Subconvexity and shifted convolution sums over number fields

by
Péter Maga

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Supervisor: Gergely Harcos

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Abstract

In this dissertation, we prove three results in the analytic theory of automorphic forms over arbitrary number fields. First we establish the spectral decomposition of shifted convolution sums of two irreducible cuspidal representations π_1, π_2 over GL_2. Secondly, as an application of the previous one, we prove a Burgess type subconvex bound for twisted GL_2 L-functions. Thirdly, we work out a semi-adelic Kuznetsov formula.

Declaration

I hereby declare that the dissertation contains no materials accepted for any other degrees in any other institution.

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I hereby declare that the dissertation contains no materials previously written and/or published by another person, except where appropriate acknowledgment is made in the form of bibliographical reference, etc.

Péter Maga

Contents

Chapter 1

Introduction

1.1 Overview of the dissertation

Automorphic theory plays an important role in mathematics. The early origins of automorphic forms date back to the first half of the nineteenth century. These special funcions are present in many branches of number theory, for example, in the investigation of representation numbers of quadratic forms. Also, automorphic L-functions are the natural generalizations of Dirichlet L-functions. Automorphic forms also appear in several other areas of mathematics, among others, in algebraic geometry, representation theory, partial differential equations and combinatorics. This dissertation is a contribution to the analytic theory of automorphic forms.

The first results on the asymptotic of the additive divisor sum $\sum_{n\leq x}\tau(n)\tau(n+a)$ (with $a\neq 0$ fixed and $\tau(n)$ standing for the number of divisors of n) were obtained by Ingham [35] and Estermann [23]. The main motivation was the investigation of the fourth moment of the Riemann zeta function, see the works of Ingham [34] and Heath-Brown [31]. Later Motohashi [50], [51] decomposed spectrally the additive divisor sum in order to obtain a better error term. Up to normalization, $\tau(n)$ is the Fourier coefficient of the derivative $(\partial/\partial s)E(z,s)|_{s=1/2}$ of the weight 0 Eisenstein series $E(z,s)$ for the full modular group $\mathrm{SL}_2(\mathbf{Z})$ (see [38, Chapter 3]). The automorphic spectrum over GL_2 consists of such Eisenstein series and the arithmetically more interesting cusp forms. The Fourier coefficients of cusp forms are proportional to Hecke eigenvalues $\lambda(n)$, so it is natural to ask for a spectral decomposition of the sum $\sum_{m-n=a}W(m,n)\lambda(m)\lambda(n)$, where W is a reasonable weight function. This was established by Blomer and Harcos in [4] and [5] for totally real fields. In Chapter 6, we prove a similar spectral decomposition over arbitrary number fields.

L-functions are among the most interesting and most mysterious objects in mathematics. Their importance is further confirmed by the fact that two of the seven Millenium Prize Problems stated by the Clay Institute concerns L-functions, namely, the Birch and Swinnerton-Dyer conjecture and the Riemann hypothesis. Although most mathematicians strongly believe that the Riemann hypothesis is true, according to leading experts of analytic number theory, there is very little hope that it will be proved in our lifetime. For automorphic L-functions, the Riemann hypothesis implies the Lindelöf hypothesis. For $\mathrm{GL}_1(\mathbf{Q})$ L-functions (that is, Dirichlet L-funtions) in the conductor aspect, this means that $L(1/2,\chi)\ll_\varepsilon q^\varepsilon$ holds for every Dirichlet character χ of conductor q, the implied constant depends only on ε. This is also largely open. By the Phragmén-Lindelöf principle, we have the convexity bound $L(1/2,\chi)\ll_\varepsilon q^{1/4+\varepsilon}$ and any bound of the form $L(1/2,\chi)\ll_\varepsilon q^{\delta+\varepsilon}$ with $\delta<1/4$ is a subconvex bound. The famous Burgess bound is the above with $\delta=3/16$ and was proved in 1963 [17]. Over $\mathrm{GL}_2(\mathbf{Q})$, let π be an automorphic cuspidal representation with conductor $C(\pi)$. The Lindelöf hypothesis $L(1/2,\pi)\ll_\varepsilon C(\pi)^\varepsilon$ is again implied by the Riemann hypothesis. On the other hand, there are several unconditional results, see for example [21]. It makes sense to investigate this question over general number fields. The general subconvex bound $L(1/2,\pi)\ll_\varepsilon C(\pi)^{\delta+\varepsilon}$ was proved (with an unspecified $\delta<1/4$) by Michel and Venkatesh [47]. Fixing π, we may twist it by GL_1 characters χ of conductor q. Then the conductor is multiplied by essentially q^2, so Phragmén-Lindelöf gives $L(1/2,\pi\otimes\chi)\ll_{\pi,\varepsilon,\chi_\infty} q^{1/2+\varepsilon}$. Of course, this is again interesting in the number field case. A Burgess type subconvex bound in this q-aspect is proved in [5] over totally real fields. In Chapter 7, we extend that result to arbitrary number fields. We note that this has been proved recently by Wu [61], using a different method.

In the theory of automorphic forms, Kuznetsov's formula is among the most important and most frequently used tools. Briefly speaking, it matches a certain weighted sum of products of Fourier coefficients (or equivalently, Hecke eigenvalues) of an orthogonal basis in the cuspidal space with a sum of Kloosterman sums, weighted by a Bessel transform. Kuznetsov [44] originally proved his formula for the modular group over the rational field $\mathbf{Q}$, which was an extension of the Petersson trace formula (which refers to holomorphic forms) to weight 0 Maass forms (we note that similar results were independently proved by Bruggeman [8]). Since then, many generalizations and reformulations were born. For totally real number fields, see the work of Bruggeman and Miatello [10], which includes the principal series representations and the discrete series representations in a single formula. For general number fields, the thesis of Venkatesh [56] gives a treatment to spherical vectors with references to [9]. In Chapter 8, we work out a formula to the more general case: non-spherical vectors are also included. In the derivation, we follow [56], borrowing the archimedean investigations from [10], [13] and [45]. We note that the content of this last chapter is more or less the same as of [46].

Now we introduce the notations we shall use later. We advise the reader to consult [59] for the arising notions.

1.2 The number field

Let F be a number field, a finite algebraic extension of $\mathbf{Q}$. Assume F has r real and s complex places, we will throughout denote the corresponding archimedean completions by $F_1, \dots, F_{r+s}$, where $F_1, \dots, F_r$ are all isomorphic to $\mathbf{R}$ and $F_{r+1}, \dots, F_{r+s}$ are all isomorphic to $\mathbf{C}$ as topological fields. Let F_∞ stand for the direct sum of these fields (as rings), $F_\infty^\times$ for its multiplicative group, $F_{\infty,+}^\times$ for the totally positive elements (which are positive at each real place), and $F_{\infty,+}^{\mathrm{diag}}$ for $\{(a_1, \dots, a_{r+s}) \in F_{\infty,+}^\times : a_1 = \dots = a_{r+s}\}$.

Denote by $\mathfrak{o}$ the ring of integers of F. The ideals and fractional ideals will be denoted by gothic characters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \dots$, the prime ideals by $\mathfrak{p}$ and we keep $\mathfrak{d}$ for the different and D_F for the discriminant of F. Each prime ideal $\mathfrak{p}$ determines a non-archimedean place and a corresponding completion $F_\mathfrak{p}$. At such a place, we denote by $\mathfrak{o}_\mathfrak{p}$ the maximal compact subring.

Write $\mathbf{A}$ for the adele ring of F. Given an adele a, a_j denotes its projection to F_j for $1 \leq j \leq r+s$, and $a_\mathfrak{p}$ the same to $F_\mathfrak{p}$ for a prime ideal $\mathfrak{p}$. We will also use the subscripts $j, \mathfrak{p}$ for the projections of other adelic objects to the place corresponding to $j, \mathfrak{p}$, respectively. The subscripts ∞ and fin stand for the projections to F_∞ and $\prod_\mathfrak{p} F_\mathfrak{p}$.

The absolute norm (module) of adeles will be denoted by $|\cdot|$, while $|\cdot|_j$ and $|\cdot|_\mathfrak{p}$ will stand for the norm (module) at single places. Sometimes we will need $|\cdot|_\infty$, which is the product of the archimedean norms. (At this point, we call the reader's attention to the notational ambiguity that for a real or complex number y, we keep the conventional $|y|$ for its ordinary absolute value. We hope this will not lead to confusion. Note that at real places, $|y|_j = |y|$, while at complex places, $|y|_j = |y|^2$.) For a fractional ideal $\mathfrak{a}$, $\mathcal{N}(\mathfrak{a})$ will denote its absolute norm, defined as $\mathcal{N}(\mathfrak{a}) = |a|^{-1}$, where a is any finite representing idele for $\mathfrak{a}$. When a is a finite idele, we may also write $\mathcal{N}(a)$ for $|a|^{-1}$.

We define an additive character ψ on $\mathbf{A}$: it is required to be trivial on F (embedded diagonally); on F_∞: $\psi_\infty(x) = \exp(2\pi i \mathrm{Tr}(x)) = \exp(2\pi i(x_1 + \dots + x_r + x_{r+1} + \overline{x_{r+1}} + \dots + x_{r+s} + \overline{x_{r+s}}))$; while on $F_\mathfrak{p}$: it is trivial on $\mathfrak{d}_\mathfrak{p}^{-1}$ but not on $\mathfrak{d}_\mathfrak{p}^{-1}\mathfrak{p}^{-1}$.

1.3 Matrix groups

Given a ring R, we define the following subgroups of $\mathrm{GL}_2(R)$:

$$Z(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in R^\times \right\}, \quad B(R) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in R^\times, b \in R \right\}, \quad N(R) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in R \right\}.$$

Assume $0 \neq \mathfrak{n}_\mathfrak{p}, \mathfrak{c}_\mathfrak{p} \subseteq \mathfrak{o}_\mathfrak{p}$. Then let

$$K_\mathfrak{p}(\mathfrak{n}_\mathfrak{p}, \mathfrak{c}_\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in \mathfrak{o}_\mathfrak{p}, b \in (\mathfrak{n}_\mathfrak{p}\mathfrak{d}_\mathfrak{p})^{-1}, c \in \mathfrak{n}_\mathfrak{p}\mathfrak{d}_\mathfrak{p}\mathfrak{c}_\mathfrak{p}, ad - bc \in \mathfrak{o}_\mathfrak{p}^\times \right\},$$

moreover in the special case $\mathfrak{n}_\mathfrak{p} = \mathfrak{o}_\mathfrak{p}$, we simply write $K_\mathfrak{p}(\mathfrak{c}_\mathfrak{p})$ instead of $K_\mathfrak{p}(\mathfrak{o}_\mathfrak{p}, \mathfrak{c}_\mathfrak{p})$. For ideals $0 \neq \mathfrak{n}, \mathfrak{c} \subseteq \mathfrak{o}_\mathfrak{p}$, let

$$K(\mathfrak{n}, \mathfrak{c}) = \prod_\mathfrak{p} K_\mathfrak{p}(\mathfrak{n}_\mathfrak{p}, \mathfrak{c}_\mathfrak{p}), \qquad K(\mathfrak{c}) = \prod_\mathfrak{p} K_\mathfrak{p}(\mathfrak{c}_\mathfrak{p}),$$

and taking the archimedean places into account, let

$$K = K_\infty \times K(\mathfrak{o}) \subseteq \mathrm{GL}_2(\mathbf{A}),$$

where

$$K_\infty = \prod_{j=1}^{r} \mathrm{SO}_2(\mathbf{R}) \times \prod_{j=r+1}^{r+s} \mathrm{SU}_2(\mathbf{C}).$$

Finally, for $0 \neq \mathfrak{n}, \mathfrak{c} \subseteq \mathfrak{o}$, let

$$\Gamma(\mathfrak{n}, \mathfrak{c}) = \left\{ g_\infty \in \mathrm{GL}_2(F_\infty) : \exists g_{\mathrm{fin}} \in \prod_{\mathfrak{p}} K_{\mathfrak{p}}(\mathfrak{n}_{\mathfrak{p}}, \mathfrak{c}_{\mathfrak{p}}) \text{ such that } g_\infty g_{\mathrm{fin}} \in \mathrm{GL}_2(F) \right\}.$$

We note that the choice of the subgroups K is not canonical (they can be conjugated arbitrarily), our normalization follows [5].

1.3.1 Archimedean matrix coefficients

On K_∞, we define the matrix coefficients (see [16, p.8]). Again, it is more convenient to give them on the factors. At a real place, on $\mathrm{SO}_2(\mathbf{R})$, for a given even integer q, set

$$\Phi_q \left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right) = \exp(iq\theta).$$

At a complex place, on $\mathrm{SU}_2(\mathbf{C})$, we introduce the parametrization

$$\mathrm{SU}_2(\mathbf{C}) = \left\{ k[\alpha, \beta] = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbf{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Assume now that the integers p, q, l satisfy $|p|, |q| \leq l$. Then the matrix coefficient $\Phi^l_{p,q}$ is defined via

$$\sum_{|p| \leq l} \Phi^l_{p,q}(k[\alpha, \beta]) x^{l-p} = (\alpha x - \overline{\beta})^{l-q} (\beta x + \overline{\alpha})^{l+q},$$

where this equation is understood in the polynomial ring $\mathbf{C}[x]$. See [13, (3.18)] and [45, (2.28)]. Note that

$$||\Phi^l_{p,q}||_{\mathrm{SU}_2(\mathbf{C})} = \left(\int_{\mathrm{SU}_2(\mathbf{C})} |\Phi^l_{p,q}(k)|^2 dk \right)^{1/2} = \frac{1}{\sqrt{2l+1}} \binom{2l}{l-p}^{1/2} \binom{2l}{l-q}^{-1/2}$$

by [45, (2.35)], where the Haar measure on $\mathrm{SU}_2(\mathbf{C})$ is the probability measure.

1.4 Measures

On F_∞, we use the Haar measure $|D_F|^{-1/2} dx_1 \cdots dx_r |dx_{r+1} \wedge d\overline{x_{r+1}}| \cdots |dx_{r+s} \wedge d\overline{x_{r+s}}|$. On $F_{\mathfrak{p}}$, we normalize the Haar measure such that $\mathfrak{o}_{\mathfrak{p}}$ has measure 1. On $\mathbf{A}$, we use the Haar measure dx, the product of these measures, this induces a Haar probability measure on $F \backslash \mathbf{A}$ (see [59, Chapter V, Proposition 7]).

On $\mathbf{R}^\times$, we use the Haar measure $d^\times_{\mathbf{R}} y = dy/|y|$, this gives rise to a Haar measure on $\mathbf{C}^\times$ as $d^\times_{\mathbf{C}} y = d^\times_{\mathbf{R}} |y| d\theta / 2\pi$, where $\exp(i\theta) = y/|y|$. On $F^\times_\infty$, we use the product $d^\times_\infty y$ of these measures. On $F^\times_{\mathfrak{p}}$, we normalize the Haar measure such that $\mathfrak{o}^\times_{\mathfrak{p}}$ has measure 1. The product $d^\times y$ of these measures is a Haar measure on $\mathbf{A}^\times$, inducing some Haar measure on $F^\times \backslash \mathbf{A}^\times$.

On K and its factors, we use the Haar probability measures. On $Z(F_\infty) \backslash \mathrm{GL}_2(F_\infty)$, we use the Haar measure which satisfies

$$\int_{Z(F_\infty) \backslash \mathrm{GL}_2(F_\infty)} f(g) dg = \int_{(\mathbf{R}^\times)^r \times (\mathbf{R}^\times_+)^s} \int_{F_\infty} \int_{K_\infty} f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k \right) dk dx \frac{d^\times_\infty y}{\prod_{j=1}^{r+s} |y_j|}.$$

Observing further $|y|_\infty = \prod_{j=1}^{r} |y_j| \prod_{j=r+1}^{r+s} |y_j|^2$, it follows that on $F^\times_\infty$, $d^\times_\infty y = \text{const.} dy/|y|_\infty$.

On $\mathrm{GL}_2(F_\mathfrak{p})$ we normalize the Haar measure such that $K(\mathfrak{o}_\mathfrak{p})$ has measure 1. On $Z(F_\infty)\backslash\mathrm{GL}_2(\mathbf{A})$ we use the product of these measures, which, on the factor $Z(\mathbf{A})\backslash\mathrm{GL}_2(\mathbf{A})$, restricts as

$$\int_{Z(\mathbf{A})\backslash\mathrm{GL}_2(\mathbf{A})} f(g)dg = \int_{\mathbf{A}^\times}\int_{\mathbf{A}}\int_K f\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix} k\right) dk dx \frac{d^\times y}{|y|}.$$

Compare this with [5, p.6] and [28, (3.10)]. The seeming difference of the last two displays (i.e. the factor $\prod_{j=r+1}^{r+s}|y_j|$) is explained by the nontrivial intersection $\left\{\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & 1\end{smallmatrix}\right) : \theta \in \mathbf{R}\right\}$ of $\left\{\left(\begin{smallmatrix} y & 0 \\ 0 & 1\end{smallmatrix}\right) : y \in \mathbf{C}^\times\right\}$ and $Z(\mathbf{C})\mathrm{SU}_2(\mathbf{C})$.

Chapter 2

Background on automorphic theory

We review some basic facts about the automorphic theory of GL_2 that we shall use later. In the setup, we follow the work of Blomer and Harcos [5, Sections 2.2-7], even when it is not emphasized. Since our aim is to extend the main results of [5] from totally real number fields to all number fields, we will always pay special attention to the complex places.

2.1 Spectral decomposition and Eisenstein series

In this section, following [5, Section 2.2] closely, we give a short exposition of the spectral decomposition of the Hilbert space $L^2(Z(\mathbf{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A}))$. For a detailed discussion, consult [28, Sections 2-5].

First of all, $\phi \in L^2(Z(\mathbf{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A}))$ is called cuspidal if for almost every $g \in \mathrm{GL}_2(\mathbf{A})$,

$$\int_{F\backslash\mathbf{A}} \phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) dx = 0.$$

The closed subspace generated by cuspidal functions is an invariant subspace L_{cusp} decomposing into a countable sum of irreducible representations V_π, each π occuring with finite multiplicity (see [28, Section 2]). This multiplicity is in fact one, as it follows from Shalika's multiplicity-one theorem, see [39, Proposition 11.1.1] for the case GL_2. Therefore, denoting the set of cuspidal representations by $\mathcal{C}$, we may write

$$L_{\mathrm{cusp}} = \bigoplus_{\pi\in\mathcal{C}} V_\pi,$$

where the irredicble representations on the right-hand side are distinct.

Remark 1. More generally, for a Hecke character ω (referred as the central character), the Hilbert space $L^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A}),\omega)$ consists of those functions $\phi : \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ that are square-integrable on the coset space $Z(\mathbf{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A})$ and satisfy

$$\phi\left(\begin{pmatrix}z & 0\\ 0 & z\end{pmatrix} \gamma g\right) = \omega(z)\phi(g)$$

for all $z \in \mathbf{A}^\times, \gamma \in \mathrm{GL}_2(F)$. Cuspidal functions and cuspidal representations are defined as above. The results presented below hold for arbitrary unitary central character and can be proved essentially the same way. However, in order to keep the exposition as simple as possible, we will work throughout with trivial central character.

To any Hecke character χ with $\chi^2 = 1$, we can associate a one-dimensional representation V_χ generated by $g \mapsto \chi(\det g)$, these sum up to

$$L_{\mathrm{sp}} = \bigoplus_{\chi^2=1} V_\chi.$$

For details, see [28, Sections 3-4].

Now

$$L^2(Z(\mathbf{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A})) = L_{\mathrm{cusp}} \oplus L_{\mathrm{sp}} \oplus L_{\mathrm{cont}},$$

where L_{cont} can be described in terms of Eisenstein series.

Take a Hecke quasicharacter $\chi : F^\times \backslash \mathbf{A}^\times \to \mathbf{C}^\times$. Denote by $H(\chi)$ the space of functions $\varphi : \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ satisfying

$$\int_K |\varphi(k)|^2 dk < \infty$$

and

$$\varphi\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) = \chi(a)\chi^{-1}(b)\left|\frac{a}{b}\right|^{1/2} \varphi(g), \qquad x \in \mathbf{A}, a, b \in \mathbf{A}^\times. \tag{2.1}$$

In particular, $H(\chi)$ can be identified with the set of functions $\varphi \in L^2(K)$ satisfying

$$\varphi\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) = \chi(a)\chi^{-1}(b)\varphi(g), \qquad \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in K.$$

There is a unique $s \in \mathbf{C}$ such that $\chi(a) = |a|_\infty^s$ for $a \in F_{\infty,+}^{\text{diag}}$ and introduce

$$H(s) = \bigoplus_{\chi^2 = |\cdot|_\infty^{2s} \text{ on } F_{\infty,+}^{\text{diag}}} H(\chi).$$

Now regard the space $H = \int_{s \in \mathbf{C}} H(s) ds$ as a holomorphic fibre bundle over base $\mathbf{C}$. Given a section $\varphi \in H$, $\varphi(s) \in H(s)$ and $\varphi(s, g) \in \mathbf{C}$. The bundle H is trivial, since any $\varphi(0) \in H(0)$ extends to a section $\varphi \in H$ satisfying $\varphi(s, g) = \varphi(0, g) H(g)^s$, where $H(g)$ is the height function defined at [28, p.219]. (One may think of this as a deformation of the function φ such that it is invariant on K under the deformation.)

Define

$$L'_{\text{cont}} = \int_0^\infty H(iy) dy,$$

and equip it with the inner product

$$\begin{aligned} \langle \phi_1, \phi_2 \rangle &= \frac{2}{\pi} \int_0^\infty \langle \phi_1(iy), \phi_2(iy) \rangle dy \\ &= \frac{2}{\pi} \int_0^\infty \int_{F^\times \backslash \mathbf{A}^1} \int_K \phi_1\left(iy, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) \overline{\phi_2\left(iy, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right)} dk\, da\, dy, \end{aligned}$$

where $\mathbf{A}^1$ stands for the group of ideles of norm 1 (see [28, (3.15)]). Then there is an intertwining operator $S : L_{\text{cont}} \to L'_{\text{cont}}$ given by [28, (4.23)] on a dense subspace. Now combining this with the theory of Eisenstein series [28, Section 5], we obtain the spectral decomposition of L_{cont}.

For $\varphi \in H$, and for $\Re s > 1/2$, define the Eisenstein series

$$E(\varphi(s), g) = \sum_{\gamma \in B(F) \backslash \mathrm{GL}_2(F)} \varphi(s, \gamma g) \tag{2.2}$$

on $\mathrm{GL}_2(\mathbf{A})$. This is a holomorphic function which continues meromorphically to $s \in \mathbf{C}$, with no poles on the line $\Re s = 0$. Now for $y \in \mathbf{R}^\times$, consider the complex vector space

$$V(iy) = \{E(\varphi(iy)) : \varphi(iy) \in H(iy)\}$$

with the inner product

$$\langle E(\varphi_1(iy)), E(\varphi_2(iy)) \rangle = \langle \varphi_1(iy), \varphi_2(iy) \rangle.$$

As above,

$$V(iy) = \bigoplus_{\chi^2 = |\cdot|_\infty^{2iy} \text{ on } F_{\infty,+}^{\text{diag}}} V_{\chi, \chi^{-1}},$$

with

$$V_{\chi,\chi^{-1}} = \{E(\varphi(iy)) : \varphi(iy) \in H(\chi)\}.$$

Here, $V(iy) = V(-iy)$ by [28, (4.3), (4.24), (5.15)]. Therefore, we have a $\mathrm{GL}_2(\mathbf{A})$-invariant decomposition

$$L_{\text{cont}} = \int_0^\infty V(iy)dy = \int_0^\infty \bigoplus_{\chi^2 = |\cdot|_\infty^{2iy} \text{ on } F_{\infty,+}^{\text{diag}}} V_{\chi,\chi^{-1}} dy.$$

In fact, [28, (4.24), (5.15-18)] implies that for $\phi \in L_{\text{cont}}$, taking $S\phi = \varphi \in L'_{\text{cont}}$,

$$\phi(g) = \frac{1}{\pi} \int_0^\infty E(\varphi(iy), g) dy,$$

and also Plancherel holds, that is,

$$\langle \phi_1, \phi_2 \rangle = \frac{1}{\pi} \int_0^\infty \langle E(\varphi(iy), g), \phi_2 \rangle dy = \frac{2}{\pi} \int_0^\infty \langle \varphi_1(iy), \varphi_2(iy) \rangle dy = \frac{2}{\pi} \int_0^\infty \langle E(\varphi_1(iy)), E(\varphi_2(iy)) \rangle dy.$$

To summarize,

$$L^2(Z(\mathbf{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A})) = \bigoplus_{\pi \in \mathcal{C}} V_\pi \oplus \bigoplus_{\chi^2 = 1} V_\chi \oplus \int_0^\infty \bigoplus_{\chi^2 = |\cdot|_\infty^{2iy} \text{ on } F_{\infty,+}^{\text{diag}}} V_{\chi,\chi^{-1}} dy, \tag{2.3}$$

a function on the left-hand side decomposes into a convergent sum and integral of functions from the spaces appearing on the right-hand side, and also Plancherel holds.

For the Eisenstein spectrum, we introduce the notation $\int_{\mathcal{E}} V_\varpi d\varpi$, where $\mathcal{E}$ is a set of Hecke characters which are nontrivial on $F_{\infty,+}^{\text{diag}}$, such that for each Hecke character χ, exactly one of χ and χ^{-1} appears in $\mathcal{E}$.

We also note that while representations in L_{sp} are one-dimensional, those occuring in L_{cusp} and L_{cont} are infinite-dimensional.

2.2 Derivations and weights

We review the action of the Lie algebra of $\mathrm{GL}_2(F_\infty)$ on the space $L^2(\mathrm{GL}_2(F)Z(\mathbf{A})\backslash \mathrm{GL}_2(\mathbf{A}))$, following [5, Sections 2.3 and 2.10] at real places, [13, Section 3] and [45, Chapter 2] at complex places.

Since the central character we are dealing with is trivial, we can restrict ourselves to the Lie algebra $\mathfrak{sl}_2(F_\infty)$. First we give a real basis such that each basis element is 0 for all but one place F_j. At this exceptional place, we use the following elements. For a real place ($j \leq r$), let

$$\mathbf{H}_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{R}_j = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{L}_j = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.4}$$

while for a complex place ($j > r$), let

$$\begin{aligned} \mathbf{H}_{1,j} &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{V}_{1,j} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{W}_{1,j} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \mathbf{H}_{2,j} &= \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \mathbf{V}_{2,j} = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \mathbf{W}_{2,j} = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \end{aligned} \tag{2.5}$$

An element $X \in \mathfrak{sl}_2(F_\infty)$ acts as a right-differentiation on a function $\phi : \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ via

$$(X\phi)(g) = \left. \frac{d}{dt}\phi(g \exp(tX)) \right|_{t=0}.$$

Let $\mathfrak{g} = \mathfrak{sl}_2(F_\infty) \otimes_{\mathbf{R}} \mathbf{C}$ be the complexified Lie algebra and set $U(\mathfrak{g})$ for its universal enveloping algebra, consisting of higher-order right-differentiations with complex coefficients.

The above-defined first-order differentiations give rise to local Casimir elements

$$\Omega_j = -\frac{1}{4}\left(\mathbf{H}_j^2 - 2\mathbf{H}_j + 4\mathbf{R}_j\mathbf{L}_j\right), \qquad \Omega_{\pm,j} = \frac{1}{8}\left((\mathbf{H}_{1,j} \mp \mathbf{H}_{2,j})^2 + (\mathbf{V}_{1,j} \mp \mathbf{W}_{2,j})^2 - (\mathbf{W}_{1,j} \mp \mathbf{V}_{2,j})^2\right) \tag{2.6}$$

at real and complex places, respectively.

On an irreducible unitary representation (π, V_π), these local Casimir elements act as scalars, that is, for $\phi \in V_\pi^\infty$, $\Omega_j\phi = \lambda_j\phi$, $\Omega_{+,j}\phi = \lambda_{+,j}\phi$, $\Omega_{-,j}\phi = \lambda_{-,j}\phi$ with

$$\lambda_j = \frac{1}{4} - \nu_j^2, \qquad \lambda_{\pm,j} = \frac{1}{8}\left((\nu_j \mp p_j)^2 - 1\right), \tag{2.7}$$

where $\nu_j \in i\mathbf{R}, p_j \in \mathbf{Z}$ for principal series representations, $|\nu_j| \leq \theta \deg[F_j : \mathbf{R}], p_j = 0$ for complementary series representations and $\nu_j \in 1/2 + \mathbf{Z}$ for discrete series representations (which may occur only if F_j is real). Here, θ is a constant towards the Ramanujan-Petersson conjecture, according to the current state of art (see [3]), $\theta = 7/64$ is admissible.

For some $\mathcal{D} \in U(\mathfrak{g})$ and a smooth vector $\phi \in L^2(\mathrm{GL}_2(F)Z(\mathbf{A})\backslash\mathrm{GL}_2(\mathbf{A}))$, recalling the spectral decomposition (2.3),

$$\phi = \sum_{\pi\in\mathcal{C}} \phi_\pi + \sum_{\chi^2=1} \phi_\chi + \int_{\mathcal{E}} \phi_\varpi d\varpi,$$

we have

$$||\mathcal{D}\phi||^2 = \sum_{\pi\in\mathcal{C}} ||\mathcal{D}\phi_\pi||^2 + \sum_{\chi^2=1} ||\mathcal{D}\phi_\chi||^2 + \int_{\mathcal{E}} ||\mathcal{D}\phi_\varpi||^2 d\varpi, \tag{2.8}$$

see [19, Sections 1.2-4] with references to [20]. Compare (2.8) also with [4, (33)] and [5, (84)].

The local maximal connected compact subgroups are $\mathrm{SO}_2(\mathbf{R})$ (for $j \leq r$) and $\mathrm{SU}_2(\mathbf{C})$ (for $j > r$). The corresponding Lie algebras are $\mathfrak{so}_2(\mathbf{R})$ and $\mathfrak{su}_2(\mathbf{C})$, and define

$$\Omega_{\mathfrak{k},j} = \mathbf{R}_j - \mathbf{L}_j, \qquad \Omega_{\mathfrak{k},j} = -\frac{1}{2}(\mathbf{H}_{2,j}^2 + \mathbf{W}_{1,j}^2 + \mathbf{W}_{2,j}^2), \tag{2.9}$$

at real and complex places, respectively. At a complex place, $\Omega_{\mathfrak{k},j}$ is the Casimir element (see [55, Definition 9 on p.72]).

We now define the weight set $W(\pi)$. For $j \leq r$, let q_j be any even integer with the only restriction $|q_j| \geq 2|\nu_j|+1$ in the discrete series. For $j > r$, let (l_j, q_j) be any pair of integers satisfying $|q_j| \leq l_j \geq |p_j|$. Now set

$$\mathbf{w} = (q_1, \ldots, q_r, (l_{r+1}, q_{r+1}), \ldots, (l_{r+s}, q_{r+s})) \tag{2.10}$$

and denote by $W(\pi)$ the set of $\mathbf{w}$'s satisfying the above condition.

For a given $\mathbf{w} \in W(\pi)$, we say that $\phi : \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ is of weight $\mathbf{w}$, if for $j \leq r$,

$$\Omega_{\mathfrak{k},j}\phi = iq_j\phi \tag{2.11}$$

and for $j > r$,

$$\mathbf{H}_{2,j}\phi = -iq_j\phi, \qquad \Omega_{\mathfrak{k},j}\phi = \frac{1}{2}(l_j^2 + l_j)\phi, \tag{2.12}$$

for the action of $\Omega_{\mathfrak{k},j}$ at complex places, see [55, Chapter II, Proposition 5.15].

Note that $W(\pi)$, through $(q_1, \ldots, q_r, l_{r+1}, \ldots, l_{r+s})$, lists all irreducible representations of K_∞ occuring in π, while $(q_{r+1}, \ldots, q_{r+s})$ is to single out a one-dimensional space from each such representation.

Similarly, introduce the notation

$$\mathbf{r} = (\nu_1 \ldots, \nu_r, (\nu_{r+1}, p_{r+1}), \ldots, (\nu_{r+s}, p_{r+s})), \tag{2.13}$$

and also its norm

$$\mathcal{N}(\mathbf{r}) = \prod_{j=1}^{r}(1 + |\nu_j|) \prod_{j=r+1}^{r+s}(1 + |\nu_j| + |p_j|)^2, \tag{2.14}$$

compare this with [47, Section 3.1.8].

2.3 Cuspidal spectrum

2.3.1 Analytic conductor, newforms and oldforms

Let V_π be a cuspidal representation occuring in $L^2(Z(\mathbf{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}))$. By the tensor product theorem (see [15, Section 3.4] or [24]),

$$V_\pi = \bigotimes_v V_{\pi_v} \tag{2.15}$$

as a restricted tensor product with respect to the family $\{K_\mathfrak{p}(\mathfrak{o}_\mathfrak{p})\}$ (by [15, Theorem 3.3.4], irreducible cuspidal representations are admissible).

For an ideal $\mathfrak{c} \subseteq \mathfrak{o}$, let

$$V_\pi(\mathfrak{c}) = \left\{ \phi \in V_\pi : \phi\left(g \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \phi(g), \text{ if } g \in \mathrm{GL}_2(\mathbf{A}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K(\mathfrak{c}) \right\}.$$

Obviously, $\mathfrak{c}' \subseteq \mathfrak{c}$ implies $V_\pi(\mathfrak{c}') \supseteq V_\pi(\mathfrak{c})$.

By [48, Corollary 2(a) of Theorem 2], there is a nonzero ideal $\mathfrak{c}_\pi$ such that $V_\pi(\mathfrak{c})$ is nontrivial if and only if $\mathfrak{c} \subseteq \mathfrak{c}_\pi$. Now the analytic conductor of the representation is defined as

$$C(\pi) = \mathcal{N}(\mathfrak{c}_\pi)\mathcal{N}(\mathbf{r}). \tag{2.16}$$

Introducing also

$$V_{\pi,\mathbf{w}}(\mathfrak{c}) = \{\phi \in V_\pi(\mathfrak{c}) : \phi \text{ is of weight } \mathbf{w}\}$$

for $\mathbf{w} \in W(\pi)$, [48, Corollary 2(b) of Theorem 2] states that for any $\mathbf{w} \in W(\pi)$, $V_{\pi,\mathbf{w}}(\mathfrak{c}_\pi)$ is one-dimensional, that is, restricting $V_\pi(\mathfrak{c}_\pi)$ to K_∞, each irreducible representation of K_∞ listed in $W(\pi)$ appears with multiplicity one. A nontrivial element of $V_{\pi,\mathbf{w}}(\mathfrak{c}_\pi)$ is called a newform of weight $\mathbf{w}$.

Now consider an ideal $\mathfrak{c} \subseteq \mathfrak{c}_\pi$, and take any ideal $\mathfrak{t}$ such that $\mathfrak{t}\mathfrak{c}_\pi \supseteq \mathfrak{c}$. Fixing some finite idele $t \in \mathbf{A}_{\mathrm{fin}}^\times$ representing $\mathfrak{t}$, we obtain an isometric embedding

$$R_\mathfrak{t} : V_\pi(\mathfrak{c}_\pi) \hookrightarrow V_\pi(\mathfrak{c}), \qquad (R_\mathfrak{t}\phi)(g) = \phi\left(g \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right). \tag{2.17}$$

Then combining [48, Corollary 2(c) of Theorem 2] with [18, Corollary on p.306] and (2.15), we see the decompositions

$$V_\pi(\mathfrak{c}) = \bigoplus_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}} R_\mathfrak{t} V_\pi(\mathfrak{c}_\pi), \qquad V_{\pi,\mathbf{w}}(\mathfrak{c}) = \bigoplus_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}} R_\mathfrak{t} V_{\pi,\mathbf{w}}(\mathfrak{c}_\pi),$$

which are not orthogonal in general. However, in Section 3.2 we will prove that for ideals $\mathfrak{t}_1, \mathfrak{t}_2$, $\langle R_{\mathfrak{t}_1}\phi_1, R_{\mathfrak{t}_2}\phi_2 \rangle = \langle \phi_1, \phi_2 \rangle C(\mathfrak{t}_1, \mathfrak{t}_2, \pi)$, with the constant factor $C(\mathfrak{t}_1, \mathfrak{t}_2, \pi)$ depending only on $\mathfrak{t}_1, \mathfrak{t}_2, \pi$, but not on $\mathbf{w}$. This allows us to use the Gram-Schmidt method, obtaining complex numbers $\alpha_{\mathfrak{t},\mathfrak{s}}$ (with $\alpha_{\mathfrak{o},\mathfrak{o}} = 1$) for any pair of ideals $\mathfrak{s}|\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}$ such that the isometries

$$R^\mathfrak{t} = \sum_{\mathfrak{s}|\mathfrak{t}} \alpha_{\mathfrak{t},\mathfrak{s}} R_\mathfrak{s} : V_\pi(\mathfrak{c}_\pi) \hookrightarrow V_\pi(\mathfrak{c}), \qquad \mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}, \tag{2.18}$$

give rise to the orthogonal decompositions

$$V_\pi(\mathfrak{c}) = \bigoplus_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}} R^\mathfrak{t} V_\pi(\mathfrak{c}_\pi), \qquad V_{\pi,\mathbf{w}}(\mathfrak{c}) = \bigoplus_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}} R^\mathfrak{t} V_{\pi,\mathbf{w}}(\mathfrak{c}_\pi). \tag{2.19}$$

2.3.2 Whittaker functions and the Fourier-Whittaker expansion

For a given $\mathbf{r}, \mathbf{w}$ (recall (2.10) and (2.13)), we define the Whittaker function as the product of Whittaker functions at archimedean places. The important property of these functions is that they are the exponentially decaying eigenfunctions of the Casimir operators $\Omega, \Omega_\pm$, therefore, they emerge in the Fourier expansion of automorphic forms (see [15, Section 3.5]).

At real places,

$$\mathcal{W}_{q,\nu}(y) = \frac{i^{\mathrm{sign}(y)\frac{q}{2}} W_{\mathrm{sign}(y)\frac{q}{2},\nu}(4\pi|y|)}{(\Gamma(\frac{1}{2} - \nu + \mathrm{sign}(y)\frac{q}{2})\Gamma(\frac{1}{2} + \nu + \mathrm{sign}(y)\frac{q}{2}))^{1/2}}, \tag{2.20}$$

W denoting the classical Whittaker function (see [60, Chapter XVI]). This is taken from [5, (23)].

At complex places, let

$$\mathcal{W}_{(l,q),(\nu,p)}(y) = \frac{\sqrt{8(2l+1)}}{(2\pi)^{\Re\nu}} \binom{2l}{l-q}^{\frac{1}{2}} \binom{2l}{l-p}^{-\frac{1}{2}} \sqrt{\left|\frac{\Gamma(l+1+\nu)}{\Gamma(l+1-\nu)}\right|} \cdot (-1)^{l-p}(2\pi)^{\nu} i^{-p-q} w_q^l(\nu,p;|y|) \left(\frac{y}{|y|}\right)^{-q}, \tag{2.21}$$

where

$$w_q^l(\nu,p;|y|) = \sum_{k=0}^{l-\frac{1}{2}(|q+p|+|q-p|)} (-1)^k \xi_p^l(q,k) \frac{(2\pi|y|)^{l+1-k}}{\Gamma(l+1+\nu-k)} K_{\nu+l-|q+p|-k}(4\pi|y|), \tag{2.22}$$

K denoting the K-Bessel function, and

$$\xi_p^l(q,k) = \frac{k!(2l-k)!}{(l-p)!(l+p)!} \binom{l-\frac{1}{2}(|q+p|+|q-p|)}{k} \binom{l-\frac{1}{2}(|q+p|-|q-p|)}{k}. \tag{2.23}$$

This definition is borrowed from [13, Section 5] and [45, Section 4.1], apart from the first line, which is a normalization to gain the right L^2-norm.

In both cases, the occuring numbers ν, p, q, l are those given by the representation and weight data, encoded in the action of the elements $\Omega, \Omega_\pm, \Omega_\mathfrak{k}, \mathbf{H}_2$ (recall (2.5), (2.6), (2.7), (2.9), (2.10), (2.13)).

Finally, define the archimedean Whittaker function as

$$\mathcal{W}_{\mathbf{w},\mathbf{r}}(y) = \prod_{j\leq r} \mathcal{W}_{q_j,\nu_j}(y_j) \prod_{j>r} \mathcal{W}_{(l_j,q_j),(\nu_j,p_j)}(y_j).$$

With the given normalization, for a fixed $\mathbf{r}$,

$$\int_{F_\infty^\times} \mathcal{W}_{\mathbf{w},\mathbf{r}}(y)\overline{\mathcal{W}_{\mathbf{w}',\mathbf{r}}}(y) d_\infty^\times y = \delta_{\mathbf{w},\mathbf{w}'}. \tag{2.24}$$

This can be seen as the product of the analogous results at single places. For real places, see [5, (25)] and [14, Section 4]. As for complex places, this will be proved in Lemma 8.1 and Lemma 8.2. The normalization, i.e. $\int_{\mathbf{C}^\times} |\mathcal{W}_{(l,q),(\nu,p)}(y)|^2 d_\mathbf{C}^\times y = 1$ is already proved in [46, Lemma 2], now we need also that $\int_{\mathbf{C}^\times} \mathcal{W}_{(l,q),(\nu,p)}(y)\overline{\mathcal{W}_{(l',q'),(\nu,p)}(y)} d_\mathbf{C}^\times y = 0$, if $(l,q) \neq (l',q')$.

Now we extend [5, Section 2.5] to our more general situation.

Proposition 2.1. *Let $\pi \in \mathcal{C}$ and $\mathfrak{c} \subseteq \mathfrak{c}_\pi$. Then any function $\phi \in V_{\pi,\mathbf{w}}(\mathfrak{c})$ can be expanded into Fourier series as follows. There exists a character $\varepsilon_\pi : \{\pm1\}^r \to \{\pm1\}$ depending only on π such that*

$$\phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{t\in F^\times} \rho_\phi(ty_{\mathrm{fin}})\varepsilon_\pi(\mathrm{sign}(ty_\infty))\mathcal{W}_{\mathbf{w},\mathbf{r}}(ty_\infty)\psi(tx). \tag{2.25}$$

Proof. From the discussion above, the existence, the uniqueness and the factorization of the Whittaker model (see [15, Section 3.5]), we have

$$\begin{aligned}\phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) &= \sum_{t\in F^\times} \rho_\phi(ty_{\mathrm{fin}}) \left(\prod_{j\leq r} c_j(\mathrm{sign}(ty_j))\mathcal{W}_{q_j,\nu_j}(ty_j) \prod_{j>r} \mathcal{W}_{(l_j,q_j),(\nu_j,p_j)}(ty_j)\right)\psi(tx) \\ &= \sum_{t\in F^\times} \rho_\phi(ty_{\mathrm{fin}}) \left(\prod_{j\leq r} c_j(\mathrm{sign}(y_j))\right) \mathcal{W}_{\mathbf{w},\mathbf{r}}(ty_\infty)\psi(tx).\end{aligned}$$

Now we have to prove that we can take $c_j(-1) = \pm c_j(1)$. Fix some $j' \leq r$. If we are in the discrete series, $\mathcal{W}_{q_{j'},\nu_{j'}}(ty_{j'})$ is constant 0 either on $\mathbf{R}_+$ or $\mathbf{R}_-$ (recall (2.20)), so in this case, there is nothing to prove. If we are in the principal or the complementary series, then $c_{j'}(\pm1) \neq 0$, and so their quotient is well-defined. Assume that $y_{j'} > 0$ and $q_{j'} \geq 0$. Let X be the matrix, which is $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ at the place j', and the identity at other places. Define moreover the Maass operators Λ_k at place j' and the normalizing constant $\delta(\nu,k)$ for each even integer k (again, at other places, let them act trivially)

$$\Lambda_k = \frac{k}{2} + y\left(i\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right), \qquad \delta(\nu,k) = \frac{\Gamma\left(\frac{1}{2}+\nu-\frac{k}{2}\right)}{\Gamma\left(\frac{1}{2}+\nu+\frac{k}{2}\right)}.$$

These are taken from [22, (4.4) and (4.62)]. Now introducing

$$P = \delta(\nu_{j'}, q_{j'})\Lambda_{-q_{j'}+2}\circ\Lambda_{-q_{j'}+4}\circ\ldots\circ\Lambda_{q_{j'}-2}\circ\Lambda_{q_{j'}}, \qquad Q = X\circ P,$$

by [22, Proposition 4.5], Q is an involution on $V_{\pi,\mathbf{w}}$. By our normalization of $\mathcal{W}_{q_{j'},\nu_{j'}}$, $P\mathcal{W}_{q_{j'},\nu_{j'}} = \mathcal{W}_{-q_{j'},\nu_{j'}}$ (see [22, (4.27) and (4.59)]). Now let $\phi\in V_{\pi,\mathbf{w}}$, and consider $\phi' = P\phi$. Then with the abbreviation $W'(t,y,x) = \rho_\phi(ty_{\mathrm{fin}})\left(\prod_{j'\neq j\leq r} c_j(\mathrm{sign}(ty_j))\mathcal{W}_{q_j,\nu_j}(ty_j)\prod_{j>r}\mathcal{W}_{(l_j,q_j),(\nu_j,p_j)}(ty_j)\right)\psi(tx)$,

$$\begin{aligned}
\phi\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right) &= \sum_{t\in F^\times} c_{j'}(\mathrm{sign}(ty_{j'}))\mathcal{W}_{q_{j'},\nu_{j'}}(ty_{j'})W'(t,y,x),\\
P\phi\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right) &= \phi'\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right) = \sum_{t\in F^\times} c_{j'}(\mathrm{sign}(ty_{j'}))\mathcal{W}_{-q_{j'},\nu_{j'}}(ty_{j'})W'(t,y,x),\\
Q\phi\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right) &= X\phi'\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right) = \sum_{t\in F^\times} c_{j'}(-\mathrm{sign}(ty_{j'}))\mathcal{W}_{-q_{j'},\nu_{j'}}(-ty_{j'})W'(t,y,x).
\end{aligned}$$

Since Q is an involution, $Q\phi = \pm\phi$, showing $c_{j'}(-1) = \pm c_{j'}(1)$.

We remark that this is the same analysis as in [22, Section 4] or in [41, Sections 1-3]. Note that ε_π is not uniquely determined, if we are in the discrete series and that the coefficient $\varrho(ty_{\mathrm{fin}})$ depends only on the fractional ideal generated by ty_{fin} and it is zero, if this fractional ideal is nonintegral. □

Now assume that $\mathfrak{c} = \mathfrak{c}_\pi$, i.e. ϕ is a newform of weight $\mathbf{w}$. In this case, the coefficients $\varrho_\pi(\mathfrak{m})$ are proportional to the Hecke eigenvalues $\lambda_\pi(\mathfrak{m})$:

$$\varrho_\phi(\mathfrak{m}) = \frac{\lambda_\pi(\mathfrak{m})}{\sqrt{\mathcal{N}(\mathfrak{m})}}\varrho_\phi(\mathfrak{o}).$$

We record

$$\lambda_\pi(\mathfrak{m}) \ll_\varepsilon \mathcal{N}(\mathfrak{m})^{\theta+\varepsilon} \tag{2.26}$$

with $\theta = 7/64$ [3], while according to the Ramanujan-Petersson conjecture, $\theta = 0$ is admissible. Also note the multiplicativity relation

$$\lambda_\pi(\mathfrak{m})\lambda_\pi(\mathfrak{n}) = \sum_{\mathfrak{a}\mid\gcd(\mathfrak{m},\mathfrak{n})}\lambda_\pi(\mathfrak{m}\mathfrak{n}\mathfrak{a}^{-2}). \tag{2.27}$$

Setting

$$W_\phi(y) = \varrho_\phi(\mathfrak{o})\varepsilon_\pi(\mathrm{sign}(y))\mathcal{W}_{\mathbf{w},\mathbf{r}}(y), \qquad y\in F_\infty^\times, \tag{2.28}$$

we obtain

$$\phi\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right) = \sum_{t\in F^\times}\frac{\lambda_\pi(ty_{\mathrm{fin}})}{\sqrt{\mathcal{N}(ty_{\mathrm{fin}})}}W_\phi(ty_\infty)\psi(tx). \tag{2.29}$$

2.3.3 The archimedean Kirillov model

Now fixing $y_{\mathrm{fin}} = (1,1,\ldots)$, we can single out the term corresponding to $t=1$:

$$W_\phi(y) = \int_{F\backslash\mathbf{A}}\phi\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right)\psi(-x)dx. \tag{2.30}$$

In the case of arbitrary (i.e. non-necessarily pure weight) smooth functions in $V_\pi(\mathfrak{c}_\pi)$, this latter formula can be considered as the definition of the mapping $\phi\mapsto W_\phi$, the image is a subspace in $L^2(F_\infty^\times, d_\infty^\times y)$.

Proposition 2.2. *The image in fact is a dense subspace in $L^2(F_\infty^\times, d_\infty^\times y)$. Moreover, there is a positive constant C_π depending only on π such that*

$$\langle \phi_1, \phi_2 \rangle = C_\pi \langle W_{\phi_1}, W_{\phi_2} \rangle, \tag{2.31}$$

where the scalar product on the left-hand side is the scalar product in $L^2(Z(\mathbf{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}))$, while on the right-hand side, it is the scalar product in $L^2(F_\infty^\times, d_\infty^\times y)$. The map $\phi \mapsto W_\phi$ is therefore surjective from $V_\pi(\mathfrak{c}_\pi)$ to $L^2(F_\infty^\times, d_\infty^\times y)$.

Proof. On the space $L^2(F_\infty^\times, d_\infty^\times y)$, the Borel subgroup $B(F_\infty)$ acts through the Kirillov model action

$$\left(\begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} W_\phi \right)(y) = \psi_\infty(x'y) W_\phi(y'y). \tag{2.32}$$

This action is irreducible on a single $L^2(\mathbf{R}^\times, d_{\mathbf{R}}^\times y)$ or $L^2(\mathbf{C}^\times, d_{\mathbf{C}}^\times y)$. Indeed, by [43, Propositions 2.6 and 2.7], the Borel subgroup $B(\mathbf{R})$ or $B(\mathbf{C})$ has a unitary, infinite-dimensional irreducible representation, then by [42, p.197], it must be equivalent to the representation induced from the unipotent subgroup $N(\mathbf{R})$ or $N(\mathbf{C})$, which can be computed to be (2.32). Therefore the action (2.32) is irreducible on the tensor product

$$L^2(F_\infty^\times, d_\infty^\times y) = \overline{\bigotimes_{j=1}^{r+s} L^2(F_j^\times, d_{F_j}^\times y_j)}.$$

Then taking some $\phi \in V_\pi^\infty(\mathfrak{c}_\pi)$ such that W_ϕ is not identically zero, a closed, invariant subspace containing W_ϕ must equal $L^2(F_\infty^\times, d_\infty^\times y)$, because of irreducibility (the existence of such a ϕ follows from the Fourier-Whittaker expansion, which includes harmonics with nonzero coefficients).

As for the existence of C_π, we refer to Section 3.2. We will prove there that if $\phi_1, \phi_2 \in V_{\pi,\mathbf{w}}(\mathfrak{c}_\pi)$, then

$$\langle \phi_1, \phi_2 \rangle = C_\pi \langle W_{\phi_1}, W_{\phi_2} \rangle. \tag{2.33}$$

If moreover ϕ_1, ϕ_2 are of different weights $\mathbf{w}_1 \neq \mathbf{w}_2$, then both sides are 0, since for pure weight forms, the associated Kirillov vectors are proportional to $\mathcal{W}_{\mathbf{r},\mathbf{w}_{1,2}}$ (recall (2.28)), which are orthogonal by (2.24). Then the orthogonal decomposition

$$V_\pi(\mathfrak{c}_\pi) = \bigoplus_{\mathbf{w} \in W(\pi)} V_{\pi,\mathbf{w}}(\mathfrak{c}_\pi)$$

completes the proof. □

Now turn to the general case $\mathfrak{c} \subseteq \mathfrak{c}_\pi$. Using the isometries $R^\mathfrak{t}$, (2.29) gives rise to, for every $\phi \in R^\mathfrak{t} V_\pi(\mathfrak{c}_\pi)$,

$$\phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) = \sum_{t \in F^\times} \frac{\lambda_\pi^\mathfrak{t}(t y_{\mathrm{fin}})}{\sqrt{\mathcal{N}(t y_{\mathrm{fin}})}} W_\phi(t y_\infty) \psi(tx), \tag{2.34}$$

with

$$W_\phi = W_{(R^\mathfrak{t})^{-1}\phi}, \qquad \lambda_\pi^\mathfrak{t}(\mathfrak{m}) = \sum_{\mathfrak{s} \mid \gcd(\mathfrak{t},\mathfrak{m})} \alpha_{\mathfrak{t},\mathfrak{s}} \mathcal{N}(\mathfrak{s})^{1/2} \lambda_\pi(\mathfrak{m}\mathfrak{s}^{-1}). \tag{2.35}$$

2.4 Eisenstein spectrum

In this section, we develop the theory of Eisenstein series. From now on, let $\chi \in \mathcal{E}$ be a Hecke character which is nontrivial on $F_{\infty,+}^{\mathrm{diag}}$.

2.4.1 Analytic conductor, newforms and oldforms

Similarly to the cuspidal case, for any ideal $\mathfrak{c} \subseteq \mathfrak{o}$, define

$$V_{\chi,\chi^{-1}}(\mathfrak{c}) = \left\{ \phi \in V_{\chi,\chi^{-1}} : \phi\left(g \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \phi(g), \text{ if } g \in \mathrm{GL}_2(\mathbf{A}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K(\mathfrak{c}) \right\}.$$

Using that $V_{\chi,\chi^{-1}}$ and $H(\chi)$ are isomorphic as $\mathrm{GL}_2(\mathbf{A})$-representations, we have

$$V_{\chi,\chi^{-1}}(\mathfrak{c}) = \{E(\varphi(iy),\cdot) \in V_{\chi,\chi^{-1}} : \varphi \in H(\chi,\mathfrak{c})\}$$

with

$$H(\chi,\mathfrak{c}) = \left\{\varphi \in H(\chi) : \varphi\left(g\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) = \varphi(g),\ \text{if } g \in \mathrm{GL}_2(\mathbf{A}), \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in K(\mathfrak{c})\right\}.$$

Analogously to (2.15), we have

$$H(\chi) = \bigotimes_v H_v(\chi),$$

a restricted tensor product with respect to the family $\{K_\mathfrak{p}(\mathfrak{o}_\mathfrak{p})\}$ again, the admissibility of $H(\chi)$ is straight-forward.

Assume χ has conductor $\mathfrak{c}_\chi$. The following is taken from [5, Section 2.6].

Proposition 2.3. *For any non-archimedean place $\mathfrak{p}$, set $d = v_\mathfrak{p}(\mathfrak{d})$ and $m = v_\mathfrak{p}(\mathfrak{c}_\chi)$, and fix some ϖ such that $v_\mathfrak{p}(\varpi) = 1$. Then for any integer $n \geq 0$, the complex vector space $H_\mathfrak{p}(\chi,\mathfrak{p}^n)$ has dimension* $\max(0, n-2m+1)$. *For $n \geq 2m$, an orthogonal basis is $\{\varphi_{\mathfrak{p},j} : 0 \leq j \leq n-2m\}$ with functions $\varphi_{\mathfrak{p},j}$ defined as follows.*

If $m = 0$ and $k = \begin{pmatrix} * & * \\ b\varpi^d & *\end{pmatrix} \in K_\mathfrak{p}(\mathfrak{o}_\mathfrak{p})$, let

$$\varphi_{\mathfrak{p},0}(k) = 1; \qquad \varphi_{\mathfrak{p},1}(k) = \begin{cases} \mathcal{N}(\mathfrak{p})^{-1/2}, & \text{if } v_\mathfrak{p}(b) = 0, \\ -\mathcal{N}(\mathfrak{p})^{1/2}, & \text{if } v_\mathfrak{p}(b) \geq 1; \end{cases}$$

while for $j \geq 2$,

$$\varphi_{\mathfrak{p},j}(k) = \begin{cases} 0, & v_\mathfrak{p}(b) \leq j-2, \\ -\mathcal{N}(\mathfrak{p})^{j/2-1}, & \text{if } v_\mathfrak{p}(b) = j-1, \\ \mathcal{N}(\mathfrak{p})^{j/2}\left(1 - \frac{1}{\mathcal{N}(\mathfrak{p})}\right), & \text{if} v_\mathfrak{p}(b) \geq j. \end{cases}$$

If $m > 0$ and $k = \left(\begin{smallmatrix} a & * \\ b\varpi^d & *\end{smallmatrix}\right) \in K_\mathfrak{p}(\mathfrak{o}_\mathfrak{p})$, let

$$\varphi_{\mathfrak{p},j}(k) = \begin{cases} \mathcal{N}(\mathfrak{p})^{(m+j)/2}\chi_\mathfrak{p}(ab^{-1}), & \text{if } v_\mathfrak{p}(b) = m+j, \\ 0, & \text{if } v_\mathfrak{p}(b) \neq m+j. \end{cases}$$

Moreover,

$$1 - \frac{1}{\mathcal{N}(\mathfrak{p})} \leq ||\varphi_{\mathfrak{p},j}|| \leq 1.$$

Proof. See [5, Lemma 1 and Remark 7]. □

This shows that $\mathfrak{c}_{\chi,\chi^{-1}} = (\mathfrak{c}_\chi)^2$ is the maximal ideal $\mathfrak{c}$ such that $V_{\chi,\chi^{-1}}(\mathfrak{c})$ and $H(\chi,\mathfrak{c})$ are nontrivial.

Now we turn our attention to the archimedean quasifactors $H_j(\chi)$. They are always principal series representations and their parameter $\mathbf{r}$ is the following. At real places, $\nu_j \in i\mathbf{R}$ of (2.7) is the one satisfying $\chi_j(a) = a^{\nu_j}$ for $a \in \mathbf{R}_+$ (see [14, p.83]). At complex places, $\nu_j \in i\mathbf{R}$ and $p_j \in \mathbf{Z}$ of (2.7) are those satisfying $\chi_j(ae^{i\theta}) = a^{\nu_j}e^{-ip_j\theta}$ for $a \in \mathbf{R}_+, \theta \in \mathbf{R}$ (see [13, Section 3] or [45, Section 2.3]). Now these give rise to the set $W(\chi,\chi^{-1})$ of weights (those occuring in $H_j(\chi)$): the only condition is $|q_j| \leq l_j \geq |p_j|$ at complex places.

The analytic conductor is again defined as

$$C(\chi,\chi^{-1}) = \mathcal{N}(\mathfrak{c}_{\chi,\chi^{-1}})\mathcal{N}(\mathbf{r}). \tag{2.36}$$

We can now give an orthogonal basis of $H(\chi,\mathfrak{c})$ for any $\mathfrak{c} \subseteq \mathfrak{c}_\chi^2$. Given $\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\chi^{-2}$ and any weight $\mathbf{w} \in W(\chi,\chi^{-1})$, let $\varphi^{\mathfrak{t},\mathbf{w}}$ be the tensor product of the following local functions. At the archimedean places, let $\varphi_j^{\mathfrak{t},\mathbf{w}} = \Phi_{q_j}(k)$, $\varphi_j^{\mathfrak{t},\mathbf{w}}(k) = \Phi_{p_j,q_j}^{l_j}(k)/||\Phi_{p_j,q_j}^{l_j}||_{\mathrm{SU}_2(\mathbf{C})}$ for $k \in K_j$ with $j \leq r$, $j > r$, respectively. At non-archimedean places, let $\varphi_\mathfrak{p}^{\mathfrak{t},\mathbf{w}} = \varphi_{\mathfrak{p},v_\mathfrak{p}(\mathfrak{t})}$. The global functions form an orthogonal basis of $H(\chi,\mathfrak{c})$ and this gives rise to an orthogonal basis in $V_{\chi,\chi^{-1}}$ via the corresponding Eisenstein series $\phi^{\mathfrak{t},\mathbf{w}} = E(\varphi^{\mathfrak{t},\mathbf{w}})$. Finally, defining $R^\mathfrak{t} : V_{\chi,\chi^{-1}}(\mathfrak{c}_\chi^2) \hookrightarrow V_{\chi,\chi^{-1}}(\mathfrak{c})$ as $\phi^{\mathfrak{o},\mathbf{w}}/||\phi^{\mathfrak{o},\mathbf{w}}|| \mapsto \phi^{\mathfrak{t},\mathbf{w}}/||\phi^{\mathfrak{t},\mathbf{w}}||$ for all $\mathbf{w}$, we obtain the orthogonal decomposition

$$V_{\chi,\chi^{-1}}(\mathfrak{c}) = \bigoplus_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\chi^{-2}} R^\mathfrak{t} V_{\chi,\chi^{-1}}(\mathfrak{c}_\chi^2). \tag{2.37}$$

2.4.2 The Fourier-Whittaker expansion and the archimedean Kirillov model

Similarly to cusp forms, Eisenstein series can also be expanded into Fourier-Whittaker series. We may assume φ is one of the pure tensors defined above and $\phi = E(\varphi)$, where we dropped $\mathfrak{t}$ and $\mathbf{w}$ from the notation. We will insert the original definition of Eisenstein series, that is, the formal computation below is made precise by performing it on the domain $\Re s > 1/2$ and using the meromorphic continuation to $\Re s \geq 0$.

Now the Fourier expansion with respect to the left action of $N(\mathbf{A})$ admits

$$E\left(\varphi, \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{t\in F} \int_{F\backslash\mathbf{A}} E\left(\varphi, \begin{pmatrix} y & \xi \\ 0 & 1 \end{pmatrix}\right) \psi(-t\xi) d\xi \psi(tx).$$

It is easy to check that $B(F)\backslash \mathrm{GL}_2(F)$ has a complete set of representatives

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix} : d \in F\right\}.$$

By (2.1), φ is left $N(\mathbf{A})$-invariant, therefore

$$\sum_{t\in F} \int_{F\backslash\mathbf{A}} \varphi\left(\begin{pmatrix} y & \xi \\ 0 & 1 \end{pmatrix}\right) \psi(-t\xi) d\xi \psi(tx) = \varphi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Inserting these, we obtain

$$\begin{aligned} E\left(\varphi, \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) &= \varphi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) + \sum_{t\in F} \int_{F\backslash\mathbf{A}} \sum_{d\in F} \varphi\left(\begin{pmatrix} 0 & -1 \\ y & \xi + d \end{pmatrix}\right) \psi(-t\xi) d\xi \psi(tx) \\ &= \varphi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) + \sum_{t\in F} \int_{\mathbf{A}} \varphi\left(\begin{pmatrix} 0 & -1 \\ y & \xi \end{pmatrix}\right) \psi(-t\xi) d\xi \psi(tx). \end{aligned}$$

On the right-hand side, the first term together with the term corresponding to $t = 0$ give the constant term of $E(\varphi)$,

$$\varrho_{E(\varphi),0}(y) = \varphi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) + \int_{\mathbf{A}} \varphi\left(\begin{pmatrix} 0 & -1 \\ y & \xi \end{pmatrix}\right) d\xi.$$

Turn to the rest of the sum, and compute a typical term corresponding to $t \in F^\times$. Fix $\delta \in \mathbf{A}^\times$ as $\delta_\infty = (1, \dots, 1)$ and $\delta_{\mathfrak{p}} = \varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{d})}$. By the change of variable $\xi \mapsto y\delta^{-1}\xi$, we obtain

$$\int_{\mathbf{A}} \varphi\left(\begin{pmatrix} 0 & -1 \\ y & \xi \end{pmatrix}\right) \psi(-t\xi) d\xi = \chi^2(\delta)\chi^{-1}(ty)|ty|^{1/2} \int_{\mathbf{A}} \varphi\left(\begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \xi \end{pmatrix}\right) \psi(-ty\delta^{-1}\xi) d\xi,$$

using (2.1) and the fact that $\chi(t) = |t| = 1$ (as $t \in F^\times$). Since φ is a pure tensor, we may compute the integral on the right-hand side as the product of the local factors.

For $v \cong \mathbf{R}$, the local contribution is (see [5, (55)])

$$\chi_j(\mathrm{sign}(t_jy_j))|t_jy_j|^{1/2-\nu_j} \int_{\mathbf{R}} \varphi_j\left(\begin{pmatrix} 0 & -1 \\ 1 & \xi \end{pmatrix}\right) \exp(-2\pi i t_jy_j\xi) d\xi = \eta_j\chi_j(\mathrm{sign}(t_jy_j))\mathcal{W}_{q_j,\nu_j}(t_jy_j),$$

where $\eta_j \in \mathbf{C}$ is a constant (depending on q_j and χ_j) of absolute value $\pi^{1/2}$.

For $v \cong \mathbf{C}$, the local contribution is

$$\arg(t_jy_j)^{p_j}|t_jy_j|^{1-\nu_j} \int_{\mathbf{C}} \varphi_j\left(\begin{pmatrix} 0 & -1 \\ 1 & \xi \end{pmatrix}\right) \exp(-2\pi i(t_jy_j\xi + \overline{t_jy_j\xi})) d\xi.$$

The computation of [45, Section 4.1] in our normalization means that this equals

$$\frac{1}{\sqrt{2}}\mathcal{W}_{(l_j,q_j),(\nu_j,p_j)}(t_jy_j).$$

Altogether, the archimedean contribution is

$$\eta_\infty\chi_\infty(\mathrm{sign}(ty_\infty))\mathcal{W}_{\mathbf{w},\mathbf{r}}(ty_\infty),$$

where $\eta_\infty = \eta_\infty(\mathbf{w}, \chi_\infty) \in \mathbf{C}$ is a constant of absolute value $\pi^{r/2}2^{-s/2}|D_F|^{-1/2}$.

For a non-archimedean place $\mathfrak{p}$, we collect the results of [5, pp.18-20]. Introduce the notations $m = v_\mathfrak{p}(\mathfrak{c}_\chi) \geq 0$, $n = v_\mathfrak{p}(ty) \geq 0$ and let ϖ as above.

If $m = 0$ and $v_\mathfrak{p}(\mathfrak{t}) = 0$, then the local factor is

$$|ty|_\mathfrak{p}^{1/2}\chi_\mathfrak{p}^2(\delta)\left(1 - \frac{\chi_\mathfrak{p}^2(\varpi)}{\mathcal{N}(\mathfrak{p})}\right)\sum_{j=0}^{n}\chi_\mathfrak{p}(\varpi^{2j-n}).$$

If $m = 0$ and $v_\mathfrak{p}(\mathfrak{t}) = 1$, then in the case $\chi_\mathfrak{p}^2(\varpi) \neq -1$, the local factor has absolute value equal to $|1 + \chi_\mathfrak{p}^2(\varpi)|\mathcal{N}(\mathfrak{p})^{-1/2}$ for $n = 0$ and not exceeding $(n+1)\mathcal{N}(\mathfrak{p})^{(1-n)/2}$ in general. For $\chi_\mathfrak{p}^2(\varpi) = -1$, the local factor is

$$|ty|_\mathfrak{p}^{1/2}\chi_\mathfrak{p}^2(\delta)\chi_\mathfrak{p}^{-1}(\varpi)\mathcal{N}(\mathfrak{p})^{1/2}\left(1 - \frac{\chi_\mathfrak{p}^2(\varpi)}{\mathcal{N}(\mathfrak{p})}\right)\sum_{j=0}^{n-1}\chi_\mathfrak{p}(\varpi^{2j-n+1}).$$

If $m = 0$ and $v_\mathfrak{p}(\mathfrak{t}) \geq 2$, then the local factor vanishes for $n \leq v_\mathfrak{p}(\mathfrak{t}) - 3$, has absolute value equal to $\mathcal{N}(\mathfrak{p})^{-1}$ for $n = v_\mathfrak{p}(\mathfrak{t}) - 2$ and not exceeding $(n - v_\mathfrak{p}(\mathfrak{t}) + 3)\mathcal{N}(\mathfrak{p})^{(v_\mathfrak{p}(\mathfrak{t})-n)/2}$ for $n \geq v_\mathfrak{p}(\mathfrak{t}) - 1$.

If $m > 0$, then the local factor vanishes for $v_\mathfrak{p}(ty) \neq v_\mathfrak{p}(\mathfrak{t})$ and has absolute value 1 for $v_\mathfrak{p}(ty) = v_\mathfrak{p}(\mathfrak{t})$.

Altogether, in the Fourier expansion

$$E\left(\varphi, \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \varrho_{E(\varphi),0}(y) + \sum_{t\in F^\times}\varrho_{E(\varphi)}(ty_{\mathrm{fin}})\chi_\infty(ty_\infty)\mathcal{W}_{\mathbf{w},\mathbf{r}}(ty_\infty)\psi(tx),$$

we see that $\varrho_{E(\varphi)}(\mathfrak{m})$ is supported on ideals divisible by

$$\mathfrak{t}_\chi = \prod_{\mathfrak{p}|\mathfrak{t},\mathfrak{p}\nmid\mathfrak{c}_\chi,v_\mathfrak{p}(\mathfrak{t})=1,\chi_\mathfrak{p}^2(\varpi_\mathfrak{p})=-1}\mathfrak{p}\prod_{\mathfrak{p}|\mathfrak{t},\mathfrak{p}\nmid\mathfrak{c}_\chi,v_\mathfrak{p}(\mathfrak{t})\geq 3}\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{t})-2}\prod_{\mathfrak{p}|\mathfrak{t},\mathfrak{p}|\mathfrak{c}_\chi}\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{t})},$$

noting that

$$\mathfrak{t}\mathfrak{t}_\chi^{-1} = \prod_{\mathfrak{p}|\mathfrak{t},\mathfrak{p}\nmid\mathfrak{c}_\chi,v_\mathfrak{p}(\mathfrak{t})=1,\chi_\mathfrak{p}^2(\varpi_\mathfrak{p})\neq-1}\mathfrak{p}\prod_{\mathfrak{p}|\mathfrak{t},\mathfrak{p}\nmid\mathfrak{c}_\chi,v_\mathfrak{p}(\mathfrak{t})\geq 2}\mathfrak{p}^2.$$

With the notation

$$F_{\chi,\mathfrak{t}} = \prod_{\mathfrak{p}|\mathfrak{t},\mathfrak{p}\nmid\mathfrak{c}_\chi,v_\mathfrak{p}(\mathfrak{t})=1,\chi_\mathfrak{p}^2(\varpi_\mathfrak{p})\neq-1}|1 + \chi_\mathfrak{p}^2(\varpi_\mathfrak{p})|^{-1},$$

we may write

$$|\varrho_{E(\varphi)}(\mathfrak{t}_\chi)| = \frac{\pi^{r/2}2^{-s/2}|D_F|^{-1/2}}{|L^{\mathfrak{t}\mathfrak{t}_\chi^{-1}}(1,\chi^2)|\mathcal{N}(\mathfrak{t}\mathfrak{t}_\chi^{-1})^{1/2}F_{\chi,\mathfrak{t}}},$$

where $L^{\mathfrak{t}\mathfrak{t}_\chi^{-1}}(\cdot,\chi^2)$ stands for a partial Hecke L-function, which is holomorphic and nonzero at $s = 1$ (since χ^2 is a nontrivial Hecke character).

Normalize the coefficients as

$$\varrho_{E(\varphi)}(\mathfrak{m}\mathfrak{t}_\chi) = \frac{\lambda_{\chi,\mathfrak{t}}(\mathfrak{m})}{\sqrt{\mathcal{N}(\mathfrak{m})}}\varrho_{E(\varphi)}(\mathfrak{t}_\chi).$$

Then (see [5, (65) and the display above it]) $\lambda_{\chi,\mathfrak{t}}$ is a multiplicative function on nonzero ideals satisfying

$$\lambda_{\chi,\mathfrak{t}}(\mathfrak{m}) = \begin{cases} \sum_{\mathfrak{a}\mathfrak{b}=\mathfrak{m}}\chi(\mathfrak{a}\mathfrak{b}^{-1}), & \gcd(\mathfrak{m},\mathfrak{t}\mathfrak{t}_\chi^{-1}\mathfrak{c}_\chi) = \mathfrak{o}, \\ 0, & \gcd(\mathfrak{m},\mathfrak{c}_\chi) \neq \mathfrak{o}, \end{cases}$$

and

$$|\lambda_{\chi,\mathfrak{t}}(\mathfrak{m})| \leq \tau(\mathfrak{t})F_{\chi,\mathfrak{t}}\mathcal{N}(\mathfrak{t}\mathfrak{t}_\chi^{-1})^{1/2}\mathcal{N}(\gcd(\mathfrak{t}\mathfrak{t}_\chi^{-1},\mathfrak{m}))\tau(\mathfrak{m}),$$

where $\tau(\mathfrak{n})$ stands for the number of ideals dividing $\mathfrak{n}$.

Defining

$$\varepsilon_{\chi,\chi^{-1}}(\mathrm{sign}(y)) = \chi_\infty(\mathrm{sign}(y)),$$

$$\lambda_{\chi,\chi^{-1}}^{\mathfrak{t}}(\mathfrak{m}) = \begin{cases} \tau(\mathfrak{t})^{-1}F_{\chi,\mathfrak{t}}^{-1}\mathcal{N}(\mathfrak{t})^{-1/2}\mathcal{N}(\mathfrak{t}_\chi)\lambda_{\chi,\mathfrak{t}}(\mathfrak{m}\mathfrak{t}_\chi^{-1}), & \mathfrak{t}_\chi|\mathfrak{m}, \\ 0, & \text{otherwise}, \end{cases}$$

and finally

$$W_{E(\varphi)}(y) = \tau(\mathfrak{t})F_{\chi,\mathfrak{t}}\mathcal{N}(\mathfrak{t}\mathfrak{t}_\chi^{-1})^{1/2}\varrho_{E(\varphi)}(\mathfrak{t}_\chi)\varepsilon_{\chi,\chi^{-1}}(\mathrm{sign}(y))\mathcal{W}_{\mathbf{w},\mathbf{r}}(y)$$

for $y \in F_\infty^\times$, we obtain the Fourier-Whittaker expansion of Eisenstein series

$$\phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \varrho_{E(\varphi),0}(y) + \sum_{t \in F^\times} \frac{\lambda_{\chi,\chi^{-1}}^{\mathfrak{t}}(ty_{\mathrm{fin}})}{\sqrt{\mathcal{N}(ty_{\mathrm{fin}})}} W_{E(\varphi)}(ty_\infty)\psi(tx). \tag{2.38}$$

We also obtain

$$\lambda_{\chi,\chi^{-1}}^{\mathfrak{t}}(\mathfrak{m}) \ll_{F,\varepsilon} \mathcal{N}(\gcd(\mathfrak{t},\mathfrak{m}))\mathcal{N}(\mathfrak{m})^\varepsilon, \tag{2.39}$$

for all $\mathfrak{m} \subseteq \mathfrak{o}$ and

$$||W_{E(\varphi)}|| \ll_{F,\varepsilon} \mathcal{N}(\mathfrak{t})^\varepsilon C(\chi,\chi^{-1})^\varepsilon ||\varphi||, \tag{2.40}$$

where the norms are understood in the spaces $L^2(F_\infty^\times, d_\infty^\times y)$ and $L^2(K)$ (recall also (2.36)). Compare these with [5, (48-50)].

We also see that $E(\varphi) \mapsto W_{E(\varphi)}$ has similar properties as in the cuspidal spectrum. In the special case $\mathfrak{c} = \mathfrak{c}_\chi^2$, $\mathfrak{t} = \mathfrak{t}_\chi = \mathfrak{o}$, $E(\varphi)$ spans the space $V_{\chi,\chi^{-1},\mathbf{w}}(\mathfrak{c}_\chi^2)$ of newforms of weight $\mathbf{w}$. In this case, we have the alternative definition

$$W_{E(\varphi)}(y) = \int_{F\backslash\mathbf{A}} E(\varphi)\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)\psi(-x)dx, \tag{2.41}$$

where $y_{\mathrm{fin}} = (1,1,\ldots)$. Also, $\lambda_{\chi,\chi^{-1}}$ specialize to Hecke eigenvalues. Finally, for $\phi_1, \phi_2 \in V_{\chi,\chi^{-1}}(\mathfrak{c}_\chi^2)$, we have

$$\langle \phi_1, \phi_2 \rangle = C_{\chi,\chi^{-1}} \langle W_{\phi_1}, W_{\phi_2} \rangle \tag{2.42}$$

with some positive constant $C_{\chi,\chi^{-1}} \gg_{F,\varepsilon} C(\chi,\chi^{-1})^{-\varepsilon}$ depending only on χ.

Chapter 3

L-functions

3.1 The constant term of an Eisenstein series

In this section, we follow [5, Section 2.8]. Again, we pay special attention to the complex places, which is not covered there.

For some $s \in \mathbf{C}$, consider the Hecke quasicharacter $\chi(y) = |y|^s$ for $y \in \mathbf{A}^\times$. Taking also some nonzero ideal $\mathfrak{c} \subseteq \mathfrak{o}$, define the function $\varphi(s) \in H(\chi)$ as

$$\varphi\left(s, \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k\right) = \begin{cases} |a/b|^{1/2+s}, & k \in K_\infty \times K(\mathfrak{c}), \\ 0, & k \in K \setminus (K_\infty \times K(\mathfrak{c})). \end{cases}$$

The constant term [28, p.220] of the corresponding Eisenstein series $E(\varphi(s), g)$ is

$$E_0(\varphi(s), g) = \varphi(s, g) + \int_{\mathbf{A}} \varphi\left(s, \begin{pmatrix} 0 & -1 \\ 1 & \xi \end{pmatrix} g\right) d\xi. \tag{3.1}$$

Proposition 3.1.

$$\int_{\mathbf{A}} \varphi\left(s, \begin{pmatrix} 0 & -1 \\ 1 & \xi \end{pmatrix} g\right) d\xi = \frac{\Lambda_F(2s)}{\Lambda_F(2s+1)} H(s, g),$$

where

$$\Lambda_F(s) = |D_F|^{s/2} \prod_{v \cong \mathbf{R}} \left(\pi^{-s/2}\Gamma(s/2)\right) \prod_{v \cong \mathbf{C}} \left(2(2\pi)^{-s}\Gamma(s)\right) \prod_{\mathfrak{p}} (1 - \mathcal{N}(p)^{-s})^{-1}$$

for $\Re s > 1$, and $H(s, g)$ is a meromorphic function of s, its zeros lie on $\Re s = 0$, its poles on $\Re s = -1/2$ and it is constant at $s = 1/2$:

$$H(1/2, g) = |\delta| \mathcal{N}(\mathfrak{c})^{-1} \prod_{\mathfrak{p} | \mathfrak{c}} (1 + \mathcal{N}(\mathfrak{p})^{-1})^{-1} = |D_F|^{-1} [K(\mathfrak{o}) : K(\mathfrak{c})]^{-1}.$$

Proof. We may write

$$g = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} h, \qquad x \in \mathbf{A}, a, b \in \mathbf{A}^\times, h \in \mathrm{GL}_2(\mathbf{A}),$$

where $h_\infty \in K_\infty$, $h_\mathfrak{p} \in K(\mathfrak{o}_\mathfrak{p})$ for $\mathfrak{p} \nmid \mathfrak{c}$ and for $\mathfrak{p} | \mathfrak{c}$, $h_\mathfrak{p} \in \mathrm{GL}_2(F_\mathfrak{p})$ is of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & -\delta_\mathfrak{p}^{-1} \\ \delta_\mathfrak{p} & \eta_\mathfrak{p} \end{smallmatrix}\right)$.

Then our integral becomes

$$\left|\frac{a}{b}\right|^{1/2-s} |\delta|^{2s} \int_{\mathbf{A}} \varphi\left(s, \begin{pmatrix} 0 & -\delta^{-1} \\ \delta & \xi \end{pmatrix} h\right) d\xi,$$

which can be computed as the product of the corresponding local integrals. These are given at [5, pp.22-24] for real and non-archimedean places.

First assume $\mathfrak{p}$ is a non-archimedean place. If $\mathfrak{p} \nmid \mathfrak{c}$, then the local integral is

$$\int_{F_\mathfrak{p}} \varphi_\mathfrak{p}\left(s, \begin{pmatrix} 0 & -\delta_\mathfrak{p}^{-1} \\ \delta_\mathfrak{p} & \xi \end{pmatrix} h_\mathfrak{p}\right) d\xi = \frac{1 - \mathcal{N}(\mathfrak{p})^{-1-2s}}{1 - \mathcal{N}(\mathfrak{p})^{-2s}}.$$

If $\mathfrak{p}|\mathfrak{c}$, then we have two cases:

$$\int_{F_\mathfrak{p}} \varphi_\mathfrak{p}\left(s, \begin{pmatrix} 0 & -\delta_\mathfrak{p}^{-1} \\ \delta_\mathfrak{p} & \xi \end{pmatrix}\right) d\xi = \mathcal{N}(\mathfrak{p})^{-2sv_\mathfrak{p}(\mathfrak{c})} \frac{1-\mathcal{N}(\mathfrak{p})^{-1}}{1-\mathcal{N}(\mathfrak{p})^{-2s}},$$

and

$$\int_{F_\mathfrak{p}} \varphi_\mathfrak{p}\left(s, \begin{pmatrix} 0 & -\delta_\mathfrak{p}^{-1} \\ \delta_\mathfrak{p} & \xi \end{pmatrix}\begin{pmatrix} 0 & -\delta_\mathfrak{p}^{-1} \\ \delta_\mathfrak{p} & \eta_\mathfrak{p} \end{pmatrix}\right) d\xi = \mathcal{N}(\mathfrak{p})^{-v_\mathfrak{p}(\mathfrak{c})}.$$

Now assume v is a real place, then the local integral is

$$\int_\mathbf{R} \varphi_j\left(s, \begin{pmatrix} 0 & -1 \\ 1 & \xi \end{pmatrix} h_j\right) d\xi = \frac{\Gamma(1/2)\Gamma(s)}{\Gamma(1/2+s)}.$$

Finally, assume $v \cong \mathbf{C}$. Using the Iwasawa decomposition

$$\begin{pmatrix} 0 & -1 \\ 1 & \xi \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{|\xi|^2+1}} & \frac{-\bar{\xi}}{\sqrt{|\xi|^2+1}} \\ 0 & \sqrt{|\xi|^2+1} \end{pmatrix} \begin{pmatrix} \frac{\bar{\xi}}{\sqrt{|\xi|^2+1}} & \frac{-1}{\sqrt{|\xi|^2+1}} \\ \frac{1}{\sqrt{|\xi|^2+1}} & \frac{\xi}{\sqrt{|\xi|^2+1}} \end{pmatrix},$$

we see that the local integral is

$$\begin{aligned} \int_\mathbf{C} \frac{1}{(1+|\xi|^2)^{1+2s}} d\xi &= 2\int_\mathbf{C} \frac{1}{(1+|\xi|^2)^{1+2s}} d\Re\xi d\Im\xi = 2\int_0^{2\pi}\int_0^\infty \frac{r}{(1+r^2)^{1+2s}} dr d\theta \\ &= \frac{\pi}{s} = 2\pi \frac{\Gamma(1)\Gamma(2s)}{\Gamma(1+2s)}. \end{aligned}$$

As for

$$\mathcal{N}(\mathfrak{c})^{-1} \prod_{\mathfrak{p}|\mathfrak{c}} (1+\mathcal{N}(\mathfrak{p})^{-1})^{-1} = [K(\mathfrak{o}) : K(\mathfrak{c})]^{-1}, \tag{3.2}$$

consult [37, Proposition 2.5]. Collecting these, the proof is complete. □

3.2 A Rankin-Selberg convolution

Earlier, we referred to this section twice: in the construction of the isometries $R^\mathfrak{t}$ (2.18) and in the proof of Proposition 2.2. Now we borrow the Rankin-Selberg method from [5, pp.25-26] in order to prove the essential equivalence of the Kirillov model promised earlier (i.e. to complete the proof of Proposition 2.2), and also to relate the proportionality constant to the residue of a certain $\mathrm{GL}_2 \times \mathrm{GL}_2$ L-function. We will also obtain that for $\phi_1, \phi_2 \in V_{\pi,\mathbf{w}}(\mathfrak{c}_\pi)$, $\langle R_{\mathfrak{t}_1}\phi_1, R_{\mathfrak{t}_2}\phi_2\rangle = \langle \phi_1, \phi_2\rangle C(\mathfrak{t}_1, \mathfrak{t}_2, \pi)$ with a constant $C(\mathfrak{t}_1, \mathfrak{t}_2, \pi)$ independent of $\mathbf{w}$, this was the fact used in the construction of $R^\mathfrak{t}$.

Let $\phi_1, \phi_2 \in V_{\pi,\mathbf{w}}$ be newforms of some weight $\mathbf{w} \in W(\pi)$ and let $\mathfrak{t}_1, \mathfrak{t}_2 \subseteq \mathfrak{o}$ be nonzero ideals. If $\mathfrak{c}$ is a nonzero ideal divisible by $\mathfrak{t}_1\mathfrak{c}_\pi$, $\mathfrak{t}_2\mathfrak{c}_\pi$, then $\psi_1 = R_{\mathfrak{t}_1}\phi_1$, $\psi_2 = R_{\mathfrak{t}_2}\phi_2$ are elements in $V_{\pi,\mathbf{w}}(\mathfrak{c})$.

Define

$$F(s) = \int_{\mathrm{GL}_2(F)Z(\mathbf{A})\backslash \mathrm{GL}_2(\mathbf{A})} \psi_1(g)\overline{\psi_2(g)} E(\varphi(s), g) dg,$$

where $\varphi(s,g)$ is defined in the previous section. It follows from the theory of Eisenstein series that this integral is absolutely convergent for all s which is not a pole of $E(\varphi(s), g)$ (see [28, Section 5]), and also that the possible residue comes from the residue of the constant term (3.1). Now we compute $\mathrm{res}_{s=1/2}F(s)$ in two ways.

On the one hand, using the results of the previous section,

$$\mathrm{res}_{s=1/2}F(s) = C_F \frac{\langle R_{\mathfrak{t}_1}\phi_1, R_{\mathfrak{t}_2}\phi_2\rangle}{[K(\mathfrak{o}) : K(\mathfrak{c})]}, \qquad C_F = \frac{\mathrm{res}_{s=1/2}\Lambda_F(2s)}{|D_F|\Lambda_F(2)}. \tag{3.3}$$

On the other hand, assume first $\Re s > 1/2$ for the absolute convergence of (2.2) (see [15, p.372]) and unfold the integral

$$\begin{aligned}
F(s) &= \int_{B(F)Z(\mathbf{A})\backslash \mathrm{GL}_2(\mathbf{A})} \psi_1(g)\overline{\psi_2(g)}\varphi(s,g)dg \\
&= \int_{F^\times\backslash\mathbf{A}^\times}\int_{F\backslash\mathbf{A}}\int_K \psi_1\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}k\right)\overline{\psi_2\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}k\right)}\varphi\left(s,\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}k\right)dkdx\frac{d^\times y}{|y|} \\
&= \int_{F^\times\backslash\mathbf{A}^\times}\int_{F\backslash\mathbf{A}}\int_{K_\infty\times K(\mathfrak{c})} \psi_1\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}k\right)\overline{\psi_2\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}k\right)}|y|^{s-1/2}dkdxd^\times y.
\end{aligned}$$

Here, the integral over $K_\infty \times K(\mathfrak{c})$ is $[K(\mathfrak{o}) : K(\mathfrak{c})]^{-1}$. To see this, observe that $\psi_1\overline{\psi_2}$ is invariant at real and non-archimedean places, while at complex places, we apply the more general [43, Corollary 1.10(b)]. Therefore,

$$F(s) = \frac{1}{[K(\mathfrak{o}) : K(\mathfrak{c})]}\int_{F^\times\backslash\mathbf{A}^\times}\int_{F\backslash\mathbf{A}} \psi_1\overline{\psi_2}\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right)|y|^{s-1/2}dxd^\times y.$$

Take now finite representing ideles t_1, t_2 of the ideals $\mathfrak{t}_1, \mathfrak{t}_2$, respectively. The Fourier-Whittaker expansion (2.29), the definition of $R_\mathfrak{t}$ (2.17) and $\mathrm{vol}(F\backslash\mathbf{A}) = 1$ give rise to

$$\begin{aligned}
F(s) &= \frac{\mathcal{N}(\mathfrak{t}_1\mathfrak{t}_2)^{1/2}}{[K(\mathfrak{o}) : K(\mathfrak{c})]}\int_{F^\times\backslash\mathbf{A}^\times}\sum_{t\in F^\times}\frac{\lambda_\pi(ty_{\mathrm{fin}}t_1^{-1})\overline{\lambda_\pi(ty_{\mathrm{fin}}t_2^{-1})}}{\mathcal{N}(ty_{\mathrm{fin}})}W_{\phi_1}(ty_\infty)\overline{W_{\phi_2}(ty_\infty)}|y|^{s-1/2}d^\times y \\
&= \frac{\mathcal{N}(\mathfrak{t}_1\mathfrak{t}_2)^{1/2}}{[K(\mathfrak{o}) : K(\mathfrak{c})]}\int_{\mathbf{A}^\times}\frac{\lambda_\pi(y_{\mathrm{fin}}t_1^{-1})\overline{\lambda_\pi(y_{\mathrm{fin}}t_2^{-1})}}{\mathcal{N}(y_{\mathrm{fin}})}W_{\phi_1}(y_\infty)\overline{W_{\phi_2}(y_\infty)}|y|^{s-1/2}d^\times y \\
&= \frac{\mathcal{N}(\mathfrak{t}_1\mathfrak{t}_2)^{1/2}}{[K(\mathfrak{o}) : K(\mathfrak{c})]}\int_{F_\infty^\times}W_{\phi_1}(y_\infty)\overline{W_{\phi_2}(y_\infty)}|y_\infty|^{s-1/2}d^\times y_\infty\int_{\mathbf{A}_{\mathrm{fin}}^\times}\frac{\lambda_\pi(y_{\mathrm{fin}}t_1^{-1})\overline{\lambda_\pi(y_{\mathrm{fin}}t_2^{-1})}}{\mathcal{N}(y_{\mathrm{fin}})^{1/2+s}}d^\times y_{\mathrm{fin}}.
\end{aligned}$$

Let now $s \to 1/2$ from above, then the first integral is $\langle W_{\phi_1}, W_{\phi_2}\rangle$, where the inner product is understood in $L^2(F_\infty^\times, d_\infty^\times y)$. In the second integral, define $\mathfrak{t}_1' = \mathfrak{t}_1\gcd(\mathfrak{t}_1,\mathfrak{t}_2)^{-1}$, $\mathfrak{t}_2' = \mathfrak{t}_2\gcd(\mathfrak{t}_1,\mathfrak{t}_2)^{-1}$, we obtain

$$\mathrm{res}_{s=1/2}F(s) = \frac{\langle W_{\phi_1}, W_{\phi_2}\rangle}{\mathcal{N}(\mathfrak{t}_1'\mathfrak{t}_2')^{1/2}[K(\mathfrak{o}) : K(\mathfrak{c})]}\mathrm{res}_{s=1}\sum_{0\neq\mathfrak{m}\subseteq\mathfrak{o}}\frac{\lambda_\pi(\mathfrak{m}\mathfrak{t}_2')\overline{\lambda_\pi(\mathfrak{m}\mathfrak{t}_1')}}{\mathcal{N}(\mathfrak{m})^s} \tag{3.4}$$

by a linear change of variable $\mathfrak{m} = y_{\mathrm{fin}}t_1t_2\gcd(\mathfrak{t}_1,\mathfrak{t}_2)^{-1}$.

For arbitrary ideals $\mathfrak{t}_1, \mathfrak{t}_2$, this gives

$$\langle R_{\mathfrak{t}_1}\phi_1, R_{\mathfrak{t}_2}\phi_2\rangle = \langle\phi_1,\phi_2\rangle C(\mathfrak{t}_1,\mathfrak{t}_2,\pi),$$

where $C(\mathfrak{t}_1,\mathfrak{t}_2,\pi)$ is a constant not depending on the weight $\mathbf{w}$. This independence of the weight is essential in the construction of $R^\mathfrak{t}$ (2.18) as we indicated it earlier.

Using the equations (3.3) and (3.4) about $\mathrm{res}_{s=1/2}F(s)$, and taking $\mathfrak{t}_1 = \mathfrak{t}_2 = \mathfrak{o}$, we obtain (2.33) with

$$C_\pi = \frac{|D_F|\Lambda_F(2)}{\mathrm{res}_{s=1/2}\Lambda_F(2s)}\mathrm{res}_{s=1}\sum_{0\neq\mathfrak{m}\subseteq\mathfrak{o}}\frac{\lambda_\pi(\mathfrak{m})\overline{\lambda_\pi(\mathfrak{m})}}{\mathcal{N}(\mathfrak{m})^s}.$$

Here, the first factor $|D_F|\Lambda_F(2)/\mathrm{res}_{s=1/2}\Lambda_F(2s)$ is a positive constant depending only on F, while

$$L^{\mathfrak{c}_\pi}(s,\pi\times\pi)\zeta_F(2s)\sum_{0\neq\mathfrak{m}\subseteq\mathfrak{o}}\frac{\lambda_\pi(\mathfrak{m})\overline{\lambda_\pi(\mathfrak{m})}}{\mathcal{N}(\mathfrak{m})^s} = L(s,\pi\times\pi)$$

with $L(s,\pi\times\pi)$ defined in [27, Sections 1-2] and $L^{\mathfrak{c}_\pi}(s,\pi\times\pi)$ is a finite Euler product over places dividing $\mathfrak{c}_\pi$, the number of such places is $O_{F,\varepsilon}(\mathcal{N}(\mathfrak{c}_\pi)^\varepsilon)$. Checking the cases from [27, Section 1] and [39, Chapter I, §§2-3]), we obtain

$$\mathcal{N}(\mathfrak{c}_\pi)^{-\varepsilon}\mathrm{res}_{s=1}L(s,\pi\times\pi) \ll_{F,\varepsilon} C_\pi \ll_{F,\varepsilon} \mathcal{N}(\mathfrak{c}_\pi)^\varepsilon\mathrm{res}_{s=1}L(s,\pi\times\pi). \tag{3.5}$$

Proposition 3.2. *We have*

$$C(\pi)^{-\varepsilon} \ll_{F,\varepsilon} \mathrm{res}_{s=1} L(s, \pi \times \pi) \ll_{F,\varepsilon} C(\pi)^{\varepsilon},$$

recall (2.16).

Proof. We repeat the proof of [5, Lemma 3], however, our references differ a little.

First we prove the lower bound. By [33, Lemma b], we have a constant B depending only on F such that

$$C(\pi)^{-B} \leq C(\pi \times \pi) \leq C(\pi)^{B} \tag{3.6}$$

holds for the analytic conductors. We also see by [33, Lemma a] that $L(s, \pi \times \pi)$ has nonnegative coefficients. Record the factorization

$$L(s, \pi \times \pi) = \zeta_F(s) L(s, \mathrm{sym}^2\pi), \tag{3.7}$$

where $L(s, \mathrm{sym}^2\pi)$ is the Gelbart-Jacquet lift of π, which is known to exist for all automorphic cuspidal π (see [27, Section 3, (3.6), (3.7) and Theorem (9.3)]).

Case 1: $\mathrm{sym}^2\pi$ is cuspidal. Now [32, Proposition 1.1] gives the statement, as soon as we can prove that there is no Siegel zero of $L(s, \pi \times \pi)$ in the sense of [33, p.284] and [1, p.345]. The factors in (3.7) do not admit any Siegel zero: this is obvious for $\zeta_F(s)$, while it follows from [1, Corollary 4 and Theorem 5] for $L(s, \mathrm{sym}^2\pi)$.

Case 2: $\mathrm{sym}^2\pi$ is not cuspidal. Then $\pi \cong \pi \otimes \chi$, where χ is some nontrivial Hecke character on $F^\times \backslash \mathbf{A}^\times$ with $\chi^2 = 1$ (see [27, Theorem (9.3)]). The conductor of this χ is bounded by $\ll_F C(\pi)$. In this case, $L(s, \mathrm{sym}^2\pi)$ can be factored as the product of two GL_1 L-functions, see [27, (3.7) and Remark (9.9)]:

$$L(s, \mathrm{sym}^2\pi) = L(s, \chi) L_{F'}(s, \Omega),$$

where F' is the quadratic extension of F corresponding to χ and Ω is a Hecke character over F', each conductor is $\ll_F C(\pi)$. In the first factor, by [25], there is no zero of $L(s, \chi)$ on $[1 - \mathrm{const.}(F, \varepsilon) C(\pi)^{-\varepsilon}, 1]$. As for the second factor, Ω might be quadratic or not. In both cases we apply [25] again, noting also that the discriminant of F' is $\ll_F C(\pi)$ (see also [7, Theorem 2]). We remark, however, that if Ω is not quadratic, we can again guarantee a bigger zero-free interval $[1 - \mathrm{const.}(F, \varepsilon)(\log C(\pi))^{-1}, 1]$ (see [49, Theorem 11.3], for example). Altogether, we may apply again [32, Proposition 1.1] by noting that in this case, the known zero-free interval is smaller than before, and we have to replace $\log C(\pi)$ by $C(\pi)^{\varepsilon}$.

We see that our bound is weaker in the case when π is a lift of a GL_1 form. This is analogous to [32, Main Theorem of Appendix].

As for the upper bound, the method of [36, pp.72-73] goes through (see also [32]). For later references, we also record

$$\sum_{\mathcal{N}(\mathfrak{m}) \leq x} \frac{|\lambda_\pi(\mathfrak{m})|^2}{\mathcal{N}(\mathfrak{m})} \ll_{F,\varepsilon} C(\pi)^{B'} x^{\varepsilon} \tag{3.8}$$

with some B' depending only on F, which follows from the upper bound of (3.6) by a contour integration similar to the one in [32, Proof of Lemma 2.1]. □

3.3 An upper bound on the central value $L(1/2, \pi \otimes \chi)$

Proposition 3.3. *There is a constant $c = c(\pi, \chi_\infty, \varepsilon) > 0$ and a smooth function $V : (0, \infty) \to \mathbf{C}$ supported on $[1/2, 2]$, satisfying $V^{(j)}(y) \ll_{\pi, \chi_\infty, j} 1$ for each nonnegative integer j, such that*

$$L(1/2, \pi \otimes \chi) \ll_{\pi, \chi_\infty, \varepsilon} \mathcal{N}(\mathfrak{q})^{\varepsilon} \max_{Y \leq c\mathcal{N}(\mathfrak{q})^{1+\varepsilon}} \left| \sum_{0 \neq \mathfrak{m} \subseteq \mathfrak{o}} \frac{\lambda_\pi(\mathfrak{m})\chi(\mathfrak{m})}{\sqrt{\mathcal{N}(\mathfrak{m})}} V\left(\frac{\mathcal{N}(\mathfrak{m})}{Y}\right) \right|. \tag{3.9}$$

Proof. This is [5, (75)] (see also [6, Section 5.1]), for completeness, we decided to give the proof.

Our starting point is the approximate functional equation [30, Theorem 2.1]

$$L(1/2, \pi \otimes \chi) = \Sigma + \eta \overline{\Sigma}, \qquad \Sigma = \sum_{0 \neq \mathfrak{m} \subseteq \mathfrak{o}} \frac{\lambda_{\pi \otimes \chi}(\mathfrak{m})}{\sqrt{\mathcal{N}(\mathfrak{m})}} V_0\left(\frac{\mathcal{N}(\mathfrak{m})}{\sqrt{C(\pi \otimes \chi)}}\right),$$

where $\eta \in \mathbf{C}$ and $V_0 : (0,\infty) \to \mathbf{C}$ depend only on the archimedean parameters of $\pi \otimes \chi$ and satisfy the following properties. The smooth function V_0 and all its derivatives tend to zero faster then any negative power of the identity; $|\eta| = 1$. Since the possible values of $\mathcal{N}(\mathfrak{m})$ can be bounded away from 0, we may assume moreover that V_0 vanishes in a neighborhood of 0.

Now

$$\sum_{0\neq\mathfrak{m}\subseteq\mathfrak{o}} \frac{\lambda_{\pi\otimes\chi}(\mathfrak{m})}{\sqrt{\mathcal{N}(\mathfrak{m})}} V_0\left(\frac{\mathcal{N}(\mathfrak{m})}{\sqrt{C(\pi\otimes\chi)}}\right) = \sum_{\mathfrak{c}|(\mathfrak{c}_\pi\mathfrak{q})^\infty} \frac{a(\mathfrak{c})}{\sqrt{\mathcal{N}(\mathfrak{c})}} \sum_{0\neq\mathfrak{m}\subseteq\mathfrak{o}} \frac{\lambda_\pi(\mathfrak{m})\chi(\mathfrak{m})}{\sqrt{\mathcal{N}(\mathfrak{m})}} V_0\left(\frac{\mathcal{N}(\mathfrak{c}\mathfrak{m})}{\sqrt{C(\pi\otimes\chi)}}\right),$$

where $a(\mathfrak{c})$'s are the coefficients of the Dirichlet series defined via

$$\sum_{\mathfrak{c}|(\mathfrak{c}_\pi\mathfrak{q})^\infty} a(\mathfrak{c})\mathcal{N}(\mathfrak{c})^{-s} = \prod_{\mathfrak{p}|\mathfrak{c}_\pi\mathfrak{q}} \frac{1-\lambda_\pi(\mathfrak{p})\chi(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s}+\chi(\mathfrak{p}^2)\mathcal{N}(\mathfrak{p})^{-2s}}{L_\mathfrak{p}^{-1}(s,\pi\otimes\chi)}$$

on $\Re s > 1$.

Using that $\lambda_\pi(\mathfrak{m}), \lambda_{\pi\otimes\chi}(\mathfrak{m}) \ll_\varepsilon \mathcal{N}(\mathfrak{m})^{\theta+\varepsilon}$, this implies that

$$\sum_{0\neq\mathfrak{m}\subseteq\mathfrak{o}} \frac{\lambda_{\pi\otimes\chi}(\mathfrak{m})}{\sqrt{\mathcal{N}(\mathfrak{m})}} V_0\left(\frac{\mathcal{N}(\mathfrak{m})}{\sqrt{C(\pi\otimes\chi)}}\right) \ll_\varepsilon \sum_{\mathfrak{c}|(\mathfrak{c}_\pi\mathfrak{q})^\infty} \frac{1}{\mathcal{N}(\mathfrak{c})^{1/2-\theta-\varepsilon}} \left|\sum_{0\neq\mathfrak{m}\subseteq\mathfrak{o}} \frac{\lambda_\pi(\mathfrak{m})\chi(\mathfrak{m})}{\sqrt{\mathcal{N}(\mathfrak{m})}} V_0\left(\frac{\mathcal{N}(\mathfrak{c}\mathfrak{m})}{\sqrt{C(\pi\otimes\chi)}}\right)\right|.$$

Since V_0 decays rapidly, the contribution of $\mathcal{N}(\mathfrak{c}\mathfrak{m}) > C(\pi\otimes\chi)^{1+\varepsilon}$ to the inner summation is $O_\varepsilon(1)$. For the rest, we apply a smooth dyadic partition of unity: let $U : (0,\infty) \to (0,\infty)$ be a smooth function (fixed once for all, independently of F, π, χ) supported in the interior of $[1/2, 2]$ satisfying $\sum_{n\in\mathbf{Z}} U(2^n x) = 1$ for all $x \in (0,\infty)$. Writing now $V(x) = V_0(x)U(x)$, the inner summation over $\mathfrak{m}$ is splitting up according to the magnitude of $\mathcal{N}(\mathfrak{c}\mathfrak{m})$ on the logarithmic scale. The number of terms is $O(\log C(\pi\otimes\chi)) = O_{\pi,\chi_\infty}(\log\mathcal{N}(\mathfrak{q})) \ll_{\pi,\varepsilon} \mathcal{N}(\mathfrak{q})^\varepsilon$. Also the outer summation over $\mathfrak{c}$ gives a factor $\ll_{\pi,\varepsilon} \mathcal{N}(\mathfrak{q})^\varepsilon$. Altogether, we obtain the statement. $\square$

Chapter 4

Sobolev norms

Assume that we are given a smooth automorphic vector ϕ appearing in an automorphic representation. The aim of this chapter is to give a pointwise estimate for the associated Kirillov vector W_ϕ, and, when ϕ is a cuspidal newform, the supremum norm of ϕ, both in terms of some Sobolev norm of ϕ.

Let $d \geq 0$ be an integer. Assume that $\phi \in L^2(\mathrm{GL}_2(F)Z(\mathbf{A})\backslash\mathrm{GL}_2(\mathbf{A}))$ is a function such that $X_1 \dots X_d\phi$ exists for every sequence $X_1 \dots X_d$, where each X_k is one of those differential operators given in (2.4) and (2.5). Then the Sobolev norm $||\phi||_{S_d}$ of ϕ is defined via

$$||\phi||_{S_d}^2 = \sum_{k=0}^{d} \sum_{\{X_1,\dots,X_k\}\in\{\mathbf{H}_j,\mathbf{R}_j,\mathbf{L}_j,\mathbf{H}_{1,j},\mathbf{H}_{2,j},\mathbf{V}_{1,j},\mathbf{V}_{2,j},\mathbf{W}_{1,j},\mathbf{W}_{2,j}\}^k} ||X_1 \dots X_k\phi||^2.$$

4.1 Bounds on Bessel functions

The first lemma is an estimate on the classical J-Bessel function.

Lemma 4.1. *Denote by J the J-Bessel function. Let $p \in \mathbf{Z}$. Then*

$$|J_{2p}(x)| \leq 1 \text{ for all } x \in (0,\infty), \qquad |J_{2p}(x)| \ll x^{-1/2} \text{ for all } x \in (\max(1/2,(2p)^2),\infty). \tag{4.1}$$

Proof. The first inequality is obvious from [58, 2.2(1)]. The second one follows from [29, 8.451(1)] by writing '$n = p$' there, the error term is estimated in [29, 8.451(7-8)]. □

In the next lemma, we define and estimate a function j that later will turn out to be the Bessel function of a certain representation (after a simple transformation of the argument).

Lemma 4.2. *Assume $\nu \in \mathbf{C}$ and $p \in \mathbf{Z}$ are given such that either $\Re\nu = 0$ (principal series) or $\Re\nu \neq 0$, $\Im\nu = 0$, $|\nu| \leq 2\theta = 7/32$, $p = 0$ (complementary series). Define*

$$j(t) = (-1)^p 4\pi|t|^2 \int_0^\infty y^{2\nu} \left(\frac{yt + y^{-1}\bar{t}}{|yt + y^{-1}\bar{t}|}\right)^{2p} J_{2p}(2\pi|yt + y^{-1}\bar{t}|) d_{\mathbf{R}}^\times y. \tag{4.2}$$

Then $j(t)$ is an even function of $t \in \mathbf{C}^\times$ satisfying the bound

$$j(t) \ll |t|^2(1 + |t|^{-1/2})(1 + |p|). \tag{4.3}$$

Proof. It is clear that $j(t) = j(-t)$, so we are left to prove (4.3). Assume first that $p \neq 0$, which implies that we are in the principal series. Then trivially

$$j(t) \ll |t|^2 \int_0^\infty |J_{2p}(2\pi|yt + y^{-1}\bar{t}|)| d_{\mathbf{R}}^\times y.$$

The integral is invariant under $y \leftrightarrow 1/y$, so we have

$$j(t) \ll |t|^2 \int_1^\infty |J_{2p}(2\pi|yt + y^{-1}\bar{t}|)| d_{\mathbf{R}}^\times y.$$

Here

$$\int_1^2 |J_{2p}(2\pi|yt + y^{-1}\overline{t}|)| d_{\mathbf{R}}^{\times} y \ll 1$$

and

$$\int_2^{\max\left(\frac{4p^2}{\pi|t|},2\right)} |J_{2p}(2\pi|yt + y^{-1}\overline{t}|)| d_{\mathbf{R}}^{\times} y \ll \max\left(\log\left(\frac{4p^2}{\pi|t|}\right), 0\right)$$

by $|J_{2p}(x)| \leq 1$ of (4.1). On the remaining domain, $y \geq 2$, hence $|yt + y^{-1}\overline{t}| \geq y|t|/2$. Moreover, since $y \geq 4p^2/(\pi|t|)$, we have $2\pi|yt+y^{-1}\overline{t}| \geq (2p)^2 > 1/2$, so we may apply $|J_{2p}(x)| \ll x^{-1/2}$ of (4.1), obtaining

$$\begin{aligned}
\int_{\max\left(\frac{4p^2}{\pi|t|},2\right)}^{\infty} |J_{2p}(2\pi|yt + y^{-1}\overline{t}|)| d_{\mathbf{R}}^{\times} y &\ll \int_{\max\left(\frac{4p^2}{\pi|t|},2\right)}^{\infty} (2\pi|yt + y^{-1}\overline{t}|)^{-1/2} d_{\mathbf{R}}^{\times} y \\
&\ll \int_{\max\left(\frac{4p^2}{\pi|t|},2\right)}^{\infty} (y|t|)^{-1/2} d_{\mathbf{R}}^{\times} y \\
&\ll \int_{\frac{4p^2}{\pi|t|}}^{\infty} (y|t|)^{-1/2} d_{\mathbf{R}}^{\times} y + \int_2^{\infty} (y|t|)^{-1/2} d_{\mathbf{R}}^{\times} y \\
&\ll 1 + |t|^{-1/2}.
\end{aligned}$$

Altogether,

$$j(t) \ll |t|^2 \left(1 + |t|^{-1/2} + \max\left(\log\left(\frac{4p^2}{\pi|t|}\right), 0\right)\right),$$

which obviously implies

$$j(t) \ll |t|^2 (1 + |t|^{-1/2})(1 + |p|). \tag{4.4}$$

If $p = 0$, in particular, in the complementary series, we have

$$j(t) \ll |t|^2 \int_0^{\infty} y^{2\Re\nu} |J_0(2\pi|yt + y^{-1}\overline{t}|)| d_{\mathbf{R}}^{\times} y \ll |t|^2 \int_1^{\infty} y^{2|\Re\nu|} |J_0(2\pi|yt + y^{-1}\overline{t}|)| d_{\mathbf{R}}^{\times} y,$$

since under $y \leftrightarrow 1/y$, $J_0(2\pi|yt + y^{-1}\overline{t}|)$ and $d_{\mathbf{R}}^{\times} y$ are invariant, while $y^{2\Re\nu} \geq (1/y)^{2\Re\nu}$ if and only if $\log y/\Re\nu \geq 0$. Then

$$\int_1^2 y^{2|\Re\nu|} |J_0(2\pi|yt + y^{-1}\overline{t}|)| d_{\mathbf{R}}^{\times} y \ll 1$$

and

$$\int_2^{\max\left(\frac{1}{|t|},2\right)} y^{2|\Re\nu|} |J_0(2\pi|yt + y^{-1}\overline{t}|)| d_{\mathbf{R}}^{\times} y \ll |t|^{-4\theta},$$

again by $|J_{2p}(x)| \leq 1$ of (4.1) and $2|\Re\nu| \leq 4\theta = 7/16$. On the remaining domain, $y \geq 2$ implies $|yt + y^{-1}\overline{t}| \geq y|t|/2$, so we may apply $|J_{2p}(x)| \ll x^{-1/2}$, since $y \geq 1/|t|$. Then

$$\begin{aligned}
\int_{\max\left(\frac{1}{|t|},2\right)}^{\infty} y^{2|\Re\nu|} |J_0(2\pi|yt + y^{-1}\overline{t}|)| d_{\mathbf{R}}^{\times} y &\ll \int_{\max\left(\frac{1}{|t|},2\right)}^{\infty} y^{2|\Re\nu|} (2\pi|yt + y^{-1}\overline{t}|)^{-1/2} d_{\mathbf{R}}^{\times} y \\
&\ll \int_{\max\left(\frac{1}{|t|},2\right)}^{\infty} y^{2|\Re\nu|} (y|t|)^{-1/2} d_{\mathbf{R}}^{\times} y \\
&\ll \int_{\frac{1}{|t|}}^{\infty} y^{2|\Re\nu|} (y|t|)^{-1/2} d_{\mathbf{R}}^{\times} y + \int_2^{\infty} y^{2|\Re\nu|} (y|t|)^{-1/2} d_{\mathbf{R}}^{\times} y \\
&\ll |t|^{-2|\Re\nu|} + |t|^{-1/2},
\end{aligned}$$

where we used again that $2|\Re\nu| \leq 7/16$. Therefore in this case, we obtain

$$j(t) \ll |t|^2 (1 + |t|^{-1/2}). \tag{4.5}$$

Collecting the bounds (4.4), (4.5), we arrive at (4.3). □

4.2 Bounds on Whittaker functions

We would like to give estimates on the Whittaker functions defined in (2.20) and (2.21). At real places, we refer to [4].

Lemma 4.3. *For all ν,*

$$\mathcal{W}_{q,\nu}(y) \ll |y|^{1/2} \left(\frac{|y|}{|q|+|\nu|+1} \right)^{-1-|\Re\nu|} \exp\left(-\frac{|y|}{|q|+|\nu|+1} \right). \tag{4.6}$$

For $\nu \in (1/2+\mathbf{Z}) \cup i\mathbf{R}$ and for any $0<\varepsilon<1/4$,

$$\mathcal{W}_{q,\nu}(y) \ll_\varepsilon |y|^{1/2-\varepsilon}(|q|+|\nu|+1). \tag{4.7}$$

For $\nu \in (-1/2, 1/2)$ and for any $0<\varepsilon<1$,

$$\mathcal{W}_{q,\nu}(y) \ll_\varepsilon |y|^{1/2-|\nu|-\varepsilon}(|q|+|\nu|+1)^{1+|\nu|}. \tag{4.8}$$

Proof. See [4, (24-26)] (and also [5, (26-28)]). ☐

At complex places, introduce

$$\mathbf{J}_{(l,q),(\nu,p)}(y) = \mathcal{W}_{(l,q),(\nu,p)}(y) \left(\frac{\sqrt{8(2l+1)}}{(2\pi)^{\Re\nu}} \binom{2l}{l-q}^{\frac{1}{2}} \binom{2l}{l-p}^{-\frac{1}{2}} \sqrt{\left| \frac{\Gamma(l+1+\nu)}{\Gamma(l+1-\nu)} \right|} \right)^{-1}, \tag{4.9}$$

the unnormalized Whittaker function appearing in [13, Section 5] and [45, Section 4.1]; our function $\mathbf{J}_{(l,q),(\nu,p)}(y)$ is the same as $\mathbf{J}_1\varphi_{l,q}(\nu,p)(a(y))$ in [45]. The advantage of this unnormalized function is its regularity in ν. Note that $\mathbf{J}_{(l,q),(\nu,p)}$ is nothing else but (2.21) without its first line.

Lemma 4.4. *For $0<|y|\le 1$ and $\varepsilon>0$,*

$$\mathcal{W}_{(l,q),(\nu,p)}(y) \ll_\varepsilon |y|^{1-|\Re\nu|-\varepsilon}(1+|p|+l)^{1+|p|/2}. \tag{4.10}$$

For $|y| \ge (l^4+1)(|\nu|^2+1)$,

$$\mathcal{W}_{(l,q),(\nu,p)}(y) \ll \exp\left(-\frac{|y|}{|\nu|+l+1} \right). \tag{4.11}$$

Proof. It is clear from the definition and the fact $|\Re\nu| \le 7/32$ that

$$\mathcal{W}_{(l,q),(\nu,p)}(y) \ll \mathbf{J}_{(l,q),(\nu,p)}(y)(1+l) \binom{2l}{l-q}^{\frac{1}{2}} \binom{2l}{l-p}^{-\frac{1}{2}} \ll \mathbf{J}_{(l,q),(\nu,p)}(y)(1+|p|+l)^{1+|p|/2},$$

since

$$\binom{2l}{l-q}^{\frac{1}{2}} \binom{2l}{l-p}^{-\frac{1}{2}} = \left(\frac{(l-p)!(l+p)!}{(l-q)!(l+q)!} \right)^{1/2} \le \left(\frac{(l+|p|)!}{l!} \right)^{1/2} \le (1+|p|+l)^{|p|/2}.$$

Together with [45, (4.28)], this shows the bound (4.10). As for (4.11), take $|y| \ge (l^4+1)(|\nu|^2+1) \ge 1$. We first estimate $\mathbf{J}_{(l,q),(\nu,p)}$ from its expression in terms of K-Bessel functions (recall (2.22) and (2.23)). We estimate the contribution of the binomial factor trivially:

$$\xi_p^l(q,k) \le \binom{2l}{l}^2 \binom{l}{\lfloor l/2 \rfloor} \le 32^l \ll e^{|y|/(3(|\nu|+l+1))},$$

since

$$\frac{|y|}{3(|\nu|+l+1)} \ge \frac{|y|}{3(|\nu|+1)(l+1)} \gg l^3+1.$$

Also trivially $C|y|^{1/4} > l+1$ and $|y|/(3(|\nu|+l+1)) > y^{1/2}/C$ with some absolute constant C, therefore

$$(2\pi|y|)^{l+1-k} \le (2\pi|y|)^{l+1} \le (2\pi|y|)^{C|y|^{1/4}} \ll e^{y^{1/2}/C} \le e^{|y|/(3(|\nu|+l+1))}.$$

At this point, we record that the summation over k and transition factor from $\mathbf{J}_{(l,q),(\nu,p)}$ to $\mathcal{W}_{(l,q),(\nu,p)}$ is also estimated similarly, since

$$(1+l)(1+|p|+l)^{1+|p|/2} \ll (2l+1)^{l/2+2} \ll (2\pi|y|)^{l+1} \ll e^{|y|/(3(|\nu|+l+1))}.$$

Now we would like to estimate

$$\frac{K_{\nu+l-|q+p|-k}(4\pi|y|)}{\Gamma(l+1+\nu-k)},$$

where $0 \le k \le l - \max(|p|,|q|)$. Instead of this, we may write

$$\frac{K_{\nu+a}(4\pi|y|)}{\Gamma(b+1+\nu)},$$

where $0 \le a \le b \le l$: in the principal series $\Re\nu = 0$, this is justified by $K_s(x) = K_{-s}(x)$ (see [58, 3.7(6)]) and $|\Gamma(x)| = |\Gamma(\overline{x})|$, hence take $b = l-k$, then $a = |l-k-|q+p||$ (and we conjugate ν, if $l-k < |q+p|$), $0 \le a \le b \le l$ follows from the constraint on k; while in the complementary series, $p=0$ implies $l-|q+p|-k \ge 0$, from which $0 \le a \le b \le l$ is satisfied by setting $b = l-k$, $a = l-k-|q+p|$. By Basset's integral [58, §6.16],

$$\frac{K_{\nu+a}(4\pi|y|)}{\Gamma(b+1+\nu)} = \frac{\Gamma(\nu+a+1/2)}{\Gamma(\nu+b+1)} \frac{1}{2\sqrt{\pi}(2\pi|y|)^{\nu+a}} \int_{-\infty}^{\infty} \frac{e^{-i4\pi|y|t}}{(1+t^2)^{\nu+a+1/2}} dt.$$

From Stirling's formula, we see that the quotient of the Γ-factors is $O(1)$. As for the rest, integrating by parts, then shifting the contour to $\Im t = -(|\nu|+a+2)^{-1}$ (similarly as in [14, (4.2-5)]),

$$\begin{aligned}
\frac{1}{2\sqrt{\pi}(2\pi|y|)^{\nu+a}} \int_{-\infty}^{\infty} \frac{e^{-i4\pi|y|t}}{(1+t^2)^{\nu+a+1/2}} dt &\ll \frac{|\nu|+a+1}{|y|^{\nu+a-1}} \int_{-\infty}^{\infty} \frac{te^{-i4\pi|y|t}}{(1+t^2)^{\nu+a+3/2}} dt \\
&= \frac{|\nu|+a+1}{|y|^{\nu+a-1}} \int_{-i(|\nu|+a+2)^{-1}-\infty}^{-i(|\nu|+a+2)^{-1}+\infty} \frac{te^{-i4\pi|y|t}}{(1+t^2)^{\nu+a+3/2}} dt \\
&\ll \frac{|\nu|+a+1}{|y|^{\nu+a-1}} \exp\left(\frac{-(3+1/3)\pi|y|}{|\nu|+a+1}\right).
\end{aligned}$$

Here, $|\nu|+a+1 \ll |y|^{1/2}$, so as above,

$$|\nu|+a+1 \ll e^{|y|/(|\nu|+a+1)}, \qquad |y|^{-\nu-a+1} \ll e^{|y|/(3(|\nu|+a+1))},$$

giving

$$\frac{K_{\nu+l-|q+p|-j}(4\pi|y|)}{\Gamma(l+1+\nu-j)} \ll \exp\left(-\frac{2|y|}{|\nu|+l+1}\right).$$

Altogether

$$\mathcal{W}_{(l,q),(\nu,p)}(y) \ll \exp\left(-\frac{|y|}{|\nu|+l+1}\right)$$

as claimed. □

Now borrowing an idea from [4, p.330], we give a further bound on $\mathcal{W}_{(l,q),(\nu,p)}$.

Lemma 4.5. *For all $y \in \mathbf{C}^\times$,*

$$\mathcal{W}_{(l,q),(\nu,p)}(y) \ll (|y|^{3/4}+|y|)(l^4+1)(|\nu|^2+1)(|p|+1). \tag{4.12}$$

Proof. Our starting point is a special Jacquet-Langlands functional equation

$$\mathcal{W}_{(l,q),(\nu,p)}(y) = (-1)^{l+q}\pi \int_{\mathbf{C}^\times} j(\sqrt{t})\mathcal{W}_{(l,-q),(\nu,p)}(t/y) d^\times_{\mathbf{C}} t, \tag{4.13}$$

where j is defined in (4.2). This is proved in [12, Theorem 2 and (3)] in a different formulation, one is straight-forward from the other using [45, (2.30), (2.43) and (4.2)]. Note that in [12], it is stated only for the principal series, but it extends to the complementary series by analytic continuation, using the regularity of $\mathbf{J}_{(l,q),(\nu,p)}$ in ν and then the homogenity of (4.13). Also note that $j(\sqrt{t})$ does not lead to confusion, since $j(t)$ is an even function of t (by Lemma 4.2).

In (4.13), split up the integral as

$$\mathcal{W}_{(l,q),(\nu,p)}(y) \ll \overbrace{\int_{0<|t|<|y|(l^4+1)(|\nu|^2+1)} j(\sqrt{t})\mathcal{W}_{(l,-q),(\nu,p)}(t/y)d_{\mathbf{C}}^{\times}t}^{\mathrm{I}} + \overbrace{\int_{|t|\geq|y|(l^4+1)(|\nu|^2+1)} j(\sqrt{t})\mathcal{W}_{(l,-q),(\nu,p)}(t/y)d_{\mathbf{C}}^{\times}t}^{\mathrm{II}}.$$

First estimate I. Using Cauchy-Schwarz,

$$\mathrm{I} \ll \left(\int_{0<|t|<|y|(l^4+1)(|\nu|^2+1)} |j(\sqrt{t})|^2 d_{\mathbf{C}}^{\times}t\right)^{1/2} \left(\int_{0<|t|<|y|(l^4+1)(|\nu|^2+1)} |\mathcal{W}_{(l,-q),(\nu,p)}(t/y)|^2 d_{\mathbf{C}}^{\times}t\right)^{1/2}.$$

The second factor is at most 1, since the Whittaker functions have L^2-norm 1 (recall (2.24) and the remark after that). In the first factor, we may apply (4.3):

$$\begin{aligned}\mathrm{I} &\ll \left(\int_{0<|t|<|y|(l^4+1)(|\nu|^2+1)} |j(\sqrt{t})|^2 d_{\mathbf{C}}^{\times}t\right)^{1/2}\\ &\ll \left((1+p^2)\int_0^{|y|(l^4+1)(|\nu|^2+1)} r^2(1+r^{-1/2})d_{\mathbf{R}}^{\times}r\right)^{1/2}\\ &\ll \max(|y|,|y|^{3/4})(l^4+1)(|\nu|^2+1)(|p|+1).\end{aligned}$$

In the second term II, we apply Lemma 4.4 together with (4.3). As above,

$$\begin{aligned}\mathrm{II} &\ll (1+|p|)\int_{|t|>|y|(l^4+1)(|\nu|^2+1)} |t|(1+|t|^{-1/4})\exp\left(-\frac{|t|}{|y|(|\nu|+l+1)}\right)d_{\mathbf{C}}^{\times}t\\ &\ll (1+|p|)\int_{|y|(l^4+1)(|\nu|^2+1)}^{\infty} r(1+r^{-1/4})\exp\left(-\frac{r}{|y|(|\nu|+l+1)}\right)d_{\mathbf{R}}^{\times}r\\ &\ll (1+|p|)\int_{1/2}^{\infty}(|y|+|y|^{3/4})(|\nu|+l+1)s\exp(-s)d_{\mathbf{R}}^{\times}s\\ &\ll \max(|y|,|y|^{3/4})(|\nu|+l+1)(|p|+1)\end{aligned}$$

with the change of variable $r = s|y|(|\nu|+l+1)$. Summing up, we arrive at (4.12). □

From (2.24), we know that the square-integral of a Whittaker function is 1. The next lemma encapsulates the fact that a Whittaker function cannot concentrate to a neighborhood of 0 or ∞. To formulate it properly, we introduce the notation, for any $a \in \mathbf{R}^{r+s}$,

$$S(a) = \left\{ y = (y_1, \dots, y_{r+s}) \in \mathbf{R}^{r+s} : \left\{ \begin{array}{ll} |y_j| > |a_j|, & \text{for all } j \leq r, \\ y_j > |a_j|, & \text{for all } j > r \end{array} \right\} \right\}.$$

Lemma 4.6. *There exist some positive constants C_0, C_1 depending only on F and $\mathbf{r}$ with the following property. For any $t \in F_\infty^\times$ and $\mathbf{w} \in W(\pi)$ (where π is an automorphic representation with spectral parameter $\mathbf{r}$), we have*

$$\int_{S(\varepsilon/t)} |\mathcal{W}_{\mathbf{w},\mathbf{r}}(ty)|^2 \frac{dy}{\prod_{j\leq r}|y_j|^2\prod_{j>r}|y_j|^3} > C_1|t|_\infty \left(\prod_{j\leq r}(1+q_j^2)\prod_{j>r}(1+l_j^4)\right)^{-1}, \tag{4.14}$$

if ε is chosen such that its archimedan images satisfy $\varepsilon_j \leq C_0(1+q_j^8)^{-1}$ at real, and $\varepsilon_j \leq C_0(1+l_j^{16})^{-1}$ at complex places.

Proof. Observe that the integral on the left-hand side of (4.14) can be written as

$$|t|_\infty \int_{S(\varepsilon)} |\mathcal{W}_{\mathbf{w},\mathbf{r}}(y)|^2 \frac{dy}{\prod_{j\le r}|y_j|^2\prod_{j>r}|y_j|^3},$$

so we are left to estimate this. By (2.24), we have a positive constant A depending only on F such that

$$\int_{S(0)} |\mathcal{W}_{\mathbf{w},\mathbf{r}}(y)|^2 \frac{dy}{\prod_{j\le r}|y_j|\prod_{j>r}|y_j|^2} = A.$$

Now observe that by (4.7), (4.8) and (4.12), for all $0<\varepsilon<1$,

$$\left(\int_{-\varepsilon}^0 + \int_0^\varepsilon\right) |\mathcal{W}_{q,\nu}(y)|^2 \frac{dy}{|y|} \ll_{F,\nu} \varepsilon^{1/2}(1+q^4), \qquad \int_0^\varepsilon |\mathcal{W}_{(l,q),(\nu,p)}(y)|^2 \frac{dy}{|y|^2} \ll_{F,\nu,p} \varepsilon^{1/2}(1+l^8)$$

at real and complex places, respectively (in the real case, use also that $|\Re\nu|\le 7/64$). Also by (4.6) and (4.11),

$$\left(\int_{-\infty}^{-B(1+q^2)} + \int_{B(1+q^2)}^\infty\right) |\mathcal{W}_{q,\nu}(y)|^2 \frac{dy}{|y|} < \frac{1}{2(r+s)}, \qquad \int_{B(1+l^4)}^\infty |\mathcal{W}_{(l,q),(\nu,p)}(y)|^2 \frac{dy}{|y|^2} < \frac{1}{2(r+s)}$$

for some positive constant B depending on F and $\mathbf{r}$. Altogether,

$$\int_{\substack{y\in S(\varepsilon)\\ |y_j|<A(1+q_j^2)\ (j\le r)\\ y_j<A(1+l_j^4)\ (j>r)}} |\mathcal{W}_{\mathbf{w},\mathbf{r}}(y)|^2 \frac{dy}{\prod_{j\le r}|y_j|\prod_{j>r}|y_j|^2} > C_1 A^{r+s}.$$

with some positive number C_1 (depending only on F and $\mathbf{r}$), if ε is small enough (as in the statement, with an appropriate C_0). From this, the statement is obvious. □

4.3 Transition between adelic and classical functions

In this section, we match the adelic automorphic functions with classical ones. By a classical automorphic function with respect to $\Gamma(\mathfrak{n},\mathfrak{c})$, we mean a funtion $f:\mathrm{GL}_2(F_\infty)\to\mathbf{C}$, which is left $\Gamma(\mathfrak{n},\mathfrak{c})Z(F_\infty)$-invariant, where $0\ne\mathfrak{n},\mathfrak{c}\subseteq\mathfrak{o}$. We borrow the transition from [5, Section 2.12] and also slightly generalize it.

For $m\in\mathrm{GL}_2(\mathbf{A}_{\mathrm{fin}})$, introduce the notation $K_m(\mathfrak{c})=m^{-1}K(\mathfrak{c})m$. Given the ideals $\mathfrak{n},\mathfrak{c}$ as above, let $\eta\in\mathbf{A}_{\mathrm{fin}}^\times$ be a finite idele representing $\mathfrak{n}$. It is easy to check that

$$\Gamma(\mathfrak{n},\mathfrak{c})Z(F_\infty)g\mapsto \mathrm{GL}_2(F)Z(F_\infty)g\begin{pmatrix}\eta^{-1}&0\\0&1\end{pmatrix}mK_m(\mathfrak{c}), \qquad g\in\mathrm{GL}_2(F_\infty)$$

gives an embedding

$$\Gamma(\mathfrak{n},\mathfrak{c})Z(F_\infty)\backslash\mathrm{GL}_2(F_\infty)\hookrightarrow \mathrm{GL}_2(F)Z(F_\infty)\backslash\mathrm{GL}_2(\mathbf{A})/K_m(\mathfrak{c}).$$

Using now strong approximation [15, Theorem 3.3.1], and taking ideal class representatives $\mathfrak{n}_1,\dots,\mathfrak{n}_h$, we obtain a decomposition

$$\mathrm{GL}_2(F)Z(F_\infty)\backslash\mathrm{GL}_2(\mathbf{A})/K_m(\mathfrak{c})\cong\coprod_{j=1}^h\Gamma(\mathfrak{n}_j,\mathfrak{c})Z(F_\infty)\backslash\mathrm{GL}_2(F_\infty) \tag{4.15}$$

for each pair $m\in\mathrm{GL}_2(\mathbf{A}_{\mathrm{fin}}),\mathfrak{c}\subseteq\mathfrak{o}$.

Lemma 4.7. *Using the measures induced by those we defined earlier, for any Borel set U in the decomposition (4.15),*

$$\mathrm{measure}_{\mathrm{LHS}}(U)=[K(\mathfrak{o}):K(\mathfrak{c})]^{-1}\mathrm{measure}_{\mathrm{RHS}}(U).$$

Proof. See [5, p.34]. □

4.4 A bound on the supremum norm of a cusp form

The aim of this section is to give a bound of the form $||\phi||_{\sup} \ll_{F,\pi} ||\phi||_{S_d}$, where ϕ is a sufficiently smooth newform in the cuspidal representation π, and the order d depends only on F. We need some preparatory lemmas.

Lemma 4.8. *Assume (π, V_π) is an irreducible cuspidal representation in $L^2(\mathrm{GL}_2(F)Z(\mathbf{A})\backslash\mathrm{GL}_2(\mathbf{A}))$, and $\phi \in V_\pi$ is of pure weight $\mathbf{w}$. Then for any $k_\infty \in K_\infty$ and $g \in \mathrm{GL}_2(\mathbf{A})$*

$$|\phi(gk_\infty)| \ll_F |\phi(g)| \prod_{j=r+1}^{r+s} (l_j+1)^7.$$

Proof. We may assume $||\phi|| = 1$. First observe that $\phi'(g) = \phi(gk_\infty)$ is in the same irreducible representation of K_∞ as ϕ, therefore, we may write

$$\phi'(g) = \phi(gk_\infty) = \phi(g) \sum_{|q_{r+1}|\le l_{r+q},\dots,|q_{r+s}|\le l_{r+s}} \alpha(g; q_1, \dots, q_{r+s}) \prod_{j=1}^{r} \Phi_{q_j}(k_j) \prod_{j=r+1}^{r+s} \frac{\Phi^{l_j}_{p_j,q_j}(k_j)}{||\Phi^{l_j}_{p_j,q_j}||_{\mathrm{SU}_2(\mathbf{C})}},$$

where for each g,

$$\sum_{|q_{r+1}|\le l_{r+q},\dots,|q_{r+s}|\le l_{r+s}} |\alpha(g; q_1, \dots, q_{r+s})|^2 = 1,$$

in particular, each $|\alpha(g; q_1, \dots, q_{r+s})| \le 1$. Since the sum has $\ll_F \prod_{j=r+1}^{r+s}(l_j+1)$ terms, it suffices to prove

$$\frac{|\Phi^l_{p,q}(k)|}{||\Phi^l_{p,q}||_{\mathrm{SU}_2(\mathbf{C})}} \ll (l+1)^6.$$

This follows from [2, Lemma on p.348 and Corollary on p.349] with $n = 4$ by the standard quaternion representation of $\mathrm{SU}_2(\mathbf{C})$. Each derivation gives a factor $\ll (l+1)^{3/2}$, see [45, (2.19), (2.31)]. □

Lemma 4.9. *Let $N = 2^r h$, where h is the class number of F. There are finitely many elements $a_1, \dots, a_N \in \mathrm{GL}_2(F)$ regarded as elements of $\mathrm{GL}_2(F_\infty)$ and some $\delta > 0$ such that for any $g \in \mathrm{GL}_2(F_\infty)$, there exist elements $z \in Z(F_\infty)$, $\gamma \in \mathrm{SL}_2(\mathfrak{o})$ (regarded as an element of $\mathrm{GL}_2(F_\infty)$) and $k \in K_\infty$ such that*

$$g = z\gamma a_j \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k,$$

for some $1 \le j \le N$, where $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in B(F_\infty)$ satisfies $y_1, \dots, y_{r+s} > \delta$ (in particular, all of them are real).

Proof. It is proved in [26, Theorem 3.6] that there are finitely many elements $b_1, \dots, b_h \in \mathrm{SL}_2(F)$ and some $\delta' > 0$ such that for any $g \in \mathrm{SL}_2(F_\infty)$, there exist elements $z \in Z(F_\infty)$, $\gamma \in \mathrm{SL}_2(\mathfrak{o})$ and $k \in K_\infty$ such that in $\mathrm{GL}_2(F_\infty)$,

$$g = z\gamma b_j \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k,$$

for some $1 \le j \le h$, where $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in B(F_\infty)$ satisfies $y_1, \dots, y_{r+s} > \delta'$. Note that [26] works with totally real fields, the general case is straight-forward using the same technique.

Therefore, we are left to work out the transition from SL_2 to GL_2. Take diagonal matrices $s_1, \dots, s_{2^r} \in \mathrm{GL}_2(F)$ (regarded as matrices in $\mathrm{GL}_2(F_\infty)$) with lower-right entry 1, upper-left entry u_j ($1 \le j \le 2^r$), such that for any given sign $e = (\pm 1)^r$, there is a j such that $\mathrm{sign}(u_j) = e$. Then let $g \in \mathrm{GL}_2(F_\infty)$ be given. We can assume that its determinant is 1 at all complex places and it is ± 1 at all real places. Denote by s the matrix which is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ at those places where $\det g$ is 1, and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where $\det g$ is -1. Now by the result for SL_2, in $\mathrm{GL}_2(F_\infty)$,

$$gs = z\gamma b_j \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k$$

for some $1 \le j \le h$, where $z \in Z(F_\infty)$, $\gamma \in \mathrm{SL}_2(\mathfrak{o})$, $y_1 \dots, y_{r+s} > \delta'$, $x \in F_\infty$, $k \in K_\infty$. Then

$$g = z\gamma b_j ss \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} ssks = z\gamma b_j s \begin{pmatrix} y & \mathrm{sign}(\det s)x \\ 0 & 1 \end{pmatrix} k',$$

with $k' = sks \in K_\infty$. This is almost appropriate, the only problem is that s is not necessarily in $\mathrm{GL}_2(F)$. To remedy this, we take the s_i that admits the same sign as s, write $b_j s = b_j s_i s_i^{-1} s$, and observe that

$$s_i^{-1} s \begin{pmatrix} y & \mathrm{sign}(\det s) x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} |u_i|^{-1} y & \mathrm{sign}(\det s)|u_i|^{-1} x \\ 0 & 1 \end{pmatrix}.$$

This shows that the collection $(b_j s_i)_{i,j}$ does the job with $\delta = \delta' \min_i(|u_i|^{-1}) > 0$. □

Proposition 4.10. *Let (π, V_π) be an irreducible cuspidal representation in $L^2(\mathrm{GL}_2(F)Z(\mathbf{A})\backslash \mathrm{GL}_2(\mathbf{A}))$. Assume $\phi \in V_\pi(\mathfrak{c}_\pi)$ such that $||\phi||_{S_{2(7r+18s)}}$ exists. Then*

$$||\phi||_\infty = \sup_{g \in \mathrm{GL}_2(\mathbf{A})} |\phi(g)| \ll_{F,\pi} ||\phi||_{S_{2(7r+18s)}}.$$

Proof. We follow the proof of [5, Lemma 5]. Note that there is a correction made later in its erratum, which we also incorporate. First assume $\phi \in V_\pi(\mathfrak{c}_\pi)$ is of pure weight $\mathbf{w}$. Let $\eta_1, \dots, \eta_h \in \mathbf{A}_{\mathrm{fin}}^\times$ be finite ideles representing the ideal classes. By strong approximation [15, Theorem 3.3.1], there exist $\gamma \in \mathrm{GL}_2(F)$, $g' \in \mathrm{GL}_2(F_\infty)$, $k \in K(\mathfrak{o})$ such that for some $1 \le j \le h$,

$$g = \gamma \left(\overbrace{g'}^{\in \mathrm{GL}_2(F_\infty)} \times \overbrace{\begin{pmatrix} \eta_j^{-1} & 0 \\ 0 & 1 \end{pmatrix} k}^{\in \mathrm{GL}_2(\mathbf{A}_{\mathrm{fin}})} \right).$$

Now decompose g' in the sense of Lemma 4.9 as

$$g' = z\gamma' a_{j'} \begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} k',$$

where $a_{j'} \in \mathrm{GL}_2(F)$ (regarded as an element of $\mathrm{GL}_2(F_\infty)$) is from the fixed set $\{a_1, \dots, a_{2^r h}\}$, $y' > \delta$ at all archimedean places, where $\delta > 0$ is fixed (depending only on F), $z \in Z(F_\infty)$, $\gamma' \in \mathrm{SL}_2(\mathfrak{o})$, $k' \in K_\infty$. From now on, we regard z as an element in $Z(\mathbf{A})$, therefore we have

$$g = z\gamma\gamma' a_{j'} \left(\overbrace{\begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} k'}^{\in \mathrm{GL}_2(F_\infty)} \times \overbrace{a_{j'}^{-1} \gamma'^{-1} \begin{pmatrix} \eta_j^{-1} & 0 \\ 0 & 1 \end{pmatrix} k}^{\in \mathrm{GL}_2(\mathbf{A}_{\mathrm{fin}})} \right).$$

Here, $a_{j'}^{-1}\gamma'^{-1}\left(\begin{smallmatrix} \eta_j^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)k$ lies in a fixed compact subset of $\mathrm{GL}_2(\mathbf{A}_{\mathrm{fin}})$, which can be covered with finitely many left cosets of the open subgroup $K(\mathfrak{c}_\pi)$. Therefore

$$g = z\gamma^* \left(\overbrace{\begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix}}^{\in \mathrm{GL}_2(F_\infty)} \times \overbrace{m}^{\in \mathrm{GL}_2(\mathbf{A}_{\mathrm{fin}})} \right) \left(\overbrace{k_\infty^*}^{\in \mathrm{GL}_2(F_\infty)} \times \overbrace{k_{\mathrm{fin}}^*}^{\mathrm{GL}_2(\mathbf{A}_{\mathrm{fin}})} \right),$$

where $\gamma^* \in \mathrm{GL}_2(F)$, $k^* = k_\infty^* \times k_{\mathrm{fin}}^* \in K_\infty \times K(\mathfrak{c}_\pi)$, and $m \in \mathrm{GL}_2(\mathbf{A}_{\mathrm{fin}})$ runs through a finite set depending only on F and $\mathfrak{c}_\pi$, $y' > \delta$ at all archimedean places.

Now let $\phi_m(g) = \phi(gm)$. Obviously, ϕ and ϕ_m have the same supremum and Sobolev norms, and when g decomposes as above,

$$|\phi(g)| = \left| \phi_m \left(\overbrace{\begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} k_\infty^*}^{\in \mathrm{GL}_2(F_\infty)} \times \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{\in \mathrm{GL}_2(\mathbf{A}_{\mathrm{fin}})} \right) \right| \ll_F \left| \phi_m \left(\overbrace{\begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix}}^{\in \mathrm{GL}_2(F_\infty)} \times \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{\in \mathrm{GL}_2(\mathbf{A}_{\mathrm{fin}})} \right) \right| \prod_{j=r+1}^{r+s} (l_j + 1)^7, \quad (4.16)$$

where we applied Lemma 4.8 in the last estimate.

It is easy to check that ϕ_m is right $K_{m^{-1}}(\mathfrak{c}_\pi)$-invariant, so we may apply the adelic-classical transition. It implies that there is a fractional ideal $\mathfrak{f}$ (regarded as a lattice in F_∞) depending only on F and π, such that if $x \in \mathfrak{f}'$ (the dual of $\mathfrak{f}$),

$$\phi_m\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y' & x'\\ 0 & 1\end{pmatrix}\right) = \phi_m\left(\begin{pmatrix}y' & x'\\ 0 & 1\end{pmatrix}\right),$$

where we did not indicate the finite part.

Therefore, analogously to (2.25), we see that ϕ_m can be expanded into Fourier series

$$\phi_m\left(\begin{pmatrix}y' & x'\\ 0 & 1\end{pmatrix}\right) = \sum_{0\neq t\in\mathfrak{f}} a(t)\mathcal{W}_{\mathbf{w},\mathbf{r}}(ty')\psi_\infty(tx'). \tag{4.17}$$

We need some bound on the Fourier-Whittaker coefficients, which we work out in the following lemma.

Lemma 4.11.

$$a(t) \ll_{F,\pi} ||\phi|| \prod_{j\le r}(1+|q_j|^5)\prod_{j>r}(1+l_j^{10}).$$

Proof. By Plancherel's formula,

$$\sum_{0\neq t\in\mathfrak{f}} |a(t)\mathcal{W}_{\mathbf{w},\mathbf{r}}(ty')|^2 = \text{const.}(F,\pi)\int_{F_\infty/\mathfrak{f}'}\left|\phi_m\left(\begin{pmatrix}y' & x'\\ 0 & 1\end{pmatrix}\right)\right|^2 dx'.$$

Take only a single term on the left-hand side. Choose C_0 as in Lemma 4.6 and then take ε to be the largest which is allowed there. Integrate both sides on the domain $S(\varepsilon/t)$ with respect to the measure $dy'/(\prod_{j\le r}|y_j'|^2\prod_{j>r}|y_j'|^3)$ (note that $dx'dy'/(\prod_{j\le r}|y_j'|^2\prod_{j>r}|y_j'|^3)$ is the invariant measure on the symmetric space $\mathrm{GL}_2(F_\infty)/K_\infty$). By Lemma 4.6, we obtain

$$|a(t)|^2|t|_\infty\left(\prod_{j\le r}(1+q_j^2)\prod_{j>r}(1+l_j^4)\right)^{-1} \ll_{F,\pi} \int_{F_\infty/\mathfrak{f}'\times S(\varepsilon/t)}\left|\phi_m\left(\begin{pmatrix}y' & x'\\ 0 & 1\end{pmatrix}\right)\right|^2\frac{dx'dy'}{\prod_{j\le r}|y_j'|^2\prod_{j>r}|y_j'|^3}.$$

Here, the Siegel domain $F_\infty/\mathfrak{f}'\times S(\varepsilon/t)$ covers each point of $Z(\mathbf{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A})/K_{m^{-1}}(\mathfrak{c}_\pi)$ at most $O_{F,\pi}(|t/\varepsilon|_\infty)$ times (see [38, Lemma 2.10]), which, together with the choice of ε, gives

$$|a(t)|^2 \ll_{F,\pi} ||\phi||^2\prod_{j\le r}(1+q_j^{10})\prod_{j>r}(1+l_j^{20}),$$

and the claim follows. $\square$

Now (4.16) and (4.17) give

$$|\phi(g)| \ll_{F,\pi} ||\phi||\prod_{j\le r}(1+|q_j|^5)\prod_{j>r}(1+l_j^{17})\sum_{0\neq t\in\mathfrak{f}}|\mathcal{W}_{\mathbf{w},\mathbf{r}}(ty')|. \tag{4.18}$$

We turn our attention to $\sum_{0\neq t\in\mathfrak{f}}|\mathcal{W}_{\mathbf{w},\mathbf{r}}(ty')|$.

The pointwise bounds for local Whittaker functions

$$\begin{aligned}
&\mathcal{W}_{q,\nu}(y) \ll_{F,\pi} |q|^3+1, && \text{if } |y|<(q^2+1)(|\nu|^2+1),\\
&\mathcal{W}_{q,\nu}(y) \ll_{F,\pi} \exp\left(-\tfrac{|y|}{2(q^2+1)(|\nu|^2+1)}\right), && \text{if } |y|\ge(q^2+1)(|\nu|^2+1),\\
&\mathcal{W}_{(l,q),(\nu,p)}(y) \ll_{F,\pi} l^8+1, && \text{if } |y|<(l^4+1)(|\nu|^2+1),\\
&\mathcal{W}_{(l,q),(\nu,p)}(y) \ll_{F,\pi} \exp\left(-\tfrac{|y|}{2(l^4+1)(|\nu|^2+1)}\right), && \text{if } |y|\ge(l^4+1)(|\nu|^2+1)
\end{aligned}$$

follow easily from (4.6), (4.7), (4.8), (4.11) and (4.12). From these, we see that

$$\begin{aligned}
\mathcal{W}_{q,\nu}(y) &\ll_{F,\pi} (|q|^3+1)\exp\left(-\frac{|y|}{2(q^2+1)(|\nu|^2+1)}\right),\\
\mathcal{W}_{(l,q),(\nu,p)}(y) &\ll_{F,\pi} (l^8+1)\exp\left(-\frac{|y|}{2(l^4+1)(|\nu|^2+1)}\right)
\end{aligned} \tag{4.19}$$

holds for all $y \neq 0$, at real and complex places, respectively.

With the notation $A_j = |q_j|^3+1, B_j = 2(q_j^2+1)(|\nu_j|^2+1)$ at real, $A_j = l_j^8+1, B_j = 2(l_j^4+1)(|\nu_j|^2+1)$ at complex places, (4.19) implies that $\mathcal{W}_{\mathbf{w},\mathbf{r}}(y) \ll_{F,\pi} \prod_{j=1}^{r+s} A_j \exp(-|y_j|/B_j)$. Then

$$\begin{aligned}\sum_{0\neq t\in\mathfrak{f}} |\mathcal{W}_{\mathbf{w},\mathbf{r}}(ty')| &\ll_{F,\pi} \prod_{j=1}^{r+s} A_j \sum_{0\neq t\in\mathfrak{f}} \prod_{j=1}^{r+s} \exp(-|t_jy_j'|/B_j) \\ &\leq \prod_{j=1}^{r+s} A_j \sum_{N=1}^{\infty} \sum_{\substack{0\neq t\in\mathfrak{f} \\ (N-1)\leq \max_j(|t_jy_j'|/B_j)<N}} \exp(-\max_j(|t_jy_j'|/B_j)) \\ &\ll_{F,\pi} \prod_{j=1}^{r+s} A_j \sum_{N=1}^{\infty} e^{-N} \prod_{j=1}^{r} NB_j \prod_{j=r+1}^{r+s} N^2B_j^2 \\ &\ll \prod_{j=1}^{r+s} A_j B_j^{\deg[F_j:\mathbf{R}]}.\end{aligned}$$

Here we used that $|y_j'| > \delta$ at all places, and also the fact that a lattice L in F_∞ contains $O_L(N^{r+2s})$ points of supremum norm $\leq N$.

Therefore,

$$\sum_{0\neq t\in\mathfrak{f}} |\mathcal{W}_{\mathbf{w},\mathbf{r}}(ty')| \ll_{F,\pi} \prod_{j=1}^{r}(|q_j|^5+1) \prod_{j=r+1}^{r+s}(l_j^{16}+1),$$

which, together with (4.18), give rise to

$$|\phi(g)| \ll_{F,\pi} ||\phi|| \prod_{j\leq r}(1+q_j^{10}) \prod_{j>r}(1+l_j^{33}). \tag{4.20}$$

Assume now a sufficiently smooth $\phi \in V_\pi$ is not necessarily of pure weight. We may decompose it as

$$\phi = \sum_{\mathbf{w}\in W(\pi)} b_{\mathbf{w}}\phi_{\mathbf{w}}, \tag{4.21}$$

where $\phi_{\mathbf{w}}$ is a weight $\mathbf{w}$ function of norm 1 in V_π. Let us follow the common practice and using the smoothness of ϕ, estimate $b_{\mathbf{w}}$ in terms of $\sup \mathbf{w} = \max(|q_1|, \dots, |q_r|, l_{r+1}, \dots, l_{r+s})$. Using Parseval, then (2.11) and (2.12), we find, for any nonnegative integer k,

$$b_{\mathbf{w}} = \langle \phi, \phi_{\mathbf{w}}\rangle \ll_k \frac{1}{(1+(\sup \mathbf{w}))^{2k}} \langle \Omega_{\mathfrak{k},j}^k \phi, \phi_{\mathbf{w}}\rangle \ll_k \frac{1}{(1+(\sup \mathbf{w}))^{2k}} ||\phi||_{S_{2k}}, \tag{4.22}$$

where j is the index of an archimedean place, where the maximum (in the definition of $\sup \mathbf{w}$) is attained. Together with (4.20) and (4.21), this implies

$$\begin{aligned}|\phi(g)| &\ll_{F,\pi,k} \sum_{\mathbf{w}\in W(\pi)} \frac{1}{1+(\sup \mathbf{w})^{2k}} ||\phi||_{S_{2k}} \prod_{j<r}(1+q_j^{10}) \prod_{j\geq r}(1+l_j^{33}) \\ &\ll_{F,\pi,k} \sum_{\mathbf{w}\in W(\pi)} (1+\sup \mathbf{w})^{10r+33s-2k} ||\phi||_{S_{2k}}.\end{aligned}$$

Here, choosing $k = 7r+18s$, we obtain the statement by noting that $\sup \mathbf{w}$ attains the positive integer N on a set of cardinality $O_F(N^{r+2s-1})$. □

4.5 A bound on Kirillov vectors

Proposition 4.12. *Let (π, V_π) be an automorphic representation occuring in $L^2(\mathrm{GL}_2(F)Z(\mathbf{A})\backslash\mathrm{GL}_2(\mathbf{A}))$. Let $\mathfrak{t} \subseteq \mathfrak{o}$ be an ideal, a, b, c be nonnegative integers, $0 < \varepsilon < 1/4$. Let $P \in \mathbf{C}[x_1, \dots, x_{r+2s}]$ be a polynomial of degree at most a in each variable. Set then*

$$\mathcal{D} = P\left(\left(y_j\frac{\partial}{\partial y_j}\right)_{j\leq r}, \left(y_j\frac{\partial}{\partial y_j}\right)_{j>r}, \left(\overline{y_j}\frac{\partial}{\partial \overline{y_j}}\right)_{j>r}\right).$$

Assume $\phi \in R^{\mathfrak{t}}V_\pi(\mathfrak{c}_\pi)$ such that $||\phi||_{S_{2(3r+4s+2)+(r+s)(a+b+2c)}}$ exists. Then $\mathcal{D}W_\phi$ exists and

$$\mathcal{D}W_\phi(y) \ll_{a,b,c,P,F,\varepsilon} ||\phi||_{S_{2(3r+4s+2)+(r+s)(a+b+2c)}} \mathcal{N}(\mathfrak{t})^\varepsilon \mathcal{N}(\mathfrak{c}_\pi)^\varepsilon \mathcal{N}(\mathbf{r})^{-c} \prod_{j=1}^{r}(|y_j|^{1/2-\varepsilon} + |y_j|^{1/2-\theta-\varepsilon})(\min(1,|y_j|^{-b})) \prod_{j=r+1}^{r+s}(|y_j|^{3/4} + |y_j|)(\min(1,|y_j|^{-b})).$$

Proof. We follow the proof of [5, Lemma 4]. First assume $\phi \in R^{\mathfrak{t}}V_\pi(\mathfrak{c}_\pi)$ is of pure weight $\mathbf{w}$. Then we may write

$$|W_\phi(y)| = ||W_\phi|| \cdot |\mathcal{W}_{\mathbf{w},\mathbf{r}}(y)|.$$

Using Proposition 2.2, (2.31), (2.40), (2.42), the remark after that, (3.5), Proposition 3.2, and the estimates (4.7), (4.8), (4.12), we have, for $0 < \varepsilon < 1/4$,

$$W_\phi(y) \ll_{F,\varepsilon} ||\phi|| \mathcal{N}(\mathfrak{t})^\varepsilon \mathcal{N}(\mathfrak{c}_\pi)^\varepsilon \mathcal{N}(\mathbf{r})^\varepsilon \prod_{j=1}^{r}(1+|\nu_j|+|q_j|)^{1+\theta}(|y_j|^{1/2-\varepsilon} + |y_j|^{1/2-\theta-\varepsilon}) \cdot \prod_{j=r+1}^{r+s}(1+|p_j|)(1+|\nu_j|^2)(1+l_j^4)(|y_j|^{3/4}+|y_j|).$$

This gives

$$W_\phi(y) \ll_{F,\varepsilon} ||\phi|| \mathcal{N}(\mathfrak{t})^\varepsilon \mathcal{N}(\mathfrak{c}_\pi)^\varepsilon \mathcal{N}(\mathbf{r})^2 \prod_{j=1}^{r}(1+|q_j|)^{1+\theta}(|y_j|^{1/2-\varepsilon} + |y_j|^{1/2-\theta-\varepsilon}) \cdot \prod_{j=r+1}^{r+s}(1+l_j^4)(|y_j|^{3/4}+|y_j|).$$

Now take an arbitrary $\phi \in R^{\mathfrak{t}}V_\pi(\mathfrak{c}_\pi)$, which is sufficiently smooth. Then recalling (4.21) and (4.22), in

$$\phi = \sum_{\mathbf{w}\in W(\pi)} b_{\mathbf{w}}\phi_{\mathbf{w}}, \qquad ||\phi_{\mathbf{w}}|| = 1,$$

we have

$$b_{\mathbf{w}} \ll_k \frac{1}{(1+(\sup \mathbf{w}))^{2k}} ||\phi||_{S_{2k}}.$$

Now choosing $k = 3r+4s$, we obtain

$$W_\phi(y) \ll_{F,\varepsilon} ||\phi||_{S_{2(3r+4s)}} \mathcal{N}(\mathfrak{t})^\varepsilon \mathcal{N}(\mathfrak{c}_\pi)^\varepsilon \mathcal{N}(\mathbf{r})^2 \prod_{j=1}^{r}(|y_j|^{1/2-\varepsilon} + |y_j|^{1/2-\theta-\varepsilon}) \prod_{j=r+1}^{r+s}(|y_j|^{3/4}+|y_j|). \tag{4.23}$$

The differential operators given in (2.4) and (2.5) act on the sufficiently smooth Kirillov vectors. We record the action of some of them (neglecting some absolute scalars for simplicity). Of course, $\Omega_{j(,\pm)}$ act by $\lambda_{(\pm)}$. From (2.30) and (2.41), it is easy to derive that $\mathbf{R}_j$, $\mathbf{V}_{1,j}+\mathbf{W}_{1,j}$, $\mathbf{V}_{2,j}+\mathbf{W}_{2,j}$ act via a multiplication by y_j, $\Re y_j$, $\Im y_j$, respectively; finally $\mathbf{H}_j$ by $y_j(\partial/\partial y_j)$, and $\mathbf{H}_{1,j}$, $\mathbf{H}_{2,j}$ by $y_j(\partial/\partial y_j) + \overline{y_j}(\partial/\partial\overline{y_j})$, $iy_j(\partial/\partial y_j) - i\overline{y_j}(\partial/\partial\overline{y_j})$, respectively.

Now assume given a, b, c and the polynomial P as in the statement. Then

$$\mathcal{D} = \mathrm{const.}_F P\Big((\mathbf{H}_j)_{j\le r}, ((\mathbf{H}_{1,j} - i\mathbf{H}_{2,j})/2)_{j>r}, ((\mathbf{H}_{1,j} + i\mathbf{H}_{2,j})/2)_{j>r}\Big),$$

and define the differential operator

$$\mathcal{D}' = \left(\prod_{j\le r}\Omega_j^{c+2}\prod_{j>r}\Omega_{j,+}^{c+2}\right)\left(\prod_{\substack{1\le j\le r\\ |y_j|\ge 1}}\mathbf{R}_j^b\right)\left(\prod_{\substack{r+1\le j\le r+s\\ |y_j|\ge 1\\ |\Re y_j|\ge|\Im y_j|}}(\mathbf{V}_{1,j}+\mathbf{W}_{1,j})^b\right)\left(\prod_{\substack{r+1\le j\le r+s\\ |y_j|\ge 1\\ |\Re y_j|<|\Im y_j|}}(\mathbf{V}_{2,j}+\mathbf{W}_{2,j})^b\right).$$

Applying (4.23) to $\mathcal{D}'\mathcal{D}\phi$, we obtain the statement. □

Chapter 5

The density of the spectrum

In this chapter, we estimate the density of the Eisenstein and the cuspidal spectrum in terms of the spectral parameters. These are the extensions of [5, Lemma 2 and Lemma 6]. After the suitable modifications, the proofs given there apply in the more general situation.

First of all, we introduce some notations. Given an ideal $\mathfrak{c}$, let

$$\mathcal{C}(\mathfrak{c}) = \{\pi \in \mathcal{C} \mid \mathfrak{c} \subseteq \mathfrak{c}_\pi\}, \qquad \mathcal{E}(\mathfrak{c}) = \{\chi \in \mathcal{E} \mid \mathfrak{c} \subseteq \mathfrak{c}_{\chi,\chi^{-1}}\}.$$

5.1 Density of the Eisenstein spectrum

Lemma 5.1. *Let $\mathfrak{c}_1^2\mathfrak{c}_2 = \mathfrak{c} \subseteq \mathfrak{o}$, where $\mathfrak{c}_2$ is squarefree. Then for $1 \leq X \in \mathbf{R}$, $1 \leq P \in \mathbf{Z}$,*

$$\int_{\substack{\varpi \in \mathcal{E}(\mathfrak{c}) \\ |\nu_{\varpi,j}| \leq X \\ |p_{\varpi,j}| \leq P}} 1 d\varpi \ll_F X^{r+s} P^s \mathcal{N}(\mathfrak{c}_1).$$

Proof. Any Hecke character χ can be factorized as $\chi = \chi_\infty \chi_{\mathrm{fin}}$. Here, $\chi_{\mathrm{fin}}|_{\prod_\mathfrak{p} \mathfrak{o}_\mathfrak{p}^\times}$ is a character of $\prod_\mathfrak{p} \mathfrak{o}_\mathfrak{p}^\times$. By Proposition 2.3, $\mathfrak{c}_\chi | \mathfrak{c}_1$, so there are at most $\varphi(\mathfrak{c}_1)$ possibilities for this restriction. Assume given a character ξ of $\prod_\mathfrak{p} \mathfrak{o}_\mathfrak{p}^\times$, we estimate the measure of the set S of those Hecke characters χ for which $\chi_{\mathrm{fin}}|_{\prod_\mathfrak{p} \mathfrak{o}_\mathfrak{p}^\times} = \xi$. If $S = \emptyset$, the measure is 0. If $S \neq \emptyset$, fix some $\chi_0 \in S$. Then to any χ in S, associate $\chi' = \chi\chi_0^{-1}$. From the non-archimedan part, we see χ' is trivial on $\prod_\mathfrak{p} \mathfrak{o}_\mathfrak{p}^\times$. From the archimedan part, we see that for $a \in F_{\infty,+}^\times$, $\chi'(a_j) = |a_j|^{t_j}$, if $j \leq r$, and $\chi'(a_j) = |a_j|^{t_j}(a_j/|a_j|)^{p_j}$, if $j > r$, where $t_j \in i[-2X, 2X]$, and $p_j \in [-2P, 2P] \cap \mathbf{Z}$. Fix the vector $(p_j)_{j>r} \in [-2P, 2P]^s \cap \mathbf{Z}^s$.

Now χ'_∞ is trivial on the group U^+ of totally positive units embedded in $F_{\infty,+}^\times$. Fix a generating set $\{u_1, \ldots, u_{r+s-1}\}$ for the torsion-free part of U^+. Then by the notation of [5], take

$$M = \begin{pmatrix} \deg[F_1 : \mathbf{R}] & \ldots & \deg[F_{r+s} : \mathbf{R}] \\ \deg[F_1 : \mathbf{R}] \log|u_{1,1}| & \ldots & \deg[F_{r+s} : \mathbf{R}] \log|u_{1,r+s}| \\ \vdots & & \vdots \\ \deg[F_1 : \mathbf{R}] \log|u_{r+s-1,1}| & \ldots & \deg[F_{r+s} : \mathbf{R}] \log|u_{r+s-1,r+s}| \end{pmatrix} \in \mathbf{R}^{(r+s)\times(r+s)}.$$

Then the column vector $t = (t_j)_j \in i[-2X, 2X]^{r+s}$ with $iT = \sum_j \deg[F_j : \mathbf{R}] t_j$ satisfies $Mt \in i\{T\} \times (2\pi i \mathbf{Z})^{r+s-1}$. Using that M is invertible and depends only on F, we see

$$\int_{-2(r+2s)X}^{2(r+2s)X} \#((\{T\} \times (2\pi i \mathbf{Z})^{r+s-1}) \cap Mi[-2X, 2X]^{r+s}) dT \ll_F X^{r+s},$$

since the integrand is $O_F(X^{r+s-1})$. Taking into account the finiteness of the torsion subgroup of U^+ and of $F^\times F_{\infty,+}^\times \prod_\mathfrak{p} \mathfrak{o}_\mathfrak{p}^\times \backslash \mathbf{A}^\times$, finally summing over $(p_j)_{j>r} \in [-2P, 2P]^s$, we obtain the statement. $\square$

Lemma 5.2. *Let $\mathfrak{c}_1^2\mathfrak{c}_2 = \mathfrak{c} \subseteq \mathfrak{o}$, where $\mathfrak{c}_2$ is squarefree. Then for $1 \leq X \in \mathbf{R}$,*

$$\int_{\substack{\varpi\in\mathcal{E}(\mathfrak{c})\\ j\leq r:|\nu_{\varpi,j}|\leq X\\ j>r:|\nu_{\varpi,j}^2-p_{\varpi,j}^2|\leq X^2}} 1 d\varpi \ll_F X^{r+2s}\mathcal{N}(\mathfrak{c}_1).$$

Proof. Set $P = X$ in the previous lemma. □

5.2 Density of the cuspidal spectrum

In the estimate of the density of the cuspidal spectrum, we use the Kuznetsov formula (see [46, Theorem 1] or Theorem 3 in Chapter 8). In our notation, for a weight function h of the form described below,

$$[K(\mathfrak{o}) : K(\mathfrak{c})]^{-1} \sum_{\pi\in\mathcal{C}(\mathfrak{c})} C_\pi^{-1} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}} h(\mathbf{r}_\pi)\lambda_\pi^{\mathfrak{t}}(\alpha\mathfrak{a}^{-1})\overline{\lambda_\pi^{\mathfrak{t}}(\alpha'\mathfrak{a}'^{-1})} + CSC =$$

$$\text{const.}_F \Delta(\alpha\mathfrak{a}^{-1}, \alpha'\mathfrak{a}'^{-1}) \int h(\mathbf{r})d\mu +$$

$$\text{const.}_F \sum_{\mathfrak{m}\in C} \sum_{c\in\mathfrak{a}\mathfrak{m}\mathfrak{c}} \sum_{\epsilon\in\mathfrak{o}_+^\times/\mathfrak{o}^{2\times}} \frac{KS(\epsilon\alpha, \mathfrak{a}^{-1}\mathfrak{d}^{-1}; \alpha'\gamma_\mathfrak{m}, \mathfrak{a}'^{-1}\mathfrak{d}^{-1}; c, \mathfrak{a}^{-1}\mathfrak{m}^{-1}\mathfrak{d}^{-1})}{\mathcal{N}(c\mathfrak{a}^{-1}\mathfrak{m}^{-1})} \int \mathcal{B}h_{(\mathbf{r})}\left(4\pi\frac{(\alpha\alpha'\gamma_\mathfrak{m}\epsilon)^{\frac{1}{2}}}{c}\right) d\mu, \tag{5.1}$$

where KS is a Kloosterman sum, $\mathcal{B}$ is a certain Bessel function, and $d\mu$ is a certain measure of the space of the spectral parameters $\mathbf{r}$. We explain the notation and the conditions: $\mathfrak{a}^{-1}$ and $\mathfrak{a}'^{-1}$ are nonzero fractional ideals; $\alpha \in \mathfrak{a}, \alpha \in \mathfrak{a}'$ such that $\alpha\alpha'$ is totally positive; C is a fixed set of narrow ideal class representatives $\mathfrak{m}$, for which $\mathfrak{m}^2\mathfrak{a}\mathfrak{a}'^{-1}$ is a principal ideal generated by a totally positive element $\gamma_\mathfrak{m}$; $\Delta(\alpha\mathfrak{a}^{-1}, \alpha'\mathfrak{a}'^{-1})$ is 1 if $\alpha\mathfrak{a}^{-1} = \alpha'\mathfrak{a}'^{-1}$, otherwise it is 0; CSC is an analogous integral over the Eisenstein spectrum. For the sake of completeness, we will discuss the details in Chapter 8.

The weight function h we will use is of the form $h = \prod_j h_j$, where h_j's are defined as follows. Let $a_j, b_j > 1, a'_j \in \mathbf{R}$ be given. Then at real places

$$h_j(\nu_j) = \begin{cases} e^{(\nu_j^2-\frac{1}{4})/a_j}, & \text{if } |\Re\nu_j| < \frac{2}{3}, \\ 1, & \text{if } \nu_j \in \frac{1}{2}+\mathbf{Z}, \frac{3}{2} \leq |\nu_j| \leq b_j, \\ 0 & \text{otherwise,} \end{cases} \tag{5.2}$$

while at complex places

$$h_j(\nu_j, p_j) = \begin{cases} e^{(\nu_j^2+a'_jp_j^2-1)/a_j}, & \text{if } |\Re\nu_j| < \frac{2}{3}, p_j \in \mathbf{Z}, |p_j| \leq b_j, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

In the estimate of the density of the cuspidal spectrum, we choose our parameters as follows. At each place, $a_j > 1$ is arbitrary, then set $b_j = \sqrt{a_j}$. Furthermore, at complex places, we use $a'_j = -1$. In this setup, we have the following lemma about the integrals on the geometric side.

Lemma 5.3. *At real places,*

$$\int h_j(\nu_j)d\mu_j \ll a_j, \qquad \int (\mathcal{B}_jh_j)_{\nu_j}(t)d\mu_j \ll a_j \min(1, |t|^{1/2}).$$

At complex places,

$$\int h_j(\nu_j, p_j)d\mu_j \ll a_j^2, \qquad \int (\mathcal{B}_jh_j)_{(\nu_j,p_j)}(t)d\mu_j \ll a_j \min(1, |t|).$$

Proof. The bounds at real places are taken from [11, pp.124-126]. As for complex places, the first bound follows from trivial estimates: by (8.9),

$$\begin{aligned}\int h_j(\nu_j, p_j)d\mu_j &\ll \sum_{|p_j|\leq b_j} \int_{(0)} e^{(\nu_j^2-p_j^2)/a_j}(p_j^2 - \nu_j^2)d\nu_j \\ &\ll \sum_{|p_j|\leq b_j} \int_0^\infty e^{-u^2/a_j}u^2du + \sum_{|p_j|\leq b_j} \int_0^\infty e^{-u^2/a_j}p_j^2du \\ &\ll b_ja_j^{3/2} + b_j^3a_j^{1/2} = 2a_j^2.\end{aligned}$$

The second bound follows the same way as [13, (10.16)]. The only difference is that in our case, the sum over p_j (which is the same as p in [13, (10.18-19)]) is restricted to $|p_j| \le b_j$. It is easy to check that the difference coming from the terms $|p_j| > b_j$ is majorized by the right-hand side of [13, (10.20)]. ☐

Now we can estimate the density of the cuspidal spectrum.

Lemma 5.4. *Let $\mathfrak{c} \subseteq \mathfrak{o}$ be an ideal. Then for $1 \le X_j \in \mathbf{R}^{r+s}$,*

$$\sum_{\substack{\varpi \in \mathcal{C}(\mathfrak{c}) \\ j\le r: |\nu_{\varpi,j}| \le X_j \\ j>r: |\nu_{\varpi,j}^2 - p_{\varpi,j}^2| \le X_j^2}} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_\varpi^{-1}} |\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m})|^2 \ll_{F,\varepsilon}$$
$$\left(\prod_{j\le r} X_j^{2+\varepsilon}\right)\left(\prod_{j>r} X_j^{4+\varepsilon}\right)\mathcal{N}(\mathfrak{c})^{1+\varepsilon} + \left(\prod_j X_j^{2+\varepsilon}\right)(\mathcal{N}(\gcd(\mathfrak{m},\mathfrak{c})))^{1/2}\mathcal{N}(\mathfrak{m})^{1/2+\varepsilon}.$$

Proof. This is the generalization of [5, Lemma 6], we can repeat its proof. Choose a narrow class representative $\mathfrak{n}$ of $\mathfrak{m}^{-1}$ from a fixed set of narrow class representatives. Then for some $\alpha \in F^\times$, $\mathfrak{m} = \alpha\mathfrak{n}^{-1}$, and $1 \ll_F \mathcal{N}((\alpha))/\mathcal{N}(\mathfrak{m}) \ll_F 1$. We apply the Kuznetsov formula (5.1) with $\alpha = \alpha'$, $\mathfrak{a} = \mathfrak{a}' = \mathfrak{n}$, and the weight function is the one described above, setting $a_j = X_j^2$, $b_j = X_j$ at each archimedean place. On the spectral side of the Kuznetsov formula, we obtain an upper bound on the left-hand side of the statement, since the contribution of the Eisenstein spectrum is nonnegative. For $\varpi \in \mathcal{C}(\mathfrak{c})$, by (3.2), (3.5) and Proposition 3.2, $[K(\mathfrak{o}) : K(\mathfrak{c})]C_\varpi \ll_{F,\varepsilon} (\prod_j X_j)^\varepsilon \mathcal{N}(\mathfrak{c})^{1+\varepsilon}$. Then by the previous lemma, the delta term gives $\ll_{F,\varepsilon} \left(\prod_{j\le r} X_j^{2+\varepsilon}\right)\left(\prod_{j>r} X_j^{4+\varepsilon}\right)\mathcal{N}(\mathfrak{c})^{1+\varepsilon}$. As for the Kloosterman term, we use Weil's bound [56, (13)] together with the previous lemma to see it is

$$\ll_{F,\varepsilon} \left(\prod_j X_j^{2+\varepsilon}\right)\mathcal{N}(\mathfrak{c})^{1+\varepsilon} \max_{\mathfrak{a}\in C} \sum_{0\ne c\in\mathfrak{n}\mathfrak{a}\mathfrak{c}} \frac{\mathcal{N}((\gcd(\mathfrak{m}, c\mathfrak{n}^{-1}\mathfrak{a}^{-1})))^{1/2}}{\mathcal{N}(c\mathfrak{n}^{-1}\mathfrak{a}^{-1})^{1/2-\varepsilon}} \cdot \prod_{j\le r}\min(1, |\alpha_j/c_j|^{1/2}) \prod_{j>r}\min(1,|\alpha_j/c_j|), \tag{5.4}$$

where C is a fixed set of narrow class representatives (depending only on F) such that $\mathfrak{a}^2$ is a totally positive principal ideal for each $\mathfrak{a} \in C$. Then the sum over the elements c can be rewritten as a sum over the principal ideals (c), the sum over the units is estimated in [9, Lemma 8.1]. Then the above display is

$$\ll_{F,\varepsilon} \left(\prod_j X_j^{2+\varepsilon}\right)\mathcal{N}(\mathfrak{c})^{1+\varepsilon} \max_{\mathfrak{a}\in C} \sum_{0\ne (c)\subseteq\mathfrak{n}\mathfrak{a}\mathfrak{c}} \frac{\mathcal{N}(\gcd(\mathfrak{m}, c\mathfrak{n}^{-1}\mathfrak{a}^{-1}))^{1/2}}{\mathcal{N}(c\mathfrak{n}^{-1}\mathfrak{a}^{-1})^{1/2-\varepsilon}} \cdot (1 + |\log(\mathcal{N}((\alpha/c)))|^{r+s-1})\min(1, \mathcal{N}((\alpha/c))).$$

This is obviously

$$\ll_{F,\varepsilon} \left(\prod_j X_j^{2+\varepsilon}\right)\mathcal{N}(\mathfrak{c})^{1+\varepsilon}\mathcal{N}(\mathfrak{m})^{1/2+2\varepsilon} \max_{\mathfrak{a}\in C} \sum_{0\ne (c)\subseteq\mathfrak{n}\mathfrak{a}\mathfrak{c}} \frac{\mathcal{N}((\gcd(\mathfrak{m}, c\mathfrak{n}^{-1}\mathfrak{a}^{-1})))^{1/2}}{\mathcal{N}((c))^{1/2-\varepsilon}} \frac{1}{\mathcal{N}((c))^{1/2+2\varepsilon}}.$$

We estimate now the sum. First extend it to all nonzero ideals contained in $\mathfrak{n}\mathfrak{a}\mathfrak{c}$ (parametrized as $\mathfrak{b}\mathfrak{n}\mathfrak{a}\mathfrak{c}$, where $0 \ne \mathfrak{b} \subseteq \mathfrak{o}$), then factorize out $\mathcal{N}(\gcd(\mathfrak{m},\mathfrak{c}))^{1/2}$. We obtain

$$\begin{aligned}
\frac{1}{\mathcal{N}(\mathfrak{n}\mathfrak{a}\mathfrak{c})^{1+\varepsilon}} \sum_{\mathfrak{b}\subseteq\mathfrak{o}} \frac{\mathcal{N}(\gcd(\mathfrak{m},\mathfrak{c}\mathfrak{b}))^{1/2}}{\mathcal{N}(\mathfrak{b})^{1+\varepsilon}} &\le \frac{\mathcal{N}(\gcd(\mathfrak{m},\mathfrak{c}))^{1/2}}{\mathcal{N}(\mathfrak{n}\mathfrak{a}\mathfrak{c})^{1+\varepsilon}} \sum_{\mathfrak{b}\subseteq\mathfrak{o}} \frac{\mathcal{N}(\gcd(\mathfrak{m},\mathfrak{b}))^{1/2}}{\mathcal{N}(\mathfrak{b})^{1+\varepsilon}} \\
&\le \frac{\mathcal{N}(\gcd(\mathfrak{m},\mathfrak{c}))^{1/2}}{\mathcal{N}(\mathfrak{n}\mathfrak{a}\mathfrak{c})^{1+\varepsilon}} \sum_{\mathfrak{m}'|\mathfrak{m}} \sum_{\mathfrak{b}'\subseteq\mathfrak{o}} \frac{\mathcal{N}(\mathfrak{m}')^{1/2}}{\mathcal{N}(\mathfrak{m}'\mathfrak{b}')^{1+\varepsilon}} \\
&\le \frac{\mathcal{N}(\gcd(\mathfrak{m},\mathfrak{c}))^{1/2}}{\mathcal{N}(\mathfrak{n}\mathfrak{a}\mathfrak{c})^{1+\varepsilon}} \sum_{\mathfrak{m}'|\mathfrak{m}} \zeta_F(1+\varepsilon) \\
&\ll_{F,\varepsilon} \frac{\mathcal{N}(\gcd(\mathfrak{m},\mathfrak{c}))^{1/2}}{\mathcal{N}(\mathfrak{n}\mathfrak{a}\mathfrak{c})^{1+\varepsilon}}\mathcal{N}(\mathfrak{m})^\varepsilon.
\end{aligned}$$

Altogether, the contribution of the Kloosterman term is

$$\ll_{F,\varepsilon} \left(\prod_j X_j^{2+\varepsilon} \right) \mathcal{N}(\gcd(\mathfrak{m}, \mathfrak{c}))^{1/2} \mathcal{N}(\mathfrak{m})^{1/2+\varepsilon}. \tag{5.5}$$

Recalling that the contribution of the Eisenstein spectrum is nonnegative, the proof is complete. □

Lemma 5.5. *Let* $\mathfrak{c} \subseteq \mathfrak{o}$ *be an ideal. Then for* $1 \leq X_j \in \mathbf{R}^{r+s}$,

$$\sum_{\substack{\varpi \in \mathcal{C}(\mathfrak{c}) \\ j \leq r : |\nu_{\varpi,j}| \leq X_j \\ j > r : |\nu_{\varpi,j}^2 - p_{\varpi,j}^2| \leq X_j^2}} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_\pi^{-1}} 1 \ll_{F,\varepsilon} \left(\prod_{j \leq r} X_j^{2+\varepsilon} \right) \left(\prod_{j > r} X_j^{4+\varepsilon} \right) \mathcal{N}(\mathfrak{c})^{1+\varepsilon}.$$

Proof. Take $\mathfrak{m} = \mathfrak{o}$ in the previous lemma. □

Chapter 6

The spectral decomposition of shifted convolution sums

The aim of this chapter is to prove a variant of [5, Theorem 2] for arbitrary number fields.

We focus on the subspace $L^2(Z(\mathbf{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A})/K(\mathfrak{c}))$, its spectral decomposition is similar to (2.3), the only modification is the restriction of $\mathcal{C},\mathcal{E}$ to $\mathcal{C}(\mathfrak{c}),\mathcal{E}(\mathfrak{c})$, respectively. If f is a function of those infinite-dimensional representations, which are not orthogonal to $L^2(Z(\mathbf{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A})/K(\mathfrak{c}))$, we write

$$\int_{(\mathfrak{c})} f_\varpi d\varpi = \sum_{\pi\in\mathcal{C}(\mathfrak{c})} f_\pi + \int_{\mathcal{E}(\mathfrak{c})} f_\varpi d\varpi.$$

Theorem 1. *We have a spectral decomposition of shifted convolution sums in the sense of Part A with functions satisfying the bound in Part B.*

Part A. *Assume π_1,π_2 are irreducible cuspidal representations in $L^2(Z(\mathbf{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A}))$. Let $l_1,l_2\in\mathfrak{o}\setminus\{0\}$, and set $\mathfrak{c}=\mathrm{lcm}(l_1\mathfrak{c}_{\pi_1},l_2\mathfrak{c}_{\pi_2})$. Let moreover $W_1,W_2:F_\infty^\times\to\mathbf{C}$ be arbitrary Schwarz functions, that is, they are smooth and tend to 0 faster then any power of y^{-1} or y, as y tends to ∞ or 0, respectively. Then for any $\varpi\in\mathcal{C}(\mathfrak{c})\cup\mathcal{E}(\mathfrak{c})$ and $\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}$, there exists a function $W_{\varpi,\mathfrak{t}}:F_\infty^\times\to\mathbf{C}^\times$ depending only on $F,\pi_1,\pi_2,W_1,W_2,\varpi,\mathfrak{t}$ such that the following holds. For any $Y\in(0,\infty)^{r+s}$, any ideal $\mathfrak{n}\subseteq\mathfrak{o}$ and any $0\neq q\in\mathfrak{n}$, there is a spectral decomposition of the shifted convolution sum*

$$\sum_{l_1t_1-l_2t_2=q,0\neq t_1,t_2\in\mathfrak{n}} \frac{\lambda_{\pi_1}(t_1\mathfrak{n}^{-1})\overline{\lambda_{\pi_2}(t_2\mathfrak{n}^{-1})}}{\sqrt{\mathcal{N}(t_1t_2\mathfrak{n}^{-2})}} W_1\left(\left(\frac{(l_1t_1)_j}{Y_j}\right)_j\right)\overline{W_2\left(\left(\frac{(l_2t_2)_j}{Y_j}\right)_j\right)}$$
$$= \int_{(\mathfrak{c})}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} \frac{\lambda_\varpi^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}} W_{\varpi,\mathfrak{t}}\left(\left(\frac{q_j}{Y_j}\right)_j\right)d\varpi,$$

where $\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m})$ is given in (2.35).

Proof. First apply Proposition 2.2 to see that there exist functions $\phi_1\in V_{\pi_1}(\mathfrak{c}_{\pi_1})$, $\phi_2\in V_{\pi_2}(\mathfrak{c}_{\pi_2})$ such that $W_{\phi_1}=W_1$, $W_{\phi_2}=W_2$. Set then

$$\Phi = R_{(l_1)}\phi_1 R_{(l_2)}\overline{\phi_2}.$$

Then since $\mathfrak{c}=\mathrm{lcm}(l_1\mathfrak{c}_{\pi_1},l_2\mathfrak{c}_{\pi_2})$, we see that Φ is right $K(\mathfrak{c})$-invariant. Also, since W_1,W_2 are from the Schwarz space, ϕ_1, ϕ_2 are smooth and have finite Sobolev norms of arbitrarily large order, so does $\Phi\in L^2(Z(\mathbf{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A})/K(\mathfrak{c}))$ (use Proposition 2.2 and Proposition 4.10 together with [57, Lemma 8.4]). Then by (2.3), (2.19), (2.37) and the remark made in the beginning of this chapter, we can decompose Φ as

$$\Phi = \Phi_{\mathrm{sp}} + \int_{(\mathfrak{c})}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}} \Phi_{\varpi,\mathfrak{t}} d\varpi, \tag{6.1}$$

where $\Phi_{\varpi,\mathfrak{t}}\in R^{\mathfrak{t}}(V_\varpi(\mathfrak{c}_\varpi))$ and Φ_{sp} is the orthogonal projection of Φ to L_{sp}. Now set $W_{\varpi,\mathfrak{t}}=W_{\Phi_{\varpi,\mathfrak{t}}}$. We claim this fulfills the property stated in Part A. Given $Y\in(0,\infty)^{r+s}$, $\mathfrak{n}\subseteq\mathfrak{o}$, $0\neq q\in\mathfrak{n}$, let $(y_{\mathrm{fin}})=\mathfrak{n}$, and $y_\infty = Y$. We compute

$$\int_{F\backslash\mathbf{A}} \Phi\left(\begin{pmatrix} y^{-1} & x \\ 0 & 1 \end{pmatrix}\right)\psi(-qx)dx \tag{6.2}$$

in two ways. On the one hand, we use (6.1). Here, $q \neq 0$ implies that Φ_{sp} has zero contribution to (6.2), and we obtain

$$\int_{F\backslash\mathbf{A}} \Phi\left(\begin{pmatrix} y^{-1} & x \\ 0 & 1 \end{pmatrix}\right)\psi(-qx)dx = \int_{(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} \frac{\lambda_\varpi^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}} W_{\varpi,\mathfrak{t}}\left(\left(\frac{q_j}{Y_j}\right)_j\right) d\varpi$$

from (2.34) and (2.38). On the other hand, using (2.17) and (2.29) together with the choice of ϕ_1, ϕ_2, we obtain

$$\begin{aligned}\int_{F\backslash\mathbf{A}} \Phi\left(\begin{pmatrix} y^{-1} & x \\ 0 & 1 \end{pmatrix}\right)\psi(-qx)dx = &\sum_{l_1t_1 - l_2t_2 = q, 0 \neq t_1, t_2 \in \mathfrak{n}} \frac{\lambda_{\pi_1}(t_1\mathfrak{n}^{-1})\overline{\lambda_{\pi_2}(t_2\mathfrak{n}^{-1})}}{\sqrt{\mathcal{N}(t_1t_2\mathfrak{n}^{-2})}} \\ &\cdot W_1\left(\left(\frac{(l_1t_1)_j}{Y_j}\right)_j\right)\overline{W_2\left(\left(\frac{(l_2t_2)_j}{Y_j}\right)_j\right)}.\end{aligned}$$

The equality of the last two displays is exactly the statement. □

Part B. *Conditions as in Part A. Assume $\mathcal{D}$ is a differential operator as in Proposition 4.12. Then for any $0 < \varepsilon < 1/4$ and nonnegative integers b, c, we have, for all $y \in F_\infty^\times$,*

$$\begin{aligned}\int_{(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} (\mathcal{N}(\mathbf{r}_\varpi))^{2c} |\mathcal{D}W_{\varpi,\mathfrak{t}}(y)|^2 d\varpi &\ll_{F,\varepsilon,\pi_1,\pi_2,a,b,c,P} \mathcal{N}((l_1l_2))^\varepsilon ||W_1||_{S_\alpha}^2 ||W_2||_{S_\alpha}^2 \\ &\cdot \prod_{j=1}^{r} (|y_j|^{1-\varepsilon} + |y_j|^{1-2\theta-\varepsilon})(\min(1, |y_j|^{-2b})) \prod_{j=r+1}^{r+s} (|y_j|^{3/2} + |y_j|^2)(\min(1, |y_j|^{-2b}))\end{aligned}$$

with $\alpha = 2(3r + 4s + 2) + (r + s)(a + b + 2c) + 2(7r + 18s)$.

Proof. Let Φ be the function appearing in the proof of Part A. Then by Proposition 4.12 and a consequence of (2.8) (see [5, (85)]), we have

$$\begin{aligned}\int_{(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} (\mathcal{N}(\mathbf{r}_\varpi))^{2c} |\mathcal{D}W_{\varpi,\mathfrak{t}}(y)|^2 d\varpi &\ll_{F,\varepsilon,\pi_1,\pi_2,a,b,c,P} \mathcal{N}(l_1l_2)^\varepsilon ||\Phi||_{S_\beta}^2 \\ &\cdot \prod_{j=1}^{r} (|y_j|^{1-\varepsilon} + |y_j|^{1-2\theta-\varepsilon})(\min(1, |y_j|^{-2b})) \prod_{j=r+1}^{r+s} (|y_j|^{3/2} + |y_j|^2)(\min(1, |y_j|^{-2b}))\end{aligned}$$

with $\beta = 2(3r + 4s + 2) + (r + s)(a + b + 2c)$. For any differential operator $\mathcal{D}' \in U(\mathfrak{g})$ of order k, we have

$$||\mathcal{D}'\phi_1||_\infty \ll_{F,\pi_1} ||\phi_1||_{S_{k+2(7r+18s)}}, \qquad ||\mathcal{D}'\phi_2||_\infty \ll_{F,\pi_2} ||\phi_2||_{S_{k+2(7r+18s)}}$$

by Proposition 4.10. The operators $R_{(l_1)}, R_{(l_2)}$ do not affect Sobolev norms and $Z(\mathbf{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A})$ has finite volume, therefore $||\Phi||_{S_\beta} \ll_{F,\pi_1,\pi_2} ||\phi_1||_{S_{\beta+2(7r+18s)}} ||\phi_2||_{S_{\beta+2(7r+18s)}}$. Now Proposition 3.2 and (3.5) completes the proof. □

Remark 2. In the more general setup described briefly in Remark 1, we have to require that π_1 and π_2 are of the same central character.

From the L^2-bound presented in Theorem 1, Part B, we can easily deduce L^1-bounds, these are generalizations of [5, Remark 12].

Corollary 6.1. *Conditions as in Theorem 1. Assume $\mathcal{D}$ is a differential operator as in Proposition 4.12. Then for any $0 < \varepsilon < 1/4$ and nonnegative integers b, c', we have, for all $y \in F_\infty^\times$,*

$$\int_{\mathcal{E}(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} (\mathcal{N}(\mathbf{r}_\varpi))^{c'} |\mathcal{D}W_{\varpi,\mathfrak{t}}(y)| d\varpi \ll_{F,\varepsilon,\pi_1,\pi_2,a,b,c',P} \mathcal{N}(\mathfrak{l})^{1/4} \mathcal{N}((l_1 l_2))^{\varepsilon} ||W_1||_{S_{\alpha'}} ||W_2||_{S_{\alpha'}}$$

$$\cdot \prod_{j=1}^{r} (|y_j|^{1/2-\varepsilon} + |y_j|^{1/2-\theta-\varepsilon})(\min(1, |y_j|^{-b})) \prod_{j=r+1}^{r+s} (|y_j|^{3/4} + |y_j|)(\min(1, |y_j|^{-b}))$$

with $\alpha' = 2(3r + 4s + 2) + (r + s)(a + b + 2c' + 4(r + 2s)) + 2(7r + 18s)$, where $\mathfrak{l}$ stands for the largest square divisor of $\operatorname{lcm}((l_1), (l_2))$.

Proof. Set $c = c' + 2(r + 2s)$. Apply first Cauchy-Schwarz,

$$\left(\int_{\mathcal{E}(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} (\mathcal{N}(\mathbf{r}_\varpi))^{c'} |\mathcal{D}W_{\varpi,\mathfrak{t}}(y)| d\varpi \right)^2 \ll_F \int_{\mathcal{E}(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} (\mathcal{N}(\mathbf{r}_\varpi))^{2c} |\mathcal{D}W_{\varpi,\mathfrak{t}}(y)|^2 d\varpi$$

$$\cdot \int_{\mathcal{E}(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} (\mathcal{N}(\mathbf{r}_\varpi))^{-4(r+2s)} d\varpi.$$

Now Theorem 1, Part B estimates the first integral, while by Lemma 5.2, the second integral is $\ll_F (\mathcal{N}(\mathfrak{l}))^{1/2}$. We are done by taking square-roots. □

Corollary 6.2. *Conditions as in Theorem 1. Assume $\mathcal{D}$ is a differential operator as in Proposition 4.12. Then for any $0 < \varepsilon < 1/4$ and nonnegative integers b, c', we have, for all $y \in F_\infty^\times$,*

$$\sum_{\varpi \in \mathcal{C}(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} (\mathcal{N}(\mathbf{r}_\varpi))^{c'} |\mathcal{D}W_{\varpi,\mathfrak{t}}(y)| \ll_{F,\varepsilon,\pi_1,\pi_2,a,b,c',P} \mathcal{N}((l_1 l_2))^{1/2+\varepsilon} ||W_1||_{S_{\alpha'}} ||W_2||_{S_{\alpha'}}$$

$$\cdot \prod_{j=1}^{r} (|y_j|^{1/2-\varepsilon} + |y_j|^{1/2-\theta-\varepsilon})(\min(1, |y_j|^{-b})) \prod_{j=r+1}^{r+s} (|y_j|^{3/4} + |y_j|)(\min(1, |y_j|^{-b}))$$

with $\alpha' = 2(3r + 4s + 2) + (r + s)(a + b + 2c' + 4(r + 2s)) + 2(7r + 18s)$.

Proof. Almost the same as before, the only difference is that we use Lemma 5.5. □

Chapter 7

A Burgess type subconvex bound for twisted GL_2 L-functions

In this chapter, as an application of Theorem 1, we prove a Burgess type subconvexity for twisted GL_2 L-functions over arbitrary number fields. For totally real fields, this was proved by Blomer and Harcos in [5]. We also note that for arbitrary number fields, Wu [61] recently proved this result, using a different method. Our approach is the extension of the one in [5, Section 3.3].

Assume that π is an irreducible automorphic cuspidal representation in $L^2(Z(\mathbf{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A}))$. Let $\mathfrak{q} \subseteq \mathfrak{o}$ be an ideal, χ a Hecke character of conductor $\mathfrak{q}$. We may also think of χ as a character on the group of fractional ideals coprime to $\mathfrak{q}$, extended to be 0 on other ideals. There exists characters χ_{fin} of $(\mathfrak{o}/\mathfrak{q})^\times$ and χ_∞ of $F_\infty^\times$ satisfying $\chi((t)) = \chi_{\mathrm{fin}}(t)\chi_\infty(t)$ for all $t \in \mathfrak{o}$ coprime to $\mathfrak{q}$. The transition from one meaning to another of Hecke characters can be found at several places (see [15, Sections 1.7 and 3.1], for example). Our goal is to estimate $L(1/2, \pi \otimes \chi)$ in terms of $\mathcal{N}(\mathfrak{q})$. Fix any $\varepsilon > 0$. From now on, the implicit constants in $\ll$ are always meant to depend on $F, \varepsilon, \pi, \chi_\infty$, even if it is not emphasized in the subscript like $\ll_{F,\varepsilon,\pi,\chi_\infty}$. Fix an ideal $\mathfrak{n}$ coprime to $\mathfrak{q}$ satisfying

$$\mathcal{N}(\mathfrak{n}) \ll_{F,\varepsilon} \mathcal{N}(\mathfrak{q})^\varepsilon \tag{7.1}$$

and note that in every narrow ideal class, there is a representative $\mathfrak{n}$ with these properties.

First we introduce the following notation. For given positive real numbers $a < b$,

$$[[a, b]] = \{x \in F_{\infty,+}^\times : a \le |x_j| \le b\}. \tag{7.2}$$

Let G_0 be a smooth, compactly supported function on $F_{\infty,+}^1 = \{x \in F_{\infty,+}^\times : |x|_\infty = 1\}$ satisfying that $\sum_{u\in\mathfrak{o}_+^\times} G_0(ux) = 1$ for all $x \in F_{\infty,+}^1$. We extend this function to $F_{\infty,+}^\times$ as $G(x) = G_0(x/|x|_\infty)$, then $\sum_{u\in\mathfrak{o}_+^\times} G(ux) = 1$ for all $x \in F_{\infty,+}^\times$. Assume that G_0 is supported on $[[c_1, c_2]]$, then G is supported on $F_{\infty,+}^{\mathrm{diag}}[[c_1, c_2]]$, where c_1, c_2 are constants depending only on F (recall (7.2)). Fix moreover a compact fundamental domain $\mathcal{G}_0$ for the action of $\mathfrak{o}_+^\times$ on $F_{\infty,+}^1$ and let $\mathcal{G} = F_{\infty,+}^{\mathrm{diag}}\mathcal{G}_0$ be its extension to $F_{\infty,+}^\times$.

7.1 The amplification method

Let ξ be a character of $(\mathfrak{o}/\mathfrak{q})^\times$. Parametrized by $v = (v_1, \ldots, v_{r+s}) \in (i\mathbf{R})^{r+s}, p = (p_{r+1}, \ldots, p_{r+s}) \in \mathbf{Z}^s$, assume that $W_{v,p}$ are functions on $F_{\infty,+}^\times$ satisfying the following properties:

(i) $W_{v,p}$ is smooth and supported on $[[c_3, c_4]]$ for some $c_3 < c_1$ and $c_4 > c_2$ depending only on F;

(ii) for any differential operator $\mathcal{D}$ of the form

$$\mathcal{D} = \left(\left(\frac{\partial}{\partial y_j}\right)^{\mu_j}_{j\le r}\left(\frac{\partial}{\partial y_j}\right)^{\mu_{j,1}}_{j>r}\left(\frac{\partial}{\partial \overline{y_j}}\right)^{\mu_{j,2}}_{j>r}\right),$$

with nonnegative integers $\mu_{j,*}$, we have

$$\mathcal{D}W_{v,p}(y) \ll_{F,\mathcal{D}} \prod_{j=1}^{r}(1+|v_j|)^{\mu_j}\prod_{j=r+1}^{r+s}(1+|v_j|+|p_j|)^{\mu_{j,1}+\mu_{j,2}}.$$

Compare this with (2.13) and (2.14), and for convenience, introduce

$$\mathcal{N}(v,p) = \prod_{j=1}^{r}(1+|v_j|)\prod_{j=r+1}^{r+s}(1+|v_j|+|p_j|)^2. \tag{7.3}$$

Then set

$$\mathcal{L}_\xi(v,p) = \sum_{0<<t\in\mathfrak{n}} \frac{\lambda_\pi(t\mathfrak{n}^{-1})\xi(t)}{\sqrt{\mathcal{N}(t\mathfrak{n}^{-1})}} W_{v,p}\left(\frac{t}{Y^{1/(r+2s)}}\right). \tag{7.4}$$

The only assumption on the positive real number Y is that

$$Y \ll_\varepsilon \mathcal{N}(\mathfrak{q})^{1+\varepsilon}. \tag{7.5}$$

Introduce $\mathcal{K} = \mathfrak{n}\cap F^{\mathrm{diag}}_{\infty,+}[[c_3,c_4]]$. We see that the numbers t that give a positive contribution are all in the set $\mathfrak{n}\cap\mathcal{K}$ and also satisfy $t\in[[c_3,c_4]]Y^{1/(r+2s)}$, this latter implies $|t|_\infty \approx_F Y$.

Assume L (the amplification length) is a further parameter satisfying

$$\log L \approx \log \mathcal{N}(\mathfrak{q}). \tag{7.6}$$

Lemma 7.1. *Denote by $\Pi_{\mathfrak{q},+}(L,2L)$ the set of totally positive, principal prime ideals $\mathfrak{l}\subseteq\mathfrak{o}$ satisfying $\mathcal{N}(\mathfrak{l})\in[L,2L]$ and $\mathfrak{l}\nmid\mathfrak{q}$. Set $\pi_{\mathfrak{q},+}(L,2L) = \#\Pi_{\mathfrak{q},+}(L,2L)$. Then*

$$\pi_{\mathfrak{q},+}(L,2L) \gg_{F,\varepsilon} L\mathcal{N}(\mathfrak{q})^{-\varepsilon}.$$

Proof. This follows immediately from the results [52, Corollary 6 of Proposition 7.8 and Proposition 7.9(ii)] about the natural density of prime ideals in narrow ideal classes. (See also [53, Chapter VII, §13] for analogous statements about the Dirichlet density.) ☐

Therefore,

$$\begin{aligned}|\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)|^2 &= \frac{1}{\pi_{\mathfrak{q},+}(L,2L)^2}\left|\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)\sum_{\substack{l\in\mathfrak{o}\cap\mathcal{G}\\(l)\in\Pi_{\mathfrak{q},+}(L,2L)}} 1\right|^2\\ &\ll_\varepsilon \frac{\mathcal{N}(\mathfrak{q})^\varepsilon}{L^2}\sum_{\xi\in\widehat{(\mathfrak{o}/\mathfrak{q})^\times}}\left|\mathcal{L}_\xi(v,p)\sum_{\substack{l\in\mathfrak{o}\cap\mathcal{G}\\(l)\in\Pi_{\mathfrak{q},+}(L,2L)}}\xi(l)\overline{\chi_{\mathrm{fin}}(l)}\right|^2.\end{aligned}$$

Observe that the ξ-sum is the square integral of the Fourier transform of the function

$$(\mathfrak{o}/\mathfrak{q})^\times \ni x \mapsto \sum_{t\in\mathfrak{n}\cap\mathcal{K}}\sum_{\substack{l\in\mathfrak{o}\cap\mathcal{G}\\(l)\in\Pi_{\mathfrak{q},+}(L,2L)\\ lt\equiv x \pmod{\mathfrak{q}}}}\overline{\chi_{\mathrm{fin}}(l)}\frac{\lambda_\pi(t\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(t\mathfrak{n}^{-1})}}W_{v,p}\left(\frac{t}{Y^{1/(r+2s)}}\right),$$

so Plancherel gives

$$|\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)|^2 \ll_\varepsilon \frac{\varphi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^\varepsilon}{L^2}\sum_{x\in(\mathfrak{o}/\mathfrak{q})^\times}\left|\sum_{\substack{l\in\mathfrak{o}\cap\mathcal{G}\\(l)\in\Pi_{\mathfrak{q},+}(L,2L)}}\overline{\chi_{\mathrm{fin}}(l)}\sum_{\substack{t\in\mathfrak{n}\cap\mathcal{K}\\ lt\equiv x \pmod{\mathfrak{q}}}}\frac{\lambda_\pi(t\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(t\mathfrak{n}^{-1})}}W_{v,p}\left(\frac{t}{Y^{1/(r+2s)}}\right)\right|^2.$$

This can be further majorized using $\varphi(\mathfrak{q})\le\mathcal{N}(\mathfrak{q})$ and $(\mathfrak{o}/\mathfrak{q})^\times\subset\mathfrak{o}/\mathfrak{q}$, giving

$$\begin{aligned}|\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)|^2 \ll_\varepsilon \frac{\mathcal{N}(\mathfrak{q})^{1+\varepsilon}}{L^2} &\sum_{\substack{l_1,l_2\in\mathfrak{o}\cap\mathcal{G}\\(l_1),(l_2)\in\Pi_{\mathfrak{q},+}(L,2L)}}\overline{\chi_{\mathrm{fin}}(l_1)}\chi_{\mathrm{fin}}(l_2)\\ &\sum_{\substack{t_1,t_2\in\mathfrak{n}\cap\mathcal{K}\\ l_1t_1-l_2t_2\in\mathfrak{q}}}\frac{\lambda_\pi(t_1\mathfrak{n}^{-1})\overline{\lambda_\pi(t_2\mathfrak{n}^{-1})}}{\sqrt{\mathcal{N}(t_1t_2\mathfrak{n}^{-2})}}W_{v,p}\left(\frac{t_1}{Y^{1/(r+2s)}}\right)\overline{W_{v,p}\left(\frac{t_2}{Y^{1/(r+2s)}}\right)}.\end{aligned} \tag{7.7}$$

In (7.7), the contribution of $l_1t_1 - l_2t_2 = 0$ will be referred as the diagonal contribution DC, and that of $l_1t_1 - l_2t_2 \neq 0$ as the off-diagonal contribution ODC, that is,

$$DC = \frac{\mathcal{N}(\mathfrak{q})^{1+\varepsilon}}{L^2} \sum_{\substack{l_1,l_2 \in \mathfrak{o}\cap\mathcal{G} \\ (l_1),(l_2)\in\Pi_{\mathfrak{q},+}(L,2L)}} \overline{\chi_{\mathrm{fin}}(l_1)}\chi_{\mathrm{fin}}(l_2) \sum_{\substack{t_1,t_2\in\mathfrak{n}\cap\mathcal{K} \\ l_1t_1-l_2t_2=0}} \frac{\lambda_\pi(t_1\mathfrak{n}^{-1})\overline{\lambda_\pi(t_2\mathfrak{n}^{-1})}}{\sqrt{\mathcal{N}(t_1t_2\mathfrak{n}^{-2})}} W_{v,p}\left(\frac{t_1}{Y^{1/(r+2s)}}\right)\overline{W_{v,p}\left(\frac{t_2}{Y^{1/(r+2s)}}\right)} \tag{7.8}$$

and

$$ODC = \frac{\mathcal{N}(\mathfrak{q})^{1+\varepsilon}}{L^2} \sum_{\substack{l_1,l_2 \in \mathfrak{o}\cap\mathcal{G} \\ (l_1),(l_2)\in\Pi_{\mathfrak{q},+}(L,2L)}} \overline{\chi_{\mathrm{fin}}(l_1)}\chi_{\mathrm{fin}}(l_2) \sum_{\substack{t_1,t_2\in\mathfrak{n}\cap\mathcal{K} \\ l_1t_1-l_2t_2\in\mathfrak{q}\setminus\{0\}}} \frac{\lambda_\pi(t_1\mathfrak{n}^{-1})\overline{\lambda_\pi(t_2\mathfrak{n}^{-1})}}{\sqrt{\mathcal{N}(t_1t_2\mathfrak{n}^{-2})}} W_{v,p}\left(\frac{t_1}{Y^{1/(r+2s)}}\right)\overline{W_{v,p}\left(\frac{t_2}{Y^{1/(r+2s)}}\right)}. \tag{7.9}$$

We will estimate them separately, optimize in the choice of the parameter L (taking care of (7.6)), which will give rise to an estimate of $\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)$. Using Mellin inversion and the consequence of the approximate functional equation presented in Section 3.3, this bound on $\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)$ (with implicit parameters satisfying (7.1) and (7.5)) will give rise to a Burgess type subconvex bound on $L(1/2, \pi\otimes\chi)$.

7.2 Estimate of the diagonal contribution

First focus on (7.8). Then by Cauchy-Schwarz,

$$DC \ll_{F,\varepsilon} \frac{\mathcal{N}(\mathfrak{q})^{1+\varepsilon}}{L^2} \sum_{\substack{l\in\mathfrak{o}\cap\mathcal{G} \\ (l)\in\Pi_{\mathfrak{q},+}(L,2L)}} \sum_{\substack{t\in\mathfrak{n}\cap\mathcal{K} \\ |t|_\infty\approx_F Y}} \frac{|\lambda_\pi(t\mathfrak{n}^{-1})|^2}{\mathcal{N}(t\mathfrak{n}^{-1})} |\{(l',t')\in(\mathfrak{o}\cap\mathcal{G})\times(\mathfrak{n}\cap\mathcal{K}) : l't'=lt\}|.$$

Here, $|\{(l',t')\in(\mathfrak{o}\cap\mathcal{G})\times(\mathfrak{n}\cap\mathcal{K}) : l't'=lt\}|$ is at most the number of divisors of (lt), which is $\ll_{F,\varepsilon} \mathcal{N}((lt))^\varepsilon \ll_{F,\varepsilon} (LY)^\varepsilon$. By (3.8), (7.1) and (7.5), we see

$$\sum_{\substack{t\in\mathfrak{n}\cap\mathcal{K} \\ |t|_\infty\approx_F Y}} \frac{|\lambda_\pi(t\mathfrak{n}^{-1})|^2}{\mathcal{N}(t\mathfrak{n}^{-1})} \ll_{F,\varepsilon} \mathcal{N}(\mathfrak{q})^\varepsilon,$$

and estimate the number of prime ideals (l) trivially by $\ll_F L$. Altogether,

$$DC \ll_{F,\varepsilon} \frac{\mathcal{N}(\mathfrak{q})^{1+\varepsilon}}{L}. \tag{7.10}$$

7.3 Off-diagonal contribution: spectral decomposition and Eisenstein part

7.3.1 Spectral decomposition

The estimate of the off-diagonal contribution (7.9) requires much more work. Assume $\mathcal{G}_0$ is supported on $[[c_5, c_6]]$ for some constants c_5, c_6 depending only on F. Then only $l_1, l_2 \in [[c_5L^{1/(r+2s)}, c_6L^{1/(r+2s)}]]$ and $t_1, t_2 \in [[c_3Y^{1/(r+2s)}, c_4Y^{1/(r+2s)}]]$ have nonzero contribution to (7.9). If l_1, l_2, t_1, t_2 satisfy these constraints, then

$$l_1t_1 - l_2t_2 \in \mathcal{B} = \{x\in F_\infty : |x_j| \le c_7(LY)^{1/(r+2s)}\}$$

with $c_7 = 2c_4c_6$. Now a term on the right-hand side of (7.9) corresponding to some fixed l_1, l_2 can be written as

$$\sum_{0\neq q\in\mathfrak{q}\mathfrak{n}\cap\mathcal{B}} \sum_{\substack{l_1t_1-l_2t_2=q \\ 0\neq t_1,t_2\in\mathfrak{n}}} \frac{\lambda_\pi(t_1\mathfrak{n}^{-1})\overline{\lambda_\pi(t_2\mathfrak{n}^{-1})}}{\sqrt{\mathcal{N}(t_1t_2\mathfrak{n}^{-2})}} W_1\left(\frac{l_1t_1}{(LY)^{1/(r+2s)}}; v, p\right)\overline{W_2\left(\frac{l_2t_2}{(LY)^{1/(r+2s)}}; v, p\right)}, \tag{7.11}$$

where W_1, W_2 are smooth functions on $F_{\infty,+}^{\times}$ defined as

$$W_1(y; v, p) = W_{v,p}(yL^{1/(r+2s)}/l_1), \qquad W_2(y; v, p) = W_{v,p}(yL^{1/(r+2s)}/l_2).$$

Now by the assumptions made on $W_{v,p}$ and l_1, l_2, we have that W_1, W_2 are smooth of compact support $[[c_8, c_9]]$ (where c_8, c_9 depend on F) and for any differential operator $\mathcal{D}$ of the form

$$\mathcal{D} = \left(\left(\frac{\partial}{\partial y_j} \right)_{j \le r}^{\mu_j} \left(\frac{\partial}{\partial y_j} \right)_{j > r}^{\mu_{j,1}} \left(\frac{\partial}{\partial \overline{y_j}} \right)_{j > r}^{\mu_{j,2}} \right),$$

with nonnegative integers $\mu_{j(,1,2)}$, we have

$$\mathcal{D}W_{1,2}(y; v, p) \ll_{F,\mathcal{D}} \mathcal{N}(v, p)^{\mu}, \tag{7.12}$$

where $\mu = \max_j(\mu_{j(,1,2)})$ (recall (7.3)).

Now by Theorem 1, (7.11) can be rewritten as

$$\sum_{0 \ne q \in \mathfrak{q}\mathfrak{n} \cap \mathcal{B}} \int_{(\mathfrak{c})} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} \frac{\lambda_{\varpi}^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}} W_{\varpi,\mathfrak{t}} \left(\frac{q}{(LY)^{1/(r+2s)}}; v, p \right) d\varpi, \tag{7.13}$$

where $\mathfrak{c} = \mathfrak{c}_{\pi} \mathrm{lcm}((l_1), (l_2))$.

7.3.2 Eisenstein spectrum

First we estimate the contribution of the Eisenstein spectrum to (7.13). We use Corollary 6.1 with $\mathcal{D} = 1, a = c' = 0, b = 2$. The largest square divisor of $\mathrm{lcm}((l_1), (l_2))$ is $\mathfrak{o}$, hence

$$\int_{\mathcal{E}(\mathfrak{c})} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} |W_{\varpi,\mathfrak{t}}(y; v, p)| \ll_{F,\varepsilon,\pi} \mathcal{N}((l_1 l_2))^{\varepsilon} ||W_1||_{S_{\alpha_1}} ||W_2||_{S_{\alpha_1}}$$

with some positive integer α_1 depending only on F, uniformly in y, v, p. Moreover, by [57, Lemma 8.4] and (7.12), for any positive α,

$$||W_{1,2}||_{S^{\alpha}} \ll_{F,\pi,\alpha} \mathcal{N}(v, p)^{2\alpha} \tag{7.14}$$

giving

$$\int_{\mathcal{E}(\mathfrak{c})} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} |W_{\varpi,\mathfrak{t}}(y; v, p)| \ll_{F,\varepsilon,\pi} \mathcal{N}((l_1 l_2))^{\varepsilon} \mathcal{N}(v, p)^{4\alpha_1}.$$

Taking into account (2.39), (7.1), (7.5) and (7.6), we see that the contribution of the Eisenstein spectrum to (7.13) is

$$\ll_{F,\varepsilon} \mathcal{N}(v, p)^{4\alpha_1} \mathcal{N}(\mathfrak{q})^{\varepsilon} \sum_{0 \ne q \in \mathfrak{q}\mathfrak{n} \cap \mathcal{B}} \frac{\mathcal{N}(\gcd(\mathfrak{c}, (q)))}{\sqrt{\mathcal{N}((q))}}.$$

In the sum, each ideal (q) appears with multiplicity $\ll_{F,\varepsilon} \mathcal{N}(\mathfrak{q})^{\varepsilon}$. Indeed, each ideal $(q) \subseteq \mathfrak{o}$ has a generator q satisfying $|q_j| \ge c_5$ at each archimedean place. Hence the possible units ϵ for which $q\epsilon \in \mathcal{B}$ all satisfy $|\epsilon_j| \le c_{10}(LY)^{1/(r+2s)}$ at each place, for some constant c_{10} depending only on F. The number of such units is $\ll_{F,\varepsilon} \log(\mathcal{N}(\mathfrak{q}))^{r+s-1}$ by (7.5) and (7.6). Then the above display is

$$\ll_{F,\varepsilon} \mathcal{N}(v, p)^{4\alpha_1} \mathcal{N}(\mathfrak{q})^{2\varepsilon} \sum_{\substack{0 \ne (q) \subseteq \mathfrak{q}\mathfrak{n} \\ \mathcal{N}((q)) \ll_F LY}} \frac{\mathcal{N}(\gcd(\mathfrak{c}, (q)))}{\sqrt{\mathcal{N}((q))}}.$$

Here, the sum is $\ll_{F,\varepsilon} \mathcal{N}(\mathfrak{q})^{-1+\varepsilon}(LY)^{1/2}$, since $\gcd(\mathfrak{c}, (q)) = \gcd(\mathfrak{c}_{\pi}, (q))$, which has norm $O_{F,\pi}(1)$.

Altogether, using again (7.5), in (7.13), the Eisenstein spectrum has contribution

$$\ll_{F,\varepsilon,\pi} \mathcal{N}(v, p)^{4\alpha_1} \mathcal{N}(\mathfrak{q})^{-1/2+\varepsilon} L^{1/2}, \tag{7.15}$$

which is analogous to [5, (116)].

7.4 Off-diagonal contribution: cuspidal spectrum

Set

$$\mathcal{C}(\mathfrak{c},\varepsilon)=\{\varpi\in\mathcal{C}(\mathfrak{c}):\mathcal{N}(\mathbf{r}_\varpi)\le\mathcal{N}(\mathfrak{q})^\varepsilon\}.$$

Later we will prove that the contribution of representations outside $\mathcal{C}(\mathfrak{c},\varepsilon)$ is small. So restrict to $\mathcal{C}(\mathfrak{c},\varepsilon)$, and fix also the sign of q as follows. For any sign $\xi\in\{\pm1\}^r$, set

$$\mathcal{B}(\xi)=\{y\in\mathcal{B}:\operatorname{sign}(y)=\xi\}.$$

Then focus on the quantity

$$\sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathcal{B}(\xi)}\sum_{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}}\frac{\lambda_\varpi^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}}W_{\varpi,\mathfrak{t}}\left(\frac{q}{(LY)^{1/(r+2s)}};v,p\right). \tag{7.16}$$

We follow again [5]. Consider the Mellin transform (a function of $v'=(v_1',\dots,v_{r+s}')\in(i\mathbf{R})^{r+s},p'=(p_{r+1}',\dots,p_{r+s}')\in\mathbf{Z}^s$)

$$\widehat{W}^{\xi}_{\varpi,\mathfrak{t}}(v',p';v,p)=\int_{F^\times_{\infty,+}}W_{\varpi,\mathfrak{t}}(\xi y;v,p)\prod_{j=1}^{r+s}|y_j|^{v_j'}\prod_{j=r+1}^{r+s}\left(\frac{y_j}{|y_j|}\right)^{p_j'}d^\times_\infty y. \tag{7.17}$$

We would like to invert this. As for p', observe that $W_{\varpi,\mathfrak{t}}(y;v,p)$ is continuous on the set where each $|y_j|$ is fixed (this is the product of s circles), so the standard Fourier analysis of the circle group is applicable. Also from Corollary 6.2, we see that the set $(i\mathbf{R})^{r+s}$ (this is the product of $r+s$ lines) can be used for Mellin inversion (see [29, 17.41]). Therefore, (7.16) is

$$\ll_F\sum_{p'\in\mathbf{Z}^s}\int_{(i\mathbf{R})^{r+s}}(LY)^{(v_1'+\ldots+v_{r+s}')/(r+2s)}$$
$$\cdot\sum_{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}}\left(\widehat{W}^{\xi}_{\varpi,\mathfrak{t}}(v',p';v,p)\sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathcal{B}(\xi)}\frac{\lambda_\varpi^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}}\prod_{j=1}^{r+s}|q_j|^{-v_j'}\prod_{j=r+1}^{r+s}\left(\frac{q_j}{|q_j|}\right)^{-p_j'}\right)dv_j'.$$

By Cauchy-Schwarz, this is

$$\ll_F\sum_{p'\in\mathbf{Z}^s}\int_{(i\mathbf{R})^{r+s}}\left(\sum_{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}}\left|\widehat{W}^{\xi}_{\varpi,\mathfrak{t}}(v',p';v,p)\right|^2\right)^{1/2}$$
$$\left(\sum_{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}}\left|\sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathcal{B}(\xi)}\frac{\lambda_\varpi^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}}\prod_{j=1}^{r+s}|q_j|^{-v_j'}\prod_{j=r+1}^{r+s}\left(\frac{q_j}{|q_j|}\right)^{-p_j'}\right|^2\right)^{1/2}|dv_j'|. \tag{7.18}$$

In what follows, we estimate the Mellin part

$$\left(\sum_{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}}\left|\widehat{W}^{\xi}_{\varpi,\mathfrak{t}}(v',p';v,p)\right|^2\right)^{1/2} \tag{7.19}$$

and the arithmetic part

$$\left(\sum_{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}}\left|\sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathcal{B}(\xi)}\frac{\lambda_\varpi^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}}\prod_{j=1}^{r+s}|q_j|^{-v_j'}\prod_{j=r+1}^{r+s}\left(\frac{q_j}{|q_j|}\right)^{-p_j'}\right|^2\right)^{1/2} \tag{7.20}$$

separately.

7.4.1 Estimate of the Mellin part

Recall the definition (7.17) of the Mellin transform. Our plan is to insert differentiations (using that W's are highly differentiable) to show that the Mellin part decays fast in terms of $\mathcal{N}(v',p')$.

At real places ($j \le r$), for $v'_j \neq 0$,

$$\int_{\mathbf{R}_+^\times} W(y_j) y_j^{v'_j} d^\times_{\mathbf{R}} y_j = \int_0^\infty W(y_j) y_j^{v'_j-1} dy_j = -\frac{1}{v'_j}\int_0^\infty \frac{\partial}{\partial y_j} W(y_j) y_j^{v'_j} dy_j$$
$$= -\frac{1}{v'_j}\int_{\mathbf{R}_+^\times} y_j \frac{\partial}{\partial y_j} W(y_j) y_j^{v'_j} d^\times_{\mathbf{R}} y_j,$$

so at those real places, where $|v'_j| \ge 1$, we can gain a factor $|v'_j|^{-1}$ using the differential operator $y_j(\partial/\partial y_j)$. The complex places ($j > r$) can be handled similarly. For $v'_j \neq 0$,

$$\int_{\mathbf{C}^\times} W(y_j)|y_j|^{v'_j}\left(\frac{y_j}{|y_j|}\right)^{p'_j} d^\times_{\mathbf{C}} y_j = -\frac{1}{v'_j}\int_{\mathbf{C}^\times} |y_j| \frac{\partial}{\partial |y_j|} W(y_j)|y_j|^{v'_j}\left(\frac{y_j}{|y_j|}\right)^{p'_j} d^\times_{\mathbf{C}} y_j,$$

while for $p'_j \neq 0$,

$$\int_{\mathbf{C}^\times} W(y_j)|y_j|^{v'_j}\left(\frac{y_j}{|y_j|}\right)^{p'_j} d^\times_{\mathbf{C}} y_j = -\frac{1}{ip'_j}\int_{\mathbf{C}^\times} \frac{\partial}{\partial (y_j/|y_j|)} W(y_j)|y_j|^{v'_j}\left(\frac{y_j}{|y_j|}\right)^{p'_j} d^\times_{\mathbf{C}} y_j.$$

This means that at those complex places, where $|v'_j| \ge 1$ (or $|p'_j| \ge 1$, respectively), we can gain a factor $|v'_j|^{-1}$ (or $|p'_j|^{-1}$, respectively), by inserting the differential operator $y(\partial/\partial y)$ (or $\partial/\partial(y/|y|)$, respectively).

A simple calculation shows that for any real-differentiable complex function $f(z)$ with $z = re^{i\theta}$ ($r > 0$, $\theta \in [0, 2\pi]$), both $r\partial f/\partial r$ and $\partial f/\partial\theta$ are bounded by $\ll |z\partial f/\partial z| + |\overline{z}\partial f/\partial\overline{z}|$.

Therefore, set the differential operators

$$\mathcal{D}_{(e,f,g)} = \left(\left(\left(y_j\frac{\partial}{\partial y_j}\right)^{e_j}\right)_{j\le r}, \left(\left(y_j\frac{\partial}{\partial y_j}\right)^{f_j}\right)_{j>r}, \left(\left(\overline{y_j}\frac{\partial}{\partial \overline{y_j}}\right)^{g_j}\right)_{j>r}\right),$$

where $0 \le e_j \le 3$ ($j \le r$), $0 \le f_j \le 6$, $0 \le g_j \le 6$ ($j > r$). Then the above argument, together with (7.17) and Cauchy-Schwarz, implies that (7.19) is

$$\ll_F (\mathcal{N}(v',p'))^{-3/2} \sum_{(e,f,g)} \left(\int_{F^\times_{\infty,+}}\int_{F^\times_{\infty,+}} \left(\sum_{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} |\mathcal{D}_{(e,f,g)}W_{\varpi,\mathfrak{t}}(y;v,p)|^2\right)^{1/2}\right.$$
$$\left.\left(\sum_{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} |W_{\varpi,\mathfrak{t}}(y';v,p)|^2\right)^{1/2} d^\times_\infty y\, d^\times_\infty y'\right)^{1/2}.$$

Now we apply Theorem 1 with $a = 6, b = 2, c = 0$ in the first sum, and with $a = 0, b = 2, c = 0$ in the second sum. Together with (7.14), this implies that the integrand is

$$\ll_{F,\varepsilon} \mathcal{N}(\mathfrak{q})^\varepsilon \mathcal{N}(v,p)^{4\alpha_2} \prod_{j=1}^{r} \min(|y_j|^{1/4}, |y_j|^{-3/2}) \min(|y'_j|^{1/4}, |y'_j|^{-3/2})$$
$$\prod_{j=r+1}^{r+s} \min(|y_j|^{3/4}, |y_j|^{-1}) \min(|y'_j|^{3/4}, |y'_j|^{-1})$$

with some positive integer α_2 depending only on F. Altogether, the Mellin part (7.19) is

$$\ll_{F,\varepsilon} \mathcal{N}(\mathfrak{q})^\varepsilon \mathcal{N}(v,p)^{4\alpha_2}\mathcal{N}(v',p')^{-3/2}. \tag{7.21}$$

7.4.2 Estimate of the arithmetic part

Our next goal is to give a bound on (7.20), which is uniform in v', p'. Fix v', p' and consider

$$\sum_{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}}\left|\sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathcal{B}(\xi)}\frac{\lambda_\varpi^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}}\prod_{j=1}^{r+s}|q_j|^{-v_j'}\prod_{j=r+1}^{r+s}\left(\frac{q_j}{|q_j|}\right)^{-p_j'}\right|^2. \tag{7.22}$$

Following [5, p.45], introduce, for any ideal $\mathfrak{a}\subseteq\mathfrak{o}$,

$$f(\mathfrak{a};v',p')=\sum_{\substack{q\in\mathcal{B}(\xi)\\(q)=\mathfrak{a}\mathfrak{n}}}\prod_{j=1}^{r+s}|q_j|^{-v_j'}\prod_{j=r+1}^{r+s}\left(\frac{q_j}{|q_j|}\right)^{-p_j'}.$$

The number of possible units ϵ for which $q\epsilon\in\mathcal{B}$ is $\ll_{F,\varepsilon}\mathcal{N}(\mathfrak{q})^\varepsilon$ (recall the argument in Section 7.3.2), hence

$$|f(\mathfrak{a};v',p')|\ll_{F,\varepsilon}\mathcal{N}(\mathfrak{q})^\varepsilon. \tag{7.23}$$

With this notation, we can rewrite the innermost sum in (7.22) as

$$\sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathcal{B}(\xi)}\frac{\lambda_\varpi^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}}\prod_{j=1}^{r+s}|q_j|^{-v_j'}\prod_{j=r+1}^{r+s}\left(\frac{q_j}{|q_j|}\right)^{-p_j'}=\sum_{\mathcal{N}(\mathfrak{m})\ll LY/\mathcal{N}(\mathfrak{q}\mathfrak{n})}\frac{\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m}\mathfrak{q})}{\sqrt{\mathcal{N}(\mathfrak{m}\mathfrak{q})}}f(\mathfrak{m}\mathfrak{q};v',p'),$$

where $\ll$ in the sum means that we may choose a constant depending only on F such that this holds. Now on the right-hand side, for each occuring $\mathfrak{m}$, transfer each prime factor dividing both $\mathfrak{m}$ and $\mathfrak{q}$ from $\mathfrak{m}$ to $\mathfrak{q}$. This does not affect the summand (since it depends only on the product $\mathfrak{m}\mathfrak{q}$) and lets us write

$$\sum_{\mathcal{N}(\mathfrak{m})\ll LY/\mathcal{N}(\mathfrak{q}\mathfrak{n})}\frac{\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m}\mathfrak{q})}{\sqrt{\mathcal{N}(\mathfrak{m}\mathfrak{q})}}f(\mathfrak{m}\mathfrak{q};v',p')=\sum_{\mathfrak{q}|\mathfrak{q}'|\mathfrak{q}^\infty}\sum_{\substack{\mathcal{N}(\mathfrak{m})\ll LY/\mathcal{N}(\mathfrak{q}'\mathfrak{n})\\\gcd(\mathfrak{m},\mathfrak{q})=\mathfrak{o}}}\frac{\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m}\mathfrak{q}')}{\sqrt{\mathcal{N}(\mathfrak{m}\mathfrak{q}')}}f(\mathfrak{m}\mathfrak{q}';v',p'). \tag{7.24}$$

The following lemma expresses $\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m}\mathfrak{q}')$.

Lemma 7.2. *Assume $\mathfrak{m}$ and $\mathfrak{q}'$ are coprime ideals in $\mathfrak{o}$. Then*

$$\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m}\mathfrak{q}')=\sum_{\mathfrak{b}|\gcd(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1},\gcd(\mathfrak{q}',\mathfrak{t}))}\mu(\mathfrak{b})\lambda_\varpi(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1}\mathfrak{b}^{-1})\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m}\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{b}^{-1}).$$

Proof. We follow [6, pp.73-74]. (At [5, p.45], [6, pp.73-74] is adapted incorrectly. The corrected version can be found in the erratum of [5].)

By (2.18) and (2.35), we have

$$\begin{aligned}\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m}\mathfrak{q}')&=\sum_{\mathfrak{s}|\gcd(\mathfrak{m}\mathfrak{q}',\mathfrak{t})}\alpha_{\mathfrak{t},\mathfrak{s}}\mathcal{N}(\mathfrak{s})^{1/2}\lambda_\varpi(\mathfrak{m}\mathfrak{q}'\mathfrak{s}^{-1})\\&=\sum_{\substack{\mathfrak{s}_1|\gcd(\mathfrak{m},\mathfrak{t})\\\mathfrak{s}_2|\gcd(\mathfrak{q}',\mathfrak{t})}}\alpha_{\mathfrak{t},\mathfrak{s}_1\mathfrak{s}_2}\mathcal{N}(\mathfrak{s}_1\mathfrak{s}_2)^{1/2}\lambda_\varpi(\mathfrak{m}\mathfrak{q}'\mathfrak{s}_1^{-1}\mathfrak{s}_2^{-1})\\&=\sum_{\substack{\mathfrak{s}_1|\gcd(\mathfrak{m},\mathfrak{t})\\\mathfrak{s}_2|\gcd(\mathfrak{q}',\mathfrak{t})}}\alpha_{\mathfrak{t},\mathfrak{s}_1\mathfrak{s}_2}\mathcal{N}(\mathfrak{s}_1\mathfrak{s}_2)^{1/2}\lambda_\varpi(\mathfrak{q}'\mathfrak{s}_2^{-1})\lambda_\varpi(\mathfrak{m}\mathfrak{s}_1^{-1}),\end{aligned}$$

where the last equation holds by $\gcd(\mathfrak{m},\mathfrak{q}')=\mathfrak{o}$ and (2.27).

Inverting the multiplicativity relation (2.27), we see that

$$\begin{aligned}\lambda_\varpi(\mathfrak{q}'\mathfrak{s}_2^{-1})&=\lambda_\varpi(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1}\cdot\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{s}_2^{-1})\\&=\sum_{\mathfrak{b}|\gcd(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1},\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{s}_2^{-1})}\mu(\mathfrak{b})\lambda_\varpi(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1}\mathfrak{b}^{-1})\lambda_\varpi(\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{s}_2^{-1}\mathfrak{b}^{-1}).\end{aligned}$$

Writing this into the above display, we obtain that

$$\begin{aligned}\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m}\mathfrak{q}') = & \sum_{\mathfrak{b}|\gcd(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1},\gcd(\mathfrak{q}',\mathfrak{t}))} \mu(\mathfrak{b})\lambda_\varpi(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1}\mathfrak{b}^{-1}) \\ & \sum_{\substack{\mathfrak{s}_1|\gcd(\mathfrak{m},\mathfrak{t})\\ \mathfrak{s}_2|\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{b}^{-1}}} \alpha_{\mathfrak{t},\mathfrak{s}_1\mathfrak{s}_2}\mathcal{N}(\mathfrak{s}_1\mathfrak{s}_2)^{1/2}\lambda_\varpi(\mathfrak{m}\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{b}^{-1}\mathfrak{s}_1^{-1}\mathfrak{s}_2^{-1}) \\ = & \sum_{\mathfrak{b}|\gcd(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1},\gcd(\mathfrak{q}',\mathfrak{t}))} \mu(\mathfrak{b})\lambda_\varpi(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1}\mathfrak{b}^{-1}) \\ & \sum_{\mathfrak{s}|\gcd(\mathfrak{t},\mathfrak{m}\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{b}^{-1})} \alpha_{\mathfrak{t},\mathfrak{s}}\mathcal{N}(\mathfrak{s})^{1/2}\lambda_\varpi(\mathfrak{m}\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{b}^{-1}\mathfrak{s}^{-1}) \\ = & \sum_{\mathfrak{b}|\gcd(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1},\gcd(\mathfrak{q}',\mathfrak{t}))} \mu(\mathfrak{b})\lambda_\varpi(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1}\mathfrak{b}^{-1})\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m}\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{b}^{-1}),\end{aligned}$$

which completes the proof. □

Now we use this lemma in (7.24). By (2.26), we see

$$\lambda_\varpi(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1}\mathfrak{b}^{-1}) \ll_\varepsilon \mathcal{N}(\mathfrak{q}')^{\theta+\varepsilon}.$$

We claim $\gcd(\mathfrak{q}',\mathfrak{t})|\mathfrak{c}_\pi$. Indeed, $\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}$ with $\mathfrak{c}=\mathfrak{c}_\pi\mathrm{lcm}((l_1),(l_2))$, where l_1,l_2 are primes not dividing $\mathfrak{q}$. Altogether, the q-sum in (7.22) can be estimated as

$$\ll_{F,\varepsilon} \sum_{\mathfrak{q}|\mathfrak{q}'|\mathfrak{q}^\infty} \mathcal{N}(\mathfrak{q}')^{-1/2+\theta+\varepsilon} \sum_{\mathfrak{b}|\mathfrak{c}_\pi} \left| \sum_{\substack{\mathcal{N}(\mathfrak{m})\ll LY/\mathcal{N}(\mathfrak{q}'\mathfrak{n})\\ \gcd(\mathfrak{m},\mathfrak{q})=\mathfrak{o}}} \frac{\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m}\mathfrak{b})}{\sqrt{\mathcal{N}(\mathfrak{m})}} f(\mathfrak{m}\mathfrak{q}';v',p') \right|. \tag{7.25}$$

Similarly as in Lemma 5.4, take the function h defined in (5.2), (5.3) with $a_j=\mathcal{N}(\mathfrak{q})^{2\varepsilon}$ at real, $a_j=\mathcal{N}(\mathfrak{q})^{\varepsilon}$ at complex places, $b_j=\sqrt{a_j}$ at all archimedean places, finally $a_j'=-1$ at complex places. This has the property that it gives weight $\gg_F 1$ to representations in $\mathcal{C}(\mathfrak{c},\varepsilon)$.

$$\begin{aligned}\sum_{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} \left| \sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathcal{B}(\xi)} \frac{\lambda_\varpi^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}} \prod_{j=1}^{r+s}|q_j|^{-v_j'} \prod_{j=r+1}^{r+s}\left(\frac{q_j}{|q_j|}\right)^{-p_j'} \right|^2 & \\ \ll_{F,\varepsilon} \sum_{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} h(\mathbf{r}_\varpi) \left| \sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathcal{B}(\xi)} \frac{\lambda_\varpi^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}} \prod_{j=1}^{r+s}|q_j|^{-v_j'} \prod_{j=r+1}^{r+s}\left(\frac{q_j}{|q_j|}\right)^{-p_j'} \right|^2 & .\end{aligned}$$

In the summation over ϖ, multiply by a factor C_ϖ^{-1}, which is $\gg_{F,\varepsilon,\pi} \mathcal{N}(\mathfrak{q})^{-\varepsilon}$ by (3.5) and Proposition 3.2. We also add the analogous nonnegative contribution of the Eisenstein spectrum.

Therefore, using (7.23), (7.25) estimates the ϖ-sum of (7.22) as

$$\begin{aligned}\sum_{\varpi\in\mathcal{C}(\mathfrak{c},\varepsilon)} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} \left| \sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathcal{B}(\xi)} \frac{\lambda_\varpi^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}} \prod_{j=1}^{r+s}|q_j|^{-v_j'} \prod_{j=r+1}^{r+s}\left(\frac{q_j}{|q_j|}\right)^{-p_j'} \right|^2 & \\ \ll_{F,\varepsilon,\pi} \mathcal{N}(\mathfrak{q})^{-1+2\theta+\varepsilon} \max_{\mathfrak{b}_1,\mathfrak{b}_2|\mathfrak{c}_\pi} \sum_{\mathcal{N}(\mathfrak{m}_1),\mathcal{N}(\mathfrak{m}_2)\ll LY/\mathcal{N}(\mathfrak{q})} \frac{1}{\sqrt{\mathcal{N}(\mathfrak{m}_1\mathfrak{m}_2)}} & \\ \left| \sum_{\varpi\in\mathcal{C}(\mathfrak{c})} C_\varpi^{-1} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} h(\mathbf{r}_\varpi)\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m}_1\mathfrak{b}_1)\overline{\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m}_2\mathfrak{b}_2)} + CSC \right| & .\end{aligned} \tag{7.26}$$

We apply the Kuznetsov formula (5.1) to estimate the last line of (7.26), with $\alpha=\alpha'=1$, $\mathfrak{a}^{-1}=\mathfrak{m}_1\mathfrak{b}_1$, $\mathfrak{a}'^{-1}=\mathfrak{m}_2\mathfrak{b}_2$. The delta term is, up to a constant multiple,

$$[K(\mathfrak{o}):K(\mathfrak{c})]\Delta(\mathfrak{m}_1\mathfrak{b}_1,\mathfrak{m}_2\mathfrak{b}_2)\int h(\mathbf{r}_\varpi)d\mu.$$

Here, by Lemma 5.3, the integral of h gives $\ll_{F,\varepsilon} \mathcal{N}(\mathfrak{q})^{2(r+s)\varepsilon}$, and we also have $[K(\mathfrak{o}):K(\mathfrak{c})] \ll_{F,\varepsilon,\pi} L^2\mathcal{N}(\mathfrak{q})^{\varepsilon}$ by (3.2) and (7.6). When $\Delta(\mathfrak{m}_1\mathfrak{b}_1,\mathfrak{m}_2\mathfrak{b}_2)\neq 0$, $\mathcal{N}(\mathfrak{m}_1)\approx_{F,\pi}\mathcal{N}(\mathfrak{m}_2)$, so the sum over $\mathfrak{m}_1,\mathfrak{m}_2$ can be replaced by a sum over $\mathfrak{m}$. Using (7.5), we see that $LY/\mathcal{N}(\mathfrak{q})\ll_{\varepsilon}\mathcal{N}(\mathfrak{q})^{\varepsilon}L$, and taking into account also (7.6), we obtain that

$$\sum_{\mathcal{N}(\mathfrak{m})\ll LY/\mathcal{N}(\mathfrak{q})}\frac{1}{\mathcal{N}(\mathfrak{m})}\ll_{F,\varepsilon}\mathcal{N}(\mathfrak{q})^{\varepsilon}.$$

Altogether, the delta term of the geometric side of the Kuznetsov formula (5.1) contributes

$$\ll_{F,\varepsilon,\pi}\mathcal{N}(\mathfrak{q})^{-1+2\theta+\varepsilon}L^2 \tag{7.27}$$

to the right-hand side of (7.26).

As for the Kloosterman term, similarly to (5.4), we have to estimate

$$\max_{\mathfrak{a}\in C}\sum_{\epsilon\in\mathfrak{o}_+^\times/\mathfrak{o}^{2\times}}\sum_{0\neq c\in\mathfrak{m}_1^{-1}\mathfrak{b}_1^{-1}\mathfrak{a}\mathfrak{c}}\frac{\mathcal{N}((\gcd(\mathfrak{m}_1\mathfrak{b}_1,\mathfrak{m}_2\mathfrak{b}_2,c\mathfrak{m}_1\mathfrak{b}_1\mathfrak{a}^{-1})))^{1/2}}{\mathcal{N}(c\mathfrak{m}_1\mathfrak{b}_1\mathfrak{a}^{-1})^{1/2-\varepsilon}}$$
$$\cdot\prod_{j\leq r}\min(1,|\epsilon_j\gamma_{\mathfrak{a},j}/c_j|^{1/2})\prod_{j>r}\min(1,|\epsilon_j\gamma_{\mathfrak{a},j}/c_j|),$$

where $\gamma_{\mathfrak{a}}$ is a totally positive generator of the ideal $\mathfrak{a}^2(\mathfrak{m}_1\mathfrak{b}_1)^{-1}\mathfrak{m}_2\mathfrak{b}_2$, C is a fixed set of narrow class representatives (depending only on F and the narrow class of $(\mathfrak{m}_1\mathfrak{b}_1)^{-1}\mathfrak{m}_2\mathfrak{b}_2$) with the property that such a $\gamma_{\mathfrak{a}}$ exists for each $\mathfrak{a}\in C$. The sum over $\epsilon\in\mathfrak{o}_+^\times/\mathfrak{o}^{2\times}$ is negligible. Now take a totally positive $\beta\in\mathfrak{o}$ such that $(\beta)\supseteq\mathfrak{m}_1\mathfrak{b}_1$, $\mathcal{N}((\beta))\gg_F\mathcal{N}(\mathfrak{m}_1\mathfrak{b}_1)$, and then the above is

$$\ll_F\max_{\mathfrak{a}\in C}\sum_{0\neq c\in\mathfrak{a}\mathfrak{c}}\frac{\mathcal{N}((\gcd(\mathfrak{m}_1\mathfrak{b}_1,\mathfrak{m}_2\mathfrak{b}_2,c\mathfrak{a}^{-1})))^{1/2}}{\mathcal{N}(c\mathfrak{a}^{-1})^{1/2-\varepsilon}}$$
$$\cdot\prod_{j\leq r}\min(1,|\gamma_{\mathfrak{a},j}\beta_j|^{1/4}/|c_j|^{1/2})\prod_{j>r}\min(1,|\gamma_{\mathfrak{a},j}\beta_j|^{1/2}/|c_j|),$$

Then the same method as in the proof of Lemma 5.4 shows that the previous display can be estimated as

$$\ll_{F,\varepsilon,\pi}\mathcal{N}((\gamma_{\mathfrak{a}}\beta))^{1/4+\varepsilon}\mathcal{N}(\gcd(\mathfrak{m}_1,\mathfrak{m}_2,\mathfrak{c}))^{1/2+\varepsilon}\mathcal{N}(\mathfrak{c})^{-1-\varepsilon}.$$

The last factor $\mathcal{N}(\mathfrak{c})^{-1-\varepsilon}$ cancels $[K(\mathfrak{o}):K(\mathfrak{c})]$. Noting that $\mathcal{N}((\gamma_{\mathfrak{a}}\beta))\ll_{F,\pi}\mathcal{N}(\mathfrak{m}_1\mathfrak{m}_2)$, we see that the Kloosterman term contributes

$$\ll_{F,\varepsilon,\pi}\mathcal{N}(\mathfrak{q})^{-1+2\theta+\varepsilon}\sum_{\mathcal{N}(\mathfrak{m}_1),\mathcal{N}(\mathfrak{m}_2)\ll LY/\mathcal{N}(\mathfrak{q})}\mathcal{N}(\gcd(\mathfrak{m}_1,\mathfrak{m}_2,\mathfrak{c}))^{1/2+\varepsilon}\mathcal{N}(\mathfrak{m}_1\mathfrak{m}_2)^{-1/4+\varepsilon}$$

to the right-hand side of (7.26). Obviously

$$\mathcal{N}(\gcd(\mathfrak{m}_1,\mathfrak{m}_2,\mathfrak{c}))^{1/2}\leq\mathcal{N}(\gcd(\mathfrak{m}_1,\mathfrak{c}))^{1/4}\mathcal{N}(\gcd(\mathfrak{m}_2,\mathfrak{c}))^{1/4},$$

so the above display is (using also (7.5) and (7.6) again)

$$\ll_{F,\varepsilon}\mathcal{N}(\mathfrak{q})^{-1+2\theta+2\varepsilon}\left(\sum_{\mathcal{N}(\mathfrak{m})\ll LY/\mathcal{N}(\mathfrak{q})}\left(\frac{\mathcal{N}(\gcd(\mathfrak{m},\mathfrak{c}))}{\mathcal{N}(\mathfrak{m})}\right)^{1/4}\right)^2.$$

Here, if $\mathfrak{m}$ is divisible by l_1 or l_2, then $\mathcal{N}(\gcd(\mathfrak{m},\mathfrak{c}))\ll L$ (by (7.6), an ideal of norm $\ll\mathcal{N}(\mathfrak{q})^{\varepsilon}L$ cannot have two different prime divisors l_1,l_2), this happens at most for $\mathcal{N}(\mathfrak{q})^{\varepsilon}$ many $\mathfrak{m}$'s. For other $\mathfrak{m}$'s, $\mathcal{N}(\gcd(\mathfrak{m},\mathfrak{c}))\ll_{\pi}1$. Therefore, the Kloosterman contribution to (7.26) is

$$\ll_{F,\varepsilon,\pi}\mathcal{N}(\mathfrak{q})^{-1+2\theta+\varepsilon}L^{3/2}. \tag{7.28}$$

Taking square-roots, we obtain from (7.27) and (7.28) that the arithmetic part (7.20) is

$$\ll_{F,\varepsilon,\pi}\mathcal{N}(\mathfrak{q})^{-1/2+\theta+\varepsilon}L. \tag{7.29}$$

7.4.3 Summing up in the cuspidal spectrum

Inside $\mathcal{C}(\mathfrak{c},\varepsilon)$, (7.18), (7.21) and (7.29) show that the contribution (7.16) is

$$\ll_{F,\varepsilon} \sum_{p'\in\mathbf{Z}^s}\int_{(i\mathbf{R})^{r+s}} \mathcal{N}(v,p)^{4\alpha_2}\mathcal{N}(v',p')^{-3/2}\mathcal{N}(\mathfrak{q})^{-1/2+\theta+\varepsilon}L|dv'_j| \ll_{F,\varepsilon} \mathcal{N}(\mathfrak{q})^{-1/2+\theta+\varepsilon}L\mathcal{N}(v,p)^{4\alpha_2},$$

and this bound holds (with the implicit constant multiplied by 2^r) without rectricting the summation in (7.18) to a specific sign ξ.

Now we concentrate on representations outside $\mathcal{C}(\mathfrak{c},\varepsilon)$. First of all, from Lemma 5.4, we see that

$$\lambda^{\mathfrak{t}}_{\varpi}(q\mathfrak{n}^{-1}) \ll_{F,\varepsilon,\pi} L^{1/2+\varepsilon}\mathcal{N}((q))^{1/4+\varepsilon}\mathcal{N}(\mathbf{r}_{\varpi}),$$

therefore, with a large c' (depending on ε), we may write (using (7.6)), outside $\mathcal{C}(\mathfrak{c},\varepsilon)$,

$$\frac{\lambda^{\mathfrak{t}}_{\varpi}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}} \ll_{F,\varepsilon,\pi} L^{1/2+\varepsilon}\mathcal{N}(\mathfrak{q})^{-1/4}\mathcal{N}(\mathbf{r}_{\varpi}) \ll_{F,\varepsilon,\pi} \mathcal{N}(\mathbf{r}_{\varpi})^{c'}.$$

Now by Cauchy-Schwarz, outside $\mathcal{C}(\mathfrak{c},\varepsilon)$, the cuspidal contribution is, with some c much larger than c',

$$\ll_{F,\varepsilon,\pi}\left(\sum_{0\neq q\in q\mathfrak{n}\cap\mathcal{B}}\ \sum_{\varpi\notin\mathcal{C}(\mathfrak{c},\varepsilon)}\ \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} \mathcal{N}(\mathbf{r}_{\varpi})^{2(c'-c)}\right)^{1/2}$$
$$\left(\sum_{0\neq q\in q\mathfrak{n}\cap\mathcal{B}}\ \sum_{\varpi\notin\mathcal{C}(\mathfrak{c},\varepsilon)}\ \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} \left|\mathcal{N}(\mathbf{r}_{\varpi})^{c}W_{\varpi,\mathfrak{t}}\left(\frac{q}{(LY)^{1/(r+2s)}};v,p\right)\right|^2\right)^{1/2}.$$

The first factor is $\mathcal{N}(\mathfrak{q})^{-k}$ for any $k\in\mathbf{N}$, if $c-c'$ is large enough, as it follows from Lemma 5.5. As for the second factor, apply Theorem 1, Part B with $a=0$, $b=0$ and the above c. The number of q's in $\mathfrak{q}\mathfrak{n}\cap\mathcal{B}$ is $O_F(LY)$. Then together with (7.14), we see that the second factor is $\ll_{F,\varepsilon,\pi}\mathcal{N}(\mathfrak{q})^{-1+\varepsilon}\mathcal{N}(v,p)^{4\alpha_3}$ with some positive integer α_3. To match Y and L with $\mathcal{N}(\mathfrak{q})$, we use (7.5) and (7.6) throughout.

Altogether, the cuspidal spectrum has contribution

$$\ll_{F,\varepsilon,\pi}\mathcal{N}(v,p)^{4\max(\alpha_2,\alpha_3)}\mathcal{N}(\mathfrak{q})^{-1/2+\theta+\varepsilon}L. \tag{7.30}$$

7.5 Choice of the amplification length

Set $\alpha=\max(\alpha_1,\alpha_2,\alpha_3)$. Summing trivially over l_1,l_2, and using (7.11), (7.13), (7.15) and (7.30), we see

$$ODC \ll_{F,\varepsilon,\pi}\mathcal{N}(v,p)^{4\alpha}\mathcal{N}(\mathfrak{q})^{1/2+\theta+\varepsilon}L.$$

This estimate, together with (7.10) and through (7.7), (7.8), (7.9), gives rise to

$$|\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)|^2 \ll_{F,\varepsilon,\pi}\mathcal{N}(v,p)^{4\alpha}(\mathcal{N}(\mathfrak{q})^{1+\varepsilon}L^{-1}+\mathcal{N}(\mathfrak{q})^{1/2+\theta+\varepsilon}L),$$
$$|\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)| \ll_{F,\varepsilon,\pi}\mathcal{N}(v,p)^{2\alpha}(\mathcal{N}(\mathfrak{q})^{1/2+\varepsilon}L^{-1/2}+\mathcal{N}(\mathfrak{q})^{1/4+\theta/2+\varepsilon}L^{1/2}).$$

We see that the optimal choice is $L=\mathcal{N}(\mathfrak{q})^{1/4-\theta/2}$, which meets the condition (7.6). With this, we obtain the bound

$$|\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)| \ll_{F,\varepsilon,\pi}\mathcal{N}(v,p)^{2\alpha}\mathcal{N}(\mathfrak{q})^{3/8+\theta/4+\varepsilon}. \tag{7.31}$$

7.6 A subconvex bound for L-functions

Let us return to the point of the introduction of this chapter. Our aim is to bound $L(1/2,\pi\otimes\chi)$ in terms of $\mathcal{N}(\mathfrak{q})$, where $\mathfrak{q}$ is the conductor of χ.

Theorem 2.

$$L(1/2,\pi\otimes\chi)\ll_{F,\pi,\chi_\infty,\varepsilon}\mathcal{N}(\mathfrak{q})^{3/8+\theta/4+\varepsilon}.$$

Proof. We start out from (3.9) as follows. First of all, we split up the sum over ideals according to their narrow class (with representatives $\mathfrak{n}$ satisfying (7.1)). Then

$$L(1/2,\pi\otimes\chi)\ll_{\pi,\chi_\infty,\varepsilon}\mathcal{N}(\mathfrak{q})^{\varepsilon}\max_{Y\leq c\mathcal{N}(\mathfrak{q})^{1+\varepsilon}}\left|\sum_{0<<t\in\mathfrak{n}\ (\mathrm{mod}\ \mathfrak{o}_{+}^{\times})}\frac{\lambda_\pi(t\mathfrak{n}^{-1})\chi(t\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(t\mathfrak{n}^{-1})}}V\left(\frac{|t|_\infty}{Y}\right)\right|$$

for some $c=c(\pi,\chi_\infty,\varepsilon)$, hence (7.5) is satisfied. Here, by the partition of unity introduced in the beginning of this chapter, the sum on the right-hand side can be rewritten as

$$\sum_{0<<t\in\mathfrak{n}}\frac{\lambda_\pi(t\mathfrak{n}^{-1})\chi(t\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(t\mathfrak{n}^{-1})}}G(t_\infty)V\left(\frac{|t|_\infty}{Y}\right)W\left(\frac{t_\infty}{Y^{1/(r+2s)}}\right),$$

where W is a smooth nonnegative function which is 1 on $[[c_1,c_2]]$ and supported on $[[c_3,c_4]]$. Now introducing the Mellin transform

$$\widehat{V}(v,p)=\int_{F_{\infty,+}^{\times}}G(y)V(y)\chi_\infty(y)\prod_{j=1}^{r+s}|y_j|^{v_j}\prod_{j=r+1}^{r+s}\left(\frac{y_j}{|y_j|}\right)^{p_j}d^{\times}y,$$

we have, by Mellin inversion, that the above display is

$$\ll_F\sum_{p\in\mathbf{Z}^s}\int_{v\in(i\mathbf{R})^{r+s}}\widehat{V}(v,p)\sum_{0<<t\in\mathfrak{n}}\frac{\lambda_\pi(t\mathfrak{n}^{-1})\chi_{\mathrm{fin}}(t)}{\sqrt{\mathcal{N}(t\mathfrak{n}^{-1})}}W_{v,p}\left(\frac{t}{Y^{1/(r+2s)}}\right)dv,$$

where

$$W_{v,p}(y)=W(y)\prod_{j=1}^{r+s}|y_j|^{-v_j}\prod_{j=r+1}^{r+s}\left(\frac{y_j}{|y_j|}\right)^{-p_j}d^{\times}y.$$

Since $F(y)$, $V(y)$, $W(y)$ are all smooth and compactly supported, we see that

$$\widehat{V}(v,p)\ll_{\beta,\chi_\infty}\mathcal{N}(v,p)^{-\beta}$$

for all $\beta\in\mathbf{N}$ and also that the family of $W_{v,p}$'s satisfies (i) and (ii). Then

$$L(1/2,\pi\otimes\chi)\ll_{F,\pi,\chi_\infty,\varepsilon,\beta}\sum_{p\in\mathbf{Z}^s}\int_{v\in(i\mathbf{R})^{r+s}}\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)\mathcal{N}(v,p)^{-\beta}dv$$

with $\mathcal{L}$ of (7.4) satisfying all conditions we needed in its estimate. Now taking a β which is much larger than 2α, (7.31) completes the proof. □

Remark 3. For a cuspidal representation π of arbitrary central character (recall Remark 1), the same bound holds with the same proof, see also Remark 2.

Chapter 8

A semi-adelic Kuznetsov formula over number fields

The content of this chapter is more or less the same as of [46]. Our approach follows [56], borrowing the archimedean investigations from [9], [10], [13] and [45].

8.1 Some preliminaries about the group $\mathrm{SL}_2(\mathbf{C})$

The available literature about complex places is much smaller than that about real places, so we quote the details up to some extent. We mainly follow the works of Bruggeman, Motohashi and Lokvenec-Guleska [12], [13], [45]. For the notation, recall Chapter 1.

First record the Iwasawa decomposition: any element $g \in \mathrm{SL}_2(\mathbf{C})$ can be uniquely written in the form $g = n(x)a(y)k[\alpha,\beta]$, where

$$n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad a(y) = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix},$$

$x \in \mathbf{C}$, $y > 0$ real. This y will be referred as the height of g.

Assume some ν, p, l, q are given (in $\mathbf{r}$ and $\mathbf{w}$), then let

$$\varphi(n(x)a(y)k[\alpha,\beta]) = y^{1+\nu}\Phi^l_{p,q}(k[\alpha,\beta]).$$

When it is needed, we indicate the dependence on the weight and spectral data and write $\varphi_{l,q}(\nu,p)$.

8.1.1 Jacquet integral

Following [45, Section 4.1], for $\omega \in \mathbf{C}$, and $f \in C^\infty(\mathrm{SL}_2(\mathbf{C}))$ satisfying the growth condition

$$f(n(x)a(y)k[\alpha,\beta]) = O(y^{1+\sigma})$$

with some $\sigma > 0$, define the Jacquet integral

$$\mathbf{J}_\omega f(g) = \int_{\mathbf{C}} e^{-2\pi i(\omega x + \overline{\omega x})} f(\mathrm{w} n(x) g)\, d\Re x\, d\Im x,$$

where $\mathrm{w} = k[0,1]$ stands for the Weyl element. For $0 \neq \omega \in \mathbf{C}$ and $f = \varphi$, we drop φ from the notation and simply write $\mathbf{J}_\omega$ in place of $\mathbf{J}_\omega\varphi$. This can be computed (see [13, Section 5] and [45, Section 4.1]) to be

$$\begin{aligned}\mathbf{J}_\omega(n(x)a(y)k[\alpha,\beta]) = (-1)^{l-p}(2\pi)^\nu|\omega|^{\nu-1}e^{2\pi i(\omega x+\overline{\omega x})} \\ \cdot \sum_{|m|\le l}\left(\frac{i\omega}{|\omega|}\right)^{-p-m} w^l_m(\nu,p;|\omega|y)\Phi^l_{m,q}(k[\alpha,\beta]),\end{aligned} \tag{8.1}$$

where

$$w^l_m(\nu,p;y) = \sum_{j=0}^{l-\frac{1}{2}(|m+p|+|m-p|)} (-1)^j \xi^l_p(m,j)\frac{(2\pi y)^{l+1-j}}{\Gamma(l+1+\nu-j)} K_{\nu+l-|m+p|-j}(4\pi y), \tag{8.2}$$

K denoting the K-Bessel function, and

$$\xi_p^l(m,j) = \frac{j!(2l-j)!}{(l-p)!(l+p)!}\binom{l-\frac{1}{2}(|m+p|+|m-p|)}{j}\binom{l-\frac{1}{2}(|m+p|-|m-p|)}{j}. \tag{8.3}$$

Note that a priori we need $\Re\nu > 0$, but we can remove this condition by analytic continuation. Compare this with (2.21), (2.22), (2.23), and also with (4.9).

8.1.2 Goodman-Wallach operator

Another operator we need, is the Goodman-Wallach operator, which we specialize again to φ and obtain (see [13, Section 6] and [45, Section 4.2])

$$\mathbf{M}_\omega(n(x)a(y)k[\alpha,\beta]) = (2\pi|\omega|)^{-\nu-1}e^{2\pi i(\omega x+\overline{\omega x})} \cdot \sum_{|m|\le l}\left(\frac{-i\omega}{|\omega|}\right)^{p-m}\mu_m^l(\nu,p;|\omega|y)\Phi_{m,q}^l(k[\alpha,\beta]),$$

where

$$\mu_m^l(\nu,p;y) = \sum_{j=0}^{l-\frac{1}{2}(|m+p|+|m-p|)}\xi_p^l(m,j)\frac{(2\pi y)^{l+1-j}}{\Gamma(l+1+\nu-j)}I_{\nu+l-|m+p|-j}(4\pi y),$$

I denoting the I-Bessel function. Again, occasionally, we may write $\mathbf{M}_\omega\varphi$ or even $\mathbf{M}_\omega\varphi_{l,q}(\nu,p)$, when there is any danger of confusion.

We cite [13, Lemma 6.2], [45, Lemma 4.2.2] for the relations matching these operators. Here, we have to indicate the dependence on ν, p. Let $\omega_2 \neq 0$, $\Re\nu > 0$. Then

$$\mathbf{J}_0\mathbf{M}_{\omega_2} = \frac{\sin\pi(\nu-p)}{\nu^2-p^2}\frac{\Gamma(l+1-\nu)}{\Gamma(l+1+\nu)}\varphi(-\nu,-p). \tag{8.4}$$

and for $\omega_1 \neq 0$,

$$\mathbf{J}_{\omega_1}\mathbf{M}_{\omega_2} = \mathcal{J}_{\nu,p}^*(4\pi\sqrt{\omega_1\omega_2})\mathbf{J}_{\omega_1}, \tag{8.5}$$

with

$$\mathcal{J}_{\nu,p}^*(z) = J_{\nu-p}^*(z)J_{\nu+p}^*(\overline{z}), \tag{8.6}$$

where J_ν^* is the even entire function of z which is equal to $J_\nu(z)(z/2)^{-\nu}$ for $z > 0$ (J stands for the J-Bessel function).

8.1.3 Complex Whittaker functions form an orthonormal basis of $L^2(\mathbf{C}^\times, d_{\mathbf{C}}^\times y)$

First we compute the L^2-norm of $\mathbf{J}_1$, confirming the normalization we used in (2.21). What follows is analogous to [12, Theorem 1], the difference is that we work it out for the complementary series as well (i.e. we do not require $\Re\nu = 0$).

Lemma 8.1.

$$\int_0^\infty |\mathbf{J}_1(a(y))|^2\frac{dy}{y} = \frac{(2\pi)^{2\Re\nu}}{8(2l+1)}\binom{2l}{l-q}^{-1}\binom{2l}{l-p}\left|\frac{\Gamma(l+1-\nu)}{\Gamma(l+1+\nu)}\right|.$$

Proof. First observe that $\Phi_{m,q}^l(k[1,0]) = 0$, if $m \neq q$, and $\Phi_{q,q}^l(k[1,0]) = 1$. Hence in [45, (4.8)], we may write

$$\mathbf{J}_1(a(y)) = v_q^l(y,1),$$

where

$$v_q^l(y,1) = y^{1-\nu}\int_{\mathbf{C}}\frac{e^{-2\pi iy(z+\overline{z})}}{(1+|z|^2)^{1+\nu}}\Phi_{p,q}^l\left(k\left[\frac{\overline{z}}{\sqrt{1+|z|^2}},\frac{-1}{\sqrt{1+|z|^2}}\right]\right)d\Re z\, d\Im z.$$

So what is left is to compute $\int_0^\infty |v_q^l(y,1)|^2 dy/y$.

First let $q = l$. Then by (8.1), (8.2), (8.3), this integral is

$$(2\pi)^{\nu+\overline{\nu}} \frac{1}{\Gamma(l+1+\nu)\Gamma(l+1+\overline{\nu})} \left(\frac{(2l)!}{(l-p)!(l+p)!} \right)^2 \int_0^\infty (2\pi y)^{2l+2} |K_{\nu-p}(4\pi y)|^2 \frac{dy}{y}. \tag{8.7}$$

To compute the inner integral, we use [29, 6.576(4)]:

$$\int_0^\infty r^{2l+1} |K_\nu(r)|^2 dr = \frac{2^{2l-1}}{(2l+1)!} \Gamma\left(l+1+\frac{\nu}{2}-\frac{\overline{\nu}}{2}\right) \Gamma\left(l+1-\frac{\nu}{2}+\frac{\overline{\nu}}{2}\right) \Gamma\left(l+1+\frac{\nu}{2}+\frac{\overline{\nu}}{2}\right) \Gamma\left(l+1-\frac{\nu}{2}-\frac{\overline{\nu}}{2}\right),$$

the conditions are all satisfied by noting $|\Re\nu| < 1/2$.

First assume we are in the principal series ($\Re\nu = 0$). Then (8.7) equals

$$\frac{1}{2^{2l+2}} \frac{1}{\Gamma(l+1+\nu)\Gamma(l+1-\nu)} \left(\frac{(2l)!}{(l-p)!(l+p)!} \right)^2 \frac{2^{2l-1}}{(2l+1)!} \Gamma(l+1+\nu)\Gamma(l+1-\nu)(l+p)!(l-p)! = \frac{1}{8(2l+1)} \binom{2l}{l-p}.$$

Now assume we are in the complementary series ($\Im\nu = 0$, $\nu \neq 0$). Then $p = 0$ and for (8.7), we obtain

$$\frac{(2\pi)^{2\nu}}{2^{2l+2}} \frac{1}{\Gamma(l+1+\nu)\Gamma(l+1+\nu)} \left(\frac{(2l)!}{(l!)^2} \right)^2 \frac{2^{2l-1}}{(2l+1)!} \Gamma(l+1+\nu)\Gamma(l+1-\nu)(l!)^2 = \frac{(2\pi)^{2\nu}}{8(2l+1)} \binom{2l}{l} \frac{\Gamma(l+1-\nu)}{\Gamma(l+1+\nu)}.$$

For a general $|q| \leq l$, the identity [12, p.89]

$$\int_0^\infty |v_q^l(y)|^2 \frac{dy}{y} = \binom{2l}{l-q}^{-1} \int_0^\infty |v_l^l(y)|^2 \frac{dy}{y}$$

completes the proof. □

On the other hand, we prove that two complex Whittaker functions are orthogonal, provided that they are of the same spectral and different weight parameter.

Lemma 8.2. *For a fixed pair (ν, p), if $(l, q) \neq (l', q')$, then*

$$\int_{\mathbf{C}^\times} \mathcal{W}_{(l,q),(\nu,p)}(y) \overline{\mathcal{W}_{(l',q'),(\nu,p)}(y)} d_{\mathbf{C}}^\times y = 0.$$

Proof. This is again covered by [12, Theorem 1] for the principal series. If $q \neq q'$, it is clear from (2.21) that rewriting $\int_{\mathbf{C}^\times} \ldots d_{\mathbf{C}}^\times y$ as $(2\pi)^{-1} \int_0^\infty \int_0^{2\pi} \ldots d\theta r^{-1} dr$, the integral of $\mathcal{W}_{(l,q),(\nu,p)}(y) \overline{\mathcal{W}_{(l',q'),(\nu,p)}(y)}$ vanishes, since the two Whittaker functions are transformed via two different characters of the circle group. If $q = q'$ and $l \neq l'$, then we can repeat the proof of [12, Theorem 1], the differential operators D_q^+ and D_q^- extend to the complementary series by analytic continuation. □

8.2 Notation and the statement of the formula

8.2.1 Kloosterman sums

We quote the definition of Kloosterman sums from [56, Definition 2] (see also [56, Definition 1]). Let $\mathfrak{a}_1, \mathfrak{a}_2$ be fractional ideals of F, and $\mathfrak{c}$ be any ideal such that $\mathfrak{c}^2 \sim \mathfrak{a}_1\mathfrak{a}_2$ (i.e. they are in the same ideal class). Let then $c \in \mathfrak{c}^{-1}$, $\alpha_1 \in \mathfrak{a}_1^{-1}\mathfrak{d}^{-1}$, $\alpha_2 \in \mathfrak{a}_1\mathfrak{d}^{-1}\mathfrak{c}^{-2}$. We define the Kloosterman sum as

$$KS(\alpha_1, \mathfrak{a}_1; \alpha_2, \mathfrak{a}_2; c, \mathfrak{c}) = \sum_{x \in (\mathfrak{a}_1\mathfrak{c}^{-1}/\mathfrak{a}_1 c)^\times} \psi_\infty\left(\frac{\alpha_1 x + \alpha_2 x^{-1}}{c} \right),$$

where the summation runs through the x's which generate $\mathfrak{a}_1\mathfrak{c}^{-1}/\mathfrak{a}_1 c$ as an $\mathfrak{o}$-module, and x^{-1} is the unique element in $(\mathfrak{a}_1^{-1}\mathfrak{c}/\mathfrak{a}_1^{-1}c\mathfrak{c}^2)^\times$ such that $xx^{-1} \in 1 + c\mathfrak{c}$; ψ_∞ is the archimedean character defined as $\psi_\infty(x) = \exp(2\pi i \mathrm{Tr}(x)) = \exp(2\pi i(x_1 + \ldots + x_r + x_{r+1} + \overline{x_{r+1}} + \ldots + x_{r+s} + \overline{x_{r+s}}))$.

8.2.2 Archimedean Bessel transforms and measures

In the Kuznetsov formula, on the so-called geometric side, the weight functions are Bessel transforms. Assume $f(\mathbf{r})$ is a function of the form

$$f(\mathbf{r}) = \prod_{j=1}^{r} f_j(\nu_j) \prod_{j=r+1}^{r+s} f_j(\nu_j, p_j),$$

where f_j's are functions on the possible spectral parameter values: ν_j in the real case, (ν_j, p_j) in the complex case, these are encoded in $\mathbf{r}$ (recall (2.6), (2.7), (2.13)). Then let

$$\mathcal{B}f_{(\mathbf{r})}(z) = \prod_{j=1}^{r} (\mathcal{B}_j f_j)_{\nu_j}(z) \prod_{j=r+1}^{r+s} (\mathcal{B}_j f_j)_{(\nu_j,p_j)}(z),$$

where $\mathcal{B}_j f_j$ is defined as follows. At real places,

$$(\mathcal{B}_j f_j)_{\nu_j}(z) = f_j(\nu_j) \cdot (\mathcal{B}_j)_{\nu_j}(z),$$
$$(\mathcal{B}_j)_{\nu_j}(z) = \frac{2\pi}{\sin \pi\nu_j}(J_{-2\nu_j}(|z|) - J_{2\nu_j}(|z|)),$$

J standing for the J-Bessel function. At complex places,

$$(\mathcal{B}_j f_j)_{(\nu_j,p_j)}(z) = f_j(\nu_j, p_j) \cdot (\mathcal{B}_j)_{(\nu_j,p_j)}(z),$$
$$(\mathcal{B}_j)_{(\nu_j,p_j)}(z) = \frac{|z/2|^{-2\nu_j}(iz/|z|)^{2p_j}\mathcal{J}^*_{-\nu_j,-p_j}(z) - |z/2|^{2\nu_j}(iz/|z|)^{-2p_j}\mathcal{J}^*_{\nu_j,p_j}(z)}{\sin \pi(\nu_j - p_j)},$$

$\mathcal{J}^*$ is defined in (8.6).

Introduce moreover the measure $d\mu$ on the space of spectral parameters as follows. Again, we give it locally: $d\mu = \prod_j d\mu_j$. At real places,

$$\int f(\nu_j) d\mu_j(\nu_j) = \int_0^{i\infty} f(\nu_j)(-4\pi\nu_j) \tan \pi\nu_j \frac{d\nu_j}{2\pi i} + \sum_{2|2\nu_j+1, 1<2\nu_j+1} f(\nu_j). \tag{8.8}$$

At complex places,

$$\int f(\nu_j, p_j) d\mu_j(\nu_j, p_j) = \sum_{p_j} \int_{(0)} f(\nu_j, p_j)(p_j^2 - \nu_j^2) d\nu_j. \tag{8.9}$$

8.2.3 The Kuznetsov formula

Let $h = \prod_j h_j$, where h_j's are defined as follows. Let $a_j, b_j > 1, a'_j \in \mathbf{R}$ be given. Then at real places

$$h_j(\nu_j) = \begin{cases} e^{(\nu_j^2 - \frac{1}{4})/a_j}, & \text{if } |\Re\nu_j| < \frac{2}{3}, \\ 1, & \text{if } \nu_j \in \frac{1}{2} + \mathbf{Z}, \frac{3}{2} \le |\nu_j| \le b_j, \\ 0 & \text{otherwise.} \end{cases}$$

While at complex places

$$h_j(\nu_j, p_j) = \begin{cases} e^{(\nu_j^2 + a'_j p_j^2 - 1)/a_j}, & \text{if } |\Re\nu_j| < \frac{2}{3}, p_j \in \mathbf{Z}, |p_j| \le b_j, \\ 0 & \text{otherwise.} \end{cases}$$

Fix some fractional ideals $\mathfrak{a}^{-1}, \mathfrak{a}'^{-1}$, and some nonzero elements $\alpha \in \mathfrak{a}, \alpha' \in \mathfrak{a}'$ such that $\alpha\alpha'$ is totally positive. Let C be a fixed set of narrow ideal class representatives $\mathfrak{m}$, for which $\mathfrak{m}^2\mathfrak{a}\mathfrak{a}'^{-1}$ is a principal ideal generated by a totally positive element $\gamma_{\mathfrak{m}}$, fixed once for all.

Theorem 3. *The sum formula (5.1) holds for the weight function h, that is,*

$$[K(\mathfrak{o}):K(\mathfrak{c})]^{-1}\sum_{\pi\in\mathcal{C}(\mathfrak{c})}C_\pi^{-1}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}}h(\mathbf{r}_\pi)\lambda_\pi^{\mathfrak{t}}(\alpha\mathfrak{a}^{-1})\overline{\lambda_\pi^{\mathfrak{t}}(\alpha'\mathfrak{a}'^{-1})}+CSC=$$

$$\text{const.}\Delta(\alpha\mathfrak{a}^{-1},\alpha'\mathfrak{a}'^{-1})\int h(\mathbf{r})d\mu+$$

$$\text{const.}\sum_{\mathfrak{m}\in C}\sum_{c\in\mathfrak{a}\mathfrak{m}\mathfrak{c}}\sum_{\epsilon\in\mathfrak{o}_+^\times/\mathfrak{o}^{2\times}}\frac{KS(\epsilon\alpha,\mathfrak{a}^{-1}\mathfrak{d}^{-1};\alpha'\gamma_{\mathfrak{m}},\mathfrak{a}'^{-1}\mathfrak{d}^{-1};c,\mathfrak{a}^{-1}\mathfrak{m}^{-1}\mathfrak{d}^{-1})}{\mathcal{N}(c\mathfrak{a}^{-1}\mathfrak{m}^{-1})}\int\mathcal{B}h_{(\mathbf{r})}\left(4\pi\frac{(\alpha\alpha'\gamma_{\mathfrak{m}}\epsilon)^{\frac12}}{c}\right)d\mu,$$

where $\Delta(\alpha\mathfrak{a}^{-1},\alpha'\mathfrak{a}'^{-1})$ is 1 if $\alpha\mathfrak{a}^{-1}=\alpha'\mathfrak{a}'^{-1}$, and 0 otherwise; $\mathfrak{o}_+^\times$ stands for the group of totally positive units; for the integrals with respect to $d\mu$, recall (8.8) and (8.9). The constants denoted by const. *are nonzero and depend only on the field F and the normalization of measures.*

The abbreviation CSC stands for an analogous integral over the Eisenstein spectrum, we will not spell it out explicitly (see [56]). Note that taking the square-root in $\mathcal{B}h_{(\mathbf{r})}$ does not lead to confusion: at real places, there are positive numbers under the square-root, while at complex places, $\mathcal{B}h_{(\nu,p)}$ is even. Also record the terminology: the left-hand side is called the 'spectral side', the right-hand side is the 'geometric side'; the first term of the geometric side is the 'delta term', the second one is the 'Kloosterman term'.

8.3 Poincaré series

First of all, recall the content of Section 4.3. This time, we can fix $m=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathbf{A}_{\mathrm{fin}})$.

Define

$$\mathrm{FS}=\left\{f:\mathrm{GL}_2(\mathbf{A})\to\mathbf{C}:\int_{Z(\mathbf{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A})}|f|^2<\infty,\right.$$
$$\left.f\left(\gamma\begin{pmatrix}z&0\\0&z\end{pmatrix}gk\right)=f(g),\text{ if }\gamma\in\mathrm{GL}_2(F),z\in F_\infty^\times,k\in K(\mathfrak{c})\right\}.$$

This is a larger space of automorphic functions than one usually deals with: this is the L^2 space of the left-hand side of (4.15). Then to any $\phi\in\mathrm{FS}$, (4.15) associates classical automorphic functions, which we denote by

$$\phi^{\mathfrak{n}}(g_\infty)=\phi\left(\begin{pmatrix}\eta^{-1}&0\\0&1\end{pmatrix}g_\infty\right),\qquad g_\infty\in\mathrm{GL}_2(F_\infty).$$

See also [56, (9)].

8.3.1 Definition of Poincaré series

Fix some nonzero ideals $\mathfrak{a},\mathfrak{b}$. We define the following characters on $N(F_\infty)$. For $x\in F_\infty$, let $\psi_1(x)=\psi_\infty(\alpha x)$ and $\psi_2(x)=\psi_\infty(\alpha' x)$, where $\alpha\in\mathfrak{a}$, $\alpha'\in\mathfrak{a}\mathfrak{b}^2$ are nonzero elements with the property that $\alpha\alpha'$ it totally positive (that is, positive at all real places). They give rise naturally to characters of $N(F_\infty)$ which are trivial on $\Gamma(\mathfrak{a},\mathfrak{c})\cap N(F_\infty)$, $\Gamma(\mathfrak{a}\mathfrak{b}^2,\mathfrak{c})\cap N(F_\infty)$, respectively.

The building blocks of the Poincaré series are functions f_1,f_2 on $Z(F_\infty)\backslash\mathrm{GL}_2(F_\infty)$ with the prescribed left action of $N(F_\infty)$:

$$f_1\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}z&0\\0&z\end{pmatrix}g\right)=\psi_1(x)f_1(g),\qquad f_2\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}z&0\\0&z\end{pmatrix}g\right)=\psi_2(x)f_2(g).\tag{8.10}$$

Then the Poincaré series are defined on the left-hand side of (4.15) as

$$P_1^{\mathfrak{a}}(g)=\sum_{\gamma\in Z_\Gamma\Gamma_N(\mathfrak{a},\mathfrak{c})\backslash\Gamma(\mathfrak{a},\mathfrak{c})}f_1(\gamma g),\qquad P_2^{\mathfrak{a}\mathfrak{b}^2}(g)=\sum_{\gamma\in Z_\Gamma\Gamma_N(\mathfrak{a}\mathfrak{b}^2,\mathfrak{c})\backslash\Gamma(\mathfrak{a}\mathfrak{b}^2,\mathfrak{c})}f_2(\gamma g),\tag{8.11}$$

with Z_Γ standing for the center, Γ_N for the upper triangular unipotent subgroup (the intersection with $N(F_\infty)$) of the corresponding group Γ; this defines both P_1 and P_2 only on a single component (in the decomposition (4.15)), on other components, let them be zero.

Of course, there might be convergence problems. If we can define the building blocks such that the resulting Poincaré series are absolutely convergent, then this definition is valid. Unfortunately, when all the archimedean places are complex, we are not able to guarantee the absolute convergence, so in this case, we have to clarify, what we mean by the sums in (8.11). We will return to this problem later. Until then, we always assume $r > 0$ (i.e. F has at least one real embedding).

Our building blocks will be pure tensors, $f_i(x) = \prod_{j=1}^{r+s} f_{i,j}(x)$ for $i = 1, 2$. In the next subsections, we give the local definitions.

8.3.2 Building blocks at real places

In the construction of a real factor of our building blocks, we mainly follow [10, Section 3.2], where the authors work with the group $\mathrm{SL}_2(\mathbf{R})$.

For a given $\sigma \in (1/2, 1)$ and an even integer u, we denote by $\mathcal{T}_{u,\sigma}$ the linear space of functions η defined on the set

$$\{\nu \in \mathbf{C} : |\Re\nu| \leq \sigma\} \cup \left\{\frac{1}{2}, \frac{3}{2}, \dots\right\}$$

and satisfying the conditions

(i) η is holomorphic and even on a neighborhood of the strip $|\Re\nu| \leq \sigma$,

(ii) $\eta(\nu) \ll_A e^{-\frac{\pi}{2}|\Im\nu|}(1 + |\Im\nu|)^{-A}$ for each $A > 0$,

(iii) $\eta\left(\frac{b-1}{2}\right) = 0$, if b is an even integer such that $b > u$.

For $\eta \in \mathcal{T}_{u,\sigma}$ define the following function on the set $y > 0$ (see [10, (3.10)])

$$(\widetilde{\mathcal{L}}_u\eta)(y) = \frac{1}{4\pi i}\int_{(0)} \eta(\nu) W_{u/2,\nu}(y) \left|\frac{\Gamma\left(\frac{1}{2} + \nu - \frac{u}{2}\right)}{\Gamma(2\nu)}\right|^2 d\nu$$
$$+ \sum_{2|b, 1<b\leq u} \eta\left(\frac{b-1}{2}\right) W_{u/2,(b-1)/2}(y) \frac{b-1}{\left(\frac{u-b}{2}\right)!\left(\frac{u+b-2}{2}\right)!}.$$

Now for some fixed $q \in \mathbf{Z}, \alpha \in \mathbf{R} \setminus \{0\}$ and $\eta \in \mathcal{T}_{u,\sigma}$, we define the following function using the Iwasawa decomposition. First, if $\det g > 0$, then let

$$(\widetilde{\mathcal{L}}_q^\alpha\eta)(g) = (\widetilde{\mathcal{L}}_q^\alpha\eta)\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right) = e^{2\pi i\alpha x}(\widetilde{\mathcal{L}}_{q\mathrm{sign}(\alpha)}\eta)(4\pi|\alpha|y)e^{iq\theta}.$$

If $\det g < 0$, then $g = g'\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$ with $\det g' > 0$ and in this case we simply prescribe $(\widetilde{\mathcal{L}}_q^\alpha\eta)(g) = (\widetilde{\mathcal{L}}_q^{-\alpha}\eta)(g')$.

Now let $\eta, \theta \in \mathcal{T}_{u,\sigma}$. For real factors of f_1, f_2, choose $\widetilde{\mathcal{L}}_q^\alpha\eta, \widetilde{\mathcal{L}}_q^{\alpha'}\theta$, respectively. Of course, we may use different η's and θ's at different real places.

Before turning to complex places, note that the functions we defined are transformed transform like weight q functions on the positive domain $\det g > 0$, and like weight $-q$ functions on the negative domain $\det g < 0$.

8.3.3 Building blocks at complex places

In the construction of a complex factor of our bulding blocks, we follow [13, Section 7] and [45, Section 9.1].

Let $l > 0$ be an integer, $|q| \leq l$. Following [13, Theorem 7.1] and [45, Definition 9.1.3], for a given $\sigma \in (1, 3/2)$, we denote by $\mathcal{T}_\sigma^l$ the linear space of functions η defined on the set

$$\{(\nu, p) \in \mathbf{C} \times \mathbf{Z} : |\Re\nu| \leq \sigma, |p| \leq l\}$$

and satisfying the conditions

(i) η is holomorphic on a neighborhood of the strip $|\Re\nu| \leq \sigma$,

(ii) $\eta(\nu,p) \ll_A e^{-\frac{\pi}{2}|\Im\nu|}(1+|\Im\nu|)^{-A}$ for each $A>0$,

(iii) $\eta(\nu,p) = \eta(-\nu,-p)$.

Now for the given l,q, some fixed α and $\eta \in \mathcal{T}^l_\sigma$, let

$$(\widetilde{\mathcal{L}}^\alpha_{l,q}\eta)(g) = \frac{|\alpha|}{2\pi^3 i}\sum_{|p|\le l}\frac{(i\alpha/|\alpha|)^p}{(2l+1)^{-1/2}\binom{2l}{l-p}^{1/2}\binom{2l}{l-q}^{-1/2}} \cdot \int_{(0)} \eta(\nu,p)(2\pi|\alpha|)^{-\nu}\Gamma(l+1+\nu)\mathbf{J}_\alpha(g)\nu^{\epsilon(p)}\sin\pi(\nu-p)d\nu$$

with $\epsilon(0)=1$, $\epsilon(p)=-1$ for $p\in\mathbf{Z}\setminus\{0\}$. Note that this function differs from the function appearing in [45, Theorem 9.1.4] by the factor $|\alpha|$.

Now let $\eta,\theta\in\mathcal{T}^l_\sigma$. For a complex factor of f_1, choose $\widetilde{\mathcal{L}}^\alpha_{l,q}\eta$ and for f_2, choose $\widetilde{\mathcal{L}}^{\alpha'}_{l,q}\theta$. Of course, we may use different η's and θ's at different complex places.

8.3.4 Convergence of Poincaré series

Now we return to the question of convergence. As we mentioned, there is a technical difficulty, which does not arise if F has at least one real archimedean embedding, we assume this temporarily.

We cite a tool from [9].

Lemma 8.3. *Let $a,b\in\mathbf{R}$, $a+b>0$. Assume f is a function on $(\mathbf{R}^\times)^r\times(\mathbf{C}^\times)^s$ satisfying*

$$f(y) \ll \prod_{j=1}^{r+s}\min(|y_j|^{a\deg[F_j:\mathbf{R}]}, |y_j|^{-b\deg[F_j:\mathbf{R}]}).$$

Then

$$\sum_{\epsilon\in\mathfrak{o}^\times} f(\epsilon y) \ll (1+|\log|y||^{r+s-1})\min(|y|^a_\infty, |y|^{-b}_\infty).$$

Proof. See [9, Lemma 8.1]. □

We focus on P_1, the proof is the same for P_2. We have the local bounds:

- for $j\le r$, by [10, (3.11)],

$$(\widetilde{\mathcal{L}}^\alpha_q\eta)\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}y&0\\0&1\end{pmatrix}\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}\begin{pmatrix}\pm1&0\\0&1\end{pmatrix}\right) \ll \min(y^{\frac12+\sigma}, y^{\frac12-\sigma}),$$

- while for $j>r$, by [45, (9.18)],

$$(\widetilde{\mathcal{L}}^\alpha_{l,q}\eta)\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}y&0\\0&1\end{pmatrix}\begin{pmatrix}\beta&\gamma\\-\overline{\gamma}&\overline{\beta}\end{pmatrix}\right) \ll \begin{cases} y^{1+t}, & \text{as } y\to0 \text{ for all } t\in(0,1),\\ y^{-k}, & \text{as } y\to\infty \text{ for all } k\ge1.\end{cases}$$

Of course, the implied constants depend on the function η and in the complex case, on the chosen numbers t,k. Now in the definition (8.11) of P_1, we may focus on the component $P_1^{\mathfrak{a}}$. Rewrite it as

$$P_1^{\mathfrak{a}}(g) = \sum_{\gamma\in Z_\Gamma\Gamma_N(\mathfrak{a},\mathfrak{c})\backslash\Gamma(\mathfrak{a},\mathfrak{c})} f_1(\gamma g) = \sum_{\gamma'\in\Gamma_\infty(\mathfrak{a},\mathfrak{c})\backslash\Gamma(\mathfrak{a},\mathfrak{c})}\ \sum_{\gamma\in Z_\Gamma\Gamma_N(\mathfrak{a},\mathfrak{c})\backslash\Gamma_\infty(\mathfrak{a},\mathfrak{c})} f_1(\gamma\gamma' g) = \sum_{\gamma'\in\Gamma_\infty(\mathfrak{a},\mathfrak{c})\backslash\Gamma(\mathfrak{a},\mathfrak{c})} h(\gamma' g),$$

where

$$h(g) = \sum_{\gamma\in Z_\Gamma\Gamma_N(\mathfrak{a},\mathfrak{c})\backslash\Gamma_\infty(\mathfrak{a},\mathfrak{c})} f_1(\gamma g),$$

and Γ_∞ stands for the upper triangular subgroup of Γ. Now observe the coset space $Z_\Gamma\Gamma_N(\mathfrak{a},\mathfrak{c})\backslash\Gamma_\infty(\mathfrak{a},\mathfrak{c})$ is covered by the set

$$\left\{\begin{pmatrix}(\pm)^r\epsilon&0\\0&\epsilon^{-1}\end{pmatrix} : \epsilon\in\mathfrak{o}^\times\right\},$$

where by $(\pm)^r$, we mean that at each real place there can be some sign. This $(\pm)^r$ results only a finite, 2^r term summation and apart from this, our summation is over the units. Hence Lemma 8.3 applies and gives

$$h(g) \ll \begin{cases} |y|_\infty^{a-\varepsilon}, & (|y|_\infty \to 0), \\ |y|_\infty^{-b+\varepsilon}, & (|y|_\infty \to \infty), \end{cases}$$

where y is the height of g (the diagonal factor in the Iwasawa decomposition: the quotient of the upper-left and the lower-right entry), a is the product of $1/2+\sigma$'s at the real places and $(1+t)/2$'s at the complex places and b is the product of $\sigma - 1/2$'s and $k/2$'s. We can easily guarantee $a > 1$ and $b > 0$, and for a small $\varepsilon > 0$, $a - \varepsilon > 1$. Now

$$\sum_{\gamma' \in \Gamma_\infty(\mathfrak{a},\mathfrak{c}) \backslash \Gamma(\mathfrak{a},\mathfrak{c})} h(\gamma' g) \ll \sum_{\gamma' \in \Gamma_\infty(\mathfrak{a},\mathfrak{c}) \backslash \Gamma(\mathfrak{a},\mathfrak{c})} |y(\gamma' g)|_\infty^{a-\varepsilon},$$

where $y(\gamma' g)$ means the height of $\gamma' g$. On the right-hand side, we see an Eisenstein series, which is absolutely convergent (as $a - \varepsilon > 1$). Moreover, on the left-hand side, the contribution of the upper-triangular element is bounded (as $b > 0$), so the resulting function is also bounded, hence square-integrable. (See [10, Lemma 2.4] and [9, p.648].)

8.4 Scalar product of Poincaré series

8.4.1 Geometric description

Let $\pi_\mathfrak{b}$ be a finite idele representing $\mathfrak{b}$. With the abbreviation

$$\pi_\mathfrak{b}^{-1} P_2 = \begin{pmatrix} \pi_\mathfrak{b}^{-1} & 0 \\ 0 & \pi_\mathfrak{b}^{-1} \end{pmatrix} P_2,$$

consider the inner product $\langle \pi_\mathfrak{b}^{-1} P_2, P_1 \rangle$. The Poincaré series P_1, P_2 are defined in the space FS, and the inner product is also understood there. In what follows, we shall expand this both geometrically and spectrally, and the equation of these expressions will give rise to the Kuznetsov formula.

First we note the following consequence of strong approximation (see [56, (83)]). Given an element $g_\infty \in \mathrm{GL}_2(F_\infty)$, there exist elements $\gamma \in \mathrm{GL}_2(F)$, $\kappa_\gamma \in K(\mathfrak{c})$ such that

$$\begin{pmatrix} \pi_\mathfrak{b}^{-1} & 0 \\ 0 & \pi_\mathfrak{b}^{-1} \end{pmatrix} \begin{pmatrix} \pi_\mathfrak{a}^{-1} & 0 \\ 0 & 1 \end{pmatrix} g_\infty = \gamma^{-1} \begin{pmatrix} (\pi_\mathfrak{b}^2 \pi_\mathfrak{a})^{-1} & 0 \\ 0 & 1 \end{pmatrix} g'_\infty \kappa_\gamma.$$

Then

$$\gamma \in \mathrm{GL}_2(F) \cap \mathrm{GL}_2(F_\infty) \begin{pmatrix} (\pi_\mathfrak{b}^2 \pi_\mathfrak{a})^{-1} & 0 \\ 0 & 1 \end{pmatrix} \kappa_\gamma \begin{pmatrix} \pi_\mathfrak{a} \pi_\mathfrak{b} & 0 \\ 0 & \pi_\mathfrak{b} \end{pmatrix}.$$

We denote the set of such γ's by $\Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)$, following [56] in notation (however, our $\Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)$ is not exactly the same as Venkatesh's one, because of the different normalization of the congruence subgroup $K(\mathfrak{c})$). For $\gamma \in \Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)$, we denote by κ_γ a corresponding element from $K(\mathfrak{c})$. Fix an element $\gamma^* \in \Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)$, then $\Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2) = \gamma^* \Gamma(\mathfrak{a}, \mathfrak{c}) = \Gamma(\mathfrak{a}\mathfrak{b}^2, \mathfrak{c}) \gamma^*$.

When we compute the inner product $\langle \pi_\mathfrak{b}^{-1} P_2, P_1 \rangle$ using the decomposition (4.15), we see that only the $\mathfrak{a}$-part, that is, $\pi_\mathfrak{b}^{-1} P_2^\mathfrak{a}$ is relevant (on the other components, at least one of $\pi_\mathfrak{b}^{-1} P_2$ and P_1 is zero). The definition of P_2 and a simple computation (see [56, (85), (91)]) give

$$\pi_\mathfrak{b}^{-1} P_2^\mathfrak{a}(g) = \sum_{\gamma \in Z_\Gamma \Gamma_N(\mathfrak{a}\mathfrak{b}^2, \mathfrak{c}) \backslash \Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)} f_2(\gamma g).$$

Setting $I = \langle \pi_\mathfrak{b}^{-1} P_2, P_1 \rangle$, we can unfold the integral as

$$\begin{aligned} I &= \int_{Z(F_\infty)\Gamma_N(\mathfrak{a},\mathfrak{c}) \backslash \mathrm{GL}_2(F_\infty)} \overline{f_1(g)} \sum_{\gamma \in Z_\Gamma \Gamma_N(\mathfrak{a}\mathfrak{b}^2, \mathfrak{c}) \backslash \Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)} f_2(\gamma g) dg \\ &= \int_{Z(F_\infty)N(F_\infty) \backslash \mathrm{GL}_2(F_\infty)} \overline{f_1(g)} \int_{\Gamma_N(\mathfrak{a},\mathfrak{c}) \backslash N(F_\infty)} \overline{\psi_1(n)} \sum_{\gamma \in Z_\Gamma \Gamma_N(\mathfrak{a}\mathfrak{b}^2, \mathfrak{c}) \backslash \Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)} f_2(\gamma n g) dn dg, \end{aligned} \tag{8.12}$$

here the inner integral is essentially the Fourier coefficient of $\pi_{\mathfrak{b}}^{-1}P_2$ corresponding to the character ψ_1.

Let us split this up as $I = I_1 + I_2$ according to the small and the large Bruhat cell, that is, I_1 is the same integral as I, but in the inner summation, we let γ be upper-triangular, and I_2 corresponds to the rest.

First we compute I_1. Observe that I_1 is an empty integral unless $\mathfrak{b}$ is principal. Assume then that $\mathfrak{b}$ is generated by an element $[\mathfrak{b}]$. By [56, Lemma 14],

$$I_1 = \sum_{\epsilon\in\mathfrak{o}^\times}\int_{\Gamma_N(\mathfrak{a},\mathfrak{c})Z(F_\infty)\backslash\mathrm{GL}_2(F_\infty)} \overline{f_1(g)}f_2\left(\begin{pmatrix}[\mathfrak{b}]^{-1}\epsilon & 0\\ 0 & [\mathfrak{b}]\end{pmatrix}g\right)dg.$$

Let

$$\Delta(\alpha,\alpha'[\mathfrak{b}]^{-2}) = \begin{cases} 1, & \text{if } \exists\epsilon_0\in\mathfrak{o}^\times : \alpha = \alpha'[\mathfrak{b}]^{-2}\epsilon_0,\\ 0 & \text{otherwise.}\end{cases}$$

Take this ϵ_0 (if exists). Now let $N(F_\infty)$ act on the left, we obtain by (8.10)

$$I_1 = \mathrm{const.}\Delta(\alpha,\alpha'[\mathfrak{b}]^{-2})\mathcal{N}(\mathfrak{a}^{-1})\int_{N(F_\infty)Z(F_\infty)\backslash\mathrm{GL}_2(F_\infty)} \overline{f_1(g)}f_2\left(\begin{pmatrix}[\mathfrak{b}]^{-1}\epsilon_0 & 0\\ 0 & [\mathfrak{b}]\end{pmatrix}g\right)dg.$$

Now we can turn our attention to the large Bruhat cell. First we state the explicit Bruhat decomposition.

Lemma 8.4. *On the large Bruhat cell, we have*

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}1 & ac^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}c & 0\\ 0 & c^{-1}(ad-bc)\end{pmatrix}\begin{pmatrix}1 & dc^{-1}\\ 0 & 1\end{pmatrix}.$$

Proof. Straight-forward calculation. □

For $\tau\in Z_\Gamma\Gamma_N(\mathfrak{a}\mathfrak{b}^2,c)\backslash\Gamma(\mathfrak{a}\to\mathfrak{a}\mathfrak{b}^2)/\Gamma_N(\mathfrak{a},\mathfrak{c})$, denote by $[\tau]\in\Gamma(\mathfrak{a}\to\mathfrak{a}\mathfrak{b}^2)$ any representative. Then

$$I_2 = \int_{\Gamma_N(\mathfrak{a},\mathfrak{c})Z(F_\infty)\backslash\mathrm{GL}_2(F_\infty)}\sum_{\substack{\tau\in Z_\Gamma\Gamma_N(\mathfrak{a}\mathfrak{b}^2,\mathfrak{c})\backslash\Gamma(\mathfrak{a}\to\mathfrak{a}\mathfrak{b}^2)/\Gamma_N(\mathfrak{a},\mathfrak{c})\\ [\tau]\notin B(F_\infty)}}\sum_{\mu\in\Gamma_N(\mathfrak{a},\mathfrak{c})}\overline{f_1(g)}f_2([\tau]\mu g)dg.$$

Now folding together the integral and the μ-sum, we obtain

$$I_2 = \sum_{\substack{\tau\in Z_\Gamma\Gamma_N(\mathfrak{a}\mathfrak{b}^2,\mathfrak{c})\backslash\Gamma(\mathfrak{a}\to\mathfrak{a}\mathfrak{b}^2)/\Gamma_N(\mathfrak{a},\mathfrak{c})\\ [\tau]\notin B(F_\infty)}}\int_{Z(F_\infty)\backslash\mathrm{GL}_2(F_\infty)}\overline{f_1(g)}f_2([\tau]g)dg.$$

Let $[\tau] = n_{1,[\tau]}\mathrm{w}a_{[\tau]}n_{2,[\tau]}$ according to the Bruhat decomposition Lemma 8.4. Then by (8.10),

$$\begin{aligned} I_2 &= \sum_{\substack{\tau\in Z_\Gamma\Gamma_N(\mathfrak{a}\mathfrak{b}^2,\mathfrak{c})\backslash\Gamma(\mathfrak{a}\to\mathfrak{a}\mathfrak{b}^2)/\Gamma_N(\mathfrak{a},\mathfrak{c})\\ [\tau]\notin B(F_\infty)}}\psi_1(n_{2,[\tau]})\psi_2(n_{1,[\tau]})\cdot\int_{Z(F_\infty)\backslash\mathrm{GL}_2(F_\infty)}\overline{f_1(g)}f_2(\mathrm{w}a_{[\tau]}g)dg\\ &= \sum_{\substack{\tau\in Z_\Gamma\Gamma_N(\mathfrak{a}\mathfrak{b}^2,\mathfrak{c})\backslash\Gamma(\mathfrak{a}\to\mathfrak{a}\mathfrak{b}^2)/\Gamma_N(\mathfrak{a},\mathfrak{c})\\ [\tau]\notin B(F_\infty)}}\psi_1(n_{2,[\tau]})\psi_2(n_{1,[\tau]})\\ &\qquad\cdot\int_{N(F_\infty)Z(F_\infty)\backslash\mathrm{GL}_2(F_\infty)}\overline{f_1(g)}\int_{N(F_\infty)}\overline{\psi_1(n)}f_2(\mathrm{w}a_{[\tau]}ng)dndg.\end{aligned}$$

By Lemma 8.4,

$$n_{2,[\tau]} = \begin{pmatrix}1 & dc^{-1}\\ 0 & 1\end{pmatrix},\qquad n_{1,[\tau]} = \begin{pmatrix}1 & ac^{-1}\\ 0 & 1\end{pmatrix}.$$

Putting everything together and using again [56, Lemma 14] for the explicit description of $\Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)$ (keeping in mind that our normalization differs a little), we obtain

$$\begin{aligned} I = &\text{const.}\Delta(\alpha, \alpha'[\mathfrak{b}]^{-2})\mathcal{N}(\mathfrak{a}^{-1}) \int_{N(F_\infty)Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \overline{f_1(g)} f_2 \left(\begin{pmatrix} [\mathfrak{b}]^{-1}\epsilon_0 & 0 \\ 0 & [\mathfrak{b}] \end{pmatrix} g \right) dg \\ &+ \text{const.} \sum_{c \in \mathfrak{abc}, \epsilon \in \mathfrak{o}^\times/\mathfrak{o}^{2\times}} KS(\epsilon\alpha, \mathfrak{a}^{-1}\mathfrak{d}^{-1}; \alpha', \mathfrak{a}^{-1}\mathfrak{b}^{-2}\mathfrak{d}^{-1}; c, \mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{d}^{-1}) \\ &\cdot \int_{N(F_\infty)Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \overline{f_1(g)} \int_{N(F_\infty)} \overline{\psi_1(n)} f_2 \left(\mathrm{w} \begin{pmatrix} 1 & 0 \\ 0 & \epsilon c^{-2} \end{pmatrix} ng \right) dndg. \end{aligned} \tag{8.13}$$

See [56, (95)], for convenience, we note that $\mathfrak{a} = \mathfrak{a}_{\text{Venkatesh}}^{-1}\mathfrak{d}^{-1}$ and $\mathfrak{b} = \mathfrak{b}_{\text{Venkatesh}}^{-1}$.

8.4.2 Spectral description

On the other hand, we decompose spectrally the inner product (see (2.3)). With this aim in mind, take the following orthonormal system in the cuspidal spectrum of $L^2(Z(\mathbf{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}))$. First of all, use the decomposition

$$L_{\text{cusp}} = \bigoplus_{\pi \in \mathcal{C}} V_\pi.$$

Then in each V_π, take

$$V_\pi = V_\pi(\mathfrak{c}) \oplus V_\pi(\mathfrak{c})^\perp.$$

The whole $V_\pi(\mathfrak{c})^\perp$ is orthogonal to P_1 and $\pi_{\mathfrak{b}}^{-1}P_2$, so we may restrict to $V_\pi(\mathfrak{c})$. Then in $V_\pi(\mathfrak{c})$, take the decomposition

$$V_\pi(\mathfrak{c}) = \bigoplus_{\mathbf{w} \in W(\pi)} \bigoplus_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_\pi^{-1}} R^{\mathfrak{t}} V_{\pi, \mathbf{w}}(\mathfrak{c}_\pi).$$

On the right-hand side, the occuring spaces are one-dimensional, so we may take a vector of norm 1 in each of them (which is well-defined up to a complex scalar of modulus 1). The set of all these vectors is denoted by $\mathbf{B}(\mathfrak{c})$. However, we need a slightly larger space, since we defined P_1 and P_2 in $L^2(Z(F_\infty)\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}))$ instead of $L^2(Z(\mathbf{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}))$. That is, to obtain a basis in the cuspidal subspace of $L^2(Z(F_\infty)\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}))$, we have to twist the elements of $\mathbf{B}(\mathfrak{c})$ by the class group characters, obtaining $\mathbf{B}_{\mathrm{FS}}(\mathfrak{c})$. Altogether, we obtain

$$\langle \pi_{\mathfrak{b}}^{-1}P_2, P_1 \rangle = \sum_{\mathbf{f} \in \mathbf{B}_{\mathrm{FS}}(\mathfrak{c})} \overline{\langle P_1, \mathbf{f} \rangle} \langle \pi_{\mathfrak{b}}^{-1}P_2, \mathbf{f} \rangle + CSC,$$

where CSC stands for the contribution of the Eisenstein spectrum. Again, it is larger than in (2.3), since we have to twist by the class group characters, so we have to sum up finitely many pieces similar to the one appearing in the continuous part of (2.3). This part of the spectrum can be handled similarly to the cuspidal part, and we will not spell it out explicitly. The contribution of L_{sp} (and also its twisted variants) is 0, since $N(F_\infty)$ acts on both P_1 and $\pi_{\mathfrak{b}}^{-1}P_2$ through nontrivial characters (ψ_1 and ψ_2, respectively), and on any element of L_{sp} through the trivial character.

Then

$$\begin{aligned} \overline{\langle P_1, \mathbf{f} \rangle} &= [K(\mathfrak{o}) : K(\mathfrak{c})]^{-1/2} \int_{Z(F_\infty)\Gamma_N(\mathfrak{a},\mathfrak{c})\backslash \mathrm{GL}_2(F_\infty)} \overline{f_1(g)} \mathbf{f}^{\mathfrak{a}}(g) \\ &= \text{const.} [K(\mathfrak{o}) : K(\mathfrak{c})]^{-\frac{1}{2}} \int_{K_\infty} \int_{(\mathbf{R}^\times)^r \times (\mathbf{R}_+^\times)^s} \overline{f_1 \left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k \right)} \frac{1}{\prod_{j=1}^r |y_j| \prod_{j=r+1}^{r+s} |y_j|^2} \\ &\cdot \int_{\Gamma_N(\mathfrak{a},\mathfrak{c})\backslash N(F_\infty)} \psi_1(-x) \mathbf{f}^{\mathfrak{a}} \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k \right) dx \frac{dy}{\prod_{j=1}^{r+s} |y_j|} dk, \end{aligned}$$

the factor $[K(\mathfrak{o}) : K(\mathfrak{c})]^{-1/2}$ is explained in Lemma 4.7.

Now we may apply the K_∞-transformation properties we prescribed in the definition of f_1 (q_j's and (l_j, q_j)'s). It shows that P_1 is orthogonal to all forms of any weight except for those that are of the form

$$\mathbf{w} = (\pm q_1, \ldots, \pm q_r, (l_{r+1}, q_{r+1}), \ldots, (l_{r+s}, q_{r+s})).$$

So we may restrict to those $\mathbf{f}$'s that are of such a weight $\mathbf{w}$. Then by (2.24), (2.28), (2.31), (2.34),

$$\overline{\langle P_1, \mathbf{f}\rangle} = \text{const.}[K(\mathfrak{o}) : K(\mathfrak{c})]^{-1/2} C_\pi^{-1/2} \lambda_\pi^{\mathfrak{t}}(\alpha\mathfrak{a}^{-1}) \frac{\sqrt{\mathcal{N}(\mathfrak{a}^{-1})}}{\sqrt{\mathcal{N}((\alpha))}}$$

$$\prod_{j=1}^{r} \left\{ \int_0^\infty \overline{f_1\left(\begin{pmatrix} y_j & 0 \\ 0 & 1 \end{pmatrix}\right)} \mathcal{W}_{q_j,\nu_j}(\alpha_j y_j) |y_j|^{-1} d_{\mathbf{R}}^\times y_j \right.$$

$$\left. \text{OR} \int_0^\infty (-1)^{\varepsilon_{\pi,j}} \overline{f_1\left(\begin{pmatrix} y_j & 0 \\ 0 & 1 \end{pmatrix}\right)} \mathcal{W}_{-q_j,\nu_j}(\alpha_j y_j) |y_j|^{-1} d_{\mathbf{R}}^\times y_j \right\}$$

$$\prod_{j=r+1}^{r+s} \int_0^\infty \overline{f_1\left(\begin{pmatrix} y_j & 0 \\ 0 & 1 \end{pmatrix}\right)} \mathcal{W}_{(l_j,q_j),(\nu_j,p_j)}(\alpha_j y_j) |y_j|^{-2} d_{\mathbf{R}}^\times y_j,$$

where $q_j, (l_j, q_j), \nu_j, (\nu_j, p_j), \varepsilon_{\pi,j}$ are all encoded in $\mathbf{f}$, 'OR' indicates that we may choose opposite weight at real places. Now repeating the above computation with $\langle \pi_{\mathfrak{b}}^{-1} P_2, \mathbf{f}\rangle$, we obtain that

$$\langle \pi_{\mathfrak{b}}^{-1} P_2, P_1\rangle = \text{const.}\,[K(\mathfrak{o}) : K(\mathfrak{c})]^{-1} \frac{\mathcal{N}(\mathfrak{a}^{-1}\mathfrak{b}^{-1})}{\sqrt{\mathcal{N}((\alpha\alpha'))}} \sum_{\pi \in \mathcal{C}_{\mathrm{FS}}(\mathfrak{c})} C_\pi^{-1} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_\pi^{-1}} \omega_\pi(\pi_{\mathfrak{b}}^{-1}) \lambda_\pi^{\mathfrak{t}}(\alpha\mathfrak{a}^{-1}) \overline{\lambda_\pi^{\mathfrak{t}}(\alpha'\mathfrak{a}^{-1}\mathfrak{b}^{-2})}$$

$$\prod_{j=1}^{r} \left(\int_0^\infty \overline{f_1\left(\begin{pmatrix} y_j & 0 \\ 0 & 1 \end{pmatrix}\right)} \mathcal{W}_{q_j,\nu_j}(\alpha_j y_j) |y_j|^{-1} d_{\mathbf{R}}^\times y_j \int_0^\infty f_2\left(\begin{pmatrix} y_j & 0 \\ 0 & 1 \end{pmatrix}\right) \overline{\mathcal{W}_{q_j,\nu_j}(\alpha_j' y_j)} |y_j|^{-1} d_{\mathbf{R}}^\times y_j \right.$$

$$\left. + \int_0^\infty \overline{f_1\left(\begin{pmatrix} y_j & 0 \\ 0 & 1 \end{pmatrix}\right)} \mathcal{W}_{-q_j,\nu_j}(\alpha_j y_j) |y_j|^{-1} d_{\mathbf{R}}^\times y_j \int_0^\infty f_2\left(\begin{pmatrix} y_j & 0 \\ 0 & 1 \end{pmatrix}\right) \overline{\mathcal{W}_{-q_j,\nu_j}(\alpha_j' y_j)} |y_j|^{-1} d_{\mathbf{R}}^\times y_j \right)$$

$$\prod_{j=r+1}^{r+s} \left(\int_0^\infty \overline{f_1\left(\begin{pmatrix} y_j & 0 \\ 0 & 1 \end{pmatrix}\right)} \mathcal{W}_{(l_j,q_j),(\nu_j,p_j)}(\alpha_j y_j) |y_j|^{-2} d_{\mathbf{R}}^\times y_j \right.$$

$$\left. \int_0^\infty f_2\left(\begin{pmatrix} y_j & 0 \\ 0 & 1 \end{pmatrix}\right) \overline{\mathcal{W}_{(l_j,q_j),(\nu_j,p_j)}(\alpha_j' y_j)} |y_j|^{-2} d_{\mathbf{R}}^\times y_j \right) + CSC, \tag{8.14}$$

where ω_π stands for the central character of π.

8.5 Archimedean computations

In this section, we compute the local contributions to integrals given in the previous section of functions defined earlier.

8.5.1 The real case

We introduce the following notation: for any g, let

$$g^* = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We start with the evaluation of the geometric side (8.13). For the small Bruhat cell, we need to investigate the function

$$\widetilde{\mathcal{L}}_q^{\alpha_j'} \theta\left(\begin{pmatrix} [\mathfrak{b}]_j^{-2}\epsilon_{0_j} & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

In the case of $\epsilon_{0_j} < 0$, we can rewrite this as

$$\widetilde{\mathcal{L}}_q^{\alpha_j'} \theta\left(\begin{pmatrix} [\mathfrak{b}]_j^{-2}|\epsilon_{0_j}| & 0 \\ 0 & 1 \end{pmatrix} g^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \widetilde{\mathcal{L}}_q^{-\alpha_j'} \theta\left(\begin{pmatrix} [\mathfrak{b}]_j^{-2}|\epsilon_{0_j}| & 0 \\ 0 & 1 \end{pmatrix} g^*\right).$$

This is of weight $-q$. Assuming $q \neq 0$, we see that integrating this against $\widetilde{\mathcal{L}}_q^{\alpha_j}$ on $N(\mathbf{R})Z(\mathbf{R})\backslash \mathrm{GL}_2(\mathbf{R})$, we obtain 0. So this term vanishes unless $\epsilon_{0_j} > 0$ (under the condition $q \neq 0$). If $\epsilon_{0_j} > 0$, we have

$$\widetilde{\mathcal{L}}_q^{\alpha_j'} \theta \left(\begin{pmatrix} [\mathfrak{b}]_j^{-2} \epsilon_{0_j} & 0 \\ 0 & 1 \end{pmatrix} g \right) = \widetilde{\mathcal{L}}_q^{\alpha_j} \theta(g).$$

Now we may apply [10, Corollary 3.6] by noting

$$\langle \widetilde{\mathcal{L}}_q^{\alpha} \theta, \widetilde{\mathcal{L}}_q^{\alpha} \eta \rangle_{N(\mathbf{R})Z(\mathbf{R})\backslash \mathrm{GL}_2(\mathbf{R})} = \langle \widetilde{\mathcal{L}}_q^{\alpha} \theta, \widetilde{\mathcal{L}}_q^{\alpha} \eta \rangle_{N(\mathbf{R})\backslash Z(\mathbf{R})\mathrm{GL}_2^+(\mathbf{R})} + \langle \widetilde{\mathcal{L}}_q^{-\alpha} \theta, \widetilde{\mathcal{L}}_q^{-\alpha} \eta \rangle_{N(\mathbf{R})\backslash Z(\mathbf{R})\mathrm{GL}_2^+(\mathbf{R})}, \tag{8.15}$$

where $\mathrm{GL}_2^+(\mathbf{R})$ stands for the elements of $\mathrm{GL}_2(\mathbf{R})$ with positive determinant. We obtain

$$\begin{aligned} \langle \widetilde{\mathcal{L}}_q^{\alpha_j} \theta, \widetilde{\mathcal{L}}_q^{\alpha_j} \eta \rangle_{N(\mathbf{R})Z(\mathbf{R})\backslash \mathrm{GL}_2(\mathbf{R})} &= \mathrm{const.} |\alpha_j| \left(\int_0^{i\infty} \theta(\nu) \overline{\eta(\nu)} \sum_{\pm} \left| \frac{\Gamma\left(\frac{1}{2} + \nu \pm \frac{\mathrm{sign}(\alpha_j) q}{2}\right)}{\Gamma(2\nu)} \right|^2 \frac{d\nu}{2\pi i} \right. \\ &\left. + \sum_{1 < b \leq |\mathrm{sign}(\alpha_j) q|} \theta\left(\frac{b-1}{2}\right) \overline{\eta\left(\frac{b-1}{2}\right)} \sum_{\pm} \frac{b-1}{\left(\frac{\pm \mathrm{sign}(\alpha_j) q - b}{2}\right)! \left(\frac{\pm \mathrm{sign}(\alpha_j) q + b - 2}{2}\right)!} \right). \end{aligned} \tag{8.16}$$

By the further notation

$$\lambda(\nu, q) = \sum_{\pm} \frac{1}{\Gamma\left(\frac{1}{2} - \nu \pm \frac{\mathrm{sign}(\alpha_j) q}{2}\right) \Gamma\left(\frac{1}{2} + \nu \pm \frac{\mathrm{sign}(\alpha_j') q}{2}\right)},$$

this equals

$$\begin{aligned} &\mathrm{const.} |\alpha_j| \left(\int_0^{i\infty} \overline{\eta(\nu)} \theta(\nu) \lambda(\nu, q) (-4\pi\nu) \tan(\pi\nu) \frac{d\nu}{2\pi i} \right. \\ &\left. + \sum_{1 < b \leq |\mathrm{sign}(\alpha_j) q|} \overline{\eta\left(\frac{b-1}{2}\right)} \theta\left(\frac{b-1}{2}\right) \lambda\left(\frac{b-1}{2}, q\right) \right). \end{aligned}$$

On the large Bruhat cell, we need to compute

$$\int_{-\infty}^{\infty} e^{-2\pi i \alpha_j x} \widetilde{\mathcal{L}}_q^{\alpha_j'} \theta \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & c_j^{-2} \epsilon_j \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx.$$

Again, if $\epsilon_j < 0$, then

$$\begin{aligned} \int_{N(\mathbf{R})Z(\mathbf{R})\backslash \mathrm{GL}_2(\mathbf{R})} \overline{(\widetilde{\mathcal{L}}_q^{\alpha} \eta)(g)} \int_{-\infty}^{\infty} e^{-2\pi i \alpha_j x} \widetilde{\mathcal{L}}_q^{\alpha_j'} \theta \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & c_j^{-2} \epsilon_j \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx dg = \\ \int_{Z(\mathbf{R})\backslash \mathrm{GL}_2(\mathbf{R})} \overline{(\widetilde{\mathcal{L}}_q^{\alpha_j} \eta)(g)} \widetilde{\mathcal{L}}_q^{\alpha_j'} \theta \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & c_j^{-2} \epsilon_j \end{pmatrix} g \right) dg = 0, \end{aligned}$$

as we integrate a weight q function against a weight $-q$ function like before (assuming again $q \neq 0$). So we may assume $\epsilon_j > 0$. If $\det g > 0$, by [10, Theorem 3.8, (3.34-35)], we obtain that this integral equals

$$\int_{-\infty}^{\infty} e^{-2\pi i \alpha_j x} \widetilde{\mathcal{L}}_q^{\alpha_j'} \theta \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & c_j^{-2} \epsilon_j \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = \widetilde{\mathcal{L}}_q^{\alpha_j} \widetilde{\theta}(g)$$

with

$$\widetilde{\theta}(\nu) = \theta(\nu) \frac{1}{2|\alpha_j|} \frac{\Gamma\left(\frac{1}{2} + \nu + \frac{\mathrm{sign}(\alpha_j) q}{2}\right)}{\Gamma\left(\frac{1}{2} + \nu + \frac{\mathrm{sign}(\alpha_j') q}{2}\right)} \mathcal{B}_\nu \left(4\pi \frac{(\alpha_j \alpha_j' \epsilon_j)^{\frac{1}{2}}}{c_j} \right) \frac{|\alpha_j \alpha_j' \epsilon_j|^{\frac{1}{2}}}{|c_j|}$$

with $\mathcal{B}$ defined in 8.2.2. The holomorphy condition of [10, Theorem 3.8] is satisfied by our condition $\mathrm{sign}(\alpha_j) = \mathrm{sign}(\alpha'_j)$. The case $\det g < 0$ can be reduced as before (use (8.15) again). By (8.16), up to some constant, the contribution of the large cell is

$$\begin{aligned} &\frac{|\alpha_j\alpha'_j\epsilon_j|^{\frac12}}{|c_j|}\left(\int_0^{i\infty}\overline{\eta(\nu)}\theta(\nu)\lambda(\nu,q)(-4\pi\nu)\tan(\pi\nu)\mathcal{B}_\nu\left(4\pi\frac{(\alpha_j\alpha'_j\epsilon_j)^{\frac12}}{c_j}\right)\frac{d\nu}{2\pi i}\right.\\ &\qquad\left.+\sum_{1<b\leq \mathrm{sign}(\alpha_j)q}\overline{\eta\left(\frac{b-1}{2}\right)}\theta\left(\frac{b-1}{2}\right)\lambda\left(\frac{b-1}{2},q\right)\mathcal{B}_{\frac{b-1}{2}}\left(4\pi\frac{(\alpha_j\alpha'_j\epsilon_j)^{\frac12}}{c_j}\right)\right). \end{aligned} \tag{8.17}$$

On the spectral side (8.14), we need to integrate our building block against our normalized Whittaker function. Using [10, Corollary 3.5],

$$\int_0^\infty \widetilde{\mathcal{L}}_q^{\alpha_j}\eta(y)\overline{\mathcal{W}_{q_j,\nu_j}(\alpha_j y)}|y|^{-1}d^\times_{\mathbf{R}}y = \\ \mathrm{const.}|\alpha_j|\eta(\nu)\frac{-i^{\mathrm{sign}(\alpha_j)\frac{q}{2}}}{\{\Gamma(1/2-\nu_j+\mathrm{sign}(\alpha_j)q_j/2)\Gamma(1/2+\nu_j+\mathrm{sign}(\alpha_j)q_j/2)\}^{1/2}}.$$

As the inner product of such two, at a real place in (8.14) we obtain

$$\begin{aligned} &\int_0^\infty \overline{\widetilde{\mathcal{L}}_q^{\alpha_j}\eta(y)}\mathcal{W}_{q_j,\nu_j}(\alpha_j y)|y|^{-1}d^\times_{\mathbf{R}}y\int_0^\infty \widetilde{\mathcal{L}}_q^{\alpha'_j}\theta(y)\overline{\mathcal{W}_{q_j,\nu_j}(\alpha'_j y)}|y|^{-1}d^\times_{\mathbf{R}}y\\ &+\int_0^\infty \overline{\widetilde{\mathcal{L}}_q^{\alpha_j}\eta(y)}\mathcal{W}_{-q_j,\nu_j}(\alpha_j y)|y|^{-1}d^\times_{\mathbf{R}}y\int_0^\infty \widetilde{\mathcal{L}}_q^{\alpha_j}\theta(y)\overline{\mathcal{W}_{-q_j,\nu_j}(\alpha'_j y)}|y|^{-1}d^\times_{\mathbf{R}}y =\\ &\mathrm{const.}|\alpha_j\alpha'_j|\overline{\eta(\nu)}\theta(\nu)\lambda(\nu,q). \end{aligned} \tag{8.18}$$

8.5.2 The complex case

We execute the complex analog of the above procedure. On the small cell, in our normalization [45, display between (10.22-23)] gives

$$\widetilde{\mathcal{L}}_{l,q}^{\alpha'_j}\theta\left(\begin{pmatrix}[\mathfrak{b}]_j^{-2}\epsilon_{0_j} & 0\\ 0 & 1\end{pmatrix}g\right) = \widetilde{\mathcal{L}}_{l,q}^{\alpha_j}\theta(g).$$

We have to integrate this against $\widetilde{\mathcal{L}}_{l,q}^{\alpha_j}\eta$. Using [45, Lemma 9.1.5],

$$\langle\widetilde{\mathcal{L}}_{l,q}^{\alpha_j}\theta,\widetilde{\mathcal{L}}_{l,q}^{\alpha_j}\eta\rangle_{N\backslash G} = \mathrm{const.}|\alpha_j|^2\sum_{|p|\leq l}\int_{(0)}\overline{\eta(\nu,p)}\theta(\nu,p)\Gamma(l+1-\nu)\Gamma(l+1+\nu)\frac{\sin^2\pi(\nu-p)}{p^2-\nu^2}\nu^{2\epsilon(p)}d\nu$$

with $\epsilon(0)=1$, $\epsilon(p)=-1$ for $p\in\mathbf{Z}\setminus\{0\}$. Introducing

$$\lambda_l(\nu,p) = \Gamma(l+1-\nu)\Gamma(l+1+\nu)\frac{\sin^2\pi(\nu-p)}{(p^2-\nu^2)^2}\nu^{2\epsilon(p)},$$

we get

$$\langle\widetilde{\mathcal{L}}_{l,q}^{\alpha_j}\theta,\widetilde{\mathcal{L}}_{l,q}^{\alpha_j}\eta\rangle_{N\backslash G} = \mathrm{const.}|\alpha_j|^2\sum_{|p|\leq l}\int_{(0)}\overline{\eta(\nu,p)}\theta(\nu,p)\lambda_l(\nu,p)(p^2-\nu^2)d\nu. \tag{8.19}$$

Note that $\lambda_l(\nu,p)$ is nonzero.

On the large Bruhat cell, the corresponding integral is

$$\int_{\mathbf{C}} e^{-2\pi i\alpha_j(x+\overline{x})}\widetilde{\mathcal{L}}_{l,q}^{\alpha'_j}\theta\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&0\\0&c_j^{-2}\epsilon_j\end{pmatrix}\begin{pmatrix}1&x\\0&1\end{pmatrix}g\right)d\Re x d\Im x.$$

Now using [45, Lemma 9.1.8], we obtain

$$\mathrm{const.}\left|\frac{\alpha'_j\epsilon_j}{\alpha_j c_j^2}\right|\widetilde{\mathcal{L}}_{l,q}^{\alpha_j}\left(\mathcal{B}_{(\nu,p)}\left(4\pi\frac{(\alpha_j\alpha'_j\epsilon_j)^{\frac12}}{c_j}\right)\theta(\nu,p)\right)(g),$$

where

$$\mathcal{B}_{(\nu,p)}(z)=\frac{1}{\sin\pi(\nu-p)}\{|z/2|^{-2\nu}(iz/|z|)^{2p}\mathcal{J}^*_{-\nu,-p}(z)-|z/2|^{2\nu}(iz/|z|)^{-2p}\mathcal{J}^*_{\nu,p}(z)\},$$

with $\mathcal{J}^*$ defined in (8.6). Now by (8.19),

$$\text{const.}\left|\frac{\alpha_j\alpha_j'\epsilon_j}{c_j^2}\right|\sum_{|p|\le l}\int_{(0)}\overline{\eta(\nu,p)}\theta(\nu,p)\lambda_l(\nu,p)\mathcal{B}_{(\nu,p)}\left(4\pi\frac{(\alpha_j\alpha_j'\epsilon_j)^{\frac{1}{2}}}{c_j}\right)(p^2-\nu^2)d\nu. \tag{8.20}$$

We are left to work with the spectral side. To deliver the computation at a complex place of (8.14), we use [45, (10.4-7)], obtaining

$$\int_0^\infty \widetilde{\mathcal{L}}^{\alpha_j}_{l,q}\eta\left(\begin{pmatrix}y&0\\0&1\end{pmatrix}\right)\overline{\mathcal{W}_j(\alpha_j y)}|y|^{-2}d^\times_{\mathbf{R}}y=$$
$$\text{const.}|\alpha_j|^2 i^p\Gamma(l+1-\overline{\nu})\frac{\sin\pi(\overline{\nu}-p)}{\overline{\nu}^2-p^2}\overline{\nu}^{\varepsilon(p)}\eta(-\overline{\nu},p)\sqrt{\left|\frac{\Gamma(l+1+\nu)}{\Gamma(l+1-\nu)}\right|}.$$

Note that the last factor is 1, unless we are in the complementary series and in this case, $p=0$. Now taking the inner product of such two, we obtain

$$\begin{aligned}&\int_0^\infty\overline{\widetilde{\mathcal{L}}^{\alpha_j}_{l,q}\eta(y)}\mathcal{W}_j(\alpha_j y)|y|^{-2}d^\times_{\mathbf{R}}y\int_0^\infty\widetilde{\mathcal{L}}^{\alpha_j'}_{l,q}\theta(y)\overline{\mathcal{W}_j(\alpha_j' y)}|y|^{-2}d^\times_{\mathbf{R}}y=\\&=\text{const.}|\alpha_j\alpha_j'|^2\overline{\eta(\nu,p)}\theta(\nu,p)\lambda_l(\nu,p).\end{aligned} \tag{8.21}$$

Observe the similar behaviour of the real and the complex cases, even the factors coming from α,α' are the same by noting that for the complex modulus $|\cdot|_{\mathbf{C}}$, $|z|_{\mathbf{C}}=|z|^2$. Of course, the applied integral transforms and hence the integral kernels in the final formulas show some difference.

8.6 Derivation of the sum formula

8.6.1 Preliminary sum formulas

In this section we state some preliminary versions of the sum formula.

Lemma 8.5. *With the above notation, assuming that $\alpha\alpha'$ is totally positive,*

$$\begin{aligned}&[K(\mathfrak{o}):K(\mathfrak{c})]^{-1}\sum_{\pi\in\mathcal{C}_{\mathrm{FS}}(\mathfrak{c})}C_\pi^{-1}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}}\omega_\pi(\pi_{\mathfrak{b}}^{-1})(\overline{\eta}\theta\lambda)\lambda_\pi^{\mathfrak{t}}(\alpha\mathfrak{a}^{-1})\overline{\lambda_\pi^{\mathfrak{t}}(\alpha'\mathfrak{a}^{-1}\mathfrak{b}^{-2})}+CSC=\\&\text{const.}\Delta(\alpha,\alpha'[\mathfrak{b}]^{-2})\int(\overline{\eta}\theta\lambda)d\mu+\\&\text{const.}\sum_{c\in\mathfrak{a}\mathfrak{b}\mathfrak{c}}\sum_{\epsilon\in\mathfrak{o}_+^\times/\mathfrak{o}^{2\times}}\frac{KS(\epsilon\alpha,\mathfrak{a}^{-1}\mathfrak{d}^{-1};\alpha',\mathfrak{a}^{-1}\mathfrak{b}^{-2}\mathfrak{d}^{-1};c,\mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{d}^{-1})}{\mathcal{N}(c\mathfrak{a}^{-1}\mathfrak{b}^{-1})}\int\mathcal{B}_{(\mathbf{r})}\left(4\pi\frac{(\alpha\alpha'\epsilon)^{\frac{1}{2}}}{c}\right)(\overline{\eta}\theta\lambda)d\mu.\end{aligned} \tag{8.22}$$

Proof. Immediate from (8.13), (8.14), (8.16), (8.17), (8.18), (8.19), (8.20) and (8.21). □

In an actual application, some ideals $\mathfrak{a}^{-1},\mathfrak{a}'^{-1}$ are given. If there is some ideal $\mathfrak{b}$ such that $\mathfrak{a}'^{-1}$ equals $\mathfrak{a}^{-1}\mathfrak{b}^{-2}$ up to a totally positive principal ideal, that is, $\mathfrak{a}\mathfrak{a}'^{-1}$ is a square in the narrow class group, then adjusting α' in (8.22), we obtain a formula including $\lambda_\pi^{\mathfrak{t}}(\alpha\mathfrak{a}^{-1})\overline{\lambda_\pi^{\mathfrak{t}}(\alpha'\mathfrak{a}'^{-1})}$. Denote by C a fixed set of narrow class representatives $\mathfrak{m}$ for which $\mathfrak{m}^2\mathfrak{a}\mathfrak{a}'^{-1}$ is a principal ideal generated by a totally positive element $\gamma_{\mathfrak{m}}$, fixed once for all, and let C' be a set of representatives for the rest of ideals.

Lemma 8.6. *For all* $\mathfrak{m}\in C$ *and* $\alpha\in\mathfrak{a}$, $\alpha\in\mathfrak{a}'$ *such that* $\alpha\alpha'$ *is totally positive, we have*

$$[K(\mathfrak{o}):K(\mathfrak{c})]^{-1}\sum_{\pi\in\mathcal{C}_{\mathrm{FS}}(\mathfrak{c})}C_\pi^{-1}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}}\omega_\pi(\pi_{\mathfrak{m}}^{-1})(\overline{\eta}\theta\lambda)\lambda_\pi^{\mathfrak{t}}(\alpha\mathfrak{a}^{-1})\overline{\lambda_\pi^{\mathfrak{t}}(\alpha'\mathfrak{a}'^{-1})}+CSC=$$

$$\text{const.}\Delta(\alpha\mathfrak{a}^{-1},\alpha'\mathfrak{a}'^{-1})\int(\overline{\eta}\theta\lambda)d\mu+$$

$$\text{const.}\sum_{c\in\mathfrak{a}\mathfrak{m}\mathfrak{c}}\sum_{\epsilon\in\mathfrak{o}_+^\times/\mathfrak{o}^{2\times}}\frac{KS(\epsilon\alpha,\mathfrak{a}^{-1}\mathfrak{d}^{-1};\alpha'\gamma_{\mathfrak{m}},\mathfrak{a}'^{-1}\mathfrak{d}^{-1};c,\mathfrak{a}^{-1}\mathfrak{m}^{-1}\mathfrak{d}^{-1})}{\mathcal{N}(c\mathfrak{a}^{-1}\mathfrak{m}^{-1})}\int\mathcal{B}_{(\mathbf{r})}\left(4\pi\frac{(\alpha\alpha'\gamma_{\mathfrak{m}}\epsilon)^{\frac12}}{c}\right)(\overline{\eta}\theta\lambda)d\mu. \tag{8.23}$$

Proof. Immediate from the previous lemma. □

Lemma 8.7. *Let* $\mathfrak{m}$ *be a narrow class representative from either* C *or* C'. *Denote by* Ξ *the dual of the narrow class group. Assume* $\alpha\in\mathfrak{a}$, $\alpha'\in\mathfrak{a}'$ *such that* $\alpha\alpha'$ *is totally positive. Then*

$$\begin{aligned}\sum_{\pi\in\mathcal{C}_{\mathrm{FS}}(\mathfrak{c})}C_\pi^{-1}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}}\omega_\pi(\pi_{\mathfrak{m}}^{-1})(\overline{\eta}\theta\lambda)\lambda_\pi^{\mathfrak{t}}(\alpha\mathfrak{a}^{-1})\overline{\lambda_\pi^{\mathfrak{t}}(\alpha'\mathfrak{a}'^{-1})}\\ =\frac{1}{|\Xi|}\sum_{\chi\in\Xi}\sum_{\pi\in\mathcal{C}_{\mathrm{FS}}(\mathfrak{c})}C_{\pi\otimes\chi}^{-1}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}}\omega_{\pi\otimes\chi}(\pi_{\mathfrak{m}}^{-1})(\overline{\eta}\theta\lambda)\lambda_{\pi\otimes\chi}^{\mathfrak{t}}(\alpha\mathfrak{a}^{-1})\overline{\lambda_{\pi\otimes\chi}^{\mathfrak{t}}(\alpha'\mathfrak{a}'^{-1})}.\end{aligned}\tag{8.24}$$

The analogous identity holds for CSC. *Moreover, if* $\mathfrak{m}\in C'$, *the sum is* 0 *(and so is* CSC*).*

Proof. First note that if $\pi\in\mathcal{C}_{\mathrm{FS}}(\mathfrak{c})$, then for $\chi\in\Xi$, $\lambda_{\pi\otimes\chi}(\mathfrak{b})=\chi(\mathfrak{b})\lambda_\pi(\mathfrak{b})$ holds for the Fourier coefficients, which in particular implies $C_\pi=C_{\pi\otimes\chi}$.

Observe that for any narrow class group character χ,

$$\begin{aligned}\sum_{\pi\in\mathcal{C}_{\mathrm{FS}}(\mathfrak{c})}C_\pi^{-1}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}}\omega_\pi(\pi_{\mathfrak{m}}^{-1})(\overline{\eta}\theta\lambda)\lambda_\pi^{\mathfrak{t}}(\alpha\mathfrak{a}^{-1})\overline{\lambda_\pi^{\mathfrak{t}}(\alpha'\mathfrak{a}'^{-1})}=\\ \sum_{\pi\in\mathcal{C}_{\mathrm{FS}}(\mathfrak{c})}C_{\pi\otimes\chi}^{-1}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}}\omega_{\pi\otimes\chi}(\pi_{\mathfrak{m}}^{-1})(\overline{\eta}\theta\lambda)\lambda_{\pi\otimes\chi}^{\mathfrak{t}}(\alpha\mathfrak{a}^{-1})\overline{\lambda_{\pi\otimes\chi}^{\mathfrak{t}}(\alpha'\mathfrak{a}'^{-1})}.\end{aligned}$$

Indeed, the central character of each π is multiplied by χ^2, which is trivial on the archimedean ideles, and also the archimedean parameters $(q,(l,q),\nu,p)$ are invariant under these twists. From this, (8.24) is clear.

Moreover,

$$\begin{aligned}\frac{1}{|\Xi|}\sum_{\chi\in\Xi}\sum_{\pi\in\mathcal{C}_{\mathrm{FS}}(\mathfrak{c})}C_\pi^{-1}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}}\omega_{\pi\otimes\chi}(\pi_{\mathfrak{m}}^{-1})(\overline{\eta}\theta\lambda)\lambda_{\pi\otimes\chi}^{\mathfrak{t}}(\alpha\mathfrak{a}^{-1})\overline{\lambda_{\pi\otimes\chi}^{\mathfrak{t}}(\alpha'\mathfrak{a}'^{-1})}=\\ \frac{1}{|\Xi|}\sum_{\pi\in\mathcal{C}_{\mathrm{FS}}(\mathfrak{c})}C_\pi^{-1}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}}\omega_\pi(\pi_{\mathfrak{m}}^{-1})(\overline{\eta}\theta\lambda)\lambda_\pi^{\mathfrak{t}}(\alpha\mathfrak{a}^{-1})\overline{\lambda_\pi^{\mathfrak{t}}(\alpha'\mathfrak{a}'^{-1})}\sum_{\chi\in\Xi}\chi(\mathfrak{m}^{-2}\mathfrak{a}^{-1}\mathfrak{a}').\end{aligned}$$

By definition, the inner sum is $|\Xi|$ if $\mathfrak{m}\in C$, and 0 if $\mathfrak{m}\in C'$.

The same argument works for CSC. □

Lemma 8.8. *We have the preliminary sum formula*

$$[K(\mathfrak{o}):K(\mathfrak{c})]^{-1}\sum_{\pi\in\mathcal{C}(\mathfrak{c})}C_\pi^{-1}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}}(\overline{\eta}\theta\lambda)\lambda_\pi^{\mathfrak{t}}(\alpha\mathfrak{a}^{-1})\overline{\lambda_\pi^{\mathfrak{t}}(\alpha'\mathfrak{a}'^{-1})}+CSC=$$

$$\text{const.}\Delta(\alpha\mathfrak{a}^{-1},\alpha'\mathfrak{a}'^{-1})\int(\overline{\eta}\theta\lambda)d\mu+$$

$$\begin{aligned}\text{const.}\sum_{\mathfrak{m}\in C}\sum_{c\in\mathfrak{a}\mathfrak{m}\mathfrak{c}}\sum_{\epsilon\in\mathfrak{o}_+^\times/\mathfrak{o}^{2\times}}\\ \frac{KS(\epsilon\alpha,\mathfrak{a}^{-1}\mathfrak{d}^{-1};\alpha'\gamma_{\mathfrak{m}},\mathfrak{a}'^{-1}\mathfrak{d}^{-1};c,\mathfrak{a}^{-1}\mathfrak{m}^{-1}\mathfrak{d}^{-1})}{\mathcal{N}(c\mathfrak{a}^{-1}\mathfrak{m}^{-1})}\int\mathcal{B}_{(\mathbf{r})}\left(4\pi\frac{(\alpha\alpha'\gamma_{\mathfrak{m}}\epsilon)^{\frac12}}{c}\right)(\overline{\eta}\theta\lambda)d\mu.\end{aligned}\tag{8.25}$$

Proof. Using that $\mathcal{C}(\mathfrak{c})$ consists of those elements of $\mathcal{C}_{\mathrm{FS}}(\mathfrak{c})$ on which $Z(\mathbf{A})$ acts trivially, we can rewrite the left-hand side as

$$[K(\mathfrak{o}):K(\mathfrak{c})]^{-1}\frac{1}{|C\cup C'|}\sum_{\mathfrak{m}\in C\cup C'}\sum_{\pi\in\mathcal{C}_{\mathrm{FS}}(\mathfrak{c})}C_\pi^{-1}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}}\omega_\pi(\pi_{\mathfrak{m}}^{-1})(\overline{\eta}\theta\lambda)\lambda_\pi^{\mathfrak{t}}(\alpha\mathfrak{a}^{-1})\overline{\lambda_\pi^{\mathfrak{t}}(\alpha'\mathfrak{a}'^{-1})}+CSC.$$

Now the contribution of $\mathfrak{m}\in C$ is given in (8.23), while the contribution of $\mathfrak{m}\in C'$ is 0 by Lemma 8.7. Note that $|C|$ does not depend on $\mathfrak{a},\mathfrak{a}'$, since C is a coset of the squares in the narrow class group. □

We close this section by noting that as c runs through $\mathfrak{amc}$, the weighted sum of Kloosterman sums is absolutely convergent: combine the estimates [10, Lemma 3.13] and [45, Lemma 11.1.2] at real and complex places, respectively, with Weil's bound [56, (13)].

8.6.2 Extension of the preliminary sum formula

Now we are in the position to prove Theorem 3. Observe that (8.25) resembles (5.1), except for the weight function, which is a triple product $\overline{\eta}\theta\lambda$ of functions in the preliminary sum formula, and a single function in the Kuznetsov formula.

Proof of Theorem 3 in the case of $r\neq 0$. Set $q_j>\max(2,a_j,b_j)$ at real, $\min(l_j,q_j)>\max(2,a_j,b_j)$ at complex places. Choose a small $\delta>0$. Then let $\eta(\nu,p)=e^{\delta\nu^2}$ on $|\Re\nu|\leq 2/3$ and in the discrete series at real places, let $\eta(\nu)=1$, if $\nu\in 1/2+\mathbf{Z}$ and $3/2\leq|\nu|\leq b_j$.

Recall that $\lambda_j\neq 0$ at complex places. Unfortunately, at real places, $\lambda(\nu)$ might vanish. We claim that

$$\lambda(\nu)=\sum_{\pm}\frac{1}{\Gamma\left(\frac12-\nu\pm\frac q2\right)\Gamma\left(\frac12+\nu\pm\frac q2\right)}\neq 0$$

on the domain

$$D=\{\nu\in\mathbf{C}:\Re\nu=0\}\cup\left(-\frac12,\frac12\right)\cup\left\{\nu\in\frac12+\mathbf{Z}:|\nu|\leq\frac{q-1}{2}\right\}.$$

Indeed, for $\Im\nu=0$, $3/2\leq q/2+1/2\in 1/2+\mathbf{Z}$. This shows that $\Gamma(1/2+q/2+\nu),\Gamma(1/2+q/2-\nu)$ are both positive, so the term '$+q/2$' gives a positive number. Similarly, it is easy to see that in the term '$-q/2$', $\Gamma(1/2-q/2+\nu),\Gamma(1/2-q/2-\nu)$ are either of the same sign or both show a pole (for $\nu\in 1/2+\mathbf{Z}$). In any case, they give a nonnegative contribution. For $\Re\nu=0$, the positivity is clear, as there are complex norms in the denominators.

Moreover, the recursion $x\Gamma(x)=\Gamma(x+1)$ implies that if q is large enough, $\lambda(\nu)$ does not vanish on $D'=|\Re\nu|<1/3$. Adjust q to satisfy this.

Fix a positive integer $N>(q-1)/2$. Given $\varepsilon>0$, from the Mergelyan-Bishop approximaton theorem (see e.g. [54, Theorem 20.5] for the original version and [40, Theorem 1.11.5] for its extension to Riemann surfaces), we see that there is a rational function λ_ε on $\mathbf{C}\cup\{\infty\}$ with possible poles in $\{\pm N\}$ (which is disjoint from $|\Re\nu|\leq(q-1)/2$ by the choice of N) such that $|\lambda_\varepsilon(\nu)-1/\lambda(\nu)|<\varepsilon\min(1,|\nu|^{-3})$ for all $\nu\in D\cup D'\cup\{\infty\}$. Since λ is real on $\Re\nu=0$ and even, we may assume these hold for λ_ε.

At complex places, let $\lambda_\varepsilon=1/\lambda$.

We have already given η. Let $\theta(\nu,p)=h(\nu,p)e^{-\delta\nu^2}\lambda_\varepsilon(\nu,p)$. By construction, this can be chosen to be a test function, if δ is small enough (independently of ε).

Now the triple product gives $\overline{e^{\delta\nu^2}}e^{-\delta\nu^2}h(\nu,p)\lambda(\nu,p)\lambda_\varepsilon(\nu,p)$. Observe that in (5.1) and (8.25) the relevant ν's come from D, where we know the uniform convergence $h(\nu,p)\lambda(\nu,p)\lambda_\varepsilon(\nu,p)\to h(\nu,p)$ as $\varepsilon\to 0$ and also that the sum formula holds for $h\lambda_\epsilon\lambda$. Now [10, Lemmas 3.16 and 3.17] completes the proof (we note that the domain D' is introduced to obtain functions such that their Bessel transform have the right order of magnitude for the absolute convergence in the Kloosterman term: recall the remark made at the end of Section 8.6.1) by observing that $\overline{e^{\delta\nu^2}}e^{-\delta\nu^2}=1$ on D. □

8.7 The proof in the case of $r=0$

8.7.1 The definition of Poincaré series

If all the archimedean places are complex, the inverse Lebedev transform $\widetilde{\mathcal{L}}_{l,q}^{\alpha}\eta$ does not tend to 0 fast enough as the height goes to 0, so the earlier definition of Poincaré series is not exact.

In this case we have a little technical simplification, as $\mathrm{PSL}_2(\mathbf{C}) = \mathrm{PGL}_2(\mathbf{C})$, so we can assume that all occuring complex matrices have determinant 1.

To remedy our problem, we follow the argument of Bruggeman and Motohashi [13, Section 9]. We will also refer to the thesis of Lokvenec-Guleska [45].

Let

$$B(\eta) = 2\pi l \cdot l!\eta(0,1)|\alpha|^2\sqrt{2l+1}\binom{2l}{l-1}^{-\frac{1}{2}}\binom{2l}{l-q}^{\frac{1}{2}}.$$

Then [13, (7.14-15)], [45, (9.16-17)] can be written as

$$(\widetilde{\mathcal{L}}^{\alpha}_{l,q}\eta)(g) = B(\eta)\mathbf{M}_\alpha\varphi_{l,q}(1,0)(g) + O(|y|_\infty^{(1+\sigma)/2}), \tag{8.26}$$

where y is the height of g. Note that $\sigma > 1$, and here we indicated the function φ and its weight and spectral data, the latter evaluated at $(1,0)$ and we may assume that $l > 0$ (if $l = 0$, then this gives 0, and we have the order of magnitude needed to give a similar argument to that in Section 8.3.4).

Use $\prod \mathbf{M}_\alpha$ as the building block (where we dropped j from $\prod_j$), and let (see [13, (9.1)])

$$P\prod \mathbf{M}_\alpha\varphi_{l,q}(\nu,p)(g) = \sum_{\gamma\in Z_\Gamma\Gamma_N(\mathfrak{a},\mathfrak{c})\backslash\Gamma(\mathfrak{a},\mathfrak{c})} \prod \mathbf{M}_\alpha\varphi_{l,q}(\nu,p)(\gamma g).$$

This is absolutely convergent for $\Re\nu > 1$ (see [45, (4.54)], then an Eisenstein series again majorizes our sum). Note that when ν is a vector in $\mathbf{C}^s$, by $\Re\nu > 1$ we mean $\Re\nu_j > 1$ for all j. In order to keep notations as simple as it is possible, we will use similar abbreviation from now on, not only for ν, but for p, l, q as well. For example, $\nu = 1$, $p = 0$ means that $\nu_j = 1$, $p_j = 0$ for all j.

Now take any building block f which is a pure tensor, and follow [13, (5.1-6)]. Using the Bruhat decomposition, the Poincaré series Pf can be written formally as

$$\begin{aligned} Pf(g) = &\sum_{\epsilon\in\mathfrak{o}^\times/\mathfrak{o}^{2\times}} f\left(\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon^{-1}\end{pmatrix} g\right)\\ &+\sum_{0\neq c\in\mathfrak{a}\mathfrak{c}\mathfrak{d}}\sum_{d}\sum_{\omega\in(\mathfrak{a}\mathfrak{d})^{-1}} f\left(\begin{pmatrix}1 & d'c^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}c & 0\\ 0 & c^{-1}\end{pmatrix}\begin{pmatrix}1 & dc^{-1}+\omega\\ 0 & 1\end{pmatrix} g\right), \end{aligned}$$

where $d' \in \mathfrak{o}$ is the element modulo $(\mathfrak{a}\mathfrak{d})^{-1}c$ such that $dd' \equiv 1$ modulo $(\mathfrak{a}\mathfrak{d})^{-1}c$ and we sum over those d's modulo $(\mathfrak{a}\mathfrak{d})^{-1}c$ for which such a $d' \in \mathfrak{o}$ exists, that is, d generates $\mathfrak{o}/(\mathfrak{a}\mathfrak{d})^{-1}c$ as an $\mathfrak{o}$-module. Now applying Poisson summation, we see that the ω-sum can be rewritten as

$$\text{const.}\mathcal{N}(\mathfrak{a})\sum_{\omega\in\mathfrak{a}}\psi_\infty(d\omega/c)\int_{F_\infty}\psi_\infty(\omega x)^{-1} f\left(\begin{pmatrix}1 & d'c^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}c & 0\\ 0 & c^{-1}\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) dx.$$

Now assume that for some $\omega' \in \mathfrak{a}$,

$$f(n(x)g) = \psi_\infty(\omega' x) f(g).$$

Therefore we obtain

$$\begin{aligned} Pf(g) = &\sum_{\epsilon\in\mathfrak{o}^\times/\mathfrak{o}^{2\times}} f\left(\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon^{-1}\end{pmatrix} g\right)\\ &+\sum_{\omega\in\mathfrak{a}}\sum_{0\neq c\in\mathfrak{a}\mathfrak{c}\mathfrak{d}} KS(\omega,\mathfrak{a}^{-1}\mathfrak{d}^{-1};\omega',\mathfrak{a}^{-1}\mathfrak{d}^{-1};c,\mathfrak{a}^{-1}\mathfrak{d}^{-1})\mathbf{J}_\omega\left(f\left(\begin{pmatrix}1/c & 0\\ 0 & c\end{pmatrix} g\right)\right), \end{aligned}$$

noting that by $\mathbf{J}_\omega$, we mean the product of the local integrals (we have chosen f to be a pure tensor).

If ω, ω', $\mathfrak{a}$ are all fixed and only c varies, Weil's bound (see [56, (13)]) gives

$$KS(\omega,\mathfrak{a}^{-1}\mathfrak{d}^{-1};\omega',\mathfrak{a}^{-1}\mathfrak{d}^{-1};c,\mathfrak{a}^{-1}\mathfrak{d}^{-1}) \ll_{\varepsilon,\omega,\omega',\mathfrak{a}} |c|_\infty^{1/2+\varepsilon}. \tag{8.27}$$

Now specializing the above to $\prod \mathbf{M}_\alpha$, α taking the place of ω' (which is indeed in $\mathfrak{a}$), we see (still for $\Re\nu > 1$)

$$\begin{aligned} P\prod \mathbf{M}_\alpha \varphi_{l,q}(\nu,p)(g) = & \sum_{\epsilon\in\mathfrak{o}^\times/\mathfrak{o}^{2\times}} \prod \mathbf{M}_\alpha \varphi_{l,q}(\nu,p)\left(\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon^{-1}\end{pmatrix} g\right) \\ & + \sum_{0\neq c\in\mathfrak{acd}} KS(0,\mathfrak{a}^{-1}\mathfrak{d}^{-1};\alpha,\mathfrak{a}^{-1}\mathfrak{d}^{-1};c,\mathfrak{a}^{-1}\mathfrak{d}^{-1})\mathbf{J}_0\left(\prod \mathbf{M}_\alpha\varphi_{l,q}(\nu,p)\left(\begin{pmatrix}1/c & 0\\ 0 & c\end{pmatrix} g\right)\right) \\ & + \sum_{0\neq\omega\in\mathfrak{a}}\sum_{0\neq c\in\mathfrak{acd}} KS(\omega,\mathfrak{a}^{-1}\mathfrak{d}^{-1};\alpha,\mathfrak{a}^{-1}\mathfrak{d}^{-1};c,\mathfrak{a}^{-1}\mathfrak{d}^{-1})\mathbf{J}_\omega\left(\prod \mathbf{M}_\alpha\varphi_{l,q}(\nu,p)\left(\begin{pmatrix}1/c & 0\\ 0 & c\end{pmatrix} g\right)\right). \end{aligned} \tag{8.28}$$

Here, the first term continues analytically to $\nu\in\mathbf{C}^s$. In the second term, we apply (8.4) together with (8.27), this continues to $\Re\nu > 0$. In the last term, apply (8.5), together with the explicit form of $\mathbf{J}_\omega$, this gives (using also [45, (4.52)])

$$\sum_{0\neq\omega\in\mathfrak{a}} \mathbf{J}_\omega\varphi_{l,q}(\nu,p)(g)$$
$$\left(\sum_{0\neq c\in\mathfrak{acd}} KS(\omega,\mathfrak{a}^{-1}\mathfrak{d}^{-1};\alpha,\mathfrak{a}^{-1}\mathfrak{d}^{-1};c,\mathfrak{a}^{-1}\mathfrak{d}^{-1})\prod_j \frac{1}{|c_j|^{2(1+\nu_j)}}\left(\frac{c_j}{|c_j|}\right)^{2p_j}\mathcal{J}^*_{\nu_j,p_j}\left(\frac{4\pi}{c_j}\sqrt{\alpha_j\omega_j}\right)\right).$$

Here, the inner sum continues analytically to $\Re\nu > 1/2$ by (8.27). The resulting function is of order $\exp(C\sqrt{|\omega|})$ in ω with C depending on $\nu,p,\alpha,\mathfrak{a},\mathfrak{c}$, so the ω-sum gives an analytic function (as the K-Bessel function appearing in $\mathbf{J}$ has exponential decay $\exp(-c|\omega|)$ at infinity).

Now $P\prod\mathbf{M}_\alpha$ is a well-defined Poincaré series, however, it fails to be square-integrable. Fix some $0 < A\in\mathbf{R}$. Let ρ be a funcion on $\mathbf{R}^s$ such that it is smooth, $\rho(y) = 1$ if $\prod_j |y_j| \leq A$, and $\rho(y) = 0$ if $\prod_j |y_j| \geq A+1$. This extends to $(\mathrm{SL}_2(\mathbf{C}))^s$ via the Iwasawa decomposition $na(y)k$ by making it independent of n and k.

Let $\prod\mathbf{M}'_\alpha(g) = \rho(g)\prod\mathbf{M}_\alpha(g)$, then we have, for $\Re\nu > 1/2$,

$$P\prod\mathbf{M}'_\alpha(g) = P\prod\mathbf{M}_\alpha(g) + (\rho(y)-1)\sum_{\epsilon\in\mathfrak{o}^\times/\mathfrak{o}^{2\times}}\prod\mathbf{M}_\alpha\left(\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon^{-1}\end{pmatrix} g\right), \tag{8.29}$$

if A is large enough, $y = y(g)$ stands for the height of $g = na(y)k$. Now observe that as $\prod_j |y_j|\to\infty$, the magnitude of $P\prod\mathbf{M}'_\alpha$ is determined by the second line of (8.28) and it is $\ll \prod_j |y_j|^{1-\Re\nu_j}$, so it is bounded according to (8.4) at $\nu = 1$, $p = 0$, and also bounded uniformly in ν on a right-neighborhood $\nu\in[1,1+\delta)$ (with a small $\delta > 0$).

Now returning to (8.26), define $E\eta$ via

$$(\widetilde{\mathcal{L}}^\alpha_{l,q}\eta)(g) = B(\eta)\mathbf{M}'_\alpha\varphi_{l,q}(1,0)(g) + E\eta(g).$$

On the product this gives

$$\prod(\widetilde{\mathcal{L}}^\alpha_{l,q}\eta)(g) = \sum_{S\subseteq\{1,\dots,s\}}\prod_{\{1,\dots,s\}\setminus S} B(\eta)\mathbf{M}'_\alpha\varphi_{l,q}(1,0)(g)\prod_S E\eta(g),$$

where $\prod_S$ means $\prod_{j\in S}$. Now define

$$P\prod(\widetilde{\mathcal{L}}^\alpha_{l,q}\eta)(g) = P\prod B(\eta)\mathbf{M}'_\alpha\varphi_{l,q}(1,0)(g) + P\sum_{\emptyset\neq S\subseteq\{1,\dots,s\}}\left(\prod_{\{1,\dots,s\}\setminus S} B(\eta)\mathbf{M}'_\alpha\varphi_{l,q}(1,0)\prod_S E\eta\right)(g). \tag{8.30}$$

By our construction, $E\eta(g)\ll |y|_\infty^{1+\varepsilon}$ for some $\varepsilon > 0$ as $|y|_\infty\to 0$, and $E\eta(g)\ll |y|_\infty^{-k}$ for all $k\in\mathbf{N}$ as $|y|_\infty\to\infty$. Hence the second Poincaré series on the right-hand side is absolutely convergent and gives a bounded function (the argument of Section 8.3.4 goes through). This, together with the boundedness of $P\prod \mathbf{M}'$ at $\nu=1, p=0$, gives that defining our Poincaré series this way, it is bounded, hence square-integrable, $B(\eta)$ and $\prod B(\eta)$ are constants (for α, l, q, η fixed).

Altogether, in case of $r=0$, we use (8.30) as the definition of the Poincaré series, where the first term $P\prod B(\eta)\mathbf{M}'\varphi_{l,q}(1,0)$ is understood via analytic continuation as explained above:

$$P\prod B(\eta)\mathbf{M}'\varphi_{l,q}(1,0)=\prod B(\eta)\lim_{\nu\to 1+}P\prod \mathbf{M}'\varphi_{l,q}(\nu,0), \tag{8.31}$$

where inside the limit, there is an absolutely convergent Poincaré series for all $\Re\nu>1$. Finally, we also record what we obtain from (8.30), (8.31):

$$\begin{aligned} P\prod(\widetilde{\mathcal{L}}^{\alpha}_{l,q}\eta)(g)=&\lim_{\nu\to 1+}P\prod B(\eta)\mathbf{M}'_{\alpha}\varphi_{l,q}(\nu,0)(g)\\ &+P\sum_{\emptyset\neq S\subseteq\{1,\dots,s\}}\left(\prod_{\{1,\dots,s\}\setminus S}B(\eta)\mathbf{M}'_{\alpha}\varphi_{l,q}(1,0)\prod_{S}E\eta\right)(g). \end{aligned} \tag{8.32}$$

8.7.2 The complement of the original argument

From this point, we may modify the argument given in the case of $r\neq 0$ as follows. In place of each occurence of $P\widetilde{\mathcal{L}}^{\alpha}_{l,q}\eta(g)$, write (8.32): use the Poincaré series obtained from

$$\left(\prod B(\eta)\mathbf{M}'_{\alpha}\varphi_{l,q}(\nu,p)+\sum_{\emptyset\neq S\subseteq\{1,\dots,s\}}\left(\prod_{\{1,\dots,s\}\setminus S}B(\eta)\mathbf{M}'_{\alpha}\varphi_{l,q}(1,0)\prod_{S}E\eta\right)\right)(g)$$

with some $\nu>1$, and then let $\lim_{\nu\to 1+}$. That is, start out from (8.32) as

$$\begin{aligned} f_1^{\nu}(g)&=\left(\prod B(\eta)\mathbf{M}'_{\alpha}\varphi_{l,q}(\nu,p)+\sum_{\emptyset\neq S\subseteq\{1,\dots,s\}}\left(\prod_{\{1,\dots,s\}\setminus S}B(\eta)\mathbf{M}'_{\alpha}\varphi_{l,q}(1,0)\prod_{S}E\eta\right)\right)(g),\\ f_2^{\nu'}(g)&=\left(\prod B(\eta)\mathbf{M}'_{\alpha'}\varphi_{l,q}(\nu',p)+\sum_{\emptyset\neq S\subseteq\{1,\dots,s\}}\left(\prod_{\{1,\dots,s\}\setminus S}B(\eta)\mathbf{M}'_{\alpha'}\varphi_{l,q}(1,0)\prod_{S}E\theta\right)\right)(g), \end{aligned}$$

then define $P_1^{\nu}=Pf_1^{\nu}$, $P_2^{\nu'}=Pf_2^{\nu'}$ in the sense of (8.11). Finally, let

$$P_1=\lim_{\nu\to 1+}P_1^{\nu},\qquad P_2=\lim_{\nu'\to 1+}P_2^{\nu'}.$$

Now assume ϕ is an integrable function over $Z(F_\infty)\Gamma(\mathfrak{a},\mathfrak{c})\backslash \mathrm{GL}_2(F_\infty)$ such that its Fourier coefficient

$$\int_{\Gamma_N(\mathfrak{a},\mathfrak{c})\backslash N(F_\infty)}\psi_1(-n)\phi(ng)dn=\begin{cases} O(|y|_\infty^{\varepsilon}), & |y|_\infty\to 0,\\ O(|y|_\infty^{-\varepsilon}), & |y|_\infty\to\infty \end{cases}$$

for some $\varepsilon>0$, y stands for the height of g. This holds both for the basis elements $\mathbf{f}$ in the cuspidal spectrum and for Eisenstein series (recall the notation of Section 8.4.2), as it follows from their Fourier-Whittaker expansions (2.25), (2.38). Then by unfolding,

$$\begin{aligned} &\int_{Z(F_\infty)\Gamma(\mathfrak{a},\mathfrak{c})\backslash \mathrm{GL}_2(F_\infty)}\lim_{\nu\to 1+}\overline{P_1^{\nu}(g)}\phi(g)dg\\ &=\lim_{\nu\to 1+}\int_{Z(F_\infty)\Gamma(\mathfrak{a},\mathfrak{c})\backslash \mathrm{GL}_2(F_\infty)}\overline{P_1^{\nu}(g)}\phi(g)dg\\ &=\lim_{\nu\to 1+}\int_{Z(F_\infty)\Gamma(\mathfrak{a},\mathfrak{c})\backslash \mathrm{GL}_2(F_\infty)}\sum_{\gamma\in Z_\Gamma\Gamma_N(\mathfrak{a},\mathfrak{c})\backslash\Gamma(\mathfrak{a},\mathfrak{c})}\overline{f_1^{\nu}(\gamma g)}\phi(\gamma g)dg\\ &=\lim_{\nu\to 1+}\int_{Z(F_\infty)N(F_\infty)\backslash \mathrm{GL}_2(F_\infty)}\overline{f_1^{\nu}(g)}\int_{\Gamma_N(\mathfrak{a},\mathfrak{c})\backslash N(F_\infty)}\psi_1(-n)\phi(ng)dndg\\ &=\int_{Z(F_\infty)N(F_\infty)\backslash \mathrm{GL}_2(F_\infty)}\lim_{\nu\to 1+}\overline{f_1^{\nu}(g)}\int_{\Gamma_N(\mathfrak{a},\mathfrak{c})\backslash N(F_\infty)}\psi_1(-n)\phi(ng)dndg\\ &=\int_{Z(F_\infty)N(F_\infty)\backslash \mathrm{GL}_2(F_\infty)}\overline{\widetilde{\mathcal{L}}^{\alpha}_{l,q}\eta(g)}\int_{\Gamma_N(\mathfrak{a},\mathfrak{c})\backslash N(F_\infty)}\psi_1(-n)\phi(ng)dndg. \end{aligned}$$

We made two interchanges between limit and integral. Both are explained by dominated convergence as follows. First, we can split up P_1 and f_1 according to (8.32), in which we have to explain the interchange in the first term (there is no ν in the second one), restrict to this. In the first interchange, the function ϕ is integrable where it is integrated, and it is multiplied by a function which is bounded (with a bound independent of ν in a small neighborhood of $\nu = 1$: see the remark made after (8.29)), therefore, the function ϕ itself serves as an integrable majorant. In the second interchange, we may use $|y|_\infty \int_{\Gamma_N(\mathfrak{a},\mathfrak{c})\backslash N(F_\infty)} \psi_1(-n)\phi(ng)dn$ as an integrable majorant, where y is the height of g: it majorizes by [45, (4.54-55)] and it is integrable by the condition made on $\int_{\Gamma_N(\mathfrak{a},\mathfrak{c})\backslash N(F_\infty)} \psi_1(-n)\phi(ng)dn$. Now the content of Section 8.4.2 is fully explained in this case.

As for the geometric side, we start with moving the limits outside

$$\int_{Z(F_\infty)\Gamma(\mathfrak{a},\mathfrak{c})\backslash \mathrm{GL}_2(F_\infty)} \lim_{\nu\to 1+} \overline{P_1^\nu(g)} \lim_{\nu'\to 1+} P_2^\nu(g)dg = \lim_{\nu,\nu'\to 1+} \int_{Z(F_\infty)\Gamma(\mathfrak{a},\mathfrak{c})\backslash \mathrm{GL}_2(F_\infty)} \overline{P_1^\nu(g)} P_2^\nu(g),$$

which is justified by noting that $P_1, P_2, P_1^\nu, P_2^{\nu'}$ are all bounded uniformly in $\nu, \nu' \in [1, 1+\delta)$ for a small $\delta > 0$. Then inside the limit, we proceed as in Section 8.4.1 to the point (8.13), which, in this case, is

$$\begin{aligned} I = \lim_{\nu,\nu'\to 1+} \Bigg(& \mathrm{const.}\Delta(\alpha, \alpha'[\mathfrak{b}]^{-2})\mathcal{N}(\mathfrak{a}^{-1}) \int_{N(F_\infty)Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \overline{f_1^\nu(g)} f_2^{\nu'} \left(\begin{pmatrix} [\mathfrak{b}]^{-1}\epsilon_0 & 0 \\ 0 & [\mathfrak{b}] \end{pmatrix} g \right) dg \\ & + \mathrm{const.} \sum_{c\in\mathfrak{abc}, \epsilon\in\mathfrak{o}^\times/\mathfrak{o}^{2\times}} KS(\epsilon\alpha, \mathfrak{a}^{-1}\mathfrak{d}^{-1}; \alpha', \mathfrak{a}^{-1}\mathfrak{b}^{-2}\mathfrak{d}^{-1}; c, \mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{d}^{-1}) \\ & \cdot \int_{N(F_\infty)Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \overline{f_1^\nu(g)} \int_{N(F_\infty)} \overline{\psi_1(n)} f_2^{\nu'} \left(\mathrm{w} \begin{pmatrix} 1 & 0 \\ 0 & \epsilon c^{-2} \end{pmatrix} ng \right) dndg \Bigg). \end{aligned}$$

First concenctrate on the delta term, we claim there that

$$\begin{aligned} & \lim_{\nu,\nu'\to 1+} \int_{N(F_\infty)Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \overline{f_1^\nu(g)} f_2^{\nu'} \left(\begin{pmatrix} [\mathfrak{b}]^{-1}\epsilon_0 & 0 \\ 0 & [\mathfrak{b}] \end{pmatrix} g \right) dg \\ & = \int_{N(F_\infty)Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \lim_{\nu\to 1+} \overline{f_1^\nu(g)} \lim_{\nu'\to 1+} f_2^{\nu'} \left(\begin{pmatrix} [\mathfrak{b}]^{-1}\epsilon_0 & 0 \\ 0 & [\mathfrak{b}] \end{pmatrix} g \right) dg \\ & = \int_{N(F_\infty)Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \overline{\widetilde{\mathcal{L}}_{l,q}^\alpha \eta(g)} \widetilde{\mathcal{L}}_{l,q}^{\alpha'} \theta(g) \left(\begin{pmatrix} [\mathfrak{b}]^{-1}\epsilon_0 & 0 \\ 0 & [\mathfrak{b}] \end{pmatrix} g \right) dg. \end{aligned}$$

We have to check the first equality, where we can focus again on the part corresponding to the first term of (8.32), where $\rho(g)|y|_\infty^2$ (y is still the height of g) is an integrable majorant: it is trivially integrable and majorizes by [45, (4.54-55)].

As for the Kloosterman term, we claim analogously

$$\begin{aligned} & \lim_{\nu,\nu'\to 1+} \sum_{c\in\mathfrak{abc}, \epsilon\in\mathfrak{o}^\times/\mathfrak{o}^{2\times}} KS(\epsilon\alpha, \mathfrak{a}^{-1}\mathfrak{d}^{-1}; \alpha', \mathfrak{a}^{-1}\mathfrak{b}^{-2}\mathfrak{d}^{-1}; c, \mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{d}^{-1}) \\ & \quad \cdot \int_{N(F_\infty)Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \overline{f_1^\nu(g)} \int_{N(F_\infty)} \overline{\psi_1(n)} f_2^{\nu'} \left(\mathrm{w} \begin{pmatrix} 1 & 0 \\ 0 & \epsilon c^{-2} \end{pmatrix} ng \right) dndg \\ & = \sum_{c\in\mathfrak{abc}, \epsilon\in\mathfrak{o}^\times/\mathfrak{o}^{2\times}} KS(\epsilon\alpha, \mathfrak{a}^{-1}\mathfrak{d}^{-1}; \alpha', \mathfrak{a}^{-1}\mathfrak{b}^{-2}\mathfrak{d}^{-1}; c, \mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{d}^{-1}) \\ & \quad \cdot \int_{N(F_\infty)Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \lim_{\nu\to 1+} \overline{f_1^\nu(g)} \int_{N(F_\infty)} \overline{\psi_1(n)} \lim_{\nu'\to 1+} f_2^{\nu'} \left(\mathrm{w} \begin{pmatrix} 1 & 0 \\ 0 & \epsilon c^{-2} \end{pmatrix} ng \right) dndg \\ & = \sum_{c\in\mathfrak{abc}, \epsilon\in\mathfrak{o}^\times/\mathfrak{o}^{2\times}} KS(\epsilon\alpha, \mathfrak{a}^{-1}\mathfrak{d}^{-1}; \alpha', \mathfrak{a}^{-1}\mathfrak{b}^{-2}\mathfrak{d}^{-1}; c, \mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{d}^{-1}) \\ & \quad \cdot \int_{N(F_\infty)Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \overline{\widetilde{\mathcal{L}}_{l,q}^\alpha \eta(g)} \int_{N(F_\infty)} \overline{\psi_1(n)} \widetilde{\mathcal{L}}_{l,q}^{\alpha'} \eta \left(\mathrm{w} \begin{pmatrix} 1 & 0 \\ 0 & \epsilon c^{-2} \end{pmatrix} ng \right) dndg. \end{aligned}$$

Again, we have to check the interchangeability, and we may restrict to the first term according to (8.32). The integrable majorant is $\rho(g)|y|_\infty^{3/2}|c|_\infty^{-3}$ (y is still the height of g), its integrability is obvious, and it majorizes by [13, (9.25)] (for the integral over $N(F_\infty)$), [45, (4.54)] (for the integral over

$N(F_\infty)Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)$) and finally (8.27) (for the sum of Kloosterman sums). This altogether explains Section 8.4.1 in this case.

Proof of Theorem 3 in the case of $r = 0$. The above argument shows that this definition of the Poincaré series leads to the same scalar product, both on the geometric and on the spectral side, as in Section 8.4. Now we continue as in Section 8.5 and Section 8.6, everything goes through with the simplification that we do not need the approximation in the final step, since the function λ cannot vanish at complex places. □

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