, states that after Lévy-collapsing an inaccessible $\lambda > \kappa$ to κ^+, $\mathrm{OCA}^*_\kappa(X)$ holds for all κ-analytic (i.e., $\mathbf{\Sigma}^1_1(\kappa)$) subsets $X \subseteq {}^\kappa\kappa$ (and therefore so does $\mathrm{OCA}_\kappa(X)$).

Theorem 3.14. *Suppose that κ is an uncountable regular cardinal, $\lambda > \kappa$ is inaccessible, and G is $\mathrm{Col}(\kappa, {<}\lambda)$-generic. Then $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ holds in $V[G]$.*

We remark that in the original $\kappa = \omega$ case, $\mathrm{OCA}^*(\mathbf{\Sigma}^1_1)$ already holds in ZFC [7, Theorem 1.1].

Note that $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ implies $\mathrm{PSP}(\mathbf{\Sigma}^1_1(\kappa))$, i.e., that the κ-perfect set property holds for $\mathbf{\Sigma}^1_1(\kappa)$ subsets of the κ-Baire space. This latter statement is equiconsistent with the existence of an inaccessible cardinal above κ [9, 19, 39] (see Remark 2.14). Thus, our Theorem 3.14 implies that $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ is also equiconsistent with the existence of an inaccessible cardinal above κ.

Recall that $\mathcal{C}_\kappa$ denotes the family of all closed subsets of the κ-Baire space. We begin this section by describing two examples which show that the "dual version" of $\mathrm{OCA}_\kappa(\mathcal{C}_\kappa)$ does not hold, and neither does the analogue of $\mathrm{OCA}_\kappa(\mathcal{C}_\kappa)$ for $n > 2$.

The first example, Example 3.3, shows that the "dual" of $\mathrm{OCA}_\kappa(\mathcal{C}_\kappa)$ does not hold. That is,

> there exists a closed set $X \subseteq {}^\kappa\kappa$ and a partition $[X]^2 = R_0 \cup R_1$ such that R_0 is a *closed* subset of $[X]^2$, every R_0-homogeneous subset of X is of cardinality $\leq\kappa$, but X is not a union of κ many R_1-homogeneous sets.

Example 3.3 is generalized from [20, Exercise 29.9]. We note that in the original $\kappa = \omega$

case, [20, Exercise 29.9] gives an example of a $\mathbf{\Pi}^1_1$ subset $X \subseteq {}^{\omega}\omega$ for which the dual of $\mathrm{OCA}(X)$ does not hold. The uncountable analogue given in Example 3.3 below, however, provides a closed subset $X \subseteq {}^{\kappa}\kappa$ as a counterexample. See [49, Proposition 10.1] for an example of a closed coloring of the whole Baire space ${}^{\omega}\omega$ showing that the dual of $\mathrm{OCA}({}^{\omega}\omega)$ does not hold.

For a partially ordered set $\mathcal{Q}$ and regular cardinals μ, ν, say that $\mathcal{Q}$ *has a* (μ,ν)*-gap* if there exist sequences $\langle a_\alpha : \alpha < \mu\rangle$ and $\langle b_\alpha : \alpha < \nu\rangle$ in $\mathcal{Q}$ such that

(i) for all $\alpha < \alpha' < \mu$ and $\beta < \beta' < \nu$ we have $a_\alpha <_{\mathcal{Q}} a'_\alpha <_{\mathcal{Q}} b'_\beta <_{\mathcal{Q}} b_\beta$,

(ii) but there is no $c \in \mathcal{Q}$ such that $a_\alpha <_{\mathcal{Q}} c <_{\mathcal{Q}} b_\beta$ for all $\alpha < \mu$ and $\beta < \nu$

In particular, $\mathcal{Q}$ has no $(1,1)$-gaps iff $\mathcal{Q}$ is dense. If μ is any regular cardinal, then $\mathcal{Q}$ has no $(\mu,0)$-gaps iff the cofinality of $\mathcal{Q}$ is $\geq \mu$.

Notice that if $\mathcal{Q}$ is a linear order, then $\mathcal{Q}$ has no (μ,ν)-gaps for any two regular cardinals $\mu,\nu < \kappa$ if and only if $\mathcal{Q}$ is a κ-saturated dense linear order without endpoints. Thus, by the assumption $\kappa^{<\kappa} = \kappa$, there exists a linear order $\mathcal{Q} = \langle Q, \leq_{\mathcal{Q}}\rangle$ of size κ such that $\mathcal{Q}$ has no (μ,ν)-gaps for any two regular cardinals $\mu,\nu < \kappa$.

Example 3.3. Let $\mathcal{Q} = \langle Q, \leq_{\mathcal{Q}}\rangle$ be a linear order of size $|Q| = \kappa$ which has no (μ,ν)-gaps for any regular cardinals $\mu,\nu < \kappa$. Then $\mathcal{P}(Q)$ can be identified with the κ-Cantor space ${}^{\kappa}2$ in a natural way. Consider

$$X = \{y \in \mathcal{P}(Q) : \leq_{\mathcal{Q}}\restriction y \times y \text{ is a well-order.}\}$$

Because $\kappa = \kappa^{<\kappa}$ is uncountable, X is a closed subset of $\mathcal{P}(Q)$.

Define a partition $[X]^2 = R_0 \cup R_1$ by letting, for all $\{x,y\} \in [X]^2$,

$$\{x,y\} \in R_0 \quad \text{iff} \quad x \subseteq y \text{ or } y \subseteq x$$

and letting $R_1 = [X]^2 - R_0$. Clearly, R_0 is a closed subset of $[X]^2$, and X does not have an R_0-homogeneous subset of size κ^+.

Assume, seeking a contradiction, that $X = \bigcup_{\alpha<\kappa} X_\alpha$ for R_1-homogeneous subsets X_α. Using that $\mathcal{Q}$ has no (μ,ν)-gaps whenever $\mu,\nu < \kappa$, one can recursively define a $\subset$-increasing chain $\langle x_\alpha \in X : \alpha < \kappa\rangle$ and a $<_{\mathcal{Q}}$-decreasing chain $\langle q_\alpha \in Q : \alpha < \kappa\rangle$ such that $\sup x_\alpha < q_\alpha$ and, whenever possible, $x_\alpha \in X_\alpha$.

Let $x = \bigcup_{\alpha<\kappa} x_\alpha$. By our assumption, $x \in X_\beta$ for some $\beta < \kappa$. Then $x_\beta \in X_\beta$ by the construction. But we also have $\{x_\beta, x\} \in R_0$, implying that X_β is not R_1-homogeneous after all.

Example 3.4 shows that the 3-dimensional analogue of $\mathrm{OCA}_\kappa({}^\kappa 2)$ fails. That is,

> there exists an open partition $[{}^\kappa 2]^3 = R_0 \cup R_1$ such that every R_0-homogeneous set is of cardinality $\leq\kappa$, but ${}^\kappa 2$ is not a union of κ many R_1-homogeneous sets.

In fact, in the example below, every R_0 homogeneous set has at most 4 elements. This example is the uncountable analogue of an example on [49, p. 80].

Example 3.4. Consider the set

$$C = \{x \in {}^\kappa 2 : x(\alpha) = 0 \text{ for all } \alpha \in \kappa \cap \mathrm{Lim} \text{ and for } \alpha = 0\}.$$

Then C is a closed subset of ${}^\kappa 2$ which is homeomorphic to ${}^\kappa 2$.

We use the following notation for the purposes of this example: if $x, y \in {}^\kappa\kappa$ and $x \neq y$, then let

$$\Delta(x, y) = \min\{\alpha < \kappa : x(\alpha) \neq y(\alpha)\}.$$

If $x, y \in C$ and $x \neq y$, then $\Delta(x, y) \in \mathrm{Succ}$, by definition, and therefore $\Delta(x, y) - 1$ is defined. Observe that for all pairwise distinct $x, y, z \in C$, the set $\{\Delta(x, y), \Delta(y, z), \Delta(z, x)\}$ contains exactly 2 elements and if $\Delta(x, y) = \Delta(y, z) = \alpha$, then $\alpha < \Delta(z, x)$.

Define a partition $[C]^3 = R_0 \cup R_1$ by letting, for all $\{x, y, z\} \in [C]^3$,

$$\{x, y, z\} \in R_0 \quad \text{iff} \quad \Big|\Big\{x\big(\Delta(x, y) - 1\big),\ y\big(\Delta(y, z) - 1\big),\ z\big(\Delta(z, x) - 1\big)\Big\}\Big| = 2.$$

Observe that R_0 is a relatively clopen and dense subset of $[C]^3$. This fact implies that any R_1-homogeneous set H is a nowhere dense subset of C. To see the last statement, suppose $H \subseteq C$ is R_1 homogeneous, and let $s \in T_C$ (i.e., s is an initial segment of an element of C). We need to find a node $s' \in T_C$ extending s such that $N_{s'} \cap H = \emptyset$. Because R_0 is a relatively open dense subset of $[C]^3$, there exist $s_0, s_1, s_2 \in T_C$ extending s such that $N_{s_0} \times N_{s_1} \times N_{s_2} \subseteq K_0$. By the R_1-homogeneity of H, there exists $i < 3$ such that $N_{s_i} \cap H = \emptyset$.

Note that the κ-Baire category theorem holds for C (because C is homeomorphic to 2^κ, and by $\kappa^{<\kappa} = \kappa$). Thus, by the observation above, C cannot be a union of κ many R_1-homogeneous subsets.

Now, assume that H is a subset of C which has at least 5 elements. We will show that H is not R_0-homogeneous. Indeed, because $|H| \geq 5$, there exist pairwise distinct elements $x_0, x_1, x_2, x_3 \in H$ and ordinals $\alpha_1 > \alpha_2 > \alpha_3$ such that

$$\begin{aligned} \alpha_1 &= \Delta(x_0, x_1), \\ \alpha_2 &= \Delta(x_0, x_2) = \Delta(x_1, x_2), \\ \alpha_3 &= \Delta(x_0, x_3) = \Delta(x_1, x_3) = \Delta(x_2, x_3). \end{aligned}$$

Let $0 < i < j \leq 3$ be such that $x_0(\alpha_i - 1) = x_0(\alpha_j - 1)$. It is easy to check that $\{x_0, x_i, x_j\} \in K_1$. This witnesses that H is not R_0-homogeneous.

In the remainder of this section, we will work with the following equivalent version of $\mathrm{OCA}^*_\kappa(X)$:

> if R is a closed symmetric binary relation on X, then either X is a union of κ many R-homogeneous sets, or there exists a κ-perfect R-independent set.

One can also reformulate $\mathrm{OCA}_\kappa(X)$ in an analogous manner.

Note that the notions of κ-perfectness and strong κ-perfectness are interchangeable in $\mathrm{OCA}^*_\kappa(X)$ (see Corollary 2.10).

Lemma 3.5. *Let $X, Y \subseteq {}^\kappa\kappa$. Suppose $f : {}^\kappa\kappa \to {}^\kappa\kappa$ is continuous and $f[X] = Y$.*

(1) If $\mathrm{OCA}_\kappa(X)$ holds, then so does $\mathrm{OCA}_\kappa(Y)$.
(2) If $\mathrm{OCA}^*_\kappa(X)$ holds, then so does $\mathrm{OCA}^*_\kappa(Y)$.

Specifically, if $\mathrm{OCA}^*_\kappa(\mathcal{C}_\kappa)$ holds, then so does $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$.

Proof. For a binary relation R on Y, let

$$R' = \{(x, y) \in X \times X : (f(x), f(y)) \in R \text{ or } f(x) = f(y)\}.$$

That is, R' is the inverse image of $R \cup \mathrm{id}_Y$ under the continuous function $X \times X \to Y \times Y$, $(x, y) \mapsto (f(x), f(y))$. Thus, R' is a closed symmetric relation on X whenever R is a closed symmetric relation on Y.

The image $f[Z]$ of any R'-homogeneous set Z is R-homogeneous. If $Z \subseteq X$ is R'-independent then $f[Z]$ is R-independent, and, by the definition of R', $f \restriction Z$ is injective. These observations imply item (1) immediately.

To see item (2), suppose X has a κ-perfect R'-independent subset. Then (by Corollary 2.10) there exists a continuous injection $g : {}^\kappa 2 \to X$ whose image is R'-independent. By the above observations, $f \circ g$ is a continuous injection of ${}^\kappa 2$ into Y whose image is R-independent, and therefore Y has a κ-perfect subset (again using Corollary 2.10). □

Recall from pages 16 and 16 that given $1 \leq n < \omega$ and $R \subseteq {}^n({}^\kappa\kappa)$, R is a closed n-ary relation on ${}^\kappa\kappa$ if and only if $R = [S]$ for a subtree S of $({}^{<\kappa}\kappa)^{\otimes n}$ and if and only if $R = [S]$ for a pruned subtree S of $({}^{<\kappa}\kappa)^{\otimes n}$.

We will use either the following well-known result or the idea behind its proof a number of times in our later arguments.

Lemma 3.6. *Let $\mathbb{P}$ be a $<\kappa$-strategically closed partial order. Suppose that S is a subtree of $({}^{<\kappa}\kappa)^{\otimes n+1}$, where $n < \omega$. Let T be a subtree of ${}^{<\kappa}\kappa$, and let $t_0, \ldots, t_n \in T$.*

(1) (Folklore; see [30, Proposition 7.3]) If $[S] = \emptyset$, then $\mathbb{P} \Vdash [S] = \emptyset$.
(2) If $N_{t_0} \times \ldots \times N_{t_n} \cap [S] = \emptyset$, then $\mathbb{P} \Vdash N_{t_0} \times \ldots \times N_{t_n} \cap [S] = \emptyset$.

Proof. Suppose that $p \in P$ and $\sigma_0, \ldots \sigma_n$ are $\mathbb{P}$-names such that $p \Vdash (\sigma_0, \ldots, \sigma_n) \in [S]$. Using a winning strategy τ for player **II** in the game $\mathrm{G}_\kappa(\mathbb{P})$, it is straightforward to define a strictly increasing chain $\langle (t_0^\alpha, \ldots, t_n^\alpha) : \alpha < \kappa \rangle$ of elements of S and decreasing chains $\langle p_\alpha : \alpha < \kappa \rangle$ and $\langle q_\alpha : \alpha < \kappa \rangle$ of elements of $\mathbb{P}$ below p such that the following statements hold for all $\alpha < \kappa$:

(i) $p_\alpha = \tau(\langle q_\beta : \beta < \alpha \rangle)$
(ii) $q_\alpha \Vdash t_0^{\alpha+1} \subseteq \sigma_0 \wedge \ldots \wedge t_n^{\alpha+1} \subseteq \sigma_n$;
(iii) if $\alpha \in \mathrm{Lim}$, then $t_i^\alpha = \bigcup_{\beta<\alpha} t_i^\beta$ for all $i < n$.

Note that item (i) implies that $\langle p_\alpha, q_\alpha : \alpha < \kappa \rangle$ is a run of $\mathrm{G}_\kappa(\mathbb{P})$ where player **II** uses the strategy τ.

By items (i)-(iii), we have $p_\alpha \Vdash t_i^\alpha \subseteq \sigma_i$ for all $\alpha < \kappa$ and $i < n$, and therefore $(t_0^\alpha, \ldots, t_n^\alpha) \in S$. Thus, by letting $t_i = \bigcup_{\alpha<\kappa} t_i^\alpha$ for all $i < n$, we obtain the branch $(t_0, \ldots, t_n) \in [S]$.

Item (2) is the special case of item (1) for the tree $T_{\restriction t_0} \times \ldots \times T_{\restriction t_n} \cap S$. Note that $[T_{\restriction t_0} \times \ldots \times T_{\restriction t_n} \cap S] = [T_{\restriction t_0} \times \ldots \times T_{\restriction t_n}] \cap [S] = N_{t_0} \times \ldots \times N_{t_n} \cap [S]$. (Recall that $T_{\restriction t_i}$ denotes the subtree of T which consists of the nodes comparable with t_i.) □

In the remainder of this section, T usually denotes a subtree of ${}^{<\kappa}\kappa$, and S usually denotes a subtree of ${}^{<\kappa}\kappa \otimes {}^{<\kappa}\kappa$. Thus, $[T]$ is a closed subset of the κ-Baire space and $[S]$ is a closed binary relation on the κ-Baire space (and $[S] \cap ([T] \times [T])$ is a closed binary relation on $[T]$). We often assume that S is also a symmetric binary relation on ${}^{<\kappa}\kappa$; in this case, we say that S a *symmetric subtree* of ${}^{<\kappa}\kappa \otimes {}^{<\kappa}\kappa$. This assumption implies that $[S]$ is a symmetric binary relation on ${}^\kappa\kappa$; if S is pruned, the converse also holds.

In the case of closed binary relations on closed subsets of the κ-Baire space, the existence of a κ-perfect independent subset can be characterized in terms of trees.

Recall that for any $t \in {}^{<\kappa}\kappa$, we let $[t] = \{v \in {}^{<\kappa}\kappa : v \supseteq t\}$.

Definition 3.7. Suppose $t, u \in {}^{<\kappa}\kappa$.

(i) If R is a binary relation on ${}^\kappa\kappa$, then let

$$t \perp_R u \quad \text{iff} \quad (N_t \times N_u) \cap R = \emptyset \text{ and } (N_u \times N_t) \cap R = \emptyset \text{ and } t \perp u.$$

(ii) If S is a subtree of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$, then let

$$t \perp_S u \quad \text{iff} \quad ([t] \times [u]) \cap S = \emptyset \text{ and } ([u] \times [t]) \cap S = \emptyset \text{ and } t \perp u.$$

Note that these definitions are absolute between transitive models of ZFC. If $\mathrm{ht}(t) = \mathrm{ht}(u)$, then $t \perp_S u$ holds if and only if we have $(t, u) \notin S$, $(u, t) \notin S$ and $u \neq t$. Moreover,

$$t \perp_S u \quad \text{implies} \quad t \perp_{[S]} u, \quad \text{and}$$
$$t \perp_{[S]} u \quad \text{implies} \quad t \perp_S u \quad \text{whenever } S \text{ is a pruned tree.}$$

Definition 3.8. Let R be a binary relation on ${}^\kappa\kappa$. A map $e : {}^{<\kappa}2 \to {}^{<\kappa}\kappa$ is a *perfect R-embedding* iff for all $t, u \in {}^{<\kappa}2$, we have

(i) $t \subseteq u$ implies $e(t) \subseteq e(u)$ and
(ii) $e(t^\frown 0) \perp_R e(t^\frown 1)$.

Note that perfect R-embeddings are perfect embeddings (see Definition 2.7). When $R = \emptyset$ (and more generally whenever $R \subseteq \mathrm{id}_{{}^\kappa\kappa}$), a map e is a perfect R-embedding if and only if e is a perfect embedding.

Definition 3.9. Let S be a subtree of ${}^{<\kappa}\kappa \otimes {}^{<\kappa}\kappa$. A map $e : {}^{<\kappa}2 \to {}^{<\kappa}\kappa$ is a *perfect S-embedding* iff for all $t, u \in {}^{<\kappa}2$, we have

(i) $t \subseteq u$ implies $e(t) \subseteq e(u)$ and
(ii) $e(t^\frown 0) \perp_S e(t^\frown 1)$.

If a map e is a perfect S-embedding, then e is a perfect $[S]$-embedding. The converse holds whenever $[S]$ is a pruned tree.

Recall that for a perfect embedding $e : {}^{<\kappa}2 \to {}^{<\kappa}\kappa$, we denote by T_e the (strongly) κ-perfect tree determined by e; that is

$$T_e = \{t \in {}^{<\kappa}\kappa : t \subseteq e(u) \text{ for some } u \in {}^{<\kappa}2\}.$$

Recall also, from Corollary 2.10, that if R is a finitary relation on a set $X \subseteq {}^{\kappa}\kappa$, then X has a κ-perfect R-independent subset if and only if X has a strongly κ-perfect R-independent subset. We will often use this fact without mentioning it in the remainder of the section.

Lemma 3.10. *Suppose that R is a binary relation on ${}^{\kappa}\kappa$, T is a subtree of ${}^{<\kappa}\kappa$, and S is a subtree of ${}^{<\kappa}\kappa \otimes {}^{<\kappa}\kappa$.*

(1) If $e : {}^{<\kappa}2 \to {}^{<\kappa}\kappa$ is a perfect R-embedding, then $[T_e]$ is a κ-perfect R-independent set.

(2) If $e : {}^{<\kappa}2 \to T$ is a perfect S-embedding, then

$$[T_e] \textit{ is a } \kappa\textit{-perfect } [S]\textit{-independent subset of } [T],$$

in every transitive model $M \supseteq V$ of ZFC such that $({}^{<\kappa}2)^M = ({}^{<\kappa}2)^V$.

(3) Conversely, if $[T]$ has a κ-perfect $[S]$-independent subset then there exists a perfect $[S]$-embedding $e : {}^{<\kappa}2 \to T$. If S is pruned, then e is also a perfect S-embedding.

Items (2) and (3) imply that if $[T]$ has a κ-perfect $[S]$-independent subset and S is pruned, then $[T]$ has a κ-perfect $[S]$-independent subset in every model $M \supseteq V$ of ZFC with the same ${}^{<\kappa}2$ as V.

Proof. Item (1) is clear. Item (2) follows from item (1), by noting that the formula expressing "e is a perfect S-embedding into T" is absolute between transitive models of ZFC with the same ${}^{<\kappa}2$, and so is the definition of the tree T_e. To see item (3), suppose that T' is a strongly κ-perfect tree such that $[T'] \subseteq X$ and is $[S]$-independent. Using that $[S]$ is a closed relation, it is straightforward to define a perfect $[S]$-embedding $e : {}^{<\kappa}2 \to T'$. If S is pruned, the map e is also a perfect S-embedding. □

The homogeneity of a closed set with respect to a closed finitary relation can also be characterized in terms of trees.

Definition 3.11. Let S be a subtree of $({}^{<\kappa}\kappa)^{\otimes n}$. We say that a subtree H of ${}^{<\kappa}\kappa$ is an *S-homogeneous tree* iff for all pairwise incomparable $t_0, \ldots, t_{n-1} \in H$ we have

$$([t_0] \times \ldots \times [t_{n-1}]) \cap S \neq \emptyset. \tag{3.1}$$

Notice that if $\mathrm{ht}(t_0) = \ldots = \mathrm{ht}(t_{n-1})$, then (3.1) holds if and only if $(t_0, \ldots, t_{n-1}) \notin S$. If H is an S-homogeneous tree, then the set $[H]$ is $[S]$-homogeneous. The converse of this statement holds as well whenever H is a pruned tree.

The formula expressing "H is an S-homogeneous tree" is absolute between transitive models of ZFC. Therefore, if H is an S-homogeneous tree, then

$$[H] \text{ is an } [S]\text{-homogeneous set}$$

in any transitive model of ZFC containing V.

Suppose S is a symmetric subtree of ${}^{<\kappa}\kappa \otimes {}^{<\kappa}\kappa$ (thus, S is a symmetric binary relation on ${}^{<\kappa}\kappa$). Then

$$H \text{ is } S\text{-homogeneous iff we have } h \not\perp_S h' \text{ for all } h, h' \in H.$$

We now turn to proving the main result of this section, Theorem 3.14, which states that $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ holds after Lévy-collapsing an inaccessible $\lambda > \kappa$ to κ^+. The next lemma, which is the key step in the argument, can be stated for $<\kappa$-strategically closed forcings in general.

Lemma 3.12. *Let T be a subtree of ${}^{<\kappa}\kappa$, let S be a symmetric subtree of $T \otimes T$, and suppose that $\mathbb{P}$ is a $<\kappa$-strategically closed notion of forcing. Then at least one of the following holds:*

(1) $\mathbb{P}$ forces that

$$[T] = \bigcup\{[H] : H \in V,\ H \text{ is an } S\text{-homogeneous subtree of } T\},$$

(2) $[T]$ has a κ-perfect $[S]$-independent subset, in any transitive model $M \supseteq V$ of ZFC *such that $({}^{<\kappa}2)^M = ({}^{<\kappa}2)^V$.*

If S is pruned and there exists a $p \in \mathbb{P}$ which adds an element of ${}^{\kappa}2$ then exactly one of the above items holds.

See also Proposition 4.12 below.

Proof. First, assume that $p \in \mathbb{P}$ forces ${}^{\kappa}2 \not\subseteq V$ and that S is pruned. We show that in this case, items (1) and (2) cannot both hold. Note that by [30, Lemma 7.6], our first assumption implies that $p \Vdash [T'] \not\subseteq V$ whenever T' is a κ-perfect tree. Let G be

$\mathbb{P}$-generic with $p \in G$. If item (2) holds, then by Lemma 3.10, there exists a κ-perfect subtree T' of T such that $[T']$ is $[S]$-independent in both V and $V[G]$. We show that if item (1) is also assumed, then $[T']^{V[G]} \subseteq V$, contradicting [30, Lemma 7.6].

To this end, suppose $H \in V$ is an S-homogeneous subtree of T. Then $[H]$ is $[S]$-homogeneous in both V and $V[G]$, and therefore $\left|[T'] \cap [H]\right| \leq 1$ holds in both V and $V[G]$. Using Lemma 3.6 and the fact that $[T'] \cap [H] = [T' \cap H]$, we obtain

$$V \models \left|[T'] \cap [H]\right| = 1 \quad \text{if and only if} \quad V[G] \models \left|[T'] \cap [H]\right| = 1,$$

Thus, $[T' \cap H]^{V[G]} \subseteq V$ for all S-homogeneous subtrees $H \in V$ of T. Therefore if item (1) holds, then $[T']^{V[G]} \subseteq V$.

To see the first part of the theorem, suppose that item (1) does not hold. Then there exists a $\mathbb{P}$-name σ and $p_\emptyset \in \mathbb{P}$ such that

$$p_\emptyset \Vdash \sigma \in \big([T] - \bigcup\{[H] : H \in V,\ H \subseteq T \text{ is an } S\text{-homogeneous subtree}\}\big).$$

Let τ be a winning strategy for player **II** in $\mathrm{G}_\kappa(\mathbb{P})$. We construct, recursively, sequences $\langle t_u \in T : u \in {}^{<\kappa}2\rangle$, $\langle p_u \in \mathbb{P} : u \in {}^{<\kappa}2\rangle$ and $\langle q_u \in \mathbb{P} : u \in {}^{<\kappa}2,\ \mathrm{ht}(u) \in \mathrm{Succ}\rangle$ such that the following hold for all $u, v \in {}^{<\kappa}2$:

(i) $u \subseteq v$ iff $t_u \subseteq t_v$;
(ii) $t_{u^\frown 0} \perp_S t_{u^\frown 1}$.
(iii) $p_u \Vdash t_u \subseteq \sigma$;
(iv) $q_{v^\frown 0}, q_{v^\frown 1} < p_v$ and $p_u = \tau(\langle q_{u\restriction\alpha+1} : \alpha + 1 \leq \mathrm{ht}(u)\rangle)$.

Item (iv) implies that for all $x \in {}^{\kappa}2$, $\langle p_{x\restriction\alpha}, q_{x\restriction\alpha+1} : \alpha < \kappa\rangle$ is a run of $\mathrm{G}_\kappa(\mathbb{P})$ in which player **II** uses the strategy τ.

By the first two items, the embedding ${}^{<\kappa}2 \to T;\ u \mapsto t_u$ is a perfect S-embedding. Thus, by Lemma 3.10, $[T]$ has a κ-perfect $[S]$-independent subset, in any transitive model $M \supseteq V$ of ZFC with the same ${}^{<\kappa}2$ as V.

Let $u \in {}^{<\kappa}2$ and assume that t_v, p_v have been constructed for all $v \subsetneq u$ and that q_v has been constructed whenever $\mathrm{ht}(v)$ is successor. If $\mathrm{ht}(u)$ is a limit ordinal (or if $u = \emptyset$), we can let $t_u = \bigcup\{t_v : v \subsetneq u\}$ and we can define p_u using the winning strategy τ so that (iv) holds. For all $v \subsetneq u$, we have $p_u \leq p_v$ and therefore $p_u \Vdash t_v \subseteq \sigma$. Thus, $p_u \Vdash t_u \subseteq \sigma$.

Now, assume that $\mathrm{ht}(u)$ is successor, and $u = v^\frown i$ for $v \subsetneq u$ and $i \in 2$. For an arbitrary $p \in \mathbb{P}$, define the subtree $T^{(p)}$ of T as follows:

$$T^{(p)} = \{t \in T : (\exists q \leq p) q \Vdash t \subseteq \sigma\}.$$

Notice that $\mathbb{1}_{\mathbb{P}} \Vdash \sigma \in T^{(p)}$, and therefore by our assumption, $T^{(p)}$ cannot be S-homogeneous. Furthermore, if $p \Vdash t \subseteq \sigma$ for some $t \in T$, then $T^{(p)} \subseteq T_{\restriction t}$ (i.e. all nodes in $T^{(p)}$ are comparable with t).

Specifically, $T^{(p_v)} \subseteq T_{t_v}$ and $T^{(p_v)}$ is not S-homogeneous, so there exist $t_{v^\frown 0}, t_{v^\frown 1} \in T^{(p_v)}$ and $q_{v^\frown 0}, q_{v^\frown 1} \in \mathbb{P}$ such that

$$t_{v^\frown 0} \perp_S t_{v^\frown 1} \quad \text{and} \quad t_{v^\frown i} \supseteq t_v,\ q_{v^\frown i} \leq p_v \text{ and } q_{v^\frown i} \Vdash t_{v^\frown i} \subseteq \sigma \text{ for } i < 2.$$

Finally, define $p_{v^\frown i} = \tau(\langle q_{(v^\frown i)\restriction \alpha+1} : \alpha + 1 \leq \mathrm{ht}(v) + 1 \rangle)$ for $i < 2$. Then items (i) to (iv) are satisfied by construction. □

Recall that a subset X of ${}^\kappa\kappa$ is $\mathbf{\Sigma}^1_1(\kappa)$ iff $X = \mathbf{p}Y$ for a closed subset $Y \subseteq {}^\kappa\kappa \times {}^\kappa\kappa$, where $\mathbf{p}Y$ denotes the projection of Y onto the first coordinate.

Corollary 3.13. *Suppose that $\mathbb{P}$ is a $<\kappa$-strategically closed partial order which forces $|(2^\kappa)^V| = \kappa$. Then $\mathbb{P}$ forces the following:*

> *if $T, S \in V$ are such that T is a subtree of ${}^{<\kappa}\kappa$ and S is a symmetric subtree of $({}^{<\kappa}\kappa) \otimes ({}^{<\kappa}\kappa)$, then either $\mathbf{p}[T]$ is a union of κ many $[S]$-homogeneous sets, or there exists a κ-perfect $[S]$-independent set.*

Proof. By Lemma 3.12, $\mathbb{P}$ forces the version of the above statement in which "$\mathbf{p}[T]$" is replaced by "$[T]$". More specifically, if item (1) of Lemma 3.12 holds for trees $T, S \in V$ as above, then $\mathbb{P}$ forces that $[T]$ is a union of $\kappa = |{}^\kappa 2 \cap V|$ many $[S]$-homogeneous sets. Otherwise, item (2) holds, and so $\mathbb{P}$ forces that $[T]$ has a κ-perfect $[S]$-independent subset. Therefore, by Lemma 3.5, $\mathbb{P}$ also forces the original version of the above statement. □

Theorem 3.14. *Suppose that κ is an uncountable regular cardinal, $\lambda > \kappa$ is κ-inaccessible, and G is $\mathrm{Col}(\kappa, <\lambda)$-generic. Then $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ holds in $V[G]$.*

Proof. Working in $V[G]$, suppose that R is a closed symmetric binary relation on a $\mathbf{\Sigma}^1_1(\kappa)$ subset X of ${}^\kappa\kappa$. Let T be a subtree of ${}^{<\kappa}\kappa$ such that $X = \mathbf{p}[T]$ and let S be a symmetric subtree of $({}^{<\kappa}\kappa) \otimes ({}^{<\kappa}\kappa)$ such that $R = [S] \cap (X \times X)$. Because $\mathrm{Col}(\kappa, <\lambda)$ satisfies the λ-chain condition, there exists $\gamma < \lambda$ such that T and S have $\mathrm{Col}(\kappa, <\gamma)$-names, and therefore T and S are in $V[G_\gamma]$, where $G_\gamma = G \cap \mathrm{Col}(\kappa, <\gamma)$. Thus, the conclusion of the theorem holds in the case of $X = \mathbf{p}[T]$ and $R = [S] \cap (X \times X)$ by Corollary 3.13 applied to $V[G_\gamma]$ and $\mathbb{P} = \mathrm{Col}(\kappa, [\gamma, \lambda))$. □

Let X be a subset of the κ-Baire space. Then $\mathrm{OCA}^*_\kappa(X)$ implies $\mathrm{PSP}_\kappa(X)$, i.e., the κ-perfect set property for X. (This can be seen by considering the closed binary relation $R = \mathrm{id}_X$, or equivalently, the trivial partition of $[X]^2$ where the open part of the partition is $R_0 = [X]^2$). Specifically, $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ implies that $\mathrm{PSP}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ holds.

Thus, by results in [9] and a result of Robert Solovay [19], $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ implies that κ^+ is inaccessible in L (see Remark 2.14). This fact and Theorem 3.14 leads us to the following equiconsistency result.

Corollary 3.15. *Let κ be an uncountable cardinal with $\kappa^{<\kappa} = \kappa$. The following statements are equiconsistent.*

(1) There exists an inaccessible cardinal $\lambda > \kappa$.
(2) $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$.

Question 3.16. Does $\mathrm{PSP}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ imply $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$? Does $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ imply $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$?

Recall that $\mathcal{C}_\kappa$ denotes the collection of closed subsets of the κ-Baire space. While $\mathrm{OCA}^*_\kappa(\mathcal{C}_\kappa)$ implies $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ by Proposition 3.5 (and thus also implies $\mathrm{PSP}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$), there is no reason, to the best knowledge of the author, that $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ should imply $\mathrm{PSP}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ (see [22, Question 3.35]).

In the classical case, after Lévy-collapsing an inaccessible cardinal to ω_1, $\mathrm{OCA}^*(X)$ holds for all subsets $X \subseteq {}^\omega\omega$ definable from a countable sequence of ordinals [7]. Furthermore, after Lévy-collapsing an inaccessible $\lambda > \kappa$ to κ^+, $\mathrm{PSP}_\kappa(X)$ holds for all subsets X of the κ-Baire space which are definable from a κ-sequence of ordinals [39, Theorem 2.19].

Conjecture 3.17. *If $\lambda > \kappa$ is inaccessible and G is $\mathrm{Col}(\kappa, {<}\lambda)$-generic, then in $V[G]$, $\mathrm{OCA}^*_\kappa(X)$ holds for all subsets $X \subseteq {}^\kappa\kappa$ definable from a κ-sequence of ordinals.*

Let OCA_κ denote the statement that $\mathrm{OCA}_\kappa(X)$ holds for *all* subsets X of the κ-Baire space.

Question 3.18. Is OCA_κ consistent?

If the answer to the above question is affermative, it would be interesting to see how OCA_κ influences the structure of the κ-Baire space.

3.2 Games for open colorings

In this section, we consider certain games associated to binary open colorings of subsets of the κ-Baire space. These games are natural analogues of games discussed in Chapter 2.

As mentioned at the beginning of the chapter, a binary open coloring on a set X corresponds to the closed binary relation on X which is determined by its compliment. We therefore formulate the definition of the games and our results in terms of closed binary relations instead of open colorings.

3.2.1 A cut and choose game for open colorings.

Let $X \subseteq {}^{\kappa}\kappa$. The game $\mathcal{G}^*_\kappa(X, R)$ defined below is the analogue of the κ-perfect set game $\mathcal{G}^*_\kappa(X)$ for binary relations R on X. We show that the determinacy of the games $\mathcal{G}^*_\kappa(X, R)$ for all closed binary relations R is equivalent to $\mathrm{OCA}^*_\kappa(X)$.

Recall the definition of the relation $\perp_R$ from Definition 3.7.

Definition 3.19. Let R be a binary relation on a subset X of the κ-Baire space. The game $\mathcal{G}^*_\kappa(X, R)$ of length κ is played as follows.

I		i_0		i_1	$\dots$		i_α	$\dots$
II	u^0_0, u^1_0		u^0_1, u^1_1		$\dots$	u^0_α, u^1_α		$\dots$

Player **II** starts each round by playing $u^0_\alpha, u^1_\alpha \in {}^{<\kappa}\kappa$ such that for all $\beta < \alpha$ and $i < 2$ we have $u^i_\alpha \supseteq u^{i_\beta}_\beta$. She also has to make sure that

$$u^0_\alpha \perp_R u^1_\alpha \quad \text{if } \alpha \in \mathrm{Succ}, \text{ and}$$
$$u^0_\alpha = u^1_\alpha \quad \text{if } \alpha \in \mathrm{Lim} \cup \{0\}.$$

Player **I** then chooses between u^0_α and u^1_α by playing $i_\alpha \in 2$.

The nodes $u^{i_\alpha}_\alpha$ produced during a given run define an element $x = \bigcup_{\alpha<\kappa} u^{i_\alpha}_\alpha$ of the κ-Baire space. Player **II** wins the run if $x \in X$.

In the case of $R = \emptyset$ (or more generally, when $R \subseteq \mathrm{id}_X$) $u^0_\alpha \perp_R u^1_\alpha$ is equivalent to $u^0_\alpha \perp u^1_\alpha$. Thus, $\mathcal{G}^*_\kappa(X, R)$ is equivalent to the game $\mathcal{G}^*_\kappa(X)$ in this case.

The game $\mathcal{G}^*_\kappa(X, R)$ is the uncountable version of a game of length ω, associated to closed binary relations R subsets X of the Baire space, which was studied in [7]. The next proposition is the uncountable analogue of [7, Lemmas 3.1 and 3.2].

Proposition 3.20. *Let X be a subset of the κ-Baire space, and suppose R is a closed symmetric binary relation on X.*

*(1) Player **I** has a winning strategy in $\mathcal{G}^*_\kappa(X,R)$ if and only if X is the union of κ many R-homogeneous sets.*
*(2) Player **II** has a winning strategy in $\mathcal{G}^*_\kappa(X,R)$ if and only if X has a κ-perfect R-independent set.*

Thus, $\mathrm{OCA}^*_\kappa(X)$ *is equivalent to the statement that $\mathcal{G}^*_\kappa(X,R)$ is determined for all closed binary relations R on X.*

Proof. By an argument similar to the one in Remark 2.30, a winning strategy for player **II** in $\mathcal{G}^*_\kappa(X,R)$ determines, in a natural way, a perfect R-embedding e such that $[T_e] \subseteq X$. Conversely, a perfect R-embedding e with $[T_e] \subseteq X$ determines a winning strategy for player **II**. This observation implies item (2) immediately (by Lemma 3.10).

To see item (1), suppose first that $X = \bigcup_{\alpha<\kappa} X_\alpha$ where each X_α is R-homogeneous. The strategy of player **I** is to choose $i_{\alpha+1} \in 2$ in such a way that

$$X_\alpha \cap N_{u^{i_{\alpha+1}}_{\alpha+1}} = \emptyset$$

in each successor round $\alpha+1<\kappa$. (This can be done by the homogeneity of X_α and because $u^0_{\alpha+1} \perp_R u^1_{\alpha+1}$.) Now, suppose $x = \bigcup_{\alpha<\kappa} u^{i_\alpha}_\alpha \in {}^\kappa\kappa$ is produced during a run of $\mathcal{G}^*_\kappa(X,R)$ where player **I** uses this strategy. Then for all $\alpha<\kappa$, we have $x \notin X_\alpha$ by $x \supseteq u^{i_{\alpha+1}}_{\alpha+1}$. Thus, $x \notin X$, implying that player **I** wins this run of the game.

We now prove the converse direction. The argument presented here is similar to (the uncountable version of) the arguments in [7, Lemma 3.2] and [21, Theorem 21.1], and also to the proof of [23, Lemma 7.2.2].

Suppose that ρ is a winning strategy of player **I** in $\mathcal{G}^*_\kappa(X,R)$. We say that

$$p = \left\langle (u^0_\beta, u^1_\beta),\, i_\beta : \beta<\alpha \right\rangle$$

is a *good position* iff p is a legal partial run of $\mathcal{G}^*_\kappa(X,R)$ of length $\alpha<\kappa$ in which **I** has played according to ρ. We let $l(p)$ denote the length of p, i.e., $l(p)=\alpha$. The element of ${}^{<\kappa}\kappa$ determined by the partial run p is denoted by $u(p)$, i.e.,

$$u(p) = \bigcup_{\beta<l(p)} u^{i_\beta}_\beta .$$

Note that if $l(p) = \beta + 1$, then $u(p) = u_\beta^{i_\beta}$.

Let $x \in {}^\kappa\kappa$ be arbitrary. We say that p is a *good position for x* iff p is a good position and

$$x \supseteq u(p).$$

A good position p for x is a *maximal good position for x* iff there does not exist a good position p' for x such that $p' \supsetneq p$.

Claim 3.21. *If $x \in X$, then there exists a maximal good position for x.*

Proof of Claim 3.21. The empty sequence is a good position for x, by convention. Suppose there is no maximal good position for x, i.e., every good position for x has a proper extension that is also a good position for x. Then one can define, recursively, a run of $\mathcal{G}^*_\kappa(X, R)$ where player **I** uses ρ and which produces x (i.e. if the run is $\big\langle(u^0_\beta, u^1_\beta),\ i_\beta : \beta < \kappa\big\rangle$, then $x = \bigcup_{\beta<\kappa} u_\beta^{i_\beta}$. At limit stages of the recursion, one uses the following observation. If $\langle p_\beta : \beta < \xi\rangle$ is a strictly increasing chain of good positions for y and $\xi \in \mathrm{Lim}$, then $p = \bigcup_{\beta<\xi} p_\beta$ is also a good position for y). Because ρ is a winning strategy for **I**, this implies $x \notin X$. Thus, the statement of the claim holds.

Claim 3.22. *Suppose p is a good position for x and let $\alpha = l(p)$.*

(1) If p is a maximal good position for x, then α is a successor ordinal.

*(2) p is a maximal good position for x if and only if for every legal move (u^0_α, u^1_α) of player **II** in response to p,*

$$\rho\big(p^\frown\langle(u^0_\alpha, u^1_\alpha)\rangle\big) = i_\alpha, \qquad \textit{implies} \qquad x \not\supseteq u_\alpha^{i_\alpha}.$$

Proof of Claim 3.22. To see item (1), suppose $\alpha = l(p)$ is a limit ordinal. Let $u^0_\alpha = u^1_\alpha = u(p)$ and let

$$p' = p^\frown\big\langle(u^0_\alpha, u^1_\alpha), \rho\big(p^\frown\langle(u^0_\alpha, u^1_\alpha)\rangle\big)\big\rangle.$$

Then p' is a good position for x which extends p, so p cannot be maximal.

Item (2) is clear, using the fact that $u^0_\alpha \perp u^1_\alpha$ holds for all legal moves (u^0_α, u^1_α), by item (1). This ends the proof of Claim 3.22.

For each good position p of $\mathcal{G}^*_\kappa(X, R)$, let

$$X_p = \{x \in X : \ p \text{ is a maximal good position for } x\}.$$

Then $X = \bigcup\{X_p : p \text{ is a good position}\}$, by Claim 3.21. Note that there are at most $\kappa^{<\kappa} = \kappa$ many good positions of $\mathcal{G}^*_\kappa(X, R)$.

We show that if p is a good position, then X_p is R-homogeneous. This implies, by our earlier observations, that X is a union of κ many R-homogeneous sets and therefore completes the proof of the theorem.

Suppose that p is a good position and X_p is not R-homogeneous. Let $x_0, x_1 \in X_p$ be such that $(x_0, x_1) \notin R$ and $x_0 \neq x_1$. Let $\alpha = l(p)$. Using the facts that $X_p \subseteq N_{u(p)}$ and that R is a closed and symmetric relation, we can find $u^0_\alpha, u^1_\alpha \in {}^{<\kappa}\kappa$ such that $x_i \supseteq u^i_\alpha$ for $i \in 2$ and

$$u^0_\alpha \perp_R u^1_\alpha \qquad \text{and} \qquad u^0_\alpha, u^1_\alpha \supseteq u(p)$$

that is, (u^0_α, u^1_α) is a legal response of player **II** to p. Let $i_\alpha = \rho\big(p^\frown\langle(u^0_\alpha, u^1_\alpha)\rangle\big)$. Then $x_{i_\alpha} \supseteq u^{i_\alpha}_\alpha$, implying, by item (2) of Claim 3.22, that p is not a maximal good position for x_{i_α}. This contradicts the assumption that $x_{i_\alpha} \in X_p$. □

The following corollary is the special case of Proposition 3.20 for $R = \emptyset$ (and is also stated as Proposition 2.20).

Corollary 3.23 (essentially Lemma 7.2.2 of [23] for the game $\mathcal{G}^*_\kappa(X)$)**.** *Let $X \subseteq {}^\kappa\kappa$.*

*(1) Player **I** has a winning strategy in $\mathcal{G}^*_\kappa(X)$ iff $|X| \leq \kappa$.*
*(2) Player **II** has a winning strategy in $\mathcal{G}^*_\kappa(X)$ iff X has a κ-perfect subset.*

Thus, $\mathcal{G}^*_\kappa(X)$ is determined iff the κ-perfect set property holds for X.

Proposition 3.20 and Theorem 3.14 yield the following statement.

Corollary 3.24. *Suppose that κ is an uncountable regular cardinal, $\lambda > \kappa$ is inaccessible, and G is* $\mathrm{Col}(\kappa, {<}\lambda)$*-generic. Then, the game $\mathcal{G}^*_\kappa(X, R)$ is determined for all $\mathbf{\Sigma}^1_1(\kappa)$ subsets $X \subseteq {}^\kappa\kappa$ and all closed symmetric binary relations R on X.*

3.2.2 Games for open colorings played on trees

In this subsection, we consider games $\mathcal{G}^*_{\boldsymbol{t}}(T, R)$ and $\mathcal{G}_{\boldsymbol{t}}(T, R)$ played on subtrees T of ${}^{<\kappa}\kappa$ associated to binary relations R on $[T]$. These games allow trees $\boldsymbol{t}$ without κ-branches to generalize certain ranks associated to binary relations on closed subsets of the κ-Baire space. They are the natural analogues of games considered in Subsection 2.2.2.

We prove comparison theorems for these games, the special cases of which were mentioned in Subsection 2.2.2.

Specifically, in the $\boldsymbol{t} = \kappa$ case that the games $\mathcal{G}^*_\kappa(T,R)$ and $\mathcal{G}_\kappa(T,R)$ are equivalent, as a corollary of our results. The game $\mathcal{G}^*_\kappa(T,R)$ is also a reformulation of the game $\mathcal{G}^*_\kappa([T],R)$ defined in the previous subsection. Therefore $\mathrm{OCA}^*_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$ is equivalent to the determinacy of these games.

Definition 3.25. Let T be a subtree of ${}^{<\kappa}\kappa$, and suppose that R is a binary relation on $[T]$. If $\boldsymbol{t}$ is a tree of height $\leq \kappa$, then the game $\mathcal{G}^*_{\boldsymbol{t}}(T,R)$ is played as follows.

I	t_0		i_0	t_1		i_1	$\dots$	t_α		i_α	$\dots$
II		u^0_0, u^1_0			u^0_1, u^1_1		$\dots$		u^0_α, u^1_α		$\dots$

In the first half of each round, player **I** plays a node $t_\alpha \in \boldsymbol{t}$ and in such a way that $t_\beta < t_\alpha$ for all $\beta < \alpha$. In the second half of the round, player **II** first plays $u^0_\alpha, u^1_\alpha \in T$, and player **I** then chooses between u^0_α and u^1_α by playing $i_\alpha < 2$.

Player **II** has to play in such a way that

$$u^0_\alpha, u^1_\alpha \supseteq u^{i_\beta}_\beta$$

for all $\beta < \alpha$. For successor ordinals α, we also require that

$$\begin{aligned} u^0_\alpha \perp_R u^1_\alpha \quad & \text{if } \alpha \in \mathrm{Succ}, \text{ and} \\ u^0_\alpha = u^1_\alpha \quad & \text{if } \alpha \in \mathrm{Lim} \cup \{0\}. \end{aligned}$$

The first player who cannot play legally loses the round, and the other player wins. (In other words, if player **I** cannot play t_α legally, then he loses this run and player **II** wins. If player **II** cannot play u^0_α, u^1_α legally, then she loses this run and player **I** wins.)

For a node $t \in T$, the game $\mathcal{G}^*_{\boldsymbol{t}}(T,R,t)$ is defined just like $\mathcal{G}^*_{\boldsymbol{t}}(T,R)$ except player **II** has to start the game with $u^0_0 = u^1_0 = t$.

When $R = \emptyset$ (or more generally, whenever $R \subseteq \mathrm{id}_{[T]}$), the game $\mathcal{G}^*_{\boldsymbol{t}}(T,R)$ is equivalent to $\mathcal{G}^*_{\boldsymbol{t}}(T)$.

Recall that we also denote by γ the tree which consists of a branch of length γ. In the $\boldsymbol{t} = \gamma$ case, $\mathcal{G}^*_\gamma(T,R)$ is equivalent to a game of length γ in which the two players plays u^i_α and i_α according to the above rules and player **II** wins a run iff she can play legally in all γ rounds. In particular, $\mathcal{G}^*_\gamma(T, \emptyset)$ is equivalent to the game $\mathcal{G}^*_\gamma(T)$ defined in Definition 2.22.

If $\boldsymbol{t}$ and $\boldsymbol{u}$ are trees such that $\boldsymbol{t} \leq \boldsymbol{u}$ (i.e., there exists an order preserving map $f : \boldsymbol{t} \to \boldsymbol{u}$), then $\mathcal{G}^*_{\boldsymbol{t}}(T, R)$ is easier is for player **I** to win and harder for player **II** to win than $\mathcal{G}^*_{\boldsymbol{u}}(T, R)$.

Note that if $\boldsymbol{t}$ has height $\leq \omega$, then $\mathcal{G}^*_{\boldsymbol{t}}(T, R)$ is determined by the Gale-Stewart theorem.

Proposition 3.26. *Let T be a subtree of ${}^{<\kappa}\kappa$. Then $\mathcal{G}^*_\kappa([T], R)$ is equivalent to $\mathcal{G}^*_\kappa(T, R)$, and $\mathcal{G}^*_\kappa([T], R, t)$ is equivalent to $\mathcal{G}^*_\kappa(T, R, t)$ for any node $t \in T$.*

Proposition 3.26 can be proven using the straightforward analogue of the argument in the proof of Proposition 2.24.

Definition 3.27. Let T be a subtree of ${}^{<\kappa}\kappa$, and suppose that R is a binary relation on $[T]$. If $\boldsymbol{t}$ is a tree of height $\leq \kappa$, then the game $\mathcal{G}_{\boldsymbol{t}}(T, R)$ is played as follows.

I	t_0		i_0, δ_0	t_1		i_1, δ_1	$\dots$	t_α		i_α, δ_α	$\dots$
II		u^0_0, u^1_0			u^0_1, u^1_1		$\dots$		u^0_α, u^1_α		$\dots$

In the first half of each round, player **I** plays a node $t_\alpha \in \boldsymbol{t}$ and in such a way that $t_\beta < t_\alpha$ for all $\beta < \alpha$. In the second half of the round, player **II** first plays $u^0_\alpha, u^1_\alpha \in T$, and player **I** then plays ordinals $i_\alpha < 2$ and $\delta_\alpha < \kappa$.

Player **I** has to choose δ_α so that $\delta_\alpha > \delta_\beta$ for all $\beta < \alpha$, and player **II** has to choose so that

$$u^0_\alpha, u^1_\alpha \supseteq u^{i_\beta}_\beta$$

for all $\beta < \alpha$. For successor ordinals $\alpha = \alpha' + 1$, player **II** also has to make sure that

$$u^0_{\alpha'+1} \perp_R u^1_{\alpha'+1} \quad \text{and} \quad u^0_{\alpha'+1} \restriction \delta_{\alpha'} = u^1_{\alpha'+1} \restriction \delta_{\alpha'}.$$

At limit rounds α and in round $\alpha = 0$, she has to play so that $u^0_\alpha = u^1_\alpha$. The first player who cannot play legally loses the round, and the other player wins.

For a node $t \in T$, the game $\mathcal{G}_{\boldsymbol{t}}(T, R, t)$ is defined just like $\mathcal{G}_{\boldsymbol{t}}(T, R)$ except player **II** has to start the game with $u^0_0 = u^1_0 = t$.

Definition 3.28. Suppose T is a subtree of ${}^{<\kappa}\kappa$ and R is a binary relation on $[T]$. For any tree $\boldsymbol{t}$ of height $\leq \kappa$, we let

$$\operatorname{Ker}^*_{\boldsymbol{t}}(T,R) = \{t \in T : \text{ player } \mathbf{II} \text{ has a winning strategy in } \mathcal{G}^*_{\boldsymbol{t}}(T,R,t)\};$$
$$\operatorname{Sc}^*_{\boldsymbol{t}}(T,R) = \{t \in T : \text{ player } \mathbf{I} \text{ has a winning strategy in } \mathcal{G}^*_{\boldsymbol{t}}(T,R,t)\};$$

$$\operatorname{Ker}_{\boldsymbol{t}}(T,R) = \{t \in T : \text{ player } \mathbf{II} \text{ has a winning strategy in } \mathcal{G}_{\boldsymbol{t}}(T,R,t)\};$$
$$\operatorname{Sc}_{\boldsymbol{t}}(T,R) = \{t \in T : \text{ player } \mathbf{I} \text{ has a winning strategy in } \mathcal{G}_{\boldsymbol{t}}(T,R,t)\}.$$

Observe that $\operatorname{Ker}^*_{\boldsymbol{t}}(T,R)$ and $\operatorname{Ker}_{\boldsymbol{t}}(T,R)$ are subtrees of T. Suppose that either

$$S = \operatorname{Sc}^*_{\boldsymbol{t}}(T,R) \quad \text{or} \quad S = \operatorname{Sc}_{\boldsymbol{t}}(T,R)\,.$$

If $s \in S$ and $t \in T$ extends s, then $t \in S$. We let $N(S)$ be the relatively open subset of $[T]$ determined by S, i.e.,

$$\begin{aligned} N(S) &= \bigcup\{N_s : s \in S\} \cap [T] \\ &= \{x \in [T] : \text{ there exists } s \in S \text{ such that } s \subseteq x\}. \end{aligned}$$

Note that when $R = \emptyset$ (or whenever $R \subseteq \mathrm{id}_{[T]}$), all the above concepts reduce to the analogous concepts defined in Subsection 2.2.2.

It is immediately seen that $\mathcal{G}^*_{\boldsymbol{t}}(X,R,t)$ is harder for player **I** to win and easier for player **II** to win than $\mathcal{G}_{\boldsymbol{t}}(X,R,t)$. More precisely, we have the following.

Claim 3.29. *Suppose T is a subtree of ${}^{<\kappa}\kappa$ and R is a binary relation on $[T]$. If $\boldsymbol{t}$ is a tree of height $\leq \kappa$, then*

$$\operatorname{Sc}^*_{\boldsymbol{t}}(T,R) \subseteq \operatorname{Sc}_{\boldsymbol{t}}(T,R) \qquad \textit{and} \qquad \operatorname{Ker}^*_{\boldsymbol{t}}(T,R) \supseteq \operatorname{Ker}_{\boldsymbol{t}}(T,R)\,.$$

The two games $\mathcal{G}^*_{\boldsymbol{t}}(T,R,t)$ and $\mathcal{G}_{\boldsymbol{t}}(T,R,t)$ are not neccessarily equivalent, even when $R = \emptyset$ and $\boldsymbol{t} = \gamma$, where $\gamma < \kappa$. For instance, the tree S_γ defined in Example 2.31 is such that

$$\operatorname{Ker}^*_\gamma(S_\gamma,\emptyset) = S_\gamma = \operatorname{Sc}_\gamma(S_\gamma,\emptyset)\,.$$

However, as a corollary of the next proposition (see Corollary 3.31 below), the two games are equivalent when $\boldsymbol{t} = \kappa$ and also for certain trees $\boldsymbol{t} \in \mathcal{T}_\kappa$.

Recall from Example 1.20 that the κ-fan $\boldsymbol{f}$ is the tree which consists of branches of all lengths $<\kappa$ joined at the root. That is,

$$\boldsymbol{f} = \bigoplus_{\alpha<\kappa} \alpha.$$

We use a^α_β to denote the β^{th} element of the branch of length α. Specifically, $a^\gamma_0 = a^\alpha_0$ for all $0 < \gamma < \alpha < \kappa$.

If $\boldsymbol{t}$ is a tree, then $\boldsymbol{f} \cdot \boldsymbol{t}$ denotes the tree which is obtained from $\boldsymbol{t}$ by replacing each node $t \in \boldsymbol{t}$ with a copy of $\boldsymbol{f}$ (see Definition 1.23 for a precise definition.) Nodes of $\boldsymbol{f} \cdot \boldsymbol{t}$ can be represented as

$$(g, a^\alpha_\beta, t)$$

where $\beta < \alpha < \kappa$, $t \in \boldsymbol{t}$ and $g : \operatorname{pred}_{\boldsymbol{t}}(t) \to \kappa$. We will also think of $g(t')$ as the branch of $\boldsymbol{f}$ of lenght $g(t')$.

Proposition 3.30. *Suppose T is a subtree of ${}^{<\kappa}\kappa$ and R is a binary relation on $[T]$. If $\boldsymbol{t}$ is a tree of height $\leq \kappa$, then*

$$\mathrm{Sc}_{\boldsymbol{t}}(T, R) \subseteq \mathrm{Sc}^*_{\boldsymbol{f}\cdot\boldsymbol{t}}(T, R) \qquad \textit{and} \qquad \mathrm{Ker}_{\boldsymbol{t}}(T, R) \supseteq \mathrm{Ker}^*_{\boldsymbol{f}\cdot\boldsymbol{t}}(T, R)\,.$$

In other words, $\mathcal{G}_{\boldsymbol{t}}(T, R)$ is harder for player **I** *to win and easier for player* **II** *to win than $\mathcal{G}^*_{\boldsymbol{f}\cdot\boldsymbol{t}}(T, R)$ for all $t \in T$.*

Proof. The moves in $\mathcal{G} = \mathcal{G}_{\boldsymbol{t}}(T, R, t)$ will be denoted by t_α, u^0_α, u^1_α, i_α and δ_α, as usual. The moves in $\mathcal{G}^* = \mathcal{G}^*_{\boldsymbol{f}\cdot\boldsymbol{t}}(T, R, t)$ will be denoted by t^*_α, v^0_α, v^1_α and i^*_α.

We describe the idea behind the proof first. A more precise argument can be found a few paragraphs below. This proof combines the argument proving Proposition 2.35 (which is the special case of this proposition for $\boldsymbol{t} = \kappa$ and $R = \emptyset$) with the following observation. During one round of $\mathcal{G}$ where player **I** plays $t_\alpha \in \boldsymbol{t}$, he can play an arbitrary number $\xi < \kappa$ of rounds in $\mathcal{G}^*$ by playing the nodes (in ascending order) of the branch of length ξ in the copy of $\boldsymbol{f}$ which replaces t_α. Player **I** can therefore make sure that

$$\eta_\alpha = \sup\{\delta_\beta + 1 : \beta < \alpha\}$$

many rounds of $\mathcal{G}^*$ are played while the first α rounds of $\mathcal{G}$ are being played.

Notice that if

$$(v^0_{\delta_\beta+1}, v^1_{\delta_\beta+1})$$

is a legal move for player **II** in round $\eta_{\beta+1} = \delta_\beta + 1$ of $\mathcal{G}^*$, then

$$v^0_{\delta_\beta+1} \restriction \delta_\beta = v^1_{\delta_\beta+1} \restriction \delta_\beta \qquad \text{and} \qquad v^0_{\delta_\beta+1} \perp_R v^1_{\delta_\beta+1}. \tag{3.2}$$

Thus, player **I** is able to choose his moves i^*_η in $\mathcal{G}^*$ in such a way that the moves

$$(u^0_{\beta+1}, u^1_{\beta+1}) = (v^0_{\delta_\beta+1}, v^1_{\delta_\beta+1})$$

will also be a legal moves for player **II** in rounds $\beta + 1$ of $\mathcal{G}$.

In more detail, suppose τ is a winning strategy for player **II** in $\mathcal{G}^*$. Let $\alpha < \kappa$ and suppose that player **I** has played t_β, i_β and δ_β (where $\beta < \alpha$) and t_α so far in $\mathcal{G}$. Let

$$\eta_\beta = \sup\{\delta_{\beta'} + 1 : \beta' < \beta\}$$

for all $\beta \leq \alpha$. Note that $\eta_\beta < \kappa$ for all $\beta \leq \alpha$ (by the regularity of κ and by $\delta_{\beta'} < \kappa$).

The strategy of player **II** in round $\alpha < \kappa$ in $\mathcal{G}$ is to play

$$(u^0_\alpha, u^1_\alpha) = (v^0_{\eta_\alpha}, v^1_{\eta_\alpha}),$$

where the moves $v^0_{\eta_\alpha}$ and $v^1_{\eta_\alpha}$ are obtained from a partial run of $\mathcal{G}^*$ where player **II** uses τ and player **I** plays as follows. Let $\beta < \alpha$. Player **I** plays

$$\begin{aligned} i^*_{\eta_\beta} &= i_\beta && \text{and} \\ i^*_\eta &= 0 && \text{in rounds } \eta \text{ such that } \eta_\beta < \eta < \eta_{\beta+1}. \end{aligned}$$

Let $\xi_\beta < \kappa$ be such that

$$\eta_\beta + \xi_\beta = \eta_{\beta+1}.$$

In rounds η such that $\eta_\beta \leq \eta < \eta_{\beta+1}$, player **I** also plays the nodes (in ascending order) of the branch of length ξ_β in the copy of $\boldsymbol{f}$ which replaces t_β. That is, player **I** plays

$$t^*_\eta = (g_\beta, a^{\xi_\beta}_\xi, t_\beta)$$

for all $\xi < \xi_\beta$ and $\eta = \eta_\beta + \xi$. Here, $g_\beta : \mathrm{pred}_{\boldsymbol{t}}(t_\beta) \to \kappa$ is defined by letting $g(t_{\beta'}) = \xi_{\beta'}$ for all $\beta' < \beta$. (Thus, $g(t_{\beta'})$ corresponds to the branch of length $\xi_{\beta'}$ in the copy of $\boldsymbol{f}$ replacing $t_{\beta'}$.) In round η_α, player **I** plays

$$t^*_{\eta_\alpha} = (g_\alpha, a^1_0, t_\alpha),$$

where $g_\alpha : \mathrm{pred}_{\boldsymbol{t}}(t_\alpha) \to \kappa$ is defined by letting $g(t_{\beta'}) = \xi_{\beta'}$ for all $\beta' < \alpha$.

Note that $a^1_0 = a^\xi_0$ for all $\xi < \kappa$, and therefore this strategy is well-defined. The move (u^0_α, u^1_α) is legal for player **II** in round α of $\mathcal{G}$ because (3.2) holds whenever $\alpha \in \mathrm{Succ}$, and because

$$u^i_\alpha = v^i_{\eta_\alpha} \supseteq v^{i^*_{\eta_\beta}}_{\eta_\beta} = u^{i_\beta}_\beta \tag{3.3}$$

holds for all $\beta < \alpha$ and $i < 2$. Therefore the strategy just defined is a winning strategy for player **II** in $\mathcal{G}$.

Using the same idea, we now describe a winning strategy for player one in $\mathcal{G}^*$ assuming he has a winning strategy ρ in $\mathcal{G}$. Suppose $\eta < \kappa$, and suppose that player **II** has played $\langle (v^0_\epsilon, v^1_\epsilon) : \epsilon < \eta \rangle$ in a run of $\mathcal{G}^*$ so far (in a legal way). Using ρ, player **I** can define ordinals $\alpha < \kappa$, $\langle \eta_\beta < \kappa : \beta \leq \alpha \rangle$ and $\langle \xi_\beta < \kappa : \beta < \alpha \rangle$ and a partial run

$$r = \langle t_\beta, (u^0_\beta, u^1_\beta), i_\beta, \delta_\beta : \beta < \alpha \rangle ^\frown \langle t_\alpha \rangle$$

of $\mathcal{G}$ such that the following hold. Player **I** plays according to ρ in r, we have $\eta_\beta = \sup\{\delta_{\beta'} + 1 : \beta' < \beta\}$ for all $\beta \leq \alpha$, and α is the ordinal such that

$$\eta_\beta < \eta \leq \eta_\alpha \text{ for all } \beta < \alpha.$$

(Note that the roles of η and α are slightly different here than in the proof of Proposition 2.35.) Furthermore, $(u^0_\beta, u^1_\beta) = (v^0_{\eta_\beta}, v^1_{\eta_\beta})$ and

$$\eta_\beta + \xi_\beta = \eta_{\beta+1}$$

for all $\beta < \alpha$. Observe that if $\eta < \eta_\alpha$ or $\eta \in \text{Succ}$, then $\alpha \in \text{Succ}$ by the continuity of the function $\alpha + 1 \to \kappa$; $\beta \mapsto \eta_\beta$.

The strategy of player **I** in round η of $\mathcal{G}^*$ is defined as follows. If $\eta = \eta_\alpha$, then player **I** plays

$$t^*_\eta = (g_\alpha, a^1_0, t_\alpha),$$

where $g_\alpha : \text{pred}_{\boldsymbol{t}}(t_\alpha) \to \kappa$ is defined by letting $g(t_\beta) = \xi_\beta$ for all $\beta < \alpha$.

If $\eta < \eta_\alpha$, then let $\xi < \xi_{\alpha-1}$ be such that $\eta = \eta_{\alpha-1} + \xi$ and let $g_{\alpha-1} = g_\alpha \restriction \text{pred}_T(t_{\alpha-1})$. In this case, player **I** plays

$$t^*_\eta = (g_{\alpha-1}, a^{\xi_{\alpha-1}}_\xi, t_{\alpha-1}).$$

If $\eta \in \text{Succ}$, then player **I** also plays

$$i^*_{\eta-1} = i_{\eta_{\alpha-1}} \qquad \text{if } \eta - 1 = \eta_{\alpha-1}, \text{ and}$$
$$i^*_{\eta-1} = 0 \qquad \text{if } \eta - 1 > \eta_{\alpha-1}.$$

The move t_η is well-defined and legal (i.e. $t_\eta > t'_\eta$ for all $\eta' < \eta$) because $a^1_0 = a^\xi_0$ holds for all $\xi < \kappa$ and $t'_\beta < t_\beta$ holds for the moves t_β of player **I** in rounds $\beta' < \beta \leq \alpha$ of the partial run r. The latter statement is true because player **I** plays according to his winning strategy ρ in r, and because all the moves of player **II** are legal in r (by (3.2) and because (3.3) holds by the choice of the i^*_ϵ's in rounds $\epsilon < \eta$.) Thus, the strategy just described is a winning strategy for player **I** in $\mathcal{G}^*$. □

The previous proposition and Proposition 3.26 imply the following statements.

Corollary 3.31. *Suppose* $\boldsymbol{t}$ *is a tree of height* $\leq \kappa$ *such that* $\boldsymbol{t} \equiv \boldsymbol{f} \cdot \boldsymbol{t}$. *Let* T *be a subtree of* ${}^{<\kappa}\kappa$, *and let* R *be a binary relation on* $[T]$.

Then the games $\mathcal{G}_{\boldsymbol{t}}(T, R, t)$ *and* $\mathcal{G}^*_{\boldsymbol{t}}(T, R, t)$ *are equivalent for all* $t \in T$. *That is, we have*

$$\mathrm{Sc}_{\boldsymbol{t}}(T, R) = \mathrm{Sc}^*_{\boldsymbol{t}}(T, R) \qquad \textit{and} \qquad \mathrm{Ker}_{\boldsymbol{t}}(T, R) = \mathrm{Ker}^*_{\boldsymbol{t}}(T, R)\,.$$

Specifically, for all $t \in T$, *the games* $\mathcal{G}_\kappa(T, R, t)$ *and* $\mathcal{G}^*_\kappa(T, R, t)$ *are equivalent, and are also equivalent to* $\mathcal{G}^*_\kappa([T], R, t)$.

For instance, the tree $\boldsymbol{t} = \boldsymbol{f}^\omega$ defined in Example 2.62 is such that $\boldsymbol{t} \in \mathcal{T}_\kappa$ and $\boldsymbol{t} \equiv \boldsymbol{f} \cdot \boldsymbol{t}$.

Corollary 3.32. *Suppose* T *is a subtree of* ${}^{<\kappa}\kappa$ *and* R *is a closed binary relation on* X. *Then the games*

$$\mathcal{G}_\kappa(T, R, t) \qquad \textit{and} \qquad \mathcal{G}^*_\kappa(T, R, t) \qquad \textit{and} \qquad \mathcal{G}^*_\kappa(X, R, t)$$

are all equivalent for every node $t \in T$. *Thus, we have*

$$\mathrm{Ker}_\kappa(T, R) = \mathrm{Ker}^*_\kappa(T, R) = \{t \in T : [T] \cap N_t \textit{ has a } \kappa\textit{-perfect } R\textit{-independent subset}\};$$
$$\mathrm{Sc}_\kappa(T, R) = \mathrm{Sc}^*_\kappa(T, R) = \{t \in T : [T] \cap N_t \textit{ is the union of } \kappa \textit{ many } R\textit{-homogeneous subsets}\};$$

Corollary 3.32 follows from Corollary 3.31 and Proposition 3.20. It implies (together with Proposition 3.5) that $\mathrm{OCA}^*_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$ is equivalent to the determincay of the games $\mathcal{G}^*_\kappa(T, R, t)$ and to the determinacy of the games $\mathcal{G}_\kappa(T, R, t)$. More precisely, the following corollary holds. Recall that $\mathcal{C}_\kappa$ denotes the family of closed subsets of the κ-Baire space.

Corollary 3.33. *The following statements are equivalent:*

(1) $\mathrm{OCA}^*_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$.
(2) $\mathrm{OCA}^*_\kappa(\mathcal{C}_\kappa)$.
(3) If T *is a subtree of* ${}^{<\kappa}\kappa$ *and* R *is a closed symmetric binary relation on* $[T]$, *then*

$$T = \mathrm{Ker}^*_\kappa(T, R) \cup \mathrm{Sc}^*_\kappa(T, R)\,,$$

i.e., the game $\mathcal{G}^*_\kappa(T, R, t)$ *is determined for every* $t \in T$.

(4) If T is a subtree of ${}^{<\kappa}\kappa$ and R is a closed symmetric binary relation on $[T]$, then

$$T = \mathrm{Ker}_\kappa(T, R) \cup \mathrm{Sc}_\kappa(T, R)\,,$$

i.e., the game $\mathcal{G}_\kappa(T, R, t)$ is determined for every $t \in T$.

3.2.3 Games generalizing Väänänen's perfect set game

Suppose $X \subseteq {}^\kappa\kappa$, R is a symmetric binary relation on X, and $\boldsymbol{t}$ is a tree of height $\leq \kappa$. Below, we define two possible generalizations of the game $\mathcal{V}_{\boldsymbol{t}}(X)$. These allow trees $\boldsymbol{t}$ without κ-branches to generalize two different notions of ranks associated to binary relations on subsets of the κ-Baire space, leading to two different generalized hierarchies.

We prove comparison theorems for the games defined here and in the previous subsection, showing how the levels of the corresponding generalized hierarchies are related.

The first game $\mathcal{V}^1_{\boldsymbol{t}}(X, R)$ considered in this subsection is the same game as $\mathcal{V}_{\boldsymbol{t}}(X)$, except that in in successor rounds $\alpha = \beta + 1$, player **II** also has to make sure that $(x_\beta, x_\alpha) \notin R$.

Definition 3.34. Suppose R is a symmetric binary relation on a set $X \subseteq {}^\kappa\kappa$. For any tree $\boldsymbol{t}$ of height $\leq \kappa$, the game $\mathcal{V}^1_{\boldsymbol{t}}(X, R)$ is played as follows.

I	t_0		δ_0	t_1		δ_1	$\dots$	t_α		δ_α	$\dots$	
II		x_0			x_1		$\dots$		x_α			$\dots$

In the first half of each round, player **I** plays a node $t_\alpha \in \boldsymbol{t}$ and in such a way that $t_\beta < t_\alpha$ for all $\beta < \alpha$ In the second half of the round, player **II** first plays an element $x_\alpha \in X$. Then, player **I** plays an ordinal $\delta_\alpha < \kappa$ (and thus chooses a basic open neighborhood of x_α).

Player **I** has to choose δ_α so that $\delta_\beta < \delta_\alpha$ for all $\beta < \alpha$, and player **II** has to choose x_α in such a way that for all $\beta < \alpha$,

$$x_\beta \restriction \delta_\beta = x_\alpha \restriction \delta_\beta \text{ and } x_\alpha \neq x_\beta.$$

At successor ordinals $\alpha = \beta + 1$, we also require that

$$(x_\beta, x_{\beta+1}) \notin R.$$

The first player who cannot play legally loses the run, and the other player wins.

For an arbitrary $x \in {}^\kappa\kappa$, the game $\mathcal{V}^1_{\boldsymbol{t}}(X, R, x)$ is defined just like $\mathcal{V}^1_{\boldsymbol{t}}(X, R)$, except player **II** has to start the game with $x_0 = x$ (and thus $x_0 \notin X$ is allowed).

Note that if $R \subseteq \mathrm{id}_\kappa$, then the game $\mathcal{V}^1_{\boldsymbol{t}}(X, R)$ is equivalent to $\mathcal{V}_{\boldsymbol{t}}(X)$.

The second version, $\mathcal{V}^2_{\boldsymbol{t}}(X, R)$ differs from $\mathcal{V}^1_{\boldsymbol{t}}(X, R)$ as follows. In successor rounds $\alpha = \beta + 1$, player **II** picks two distinct elements x^0_α and x^1_α from the open set determined by the partial run so far, in such a way that $(x^0_\alpha, x^1_\alpha) \notin R$ and $x^0_\alpha \neq x^1_\alpha$. At the end of the round, player **I** chooses one of these elements, by playing $i_\alpha \in 2$. Note that in this version, x^i_α can be equal to the element $x^{i_\beta}_\beta$ played in the previous round for either $i = 0$ or $i = 1$. (At limit rounds and at round 0, player **II** picks one element $x^0_\alpha = x^1_\alpha$ from the open set determined so far.)

Definition 3.35. Suppose R is a symmetric binary relation on a set $X \subseteq {}^\kappa\kappa$. For any tree $\boldsymbol{t}$ of height $\leq \kappa$, the game $\mathcal{V}^2_{\boldsymbol{t}}(X, R)$ is played as follows.

I	t_0		i_0, δ_0	t_1		i_1, δ_1	$\dots$	t_α		i_α, δ_α	$\dots$
II		x^0_0, x^1_0			x^0_1, x^1_1		$\dots$		x^0_α, x^1_α		$\dots$

In the first half of each round, player **I** plays a node $t_\alpha \in \boldsymbol{t}$ and in such a way that $t_\beta < t_\alpha$ for all $\beta < \alpha$.

In the second half of the round, player **II** first plays plays elements $x^0_\alpha, x^1_\alpha \in X$. Then, player **I** plays ordinals $i_\alpha < 2$ and $\delta_\alpha < \kappa$ (and thus chooses between x^0_α and x^1_α, and also chooses a basic open neighborhood of $x^{i_\alpha}_\alpha$). Player **I** has to choose δ_α so that $\delta_\beta < \delta_\alpha$ for all $\beta < \alpha$. Player **II** has to choose x^0_α, x^1_α in such a way that

$$x^{i_\beta}_\beta \restriction \delta_\beta = x^0_\alpha \restriction \delta_\beta = x^1_\alpha \restriction \delta_\beta.$$

In successor rounds α, player **II** also has to make sure that

$$x^0_\alpha \neq x^1_\alpha \quad \text{and} \quad (x^0_\alpha, x^1_\alpha) \notin R.$$

In limit rounds α and in round $\alpha = 0$, she has to play so that $x^0_\alpha = x^1_\alpha$. The first player who cannot move legally loses the run, and the other player wins.

For an arbitrary $x \in {}^\kappa\kappa$, the game $\mathcal{V}^2_{\boldsymbol{t}}(X, R, x)$ is defined just like $\mathcal{V}^2_{\boldsymbol{t}}(X, R)$, except player **II** has to start the game with $x^0_0 = x^1_0 = x$ (and thus $x_0 \notin X$ is allowed).

Definition 3.36. Suppose R is a symmetric binary relation on a set $X \subseteq {}^\kappa\kappa$.

$$\mathrm{Ker}^1_{\boldsymbol{t}}(X,R) = \{x \in {}^\kappa\kappa : \text{ player } \mathbf{II} \text{ has a winning strategy in } \mathcal{V}^1_{\boldsymbol{t}}(X,R,x)\}.$$
$$\mathrm{Sc}^1_{\boldsymbol{t}}(X,R) = \{x \in X : \text{ player } \mathbf{I} \text{ has a winning strategy in } \mathcal{V}^1_{\boldsymbol{t}}(X,R,x)\}.$$

$$\mathrm{Ker}^2_{\boldsymbol{t}}(X,R) = \{x \in {}^\kappa\kappa : \text{ player } \mathbf{II} \text{ has a winning strategy in } \mathcal{V}^2_{\boldsymbol{t}}(X,R,x)\}.$$
$$\mathrm{Sc}^2_{\boldsymbol{t}}(X,R) = \{x \in X : \text{ player } \mathbf{I} \text{ has a winning strategy in } \mathcal{V}^2_{\boldsymbol{t}}(X,R,x)\}.$$

Note that $\mathrm{Sc}^1_{\boldsymbol{t}}(X,R)$ is a relatively open subsets of X, and $\mathrm{Ker}^2_{\boldsymbol{t}}(X,R)$ is a closed subset of ${}^\kappa\kappa$. However, $\mathrm{Sc}^1_{\boldsymbol{t}}(X,R)$ may not be relatively open in X, and $\mathrm{Ker}^1_{\boldsymbol{t}}(X,R)$ may not be closed (even when X and R are closed), as Example 3.37 below shows. Example 3.37 also shows that it is possible to have $\mathrm{Ker}^1_{\boldsymbol{t}}(X,R) = \emptyset$ and $\mathrm{Ker}^2_{\boldsymbol{t}}(X,R) \neq \emptyset$.

The set $\mathrm{Ker}^1_{\boldsymbol{t}}(X,R)$ contains all $\boldsymbol{t}$-perfect R-independent subsets of X (and also contains all $\boldsymbol{t}$-dense in itself R-independent subsets of X. This fact follows from Corollary 2.67 and the observation that if $Y \subseteq X$ is R-independent and $y \in Y$, then the games $\mathcal{V}_{\boldsymbol{t}}(Y,y)$ and $\mathcal{V}^1_{\boldsymbol{t}}(Y,R,y)$ are equivalent). Therefore, by Proposition 3.39 below, the same statement holds for $\mathrm{Ker}^2_{\boldsymbol{t}}(X,R)$.

If $\boldsymbol{t}$ is a reflexive tree (see Definition 1.25), then $\mathrm{Ker}^2_{\boldsymbol{t}}(X,R)$ is a $\boldsymbol{t}$-perfect set and $\mathrm{Ker}^1_{\boldsymbol{t}}(X,R)$ is $\boldsymbol{t}$-dense in itself. For the first statement, we use that $\mathcal{V}_{\boldsymbol{t}}(X,x)$ is equivalent to $\mathcal{V}^2_{\boldsymbol{t}}(X,\mathrm{id}_X,x)$ by Proposition 3.46, and is therefore easier for player **II** to win than $\mathcal{V}^2_{\boldsymbol{t}}(X,R,x)$. Example 2.4 shows that these statements may not hold when $\boldsymbol{t}$ is not reflexive, even for $R = \mathrm{id}_X$.

Example 3.37. Let $x_0 \in {}^\kappa\kappa$, and let $\langle X_\alpha : \alpha < \kappa\rangle$ be a sequence of disjoint closed sets such that $X_\alpha \subseteq N_{x_0\restriction\alpha} - \{x_0\}$. Let

$$X = \{x_0\} \cup \bigcup_{\alpha<\kappa} X_\alpha.$$

Then X is a closed set, and

$$R = (X \times X) - \bigcup_{\alpha<\kappa} (X_\alpha \times X_\alpha)$$

is a closed symmetric relation on X.

(1) First, suppose that X_α is $\boldsymbol{t}$-perfect for every $\alpha < \kappa$. Then

$$\mathrm{Ker}^1_{\boldsymbol{t}}(X,R) = \bigcup_{\alpha<\kappa} X_\alpha = X - \{x_0\} \quad \text{and} \quad \mathrm{Sc}^1_{\boldsymbol{t}}(X,R) = \{x_0\}.$$

Thus, $\mathrm{Ker}^1_{\boldsymbol{t}}(X,R)$ is not closed, and $\mathrm{Sc}^1_{\boldsymbol{t}}(X,R)$ is not a relatively open subset of X. Note that

$$\mathrm{Ker}^2_{\boldsymbol{t}}(X,R) = X \quad \text{and} \quad \mathrm{Sc}^2_{\boldsymbol{t}}(X,R) = \emptyset.$$

(2) Now, suppose that $\langle \gamma_\alpha : \alpha < \kappa \rangle$ is an enumeration of the set of indecomposable ordinals $\gamma < \kappa$ such that for each indecomposable ordinal $\gamma < \kappa$, the set $\{\alpha < \kappa : \gamma_\alpha = \gamma\}$ is cofinal in κ.

Suppose that for all $\alpha < \kappa$, X_α is γ_α-perfect and $\gamma_\alpha + 1$-scattered. (For instance, for each $\alpha < \kappa$, let X_α be a closed subset of $N_{x_0 \restriction \alpha}$ which is homeomorphic to the set Z_{γ_α} defined in Example 2.4.)

Then for the κ-fan $\boldsymbol{f}$ (defined in Example 1.20), we have

$$\mathrm{Ker}^1_{\boldsymbol{f}}(X,R) = \emptyset \qquad \mathrm{Sc}^1_{\boldsymbol{f}}(X,R) = X$$
$$\mathrm{Ker}^2_{\boldsymbol{f}}(X,R) = \{x_0\} \qquad \mathrm{Sc}^2_{\boldsymbol{f}}(X,R) = X - \{x_0\}.$$

Let R be a symmetric binary relation on a set $X \subseteq {}^\kappa\kappa$. Proposition 3.38 below shows that, for $i = 1, 2$, the sets

$$\mathrm{Ker}^i_\kappa(X,R) \qquad \text{and} \quad X - \mathrm{Sc}^i_\kappa(X,R)$$

can be represented as an intersection of the levels

$$\mathrm{Ker}^i_{\boldsymbol{t}}(X,R) \qquad \text{and} \quad X - \mathrm{Sc}^i_{\boldsymbol{t}}(X,R) \qquad\qquad (\boldsymbol{t} \in \mathcal{T}_\kappa) \tag{3.4}$$

of the generalized hierarchy given by the games $\mathcal{V}^i_{\boldsymbol{t}}(X,R,x)$. Analogous statements also hold for the games $\mathcal{G}_{\boldsymbol{t}}(T,R,t)$ and $\mathcal{G}^*_{\boldsymbol{t}}(T,R,t)$ considered in the previous subsection.

Proposition 3.38. *Suppose $X \subseteq {}^\kappa\kappa$ and R is a symmetric binary relation on X. For $i \in 1, 2$, we have*

$$\mathrm{Ker}^i_\kappa(X,R) = \bigcap \left\{\mathrm{Ker}^i_{\boldsymbol{t}}(X,R) : \boldsymbol{t} \in \mathcal{T}_\kappa\right\}, \quad \textit{and} \quad \mathrm{Sc}^i_\kappa(X,R) = \bigcup \left\{\mathrm{Sc}^i_{\boldsymbol{t}}(X,R) : \boldsymbol{t} \in \mathcal{T}_\kappa\right\}.$$

Now, suppose T is a subtree of ${}^{<\kappa}\kappa$ and R is a binary relation on $[T]$. Then we have

$$\mathrm{Ker}^*_\kappa(T,R) = \mathrm{Ker}_\kappa(T,R) = \bigcap \{\mathrm{Ker}_{\boldsymbol{t}}(T,R) : \boldsymbol{t} \in \mathcal{T}_\kappa\} = \bigcap \{\mathrm{Ker}^*_{\boldsymbol{t}}(T,R) : \boldsymbol{t} \in \mathcal{T}_\kappa\},$$
$$\mathrm{Sc}^*_\kappa(T,R) = \mathrm{Sc}_\kappa(T,R) = \bigcup \{\mathrm{Sc}_{\boldsymbol{t}}(T,R) : \boldsymbol{t} \in \mathcal{T}_\kappa\} = \bigcup \{\mathrm{Sc}^*_{\boldsymbol{t}}(T,R) : \boldsymbol{t} \in \mathcal{T}_\kappa\}.$$

The proof of this proposition is the straightforward analogue of the proof of [51, Theorem 5] or of the proof of Proposition 2.47 herein. The last statements also make use of the comparison theorems from the previous subsection, and the fact that $\boldsymbol{t} \in \mathcal{T}_\kappa$ implies $\boldsymbol{f} \cdot \boldsymbol{t} \in \mathcal{T}_\kappa$ (where $\boldsymbol{f}$ is the κ-fan; see Subsection 1.1.2).

Let T be a subtree of ${}^{<\kappa}\kappa$, let $X \subseteq [T]$, and suppose R is a symmetric binary relation on $[T]$. In the rest of this subsection, we show how the levels (3.4) of the hierarchies given by the games $\mathcal{V}^1_{\boldsymbol{t}}(X, R, x)$ and $\mathcal{V}^2_{\boldsymbol{t}}(X, R, x)$ compare to each other, and to the levels

$$\mathrm{Ker}_{\boldsymbol{t}}(T, R) \qquad \text{and} \quad T - \mathrm{Sc}_{\boldsymbol{t}}(T, R) \qquad\qquad (\boldsymbol{t} \in \mathcal{T}_\kappa)$$

of the generalized hierarchy given by the games $\mathcal{G}_{\boldsymbol{t}}(T, R, t)$. These comparison results can be reformulated as statements comparing how difficult it is for player **I** or player **II** to win the games $\mathcal{V}^1_{\boldsymbol{t}}(X, R, x)$, $\mathcal{V}^2_{\boldsymbol{t}}(X, R, x)$ and $\mathcal{G}_{\boldsymbol{t}}(T, R, t)$ (when $t \subsetneq x$).

Comparison theorems for the games $\mathcal{G}_{\boldsymbol{t}}(T, R, t)$ and $\mathcal{G}^*_{\boldsymbol{t}}(T, R, t)$ were obtained in Subsection 3.2.2. At the end of this subsection, we summarize the comparison results shown here and in Subsection 3.2.2; see Corollary 3.48.

We first show in the next two propositions that $\mathcal{V}^1_{\boldsymbol{t}}(X, R, x)$ is always harder for player **II** to win and easier for player **I** to win then $\mathcal{V}^2_{\boldsymbol{t}}(X, R, x)$. When R is an equivalence relation, the converse also holds by Proposition 3.46, i.e., the two games are equivalent in this case. Example 3.37 shows that this may not be the case otherwise (even for closed X and R).

Proposition 3.39. *Suppose R is a symmetric binary relation on a set $X \subseteq {}^\kappa\kappa$ and $\boldsymbol{t}$ is a tree of height $\leq \kappa$. Then*

$$\mathrm{Ker}^1_{\boldsymbol{t}}(X, R) \subseteq \mathrm{Ker}^2_{\boldsymbol{t}}(X, R)\,,$$

*i.e., if $x \in X$ and player **II** wins $\mathcal{V}^1_{\boldsymbol{t}}(X, R, x)$, then player **II** wins $\mathcal{V}^2_{\boldsymbol{t}}(X, R, x)$.*

Proof. Suppose player **II** has a winning strategy τ in $\mathcal{V}^1 = \mathcal{V}^1_{\boldsymbol{t}}(X, R, x)$. We describe a winning strategy for player **II** in $\mathcal{V}^2 = \mathcal{V}^2_{\boldsymbol{t}}(X, R, x)$. The moves of player **I** in $\mathcal{V}^1$ will be denoted by by t'_α and δ'_α. The moves of player **II** in $\mathcal{V}^1$ will be denoted by y_α or sometimes by y'_α or y''_α. (The moves in $\mathcal{V}^2$ will be denoted by t_α, x^0_α, x^1_α, δ_α and i_α, as usual.)

We describe the first few steps of the strategy in order for the idea behind it to be more clear. A more precise description can be found a few paragraphs below.

Suppose player **I** starts $\mathcal{V}^2$ by playing t_0. Then, in $\mathcal{V}^1$, let player **I** play $t'_0 = t_0$. The first move of player **II** in $\mathcal{V}^2$ is $(x^0_0, x^0_1) = (x, x)$ by definition (and her first move in $\mathcal{V}^1$ is $y_0 = x$). If the next moves of player **I** in $\mathcal{V}^2$ are i_0, δ_0 and t_1, then in $\mathcal{V}^2$, let player **II** play

$$\begin{aligned} x^0_1 &= y_0 = \tau(\langle t_0 \rangle), \\ x^1_1 &= y'_1 = \tau(\langle t_0, \delta_0, t_1 \rangle). \end{aligned}$$

Let i_1, δ_1 and t_2 denote the next moves of player **I** in $\mathcal{V}^2$. If $i_1 = 1$, then let player **II** play

$$\begin{aligned} x^0_2 = x^{i_1}_1 &= y'_1 = \tau(\langle t_0, \delta_0, t_1 \rangle), \\ x^1_2 = \qquad &y'_2 = \tau(\langle t_0, \delta_0, t_1, \delta_1, t_2 \rangle). \end{aligned}$$

If $i_1 = 0$, then player **II** plays

$$\begin{aligned} x^0_2 = x^{i_1}_1 &= y_0 = \tau(\langle t_0 \rangle), \\ x^1_2 = \qquad &y''_1 = \tau(\langle t_0, \delta_1, t_2 \rangle). \end{aligned}$$

More generally, suppose that player **I** has played $i_{\beta+1} = 0$ in $\mathcal{V}^2$ for all successor ordinals $\beta + 1 < \alpha$. (Note that the value of i_β at limit rounds β will not be used when defining the strategy of player **II** in $\mathcal{V}^2$.) Then player **II** plays

$$\begin{aligned} x^0_\alpha = x^{i_{\alpha-1}}_{\alpha-1} &= y_0 = \tau(\langle t_0 \rangle), \\ x^1_\alpha = \qquad &y_1 = \tau(\langle t_0, \delta_{\alpha-1}, t_\alpha \rangle) \end{aligned}$$

in $\mathcal{V}^2$ in the case that $\alpha \in \mathrm{Succ}$. If $\alpha \in \mathrm{Lim}$, then player **II** plays $x^0_\alpha = x^1_\alpha = y_0$.

If, at some point, player **I** plays $i_\alpha = 1$ in a successor round α of $\mathcal{V}^2$ (and he has been playing $i_{\beta+1} = 0$ in successor rounds so far), then player **II** plays

$$\begin{aligned} x^0_{\alpha+1} = x^{i_\alpha}_\alpha &= y_1 = \tau(\langle t_0, \delta_{\alpha-1}, t_\alpha \rangle) \\ x^1_{\alpha+1} = \qquad &y_2 = \tau(\langle t_0, \delta_{\alpha-1}, t_\alpha, \delta_\alpha, t_{\alpha+1} \rangle). \end{aligned}$$

In general, player **II** obtains her move x^i_α in $\mathcal{V}^2$ from a partial run of $\mathcal{V}^1$ where she uses τ and the sequence q^i_α of the moves of player **I** is determined by the moves player **I** has played in $\mathcal{V}^2$ so far.

More precisely, suppose that the moves t_β, (x^0_β, x^1_β), δ_β, i_β (where $\beta < \alpha$) and t_α have been played so far in $\mathcal{V}^2$. In the course of the recursive construction of the strategy, partial plays q^i_β of player **I** in $\mathcal{V}^1$ have also been defined for all $\beta < \alpha$ and $i < 2$ (in such a way that $x^i_\beta = \tau(q^i_\beta)$ and $q^i_\beta \supset q^{i_{\beta'}}_{\beta'}$ for all $\beta' < \beta < \alpha$.)

The strategy of player **II** in round α of $\mathcal{V}^2$ is to play

$$x^0_\alpha = \tau(q^0_\alpha) \qquad \text{and} \qquad x^1_\alpha = \tau(q^1_\alpha),$$

where the partial play q^i_α of player **I** in $\mathcal{V}^1$ is defined as follows. If $\alpha \in \text{Succ}$, then

$$q^0_\alpha = q^{i_{\alpha-1}}_{\alpha-1} \qquad \text{and} \qquad q^1_\alpha = q^0_\alpha{}^\frown\langle \delta_{\alpha-1}, t_\alpha\rangle.$$

That is, the partial run of $\mathcal{V}^1$ which determines x^0_α is not extended from the partial run for $x^{i_\alpha-1}_{\alpha-1}$, and therefore

$$x^0_\alpha = x^{i_\alpha-1}_{\alpha-1}.$$

The partial run for x^1_α is obtained by playing one more round of $\mathcal{V}^1$ where player **I** plays $\delta_{\alpha-1}$ and t_α and player **II** uses τ. This implies that $x^0_\alpha \neq x^1_\alpha$ and $(x^0_\alpha, x^1_\alpha) \notin R$.

Suppose that $\alpha \in \text{Lim}$. If $\{\beta{+}1 < \alpha : i_{\beta+1} = 1\}$ is cofinal in α, then let $q = \bigcup_{\beta<\alpha} q^{i_\beta}_\beta$, i.e., q is the partial play of player **I** in $\mathcal{V}^1$ defined so far. Let

$$q^0_\alpha = q^1_\alpha = q^\frown\langle t_\alpha\rangle.$$

Thus, $x^0_\alpha = x^1_\alpha$ is obtained from q by player **I** also playing t_α and player **II** responding according to τ.

Otherwise, there exists $\beta < \alpha$ such that $i_{\beta'+1} = 0$ for all $\beta \leq \beta' < \alpha$. In this case, let

$$q^0_\alpha = q^1_\alpha = q^{i_\beta}_\beta.$$

In other words, the partial run for $x^0_\alpha = x^1_\alpha$ is not extended from the partial run for $x^{i_\beta}_\beta$, and therefore

$$x^0_\alpha = x^1_\alpha = x^{i_\beta}_\beta.$$

We remark that the above statement is true for x^0_α in general (i.e., in successor rounds α as well as limit rounds). That is, if player **I** has been playing $i_{\beta'+1} = 0$ in successor rounds $\beta < \beta' + 1 \leq \alpha$, then by the above construction, $q^0_\alpha = q^{i_\beta}_\beta$ and therefore

$$x^0_\alpha = x^{i_\beta}_\beta.$$

Furthermore, the partial run q^i_α of player **I** in $\mathcal{V}^1$ consists of the following moves of player **I** in $\mathcal{V}^2$:

$\delta_\beta,\ t_{\beta+1}$	for successor rounds $\beta+1<\alpha$ such that $i_{\beta+1}=1$,
t_ν	for limit rounds $\nu\leq\alpha$ such that $\{\beta+1<\nu : i_{\beta+1}=1\}$ is cofinal in ν, and
$\delta_{\alpha-1},\ t_\alpha$	if $i=1$.

Thus, because τ is a winning strategy for player **II** in $\mathcal{V}^1$, player **II** can always define $x^0_\alpha=\tau(q^0_\alpha)$ and $x^1_\alpha=\tau(q^1_\alpha)$ in a legal way based on the moves player **I** has played in $\mathcal{V}^2$ so far. (The fact that (x^0_α,x^1_α) is a legal move can be seen easily from the construction. Specifically, q^i_α was constructed in such a way that

$$q^i_\alpha\supseteq q^{i_\beta}_\beta,\quad\text{and therefore}\quad x^i_\alpha\supseteq x^{i_\beta}_\beta\restriction\delta_\beta$$

holds for all $\beta<\alpha$ and $i<2$.) This implies that the strategy just described is indeed a winning strategy for player **II** in $\mathcal{V}^2$. $\square$

We remark that the above argument is in fact similar to the argument found in the proof of Proposition 2.69. In particular, consider $\mathcal{V}^2=\mathcal{V}^2_\gamma(X,R,x)$, i.e., the $\boldsymbol{t}=\gamma$ case. Then, using the notation found in the above proof and in the proof of Proposition 2.69, the following holds for all $\alpha<\gamma$ and $j<2$:

$$x^j_\alpha=x_{s_j}=\tau\big(\langle\delta_\beta:\beta<\alpha,\ s_j(\beta)=1\rangle\big),$$

where $s_j\in{}^{\alpha}2$ is defined as follows. Let $s_j(\beta)=i_{\beta+1}$ for all β with $\beta+1<\alpha$. If $\alpha\in\mathrm{Succ}$, then let $s_j(\alpha-1)=j$. (Note that $s_0=s_1$ when $\alpha\in\mathrm{Lim}$).

Proposition 3.40. *Suppose R is a symmetric binary relation on a set $X\subseteq{}^\kappa\kappa$ and $\boldsymbol{t}$ is a tree of height $\leq\kappa$. Then*

$$\mathrm{Sc}^1_{\boldsymbol{t}}(X,R)\supseteq\mathrm{Sc}^2_{\boldsymbol{t}}(X,R)\,,$$

i.e., if $x\in X$ and player **I** *wins $\mathcal{V}^2_{\boldsymbol{t}}(X,R,x)$, then player* **I** *wins $\mathcal{V}^1_{\boldsymbol{t}}(X,R,x)$.*

In order to prove Proposition 3.40, we first define the concept of *maximal good positions* of the game $\mathcal{V}^2_{\boldsymbol{t}}(X,R,x)$ for elements $y\in X$, analogously to the $\mathcal{G}^*_\kappa(X,R)$ case (see the proof of Proposition 3.20). We also state some claims that will be needed in the proof.

In the next few paragraphs and Claims 3.41 and 3.42, let ρ denote a fixed winning strategy of player **I** in $\mathcal{V}^2_{\boldsymbol{t}}(X, R, x)$.

A *ρ-good position*, or simply a *good position*, is a legal partial or complete run of $\mathcal{V}^2_{\boldsymbol{t}}(X, R, x)$

$$p = \langle t_\beta, (x^0_\beta, x^1_\beta), i_\beta, \delta_\beta : \beta < \xi \rangle^\frown \langle t_\xi \rangle$$

in which **I** has played according to ρ. (Note that good positions can also be complete runs of the game in this case). Let $l(p)$ denote the lenght of p, i.e., $l(p) = \xi$ using the above notation. We let $u(p)$ be the sequence in ${}^{<\kappa}\kappa$ determined by p, i.e.,

$$u(p) = \bigcup_{\beta < l(p)} x^{i_\beta}_\beta \restriction \delta_\beta.$$

Note that if $l(p) = \beta + 1$, then $u(p) = x^{i_\beta}_\beta \restriction \delta_\beta$.

Let $y, y' \in {}^\kappa\kappa$ be arbitrary. We say that p is a *good position for y* iff p is a good position and $y \supseteq u(p)$. Observe that

(1) p is a good position for y iff $u(p) = y \restriction \delta_{l(p)-1}$;

(2) p is a good position for y and y' iff (y, y') is a legal move of player **II** in response to p.

A good position p for y is a *maximal good position for y* iff there does not exist a good position p' for y such that $p' \supsetneq p$. Note that a maximal good position for y can also be a full run of the game which player **I** has won (using ρ).

Claims 3.41 and 3.42 below are similar to Claims 3.21 and 3.22 (found in the proof of Proposition 3.20). They will be needed in our proof of Proposition 3.40.

Claim 3.41. *If p_0 is a good position for y, then there exists a maximal good position p for y such that $p \supseteq p_0$.*

Proof. Suppose there is no maximal good position for y extending p_0, i.e., every such good position for y has a proper extension which is also a good position for y. Then one can define, recursively, a run of $\mathcal{V}^2_{\boldsymbol{t}}(X, R, x)$ extending p_0 in which player **I** uses ρ, but which player **II** wins. (At limit stages of the recursion, one uses the following observation. Suppose $\langle p_\beta : \beta < \xi \rangle$ is a strictly increasing chain of good positions for y and $\xi \in \mathrm{Lim}$. Let $p = \bigcup_{\beta<\xi} p_\beta$. Then either p is a run of the game which player **II** has won, or there exists $t \in \boldsymbol{t}$ such that $p^\frown\langle t \rangle$ is a good position for y.) This contradicts the fact that ρ is a winning strategy for player **I**. $\square$

Claim 3.42. *Suppose p is a maximal good position for y. Then*

(1) $l(p)$ is a successor ordinal.

(2) Suppose p is a good position for y', and let

$$p' = p^\frown\langle (y, y'), i_{l(p)}, \delta_{l(p)}, t_{l(p)+1}\rangle,$$

where $i_{l(p)}$, $\delta_{l(p)}$ and $t_{l(p)+1}$ are determined by ρ. Then p' is a good position for y'.

Proof. Suppose $l(p)$ is a limit ordinal. Then (y, y) is a response of player **II** to p. Let

$$p' = p^\frown\langle (y, y), i_{l(p)}, \delta_{l(p)}, t_{l(p)+1}\rangle,$$

where $i_{l(p)}$, $\delta_{l(p)}$ and $t_{l(p)+1}$ are determined by ρ. Then p' is a good position for y which extends p, so p cannot be maximal.

Item (2) is immediate from the definitions and the fact that, because $l(p)$ is a successor ordinal, $x^0_{l(p)} \neq x^1_{l(p)}$ holds for all legal responses $(x^0_{l(p)}, u^1_{l(p)})$ of player **II** to p. $\square$

We are now ready to prove Proposition 3.40.

Proof of Proposition 3.40. Suppose player **I** has a winning strategy ρ in $\mathcal{V}^2 = \mathcal{V}^2_{\boldsymbol{t}}(X, R, x)$. Player **I** obtains his winning strategy in $\mathcal{V}^1 = \mathcal{V}^1_{\boldsymbol{t}}(X, R, x)$ by playing a run of $\mathcal{V}^2$ using ρ. The moves in $\mathcal{V}^1$ will be denoted by by t'_α, y_α and δ'_α. (The moves in $\mathcal{V}^2$ will be denoted by t_α, x^0_α, x^1_α, δ_α and i_α, as usual.)

We describe the first few steps of the strategy in order for the idea behind it to be more clear. A more precise description can be found a few paragraphs below.

Player **I** starts $\mathcal{V}^1$ by playing $t'_0 = t_0$, where t_0 is his first move in $\mathcal{V}^2$ according to ρ. By definition, the first move of player **II** in $\mathcal{V}^1$ is $y_0 = x$, and her first move in $\mathcal{V}^2$ is (x, x). Player **I** defines his next moves in $\mathcal{V}^1$ by using ρ in $\mathcal{V}^2$ and by having player **II** play in such a way that eventually, a maximal good position p_1 for $y_0 = x$ is reached. At this point, player **I** defines his next moves δ'_0 and t'_1 in $\mathcal{V}^1$ to be the last moves he played in p_1; that is, $\delta'_0 = \delta_{l(p_1)-1}$ and $t'_1 = t_{l(p_1)}$.

If player **II** plays y_1 in $\mathcal{V}^1$ next, then player **I** repeats the above method for y_1 to define δ'_1 and t'_2. However, to make sure that $t'_2 > t'_1$ and $\delta'_1 > \delta'_0$, player **I** first defines a good position $p'_2 \supsetneq p_1$ (of length $l(p'_2) = l(p_1) + 1$) by having **II** respond with (y_0, y_1) to p_1 and then using ρ. This p'_2 is a good position for y_1 by the maximality of p_1 for y_0 and Claim 3.42. Therefore, by Claim 3.41, there exists a good position $p_2 \supseteq p'_2$, and player **I** can define δ'_1 and t'_2 to be the last moves he played in p_2.

In general, player **I** obtains his winning strategy in $\mathcal{V}^1$ by repeating this method as long as player **II** can play y_α legally in $\mathcal{V}^2$.

More precisely, suppose that player **II** has played $\langle y_\beta : \beta < \alpha \rangle$ in $\mathcal{V}^1$ so far. In the course of the recursive construction, a strictly increasing chain $\langle p_\beta : \beta < \alpha \rangle$ of ρ-good positions of $\mathcal{V}^2$ has also been built.

Player **I** defines his moves t'_α and $\delta'_{\alpha-1}$ (if $\alpha \in \mathrm{Succ}$) by constructing a ρ-good position p_α of $\mathcal{V}^2$ in such a way that the following hold.

(i) p_α is a proper extension of p_β for all $\beta < \alpha$.

(ii) p_α is a maximal good position for $y_{\alpha-1}$ if $\alpha \in \mathrm{Succ}$.

(iii) Suppose y_α is a legal move for player **II** in $\mathcal{V}^1$. Then there exists a legal move for player **II** in $\mathcal{V}^2$ in response to p_α; namely,

$$\begin{aligned} (y_{\alpha-1}, y_\alpha) &\quad \text{is such a legal move if } \alpha \in \mathrm{Succ}, \text{ and} \\ (y_\alpha, y_\alpha) &\quad \text{is such a legal move if } \alpha \in \mathrm{Lim} \text{ or } \alpha = 0. \end{aligned}$$

In other words, p_α is a good position for y_α.

Once p_α has been constructed, player **I** plays the following moves in $\mathcal{V}^1$:

$$\delta'_{\alpha-1} = \delta_{l(p_\alpha)-1} \quad \text{if } \alpha \in \mathrm{Succ}, \qquad\qquad t'_\alpha = t_{l(p_\alpha)}.$$

These moves are legal by item (i). (Note that $l(p_\alpha) \in \mathrm{Succ}$ whenever $\alpha \in \mathrm{Succ}$, by item (ii) and Claim 3.42). Item (iii) implies that as long as player **II** can keep playing in a round of $\mathcal{V}^1$ where player **I** uses this strategy, she can keep playing in the run of $\mathcal{V}^2$ (where player **I** uses ρ). Thus, because ρ is a winning strategy for player **I** in $\mathcal{V}^2$, player **II** will lose the round of $\mathcal{V}^1$ eventually, and player **I** will win (if we can show that the p_α's can indeed be constructed).

We now show that p_α can be constructed, for each round α of $\mathcal{V}^1$. For $\alpha = 0$, let $p_0 = \langle t_0 \rangle$.

Suppose $\alpha = \beta + 1$. Then p_β has already been defined in such a way that items (i) to (iii) hold for p_β. By item (iii), p_β can be extended to a good position p'_α (of length $l(p'_\alpha) = l(p_\beta) + 1$) by playing one more round of $\mathcal{V}^2$ where player **II** plays

$$\begin{aligned} (y_{\beta-1}, y_\beta) &\quad \text{if } \beta \in \mathrm{Succ}, \text{ and} \\ (y_\beta, y_\beta) &\quad \text{if } \beta \in \mathrm{Lim} \text{ or } \beta = 0. \end{aligned}$$

and player **I** plays by ρ. Observe that p'_α is a good position for $y_\beta = y_{\alpha-1}$. If $\beta \in \mathrm{Succ}$, this holds by the maximality of p_β for $y_{\beta-1}$ and Claim 3.42. Thus, by Claim 3.41, there exists a maximal good position p_α for $y_\beta = y_{\alpha-1}$ such that

$$p_\alpha \supseteq p'_\alpha \supsetneq p_\beta.$$

If α is a limit ordinal, then let

$$p_\alpha = \bigcup_{\beta<\alpha} p_\beta{}^\frown\langle t\rangle,$$

where t is the response, according to ρ, of player **I** to the partial play $\bigcup_{\beta<\alpha} p_\beta$ of $\mathcal{V}^2$. (Thus, $t_{l(p_\alpha)} = t$.)

In this way, p_α can be constructed so that items (i) and (ii) hold. Item (iii) follows from item (ii) and the definition of δ'_β (for $\beta < \alpha$). In more detail, suppose y_α is a legal move of player **II** in $\mathcal{V}^1$ and let $\beta < \alpha$. Because $p_{\beta+1}$ is a good position for y_β and $\delta'_\beta = \delta_{l(p_{\beta+1})-1}$, we have

$$u(p_{\beta+1}) = y_\beta \restriction \delta_{l(p_{\beta+1})-1} = y_\beta \restriction \delta'_\beta \subseteq y_\alpha.$$

Thus, $u(p_\alpha) = \bigcup_{\beta<\alpha} u(p_{\beta+1}) \subseteq y_\alpha$, i.e., p_α is a good position for y_α. □

Recall that $\mathrm{Ker}^2_{\boldsymbol{t}}(X,R)$ is a closed subset of ${}^\kappa\kappa$ and $\mathrm{Sc}^2_{\boldsymbol{t}}(X,R)$ is a relatively open subset of X. Thus, Propositions 3.39 and 3.40 imply the following statement.

Corollary 3.43. *If R is a symmetric binary relation on a set $X \subseteq {}^\kappa\kappa$ and $\boldsymbol{t}$ is a tree of height $\leq \kappa$, then*

$$\overline{\mathrm{Ker}^1_{\boldsymbol{t}}(X,R)} \subseteq \mathrm{Ker}^2_{\boldsymbol{t}}(X,R) \qquad \textit{and} \qquad \mathrm{Int}_X\big(\mathrm{Sc}^1_{\boldsymbol{t}}(X,R)\big) \supseteq \mathrm{Sc}^2_{\boldsymbol{t}}(X,R)\,.$$

As we will see in Subsection 3.2.4, we have $\overline{\mathrm{Ker}^1_\kappa(X,R)} = \mathrm{Ker}^2_\kappa(X,R)$ for all closed $X \subseteq {}^\kappa\kappa$ and all closed symmetric binary relations R on X. Moreover, if $\mathrm{OCA}^*_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$ holds, then $\mathrm{Int}_X\big(\mathrm{Sc}^1_\kappa(X,R)\big) = \mathrm{Sc}^2_\kappa(X,R)$ holds for all closed $X \subseteq {}^\kappa\kappa$ and $R \subseteq {}^2X$.

Example 3.37 shows that neither of the above statements holds for the κ-fan $\boldsymbol{f}$ (and it is also easy to construct counterexamples for all well-founded trees $\boldsymbol{t} \in \mathcal{T}_\omega$).

Question 3.44. For which trees $\boldsymbol{t} \in \mathcal{T}_\kappa$ do either of the following statements hold or consistently hold:

(1) $\overline{\mathrm{Ker}^1_{\boldsymbol{t}}(X,R)} = \mathrm{Ker}^2_{\boldsymbol{t}}(X,R)$ for all closed $X \subseteq {}^\kappa\kappa$ and all closed symmetric binary relations R on X;

(2) $\mathrm{Int}_X\big(\mathrm{Sc}^1_{\boldsymbol{t}}(X,R)\big) = \mathrm{Sc}^2_{\boldsymbol{t}}(X,R)$ for all closed $X \subseteq {}^\kappa\kappa$ and all closed symmetric binary relations R on X?

Is it consistent that $\mathrm{Sc}^2_\kappa(X,R) \subsetneq \mathrm{Int}_X\big(\mathrm{Sc}^1_\kappa(X,R)\big)$ for some closed $X \subseteq {}^\kappa\kappa$ and closed symmetric $R \subseteq {}^2X$?

Conjecture 3.45. *Statement (1) holds for a tree $\boldsymbol{t} \in \mathcal{T}_\kappa$ if and only if $\boldsymbol{t} \equiv 1+\boldsymbol{t}$ (where $1+\boldsymbol{t}$ denotes the tree which is obtained by adding a single node r below $\boldsymbol{t}$; thus, r is the root of $1+\boldsymbol{t}$. See Definition 1.22).*

Specifically, statement (1) holds for all infinite ordinals $\gamma \leq \kappa$.

Proposition 3.46. *Suppose E is an equivalence relation on a set $X \subseteq {}^\kappa\kappa$ and $\boldsymbol{t}$ is a tree of height $\leq \kappa$. Then the games $\mathcal{V}^1_{\boldsymbol{t}}(X,E,x)$ and $\mathcal{V}^2_{\boldsymbol{t}}(X,E,x)$ are equivalent, i.e.,*

$$\mathrm{Ker}^1_{\boldsymbol{t}}(X,E) = \mathrm{Ker}^2_{\boldsymbol{t}}(X,E) \qquad \textit{and} \qquad \mathrm{Sc}^1_{\boldsymbol{t}}(X,E) = \mathrm{Sc}^2_{\boldsymbol{t}}(X,E)\,.$$

Specifically, the game $\mathcal{V}^2_{\boldsymbol{t}}(X,\mathrm{id}_X,x)$ is equivalent to $\mathcal{V}^1_{\boldsymbol{t}}(X,\mathrm{id}_X,x)$ and is therefore also equivalent to $\mathcal{V}_{\boldsymbol{t}}(X,x)$.

Consequently, this statement also holds whenever E is a symmetric and transitive binary relation.

Proof. We will prove the two inclusions that do not follow from Propositions 3.39 and 3.40.

First, we describe a winning strategy for player **II** in $\mathcal{V}^1 = \mathcal{V}^1_{\boldsymbol{t}}(X,E,x)$, assuming she has a winning strategy τ in $\mathcal{V}^2 = \mathcal{V}^2_{\boldsymbol{t}}(X,E,x)$. The idea is that in each round α of $\mathcal{V}^1$, player **II** plays one of the moves she has played in the αth round of a simultaneous run of $\mathcal{V}^2$ where she uses τ (i.e. she plays either $x_\alpha = x^0_\alpha$ or $x_\alpha = x^1_\alpha$). She can play legally in this way in each round because E is an equivalence relation.

In more detail, player **II** starts $\mathcal{V}^1$ with $x_0 = x$. To define her in strategy in round $\alpha > 0$ of $\mathcal{V}^1$, suppose that t_β (for all $\beta \leq \alpha$) and x_β, δ_β (for all $\beta < \alpha$) have been played so far in $\mathcal{V}^1$. In the recursive construction of the strategy, moves i_β for player **I** in $\mathcal{V}^2$ have also been defined for all $\beta < \alpha$. Let x^0_α and x^1_α be obtained from a partial run of $\mathcal{V}^2$ where player **II** responds according to τ to these moves of player **I**; that is,

$$(x^0_\alpha, x^1_\alpha) = \tau\big(\langle t_\beta, \delta_\beta, i_\beta : \beta < \alpha\rangle^\frown\langle t_\alpha\rangle\big).$$

If $\alpha = \beta + 1$ is a successsor ordinal, then by $(x^0_\alpha, x^1_\alpha) \notin E$ and the transitivity of E, there exists $i_\alpha < 2$ such that

$$(x_\beta, x^{i_\alpha}_\alpha) \notin E.$$

Let $x_\alpha = x^{i_\alpha}_\alpha$, and let player **I** play i_α at the end of the αth round of $\mathcal{V}^2$. If α is a limit ordinal, let $x_\alpha = x^0_\alpha$ and let player **I** play $i_\alpha = 0$ at the end of the αth round of $\mathcal{V}^2$.

Using essentially the same argument, we describe a winning strategy for player **I** in $\mathcal{V}^2$ assuming he has a winning strategy ρ in $\mathcal{V}^1$. Player **I** obtains this strategy by playing a simultaneous run of $\mathcal{V}^1$ where he uses ρ and player **II** plays $x_\beta = x^{i_\beta}_\beta$ in all rounds β. In the αth round of $\mathcal{V}^2$, player **I** plays the same moves t_α and δ_α as in the αth round of a run of $\mathcal{V}^1$, and also plays i_α in the following way. Suppose player **II** has just played x^0_α and x^1_α. Then, if $\alpha = \beta + 1$, we let i_α be such that $(x^{i_\beta}_\beta, x^{i_\alpha}_\alpha) \notin E$ (such an i_α exists by the transitivity of E and by$(x^0_\alpha, x^1_\alpha) \notin E$). If α is limit, then we let $i_\alpha = 0$. □

Suppose R is a closed binary relation on a subset $X = [T]$ of the κ-Baire space. Proposition 3.47 below gives the connection between the levels

$$\mathrm{Ker}_{\boldsymbol{t}}(T, R) \quad \text{and} \quad T - \mathrm{Sc}_{\boldsymbol{t}}(T, R)$$

of the generalized hierarchy given by the games $\mathcal{G}_{\boldsymbol{t}}(T, R, t)$ and the levels

$$\mathrm{Ker}^2_{\boldsymbol{t}}(X, R) \quad \text{and} \quad X - \mathrm{Sc}^2_{\boldsymbol{t}}(X, R)$$

of the generalized hierarchy given by the games $\mathcal{V}^2_{\boldsymbol{t}}(X, R, x)$.

Proposition 3.47. *Suppose T is a subtree of ${}^{<\kappa}\kappa$, and R is a closed symmetric binary relation on $[T]$. We have*

$$\mathrm{Ker}^2_{\boldsymbol{t}}([T], R) \subseteq [\mathrm{Ker}_{\boldsymbol{t}}(T, R)] \qquad \textit{and} \qquad \mathrm{Sc}^2_{\boldsymbol{t}}([T], R) \supseteq N(\mathrm{Sc}_{\boldsymbol{t}}(T, R))\,.$$

In other words, let $X = [T]$. Then $\mathcal{V}^2_{\boldsymbol{t}}(X, R, x)$ is easier for player **I** *to win and harder for player* **II** *to win than $\mathcal{G}_{\boldsymbol{t}}(T, R, u)$ whenever $u \subsetneq x \in X$.*

Thus, a similar connection also holds for the levels of the generalized hierarchy given by the games $\mathcal{V}^1_{\boldsymbol{t}}(X, R, x)$.

Proof. Let $\mathcal{G} = \mathcal{G}_{\boldsymbol{t}}(T, R, u)$, and let $\mathcal{V}^2 = \mathcal{V}^2_{\boldsymbol{t}}(X, R, x)$ (where $u \subsetneq x \in X$). We will denote the moves in $\mathcal{G}$ by t'_α, u^0_α, u^1_α, δ'_α and i'_α. We will denote the moves in $\mathcal{V}^2$ by t_α, x^0_α, x^1_α, δ_α and i_α, as usual.

First, we describe a winning strategy for player **II** in $\mathcal{G}$ assuming she has a winning strategy τ in $\mathcal{V}^2$. The strategy of player **II** in $\mathcal{G}$ will be to play, legally, initial segments u^i_α of her moves x^i_α in a simultaneous run of $\mathcal{V}^2$ where she uses τ.

In more detail, player **II** plays $u^0_0 = u^1_0 = u$ in round $\alpha = 0$. Suppose $\alpha > 0$, and suppose that t'_β, u^i_β, δ'_β, i'_β (where $\beta < \alpha$ and $i = 0, 1$) and t'_α have been played in $\mathcal{G}$ so far. Let x^0_α and x^1_α be obtained from a partial run of $\mathcal{V}$ in which player **II** uses τ and player **I** plays $i_\beta = i'_\beta$ and

$$\delta_\beta = \max(\delta'_\beta, \mathrm{ht}(u^0_\beta), \mathrm{ht}(u^1_\beta)).$$

for all $\beta < \alpha$ and $t_\beta = t'_\beta$ for all $\beta \leq \alpha$. The strategy of player **II** in $\mathcal{G}$ is to choose u^0_α and u^1_α so that

$$u^{i_\beta}_\beta \subset u^i_\alpha \subseteq x^i_\alpha \qquad \text{for all } \beta < \alpha \text{ and } i = 0, 1, \text{ and}$$

$$u^0_\alpha \perp_R u^1_\alpha \qquad \text{whenever } \alpha \in \mathrm{Succ}.$$

The latter condition can be ensured since R is closed and $(x^0_\alpha, x^1_\alpha) \notin R$ whenever α is a successor ordinal.

Conversely, suppose player **I** has a winning strategy ρ in $\mathcal{G}$. By essentially the same argument as the one just described, player **I** can obtain a winning strategy in $\mathcal{V}^2$ by using ρ in $\mathcal{G}$. That is, the strategy of player **I** in $\mathcal{V}^2$ is to play $t_\alpha = t'_\alpha$, $i_\alpha = i'_\alpha$ and

$$\delta_\alpha = \max(\delta'_\alpha, \mathrm{ht}(u^0_\alpha), \mathrm{ht}(u^1_\alpha)),$$

where $t'_\beta, \delta'_\beta, i'_\beta$ are the moves of player **I** in the simultaneous run of $\mathcal{G}$ where he uses ρ and in which player **II** plays (legally) initial segments u^0_β, u^1_β of the moves x^0_β, x^1_β played by **II** in $\mathcal{V}^2$. □

In the corollary below, we sum up our results from this and the previous subsection about how the levels of the different generalized hierarchies associated to binary relations (or equivalently, to binary open colorings) compare to each other. The corollary summarizes the results in Propositions 3.39, 3.40, 3.47 and Claim 3.29, in the case of closed binary relations on closed subsets of the κ-Baire space.

Corollary 3.48. *Suppose T is a subtree of ${}^{<\kappa}\kappa$, and R is a closed symmetric binary relation on $[T]$. If $\boldsymbol{t}$ is an arbitrary tree of height $\leq \kappa$, then the following hold.*

$$\mathrm{Ker}^1_{\boldsymbol{t}}([T], R) \subseteq \mathrm{Ker}^2_{\boldsymbol{t}}([T], R) \subseteq [\mathrm{Ker}_{\boldsymbol{t}}(T, R)] \subseteq [\mathrm{Ker}^*_{\boldsymbol{t}}(T, R)];$$

$$\mathrm{Sc}^1_{\boldsymbol{t}}([T], R) \supseteq \mathrm{Sc}^2_{\boldsymbol{t}}([T], R) \supseteq N(\mathrm{Sc}_{\boldsymbol{t}}(T, R)) \supseteq N(\mathrm{Sc}^*_{\boldsymbol{t}}(T, R))\,.$$

In other words, if $x \in [T]$ and $t \subsetneq x$, then each of the games

$$\mathcal{V}^1_{\boldsymbol{t}}([T], R, x)\,, \qquad \mathcal{V}^2_{\boldsymbol{t}}([T], R, x)\,, \qquad \mathcal{G}_{\boldsymbol{t}}(T, R, t)\,, \qquad \mathcal{G}^*_{\boldsymbol{t}}(T, R, t)$$

is harder for player **II** *to win and easier for player* **I** *to win than the one after it.*

Note that some of the comparisons in Proposition 3.48 hold for arbitrary binary relations on arbitrary subsets (see Propositions 3.39, 3.40), and that in some cases, the converse statement also holds (see Propositions 3.46, 3.30 and Corollary 3.31).

3.2.4 Games of length κ

As we have seen in Subsection 3.2.2, the games $\mathcal{G}_\kappa(T, R, t)$, $\mathcal{G}^*_\kappa(T, R, t)$, and $\mathcal{G}^*_\kappa([T], R, t)$ are equivalent for all subtrees of ${}^{<\kappa}\kappa$, $t \in T$ and binary relations R on $[T]$. Therefore, $\mathrm{OCA}^*_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$ is equivalent to the determinacy of the above games for all such T, t and closed binary relations R (by Proposition 3.20; see Corollaries 3.31 to 3.33).

In this subsection, we consider winning conditions for both players in the games $\mathcal{V}^1_\kappa(X, R)$ and $\mathcal{V}^2_\kappa(X, R)$, in the case of closed (symmetric) binary relations R on X. We give some further reformulations of $\mathrm{OCA}^*_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$ in terms of the above games. In particular, we show that $\mathrm{OCA}^*_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$ is equivalent to the analogue, for closed binary relations, of Jouko Väänänen's Cantor-Bendixson theorem [51, Theorem 4]; see Corollary 3.56.

Proposition 3.49. *Let $X \subseteq {}^\kappa\kappa$, and let R be a closed symmetric binary relation on $\overline{X}$. If $\overline{X}$ is a union of $\leq \kappa$ many R-homogeneous subsets, then*

$$X = \mathrm{Sc}^1_\kappa(X, R) = \mathrm{Sc}^2_\kappa(X, R)\,,$$

or equivalently, player **I** *wins the games $\mathcal{V}^1_\kappa(X, R)$ and $\mathcal{V}^2_\kappa(X, R)$.*

This proposition follows from Corollaries 3.48 and 3.32. We give a simpler direct proof below, using diagonal arguments similar to the ones in the proofs of Propositions 3.20 and 2.12.

Proof. First, observe that under the assumptions of the proposition, $\overline{X}$ can be written as the union

$$\overline{X} = \bigcup_{\alpha<\kappa} Y_\alpha$$

of *closed* R-homogeneous sets Y_α. (This is because R is a closed subset of ${}^2({}^\kappa\kappa)$. Therefore, if Y is R-homogeneous, then $\overline{Y} \times \overline{Y} = \overline{Y \times Y} \subseteq R$, i.e., $\overline{Y}$ is also R-homogeneous.)

The strategy of player **I** in $\mathcal{V}^2_\kappa(X, R)$ is to choose $i_{\alpha+1}$ and $\delta_{\alpha+1}$ in each successor round $\alpha + 1$ as follows. If player **II** has played $(x^0_{\alpha+1}, x^1_{\alpha+1})$ legally, then by the homogeneity of Y_α, there exits $i_{\alpha+1} < 2$ such that $x^{i_{\alpha+1}}_{\alpha+1} \notin Y_\alpha$. Thus, because Y_α is a closed set, player **I** can choose $\delta_{\alpha+1}$ so that

$$\text{if} \quad u_{\alpha+1} = x^{i_{\alpha+1}}_{\alpha+1} \restriction \delta_{\alpha+1}, \quad \text{then} \quad N_{u_{\alpha+1}} \cap Y_\alpha = \emptyset. \tag{3.5}$$

Suppose player **II** wins a run of $\mathcal{V}^2_\kappa(X, R)$ in which player **I** uses this strategy. Let $x = \bigcup_{\alpha<\kappa} u_{\alpha+1}$ be the element of ${}^\kappa\kappa$ produced during the given run. Then $x \in \overline{X}$. However, (3.5) implies $x \notin Y_\alpha$ for each $\alpha < \kappa$, contradiction.

The strategy of player **I** in $\mathcal{V}^1_\kappa(X, R)$ is to choose δ_α in such a way that the following conditions are satisfied, in each round α where player **II** has played x_α legally. If $x_\alpha \notin Y_\alpha$, then

$$N_{x_\alpha \restriction \delta_\alpha} \cap Y_\alpha = \emptyset$$

holds, and if $\alpha \in \mathrm{Succ}$ and $x_{\alpha-1} \in Y_{\alpha-1}$, then

$$N_{x_\alpha \restriction \delta_\alpha} \cap Y_{\alpha-1} = \emptyset$$

also holds. (These conditions can be ensured because for all $\beta < \kappa$, Y_β is a closed and R-homogeneous set and $(x_{\beta-1}, x_\beta) \notin R$ holds if x_β is a legal move for player **II**.) Assuming player **II** wins a run of $\mathcal{V}^1_\kappa(X, R)$ where player **I** uses this strategy, we obtain a contradiction similarly to the previous case: $x = \bigcup_{\alpha<\kappa} x_\alpha \restriction \delta_\alpha$ is in $\overline{X}$, but is not an element of Y_α for any $\alpha < \kappa$. □

We note that by [51, Theorem 3], the converse of Proposition 3.49 consistently fails for $\kappa = \omega_1$ (and $R = \mathrm{id}_X$). However, the converse of Proposition 3.49 is implied by $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ and is therefore consistent relative to the existence of an inaccessible $\lambda > \kappa$ (by Theorem 3.14).

Definition 3.50. For any $X \subseteq {}^\kappa\kappa$ and binary relation R on X, let

$$\begin{aligned} CP_\kappa(X, R) = \{x \in X : {} & \text{for all } \alpha < \kappa, \\ & X \cap N_{x\restriction\alpha} \text{ is } \textit{not} \text{ the union of } \kappa \text{ many } R\text{-homogeneous sets}\}. \end{aligned}$$

The following statement can be obtained from Proposition 3.49 using an argument analogous to the one found in the proof of Claim 2.17.

Corollary 3.51. *If R is a closed symmetric binary relation on a closed subset X of the κ-Baire space, then*

$$X - CP_\kappa(X,R) \subseteq \mathrm{Sc}^1_\kappa(X,R)\,,\ \mathrm{Sc}^1_\kappa(X,R)\,.$$

However, a stronger version of the above corollary follows from Corollaries 3.32 and 3.48, and the fact that $\mathrm{Sc}^2_\kappa(X,R)$ is always an open subset of X.

Corollary 3.52. *Suppose T is a subtree of ${}^{<\kappa}\kappa$ and R is a closed symmetric binary relation on $[T]$. Then*

$$\begin{aligned}\mathrm{Int}_{[T]}\big(\mathrm{Sc}^1_\kappa([T],R)\big) \supseteq \mathrm{Sc}^2_\kappa([T],R) \supseteq \\ \supseteq N(\mathrm{Sc}_\kappa(T,R)) = N(\mathrm{Sc}^*_\kappa(T,R)) = [T] - CP_\kappa([T],R)\,. \qquad (3.6)\end{aligned}$$

The set $\mathrm{Ker}^2_\kappa(X,R)$ is κ-perfect, and $\mathrm{Ker}^1_\kappa(X,R)$ is κ-dense in itself. (The analogue also holds for all reflexive trees $\boldsymbol{t}$ of height $\leq \kappa$; see the remarks in the paragraphs after Definition 3.36.) Item (1) of Example 3.37 shows that $\mathrm{Ker}^1_\kappa(X,R)$ may not be closed, even when X and R are closed.

Proposition 3.53. *If R is a closed symmetric binary relation on a set $X \subseteq {}^\kappa\kappa$, then*

$$X \cap \mathrm{Ker}^1_\kappa(X,R) = \bigcup\{Y \subseteq X : Y \textit{ is } R\textit{-independent and } \kappa\textit{-dense in itself}\}.$$

Specifically, if X is a closed set, then

$$\mathrm{Ker}^1_\kappa(X,R) = \bigcup\{Y \subseteq X : Y \textit{ is } R\textit{-independent and } \kappa\textit{-perfect}\}.$$

Note that the second statement is equivalent to the claim that $\mathrm{Ker}^1_\kappa(X,R)$ is the union of all *strongly* κ-perfect R-independent subsets of X. The first statement can also be reformulated similarly, as a claim about the *strongly* κ-dense in itself R-independent subsets of X.

Proof. Suppose that $Y \subseteq X$ is R-independent and κ-dense in itself. Then $\mathcal{V}^1_\kappa(Y,R,y)$ is equivalent to $\mathcal{V}_\kappa(Y,y)$ for all $y \in Y$. Thus, by Claim 2.67, we have $Y \subseteq \mathrm{Ker}_\kappa(Y) \cap X \subseteq \mathrm{Ker}^1_\kappa(X,R) \cap X$. This also implies that $\mathrm{Ker}^1_\kappa(X,R)$ contains all κ-perfect R-independent subsets of X.

The other direction, in both equalities, follows from the argument below. Suppose $x \in \mathrm{Ker}^1_\kappa(X,R)$. By modifying the construction in the proof of Proposition 2.69 in a straightforward manner, it is easy to define $\langle u_s, x_s, \delta_s : s \in {}^{<\kappa}2\rangle$ such that $u_s \in {}^{<\kappa}\kappa$, $x_s \in X$, and $\delta_s < \kappa$, and the following items hold for all $s, r \in {}^{<\kappa}2$.

(i) $u_s = x_s \restriction \delta_s$;

(ii) if $r \subseteq s$, then $u_r \subseteq u_s$;

(iii) $u_{r^\frown 0} \perp_R u_{r^\frown 1}$.

(iv) $x_s = \tau\big(\langle \delta_{s\restriction\beta} : \beta < \mathrm{ht}(s), s(\beta) = 1\rangle\big)$.

Let $Y = \{x_s : s \in {}^{<\kappa}2\}$. Then $e : {}^{<\kappa}2 \to {}^{<\kappa}\kappa;\ s \mapsto u_s$ is a perfect R-embedding with $T_e = T_Y$. (Recall that by definition, T_Y is the tree of initial segments of elements of Y and T_e is the strongly κ-perfect tree of initial segments of elements of $\mathrm{ran}(e)$.) Thus, Y is a κ-dense in itself set, and $\overline{Y} = [T_e]$ is R-independent (and strongly κ-perfect). Furthermore, $x \in \overline{Y} \subseteq \overline{X}$, and if $x \in X$, then $Y \subseteq X$. This shows the required direction in both statements of the proposition. □

Proposition 3.54. *Suppose T is a subtree of ${}^{<\kappa}\kappa$ and R is a R is a closed symmetric binary relation on $[T]$. Then*

$$\begin{aligned}\overline{\mathrm{Ker}^1_\kappa([T],R)} &= \mathrm{Ker}^2_\kappa([T],R) = \\ &= [\mathrm{Ker}_\kappa(T,R)] = [\mathrm{Ker}^*_\kappa(T,R)] \subseteq CP_\kappa(X,R)\,. \end{aligned} \tag{3.7}$$

Proof. All of the inclusions $\subseteq$ in the first three equalities hold by Corollary 3.48. Thus, it is enough to show that

$$[\mathrm{Ker}^*_\kappa(T,R)] \subseteq \overline{\mathrm{Ker}^1_\kappa([T],R)} \qquad \text{and} \qquad [\mathrm{Ker}^*_\kappa(T,R)] \subseteq CP_\kappa([T],R)$$

Suppose $x \in [\mathrm{Ker}^*_\kappa(T,R)]$. Then for all $\alpha < \kappa$, there exists a perfect R-embedding $e_\alpha : {}^{<\kappa}2 \to T$ such that $e_\alpha(\emptyset) = x\restriction\alpha$. This implies that $N_{x\restriction\alpha} \cap [T]$ has a a κ-perfect R-independent subset $X_\alpha = [T_{e_\alpha}]$ for all $\alpha < \kappa$. Thus, by Proposition 3.53, we have $x \in \overline{\mathrm{Ker}^1_\kappa([T],R)}$.

For all $\alpha < \kappa$, X_α is not the union of κ-many R-homogeneous subsets, and therefore $x \in CP_\kappa([T],R)$. □

In the next two corollaries, we give some equivalent reformulations of $\mathrm{OCA}^*_\kappa(\boldsymbol{\Sigma}^1_1\kappa)$ which are implied by the results in this subsection and by Corollary 3.33.

Recall that $\mathcal{C}_\kappa$ denotes the collection of closed subsets of the κ-Baire space.

Corollary 3.55. *The following statements are equivalent.*

(1) $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$.

(2) $\mathrm{OCA}^*_\kappa(\mathcal{C}_\kappa)$.

(3) If R is a closed symmetric binary relation on a closed subset $X \subseteq {}^\kappa\kappa$, then

$$\overline{\mathrm{Ker}^1_\kappa(X,R)} = \mathrm{Ker}^2_\kappa(X,R) = CP_\kappa(X,R)\,.$$

(4) If T is a subtree of ${}^{<\kappa}\kappa$ and R is a closed symmetric binary relation on $[T]$, then equality holds everywhere in (3.6) and in (3.7), i.e.,

$$\overline{\mathrm{Ker}^1_\kappa([T],R)} = \mathrm{Ker}^2_\kappa([T],R) = [\mathrm{Ker}_\kappa(T,R)] = [\mathrm{Ker}^*_\kappa(T,R)] = CP_\kappa([T],R)\,,$$

$$\mathrm{Int}_{[T]}\big(\mathrm{Sc}^1_\kappa([T],R)\big) = \mathrm{Sc}^2_\kappa([T],R) = N(\mathrm{Sc}_\kappa(T,R)) = N(\mathrm{Sc}^*_\kappa(T,R)) = \\ = [T] - CP_\kappa([T],R)\,.$$

Proof. By Proposition 3.5, $\mathrm{OCA}^*_\kappa(\mathcal{C}_\kappa)$ is equivalent to $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$

Now, suppose $\mathrm{OCA}^*_\kappa(\mathcal{C}_\kappa)$ holds, and let X and R be as in item (3). By Proposition 3.54, it is enough to show that $CP_\kappa(X,R) \subseteq \overline{\mathrm{Ker}^1_\kappa([T],R)}$ to see that item (3) holds. Let $x \in CP_\kappa(X,R)$. Then for all $\alpha < \kappa$, the closed set $X \cap N_{x\restriction\alpha}$ contains a κ-perfect R-independent subset by $\mathrm{OCA}^*_\kappa(\mathcal{C}_\kappa)$. This implies $x \in \overline{\mathrm{Ker}^1_\kappa([T],R)}$ by Proposition 3.53.

It is easy to see that item (3) implies item (4), using Corollary 3.52, Proposition 3.54 and the fact that e.g. the sets $\mathrm{Ker}^2_\kappa([T],R)$ and $\mathrm{Sc}^2_\kappa([T],R)$ are disjoint. □

Corollary 3.56. *The following statements are equivalent.*

(1) $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$.

(2) $\mathrm{OCA}^*_\kappa(\mathcal{C}_\kappa)$.

(3) If T is a subtree of ${}^{<\kappa}\kappa$ and R is a closed symmetric binary relation on $[T]$, then

$$T = \mathrm{Ker}_\kappa(T,R) \cup \mathrm{Sc}_\kappa(T,R)\,,$$

i.e., the game $\mathcal{G}_\kappa(T,R,t)$ is determined for all $t \in T$.

(In this statement, the role of $\mathcal{G}_\kappa(T,R)$ can also be replaced with $\mathcal{G}^*_\kappa(T,R)$).

(4) If R is a closed symmetric binary relation on a closed subset $X \subseteq {}^\kappa\kappa$, then

$$X = \mathrm{Ker}^2_\kappa(X,R) \cup \mathrm{Sc}^2_\kappa(X,R)$$

and $\mathrm{Sc}^2_\kappa(X,R)$ is the union of κ many R-homogeneous sets.

Note that if X is a closed subset of ${}^\kappa\kappa$, then $\mathrm{Ker}^2_\kappa(X,R)$ is the closure of the union of all the κ-perfect subsets of X, by Propositions 3.53 and 3.54.

Item (4) in Corollary 3.56 can be viewed as the analogue of Jouko Väänänen's Cantor-Bendixson theorem [51, Theorem 4] for closed binary relations on closed subsets of the κ-Baire space. It can also be viewed as a strong form of the determinacy of the games $\mathcal{V}^2_\kappa(X,R,x)$.

By Corollary 3.15, each of the statements in Corollaries 3.55 and 3.56 is equiconsistent with the existence of an inaccessible cardinal above κ.

Proof of Corollary 3.56. The first three statements are equivalent by Corollary 3.33. The last statement clearly implies $\mathrm{OCA}^*_\kappa(\mathcal{C}_\kappa)$, and follows from item (4) of Corollary 3.55. □

It would be interesting to see if the role of the games $\mathcal{V}^2_\kappa(X,R)$ in item (4) of Corollary 3.56 could be replaced with the role of the games $\mathcal{V}^1_\kappa(X,R)$.

Question 3.57. Does $\mathrm{OCA}^*_\kappa(\mathcal{C}_\kappa)$ imply the following statement?

If R is a closed symmetric binary relation on a closed subset $X \subseteq {}^\kappa\kappa$, then

$$X = \mathrm{Ker}^1_\kappa(X,R) \cup \mathrm{Sc}^1_\kappa(X,R)\,,$$

where $\mathrm{Sc}^1_\kappa(X,R)$ is a union of κ many R-homogeneous sets.

If not, is this statement consistent?

CHAPTER 4

Dichotomies for $\mathbf{\Sigma}^0_2(\kappa)$ relations

In the first part of the chapter, we consider the κ-Silver dichotomy for $\mathbf{\Sigma}^0_2(\kappa)$ equivalence relations on $\mathbf{\Sigma}^1_1(\kappa)$ subsets of the κ-Baire space (where κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$).

Let X be a subset of the κ-Baire space, and let Γ be a collection of binary relations on X. The *κ-Silver dichotomy for* Γ is the following statement:

> if an equivalence relation $E \in \Gamma$ has at least κ^+ many equivalence classes, then E has κ-perfectly many equivalence classes (i.e., there exists a κ-perfect set $Z \subseteq X$ such that $(y, z) \notin E$ for all distinct $y, z \in Z$).

Thus, the *κ-Silver dichotomy for $\mathbf{\Sigma}^0_2(\kappa)$ (resp. κ-Borel, $\mathbf{\Delta}^1_1(\kappa)$, etc.) (equivalence) relations on* X is the above statement in the case when Γ is the collection of $\mathbf{\Sigma}^0_2(\kappa)$ (κ-Borel, $\mathbf{\Delta}^1_1(\kappa)$, etc.) subsets of $X \times X$.

Observe right away that the κ-Silver dichotomy for $\mathbf{\Sigma}^0_2(\kappa)$ equivalence relations on ${}^\kappa\kappa$ implies the κ-perfect set property for closed subsets of ${}^\kappa\kappa$, and therefore also implies the inaccessability of κ^+ in L by arguments in [9] and arguments of Robert Solovay [19]; see Remark 2.14. (Thus, this observation also holds for κ-Borel equivalence relations).

Recently, a considerable effort has been made to investigate set theoretical conditions implying (the consistency of) the satisfaction or the failure of the κ-Silver dichotomy for Borel equivalence relations on the κ-Baire space. By [8, Theorem 5], the κ-Silver dichotomy fails for $\mathbf{\Delta}^1_1(\kappa)$ equivalence relations. Furhtermore, $V = L$ implies the failure of the κ-Silver dichotomy for κ-Borel equivalence relations in a strong sense [9, 10].

In the other direction, if κ is an inaccessible cardinal, then the κ-Silver dichotomy holds for isomorphism relations [9, Theorem 36]. By [8], the κ-Silver dichotomy for κ-Borel equivalence relations is consistent relative to the consistency of $0^{\#}$.

In Section 4.1, we show that after Lévy-collapsing an inaccessible cardinal $\lambda > \kappa$ to κ^+, the κ-Silver dichotomy holds for $\mathbf{\Sigma}^0_2(\kappa)$ equivalence relations on $\mathbf{\Sigma}^1_1(\kappa)$ subsets of the κ-Baire space; see Theorem 4.14. Thus, the κ-Silver dichotomy for $\mathbf{\Sigma}^0_2(\kappa)$ equivalence relations on $\mathbf{\Sigma}^1_1(\kappa)$ sets is equiconsistent with the existence of an inaccessible cardinal above κ.

In Section 4.2, we consider dichotomies for for families $\mathcal{R}$ of (at most) κ many finitary $\mathbf{\Sigma}^0_2(\kappa)$ relations on subsets of the κ-Baire space.

Our starting point is the following "perfect set property" for independent subsets w.r.t. families of $\leq \kappa$ many finitary $\mathbf{\Sigma}^0_2(\kappa)$ relations.

Definition 4.1. Given a subset X of the κ-Baire space, let $\mathrm{PIF}_\kappa(X)$ denote the following statement:

> $\mathrm{PIF}_\kappa(X)$: if $\mathcal{R}$ is a collection of $\leq\kappa$ many finitary $\mathbf{\Sigma}^0_2(\kappa)$ relations on X and X has an $\mathcal{R}$-independent subset of cardinality κ^+, then X has a κ-perfect $\mathcal{R}$-independent subset.

If Γ is a collection of subsets of the κ-Baire space, then $\mathrm{PIF}_\kappa(\Gamma)$ denotes the statement that $\mathrm{PIF}_\kappa(X)$ holds for every $X \in \Gamma$.

By a joint result of Jouko Väänänen and the author [46, Theorem 2.4] $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ follows from the hypothesis $\mathrm{I}^-(\kappa)$ (see Definition 2.74), and therefore is consistent relative to the existence of a measurable cardinal κ. Note that $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ implies the κ-perfect set property for closed subsets of the κ-Baire space, and therefore its consistency strength is at least that of the existence of an inaccessible $\lambda > \kappa$.

In the classical case, the countable version $\mathrm{PIF}_\omega(\mathbf{\Sigma}^1_1)$ of this dichotomy holds by a result of Martin Doležal and Wieslaw Kubiś [5]. The special case of $\mathrm{PIF}_\omega(\mathbf{\Sigma}^1_1)$ for one finitary $\mathbf{\Sigma}^0_2(\kappa)$ relation is shown in [24], and the special case for one $\mathbf{\Sigma}^0_2(\kappa)$ binary relation on a Polish space is also mentioned in [42, Remark 1.14].

In fact, in the classical case, $\mathrm{PIF}_\omega(\mathbf{\Sigma}^1_1)$ is implied by [5, Theorem 1.1], which states the following (in the special case of Polish spaces).

> *Suppose $\mathcal{R}$ is a countable family of finitary $\mathbf{\Sigma}^0_2$ relations on a Polish space X.*
>
> *If X has an $\mathcal{R}$-independent subset of Cantor-Bendixson rank $\geq \gamma$ for every countable ordinal γ, then X has a perfect $\mathcal{R}$-independent subset.*

In Section 4.2, we show that a statement which may be viewed as a κ-version of [5, Theorem 1.1] holds whenewer $\diamondsuit_\kappa$ holds or κ is inaccessible. In fact, it follows from a slightly weaker principle $\mathcal{DI}_\kappa$ than $\diamondsuit_\kappa$ which also holds whenever κ is inaccessible (see Definition 4.2 below).

Theorem 4.33 states, roughly, the following.

> *Suppose $\mathcal{DI}_\kappa$ holds and $\mathcal{R}$ is a family of $\leq\kappa$ many finitary $\mathbf{\Sigma}^0_2(\kappa)$ relations on a closed subset X of the κ-Baire space.*
>
> *If X has $\mathcal{R}$-independent subsets "on all levels of the generalized Cantor-Bendixson hierarchy for player* **II***", then X has a κ-perfect $\mathcal{R}$-independent subset.*

As a corollary of our arguments, we obtain stronger versions of the main result, Theorem 2.4, of [46]. In particular, our results imply that the consistency strength of $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ is at most that of the existence of a weakly compact cardinal above κ. (See Corallaries 4.32, 4.35 and 4.36.)

We note that the arguments presented in Section 4.2 use, in part, methods similar to those used in e.g. [18, 36, 51] and in Section 2.2.

In the last part of the chapter, Section 4.3, we obtain a model theoretic dichotomy, motivated by the spectrum problem, as a special case of $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$. The contents of this section are (essentially) the same as the contents of [46, Section 3].

Note that all the questions and results in this chapter can be reformulated in terms of homogeneous subsets w.r.t. $\mathbf{\Pi}^0_2(\kappa)$ colorings on subsets of the κ-Baire space.

Throughout this chapter, we assume that κ is a cardinal such that $\kappa^{<\kappa} = \kappa$. We now state some definitions and technical lemmas which will be used later in the chapter.

The combinatorial principle $\mathcal{DI}_\kappa$, defined below, is similar to but slightly weaker than $\diamondsuit_\kappa$. In particular, it also holds when κ is inaccessible. Recall that for any set X and any ordinal γ,

$$[X]^\gamma_{\neq} = \{\langle x_i : i < \gamma\rangle \in {}^\gamma X : x_i \neq x_j \text{ for all } i < j < \gamma\},$$
$$[X]^{<\gamma}_{\neq} = \bigcup_{\beta<\gamma} [X]^\beta_{\neq} = \{\langle x_i : i < \beta\rangle \in {}^\beta X : \beta < \gamma \text{ and } x_i \neq x_j \text{ for all } i < j < \beta\}.$$

Definition 4.2. For a regular $\kappa > \aleph_0$, we let $\mathcal{DI}_\kappa$ be the statement:

There exists a sequence $\langle A_\alpha : \alpha < \kappa\rangle$ of sets $A_\alpha \subseteq [{}^\alpha 2]^{<\omega}$ such that

(i) $|A_\alpha| < \kappa$ for all $\alpha < \kappa$, and
(ii) for all $(x_0, \ldots, x_m) \in [{}^\kappa 2]^{<\omega}$, the set

$$\{\alpha < \kappa : (x_0{\restriction}\alpha, \ldots, x_m{\restriction}\alpha) \in A_\alpha\}$$

is cofinal in κ.

The sequence $\langle A_\alpha : \alpha < \kappa\rangle$ is called a $\mathcal{DI}_\kappa$-*sequence.*

We also consider the version of this combinatorial principle where the A_α's consist of tuples of a fixed length n, where $1 \leq n < \omega$.

Definition 4.3. For a regular $\kappa > \aleph_0$ and $1 \leq n < \omega$, we let $\mathcal{DI}_\kappa(n)$ be the statement:

There exists a sequence $\langle A_\alpha : \alpha < \kappa\rangle$ of sets $A_\alpha \subseteq [{}^\alpha 2]^n$ such that

(i) $|A_\alpha| < \kappa$ for all $\alpha < \kappa$, and
(ii) for all $(x_0, \ldots, x_{n-1}) \in [{}^\kappa 2]^n$, the set

$$\{\alpha < \kappa : (x_0{\restriction}\alpha, \ldots, x_{n-1}{\restriction}\alpha) \in A_\alpha\}$$

is cofinal in κ.

The sequence $\langle A_\alpha : \alpha < \kappa\rangle$ is called a $\mathcal{DI}_\kappa(n)$-*sequence.*

Claim 4.4. *Let κ be a regular uncountable cardinal.*

(1) If $\diamondsuit_\kappa$ holds or κ is inaccessible, then $\mathcal{DI}_\kappa$ holds.

(2) $\mathcal{DI}_\kappa$ *holds if and only if* $\mathcal{DI}_\kappa(n)$ *holds for all* $1 \leq n < \omega$.

(3) $\mathcal{DI}_\kappa(n)$ *implies* $\kappa^{<\kappa} = \kappa$.

Proof. If κ is inaccessible, then the sets $A_\alpha = [{}^\alpha 2]^{<\omega}$ witness that $\mathcal{DI}_\kappa$ holds. Also, $\diamondsuit_\kappa$ implies $\mathcal{DI}_\kappa$ by [26, Chapter II, Exercise 53]. Item (2) is immediate. Item (3) can be obtained by the same argument as the one proving the analogous statement for $\diamondsuit_\kappa$. $\square$

If κ is a successor cardinal, then $\diamondsuit_\kappa$ is equivalent to $\mathcal{DI}_\kappa$ and also to $\mathcal{DI}_\kappa(n)$ for all $1 \leq n < \omega$, by [37, Theorem 4]. (The specific case for $\kappa = \aleph_1$ was shown in [3].) Furthermore, if we replace "$|A_\alpha| < \kappa$" by "$|A_\alpha| \leq \alpha$" for some $\alpha < \kappa$ in the definitions above, then we obtain principles that are also equivalent to $\diamondsuit_\kappa$ whenever κ is weakly inaccessible [32].

We also note that for successor cardinals $\kappa > \aleph_1$, the assumption $\kappa^{<\kappa} = \kappa$ implies that $\diamondsuit_\kappa$ holds [44], and therefore so does $\mathcal{DI}_\kappa$. (Thus, the equivalence of these principles also follows from [44] for successor cardinals $\kappa > \aleph_1$.)

Under the assumption $\mathcal{DI}_\kappa$, the existence of κ-perfect independent sets with respect to a family of κ-many closed relations can be characterized on the level of trees.

We use the following notation. Suppose that $\mathcal{S} = \langle S_\alpha : \alpha < \delta \rangle$ is a sequence such that for all $\alpha < \delta$, S_α is a subtree of $({}^{<\kappa}\kappa)^{\otimes n_\alpha}$ for some $1 \leq n_\alpha < \omega$ (see p. 16 for the definition of the notation $T^{\otimes n}$). We let

$$\mathcal{R}(\mathcal{S}) = \langle [S_\alpha] : \alpha < \delta \rangle.$$

Thus, $\mathcal{R}(\mathcal{S})$ is a sequence of closed finitary relations on the κ-Baire space.

If there exists $1 \leq n < \omega$ such that $n_\alpha = n$ for all $\alpha < \delta$, then we let

$$[\mathcal{S}] = \bigcup \{[S_\alpha] : \alpha < \delta\}.$$

Observe that when $\delta = \kappa$, $[\mathcal{S}]$ is a $\mathbf{\Sigma}^0_2(\kappa)$ n-ary relation on the κ-Baire space, and when $\delta < \kappa$, $[\mathcal{S}]$ is a closed n-ary relation. In general, $\bigcup\{[S_\alpha] : \alpha < \delta\}$ is a set of finitary sequences of elements of the κ-Baire space.

Definition 4.5. Assume that $\mathcal{R} = \langle R_\alpha : \alpha < \kappa \rangle$ is a sequence such that R_α is an n_α-ary relation on the κ-Baire space, where $1 \leq n_\alpha < \kappa$, for all $\alpha < \kappa$. Let $\mathcal{A} = \langle A_\alpha : \alpha < \kappa \rangle$ be a sequence such that $A_\alpha \subseteq [{}^\alpha 2]^{<\omega}$ for all $\alpha < \kappa$.

Let $\gamma \leq \kappa$ be an infinite ordinal, and let T be a subtree of ${}^{<\kappa}\kappa$. An $(\mathcal{R}, \mathcal{A})$-*embedding of height* γ into T is an embedding

$$e : {}^{<\gamma}2 \to T$$

such that the following items hold for all $t, s \in {}^{<\gamma}2$ and $\alpha < \gamma$.

(i) If $t \subseteq s$ then $e(t) \subseteq e(s)$, and if $t \perp s$, then $e(t) \perp e(s)$.
(ii) For all $\beta < \alpha$ and all tuples $(t_0, \ldots, t_{n_\beta - 1}) \in A_\alpha$ we have

$$N_{e(t_0)} \times \ldots \times N_{e(t_{n_\beta - 1})} \cap R_\beta = \emptyset.$$

Suppose $\mathcal{S} = \langle S_\alpha : \alpha < \kappa \rangle$ is a sequence such that S_α is a subtree of $({}^{<\kappa}\kappa)^{\otimes n_\alpha}$ for all $\alpha < \kappa$. Then an $(\mathcal{R}(\mathcal{S}), \mathcal{A})$-embedding is also called an $(\mathcal{S}, \mathcal{A})$-*embedding.*

When $\gamma = \kappa$, we will also say *"perfect* $(\mathcal{R}, \mathcal{A})$-*embedding"* or *"perfect* $(\mathcal{S}, \mathcal{A})$-*embedding".*

Specifically, a perfect $(\mathcal{R}, \mathcal{A})$-embedding is a perfect embedding, by item (i) of the definition. If $\mathcal{A}$ is a $\mathcal{DI}_\kappa$-sequence, then a perfect $(\mathcal{R}, \mathcal{A})$-embedding determines a κ-perfect $\mathcal{R}$-independent set in a natural way (see Lemma 4.6 below). The $\gamma < \kappa$ case of the above notion will be useful in the arguments in Section 4.2 below.

Recall that for a perfect embedding $e : {}^{<\kappa}2 \to T$, T_e denotes the (strongly) κ-perfect subtree of T defined by e; that is,

$$T_e = \{t \in {}^{<\kappa}\kappa : t \subseteq e(s) \text{ for some } s \in {}^{<\kappa}2\}.$$

Lemma 4.6. *Let $\mathcal{A} = \langle A_\alpha : \alpha < \kappa \rangle$ be a $\mathcal{DI}_\kappa$-sequence. Let $\mathcal{R} = \langle R_\alpha : \alpha < \kappa \rangle$ and $\langle n_\alpha : \alpha < \kappa \rangle$ be sequences such that $1 \leq n_\alpha < \omega$ and R_α is an n_α-ary relation on ${}^\kappa\kappa$ for all $\alpha < \kappa$. Let $\mathcal{S} = \langle S_\alpha : \alpha < \kappa \rangle$ be such that S_α is a pruned subtree of $({}^{<\kappa}\kappa)^{\otimes n_\alpha}$ for all $\alpha < \kappa$. Suppose*

$$e : {}^{<\kappa}2 \to T$$

is an embedding into a subtree T of ${}^{<\kappa}\kappa$.

(1) If e is a perfect $(\mathcal{R}, \mathcal{A})$-embedding, then $[T_e]$ is a κ-perfect $\mathcal{R}$-independent subset of $[T]$.
(2) If e is a perfect $(\mathcal{S}, \mathcal{A})$-embedding, then

$$[T_e] \textit{ is a } \kappa\textit{-perfect } \mathcal{R}(\mathcal{S})\textit{-independent subset of } [T]$$

in any transitive model $M \supseteq V$ of ZFC *such that $({}^{<\kappa}2)^M = ({}^{<\kappa}2)^V$.*

(3) *Conversely, if* $[T]$ *has a* κ*-perfect* $\mathcal{R}(\mathcal{S})$*-independent subset, then there exists a perfect* $(\mathcal{S},\mathcal{A})$*-embedding* $e : {}^{<\kappa}2 \to T$*.*

All of the above statements are also true for sequences $\langle A_\alpha : \alpha < \kappa\rangle$ *witnessing that* $\mathcal{DI}_\kappa(n)$ *holds and families* $\mathcal{R}$ *of* n*-ary relations (where* $1 \leq n < \omega$*).*

Items (2) and (3) imply that if $[T]$ has a κ-perfect $\mathcal{R}(\mathcal{S})$-independent subset and $\mathcal{S}$ consists of pruned trees, then $[T]$ has a κ-perfect $\mathcal{R}(\mathcal{S})$-independent subset in every model $M \supseteq V$ of ZFC with the same ${}^{<\kappa}2$ as V.

Proof. To see the first statement, it is enough to show that $[T_e]$ is an $\mathcal{R}$-independent set. Suppose that $\beta < \kappa$ and $(x_0, \ldots, x_{n_\beta - 1}) \in [{}^\kappa 2]^{n_\beta}$. Then there exists $\alpha > \beta$ such that $(x_0 \restriction \alpha, \ldots, x_{n_\beta-1} \restriction \alpha) \in A_\alpha$. Thus, by item (ii) of Definition 4.5, we have that $(e(x_0), \ldots, e(x_{n-1})) \notin R_\beta$ (where $e(x_i) = \bigcup_{\beta<\kappa} t_{x_i \restriction \beta}$ is the branch of $[T_e]$ defined by x_i). Therefore $[T_e]$ is indeed $\mathcal{R}$-independent.

The second statement follows from the first one and the observation that whenever $\mathcal{R} = \mathcal{R}(\mathcal{S})$, the following requirement is equivalent to item (ii) of Definition 4.5 and is absolute:

$$\big([e(t_0)] \times \ldots \times [e(t_{n_\beta-1})]\big) \cap S_\beta = \emptyset \qquad \text{for all } \beta < \alpha \text{ and } (t_0, \ldots, t_{n_\beta-1}) \in A_\alpha.$$

(where for any $u \in {}^{<\kappa}\kappa$, $[u]$ denotes the set of nodes $v \in {}^{<\kappa}\kappa$ such that $v \supseteq u\}$).

Conversely, suppose $[T]$ has a κ-perfect $\mathcal{R}(\mathcal{S})$-independent subset. Let T' be a strongly κ-perfect subtree of T such that $[T']$ is $\mathcal{R}(\mathcal{S})$-independent. Using the facts that $R_\alpha = [S_\alpha]$ is closed and $|A_\alpha| < \kappa$ for all $\alpha < \kappa$, it is straightforward to construct a perfect $(\mathcal{S},\mathcal{A})$-embedding $e : {}^{<\kappa}2 \to T'$.

The same arguments can be used in the case of $\mathcal{DI}_\kappa(n)$-sequences $\langle A_\alpha : \alpha < \kappa\rangle$ and families $\mathcal{R}$ of n-ary relations. □

Recall the definition of the dichotomy $\mathrm{PIF}_\kappa(X)$, for sets $X \subseteq {}^\kappa\kappa$, from p. 112. Note that the κ-Silver dichotomy for $\mathbf{\Sigma}^0_2(\kappa)$ equivalence relations on a X is a special case of $\mathrm{PIF}_\kappa(X)$.

Lemma 4.7. *Let* X*,* $Y \subseteq {}^\kappa\kappa$*. Suppose* $f : {}^\kappa\kappa \to {}^\kappa\kappa$ *is continuous and* $f[X] = Y$*.*

(1) *If* $\mathrm{PIF}_\kappa(X)$ *holds, then so does* $\mathrm{PIF}_\kappa(Y)$*.*
(2) *The* κ*-Silver dichotomy for* $\mathbf{\Sigma}^0_2(\kappa)$ *equivalence relations on* X *implies the* κ*-Silver dichotomy for* $\mathbf{\Sigma}^0_2(\kappa)$ *equivalence relations on* Y*.*

Specifically, $\mathrm{PIF}_\kappa(\mathcal{C}_\kappa)$ implies $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$, and the analogoue of this statement holds for the κ-Silver dichotomy for $\mathbf{\Sigma}^0_2(\kappa)$ equivalence relations. (Recall that $\mathcal{C}_\kappa$ denotes the collection of all closed subsets of the κ-Baire space.)

Proof. The following argument is similar to the proof of Proposition 3.5, and implies both items (1) and (2).

Let $\mathcal{R}$ be a family of finitary relations on Y such that $\mathrm{id}_Y \in \mathcal{R}$. If $1 \leq n < \omega$, then define, for each n-ary relation $R \in \mathcal{R}$, the following n-ary relation R' on X:

$$R' = \left\{(x_0, \ldots, x_{n-1}) \in {}^nX : \big(f(x_0), \ldots, f(x_{n-1})\big) \in R\right\}.$$

In other words, R' is the inverse image of R under the continuous function ${}^nX \to {}^nY$; $(x_0, \ldots, x_{n-1}) \mapsto (f(x_0), \ldots, f(x_{n-1}))$. Thus, R' is a $\mathbf{\Sigma}^0_2(\kappa)$ relation on X whenever R is a $\mathbf{\Sigma}^0_2(\kappa)$ relation on Y, and if R is an equivalence relation, then so is R'. Let

$$\mathcal{R}' = \{R' : R \in \mathcal{R}\}.$$

On the one hand, if $Z \subseteq Y$ is an $\mathcal{R}$-independent set of cardinality κ^+, then any $Z' \subseteq X$ such that $f[Z'] = Z$ and $f \restriction Z'$ is injective is an $\mathcal{R}'$-independent subset of X of cardinality κ^+.

On the other hand, if X has a κ-perfect $\mathcal{R}'$-independent subset, then (by Corollary 2.10) there exists a continuous injection $g : {}^\kappa 2 \to X$ whose image is $\mathcal{R}'$-independent. Notice that $f \circ g : {}^\kappa 2 \to Y$ is a continuous injection whose image is $\mathcal{R}$-independent, by the definition of $\mathcal{R}$ and the assumption $\mathrm{id}_Y \in \mathcal{R}$. Therefore Y has an $\mathcal{R}$-independent κ-perfect subset (again using Corollary 2.10). $\square$

4.1 The κ-Silver dichotomy for $\mathbf{\Sigma}^0_2(\kappa)$ equivalence relations

In this section, we show that after Lévy-collapsing an inaccessible $\lambda > \kappa$ to κ^+, the κ-Silver dichotomy holds for $\mathbf{\Sigma}^0_2(\kappa)$ equivalence relations on $\mathbf{\Sigma}^1_1(\kappa)$ subsets of the κ-Baire space. This result is proved in Theorem 4.14 below.

To establish Theorem 4.14, we first prove a series of preparatory lemmas. We state some lemmas in a more general form than needed for our main result.

If $\mathbb{P}$ is a partial order and I is an arbitrary set, then $\mathbb{P}^I$ denotes the full support product $\Pi_{i\in I}\mathbb{P}$. Note that if $\mathbb{P}$ is $<\kappa$-strategically closed, then so is $\mathbb{P}^I$. We let $\dot{g}_i$ denote the canonical name for the i^{th} coordinate of the $\mathbb{P}^I$-generic filter for any $i \in I$ (if I is not clear from the context, then we write $\dot{g}^I_i$ instead of $\dot{g}_i$).

Lemma 4.8. *Assume that $\mathcal{DI}_\kappa(n)$ holds. Let T be a subtree of ${}^{<\kappa}\kappa$. Let $n < \omega$ and let $\mathcal{S} = \langle S_\alpha : \alpha < \kappa\rangle$ be a sequence of subtrees of $T^{\otimes n}$.*

Suppose that $\mathbb{P}$ is a $<\kappa$-strategically closed notion of forcing and σ is a $\mathbb{P}$-name for a new branch of T such that

$$\mathbb{P}^n \Vdash (\sigma^{\dot{g}_0}, \ldots, \sigma^{\dot{g}_{n-1}}) \notin [\mathcal{S}].$$

Then $[T]$ has a κ-perfect $[\mathcal{S}]$-independent subset, in any transitive model $M \supseteq V$ of ZFC *such that $({}^{<\kappa}2)^M = ({}^{<\kappa}2)^V$.*

Proof. Let $\mathcal{A} = \langle A_\alpha : \alpha < \kappa\rangle$ be a $\mathcal{DI}_\kappa(n)$-sequence, and let τ be a winning strategy for player **II** in $\mathrm{G}_\kappa(\mathbb{P})$. We define recursively $\langle t_u \in T : u \in {}^{<\kappa}2\rangle$ and $\langle p_u, r_u \in \mathbb{P} : u \in {}^{<\kappa}2\rangle$ and also $\langle q_u \in \mathbb{P} : u \in {}^{<\kappa}2,\ \mathrm{ht}(u) \in \mathrm{Succ}\rangle$ such that the following hold for all $u, v \in {}^{<\kappa}2$ and $\alpha < \kappa$:

(i) if $u \subseteq v$, then $t_u \subseteq t_v$ and if $u \perp v$, then $t_u \perp t_v$;

(ii) for all $(u_0, \ldots, u_{n-1}) \in A_\alpha$ and $\gamma < \alpha$ we have

$$N_{t_{u_0}} \times \ldots \times N_{t_{u_{n-1}}} \cap [S_\gamma] = \emptyset;$$

(iii) $r_u \Vdash t_u \subseteq \sigma$;

(iv) $p_u \geq r_u > q_{u^\frown i} \geq p_v$ whenever $i \in 2$ and $u \subsetneq v$;

(v) $p_u = \tau(\langle q_{u\restriction\alpha+1} : \alpha + 1 \leq \mathrm{ht}(u)\rangle)$.

Items (iv) and (v) imply that for all $x \in {}^\kappa 2$, $\langle p_{x\restriction\alpha}, q_{x\restriction\alpha+1} : \alpha < \kappa\rangle$ is a run of $\mathrm{G}_\kappa(\mathbb{P})$ in which player **II** uses the strategy τ.

The first two items ensure that ${}^{<\kappa}T \to T;\ u \mapsto t_u$ is a perfect $(\mathcal{S}, \mathcal{A})$-embedding. Thus, by Lemma 4.6, $[T]$ has an $[\mathcal{S}]$-independent subset, in any transitive model $M \subseteq V$ of ZFC which has the same ${}^{<\kappa}2$ as V.

To see that t_u, p_u, r_u and q_u as described above can indeed be built, let $t_\emptyset = \emptyset$ and let $p_\emptyset = r_\emptyset = \tau(\emptyset)$. Now, fix $\alpha < \kappa$ and suppose that t_v, p_v, r_v and q_v have been defined for all $v \in {}^{<\alpha}2$. We first construct $\langle p_u : u \in {}^\alpha 2\rangle$, and if α is a successor ordinal, we also costruct $\langle q_u : u \in {}^\alpha 2\rangle$. Simultaneously, we construct nodes $t'_u \in T$ for all $u \in {}^\alpha 2$ such that the following hold for all $u, v \in {}^\alpha 2$ with $u \neq v$ and all $\beta < \alpha$:

$$t'_u \perp t'_v, \qquad t'_u \supseteq t_{u\restriction\beta}, \qquad p_u \Vdash t'_u \subset \sigma.$$

If α is a limit ordinal and $u \in {}^\alpha 2$, we let $t_u = \bigcup_{\beta<\alpha} t_{u\restriction\beta}$ and we define p_u using the winning strategy τ so that (iv) holds. Suppose $\alpha = \beta + 1$ and $v \in {}^\beta 2$. Because

$r_v \Vdash (t_v \subseteq \sigma \in [T]$ and $\sigma \notin V)$ by our assumptions, there exist $q_{v^\frown 0}, q_{v^\frown 1} < r_v$ and $t'_{v^\frown 0}, t'_{v^\frown 1} \supseteq t_v$ such that

$$t'_{v^\frown 0} \perp t'_{v^\frown 1}, \qquad \text{and} \qquad q_{v^\frown i} \Vdash t'_{v^\frown i} \subseteq \sigma \text{ for } i < 2.$$

We define $p_{v^\frown 0}, p_{v^\frown 1}$ using the strategy τ so that (iv) holds.

Lastly, the assumptions of the proposition and by applying Lemma 4.9 (found below) for the index set $I = {}^{\alpha}2$ and for each tree S_γ such that $\gamma < \alpha$, we have

$$\mathbb{P}^{({}^{\alpha}2)} \Vdash \{\sigma^{\dot{g}_u} : u \in {}^{\alpha}2\} \text{ is } [\mathcal{S}{\restriction}\alpha]\text{-independent},$$

where $[\mathcal{S}{\restriction}\alpha] = \bigcup\{[S_\gamma] : \gamma < \alpha\}$. Because $|A_\alpha| < \kappa$ and $[\mathcal{S} \restriction \alpha]$ is closed, there exist $r_u \in \mathbb{P}$ and $t_u \in T$ such that item (ii) holds and $r_u \leq p_u$ and $r_u \Vdash t_u \subset \sigma$ for all $u \in A_\alpha$. Finally, for all $u \in {}^{\alpha}2 \setminus A_\alpha$, we let $r_u = p_u$ and $t'_u = t_u$. This construction guarantees that all four required items are fulfilled. □

Lemma 4.9. *Let T be a subtree of ${}^{<\kappa}\kappa$ and let S be a subtree of $T^{\otimes n}$ where $n < \omega$. Suppose that $\mathbb{P}$ is a $<\kappa$-strategically closed notion of forcing and σ is a $\mathbb{P}$-name for a new branch of T. Let I be an arbitrary set.*

Then each of the following items implies the items below it:

(1) $\mathbb{P} \Vdash (\sigma, \overline{y}) \notin [S]$ *for all* $\overline{y} \in \big[[T] \cap V\big]^{n-1}$;
(2) $\mathbb{P}^n \Vdash (\sigma^{\dot{g}_0}, \ldots, \sigma^{\dot{g}_{n-1}}) \notin [S]$;
(3) $\mathbb{P}^I \Vdash \{\sigma^{\dot{g}_i} : i \in I\}$ *is* $[S]$*-independent.*

Proof. First, assume that item (1) holds. Let $\overline{p} = (p_0, \ldots, p_{n-1}) \in \mathbb{P}^n$. We have to find $\overline{r} = (r_0, \ldots, r_{n-1}) \leq \overline{p}$ which forces that $(\sigma^{\dot{g}_0}, \ldots, \sigma^{\dot{g}_{n-1}}) \notin [S]$.

Using the fact that $p_j \Vdash \sigma \in [T] \setminus V$ for all $1 \leq j < n$, we build, by recursion on $1 \leq j < n$ decreasing sequences $\langle q^j_\alpha \in \mathbb{P} : \alpha < \kappa\rangle$ and strictly increasing sequences $\langle t^j_\alpha \in T : \alpha < \kappa\rangle$ such that the following hold: $q^j_0 = p_j$, for all $\alpha < \kappa$ we have $q^j_\alpha \Vdash t^j_\alpha \subseteq \sigma$, and $t^i_j \perp t^j_j$ for all $1 \leq i < j$. (The last condition can be ensured because $p_j \Vdash \sigma \notin V$.) We let

$$y_j = \bigcup_{\alpha<\kappa} t^j_\alpha$$

for all $1 \leq j < n$. Then $y_1, \ldots, y_{n-1} \in V$ are pairwise different, and so, by the assumptions of the lemma, $p_0 \Vdash (\sigma, y_1, \ldots, y_{n-1}) \notin [S]$. Therefore we can choose $r_0 \leq p_0$ and $t_1, \ldots, t_{n-1} \in T$ such that

$$N_{t_0} \times \ldots \times N_{t_{n-1}} \cap [S] = \emptyset, \qquad r_0 \Vdash t_0 \subseteq \sigma, \qquad t_j \subseteq y_j \text{ for all } 1 \leq j < n.$$

Now, for all $1 \leq j < n$, let α_j be such that $t_j \subseteq t^j_{\alpha_j}$ and let $r_j = q^j_{\alpha_j}$. Then

$$\overline{r} = (r_0, \dots, r_{n-1}) \leq \overline{p} \qquad \text{and} \qquad \overline{r} \Vdash t_0 \subseteq \sigma^{\dot{g}_0}, \dots, t_{n-1} \subseteq \sigma^{\dot{g}_{n-1}}.$$

By item (2) of Lemma 3.6 for $\mathbb{P}^n$, we also have that $\overline{r} \Vdash N_{t_0} \times \ldots \times N_{t_{n-1}} \cap [S] = \emptyset$, and therefore $\overline{r} \Vdash (\sigma^{\dot{g}_0}, \dots, \sigma^{\dot{g}_{n-1}}) \notin [S]$, as required.

Now, assume that item (2) holds, and let I be an arbitrary set. If $|I| < n$, then the conclusion follows from the definition of $[S]$-independence. Suppose $|I| \geq n$, and let g be $\mathbb{P}^I$-generic. Suppose that $i_0, \dots, i_{n-1} \in I$ are pairwise distinct. Then, denoting by g_i the projection of g onto the i^{th} coordinate (for all $i \in I$), we have that $g_{i_0} \times \ldots \times g_{i_{n-1}}$ is $\mathbb{P}^n$-generic. Thus, by item (2) and the absoluteness of "$(x_0, \dots, x_{n-1}) \in [s]$" between transitive models of ZFC,

$$V[g] \models (\sigma^{g_{i_0}}, \dots, \sigma^{g_{i_{n-1}}}) \notin [S]$$

for all pairwise distinct $i_0, \dots, i_{n-1} \in I$, or in other words, the conclusion of item (3) holds. (Note that this proof also works for the μ-support product of copies of $\mathbb{P}$, for any infinite cardinal μ.) □

Lemmas 4.8 and 4.9 imply the following fact. Let $\mathbb{P}$ be a $<\kappa$-strategically closed forcing, and let $\mathcal{R}$ be a family of κ many closed finitary relations on a closed subset $[T]$ of the κ-Baire space. If $\mathbb{P}$ adds a new branch which is "independent from V", then there already exists a κ-perfect $\mathcal{R}$-independent set in V. More precisely, the following corollary holds.

Corollary 4.10. *Suppose $\mathcal{DI}_\kappa(n)$ holds and $T, \mathcal{S}$ and $\mathbb{P}$ are as in Lemma 4.8. If σ is a $\mathbb{P}$-name for a new branch of T such that*

$$\mathbb{P} \Vdash (\sigma, \overline{y}) \notin [\mathcal{S}] \text{ for all } \overline{y} \in \big[[T] \cap V\big]^{n-1},$$

then $[T]$ has a κ-perfect $[\mathcal{S}]$-independent subset, in any transitive model $M \supseteq V$ of ZFC *such that $({}^{<\kappa}2)^M = ({}^{<\kappa}2)^V$.*

Remark 4.11. By Lemma 4.9 and a straightforward modification of the proof of Lemma 4.8, we can obtain the analogue of the above corollary for families $\mathcal{R}$ of κ many $\mathbf{\Sigma}^0_2(\kappa)$ finitary relations (instead of just one such relation). More precisely, the following statement holds.

Assume $\mathcal{DI}_\kappa$. Let T be a subtree of ${}^{<\kappa}\kappa$ and let $\mathcal{S} = \langle S_\alpha : \alpha < \kappa\rangle$ be a sequence such that for all $\alpha < \kappa$, S_α is a subtree of $T^{\otimes n_\alpha}$, where $1 \le n_\alpha < \omega$.

If $\mathbb{P}$ is a $<\kappa$-strategically closed notion of forcing and σ is a $\mathbb{P}$-name for a new branch of T such that

$$\mathbb{P} \Vdash (\sigma, \overline{y}) \notin \bigcup_{\alpha<\kappa} [S_\alpha] \text{ for all } \overline{y} \in \big[[T] \cap V\big]^{<\omega},$$

then $[T]$ has a κ-perfect $[\mathcal{S}]$-independent subset, in any transitive model $M \supseteq V$ of ZFC *such that $({}^{<\kappa}2)^M = ({}^{<\kappa}2)^V$.*

If $\mathcal{S}$ consists of only one *closed binary* relation, then the assumption $\mathcal{DI}_\kappa(2)$ can be omitted in Lemma 4.8 and in Corollary 4.10, by the next proposition.

Proposition 4.12. *Let T be a subtree of ${}^{<\kappa}\kappa$, let S be a subtree of $T \otimes T$. Suppose that $\mathbb{P}$ is a $<\kappa$-strategically closed notion of forcing and σ is a $\mathbb{P}$-name for a new branch of T. Then each of the following items implies the items below it.*

(1) $\mathbb{P} \Vdash (\sigma, y) \notin [S]$ *for all* $y \in [T] \cap V$.
(2) $\mathbb{P}^2 \Vdash (\sigma^{\dot{g}_0}, \sigma^{\dot{g}_1}) \notin [S]$.
(3) $\mathbb{P} \Vdash \sigma \notin \bigcup\{[H] : H \in V,\ H$ *is an S-homogeneous subtree of* $T\}$.
(4) $[T]$ *has a κ-perfect $[S]$-independent subset, in any transitive model $M \supseteq V$ of* ZFC *such that $({}^{<\kappa}2)^M = ({}^{<\kappa}2)^V$.*

Furthermore, if $[S]$ is an equivalence relation, then the first three items are equivalent.

Recall that by Lemma 3.12, if $\mathbb{P}$ also forces that ${}^{\kappa}2 \not\subseteq V$, then items (3) and (4) above are equivalent.

Proof. Item (3) implies item (4) by Lemma 3.12, and item (1) implies item (2) by Lemma 4.9. Suppose that item (3) is false. Then there exists $p \in \mathbb{P}$ and an S-homogeneous subtree $H \in V$ of T such that $p \Vdash \sigma \in H$. Then (p, p) forces that $(\sigma^{\dot{g}_0}, \sigma^{\dot{g}_1}) \in [H] \times [H] \subseteq [S]$, and therefore item (2) does not hold.

Now, suppose that $[S]$ is an equivalence relation, and suppose item (1) is false. Let $y \in V$ and $p \in \mathbb{P}$ be such that $p \Vdash (\sigma, y) \in [S]$. Define the tree $S(y) \subseteq T$ by letting for all $t \in T$,

$$t \in S(y) \quad \text{iff} \quad (t, y{\restriction}\mathrm{ht}(t)) \in S.$$

Then we have $x \in [S(y)]$ iff $(x, y) \in [S]$ for all $x \in {}^{\kappa}\kappa$, in any transitive model of ZFC containing V. This implies on the one hand that $p \Vdash \sigma \in [S(y)]$. On the other hand, because $[S]$ is an equivalence relation, it implies that $[S(y)]$ is an $[S]$-homogeneous set. This means that $S(y)$ is an S-homogeneous subtree of T (because $S(y)$ is pruned). Also, $S(y) \in V$, and so it witnesses that item (3) does not hold. □

The next corollary says, roughly, that in certain forcing extensions, "the κ-Silver dichotomy holds for $\mathbf{\Sigma}^0_2(\kappa)$ binary relations E on $\mathbf{\Sigma}^1_1(\kappa)$ subsets X of the κ-Baire space such that X and E can be coded in V".

Recall that for any $Y \subseteq {}^\kappa\kappa \times {}^\kappa\kappa$, we denote by $\mathbf{p}Y$ the projection of Y onto the first coordinate.

Corollary 4.13. *Assume $\mathcal{DI}_\kappa(2)$. If $\mathbb{P}$ is a $<\kappa$-strategically closed forcing which forces that $|(2^\kappa)^V| = \kappa$, then $\mathbb{P}$ forces the following:*

> *if $T \in V$ is a subtree of ${}^{<\kappa}\kappa$, E is an equivalence relation on $\mathbf{p}[T]$ and $E = [\mathcal{S}] \cap (\mathbf{p}[T] \times \mathbf{p}[T])$ for a sequence $\mathcal{S} \in V$ of subtrees of $({}^{<\kappa}\kappa) \otimes ({}^{<\kappa}\kappa)$,*
>
> *then either E has $\leq \kappa$ many equivalence classes or E has κ-perfectly many equivalence classes.*

Proof. We use an argument similar to the proof of Corollary 3.13. By Lemma 4.7, it is enough to prove that $\mathbb{P}$ forces the version of the above statement in which "$\mathbf{p}[T]$" is replaced by "$[T]$". We can also assume that $\mathcal{S}$ is a sequence of subtrees of $T \otimes T$, and so, $E = [\mathcal{S}]$

Let $T, \mathcal{S} \in V$ be as above. If

$$\mathbb{P} \Vdash \text{for all } x \in [T] \text{ there exists } y \in [T]^V \text{ such that } (x, y) \in [\mathcal{S}],$$

then $\mathbb{P}$ forces that $[\mathcal{S}]$ has at most $|(2^\kappa)^V| = \kappa$ many equivalence classes. Otherwise, $[T]$ has a κ-perfect $[\mathcal{S}]$-independent subset in any transitive model $M \supseteq V$ of ZFC with the same ${}^{<\kappa}2$ as V, by Corollary 4.10 (applied to the partial order $\mathbb{P}_{\leq p} = \{q \in \mathbb{P} : q \leq_{\mathbb{P}} p\}$ for a suitable $p \in \mathbb{P}$). Thus, $\mathbb{P}$ forces that $[T]$ has a κ-perfect $[\mathcal{S}]$-independent subset. □

Theorem 4.14. *Suppose that κ is an uncountable regular cardinal, $\lambda > \kappa$ is inaccessible, and G is $\mathrm{Col}(\kappa, <\lambda)$-generic. Then in $V[G]$, the κ-Silver dichotomy holds for all $\mathbf{\Sigma}^0_2(\kappa)$ equivalence relations on $\mathbf{\Sigma}^1_1(\kappa)$-analytic subsets of ${}^\kappa\kappa$.*

Proof. Similarly to the proof of Theorem 3.14, let $E \in V[G]$ be an $\mathbf{\Sigma}^0_2(\kappa)$ equivalence relation on a $\mathbf{\Sigma}^1_1(\kappa)$ subset X of the κ-Baire space. Take a subtree T of ${}^{<\kappa}\kappa$ for which

$X = \mathbf{p}[T]$ and let $\mathcal{S}$ be a sequence of subtrees of $({}^{<\kappa}\kappa)\otimes({}^{<\kappa}\kappa)$ such that $E = [\mathcal{S}]\cap(X\times X)$. Observe that $\mathcal{S}$ can be coded as a subset of $\kappa \times {}^{<\kappa}\kappa$. Thus, since $\mathrm{Col}(\kappa, {<}\lambda)$ satisfies the λ-chain condition, there exists $0 < \gamma < \lambda$ such that $T, \mathcal{S} \in V[G_\gamma]$ where $G_\gamma = G\cap\mathrm{Col}(\kappa, {<}\gamma)$. One can now obtain the conclusion of the theorem in the case of $X = \mathbf{p}[T]$ and $E = [\mathcal{S}] \cap (X \times X)$ by applying Corollary 3.13 for $V[G_\gamma]$ and $\mathbb{P} = \mathrm{Col}(\kappa, [\gamma, \lambda))$. Note that $\diamondsuit_\kappa$ holds in $V[G_\gamma]$, and therefore so does $\mathcal{DI}_\kappa(2)$. □

Corollary 4.15. *Let κ be an uncountable cardinal with $\kappa^{<\kappa} = \kappa$. The following statements are equiconsistent.*

(1) There exists an inaccessible cardinal $\lambda > \kappa$.

(2) The κ-Silver dichotomy holds for all $\mathbf{\Sigma}^0_2(\kappa)$ equivalence relations on $\mathbf{\Sigma}^1_1(\kappa)$ subsets of the κ-Baire space.

Question 4.16. What is the consistency strength of the κ-Silver dichotomy for κ-Borel equivalence relations on the κ-Baire space?

By a result of Philipp Schlicht [39], $\mathrm{PSP}_\kappa(X)$ holds for all subsets X of the κ-Baire space which are definable from a κ-sequence of ordinals after Lévy-collapsing an inaccessible cardinal $\lambda > \kappa$ to κ^+. In light of this result, we ask the following question.

Question 4.17. After Lévy-collapsing an inaccessible cardinal $\lambda > \kappa$ to κ^+, does the κ-Silver dichotomy hold for $\mathbf{\Sigma}^0_2(\kappa)$-equivalence relations on subsets $X \subseteq {}^\kappa\kappa$ definable from a κ-sequence of ordinals?

4.2 A Cantor-Bendixson theorem for independent subsets of infinitely many $\mathbf{\Sigma}^0_2(\kappa)$ relations

In this section, we consider dichotomies for collections of finitary $\mathbf{\Sigma}^0_2(\kappa)$ relations on closed subsets of the κ-Baire space.

The main result of this section, Theorem 4.33 states, roughly, the following.

> *Suppose $\mathcal{DI}_\kappa$ holds and $\mathcal{R}$ is a collection of $\leq\kappa$ many finitary $\mathbf{\Sigma}^0_2(\kappa)$ relations on a closed subset X of the κ-Baire space.*
>
> *If X has $\mathcal{R}$-independent subsets "on all levels of the generalized Cantor-Bendixson hierarchy for player* **II***" (in the sense of Theorem 2.44), then X has a κ-perfect $\mathcal{R}$-independent subset.*

Theorem 4.33 is the uncountable version of a result of Martin Doležal and Wieslaw Kubiś [5].

As a corollary of our arguments, we also obtain stronger versions of the main result in [46]; see Corollaries 4.32, 4.35 and 4.36.

The arguments presented in this section, are in part based on methods used in e.g. [18, 36, 51] and in Section 2.2.

Given a subset $X \subseteq {}^{\kappa}\kappa$, we can think of its complement $({}^{\kappa}\kappa - X)$ as a unary relation on the κ-Baire space. A subset Y of the κ-Baire space is $({}^{\kappa}\kappa - X)$-independent if and only if $Y \subseteq X$. If X is closed, then $({}^{\kappa}\kappa - X)$ is open and is therefore a $\mathbf{\Sigma}^0_2(\kappa)$ unary relation on ${}^{\kappa}\kappa$. Thus, our results can be stated in terms of families $\mathcal{R}$ of finitary relations on the whole κ-Baire space.

We begin this section by showing that for arbitrary families $\mathcal{R}$ of finitary relations on the κ-Baire space, if there exist $\mathcal{R}$-independent sets $Y \subseteq {}^{\kappa}\kappa$ on all levels of the generalized Cantor-Bendixson hierarchy for player **II** (in the sense of Theorem 2.44), then there exists a κ-dense in itself $\mathcal{R}$-independent subset. A similar statement for player **I** will also be obtained.

Recall that for any ordinal ξ, $\mathcal{T}_\xi$ denotes the class of trees $\boldsymbol{t}$ such that every branch of $\boldsymbol{t}$ has length $<\xi$. If λ is a cardinal, then $\mathcal{T}_{\lambda,\xi}$ denotes the class of trees $\boldsymbol{t} \in \mathcal{T}_\xi$ of cardinality $\leq\lambda$.

Proposition 4.18. *Suppose $\mathcal{R}$ is a set of finitary relations on ${}^{\kappa}\kappa$. Then the following statements are equivalent.*

(1) There exists an $\mathcal{R}$-independent κ-dense in itself subset of ${}^{\kappa}\kappa$.
(2) There exists an $\mathcal{R}$-independent set $Y \subseteq {}^{\kappa}\kappa$ such that player **II** *wins $\mathcal{V}_\kappa(Y)$.*
(3) For all trees $\boldsymbol{t} \in \mathcal{T}_\kappa$, there exists an $\mathcal{R}$-independent set $Y \subseteq {}^{\kappa}\kappa$ such that

$$\mathrm{Ker}_{\boldsymbol{t}}(Y) \cap Y \neq \emptyset,$$

i.e., player **II** *wins $\mathcal{V}_{\boldsymbol{t}}(Y)$.*

If the third statement does not hold, then there exists a tree $\boldsymbol{t}' \in \mathcal{T}_{2^\kappa,\kappa}$ showing this.

The first two statements are equivalent by Corollary 2.72, and they clearly imply the third one. We note that (by Remark 2.68 and Fact 1.26), the third statement is also equivalent to the claim that for all $\boldsymbol{t} \in \mathcal{T}_\kappa$, there exists a $\boldsymbol{t}$-dense in itself $\mathcal{R}$-independent subset of ${}^\kappa\kappa$.

In order to prove the equivalence of the statements (2) and (3), we first define, below, the tree $\boldsymbol{t}(\mathcal{R})$ of winning strategies for player **II** in short games $\mathcal{V}_{\gamma+1}(Y)$ played on $\mathcal{R}$-independent subsets Y of ${}^\kappa\kappa$.

Definition 4.19. Suppose $\mathcal{R}$ is a set of finitary relations on ${}^\kappa\kappa$. Let $\boldsymbol{t}(\mathcal{R})$ denote the tree which consists of pairs $(\gamma+1,\, \tau)$ where $\gamma < \kappa$, and

$$\text{there exists an } \mathcal{R}\text{-independent set } Y \subseteq {}^\kappa\kappa \text{ such that } \tau \text{ is a winning strategy for player } \mathbf{II} \text{ in } \mathcal{V}_{\gamma+1}(Y).$$

We let $(\gamma+1,\, \tau) \leq (\gamma'+1,\, \tau)$ if and only if $\gamma \leq \gamma'$ and τ' agrees with τ in the first $\gamma+1$ rounds of $\mathcal{V}_{\gamma'+1}({}^\kappa\kappa)$, i.e.,

$$\tau'\left(\langle \delta_\beta : \beta < \alpha\rangle\right) = \tau\left(\langle \delta_\beta : \beta < \alpha\rangle\right)$$

for all legal partial plays $\langle \delta_\beta : \beta < \alpha\rangle$ of player **I** of length $\leq \gamma$.

Definition 4.20. Suppose $(\gamma+1,\, \tau)$ is a pair such that $\gamma < \kappa$ and τ is a winning for player **II** in $\mathcal{V}_{\gamma+1}({}^\kappa\kappa)$.

We denote by $Y_{(\gamma+1,\,\tau)}$ the set of moves that τ defines for player **II** in $\mathcal{V}_{\gamma+1}({}^\kappa\kappa)$ in response to all possible legal partial plays of player **I**. That is, let

$$Y_{(\gamma+1,\,\tau)} = \big\{\tau\left(\langle \delta_\beta : \beta < \alpha\rangle\right) : \alpha \leq \gamma \text{ and } \langle \delta_\beta : \beta < \alpha\rangle \text{ is an increasing sequence of ordinals below } \kappa.\big\}$$

Thus, $Y_{(\gamma+1,\tau)}$ is the minimal set X such that τ is a winning strategy for **II** in $\mathcal{V}_{\gamma+1}(X)$.

Claim 4.21. *Suppose that $\mathcal{R}$ is a set of finitary relations on ${}^\kappa\kappa$, and that $(\gamma+1,\, \tau)$ and $(\gamma'+1,\, \tau')$ are as in Definition 4.20. Then*

(1) $(\gamma+1,\, \tau) \in \boldsymbol{t}(\mathcal{R})$ if and only if $Y_{(\gamma+1,\,\tau)}$ is $\mathcal{R}$-independent;

(2) if $(\gamma+1,\, \tau) \leq (\gamma'+1,\, \tau)$ then $Y_{(\gamma+1,\,\tau)} \subseteq Y_{(\gamma'+1,\,\tau')}$.

When stating Claims 4.22 to 4.24, we assume $\mathcal{R}$ denotes a fixed set of finitary relations on the κ-Baire space.

Note that, by definition, $\boldsymbol{t}(\mathcal{R}) \in \mathcal{T}_{\kappa+1}$.

Claim 4.22. *The tree $\boldsymbol{t}(\mathcal{R})$ has a κ-branch if and only if player* **II** *wins $\mathcal{V}_\kappa(Y)$ for some $\mathcal{R}$-independent $Y \subseteq {}^\kappa\kappa$.*

Thus, $\boldsymbol{t}(\mathcal{R})$ has a κ-branch iff there exists an $\mathcal{R}$-independent κ-dense in itself set.

Proof. Suppose $\boldsymbol{t}(\mathcal{R})$ has a branch $b = \langle t_\alpha : \alpha < \kappa\rangle$ of length κ. Using the notation defined in Definition 4.20, the set Y_{t_α} is $\mathcal{R}$-independent for all $\alpha < \kappa$, and we have $Y_{t_\beta} \subseteq Y_{t_\alpha}$ for all $\beta < \alpha < \kappa$. This implies that the set $Y = \bigcup\{Y_{t_\alpha} : \alpha < \kappa\}$ is $\mathcal{R}$-independent, and the branch b defines a winning strategy for player **II** in $\mathcal{V}_\kappa(Y)$. The other direction is clear. □

Claim 4.23. *If $\boldsymbol{t}(\mathcal{R}) \in \mathcal{T}_\kappa$, then $|\boldsymbol{t}(\mathcal{R})| \leq 2^\kappa$.*

Proof. This statement follows directly from the definition of $\boldsymbol{t}(\mathcal{R})$ (and the assumption $\kappa^{<\kappa} = \kappa$), by counting. □

Claim 4.24. *Suppose $\boldsymbol{t}$ is a tree. If player* **II** *wins $\mathcal{V}_{\boldsymbol{t}}(Y)$ for some $\mathcal{R}$-independent set $Y \subseteq {}^\kappa\kappa$, then $\boldsymbol{t} \leq \boldsymbol{t}(\mathcal{R})$.*

Proof. Let τ be a winning strategy for player **II** in $\mathcal{V}_{\boldsymbol{t}}(Y)$, where Y is $\mathcal{R}$-independent. Analogously to the proof of Claim 2.48, we construct a map $f : \boldsymbol{t} \to \boldsymbol{t}(\mathcal{R})\,;\ t \mapsto (\gamma_t{+}1\ \tau_t)$. That is, we let γ_t be the order type of $\mathrm{pred}_{\boldsymbol{t}}(t)$, and we obtain the strategy τ_t for player **II** in $\mathcal{V}_{\gamma_t+1}(Y)$ by restricting τ to $\mathrm{pred}_{\boldsymbol{t}}(t) \cup \{t\}$. In more detail, if $\langle t_\beta : \beta \leq \alpha\rangle$ is the sequence of elements of $\mathrm{pred}_{\boldsymbol{t}}(t) \cup \{t\}$ in ascending order, then we define

$$\tau_t\big(\langle \delta_\beta : \beta < \alpha\rangle\big) = \tau\big(\langle t_\beta, \delta_\beta : \beta < \alpha\rangle^\frown\langle t_\alpha\rangle\big)$$

for all legal partial plays $\langle \delta_\beta : \beta < \alpha\rangle$ of player **I** in $\mathcal{V}_{\gamma_t+1}(Y)$. By its definition, $(\gamma_t{+}1,\ \tau_t) \in \boldsymbol{t}(\mathcal{R})$. It is also easy to check that the map f is indeed order preserving. □

Proof of Proposition 4.18. Suppose there is no $\mathcal{R}$-independent κ-dense in itself subset of ${}^\kappa\kappa$. Then, by Claims 4.22 and 4.23, we have $\boldsymbol{t}(\mathcal{R}) \in \mathcal{T}_{2^\kappa,\kappa}$. Thus, the tree

$$\boldsymbol{t}' = \sigma\left(\boldsymbol{t}(\mathcal{R})\right)$$

of ascending chains of $\boldsymbol{t}(\mathcal{R})$ is also in $\mathcal{T}_{2^\kappa,\kappa}$ (see Definition 1.15 and Fact 1.17). By Claim 4.24 and Lemma 1.16, player **II** does not win $\mathcal{V}_{\boldsymbol{t}'}(Y)$ for any $\mathcal{R}$-independent set $Y \subseteq {}^\kappa\kappa$. □

The next example shows that the converse of Claim 4.24 does not hold.

Example 4.25. Consider the closed set X and the closed binary relation $R \subseteq {}^2X$ defined in item (2) of Example 3.37, and let $\mathcal{R} = \{{}^\kappa\kappa - X,\, R\}$.

Using the notation in Example 3.37, a set $Y \subseteq X$ is $\mathcal{R}$-independent if and only if $Y \subseteq X_\alpha$ for some $\alpha < \kappa$. Recall that X_α is γ_α-perfect and $(\gamma_\alpha + 1)$-scattered, where $\langle \gamma_\alpha : \alpha < \kappa \rangle$ is an enumeration of the set of indecomposable ordinals $\gamma < \kappa$.

Therefore,

$$\boldsymbol{t}(\mathcal{R}) \equiv \boldsymbol{f},$$

but any $\mathcal{R}$-independent set Y is $\boldsymbol{f}$-scattered (i.e., player **I** wins $\mathcal{V}_{\boldsymbol{f}}(Y)$. Here, $\boldsymbol{f}$ denotes the κ-fan).

Remark 4.26. The statement $\boldsymbol{t} \leq \boldsymbol{t}(\mathcal{R})$ is equivalent, for all trees $\boldsymbol{t}$, to a slightly weaker statement than the condition in Claim 4.24.

If $\boldsymbol{t} \in \mathcal{T}_{\kappa+1}$, and $b \in \mathit{Branch}(\boldsymbol{t})$, then let $Y_{(b,\,\tau)}$ denote the set of moves that τ defines for player **II** in response to all those legal partial plays of player **I** in $\mathcal{V}_{\boldsymbol{t}}({}^\kappa\kappa)$ in which he chooses all his moves $t_\beta \in \boldsymbol{t}$ from the branch b. That is, let

$$\begin{aligned} Y_{(b,\,\tau)} = \big\{ \tau\left(\langle t_\beta, \delta_\beta : \beta < \alpha\rangle^\frown \langle t_\alpha\rangle\right) : \alpha \leq \gamma,\\ \langle \delta_\beta : \beta < \alpha\rangle \text{ is an increasing sequence of ordinals below } \kappa,\\ \langle t_\beta : \beta \leq \alpha\rangle \text{ is an increasing sequence of nodes in } b. \big\} \end{aligned}$$

Equivalently, $Y_{(b,\tau)}$ is the minimal set X such that τ is a winning strategy for **II** in $\mathcal{V}_b(X)$ (where b also denotes the subtree of $\boldsymbol{t}$ which consists of the nodes in the branch b). With this notation, the following statement holds.

If $\boldsymbol{t}$ is a tree, then $\boldsymbol{t} \leq \boldsymbol{t}(\mathcal{R})$ if and only if

$$\text{player } \mathbf{II} \text{ has a winning strategy } \tau \text{ in } \mathcal{V}_{\boldsymbol{t}}({}^\kappa\kappa) \text{ such that } Y_{(b,\,\tau)} \text{ is } \mathcal{R}\text{-independent for all } b \in \mathit{Branch}(\boldsymbol{t}). \tag{4.1}$$

The proof of Claim 4.24 shows that (4.1) implies $\boldsymbol{t} \leq \boldsymbol{t}(\mathcal{R})$, for all trees $\boldsymbol{t}$. To see the other direction, it is enough to show that (4.1) holds for the tree $\boldsymbol{t} = \boldsymbol{t}(\mathcal{R})$. We define a winning strategy for player **II** in $\mathcal{V}_{\boldsymbol{t}}({}^\kappa\kappa)$ as follows. If $p = \langle t_\beta, \delta_\beta : \beta < \alpha\rangle^\frown\langle t_\alpha\rangle$ is a legal partial play of player **I** in $\mathcal{V}_{\boldsymbol{t}}({}^\kappa\kappa)$, and $t_\alpha = \langle \gamma_{\alpha+1}, \tau_\alpha\rangle$, Then let

$$\tau(p) = \tau_\alpha(\langle \delta_\beta : \beta < \alpha\rangle).$$

Note that $\tau(p)$ is well defined because $\alpha \leq \gamma_\alpha$. Clearly, τ is a winning strategy for player **II** in $\mathcal{V}_{\boldsymbol{t}}({}^\kappa\kappa)$.

Suppose that $b = \langle t_\alpha : \alpha < \delta \rangle$ is a branch of $\boldsymbol{t} = \boldsymbol{t}(\mathcal{R})$, and $t_\alpha = (\gamma_\alpha + 1, \tau_\alpha)$. Then

$$Y_{(b,\tau)} = \bigcup_{\alpha<\delta} Y_{(\gamma_\alpha+1,\tau_\alpha)}.$$

By Claim 4.21, this implies that $Y_{(b,\tau)}$ is $\mathcal{R}$-independent. Thus, τ shows that (4.1) holds.

We show below that a stamtement analogous to Proposition 4.18 also holds for player **I**.

Proposition 4.27. *Suppose $\mathcal{R}$ is a set of finitary relations on ${}^\kappa\kappa$. Then the following statements are equivalent.*

(1) Every $\mathcal{R}$-independent subset of ${}^\kappa\kappa$ is κ-scattered.

(2) There exists a tree $\boldsymbol{s} \in \mathcal{T}_\kappa$ such that every $\mathcal{R}$-independent subset of ${}^\kappa\kappa$ is $\boldsymbol{s}$-scattered.

Note that in the classical case, a set is scattered if and only if it has no dense in itself subsets. Thus, for $\kappa = \omega$ and families of finitary relations on the Baire space, Propositions 4.18 and 4.27 are equivalent.

Proof of Proposition 4.27. It is clear that item (2) implies item (1).

To see the other direction, let $Y \subseteq {}^\kappa\kappa$ be an arbitrary $\mathcal{R}$-independent set, and fix a winning strategy $\rho(Y)$ for player **I** in $\mathcal{V}_\kappa(Y)$. We let $\boldsymbol{s}_{\rho(Y)}$ denote the tree which consists of legal partial plays $\langle x_\beta : \beta \leq \alpha \rangle$ of player **II** in $\mathcal{V}_\kappa(Y)$ against $\rho(Y)$ (that is, $\boldsymbol{s}_{\rho(Y)}$ consists of those partial plays of **II** of successor length against $\rho(Y)$ where she has not lost yet). Then, $\boldsymbol{s}_{\rho(Y)} \in \mathcal{T}_\kappa$ because otherwise, player **II** would win a run of $\mathcal{V}_\kappa(Y)$ where player **I** uses ρ. Therefore the tree

$$\boldsymbol{s}_Y = \sigma \boldsymbol{s}_{\rho(Y)}$$

of ascending chains in $\boldsymbol{s}_{\rho(Y)}$ is also in $\mathcal{T}_\kappa$. Player **I** wins $\mathcal{V}_{\boldsymbol{s}_Y}(Y)$: he obtains a winning strategy by copying the sequences of moves of player **II** into $\boldsymbol{s}_Y$, and defining his moves δ_β using $\rho(Y)$ (see the proof of Claim 2.49 for a detailed definition of the winning strategy in an analogous case).

Now, let

$$\boldsymbol{s} = \bigotimes \{\boldsymbol{s}_Y : Y \text{ is a } \mathcal{R}\text{-independent subset of } {}^\kappa\kappa\};$$

that is, $\boldsymbol{s}$ is the supremum of the trees $\boldsymbol{s}_Y$ for all $\mathcal{R}$-independent subsets $Y \subseteq {}^\kappa\kappa$ (see Definition 1.19). Then $\boldsymbol{s} \in \mathcal{T}_\kappa$, and player **I** wins $\mathcal{V}_{\boldsymbol{s}}(Y)$ for all $\mathcal{R}$-independent subsets Y of the κ-Baire space. □

We now consider the case when $\mathcal{R}$ consists of κ many finitary $\mathbf{\Sigma}^0_2(\kappa)$ relations on a closed subset X of the κ-Baire space (of arbitrary arity).

Observe that it is equivalent to assume, when considering $\mathcal{R}$-independent subsets of ${}^\kappa\kappa$, that all the relations in $\mathcal{R}$ are closed relations on the whole κ-Baire space. (If $R = \bigcup_{\alpha<\kappa} R_\alpha$ for a set of closed n-ary relations R_α, then a set $Y \subseteq {}^\kappa\kappa$ is R-independent if and only if Y is independent w.r.t. $\{R_\alpha : \alpha < \kappa\}$. Secondly, $Y \subseteq X$ if and only if Y is $({}^\kappa\kappa - X)$-independent. If X is closed, then $({}^\kappa\kappa - X)$ is open and is therefore a $\mathbf{\Sigma}^0_2(\kappa)$ unary relation on ${}^\kappa\kappa$.)

Thus, in the rest of this section, we will often be assuming that

$$\mathcal{R} = \langle R_\alpha : \alpha < \kappa\rangle$$

is a sequence of closed relations R_α on ${}^\kappa\kappa$ (of arbitrary finite arity). We will also confuse $\mathcal{R}$ with $\{R_\alpha : \alpha < \kappa\}$ at times.

Definition 4.28. Suppose that $\mathcal{R} = \langle R_\alpha : \alpha < \kappa\rangle$ is a sequence of finitary relations on ${}^\kappa\kappa$, and that $\mathcal{A} = \langle A_\alpha : \alpha < \kappa\rangle$ is a sequence such that $A_\alpha \subseteq [{}^\alpha 2]^{<\omega}$ for all $\alpha < \kappa$.

We let $\boldsymbol{u}(\mathcal{R}, \mathcal{A})$ denote the tree which consists of pairs $(\gamma+1, e)$ such that $\gamma < \kappa$ and

$$e : {}^{<\gamma+1}2 \to {}^{<\kappa}\kappa$$

is an $(\mathcal{R}, \mathcal{A})$-embedding (see Definition 4.5). The tree $\boldsymbol{u}(\mathcal{R}, \mathcal{A})$ is ordered by letting

$$(\gamma+1, e) \leq (\gamma'+1, e') \qquad \text{iff} \qquad \gamma \leq \gamma' \quad \text{and} \quad e' \upharpoonright {}^{<\gamma+1}2 = e.$$

Note that $\boldsymbol{u}(\mathcal{R}, \mathcal{A})$ is a tree of height $\leq \kappa$. Furthermore, if κ is inaccessible and $\boldsymbol{u}(\mathcal{R}, \mathcal{A})$ has no κ-branches, then $|\boldsymbol{u}(\mathcal{R}, \mathcal{A})| \leq \kappa$. (Note that this latter statement may not hold for $\boldsymbol{t}(\mathcal{R})$.)

Claim 4.29. *Suppose that $\mathcal{A}$ is a $\mathcal{DI}_\kappa$-sequence and that $\mathcal{R}$ is a sequence of length κ of closed finitary relations on ${}^\kappa\kappa$. Then*

$\boldsymbol{u}(\mathcal{R}, \mathcal{A})$ has a κ-branch iff there exists a κ-perfect $\mathcal{R}$-independent $Y \subseteq {}^\kappa\kappa$.

Proof. This statement follows from the observation that $\boldsymbol{u}(\mathcal{R}, \mathcal{A})$ has a κ-branch if and only if there exists a perfect $(\mathcal{R}, \mathcal{A})$-embedding and Lemma 4.6. □

Lemma 4.30. *If $\mathcal{A}$ is a $\mathcal{DI}_\kappa$-sequence and $\mathcal{R} = \langle R_\alpha : \alpha < \kappa\rangle$ is a sequence of closed finitary relations on ${}^\kappa\kappa$, then*

$$\boldsymbol{t}(\mathcal{R}) \leq \boldsymbol{u}(\mathcal{R}, \mathcal{A})\,.$$

Proof. Suppose that $(\gamma+1,\tau) \in \boldsymbol{t}(\mathcal{R})$. Let Y be an $\mathcal{R}$-independent set such that τ is a winning strategy for player **II** in $\mathcal{V}_{\gamma+1}(Y)$.

We define an $(\mathcal{R},\mathcal{A})$-embedding e_τ of height $\gamma+1$ by modifying (the first $\gamma+1$ stages of) the construction in the proof of Proposition 2.69. In more detail, we construct $\langle u_s, x_s, \delta_s : s \in {}^{\leq\gamma}2\rangle$ such that $u_s \in {}^{<\kappa}\kappa$, $x_s \in Y$, and $\delta_s < \kappa$, and the following items hold for all $s, r \in {}^{\leq\gamma}2$ and all $\alpha \leq \gamma$:

(i) $u_s = x_s{\restriction}\delta_s$;
(ii) if $r \subseteq s$, then $u_r \subseteq u_s$;
(iii) if $r \perp s$, then $u_r \perp u_s$;
(iv) for all $\beta < \alpha$ and all tuples $(s_0, \dots, s_{n_\beta-1}) \in A_\alpha$ (where n_β is the arity of R_β), we have
$$N_{u_{s_0}} \times \dots \times N_{u_{s_{n_\beta-1}}} \cap R_\beta = \emptyset;$$
(v) $x_s = \tau\big(\langle \delta_{s\restriction\beta} : \beta < \operatorname{ht}(s), s(\beta) = 1\rangle\big)$.

The last item states that x_s is obtained from a partial run
$$\langle x_{s\restriction\beta}, \delta_{s\restriction\beta} : \beta < \operatorname{ht}(s), s(\beta)=1\rangle^\frown\langle x_s\rangle$$
of $\mathcal{V}_\kappa(Y)$ where player **II** uses the strategy τ.

Note that condition (iv) can be guaranteed in each stage $\alpha \leq \gamma$ of the construction because the elements $\langle x_s \in Y : s \in {}^{\alpha}2\rangle$ are pairwise different (this can be guaranteed as in the proof of Proposition 2.69 at successor stages and is automatic at limit stages). Thus, by the $\mathcal{R}$-independence of Y and by $|A_\alpha| < \kappa$, we can choose the ordinals $\langle \delta_s : s \in {}^{\alpha}2\rangle$ high enough for item (iv) to hold.

Conditions (ii) to (iv) imply that the map
$$e_\tau : {}^{\leq\gamma}2 \to {}^{<\kappa}\kappa \quad \text{defined by letting } e_\tau(s) = u_s \text{ for all } s \in {}^{\leq\gamma}2$$
is an $(\mathcal{R},\mathcal{A})$-embedding. That is, $(\gamma+1, e_\tau) \in \boldsymbol{u}(\mathcal{R},\mathcal{A})$. The map
$$f : \boldsymbol{t}(\mathcal{R}) \to \boldsymbol{u}(\mathcal{R},\mathcal{A})\,; \quad (\gamma+1,\,\tau) \mapsto (\gamma+1,\, e_\tau)$$
is order preserving by items (i) and (v). □

Remark 4.31. Let $\mathcal{R}$ and $\mathcal{A}$ be as in Lemma 4.30. By Remark 2.30, the tree $\boldsymbol{u}(\mathcal{R},\mathcal{A})$ can be embedded in an order preserving way into the tree of winning strategies for player **II** in short games $\mathcal{G}^*_{\gamma+1}({}^{<\kappa}\kappa)$. In fact, denoting by $\boldsymbol{u}'(\mathcal{R},\mathcal{A})$ the image of $\boldsymbol{u}(\mathcal{R},\mathcal{A})$ under this embedding, we have

$$\boldsymbol{u}(\mathcal{R},\mathcal{A}) \equiv \boldsymbol{u}'(\mathcal{R},\mathcal{A}).$$

The tree $\boldsymbol{u}'(\mathcal{R},\mathcal{A})$ consists of pairs $(\gamma+1,\tau')$ such that $\gamma<\kappa$ and τ' is a winning strategy for player **II** in $\mathcal{G}^*_{\gamma+1}({}^{<\kappa}\kappa)$ such that τ' satisfies the statement corresponding to item (ii) in Definition 4.5 (or equivalently, to item (iv) in the proof of Lemma 4.30).

The order preserving embedding f defined in the proof of Lemma 4.30 corresponds to the order preserving embedding of $\boldsymbol{u}'(\mathcal{R},\mathcal{A})$ into $\boldsymbol{t}(\mathcal{R})$ which is determined by the proofs of Propositions 3.39 and 3.47.

Claims 4.22, 4.29 and Lemma 4.30 (and the observation above Definition 4.28) imply the following statement immediately.

Corollary 4.32. *Assume $\mathcal{DI}_\kappa$. Let $\mathcal{R}$ be a collection of $\leq\kappa$ many finitary $\mathbf{\Sigma}^0_2(\kappa)$ relations on a closed subset X of the κ-Baire space.*

If X has a κ-dense in itself $\mathcal{R}$-independent subset, then X has a κ-perfect $\mathcal{R}$-independent subset.

Let $\mathcal{R}$ be a collection of $\leq\kappa$ many $\mathbf{\Sigma}^0_2(\kappa)$ relations on the κ-Baire space. The next theorem, which is the main result of this section, states roughly that under the assumption $\mathcal{DI}_\kappa$, the existence of $\mathcal{R}$-independent sets $Y\subseteq{}^\kappa\kappa$ on all levels of the generalized Cantor-Bendixson hierarchy for player **II** implies the existence of a κ-perfect $\mathcal{R}$-independent subset.

Theorem 4.33. *Assume $\mathcal{DI}_\kappa$. Let $\mathcal{R}$ be a collection of $\leq\kappa$ many finitary $\mathbf{\Sigma}^0_2(\kappa)$ relations on a closed subset X of the κ-Baire space.*

Then exactly one of the following statements holds.

(1) X has a κ-perfect $\mathcal{R}$-independent subset.
*(2) There exists a tree $\boldsymbol{u}\in\mathcal{T}_{2^\kappa,\kappa}$ such that player **II** does not win $\mathcal{V}_{\boldsymbol{u}}(Y)$ for any $\mathcal{R}$-independent set $Y\subseteq X$.*

If κ is inaccessible and the second statement holds, then there exists a tree $\boldsymbol{u}\in\mathcal{T}_{\kappa,\kappa}$ witnessing this.

Proof. Proposition 4.18 and Corollary 4.32 imly immediately that statement (2) holds iff statement (1) does not hold.

Suppose κ is inaccessible, and let $\mathcal{A}$ denote a $\mathcal{DI}_\kappa$-sequence. As observed above Definition 4.28, we can assume that $\mathcal{R}$ consists of κ many closed finitary relations on the whole κ-Baire space. If statement (2) holds, then the tree

$$\boldsymbol{u} = \sigma\boldsymbol{u}(\mathcal{R}, \mathcal{A})$$

has no κ-branches by Claim 4.29. This implies $|\boldsymbol{u}| = |\boldsymbol{u}(\mathcal{R}, \mathcal{A})|^{<\kappa} = \kappa$. □

Question 4.34. In Theorem 4.33, can we have $|\boldsymbol{u}| \leq \kappa$ for any cardinal κ with $\kappa^{<\kappa} = \kappa$ (i.e., even when κ is not inaccessible)?

In Proposition 4.27, can we have $|\boldsymbol{s}| \leq 2^\kappa$ or even $|\boldsymbol{s}| \leq \kappa$?

The next corollary follows from Corollary 4.32 and Proposition 4.7. Recall that DISP_κ is the following statement:

every subset of ${}^\kappa\kappa$ of cardinality κ^+ has a κ-dense in itself subset.

Corollary 4.35. *Assume $\mathcal{DI}_\kappa$. If DISP_κ holds, then $\mathrm{PIF}_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$ also holds, i.e.,*

if $\mathcal{R}$ is a collection of $\leq\kappa$ many finitary $\boldsymbol{\Sigma}^0_2(\kappa)$ relations on a $\boldsymbol{\Sigma}^1_1(\kappa)$ subset X of ${}^\kappa\kappa$ and X has an $\mathcal{R}$-independent subset of cardinality κ^+, then X has a κ-perfect $\mathcal{R}$-independent subset.

The above corollary, Proposition 2.78 and Theorem 2.76 imply the following stronger version of the main result, Theorem 2.4, of [46].

Corollary 4.36. *(1) The assumptions $\mathrm{I}^{\mathsf{w}}(\kappa)$ and $\mathcal{DI}_\kappa$ imply that $\mathrm{PIF}_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$ holds.*

(2) Specifically, if $\lambda > \kappa$ is weakly compact and G is $\mathrm{Col}(\kappa, {<}\lambda)$-generic, then $\mathrm{PIF}_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$ holds in $V[G]$.

We note that [46, Theorem 2.4] states that if $\mathrm{I}^-(\kappa)$ holds and either κ is inaccessible or $\diamondsuit_\kappa$ holds, then $\mathrm{PIF}_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$ holds (or rather, the special case of $\mathrm{PIF}_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$ for one finitary $\boldsymbol{\Sigma}^0_2(\kappa)$ relation holds).

Thus, [46, Theorem 2.4] implies that the consistency of $\mathrm{PIF}_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$ follows from the existence of a measurable cardinal $\lambda > \kappa$. The above corollary shows that it is already implied by the existence of a weakly compact cardinal $\lambda > \kappa$.

Because $\mathrm{PIF}_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$ implies the κ-perfect set property for $\boldsymbol{\Sigma}^1_1(\kappa)$ sets, its consistency strength is at least that of the existence of an inaccessible cardinal above κ (see Remark 2.14).

Question 4.37. What is the consistency strength of $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$? In particular, is $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ consistent relative to the existence of an inaccessible cardinal $\lambda > \kappa$?

In the classical countable case, $\mathrm{PIF}_\omega(\mathbf{\Pi}^0_2)$ does not hold: there exists a binary $\mathbf{\Pi}^0_2$ relation R on the Cantor space ${}^\omega 2$ such that every maximal R-independent set has cardinality $\aleph_1$ but there are no perfect R-independent subsets [42]; see also [25] where a concrete example of such a binary relation is given.

Question 4.38. Is $\mathrm{PIF}_\kappa(\mathbf{\Pi}^0_2(\kappa))$ false (in ZFC)? If not, is the failure of $\mathrm{PIF}_\kappa(\mathbf{\Pi}^0_2(\kappa))$ consistent?

Question 4.39. Can $\mathcal{DI}_\kappa$ be weakened or omitted in the main results of this section?

4.3 Elementary embeddability on models of size κ

In this section, we obtain a as special a special case of $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ model theoretic dichotomy which is motivated by the spectrum problem.

Before stating the main result, Theorem 4.43, of this section, we introduce some notation and terminology to be used here. We also give some possible motivation behind Theorem 4.43, and define some concepts necessary for stating it.

Notation. We will use the following notation in the rest of this section. Recall that $\mathit{Sym}(\kappa)$ denotes the permutation group of κ, and we write $\mathit{Inj}(\kappa)$ for the monoid of all injective functions of ${}^\kappa\kappa$. We denote by t_{pr} the product topology on the set ${}^\kappa\kappa$ (where the set κ is given the discrete topology).

The symbol L denotes a fixed first order language which contains only relation symbols and is of size at most κ. However, the arguments below also work in the case of languages which have infinitary relations of arity $< \kappa$. We assume the language L has κ many variables, the sequence of which is denoted by $\langle v_i : i \in \kappa\rangle$. The symbols $\mathcal{A}$, $\mathcal{B}$, etc. are used to denote L-structures whose domains are A, B, etc. The set of all L-structures with domain κ is denoted by Mod^L_κ. Given a structure $\mathcal{A} \in \mathrm{Mod}^L_\kappa$, we identify $\mathcal{A}$-valuations with elements of ${}^\kappa\kappa$.

Given cardinals $\omega \leq \mu \leq \lambda \leq \kappa^+$ and $\mu \leq \kappa$, $L_{\lambda\mu}$ denotes the infinitary language which allows conjunctions and disjunctions of $< \lambda$ many formulas and quantification over $< \mu$ many variables (see, e.g., Definitions 1.1.2 and 1.1.3 of [4] for the precise definition of $L_{\lambda\mu}$-formulas or, alternatively, Definitions 9.12 and 9.13 of [53]). In particular, note that by definition, an $L_{\lambda\mu}$-formula contains $< \mu$ many free variables, from $\{v_i : i \in \kappa\}$. The concept of the *subformulas* of a formula $\varphi \in L_{\kappa^+\kappa}$ is defined by induction on the complexity of φ as usual (see, e.g., Definition 1.3.1 of [4] or [53, p. 234]). Note that if φ is obtained as $\varphi = \bigwedge \Phi$, then any $\phi \in \Phi$ is defined to be a subformula of φ, but $\bigwedge \Phi'$, where $\Phi' \subset \Phi$, is not a subformula.

For $\varphi \in L_{\kappa^+\kappa}$ and $h \in {}^\kappa\kappa$, we denote by $s_h\varphi$ the formula obtained from φ by simultaneously substituting, for all $i \in \kappa$, the variable $v_{h(i)}$ for the variable v_i. We say that a set F of $L_{\kappa^+\kappa}$-formulas is *closed under substitution* if for any $\varphi \in F$ and $h \in {}^\kappa\kappa$ we have $s_h\varphi \in F$. Note that since $\kappa^{<\kappa} = \kappa$, closing a nonempty set of formulas of $L_{\kappa^+\kappa}$ of size $\leq \kappa$ under substitution leads to a set of formulas of size κ.

A $\mathbf{\Sigma}^1_1(L_{\kappa^+\kappa})$ denotes the set of second order formulas of the form $\exists \overline{R}\, \varphi(\overline{R})$, where $\overline{R}$ is a set of $\leq \kappa$ many symbols disjoint from the original vocabulary and $\varphi(\overline{R})$ is an $L_{\kappa^+\kappa}$ formula in the expanded language.

Given a sentence $\psi \in \mathbf{\Sigma}^1_1(L_{\kappa^+\kappa})$, we denote by Mod^ψ_κ the set of models of ψ with domain κ.

We now give some motivation for, and state, the main result of this section. Let ψ denote a fixed sentence in $\mathbf{\Sigma}^1_1(L_{\kappa^+\kappa})$. One obtains interesting questions by considering, instead of the number of non-isomorphic models in Mod^ψ_κ, the possible sizes of sets of models in Mod^ψ_κ which are pairwise non-elementarily embeddable, as in for example [2, 41]. More generally, the role of elementary embeddings may be replaced by embeddings preserving (in the sense of (4.2) in Definition 4.41) "nice" sets of formulas, possibly of some extension of first order logic.

We consider the case when the "nice" sets of formulas to be preserved are *fragments* of $L_{\kappa^+\kappa}$; this concept is defined below.

Definition 4.40. A *fragment of* $L_{\kappa^+\kappa}$ is a set $F \subseteq L_{\kappa^+\kappa}$ of size $|F| = \kappa$ such that

(i) F contains all atomic formulas,

(ii) F is closed under negation and taking subformulas, and

(iii) F is closed under substitution of variables.

Examples of fragments of $L_{\kappa^+\kappa}$ include the set of all first order formulas, the set At of all atomic formulas and their negations, the infinitary logics $L_{\lambda\mu}$, where $\omega \leq \mu \leq \lambda \leq \kappa$, and the n-variable fragments of these logics.

In the case of fragments $F \subseteq L_{\kappa^+\omega}$ and sentences $\psi \in F$, the set of models of ψ together with F-embeddings (i.e., the embeddings preserving F) forms an abstract elementary class, and the corresponding version of the above question has been studied in, e.g., [43]. To the best knowledge of the author, this question has not been studied yet in the case fragments of $L_{\kappa^+\kappa}$ which are not subsets of $L_{\kappa^+\omega}$.

Note that if f is an embedding between elements of Mod^ψ_κ (preserving a fragment of $L_{\kappa^+\kappa}$), then $f \in \mathit{Inj}(\kappa)$. We ask what happens when, in the above questions, the role of $\mathit{Inj}(\kappa)$ is replaced by a certain subset H of $\mathit{Inj}(\kappa)$, i.e., when Mod^ψ_κ is considered up to only the embeddings which are in H (and preserve the given fragment of $L_{\kappa^+\kappa}$). Notice that when H is a subgroup of $\mathit{Sym}(\kappa)$, the above question reduces to considering models up to isomorphisms in H. By introducing the set H of "allowed embeddings" as an extra parameter, we may study explicitly the role the topological properties of H play in these questions.

Definition 4.41. Suppose F is a fragment of $L_{\kappa^+\kappa}$, $H \subseteq \mathit{Inj}(\kappa)$, and $\mathcal{A}, \mathcal{B} \in \mathrm{Mod}^L_\kappa$. We say that a map $h \in {}^\kappa\kappa$ is an *F-embedding* of $\mathcal{A}$ into $\mathcal{B}$ iff we have

$$\mathcal{A} \models \varphi[a] \quad \text{iff} \quad \mathcal{B} \models \varphi[h \circ a] \qquad \text{for all } \varphi \in F \text{ and valutations } a \in {}^\kappa\kappa. \tag{4.2}$$

We say that $\mathcal{A}$ is *(F, H)-embeddable* into $\mathcal{B}$ iff there exists an F-embedding $h \in H$ of $\mathcal{A}$ into $\mathcal{B}$.

In the special case when H is a subgroup of $\mathit{Sym}(\kappa)$, we say that $\mathcal{A}$ is *H-isomorphic* to $\mathcal{B}$ iff there exists an isomorphism $h \in H$ between $\mathcal{A}$ and $\mathcal{B}$.

Let F be an arbitrary fragment of $L_{\kappa^+\kappa}$. Note that any F-embedding $h \in {}^\kappa\kappa$ of $\mathcal{A}$ into $\mathcal{B}$ must be an embedding of $\mathcal{A}$ into $\mathcal{B}$, and in particular, we must have $h \in \mathit{Inj}(\kappa)$. If $h \in \mathit{Sym}(\kappa)$, then h is an F-embedding if and only if h is an isomorphism.

If H is a subgroup of $\mathit{Sym}(\kappa)$, then H-isomorphism is an equivalence relation on Mod^L_κ, and if H is a *submonoid* of $\mathit{Inj}(\kappa)$, then (F, H)-embeddability on Mod^L_κ is a partial order. (However, there is no reason for this to hold when H is an arbitrary subset of $\mathit{Inj}(\kappa)$).

Example 4.42. We give some examples of fragments F and the corresponding notions of (F, H)-embeddability.

- When F is the set At of atomic formulas and their negations, (F, H)-embeddability is the same as *H-embeddability*, i.e., embeddability by elements of H.

- When $F = L_{\omega\omega}$, (F, H)-embeddability is the same as H-*elementary embeddability*, i.e., elementary embeddability by elements of H.
- $F = L_{\lambda\mu}$ is a fragment, given cardinals $\omega \leq \mu \leq \lambda \leq \kappa$.
- Suppose $\omega \leq \mu \leq \lambda \leq \kappa$. By definition, the n *variable fragment* of $L_{\lambda\mu}$, or equivalently of $L_{\lambda\omega}$, consists of those formulas which use only the variables $v_0, \ldots, v_{n-1}$. In this case, the corresponding fragment F is the set of those $L_{\lambda\omega}$-formulas which contain at most n (arbitrary) variables from $\{v_i : i \in \kappa\}$. We denote this fragment F by $L^n_{\lambda\omega}$.

Recall that a subset C of a topological space is defined to be κ-*compact* iff any open cover of C has a subcover of size $< \kappa$, and C is K_κ iff it can be written as the union of at most κ many κ-compact subsets.

In the next theorem, which is the main result of this section, we obtain as a special case of $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ a dichotomy about Mod^ψ_κ up to (F, H)-embeddability, for K_κ subsets H of the κ-Baire space and fragments F of $L_{\kappa^+\kappa}$. In the case of for certain fragments, it is enough to assume that H is a K_κ subset of the product space $\langle {}^\kappa\kappa, t_{\mathrm{pr}}\rangle$ (where t_{pr} is the product topology on the set ${}^\kappa\kappa$ obtained by equipping κ with the discrete topology).

Theorem 4.43. *Suppose* $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ *holds. Let* $H \subseteq Inj(\kappa)$*, let* F *be a fragment of* $L_{\kappa^+\kappa}$ *and let* ψ *be a sentence of* $\mathbf{\Sigma}^1_1(L_{\kappa^+\kappa})$*. Suppose that either*

(1) H *is a* K_κ *subset of the* κ*-Baire space, or*
(2) H *is a* K_κ *subset of the product space* $\langle {}^\kappa\kappa, t_{\mathrm{pr}}\rangle$ *and* $F \subseteq L_{\kappa^+\omega}$.

If there are at least κ^+ *many pairwise non* (F, H)*-embeddable models in* Mod^ψ_κ*, then there are* κ*-perfectly many such models.*

Theorem 4.43 can be seen as uncountable version of [38, Theorems 5.8 and 5.9]. We note that these two cited theorems of [38] also follow from [24, Corollary 2.13] or from [42, Remark 1.14].

In order to prove Theorem 4.43, we have to first show that for any fragment F and K_κ subset H of the κ-Baire space, Mod^ψ_κ can be coded as a $\mathbf{\Sigma}^1_1(\kappa)$ subset of the κ-Cantor space *on which* (F, H)*-embeddability is a* $\mathbf{\Sigma}^0_2(\kappa)$ *binary relation* (and when $F \subseteq L_{\kappa^+\omega}$, this holds even when H is K_κ only in the product topology t_{pr}). This is done by considering the κ-Borel refinement t_F induced by F of the canonical topology used to study the deep connections between model theory and generalized descriptive set theory (see [34] and, e.g., [52] and [9]), and generalizing to the uncountable case an argument in [35]. These arguments allow us to obtain the dichotomy in Theorem 4.43 as a special case of $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$. In Theorem 4.43, the word "κ-perfect" may refer to the topology $t_{F'}$ induced by *any* fragment F' of $L_{\kappa^+\kappa}$ (see Corollary 4.54); specifically, it may also refer to the canonical topology.

We remark that when κ is a non-weakly compact cardinal, $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ implies that there are no K_κ subsets of the κ-Baire space other than those of size $\leq \kappa$. However, K_κ sets of size $> \kappa$ exist in the case of the product space $\langle {}^\kappa\kappa, t_{\mathrm{pr}}\rangle$, or in the case of the κ-Baire space when κ is weakly compact. (See Propositions 4.44 and 4.45.)

One possible motivation for investigating the above questions for K_κ subsets H, even in the case of the κ-Baire space for κ non-weakly compact, is the following. Let F be a fixed fragment of $L_{\kappa^+\kappa}$ and equip Mod^ψ_κ with the topology t_F described above. Consider, for each $H \subseteq \mathit{Inj}(\kappa)$, the (F,H)-embeddability relation R^F_H viewed as a subset of $\mathrm{Mod}^\psi_\kappa \times \mathrm{Mod}^\psi_\kappa$. Specifically, the relation $R^F_{\mathit{Inj}(\kappa)}$ of F-embeddability corresponds to the original question where the set of "allowed" embeddings "has not been restricted". Because the standard base of the space $\langle \mathrm{Mod}^\psi_\kappa, t_F\rangle$ is of size κ, it is possible to construct a subset (even a submonoid) H of $\mathit{Inj}(\kappa)$ of size $\leq \kappa$ such that R^F_H is dense in $R^F_{\mathit{Inj}(\kappa)}$. On the one hand, the density of R^F_H in $R^F_{\mathit{Inj}(\kappa)}$ may be interpreted, on an intuitive level, to mean that "the action of H on Mod^ψ_κ is locally similar to the action of $\mathit{Inj}(\kappa)$". On the other hand, $|H| \leq \kappa$ and is therefore a K_κ subset of the κ-Baire space, which implies that our model theoretic dichotomy result Theorem 4.43 is applicable in this case as well.

Proposition 4.44 (by Corollary 2.8 in [31]). *Suppose κ is not weakly compact cardinal (and $\kappa^{<\kappa} = \kappa$). Then $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ implies that the K_κ subsets of the κ-Baire space are exactly those of size at most κ.*

Thus, $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ also implies that the K_κ subsets of the κ-Baire space are exactly those of size at most κ.

Examples of subsets H of the set ${}^\kappa\kappa$ which are κ-compact subsets of the *product space* $\langle {}^\kappa\kappa, t_{\mathrm{pr}}\rangle$ but are not K_κ subsets of the κ-Baire space, even when $\mathrm{PSP}_\kappa(\mathcal{C}_\kappa)$ is not assumed, include by [31, Lemma 2.2] (and Tychonoff's theorem) the set $H = {}^\kappa 2$ and, more generally, sets of the form $H = \prod_{\alpha<\kappa} I_\alpha$, where $I_\alpha \in [\kappa]^{<\kappa}$ and I_α is finite except for $< \kappa$ many $\alpha < \kappa$. (See [29] and the references therein for when this last assumption can be weakened.)

Notice that K_κ subsets of the κ-Baire space are always K_κ subsets of $\langle {}^\kappa\kappa, t_{\mathrm{pr}}\rangle$ as well, due to the fact that the κ-Baire topology on the set ${}^\kappa\kappa$ is finer than the product topology t_{pr}. When κ is a weakly compact cardinal, the converse also holds. Proposition 4.45 below gives a characterization of the K_κ subsets of ${}^\kappa\kappa$.

A subset H of ${}^\kappa\kappa$ is *bounded* iff there exists an $x \in {}^\kappa\kappa$ such that $h(\beta) \leq x(\beta)$ for all $h \in H$ and $\beta < \kappa$.

A subset H of ${}^\kappa\kappa$ is *eventually bounded* iff there exists an $x \in {}^\kappa\kappa$ such that for all $h \in H$, we have $h \leq^* x$ (i.e., there exists an $\alpha < \kappa$ such that $h(\beta) \leq x(\beta)$ for all $\alpha \leq \beta < \kappa$).

Observe that a union of κ many bounded sets is eventually bounded.

Proposition 4.45. *Suppose κ is a weakly compact cardinal. Given any set $H \subseteq {}^\kappa\kappa$, the following statements are equivalent.*

(1) H is a K_κ subset of the κ-Baire space.
(2) H is a K_κ subset of the product space $\langle {}^\kappa\kappa, t_{\mathrm{pr}}\rangle$.
(3) H is a $\mathbf{\Sigma}^0_2(\kappa)$ subset of the κ-Baire space which is eventually bounded.

Proof. The equivalence of the first and third items follows from [31, Lemma 2.6], and, as just noted, the second item is implied by the first one.

The third item can be obtained from the second using the fact that a κ-compact subset C of $\langle {}^\kappa\kappa, t_{\mathrm{pr}}\rangle$ is closed in the κ-Baire topology and is bounded. This fact can be proven by modifying standard arguments from the countable case (see e.g. [21, Exercise 4.11]). In more detail, suppose that $C \subseteq {}^\kappa\kappa$ is κ-compact in the product topology t_{pr}. If we take any $\alpha < \kappa$ (and we denote by $N_{\{(\alpha,\gamma)\}}$ the set $\{y \in {}^\kappa\kappa : y(\alpha) = \gamma\} \in t_{\mathrm{pr}}$ for all $\gamma < \kappa$), the family

$$\{N_{\{(\alpha,\gamma)\}} : \gamma < \kappa \text{ and } C \cap N_{\{(\alpha,\gamma)\}} \neq \emptyset\}$$

is a disjoint t_{pr}-open cover of C, and must therefore be of size $< \kappa$. Thus, we can define a function $x \in {}^\kappa\kappa$ by letting $x(\alpha) = \sup\{\gamma < \kappa : C \cap N_{\{(\alpha,\gamma)\}} \neq \emptyset\}$ for all $\alpha < \kappa$, and x witnesses that C is bounded.

To see that C is closed in the κ-Baire topology, let $z \in {}^\kappa\kappa - C$. We can choose, for all $y \in C$, disjoint neighborhoods $U_y, V_y \in t_{\mathrm{pr}}$ of z and y respectively. By the κ-compactness of C, the t_{pr}-open cover $\{V_y : y \in C\}$ of C can be refined to a subcover $\{V_y : y \in I\}$ of size $< \kappa$. Then the intersection $U = \bigcap_{y \in I} U_y$ is disjoint from C, contains z, and is open in the κ-Baire topology (because the κ-Baire topology is finer than t_{pr} and is closed under intersections of size $< \kappa$). $\square$

Remark 4.46. A model of ZFC in which κ is weakly compact (in fact, supercompact) and $\mathrm{I}^{\mathsf{w}}(\kappa)$ holds (and therefore so does $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$) can be obtained, starting out from a situation in which κ is supercompact and there exists a weakly compact $\lambda > \kappa$, in the following way. Before Lévy-collapsing λ to κ^+, one first applies the Laver preparation [28] to make the supercompactness of κ indestructible by any $<\kappa$-directed closed forcing. If we would also like to have $2^\kappa > \kappa^+$ together with $\mathrm{I}^{\mathsf{w}}(\kappa)$ for a supercompact κ, we force after the Laver-preparation with $\mathrm{Col}(\kappa, {<}\lambda) \times \mathrm{Add}(\kappa, \mu)$ for some $\mu > \lambda$. We would like to thank Menachem Magidor for suggesting the arguments found in this remark.

A fragment $F \subseteq L_{\kappa^+\kappa}$ induces a topology t_F on the set Mod^L_κ in a natural way.

Definition 4.47. Given a formula φ and a valuation $a \in {}^\kappa\kappa$, we let

$$\mathrm{Mod}_\kappa(\varphi, a) = \{\mathcal{A} \in \mathrm{Mod}^L_\kappa : \mathcal{A} \models \varphi[a]\}.$$

We let t_F denote the topology on Mod^L_κ which is obtained by taking arbitrary unions of intersections of $< \kappa$ many sets from the collection

$$b_F = \{\mathrm{Mod}_\kappa(\varphi, a) : \varphi \in F, a \in {}^\kappa\kappa\},$$

and we let $\mathcal{M}od_F = \langle \mathrm{Mod}^L_\kappa, t_F\rangle$.

Given a sentence $\psi \in \mathbf{\Sigma}^1_1(L_{\kappa^+\kappa})$ we denote by $\mathcal{M}od^\psi_F$ the subspace of the $\mathcal{M}od_F$ with domain Mod^ψ_κ. In other words, $\mathcal{M}od^\psi_F$ is obtained by equipping Mod^ψ_κ with the topology t_F.

The canonical topological space used to study the connections between model theory and the generalized Baire space is $\mathcal{M}od_{\mathrm{At}}$, and it is homeomorphic to the Cantor space ${}^\kappa 2$; see [34, 52, 9].

An advantage of working with t_F instead of t_{At} is that (F, H)-embeddability induces a "t_F-continuous action" of H on Mod^ψ_κ and is therefore a $\mathbf{\Sigma}^0_2(\kappa)$ binary relation on $\mathcal{M}od^\psi_F$ (see Proposition 4.55 and the proof of Theorem 4.43 below). This fact is needed in order to obtain our model theoretic dichotomy as a special case of $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$.

As a first step towards proving Theorem 4.43, we show that for an arbitrary fragment F, the space $\mathcal{M}od_F$ is homeomorphic to a $\mathbf{\Pi}^0_2(\kappa)$ subset X_F of ${}^\kappa 2$. Our proof is essentially a generalization from the countable case of an argument in [35]. Note that a bijection between κ and F allows us to define the generalized Cantor topology on ${}^F 2$. In fact, since κ is regular (by $\kappa^{<\kappa} = \kappa$), another basis for this topology is $\{N_p : p \in {}^\Phi 2 \text{ for some } \Phi \in [F]^{<\kappa}\}$, where $N_p = \{x \in {}^F 2 : p \subseteq x\}$, and therefore this topology does not depend on the chosen bijection.

Definition 4.48. Given a fragment F of $L_{\kappa^+\kappa}$, define a map $i_F : \mathrm{Mod}^L_\kappa \longrightarrow {}^F 2$ as follows: if $\mathcal{A} \in \mathrm{Mod}^L_\kappa$, then let $i_F(\mathcal{A}) \in {}^F 2$ be such that

$$i_F(\mathcal{A})(\varphi) = 1 \quad \text{iff} \quad \mathcal{A} \models \varphi[\mathrm{id}_\kappa]$$

for all $\varphi \in F$. We denote by X_F the image of i_F.

Claim 4.49. *For any fragmet F of $L_{\kappa^+\kappa}$, the map $i_F : \mathrm{Mod}^L_\kappa \longrightarrow {}^F 2$ is injective.*

Proof. If $\mathcal{A}, \mathcal{B}$ are different structures in Mod^L_κ, then there exists a formula φ in At (the set of all atomic formulas and their negations) such that $\mathcal{A} \models \varphi[\mathrm{id}_\kappa]$ and $\mathcal{B} \not\models \varphi[\mathrm{id}_\kappa]$. Then

$$i_F(\mathcal{A})(\varphi) = 1, \text{ and } i_F(\mathcal{B})(\varphi) = 0.$$

This implies, using the fact that $\mathrm{At} \subseteq F$ by Definition 4.40, that $i_F(\mathcal{A}) \neq i_F(\mathcal{B})$. □

Proposition 4.50. *If F is a fragment of $L_{\kappa^+\kappa}$, then X_F is a $\mathbf{\Pi}^0_2(\kappa)$ subset of the Cantor space ${}^F 2$, and i_F is a homeomorphism from $\mathcal{M}od_F$ onto its image X_F (where X_F is given the subspace topology).*

Proof. Because F is closed under substitution, the collection b_F from which the topology t_F is obtained is in fact equal to $\{\mathrm{Mod}_\kappa(\varphi, \mathrm{id}_\kappa) : \varphi \in F\}$. Using this, it is not hard to see that the injection i_F is a homeomorphism between $\mathcal{M}od_F$ and its image X_F.

To see that $X_F \subseteq {}^F 2$ is $\mathbf{\Pi}^0_2(\kappa)$, we define the following subsets of ${}^F 2$. For any $h \in {}^\kappa\kappa$, we denote by $\mathrm{supp}(h)$ the set of those $\alpha \in \kappa$ for which $h(\alpha) \neq \alpha$. We let

$$X_0 = \{x \in {}^F 2 : x(\psi) = 1 \text{ iff } x(\neg\psi) = 0 \text{ for all } \psi \in F\};$$

$$X_1 = \{x \in {}^F 2 : \text{if } \psi \in F \text{ and } \psi = \bigwedge \Phi \text{ for some } \Phi \in [F]^{\leq\kappa}, \text{ then } x(\psi) = 1 \text{ iff for all } \varphi \in \Phi \text{ we have } x(\varphi) = 1\};$$

$$X_2 = \{x \in {}^F 2 : \text{if } \psi \in F \text{ and } \psi = \exists(v_\beta : \beta \in I)\varphi \text{ where } \varphi \in F \text{ and } I \in [\kappa]^{<\kappa}, \text{ then } x(\psi) = 1 \text{ iff } x(s_h\varphi) = 1 \text{ for some } h \in {}^\kappa\kappa \text{ such that } \mathrm{supp}(h) \subseteq I\};$$

$$X_3 = \{x \in {}^F 2 : \text{for all } i, j \in \kappa \text{ we have } x(v_i = v_j) = 1 \text{ iff } i = j\}.$$

Let X denote the intersection of the sets X_i for $i < 4$. We claim that the X_i's are all $\mathbf{\Pi}^0_2(\kappa)$ subsets of ${}^F 2$ and show this in detail for X_1. For $\psi = \bigwedge \Phi \in F$, the set

$$X^1_{1,\psi} = \{x \in {}^F 2 : x(\psi) = 1 \text{ and } x(\varphi) = 1 \text{ for all } \varphi \in \Phi\} = \bigcap_{\varphi \in \Phi} N_{\{(\psi,1),(\varphi,1)\}}$$

is $\mathbf{\Pi}^0_2(\kappa)$, while the set

$$X^0_{1,\psi} = \{x \in {}^F 2 : x(\psi) = 0 \text{ and } x(\varphi) = 0 \text{ for some } \varphi \in \Phi\} = \bigcup_{\varphi \in \Phi} N_{\{(\psi,0),(\varphi,0)\}}$$

is open. Therefore $X_1 = \bigcap\{X^0_{1,\psi} \cup X^1_{1,\psi} : \psi = \bigwedge \Phi \in F\}$ is also a $\mathbf{\Pi}^0_2(\kappa)$ subset of the Cantor space ${}^F 2$. That X_0, X_2 and X_3 are also $\mathbf{\Pi}^0_2(\kappa)$ can be seen similarly; in the case of X_2, one has to use the fact that $|\{s_h \varphi : h \in {}^\kappa\kappa\}| \leq \kappa$ holds because φ has $< \kappa$ variables and $\kappa^{<\kappa} = \kappa$.

Therefore the intersection X of the X_i's is also $\mathbf{\Pi}^0_2(\kappa)$, and it remains to see that $X_F = X$. It is straightforward to show that for any $\mathcal{A} \in \mathrm{Mod}^L_\kappa$, we have $i_F(\mathcal{A}) \in X$, and so $X_F \subseteq X$. For the other direction, first observe that if $x, y \in X$ and $x \restriction \mathrm{At} = y \restriction \mathrm{At}$, then $x = y$ also holds, by an easy induction on the complexity of formulas. Suppose $x \in X$ is arbitrary. We wish to define a model $\mathcal{A} \in \mathrm{Mod}^L_\kappa$ such that $x = i_F(\mathcal{A})$; by the above observation, it is enough to require that $x \restriction \mathrm{At} = i_F(\mathcal{A}) \restriction \mathrm{At}$. Clearly, the L-model $\mathcal{A}$ whose domain is κ and whose relations are defined by letting, for each n-ary relation symbol R of L and $\alpha_1, \ldots, \alpha_n \in \kappa$,

$$(\alpha_1, \ldots, \alpha_n) \in R^{\mathcal{A}} \text{ iff } x(R(v_{\alpha_1}, \ldots, v_{\alpha_n})) = 1,$$

satisfies these requirements. □

Given a sentence $\psi \in \mathbf{\Sigma}^1_1(L_{\kappa^+\kappa})$, we let

$$X^\psi_F = \{i_F(\mathcal{A}) : \mathcal{A} \in \mathrm{Mod}^\psi_\kappa\};$$

that is, X^ψ_F is the set of elements of X_F corresponding to models of ψ.

Corollary 4.51. *If ψ is a sentence of $\mathbf{\Sigma}^1_1(L_{\kappa^+\kappa})$ and F is a fragment of $L_{\kappa^+\kappa}$, then X^ψ_F is a $\mathbf{\Sigma}^1_1(\kappa)$ subset of ${}^F 2$. Furthermore, the map $i_F \restriction \mathcal{M}od^\psi_F$ is a homeomorphism from $\mathcal{M}od^\psi_F$ onto its image X^ψ_F equipped with the subspace topology.*

Proof. As we have seen at the beginning of the previous proof, it is enough to show that X^ψ_F is $\mathbf{\Sigma}^1_1(\kappa)$. First, in the case when $\psi \in F$ (and therefore is an $L_{\kappa^+\kappa}$-sentence), we have $X^\psi_F = X_F \cap N_{\{(\psi,1)\}}$. Thus, X^ψ_F is a $\mathbf{\Pi}^0_2(\kappa)$ subset of ${}^F 2$ by Proposition 4.50.

Now, in general, suppose that ψ is the sentence $\exists \overline{R}\varphi(\overline{R})$. Let F' be the fragment generated (in the expanded language) by $F \cup \{\varphi(\overline{R})\} \cup \overline{R}$. Then we have that X_F^ψ is the image of the $\mathbf{\Pi}_2^0(\kappa)$ set $X_{F'}^\varphi$ under the continuous map ${}^{F'}2 \longrightarrow {}^F2$, $x \mapsto x \restriction F$ and is therefore $\mathbf{\Sigma}_1^1(\kappa)$. (Equivalently, $\mathcal{Mod}_F^\psi$ is the image of $\mathcal{Mod}_{F'}^\varphi$ under the continuous map defined by taking the L-reducts of models for the expanded language.) □

Definition 4.52. We say that ψ has *κ-perfectly many non (F, H)-embeddable models* iff Mod_κ^ψ has a t_F-perfect subset which is independent with respect to the binary relation of (F, H)-embeddability on Mod_κ^ψ.

Note that the term "t_F-perfect" makes sense since $\mathcal{Mod}_F^\psi$ is homeomorphic to a subset of the κ-Baire space.

As Corollary 4.54 below shows, the choice of the fragment that generates the topology on Mod_κ^ψ is actually irrelevant in the above definition. That is, given any fragments F, F' of $L_{\kappa^+\kappa}$, a sentence ψ has κ-perfectly many non (F, H)-embeddable models if and only if Mod_κ^ψ has a *$t_{F'}$-perfect* subset which is independent with respect to the binary relation of (F, H)-embeddability on Mod_κ^ψ.

Below, by a t_F-Borel subset of Mod_κ^ψ, we mean a κ-Borel subset of $\mathcal{Mod}_F^\psi$, and a map $f : X \longrightarrow \mathrm{Mod}_\kappa^\psi$ (where X is a topological space) is t_F-Borel iff the inverse images of t_F-Borel subsets of Mod_κ^ψ are κ-Borel subsets of X.

Proposition 4.53. *Let F and F' be arbitrary fragments of $L_{\kappa^+\kappa}$.*

(1) A subset of Mod_κ^ψ is t_F-Borel iff it is $t_{F'}$-Borel.

(2) A map $f : {}^\kappa 2 \longrightarrow \mathrm{Mod}_\kappa^\psi$ is t_F-Borel iff it is $t_{F'}$-Borel.

Proof. An easy induction shows that for any $\varphi \in L_{\kappa^+\kappa}$ (and therefore for any $\varphi \in F'$) and valuation $a \in {}^\kappa\kappa$, the set $\mathrm{Mod}_\kappa(\varphi, a)$ is t_F-Borel. Consequently, all $t_{F'}$-Borel sets are t_F-Borel sets as well. By symmetry, we have item (1), of which item (2) is a direct consequence. □

By Corollary 2.10, the space Mod_κ^ψ has a t_F-perfect R-independent set if and only if there exists a t_F-Borel injection $\iota : {}^\kappa 2 \to \mathrm{Mod}_\kappa^\psi$ such that $\mathrm{ran}(\iota)$ is R-independent. This fact and Proposition 4.53 immediately imply that the following statement.

Corollary 4.54. *Let F and F' be arbitrary fragments of $L_{\kappa^+\kappa}$, and suppose R is a binary relation on Mod_κ^ψ. Then Mod_κ^ψ has a t_F-perfect R-independent subset if and only if it has a $t_{F'}$-perfect R-independent set.*

We obtain Theorem 4.43 as a special case of the Proposition 4.55 below.

For H a topological space, X any set and $S \subseteq H \times X \times X$, let R_S be the projection of S onto $X \times X$, i.e.,

$$R_S = \{(x, y) : (h, x, y) \in S \text{ for some } h \in H\}.$$

Specifically, for the action a of a group H on X, R_a is the orbit equivalence relation.

Proposition 4.55. *Suppose $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ holds. Let X be a $\mathbf{\Sigma}^1_1(\kappa)$ subset of the κ-Baire space (equipped with the subspace topology) and let H be an arbitrary K_κ topological space.*

(1) If $S \subseteq H \times X \times X$ is closed, then either all R_S-independent sets have size $\leq \kappa$ or there is a κ-perfect R_S-independent set.

(2) If H is a group that acts continuously on X, then there are either $\leq \kappa$ many or κ-perfectly many orbits.

Proof. A generalization of a standard argument from the countable case [13, Exercise 3.4.2] shows that if H is a κ-compact topological space, then R_S is a closed subset of $X \times X$. In more detail, let $(x, y) \in X \times X - R_S$ be arbitrary. Because S is closed, we can choose for all $h \in H$ open sets $U_h \subseteq H$ and $V_h \subseteq X \times X$ such that

$$(h, x, y) \in U_h \times V_h \subseteq H \times X \times X - S.$$

By the κ-compactness of H, there exists a set $I \in [H]^{<\kappa}$ such that $H = \bigcup_{h \in I} U_h$. Then $V = \bigcap_{h \in I} V_h$ is an open subset of $X \times X$ such that $(x, y) \in V \subseteq X \times X - R_S$.

Thus R_S is indeed closed in the case H is κ-compact, implying that if H is K_κ, then R_S is a $\mathbf{\Sigma}^0_2(\kappa)$ binary relation on X. Thus, item (1) follows from $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$. Item (2) is a special case of item (1). □

We now prove the main theorem of this section.

Theorem 4.43 *Suppose $\mathrm{PIF}_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ holds. Let $H \subseteq Inj(\kappa)$, let F be a fragment of $L_{\kappa^+\kappa}$ and let ψ be a sentence of $\mathbf{\Sigma}^1_1(L_{\kappa^+\kappa})$. Suppose that either*

(1) H is a K_κ subset of the κ-Baire space, or

(2) H is a K_κ subset of the product space $\langle {}^\kappa\kappa, t_{\mathrm{pr}} \rangle$ and $F \subseteq L_{\kappa^+\omega}$.

If there are at least κ^+ many pairwise non (F, H)-embeddable models in Mod^ψ_κ, then there are κ-perfectly many such models.

Proof. We start with the proof of item (1). Thus, we assume that H is given the subspace topology induced by the κ-Baire space and X_F^ψ is equipped with the subspace topology induced by the generalized Cantor space ${}^F 2$.

Define the subset S of $H \times X_F^\psi \times X_F^\psi$ by letting, for any $h \in H$ and $\mathcal{A}, \mathcal{B} \in \mathrm{Mod}_\kappa^\psi$,

$$(h, i_F(\mathcal{B}), i_F(\mathcal{A})) \in S \text{ iff } h \text{ witnesses that } \mathcal{A} \text{ is } (F, H)\text{-embeddable into } \mathcal{B}.$$

Then, since F is closed under substitution and negation, $(h, i_F(\mathcal{B}), i_F(\mathcal{A})) \in S$ iff for all formulas φ in F, $\mathcal{A} \models \varphi[\mathrm{id}_\kappa]$ implies that $\mathcal{B} \models \varphi[h \circ \mathrm{id}_\kappa]$, or equivalently that $\mathcal{B} \models s_h\varphi[\mathrm{id}_\kappa]$. Therefore

$$S = \{(h, y, x) \in H \times X_F^\psi \times X_F^\psi : \text{ for all } \varphi \in F,\ x(\varphi) = 1 \text{ implies } y(s_h\varphi) = 1\}.$$

Claim 4.56. *S is a closed subset of $H \times X_F^\psi \times X_F^\psi$.*

Proof of Claim 4.56. We prove that the complement $U = H \times X_F^\psi \times X_F^\psi - S$ is open. Suppose that $(h, y, x) \in U$, or in other words, there exists $\varphi \in F$ such that $x(\varphi) = 1$ and $y(s_h\varphi) = 0$. Then, since the set $\Delta(\varphi)$ of free variables of φ is of size $< \kappa$, the set $N_1 = N_{h \restriction \Delta(\varphi)} \cap H$ is an open subset of H. Furthermore, $h' \in N_1$ implies that for all $x \in X_F^\psi$, we have $x(s_{h'}\varphi) = x(s_h\varphi)$. Thus, denoting by N_2 and N_3 the open subsets of X_F^ψ determined by the conditions $z(\varphi) = 1$ and $z(s_h\varphi) = 0$, respectively, we obtain an open neighborhood $N_1 \times N_2 \times N_3$ of (h, y, x) which is also a subset of U. This completes the proof of Claim 4.56.

Clearly, the projection R_S of S onto $X_F^\psi \times X_F^\psi$ is the relation corresponding to (F, H)-embeddability on Mod_κ^ψ (i.e., $(i_F(\mathcal{B}), i_F(\mathcal{A})) \in R_S$ iff $\mathcal{A}$ is F-embeddable into $\mathcal{B}$ by H). By Corollary 4.51, X_F^ψ is a $\boldsymbol{\Sigma}_1^1(\kappa)$ subset of the generalized Cantor space ${}^F 2$, and H is a K_κ topological space by the assumption of item (1). Thus, by Proposition 4.55, we have the required conclusion.

To see item (2), we equip H with the subspace topology induced by the *product topology* t_{pr} on the set ${}^\kappa\kappa$. As before, X_F^ψ is given the subspace topology induced by the generalized Cantor space ${}^F 2$, and the set S is defined as above. Then, using the assumption $F \subseteq L_{\kappa^+\omega}$ of item (2), one can show that

$$S \text{ is a closed subset of the space } H \times X_F^\psi \times X_F^\psi.$$

This can be seen by using the argument in the proof of Claim 4.56 and taking note of the fact that, since the set of free variables of any $\varphi \in F$ is finite by the assumption $F \subseteq L_{\kappa^+\omega}$, the set denoted by N_1 in the proof of Claim 4.56 is an open subset of H even when the topology on H is inherited from the product space $\langle {}^\kappa\kappa, t_{\text{pr}}\rangle$.

Furthermore, H is a K_κ topological space by the first assumption of item (2), and X_F^ψ is a $\mathbf{\Sigma}_1^1(\kappa)$ subset of the generalized Cantor space ${}^F2$ by Corollary 4.51. Therefore Proposition 4.55 can again be applied to obtain the required conclusion. □

Remark 4.57. Suppose that Φ is an arbitrary κ-sized subset of $L_{\kappa^+\kappa}$-formulas which is closed under substitution. For a subset H of ${}^\kappa\kappa$, consider the models in Mod_κ^ψ up to maps $h \in H$ which preserve the formulas in Φ (i.e., maps $h : \mathcal{A} \longrightarrow \mathcal{B}$ such that for all $\varphi \in \Phi$ and valuations $a \in {}^\kappa A$, if $\mathcal{A} \models \varphi[a]$ then $\mathcal{B} \models \varphi[h \circ a]$. Note that in this case, such maps h need not be injective). Using the topology t_F, where F is the fragment generated by Φ, the proof of Theorem 4.43 can be generalized to yield an analogous statement about the "number of models" up to such maps. This version seems to cover all natural generalizations of Theorem 4.43.

Specifically, when Φ is the set of those atomic formulas which do not contain the $=$ symbol, a map h preserves Φ iff it is a homomorphism. Therefore $\text{PIF}_\kappa(\mathbf{\Sigma}_1^1(\kappa))$ implies the following statement: *if H is a K_κ subset of the product space $\langle {}^\kappa\kappa, t_{\text{pr}}\rangle$ and there are at least κ^+ many models in Mod_κ^ψ such that no $h \in H$ is a homomorphism from one into another, then there are κ-perfectly many such models.*

We conclude this section by mentioning some special cases of Theorem 4.43.

Corollary 4.58. *Assume $\text{PIF}_\kappa(\mathbf{\Sigma}_1^1(\kappa))$. Suppose that $H \subseteq Inj(\kappa)$, and let ψ be a sentence of $\mathbf{\Sigma}_1^1(L_{\kappa^+\kappa})$.*

(1) Suppose H is a K_κ subset of the product space $\langle {}^\kappa\kappa, t_{\text{pr}}\rangle$. If there are at least κ^+ many pairwise non H-embeddable models in Mod_κ^ψ, then there is a κ-perfect set of such models.

(2) The above also holds for H-embeddability, as well as (F, H)-embeddability when F is either $L_{\lambda\omega}$ or $L_{\lambda\omega}^n$, and $n < \omega \leq \lambda \leq \kappa$.

(3) Suppose H is a K_κ subset of the κ-Baire space. Then the same statement holds for $(L_{\lambda\mu}, H)$-embeddability, where $\omega < \mu \leq \lambda \leq \kappa$.

(4) Now, suppose that H is a subgroup of $Sym(\kappa)$ which is K_κ again in the product topology t_{pr}. If there are κ^+ many pairwise non H-isomorphic models in Mod_κ^ψ,

then there is a κ-perfect set of such models.

Question 4.59. What is the consistency strength of the dichotomies in Proposition 4.55 and Theorem 4.43?

In particular, are these dichotomies equiconsistent with the existence of an inaccessible cardinal $\lambda > \kappa$?

Do they imply, or are they equiconsistent with $\mathrm{PIF}_\kappa(\boldsymbol{\Sigma}^1_1(\kappa))$?

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