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Existence results for some differential inclusions and related problems

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Abstract

The aim of this thesis is to study various nonsmooth variational problems which are governed by set-valued maps such as the Clarke generalized gradient or the convex subdifferential.

The thesis has a strong interdisciplinary character combining results and methods from different areas such as Nonsmooth and Convex Analysis, Set-Valued Analysis, PDE's, Calculus of Variations, Mechanics of Materials and Contact Mechanics. The problems considered here can be divided into three main classes:

- *boundary value problems* involving differential operators subjected to various boundary constraints. Several existence and multiplicity results for such problems are obtained by using mainly variational methods;

- *inequality problems of variational type* whose solutions are not necessarily critical points of certain energy functionals. Existence results for some problems of this type are derived by using topological methods such as fixed point theorems for set-valued maps;

- *mathematical models* which arise in Contact Mechanics and describe the contact between a body and a foundation. Two such models are investigated. Their variational formulations lead to some hemivariational inequality systems which are solved by using our theoretical results.

Introduction

The study of nonsmooth variational problems began in the 1960's with the pioneering work of Fichera [50] who introduced variational inequalities to solve an open problem in Contact Mechanics proposed by Signorini in 1933. Few decades later, Panagiotopoulos [98, 99, 100] introduced a new class of variational inequalities, called hemivariational inequalities, by replacing the convex subdifferential with the Clarke generalized gradient and successfully used these problems to model various phenomena arising in Mechanics and Engineering. The term nonsmooth is used due to the fact that, in general, the corresponding energy functional is not differentiable.

The main purpose of the present thesis is to analyze some nonsmooth, non-standard variational problems which may be formulated in terms of differential inclusions involving the Clarke generalized gradient and/or the convex subdifferential. In dealing with such problems we employ either variational or topological methods to prove the existence of at least one solution. The study of such problems is motivated by the fact that they can serve as models for various phenomena arising in our daily life.

The thesis contains seven chapters which are briefly presented below.

Chapter 1 (Preliminaries) contains introductory notions and results from nonsmooth and set-valued analysis such as the Gâteaux differentiability of convex functions, the subdifferential of a convex function, the generalized gradient (Clarke subdifferential) of a locally Lipschitz function, properties of lower and upper semicontinuous set-valued maps. Some definitions and basic properties of various function spaces (classical Lebesgue and Sobolev spaces, variable exponent Lebesgue and Sobolev spaces and Orlicz spaces) are also recalled.

Chapter 2 (Some abstract results) contains three theorems which are useful in determining critical points of locally Lipschitz functionals. First we consider locally Lipschitz functionals defined on a real reflexive Banach space X of the form

$$\mathcal{E}_\lambda = L(u) - (J_1 \circ T)(u) - \lambda(J_2 \circ S)(u)$$

where $L : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous C^1 functional, $J_1 : Y \to \mathbb{R}$ and $J_2 : Z \to \mathbb{R}$ are locally Lipschitz functionals, $T : X \to Y$ and $S : X \to Z$ are linear and compact operators and λ is a real parameter. We provide sufficient conditions for $\mathcal{E}_\lambda$ to posses three critical points for each $\lambda > 0$ and if an additional assumption is fulfilled we prove that there exists $\lambda^* > 0$ such that $\mathcal{E}_{\lambda^*}$ has at least four critical points.

The second and the third theorem provide information concerning the Clarke subdifferentiability of integral functions defined on variable exponent Lebesgue spaces and Orlicz spaces, respectively, and can be viewed as extensions of the Aubin-Clarke theorem (Clarke [24], Theorem 2.7.5) which was formulated for integral function defined on classical Lebesgue spaces. The results presented in this chapter can be found in [37, 31, 32].

Chapter 3 (Elliptic differential inclusions depending on a real parameter) comprises three sections. In the first section (based on paper [37]) we consider a differential inclusion involving the $p(\cdot)$-Laplace operator with a Steklov type boundary condition and we prove that for each $\lambda > 0$ the problem admits at least three weak solutions, and if an additional assumption is fulfilled, there exists $\lambda^* > 0$ such that the problem possesses at least four weak solutions. The second section (based on paper [27]) is devoted to the study of a differential inclusion involving a p-Laplace-like operator with mixed boundary conditions. More exactly, we divide the boundary $\partial\Omega$ of our domain into two measurable parts Γ_1 and Γ_2 and impose a nonhomogeneous Neumann boundary condition on Γ_1, while on Γ_2 we impose a Dirichlet boundary condition. We prove that for each $\lambda > 0$ the problem has at least one weak solution. In the third section (based on paper [31]) a differential inclusion involving the $\overrightarrow{p}(\cdot)$-Laplace operator with a homogeneous Dirichlet boundary condition is analyzed. We prove that for each $\lambda > 0$ the problem possesses at least two nontrivial weak solutions

Chapter 4 (Differential inclusions in Orlicz-Sobolev spaces) is devoted to the study of an elliptic differential inclusion with homogeneous Dirichlet boundary condition in Orlicz-Sobolev spaces. The approach is variational and by means of the Direct Method in the Calculus of Variations we are able to prove that the energy functional attached to our problem has a global minimizer, hence it possesses a critical point. These results are based on the paper [32].

Chapter 5 (Variational-like inequality problems governed by set-valued operators) contains existence results for for some variational-like inequality problems, in reflexive and nonreflexive Banach spaces.When the set K, in which we seek solutions, is compact and convex, we no dot impose any monotonicity assumptions on the set-valued operator A, which appears in the formulation of the inequality problems. In the case when K is only bounded, closed, and convex, certain monotonicity assumptions are needed: we ask A to be relaxed $\eta - \alpha$ monotone for generalized variational-like inequalities and relaxed $\eta - \alpha$ quasimonotone for variational-like inequalities.We also provide sufficient conditions for the existence of solutions in the case when K is unbounded, closed, and convex. The results presented in this chapter can be found in [28].

Chapter 6 (A system of nonlinear hemivariational inequalities) comprises two sections. The first section is devoted to the study of a general class of systems of nonlinear hemivariational inequalities. Several existence results are established on bounded and unbounded closed, convex subsets of real reflexive Banach spaces. In the second section section we apply the abstract results obtained in the previous section to establish existence results of Nash generalized derivative points. These results are based on the paper [38].

Chapter 7 (Weak solvability for some contact problems) is devoted to the study of two mathematical models which describe the contact between a deformable body and a rigid obstacle called foundation. In the first section (based on the paper [38]) we consider the case of piezoelectric body and a conductive foundation. In the second section (based on the paper [26]) we analyze the case of a body whose behaviour is modelled by a monotone constitutive law and on the potential contact zone we impose nonmonotone boundary conditions. We propose a variational formulation in terms of bipotentials, whose unknown is a pair consisting of the displacement field and the Cauchy stress field.

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Contents

Chapter 1

Preliminaries

Throughout this chapter we provide some notations and fundamental results which will be used in the following chapters.

In this chapter, X denotes a real normed space and X^* is its dual. The value of a functional $\xi \in X^*$ at $u \in X$ is denoted by $\langle \xi, u \rangle_{X^* \times X}$. The norm of X is denoted by $\|\cdot\|_X$, while $\|\cdot\|_*$ stands for the norm of X^*. If there is no danger of confusion we will simply write $\langle \cdot, \cdot \rangle$ to indicate the duality pairing between a normed space and its dual and $\|\cdot\|$ to denote both the norms of X and X^*. If X is a Hilbert space, then $(\cdot, \cdot)_X$ stands for the inner product, unless $X = \mathbb{R}^N$ or $X = \mathcal{S}^N$ (the linear spaces of second order symmetric tensors on $\mathbb{R}^N$, i.e. $\mathcal{S}^N = \mathbb{R}_s^{N \times N}$), in which case the inner products and the corresponding norms are denoted by

$$u \cdot v = \sum_{i=1}^{N} u_i v_i, \quad |v| = \sqrt{v \cdot v},$$

and

$$\sigma : \tau = \sum_{i,j=1}^{N} \sigma_{ij} \tau_{ij}, \quad |\tau| = \sqrt{\tau : \tau}.$$

We use the symbol $\to$ to indicate the *strong convergence* in X and $\rightharpoonup$ for the *weak convergence* in X. The *weak-star convergence* in X^* is denoted by $\rightharpoonup^*$.

Assuming X and Y are two given normed spaces, a function $T : X \to Y$ is called *operator*. An operator taking values in $\mathbb{R} \cup \{+\infty\} = (-\infty, \infty]$ is called *functional*.

1.1 Elements of nonsmooth analysis

Definition 1.1. *Let X be a real vector space and K a subset of X. The set K is said to be convex if*

$$tu + (1-t)v \in K,$$

whenever $u, v \in K$ and $t \in (0,1)$. By convention the empty set $\emptyset$ is convex.

Definition 1.2. *A functional $\phi : K \to \mathbb{R}$ is convex if K is a convex subset of a vector space X and for each $u, v \in K$ and $0 < t < 1$*

$$\phi(tu + (1-t)v) \leq t\phi(u) + (1-t)\phi(v).$$

The functional ϕ is strictly convex if the above inequality is strict for $u \neq v$.

It is sometimes useful to work with functionals having infinite values. The *effective domain* of a functional $\phi : X \to (-\infty, \infty]$ is the set

$$\mathcal{D}(\phi) = \{u \in X : \ \phi(u) \neq \infty\}.$$

We say that ϕ is *proper* if $\mathcal{D}(\phi) \neq \emptyset$. A functional taking infinite values is convex if the restriction to $\mathcal{D}(\phi)$ is convex. If $-\phi$ is convex (resp. strictly convex), then ϕ is said to be *concave* (resp. *strictly concave*).

In the following X denotes a real Banach space.

Definition 1.3. *The functional $\phi : X \to (-\infty, +\infty]$ is said to be lower semicontinuous at $u \in X$ if*

$$\liminf_{n\to\infty} \phi(u_n) \geq \phi(u) \tag{1.1}$$

whenever $\{u_n\} \subset X$ converges to u in X. The function ϕ is lower semicontinuous if it is lower semicontinuous at every point $u \in X$.

When inequality (1.1) holds for each sequence $\{u_n\} \subset X$ that converges weakly to u, the function ϕ is said to be *weakly lower semicontinuous* at u.

A functional ϕ is said to be *upper semicontinuous* (resp. *weakly upper semicontinuous*) if $-\phi$ is lower semicontinuous (resp. weakly lower semicontinuous).

If ϕ is a continuous function then it is also lower semicontiuous. The converse is not true, as a lower semicontinuous function can be discontinuous. Since strong convergence in X implies the weak convergence, it follows that a weakly lower semicontinuous function is lower semicontinuous. Moreover, it can be shown that a proper convex function $\phi : X \to (-\infty, \infty]$ is lower semicontinuous if and only if it is weakly lower semicontinuous.

Let $K \subset X$ and consider the function $I_K : X \to (\infty, +\infty]$ defined by

$$I_K(v) = \begin{cases} 0, & \text{if } v \in K, \\ \infty, & \text{otherwise.} \end{cases}$$

The function I_K is called the *indicator function* of the set K. It can be proved that the set K is a nonempty closed convex set of X if and only if its indicator function I_K is a proper convex lower semicontinuous function.

Definition 1.4. *Let $\phi : X \to \mathbb{R}$ and let $u \in X$. Then ϕ is Gâteaux differentiable at u if there exists an element of X^*, denoted $\phi'(u)$, such that*

$$\lim_{t \downarrow 0} \frac{\phi(u + tv) - \phi(u)}{t} = \langle \phi'(u), v \rangle_{X^* \times X}, \quad \textit{for all } v \in X. \tag{1.2}$$

The element $\phi'(u)$ that satisfies (1.2) is unique and is called the *Gâteaux derivative* of ϕ at u. The functional $\phi : X \to \mathbb{R}$ is said to be *Gâteaux differentiable* if it is Gâteaux differentiable at every point of X.

The convexity of Gâteaux differentiable functions can be characterized as follows.

Proposition 1.1. *Let $\phi : X \to \mathbb{R}$ be a Gâteaux differentiable function. Then, the following statements are equivalent:*

i) ϕ is a convex functional;

ii) $\phi(v) - \phi(u) \geq \langle \phi'(u), v - u \rangle_{X^* \times X}$, for all $v \in X$;

iii) $\langle \phi'(v) - \phi'(u), v - u \rangle_{X^* \times X} \geq 0,$ *for all* $u, v \in X$.

A direct consequence of the above result is that convex and Gâteaux differentiable functions are in fact lower semicontinuous. Proposition 1.1 also suggests the following generalization of the Gâteaux derivative of a convex function.

Definition 1.5. *Let* $\phi : X \to (-\infty, +\infty]$ *be a convex function. The subdifferential of* ϕ *at a point* $x \in \mathcal{D}(\phi)$ *is the (possibly empty) set*

$$\partial\phi(u) = \{\xi \in X^* : \langle \xi, v - u \rangle_{X^* \times X} \leq \phi(v) - \phi(u), \text{ for all } v \in X\}, \tag{1.3}$$

and $\partial\phi(u) = \emptyset$ *if* $u \notin \mathcal{D}(\phi)$.

It is well known that if ϕ is convex and Gâteaux differentiable at a point $u \in \operatorname{int} \mathcal{D}(\phi)$, then $\partial\phi(u)$ contains exactly one element, namely $\phi'(u)$.

The *Fenchel conjugate* of a function $\phi : X \to (-\infty, +\infty]$ is the function $\phi^* : X^* \to (-\infty, +\infty]$ given by

$$\phi^*(\xi) = \sup_{x \in X} \{\langle \xi, u \rangle_{X^* \times X} - \phi(u)\}.$$

Proposition 1.2. *Let* $\phi : X \to (-\infty, +\infty]$ *be a proper, convex and lower semicontinuous function. Then*

(i) ϕ^* *is proper, convex and lower semicontinuous;*

(ii) $\phi(u) + \phi^*(\xi) \geq \langle \xi, u \rangle_{X^* \times X}$, *for all* $u \in X,\ \xi \in X^*$;

(iii) $\xi \in \partial\phi(u) \Leftrightarrow u \in \partial\phi^*(\xi) \Leftrightarrow \phi(u) + \phi^*(\xi) = \langle \xi, u \rangle_{X^* \times X}$.

Definition 1.6. *A bipotential is a function* $B : X \times X^* \to (-\infty, +\infty]$ *satisfying the following conditions*

(i) for any $u \in X$, *if* $\mathcal{D}(B(u, \cdot)) \neq \emptyset$, *then* $B(u, \cdot)$ *is proper and lower semicontinuous; for any* $\xi \in X^*$, *if* $\mathcal{D}(B(\cdot, \xi)) \neq \emptyset$, *then* $B(\cdot, \xi)$ *is proper, convex and lower semicontinuous;*

(ii) $B(u, \xi) \geq \langle \xi, u \rangle_{X^* \times X}$, *for all* $u \in X$, $\xi \in X^*$;

(iii) $\xi \in \partial B(\cdot,\xi)(u) \Leftrightarrow u \in \partial B(u,\cdot)(\xi) \Leftrightarrow B(u,\xi) = \langle \xi, u\rangle_{X^*\times X}$.

We recall that a functional $\phi : X \to \mathbb{R}$ is called *locally Lipschitz* if for every $u \in X$ there exist a neighborhood U of u in X and a constant $L_u > 0$ such that

$$|\phi(v) - \phi(w)| \le L_u \|v - w\|_X, \quad \text{for all } v, w \in U.$$

Definition 1.7. *Let $\phi : X \to \mathbb{R}$ be a locally Lipschitz functional. The Clarke generalized directional derivative of ϕ at a point $u \in X$, in the direction $v \in X$, denoted $\phi^0(u; v)$, is defined by*

$$\phi^0(u;v) = \limsup_{\substack{w\to u\\ t\downarrow 0}} \frac{\phi(w+tv) - \phi(w)}{t}.$$

The following proposition points out some important properties of the generalized derivatives.

Proposition 1.3. *Let $\phi, \psi : X \to \mathbb{R}$ be locally Lipschitz. Then*

i) $v \mapsto \phi^0(u; v)$ *is finite, subadditve and satisfies*

$$|\phi^0(u;v)| \le L_u \|v\|_X,$$

with $L_u > 0$ being the Lipschitz constant near $u \in X$;

ii) $(u, v) \mapsto \phi^0(u; v)$ *is upper semicontinuous;*

iii) $(-\phi)^0(u; v) = \phi^0(u; -v)$ *and* $\phi^0(u; tv) = t\phi^0(u; v)$ *for all $u, v \in X$ and all $t > 0$;*

iv) $(\phi + \psi)^0(u; v) \le \phi^0(u; v) + \psi^0(u; v)$ *for all $u, v \in X$.*

For the proof see Clarke [24], Proposition 2.1.1.

Definition 1.8. *Let $\phi : X \to \mathbb{R}$ be a locally Lipschitz functional. The generalized gradient (Clarke subdifferential) of ϕ at a point $u \in X$, denoted $\partial_C\phi(u)$, is the subset of X^* defined by*

$$\partial_C\phi(u) = \{\zeta \in X^* : \ \phi^0(u; v) \ge \langle \zeta, v\rangle_{X^*\times X}, \quad \textit{for all } v \in X\}.$$

An important property of the generalized gradient is that $\partial_C\phi(u) \neq \emptyset$ for all $u \in X$. This follows directly from the Hahn-Banach Theorem (see e.g. Brezis [13], Theorem 1.1). We also point out the fact that if ϕ is convex, then $\partial_C\phi(u)$ coincides with the subdifferential of ϕ at u, that is

$$\partial_C\phi(u) = \partial\phi(u).$$

We list below some important properties of generalized gradients that will be useful in the subsequent chapters.

Proposition 1.4. *Let $\phi : X \to \mathbb{R}$ be Lipschitz continuous on a neighborhood of a point $u \in X$. Then*

(i) $\partial_C\phi(u)$ is a convex, weak compact subset of X^* and*

$$\|\zeta\|_* \leq L_u, \quad \text{for all } \zeta \in \partial_C\phi(u),$$

where $L_u > 0$ is the Lipschitz constant of ϕ near the point u.

(ii) $\phi^0(u; v) = \max\{\langle\zeta, v\rangle_{X^*\times X} : \zeta \in \partial_C\phi(u)\}$, for all $v \in X$.

(iii) For any scalar s, one has

$$\partial_C(s\phi)(u) = s\partial_C\phi(u);$$

(iv) If u is a local extremum point of ϕ, then $0 \in \partial_C\phi(u)$;

(v) For any positive integer n, one has

$$\partial_C\left(\sum_{i=1}^{n}\phi_i\right)(u) \subset \sum_{i=1}^{n}\partial_C\phi_i(u).$$

For the proof one can consult Clarke [24], Propositions 2.1.2, 2.3.1, 2.3.2 and 2.3.3.

Definition 1.9. *A locally Lipschitz functional $\phi : X \to \mathbb{R}$ is said regular at u if, for all $v \in X$, the usual one-sided directional derivative $\phi'(u; v)$ exists and $\phi'(u; v) = \phi^0(u; v)$.*

For a function $\psi : X_1 \times \ldots \times X_n \to \mathbb{R}$ which is locally Lipschitz with respect to the k^{th} variable we denote by $\psi^0_{,k}(u_1, \ldots, u_n; v_k)$ the *partial generalized derivative* of ψ at $u_k \in X_k$ in the direction $v_k \in X_k$ and by $\partial_C^k \psi(u_1, \ldots, u_n)$ the *partial generalized gradient* of ψ with respect to the variable u_k. It is known that in general the sets $\partial_C \psi(u_1, \ldots, u_n)$ and $\partial_C^1 \psi(u_1, \ldots, u_n) \times \ldots \times \partial_C^n \psi(u_1, \ldots, u_n)$ are not contained one in the other (see e.g. Clarke, Section 2.5), but for regular functionals, the following relations hold.

Proposition 1.5. *Let $\psi : X_1 \times \ldots \times X_n \to \mathbb{R}$ be a regular, locally Lipschitz functional. Then*

(i) $\partial_C \psi(u_1, \ldots, u_n) \subseteq \partial_C^1 \psi(u_1, \ldots, u_n) \times \ldots \times \partial_C^n \psi(u_1, \ldots, u_n)$;

(ii) $\psi^0(u_1, \ldots, u_n; v_1, \ldots, v_n) \leq \sum_{i=1}^{n} \psi_{,k}^0(u_1, \ldots, u_n; v_k)$.

The following result is known in the literature as Lebourg's mean value theorem (see Lebourg [71] or Clarke [24], p. 41).

Theorem 1.1. *Let $\phi : X \to \mathbb{R}$ be locally Lipschitz and $u, v \in X$. Then there exist $t \in (0,1)$ and $\xi_t \in \partial_C \phi\,(u + t(v - u))$ such that*

$$\phi(v) - \phi(u) = \langle \xi_t, v - u \rangle_{X^* \times X}.$$

Definition 1.10. *Let $\phi : X \to \mathbb{R}$ be locally Lipschitz and $u \in X$. We say that u is a critical point of ϕ if $0 \in \partial_C \phi(u)$, that is*

$$\phi^0(u; v) \geq 0, \quad \text{for all } v \in X.$$

If u is a critical point of ϕ, then the number $c = \phi(u)$ is called *critical value* of ϕ. According to Proposition 1.4 every local extremum point is also a critical point of ϕ.

Definition 1.11. *A locally Lipschitz functional $\phi : X \to \mathbb{R}$ is said to satisfy (the nonsmooth) Palais-Smale condition at level c, $(PS)_c$-condition in short, if any sequence $\{u_n\} \subset X$ which satisfies*

- $\phi(u_n) \to c$;
- *there exists $\{\epsilon_n\} \subset \mathbb{R}$, $\epsilon_n \downarrow 0$ such that $\phi^0(u_n; v - u_n) \geq -\epsilon_n \|v - u_n\|_X$ for all $v \in X$;*

possesses a (strongly) convergent subsequence.

We present next results that will be useful in determining critical points of locally Lipschitz functionals in the sequel. The following theorem is fundamental in the Calculus of Variations as it provides sufficient conditions for a functional to posses a global minimum. For the proof see Struwe [118], Theorem 1.2.

Theorem 1.2. *Suppose X is a real reflexive Banach space and let $M \subseteq X$ be a weakly closed subset of X. Suppose $E : X \to \mathbb{R}$ satisfies:*

- *E is coercive on M with respect to X, that is, $E(u) \to +\infty$ as $\|u\|_X \to +\infty,\ u \in M$;*
- *E is weakly lower semicontinuous on M.*

Then E is bounded from below on M and attains its infimum on M.

The following theorem is the nonsmooth version of the zero-altitude Mountain Pass Theorem (see Motreanu & Varga [92]).

Theorem 1.3. *Let $E : X \to \mathbb{R}$ be locally Lipschitz which satisfies the (PS)-condition. Suppose there exist $u_1, u_2 \in X$ and $r \in (0, \|u_1 - u_2\|_X)$ such that*

$$\inf_{u \in \partial B(u_1, r)} E(u) \geq \max\{E(u_1), E(u_2)\}.$$

Then $c = \inf_{\gamma \in \Gamma(u_1,u_2)} \max_{t \in [0,1]} E(\gamma(t))$ is a critical value of E. Moreover, there exists $u_0 \in X \setminus \{u_1, u_2\}$ such that

$$E(u_0) = c \geq \max\{E(u_1), E(u_2)\}.$$

In the previous theorem we have denoted by $\partial B(u, r)$ the sphere centered at u of radius r, that is

$$\partial B(u, r) = \{v \in X : \ \|v - u\|_X = r\},$$

while $\Gamma(u_1, u_2)$ denotes the set of all continuous paths connecting the points u_1, u_2, that is

$$\Gamma(u_1, u_2) = \{\gamma \in C([0, 1], X) : \ \gamma(0) = u_1, \ \gamma(1) = u_2\}.$$

Before presenting the next result, let us recall that for a functional $\phi : X \to \mathbb{R}$, the sets of the type $\phi^{-1}((-\infty, c])$ with $c \in \mathbb{R}$ are called *sub-level sets*. The functional ϕ is said to be *quasi-concave* if the set $\phi^{-1}([c,+\infty))$ is convex for all $c \in \mathbb{R}$. The following theorem is due to Ricceri [106]. Note that no smoothness is required on the functional f.

Theorem 1.4. *Let X be a topological space, $I \subseteq \mathbb{R}$ an open interval and $f : X \times I \to \mathbb{R}$ a functional satisfying the following conditions:*

- *$\lambda \mapsto f(u,\lambda)$ is quasi-concave and continuous for all $u \in X$;*
- *$u \mapsto f(u,\lambda)$ has closed and compact sub-level sets for all $\lambda \in I$;*
- $\sup\limits_{\lambda \in I} \inf\limits_{u \in X} f(u,\lambda) < \inf\limits_{u \in X} \sup\limits_{\lambda \in I} f(u,\lambda)$.

Then there exists $\lambda^* \in I$ such that the functional $u \mapsto f(u,\lambda^*)$ admits at least two global minimizers.

1.2 Elements of set-valued analysis

Set-valued analysis deals with the study of maps whose values are sets. The need for introducing multi-valued maps was recognized in the beginning of the twentieth century, but a systematic study of such maps started in the mid 1960's and since nonsmooth analysis was born these two relatively new branches of mathematics have undergone a remarkable development and have provided each other with new tools and concepts, as maybe the most important multi-valued maps are the subdifferential of a convex functional and Clarke's generalized gradient of a locally Lipschitz functional which are main ingredients in nonsmooth analysis.

Throughout this section E and F denote Hausdorff topological spaces and for $x \in E$ we denote by $\mathcal{N}(x)$ the family of all neighborhoods of x. Let $T : X \to Y$ be a set-valued map and $C \subset E$. We use the following notations:

- $D(T) = \{x \in E : \ T(x) \neq \emptyset\}$ the domain of T;
- $G(T) = \{(x,y) \in E \times F : \ x \in E$ and $y \in T(x)\}$ the graph of T;
- $T(C) = \bigcup\limits_{x \in C} T(x)$ the image of C;

- $T^{+}(C) = \{x \in E : \; T(x) \subseteq C\}$ the strong inverse image of C;
- $T^{-}(C) = \{x \in E : \; T(x) \cap C \neq \emptyset\}$ the weak inverse image of C.

If (E, d) is a metric space, $x \in E$ and $r > 0$, then we denote by

- $B(x, r) = \{y \in E : \; d(x, y) < r\}$ the open ball centered at x of radius r;
- $\bar{B}(x, r) = \{y \in E : \; d(x, y) \leq r\}$ the closed ball centered at x of radius r
- $\partial B(x, r) = \{y \in E : \; d(x, y) = r\}$ stands for the sphere centered at x of radius r.

Definition 1.12. *Let E, F be two Hausdorff topological spaces. A set-valued map $T : E \to F$ is said to be*

(i) lower semicontinuous at a point $x_0 \in E$ (l.s.c. at x_0 for short), if for any open set $V \subseteq F$ such that $T(x_0) \cap V \neq \emptyset$ we can find $U \in \mathcal{N}(x_0)$ such that $T(x) \cap V \neq \emptyset$ for all $x \in U$. If this is true for every $x_0 \in E$, we say that T is lower semicontinuous (l.s.c for short);

(ii) upper semicontinuous at a point $x_0 \in E$ (u.s.c at x_0 for short), if for any open set $V \subseteq F$ such that $T(x_0) \subseteq V$ we can find a neighborhood U of x_0 such that $T(x) \subseteq V$ for all $x \in U$. If this is true for every $x_0 \in E$, we say that T is upper semicontinuous (u.s.c. for short);

(iii) closed, if for every net $\{x_\lambda\}_{\lambda \in I} \subset E$ converging to x and $\{y_\lambda\}_{\lambda \in I} \subset F$ converging to y such that $y_\lambda \in T(x_\lambda)$ for all $\lambda \in I$, we have $y \in T(x)$.

The following propositions are direct consequences of the above definition and provide useful characterisations of l.s.c (u.s.c, closed) set-valued maps. For the proofs, one can consult Papageorgiou & Yiallourou [101] (see Propositions 6.1.3 and 6.1.4) and Deimling [39] (see Proposition 24.1).

Proposition 1.6. *Let $T : E \to F$ be a set-valued map. The following statements are equivalent:*

(i) T is lower semicontinuous;

(ii) For every closed set $C \subseteq F$, $T^{+}(C)$ is closed in E;

(iii) If $x \in X$, $\{x_\lambda\}_{\lambda \in I}$ is a net in E such that $x_\lambda \to x$ and $V \subseteq F$ is an open set such that $T(x) \cap V \neq \emptyset$, then we can find $\lambda_0 \in I$ such that $T(x_\lambda) \cap V \neq \emptyset$ for all $\lambda \in I$ with $\lambda \geq \lambda_0$;

(iv) If $x \in X$, $\{x_\lambda\}_{\lambda \in I} \subset E$ is a net in E and $y \in T(x)$, then for every $\lambda \in I$ we can find $y_\lambda \in T(x_\lambda)$ such that $y_\lambda \to y$;

Proposition 1.7. *Let $T : E \to F$ be a set-valued map. The following statements are equivalent:*

(i) T is upper semicontinuous;

(ii) For every closed set $C \subseteq F$, $T^-(C)$ is closed in E;

(iii) If $x \in X$, $\{x_\lambda\}_{\lambda \in I}$ is a net in E such that $x_\lambda \to x$ and $V \subseteq E$ is an open set such that $T(x) \subseteq V$, then we can find $\lambda_0 \in I$ such that $T(x_\lambda) \subseteq V$ for all $\lambda \in I$ with $\lambda \geq \lambda_0$;

Proposition 1.8. *Let $T : D \subseteq E \to F$ a set-valued map such that $T(x) \neq \emptyset$ for all $x \in D$.*

(i) Let $T(x)$ be closed for all $x \in D \subseteq E$. If T is u.s.c. and D is closed, then $G(T)$ is closed. If $\overline{T(D)}$ is compact and D is closed, then T is u.s.c. if and only if $G(T)$ is closed;

(ii) If $D \subseteq E$ is compact, T is u.s.c. and $T(x)$ is compact for all $x \in D$, then $T(D)$ is compact.

Remark 1.1. *The above propositions show that if T is single-valued, i.e. $T(x) = \{y\} \subset F$, then the notions of lower and upper semicontinuity coincide with the usual notion of continuity of a map between two Hausdorff topological spaces.*

We present next some results for set-valued maps which will be useful in proving the existence of solutions for various inequality problems in the following chapters. We start by recalling that $x \in E$ is a *fixed point* of the set-valued map $T : E \to E$ if $x \in T(x)$. Also recall that set-valued map $T : E \to E$ is said to be a *KKM map* if, for every finite subset $\{x_1, \ldots, x_n\} \subset E$, $\mathrm{co}\{x_1, \ldots, x_n\} \subseteq \bigcup_{j=1}^n T(x_j)$, where $\mathrm{co}\{x_1, \ldots, x_n\}$ denotes the convex hull of $\{x_1, \ldots, x_n\}$. The following result is due to Ansari & Yao [5].

Theorem 1.5. *Let K be a nonempty closed and convex subset of a Hausdorff topological vector space E and let $S, T : K \subset E \to E$ be two set-valued maps. Assume that:*

• *for each* $x \in K$*,* $S(x)$ *is nonempty and* $\mathrm{co}\{S(x)\} \subseteq T(x)$*;*

• $K = \bigcup_{y \in K} \mathrm{int}_K S^{-1}(y)$*;*

• *if* K *is not compact, assume that there exists a nonempty compact convex subset* C_0 *of* K *and a nonempty compact subset* C_1 *of* K *such that for each* $x \in K \setminus C_1$ *there exists* $\bar{y} \in C_0$ *with the property that* $x \in \mathrm{int}_K S^{-1}(\bar{y})$*.*

Then T *has at least one fixed point.*

The following version of the KKM Theorem has been proved by Ky Fan [45].

Theorem 1.6. *Let* K *be a nonempty subset of a Hausdorff topological vector space* E *and let* $T : K \subset K \to E$ *be a set-valued map satisfying the following properties:*

• T *is a KKM map;*

• $T(x)$ *is closed in* E *for every* $x \in K$*;*

• *there exists* $x_0 \in K$ *such that* $T(x_0)$ *is compact in* E*.*

Then $\bigcap_{x \in K} T(x) \neq \emptyset$*.*

Theorem 1.7. *(Lin [73]) Let* K *be a nonempty convex subset of a Hausdorff topological vector space* E*. Let* $\mathcal{P} \subseteq K \times K$ *be a subset such that*

(i) for each $\eta \in K$ *the set* $\Lambda(\eta) = \{\zeta \in K : \ (\eta, \zeta) \in \mathcal{P}\}$ *is closed in* K*;*

(ii) for each $\zeta \in K$ *the set* $\Theta(\zeta) = \{\eta \in K : \ (\eta, \zeta) \notin \mathcal{P}\}$ *is either convex or empty;*

(iii) $(\eta, \eta) \in \mathcal{P}$ *for each* $\eta \in K$*;*

(iv) K *has a nonempty compact convex subset* K_0 *such that the set*

$$B = \{\zeta \in K : \ (\eta, \zeta) \in \mathcal{P} \text{ for all } \eta \in K_0\}$$

is compact.

Then there exists a point $\zeta_0 \in B$ *such that* $K \times \{\zeta_0\} \subset \mathcal{P}$.

Theorem 1.8. *(Mosco [88]) Let* K *be a nonempty compact and convex subset of a topological vector space* E *and let* $\phi : E \to \mathbb{R} \cup \{+\infty\}$ *be a proper convex lower semicontinuous functional such that* $\mathcal{D}(\phi) \cap K \neq \emptyset$. *Let* $\xi, \zeta : E \times E \to \mathbb{R}$ *two functionals such that:*

- $\xi(x, y) \leq \zeta(x, y)$ *for all* $x, y \in E$;
- *for each* $x \in E$ *the map* $y \mapsto \xi(x, y)$ *is lower semicontinuous;*
- *for each* $y \in E$ *the map* $x \mapsto \zeta(x, y)$ *is concave.*

Then for each $\mu \in \mathbb{R}$ *the following alternative holds true: either there exists* $y_0 \in K \cap \mathcal{D}(\phi)$ *such that* $\xi(x, y_0) + \phi(y_0) - \phi(x) \leq \mu$, *for all* $x \in E$, *or, there exists* $x_0 \in E$ *such that* $\zeta(x_0, x_0) > \mu$.

1.3 Function spaces

Throughout this section we recall some basic facts on Lebesgue and Sobolev spaces, with constant and variable exponents, and some useful definitions and properties of N-functions and Orlicz spaces. Let $\Omega \subset \mathbb{R}^N$ be an open set. For $1 \leq p < \infty$ recall that the Lebesgue space is defined by

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \,\middle|\, u \text{ is measurable and } \int_\Omega |u(x)|^p \, dx < \infty \right\},$$

and the corresponding norm is given by

$$\|u\|_p = \left[\int_\Omega |u(x)|^p \, dx \right]^{1/p}.$$

For $p = \infty$, we set

$$L^\infty(\Omega) = \{ u : \Omega \to \mathbb{R} \,|\, u \text{ is measurable and } \operatorname{ess\,sup}_{x\in\Omega} |u(x)| < \infty \},$$

and the corresponding norm is given by

$$\|u\|_\infty = \inf \{ C > 0 \,|\, |u(x)| \leq C \text{ a.e. on } \Omega \}.$$

For $1 \leq p \leq \infty$ we define

$$L^p_{\text{loc}}(\Omega) = \{u : \Omega :\to \mathbb{R} \,|\, u \in L^p(\omega) \text{ for each } \omega \subset\subset \Omega\}.$$

The following results will be useful in the sequel.

Theorem 1.9. *(Fatou's Lemma) Let $\{u_n\}_{n\geq 1}$ be a sequence in $L^1(\Omega)$ such that $u_n \geq 0$ a.e. in Ω. Then*

$$\int_\Omega \liminf_{n\to\infty} u_n(x)\, dx \leq \liminf_{n\to\infty} \int_\Omega u_n(x)\, dx.$$

For any $1 \leq p \leq \infty$ we denote by p' the *conjugate exponent* of p, that is

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 1.10. *(Hölder's inequality) Assume that $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ with $1 \leq p \leq \infty$. Then $uv \in L^1(\Omega)$ and*

$$\int_\Omega uv\, dx \leq \|u\|_p \|v\|_{p'}.$$

Theorem 1.11. *(Fischer-Riesz) $(L^p(\Omega), \|\cdot\|_p)$ is a Banach space for any $1 \leq p \leq \infty$. Moreover, $L^p(\Omega)$ is reflexive for any $1 < p < \infty$ and separable for any $1 \leq p < \infty$.*

For a function $u \in L^1_{\text{loc}}(\Omega)$ the function $v_\alpha \in L^1_{\text{loc}}(\Omega)$ for which

$$\int_\Omega u(x) D^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega v_\alpha(x)\varphi(x)\, dx, \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

is called the *weak derivative of order* α of u and will be denoted by $D^\alpha u$. Here, $\alpha = (\alpha_1, \ldots, \alpha_N)$, with α_i nonnegative integers, $|\alpha| = \alpha_1 + \ldots + \alpha_N$ and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}.$$

It is obvious that if such a v_α exists, it is unique up to sets of zero measure.

For a nonnegative integer m and $1 \leq p \leq \infty$, we define $\|\cdot\|_{m,p}$ as follows

$$\|u\|_{m,p} = \left[\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha u|^p\, dx\right]^{1/p}, \quad \text{if } 1 \leq p < \infty,$$

and

$$\|u\|_{m,\infty} = \max_{|\alpha|\leq m} \sup_{\Omega} |D^\alpha u|.$$

We define the Sobolev spaces

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : \ D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leq m\}.$$

We point out the fact that $(W^{m,p}(\Omega), \|\cdot\|_{m,p})$ is a real Banach space. The closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{m,p}$ is denoted by $W_0^{m,p}(\Omega)$. In general, $W_0^{m,p}(\Omega)$ is strictly included in $W^{m,p}(\Omega)$. In the case $p = 2$ we use the notation

$$H^m(\Omega) = W^{m,2}(\Omega) \text{ and } H_0^m(\Omega) = W_0^{m,2}(\Omega).$$

These are Hilbert spaces with respect to the following scalar product

$$(u,v)_m = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u(x) D^\alpha v(x)\, dx,$$

where, as usual, $D^0 u = u$. If Ω is an open bounded subset of $\mathbb{R}^N$, with sufficiently smooth boundary $\partial\Omega$, then

$$H_0^1(\Omega) = \left\{ u \in H^1(\Omega) : \ \text{ the trace of } u \text{ on } \partial\Omega \text{ vanishes } \right\}.$$

The following theorem, known in the literature as the *Sobolev embedding theorem*, is of particular interest in the variational and qualitative analysis of differential inclusions and partial differential equations. We recall that, if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two Banach spaces, then X is *continuously embedded* into Y if there exists an injective linear map $i : X \to Y$ and a constant $C > 0$ such that $\|iu\|_Y \leq C\|u\|_X$ for all $u \in X$. We say that X is *compactly embedded* into Y if i is a compact map, that is, i maps bounded subsets of X into relatively compact subsets of Y.

Theorem 1.12. *Assume $\Omega \subset \mathbb{R}^N$ is a bounded open set with Lipschitz boundary. Then*

(i) If $mp < N$, then $W^{m,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$ for each $1 \leq q \leq \frac{Np}{N-mp}$. The embedding is compact for $q < \frac{Np}{N-mp}$;

(ii) If $0 \leq k < m - \frac{N}{p} < k+1$*, then* $W^{m,p}(\Omega)$ *is continuously embedded into* $C^{k,\beta}(\Omega)$*, for* $0 \leq \beta \leq m - k - \frac{k}{p}$*. The embedding is compact for* $\beta < m - k - \frac{k}{p}$*.*

If $\Omega \subset \mathbb{R}^N$ is a bounded open set with Lipschitz boundary then the Poincaré inequality holds

$$\|u\|_p \leq C\|\nabla u\|_p, \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

where $C = C(\Omega)$ is a constant not depending on u. Hence

$$\|u\| = \|\nabla u\|_p,$$

defines a norm on $W_0^{1,p}(\Omega)$ which equivalent to the norm $\|\cdot\|_{1,p}$.

Let us recall next some definitions and basic properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W_0^{1,p(\cdot)}(\Omega)$ and $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Assume Ω is a bounded open subset of $\mathbb{R}^N$, with sufficiently smooth boundary. We consider the set

$$C_+(\bar{\Omega}) = \left\{ p \in C(\bar{\Omega}) : \min_{x \in \bar{\Omega}} p(x) > 1 \right\}$$

and for each $p \in C_+(\bar{\Omega})$ we denote

$$p^- = \inf_{x \in \Omega} p(x) \quad \text{and} \quad p^+ = \sup_{x \in \Omega} p(x).$$

Moreover, let

$$p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)} & \text{if } p(x) < n, \\ +\infty & \text{otherwise.} \end{cases}$$

For a function $p \in C_+(\bar{\Omega})$ the *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} : \ u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \right\},$$

and can be endowed with the norm (called *Luxemburg norm*) defined by

$$\|u\|_{p(\cdot)} = \inf \left\{ \zeta > 0 : \int_\Omega \left| \frac{u(x)}{\zeta} \right|^{p(x)} dx \leq 1 \right\}.$$

It can be proved that $\left(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}\right)$ is a reflexive and separable Banach space (see, e.g., Kováčik and Rákosník [69]). If we denote by $p'(x) = \frac{p(x)}{p(x)-1}$ the pointwise conjugate exponent of $p(x)$, then for all $u \in L^{p(\cdot)}(\Omega)$ and all $v \in L^{p'(\cdot)}(\Omega)$ the following Hölder-type inequality holds

$$\left|\int_\Omega u(x)v(x)\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

We also remember the definition of the $p(\cdot)$-*modular* of the space $L^{p(\cdot)}(\Omega)$, which is the application $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx.$$

This application is extremely useful in manipulating the variable exponent Lebesgue-Sobolev spaces as it satisfies the following relations

$$\|u\|_{p(\cdot)} > 1 (< 1;\ = 1) \text{ if and only if } \rho_{p(\cdot)}(u) > 1 (< 1;\ = 1), \tag{1.4}$$

$$\|u\|_{p(\cdot)} > 1 \text{ implies } \|u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+}, \tag{1.5}$$

$$\|u\|_{p(\cdot)} < 1 \text{ implies } \|u\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-}. \tag{1.6}$$

Clearly, if $p(x) = p_0$ for all $x \in \bar{\Omega}$, then the Luxemburg norm reduces to norm of the classical Lebesgue space $L^{p_0}(\Omega)$, that is

$$\|u\|_{p_0} = \left[\int_\Omega |u(x)|^{p_0}\,dx\right]^{1/p_0}.$$

For a $p \in C_+(\bar{\Omega})$ the *(isotropic) variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ can be defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)} :\ \partial_i u \in L^{p(\cdot)}(\Omega) \text{ for all } i \in \{1, \ldots, n\}\right\},$$

and endowed with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)},$$

becomes a separable and reflexive Banach space. Moreover, if p is log-Hölder continuous, that is, there exists $M > 0$ such that $|p(x) - p(y)| \leq \frac{-M}{\log(|x-y|)}$, for all $x, y \in \Omega$ satisfying $|x - y| < 1/2$, then the space $C^\infty(\bar{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ and we can define the Sobolev space with zero boundary values $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\| \cdot \|_{1,p(\cdot)}$. Note that if $q \in C_+(\bar{\Omega})$ is a function such that $q(x) < p^*(x)$ for all $x \in \bar{\Omega}$, then $W_0^{1,p(\cdot)}(\Omega)$ is compactly embedded into $L^{q(\cdot)}(\Omega)$.

We recall now the definition of the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, where $\overrightarrow{p} : \bar{\Omega} \to \mathbb{R}^n$ is of the form

$$\overrightarrow{p}(x) = (p_1(x), \ldots, p_n(x)), \quad \text{for all } x \in \bar{\Omega},$$

and for each $i \in \{1, \ldots, n\}$, $p_i : \bar{\Omega} \to \mathbb{R}$ is a log-Hölder continuous function. The space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{p}(\cdot)} = \sum_{i=1}^{n} \|\partial_i u\|_{p_i(\cdot)},$$

and this space is a reflexive Banach space with respect to the above norm (see, e.g., Mihăilescu, Pucci and Rădulescu [85]).

For an easy manipulation of the space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ we introduce $p_M, p_m : \bar{\Omega} \to \mathbb{R}$ and $P^* \in \mathbb{R}$ as follows

$$p_M(x) = \max_{1 \leq i \leq n} p_i(x), \quad p_m(x) = \min_{1 \leq i \leq n} p_i(x), \quad P^* = n \left(\sum_{i=1}^{n} \frac{1}{p_i^-} - 1 \right)^{-1}.$$

The following result, due to Mihăilescu, Pucci and Rădulescu [85], provides useful information concerning the embedding of $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$.

Theorem 1.13. *Assume $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is an open bounded set having smooth boundary and, for each $i \in \{1, \ldots, n\}$, $p_i : \bar{\Omega} \to \mathbb{R}$ is a log-Hölder continuous function such that the following relation holds true*

$$\sum_{i=1}^{n} \frac{1}{p_i^-} > 1.$$

Then, for any $q \in C_+(\bar{\Omega})$ satisfying $1 < q(x) < \max\{p_m^+, P^*\}$ for all $x \in \bar{\Omega}$, $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is compactly embedded into $L^{q(\cdot)}(\Omega)$.

We recall below some basic notions and properties of N-functions and Orlicz spaces. For more details one can consult [2, 25, 52, 63].

Definition 1.13. *A continuous function $\Phi : \mathbb{R} \to [0, \infty)$ is called N-function if it satisfies the following properties*

(N1) Φ is a convex and even function;

(N2) $\Phi(t) = 0$ if and only if $t = 0$;

(N3) $\lim\limits_{t\to 0} \frac{\Phi(t)}{t} = 0$ and $\lim\limits_{t\to\infty} \frac{\Phi(t)}{t} = \infty$.

It is well known that a convex function $\Phi : \mathbb{R} \to [0, \infty)$ which satisfies $\Phi(0) = 0$ can be represented as

$$\Phi(t) = \int_0^t \varphi(s)ds,$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is right-continuous and non-decreasing (see e.g. Krasnosel'skiĭ & Rutickiĭ [63], Theorem 1.1). If, in addition, the function φ satisfies

(φ_1) $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$;

(φ_2) $\lim\limits_{t\to\infty} \varphi(t) = \infty$,

then the corresponding function Φ is an N-function. For a given function $\varphi : \mathbb{R} \to \mathbb{R}$ which is right-continuous, non-decreasing and satisfies $(\varphi_1) - (\varphi_2)$ we define

$$\tilde{\varphi}(s) = \sup_{\varphi(t)\le s} t.$$

One can easily see that φ can be recovered from $\tilde{\varphi}$ via

$$\varphi(t) = \sup_{\tilde{\varphi}(s)\le t} s.$$

Moreover, if φ is strictly increasing, then $\tilde{\varphi} = \varphi^{-1}$. The function

$$\Phi^*(s) = \int_0^s \tilde{\varphi}(\tau)d\tau,$$

is also an N-function and Φ, Φ^* are called *complementary functions*. They satisfy Young's inequality

$$st \leq \Phi(t) + \Phi^*(s), \quad \text{for all } s, t \in \mathbb{R}, \tag{1.7}$$

which holds with equality if $s = \varphi(t)$ or $t = \tilde{\varphi}(s)$. An important role in the embeddings of Orlicz-Sobolev spaces is played by the *Sobolev conjugate function of* Φ, denoted Φ_*, which can be defined by

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds.$$

Definition 1.14. *Let Φ and Ψ be N-functions. We say that*

- *Ψ dominates Φ at infinity (we write $\Phi \prec \Psi$) if there exist $t_0 > 0$ and $k > 0$ such that*

$$\Phi(t) \leq \Psi(kt), \quad \textit{for all } t \geq t_0;$$

- *Φ and Ψ are equivalent (we write $\Phi \sim \Psi$) if $\Phi \prec \Psi$ and $\Psi \prec \Phi$;*

- *Φ increases essentially slower than Ψ (we write $\Phi \prec\prec \Psi$) if*

$$\lim_{t \to \infty} \frac{\Phi(kt)}{\Psi(t)} = 0, \quad \textit{for all } k > 0.$$

The *Orlicz class* $K^\Phi(\Omega)$ is defined as the set of functions

$$K^\Phi(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega \Phi(|u(x)|) dx < \infty \right\}$$

It is a known fact that Orlicz classes are convex sets but not necessarily linear spaces. We are now in position to define the *Orlicz spaces* $L^\Phi(\Omega)$ *and* $E^\Phi(\Omega)$ as follows

$$L^\Phi(\Omega) = \text{ the linear space generated by } K^\Phi(\Omega),$$

$$E^\Phi(\Omega) = \text{ the maximal linear subspace of } K^\Phi(\Omega).$$

Obviously we have

$$E^\Phi(\Omega) \subseteq K^\Phi(\Omega) \subseteq L^\Phi(\Omega),$$

with equality if and only if $K^\Phi(\Omega)$ is a linear space. The latter reduces to the fact that Φ satisfies the Δ_2-*condition at infinity*, i.e. there exist $t_0 > 0$ and $k > 0$ such that

$$\Phi(2t) \leq k\Phi(t), \quad \text{for all } t \geq t_0.$$

On the Orlicz space $L^\Phi(\Omega)$ we can define the so-called *Luxemburg norm* by

$$|u|_\Phi = \inf\left\{\mu > 0 : \int_\Omega \Phi\left(\frac{|u|}{\mu}\right) dx \leq 1\right\}.$$

It is a fact that $\left(L^\Phi(\Omega), |\cdot|_\Phi\right)$ is a Banach space (see e.g. Adams [2]). Moreover, $E^\Phi(\Omega)$ coincides with the closure of bounded functions in $L^\Phi(\Omega)$ and it is complete and separable. An important role in manipulating Orlicz spaces is played by the following Hölder-type inequality

$$\left|\int_\Omega uv\ dx\right| \leq 2|u|_\Phi |v|_{\Phi^*}, \quad \text{for all } u \in L^\Phi(\Omega),\ v \in L^{\Phi^*}(\Omega).$$

Hence, for each $v \in L^{\Phi^*}(\Omega)$ one can define $R_v : L^\Phi(\Omega) \to \mathbb{R}$ by

$$R_v(u) = \int_\Omega uv\ dx,$$

which is linear and bounded, so $R_v \in \left(L^\Phi(\Omega)\right)^*$. Thus, we can define the norm

$$\|v\|_{\Phi^*} := \|R_v\|_{(L^\Phi(\Omega))^*} = \sup_{|u|_\Phi \leq 1} \left|\int_\Omega uv\ dx\right|,$$

which is called the Orlicz norm on $L^{\Phi^*}(\Omega)$. Analogously, we can define the Orlicz norm on $L^\Phi(\Omega)$. Clearly, the Luxemburg and Orlicz norms are equivalent as

$$|u|_\Phi \leq \|u\|_\Phi \leq 2|u|_\Phi.$$

Proposition 1.9. *Let Φ and Φ^* be complementary N-functions. Then,*

$$L^\Phi(\Omega) = \left(E^{\Phi^*}(\Omega)\right)^* \text{ and } L^{\Phi^*}(\Omega) = \left(E^\Phi(\Omega)\right)^*.$$

Moreover, $L^\Phi(\Omega)$ is reflexive if and only if Φ and Φ^* satisfy the Δ_2-condition.

The *Orlicz-Sobolev space* $W^1L^\Phi(\Omega)$ can be defined by setting

$$W^1L^\Phi(\Omega) = \left\{u \in L^\Phi(\Omega) : \partial_i u \in L^\Phi(\Omega),\ 1 \leq i \leq N\right\},$$

which a Banach space with respect to the norm

$$|u|_{1,\Phi} = |u|_\Phi + |\,|\nabla u|\,|_\Phi\,.$$

The space $W^1E^\Phi(\Omega)$ is defined analogously and it is separable. The Orlicz-Sobolev space of functions vanishing on the boundary $W_0^1E^\Phi$ is the closure of $C_0^\infty(\Omega)$ in $W^1L^\Phi(\Omega)$ with respect to the norm $|\cdot|_{1,\Phi}$. Define $W_0^1L^\Phi(\Omega)$ as the weak* closure of $C_0^\infty(\Omega)$ in $W^1L^\Phi(\Omega)$; hence by Proposition 1.9, $W_0^1L^\Phi(\Omega)$ is the weak* closure of the dual of a separable space. The following Poincaré-type inequality holds

$$\int_\Omega \Phi(|u|)\,dx \leq \int_\Omega \Phi(d|\nabla u|)\,dx, \text{ for all } u \in W_0^1L^\Phi(\Omega),$$

where $d = 2\mathrm{diam}(\Omega)$, hence

$$\|u\| = |\,|\nabla u|\,|_\Phi$$

defines a norm equivalent to $|\cdot|_{1,\Phi}$ on $W_0^1L^\Phi(\Omega)$.

The following result points out the relation between $W^1L^\Phi(\Omega)$ and $L^\Psi(\Omega)$ when Φ and Ψ are N-functions.

Theorem 1.14. *Let Φ and Ψ be N-functions and let Φ_* be the Sobolev conjugate function of Φ.*

(a) If $\Psi \prec\prec \Phi_*$ and

$$\int_1^\infty \frac{\Phi^{-1}(t)}{t^{\frac{N+1}{N}}}\,dt = \infty,$$

then $W^1L^\Phi(\Omega)$ is compactly embedded into $L^\Psi(\Omega)$ and $W^1L^\Phi(\Omega)$ is continuously embedded into $L^{\Phi_*}(\Omega)$.

(b) If

$$\int_1^\infty \frac{\Phi^{-1}(t)}{t^{\frac{N+1}{N}}}\,dt < \infty,$$

then $W^1L^\Phi(\Omega)$ is compactly embedded into $L^\Psi(\Omega)$ and $W^1L^\Phi(\Omega)$ is continuously embedded into$L^\infty(\Omega)$.

A particular case of interest is when $\Psi = \Phi$ as it is known that $\Phi \prec\prec \Phi_*$ whenever the latter is defined as an N-function (see e.g. García-Huidobro, Le, Manásevich & Schmitt [52], Proposition 2.1).

Chapter 2

Some abstract results

In this chapter we prove three theorems that will play a key role in the proof of the main results of the subsequent chapters. The first result represents a multiplicity theorem for the critical points of a locally Lipschitz functional depending on a real parameter and extends a recent result of Ricceri, while the second and third theorem provide information regarding the subdifferentiability of integral functionals defined on variable exponent Lebesgue spaces and Orlicz spaces, respectively. These results extend the well-known Aubin-Clarke theorem which was formulated for L^p spaces.

2.1 A four critical points theorem for parametrized locally Lipschitz functionals

Let X be a real reflexive Banach space and Y, Z two Banach spaces such that there exist $T : X \to Y$ and $S : X \to Z$ linear and compact. Let $L : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous C^1 functional such that $L' : X \to X^*$ has the $(\mathcal{S})_+$ property, i.e. if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} \langle L'(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$. Assume in addition that $J_1 : Y \to \mathbb{R}$, $J_2 : Z \to \mathbb{R}$ are two locally Lipschitz functionals.

We are interested in studying the existence of critical points for functionals $\mathcal{E}_\lambda : X \to \mathbb{R}$ of the following type

$$\mathcal{E}_\lambda(u) := L(u) - (J_1 \circ T)(u) - \lambda(J_2 \circ S)(u), \tag{2.1}$$

where $\lambda > 0$ is a real parameter.

We point out the fact that it makes sense to talk about critical points for the functional defined in (2.1) as $\mathcal{E}_\lambda$ is locally Lipschitz. In order to see this, let us fix $u \in X$, $\lambda > 0$ and $r > 0$ and choose $v, w \in \bar{B}(u; r)$. Since $L \in C^1(X; \mathbb{R})$ we have

$$|L(w) - L(v)| = |\langle L'(z), w - v\rangle| \leq \|L'(z)\|_{X^*}\|w - v\|_X,$$

where $z = tw + (1-t)v$ for some $t \in (0,1)$. But, $\bar{B}(u; r)$ is weakly compact thus there exists $M > 0$ such that $\|L'(z)\|_{X^*} \leq M$ on $\bar{B}(u; r)$. Using the fact that J_1, J_2 are locally Lipschitz functionals we get

$$\begin{aligned}
|\mathcal{E}_\lambda(w) - \mathcal{E}_\lambda(v)| &\leq |L(w) - L(v)| + |(J_1 \circ T)(w) - (J_1 \circ T)(v)| + \lambda|(J_2 \circ S)(w) - (J_2 \circ S)(v)| \\
&\leq M\|w - v\|_X + m_1\|Tw - Tv\|_Y + \lambda m_2\|Sw - Sv\|_Z \\
&\leq \left[M + m_1\|T\|_{\mathcal{L}(X,Y)} + \lambda m_2\|S\|_{\mathcal{L}(X,Z)}\right]\|w - v\|_X,
\end{aligned}$$

which shows that $\mathcal{E}_\lambda$ is locally Lipschitz.

We also point out the fact that the functional $\mathcal{E}_\lambda$ is sequentially weakly lower semicontinuous since we assumed L to be sequentially weakly lower semicontinuous and T, S to be compact operators.

In order to prove our main result we shall assume the following conditions are fulfilled:

$(\mathcal{H}_1)$ there exists $u_0 \in X$ such that u_0 is a strict local minimum for L and

$$L(u_0) = (J_1 \circ T)(u_0) = (J_2 \circ S)(u_0) = 0;$$

$(\mathcal{H}_2)$ for each $\lambda > 0$ the functional $\mathcal{E}_\lambda$ is coercive and there exists $u_\lambda^0 \in X$ such that $\mathcal{E}_\lambda(u_\lambda^0) < 0$;

$(\mathcal{H}_3)$ there exists $R_0 > 0$ such that

$$(J_1 \circ T)(u) \leq L(u) \quad \text{and} \quad (J_2 \circ S)(u) \leq 0, \quad \text{for all } u \in \bar{B}(u_0; R_0) \setminus \{u_0\};$$

$(\mathcal{H}_4)$ there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda>0} \inf_{u\in X} \{\lambda [L(u) - (J_1 \circ T)(u) + \rho] - (J_2 \circ S)(u)\} < \inf_{u\in X} \sup_{\lambda>0} \{\lambda [L(u) - (J_1 \circ T)(u) + \rho] - (J_2 \circ S)(u)\}.$$

The following theorem extends the result obtained recently by B. Ricceri (see [107], Theorem 1) to the case of non-differentiable locally Lipschitz functionals.

Theorem 2.1. (N.C. & C. VARGA [37]) *Assume that conditions $(\mathcal{H}_1) - (\mathcal{H}_3)$ are fulfilled. Then for each $\lambda > 0$ the functional $\mathcal{E}_\lambda$ defined in (2.1) has at least three critical points. If in addition $(\mathcal{H}_4)$ holds, then there exists $\lambda^* > 0$ such that $\mathcal{E}_{\lambda^*}$ has at least four critical points.*

Proof. The proof of Theorem 2.1 will be carried out in four steps an relies essentially on the zero altitude mountain pass theorem for locally Lipschitz functionals (see Theorem 1.3) combined with Theorem 1.4. Let us first fix $\lambda > 0$ and assume that $(\mathcal{H}_1) - (\mathcal{H}_3)$ are fulfilled.

STEP 1. u_0 is a critical point for $\mathcal{E}_\lambda$.

Since $u_0 \in X$ is a strict local minimum for L there exists $R_1 > 0$ such that

$$L(u) > 0, \quad \text{for all } u \in \bar{B}(u_0; R_1) \setminus \{u_0\}. \tag{2.2}$$

From $(\mathcal{H}_3)$ we deduce that

$$\frac{(J_1 \circ T)(u) + \lambda (J_2 \circ S)(u)}{L(u)} \leq 1, \quad \text{for all } u \in \bar{B}(u_0; R_0) \setminus \{u_0\}. \tag{2.3}$$

Taking $R_2 = \min\{R_0, R_1\}$ from (2.2) and (2.3) we have

$$\mathcal{E}_\lambda(u) = L(u) - (J_1 \circ T)(u) - \lambda (J_2 \circ S)(u) \geq 0 = \mathcal{E}_\lambda(u_0), \quad \text{for all } u \in \bar{B}(u_0; R_2) \setminus \{u_0\}. \tag{2.4}$$

We have proved thus that $u_0 \in X$ is a local minimum for $\mathcal{E}_\lambda$, therefore it is a critical point for this functional.

STEP 2. The functional $\mathcal{E}_\lambda$ admits a global minimum point $u_1 \in X \setminus \{u_0\}$.

Indeed, such a point exists since the functional $\mathcal{E}_\lambda$ is sequentially weakly lower semicontinuous and coercive, therefore it admits a global minimizer denoted u_1. Moreover, from $(\mathcal{H}_2)$ we deduce that $\mathcal{E}_\lambda(u_1) < 0$, hence $u_1 \neq u_0$.

STEP 3. There exists $u_2 \in X \setminus \{u_0, u_1\}$ such that u_2 is a critical point for $\mathcal{E}_\lambda$.

Using the coercivity of $\mathcal{E}_\lambda$ and the fact that L' has the $(\mathcal{S})_+$ property we are able to show that our functional satisfies the (PS)-condition.

According to STEP 2 there exists $u_1 \in X$ such that $\mathcal{E}_\lambda(u_1) < 0$. On the other hand, $\mathcal{E}_\lambda(u_0) = 0$ and we can choose $0 < r < \min\{R_2, \|u_1 - u_0\|_X\}$ such that

$$\mathcal{E}_\lambda(u) \geq \max\{\mathcal{E}_\lambda(u_0), \mathcal{E}_\lambda(u_1)\} = 0, \quad \text{for all } u \in \partial\bar{B}(u_0; r).$$

Applying Theorem 1.3 we conclude that there exists a critical point $u_2 \in X \setminus \{u_0, u_1\}$ for $\mathcal{E}_\lambda$ and $\mathcal{E}_\lambda(u_1) \geq 0$. This completes the proof of the first part of the theorem, i.e. the functional $\mathcal{E}_\lambda$ has at least three distinct critical points.

STEP 4. If in addition $(\mathcal{H}_4)$ holds, then there exists $\lambda^* > 0$ such that $\mathcal{E}_{\lambda^*}$ has two global minima.

Let us consider the functional $f : X \times (0, \infty) \to \mathbb{R}$ defined by

$$f(u, \mu) = \mu\left[L(u) - (J_1 \circ T)(u) + \rho\right] - (J_2 \circ S)(u) = \mu\mathcal{E}_{1/\mu}(u) + \mu\rho,$$

where $\rho \in \mathbb{R}$ is the number from $(\mathcal{H}_4)$.

We observe that for each $u \in X$ the functional $\mu \mapsto f(u, \mu)$ is affine, therefore it is quasi-concave. We also note that for each $\mu > 0$ the mapping $u \mapsto f(u, \mu)$ is sequentially weakly lower semicontinuous. Therefore for each $\mu > 0$, the sub-level sets of $u \mapsto f(u, \mu)$ are sequentially weakly closed.

Let us consider now the set $S^\mu(c) = \{u \in X : \; f(u, \mu) \leq c\}$ for some $c \in \mathbb{R}$ and a sequence $\{u_n\} \subset S^\mu(c)$. Obviously $\{u_n\}$ is bounded due to the fact that the functional $u \mapsto f(u, \mu)$ is coercive, which is clear since $f(u, \mu) = \mu\mathcal{E}_{1/\mu}(u) + \mu\rho$, $\mathcal{E}_{1/\mu}$ is coercive and $\mu > 0$. According to the Eberlein-Smulyan Theorem $\{u_n\}$ admits a subsequence, still denoted $\{u_n\}$, which converges weakly to some $u \in X$. Keeping in mind that $u_n \in S^\mu(c)$ for $n > 0$ we deduce that

$$\mathcal{E}_{1/\mu}(u_n) \leq \frac{c - \mu\rho}{\mu}, \quad \text{for all } n > 0.$$

Combining the above relation with the fact that $\mathcal{E}_{1/\mu}$ is sequentially weakly lower semicontinuous we get

$$\mathcal{E}_{1/\mu}(u) \leq \liminf_{n\to\infty} \mathcal{E}_{1/\mu}(u_n) \leq \frac{c-\mu\rho}{\mu},$$

which shows that $f(u,\mu) \leq c$, therefore the set $S^{\mu}(c)$ is a sequentially weakly compact subset of X. We have proved thus that, for each $\mu > 0$, the sub-level sets of $u \mapsto f(u,\mu)$ are sequentially weakly compact. Taking into account Remark 1 in [106] which states that we can replace "closed and compact" by "sequentially closed and sequentially compact" in Theorem 1.4 and using condition $(\mathcal{H}_4)$ we can apply Theorem 1.4 for the weak topology of X and conclude that there exists $\mu^* > 0$ for which $f(u,\mu^*) = \mu^*\mathcal{E}_{1/\mu^*}(u) + \mu^*\rho$ has two global minima. It is easy to check that any global minimum point of $f(u,\mu^*)$ is also a global minimum point for $\mathcal{E}_{1/\mu^*}$, and thus we get the existence of a point $u_3 \in X \setminus \{u_1\}$ such that

$$\mathcal{E}_{1/\mu^*}(u_1) = \mathcal{E}_{1/\mu^*}(u_3) \leq \mathcal{E}_{1/\mu^*}(u^0_{1/\mu^*}) < 0 = \mathcal{E}_{1/\mu^*}(u_0) \leq \mathcal{E}_{1/\mu^*}(u_2),$$

which shows that $u_3 \in X \setminus \{u_0, u_1, u_2\}$. Taking $\lambda^* = 1/\mu^*$ completes the proof.

□

2.2 Extensions of the Aubin-Clarke Theorem

In this section we prove two extensions of the Aubin-Clarke Theorem (see Clarke [24], Theorem 2.7.5) concerning the subdifferentiability of integral functionals defined on variable exponent Lebesgue spaces or Orlicz spaces.

Let $p \in C_+(\bar{\Omega})$ and $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that $x \mapsto \varphi(x,t)$ is measurable for all $t \in \mathbb{R}$ and, in addition, suppose φ satisfies one of the following conditions

(a) there exist $m \in L^{p'(\cdot)}(\Omega)$ such that

$$|\varphi(x,t_1) - \varphi(x,t_2)| \leq m(x)|t_1 - t_2|, \quad \text{for a.e. } x \in \Omega \text{ and all } t_1, t_2 \in \mathbb{R},$$

or,

(b) the application $t \mapsto \varphi(x,t)$ is locally Lipschitz for a.e. $x \in \Omega$ and there exists $c_\varphi > 0$ such that

$$|\xi| \leq c_\varphi |t|^{p(x)-1},$$

for a.e. $x \in \Omega$, all $t \in \mathbb{R}$ and all $\xi \in \partial_C \varphi(x,t)$.

We introduce next the functional $\phi : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\phi(w) = \int_\Omega \varphi(x, w(x))\, dx, \quad \text{for all } w \in L^{p(\cdot)}(\Omega). \tag{2.5}$$

Theorem 2.2. (N.C. & G. MOROŞANU [31]) *Assume $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $x \mapsto \varphi(x,t)$ is measurable for all $t \in \mathbb{R}$ and either (a) or (b) holds. Then, the functional $\phi : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by (2.5) is locally Lipschitz and satisfies*

$$\phi^0(w; z) \leq \int_\Omega \varphi^0(x, w(x); z(x))\, dx, \quad \text{for all } w, z \in L^{p(\cdot)}(\Omega). \tag{2.6}$$

Moreover, if $\varphi(x,\cdot)$ is regular at $w(x)$ for a.e. $x \in \Omega$, then ϕ is regular at w and equality takes place in (2.6).

Proof. First we prove that ϕ is locally Lipschitz. If (a) holds, this follows directly from the Hölder-type inequality. If (b) holds, we need to use Lebourg's mean value theorem and the properties of the modular.

Let us check now that

$$\phi^0(w; z) \leq \int_\Omega \varphi^0(x, w(x); z(x))\, dx, \quad \text{for all } w, z \in L^{p(\cdot)}(\Omega).$$

We denote by $h_{\mu,\delta}(w(x), z(x))$ the difference quotient

$$h_{\mu,\delta}(w(x), z(x)) = \frac{\varphi(x, w(x) + \delta + \mu z(x)) - \varphi(x, w(x) + \delta)}{\mu}.$$

Simple computations show that we can apply Fatou's lemma to get the following estimate

$$\limsup_{\substack{\delta \to 0 \\ \mu \downarrow 0}} \int_\Omega h_{\mu,\delta}(w(x), z(x))\, dx \leq \int_\Omega \limsup_{\substack{\delta \to 0 \\ \mu \downarrow 0}} h_{\mu,\delta}(w(x), z(x))\, dx,$$

which shows that

$$\phi^0(w;z) \leq \int_\Omega \varphi^0(x,w(x);z(x))\,dx, \qquad \text{for all } w,z \in L^{p(\cdot)}(\Omega).$$

Finally, let us prove that ϕ is regular at w if $\varphi(x,\cdot)$ is regular at $w(x)$ for a.e. $x \in \Omega$. Using Fatou's lemma we have

$$\begin{aligned}
\phi^0(w,z) &\geq \liminf_{\mu\downarrow 0} \frac{\phi(w+\mu z)-\phi(w)}{\mu} \\
&\geq \int_\Omega \liminf_{\mu\downarrow 0} \frac{\varphi(z,w(x)+\mu z(x))-\varphi(x,w(x))}{\mu}\,dx \\
&\geq \int_\Omega \lim_{\mu\downarrow 0} \frac{\varphi(z,w(x)+\mu z(x))-\varphi(x,w(x))}{\mu}\,dx \\
&= \int_\Omega \varphi'(x,w(x);z(x))\,dx \\
&= \int_\Omega \varphi^0(x,w(x);z(x))\,dx \\
&\geq \phi^0(w;z).
\end{aligned}$$

Thus, everywhere above we have equality, $\phi'(w;z)$ exists for all $z \in L^{p(\cdot)}(\Omega)$ and

$$\phi'(w;z) = \int_\Omega \varphi'(x,w(z);z(x))\,dx = \int_\Omega \varphi^0(x,w(z);z(x))\,dx = \phi^0(w,z).$$

□

We will extend next the Aubin-Clarke theorem to the framework of Orlicz spaces. Following Clément, de Pagter, Sweers & de Thélin [25], we say that a function $\varphi : \mathbb{R} \to \mathbb{R}$ is *admissible* if

- $\varphi \in C(\mathbb{R},\mathbb{R})$;
- φ is odd;
- φ is strictly increasing;
- $\varphi(\mathbb{R}) = \mathbb{R}$.

In this particular case, φ has an inverse and the complementary N-function of Φ is given by

$$\Phi^*(s) = \int_0^s \varphi^{-1}(\tau)\, d\tau.$$

In addition, if we assume that

$$1 < \varphi^- \leq \varphi^+ < +\infty,$$

where

$$\varphi^- = \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \text{ and } \varphi^+ = \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)},$$

then both Φ and Φ^* satisfy the Δ_2-condition (see [25] Lemma C.6), hence $L^{\Phi}(\Omega)$ and $L^{\Phi^*}(\Omega)$ are reflexive Banach spaces and each is the dual of the other (see Proposition 1.9). Moreover, if $1 < \varphi^- < +\infty$, then the following relations between the Luxemburg norm $|\cdot|_{\Phi}$ and the integral $\int_\Omega \Phi(|\cdot|)\, dx$ can be established (see [25], Lemma C.7)

$$\int_\Omega \Phi(|u|)\, dx \leq |u|_{\Phi}^{\varphi^-}, \forall u \in L^{\Phi}(\Omega), |u|_{\Phi} < 1, \tag{2.7}$$

$$\int_\Omega \Phi(|u|)\, dx \geq |u|_{\Phi}^{\varphi^-}, \forall u \in L^{\Phi}(\Omega), |u|_{\Phi} > 1. \tag{2.8}$$

In a similar manner one can prove that if $1 < \varphi^+ < \infty$, then

$$\int_\Omega \Phi(|u|)\, dx \geq |u|_{\Phi}^{\varphi^+}, \forall u \in L^{\Phi}(\Omega), |u|_{\Phi} < 1, \tag{2.9}$$

$$\int_\Omega \Phi(|u|)\, dx \leq |u|_{\Phi}^{\varphi^+}, \forall u \in L^{\Phi}(\Omega), |u|_{\Phi} > 1. \tag{2.10}$$

Assume $\psi : \mathbb{R} \to \mathbb{R}$ is an admissible function which satisfies

$$1 < \psi^- \leq \psi^+ < \infty \tag{2.11}$$

and $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function which is measurable with respect to the first variable and satisfies one of the following conditions

$(h1)$ there exists $b \in L^{\Psi^*}(\Omega)$ such that

$$|h(x,t_1) - h(x,t_2)| \leq b(x)|t_1 - t_2|,$$

for a.e. $x \in \Omega$ and all $t_1, t_2 \in \mathbb{R}$;

$(h2)$ there exist $c > 0$ and $b \in L^{\Psi^*}(\Omega)$ such that

$$|\xi| \leq b(x) + c\psi(|t|),$$

for a.e. $x \in \Omega$, all $t \in \mathbb{R}$ and all $\xi \in \partial_C h(x,t)$.

Assume ψ satisfies (2.11), let Ψ be the corresponding N-function and define $H : L^{\Psi}(\Omega) \to \mathbb{R}$ by

$$H(w) = \int_\Omega h(x, w(x))\, dx. \tag{2.12}$$

Theorem 2.3. (N.C., G. MOROŞANU & C. VARGA [32]) *Assume either $(h1)$ or $(h2)$ holds. Then, the functional H defined in (2.12) is Lipschitz continuous on bounded domains of $L^{\Psi}(\Omega)$ and*

$$\partial_C H(w) \subseteq \left\{\zeta \in L^{\Psi^*}(\Omega) :\ \zeta(x) \in \partial_C h(x, w(x)) \text{ for a.e. } x \in \Omega\right\}. \tag{2.13}$$

Moreover, if $h(x,\cdot)$ is regular at $w(x)$ for a.e. $x \in \Omega$, then H is regular at w and (2.13) holds with equality.

Proof. Suppose w_1, w_2 belong to a bounded subset of $L^{\Psi}(\Omega)$. If we assume $(h1)$ holds, then the Hölder-type inequality for Orlicz spaces shows that

$$|H(w_1) - H(w_2)| \leq 2|b|_{\Psi^*}|w_1 - w_2|_{\Psi},$$

hence H is Lipschitz continuous.

If $(h2)$ is assumed, then by Lebourg's mean value theorem, there exists $\lambda_0 \in (0,1)$ and $\bar{\xi}(x) \in \partial_C h(x, \bar{w}(x))$ such that

$$\bar{\xi}(x)(w_1(x) - w_2(x)) = h(x, w_1(x)) - h(x, w_2(x)), \text{ for a.e. } x \in \Omega,$$

with $\bar{w}(x) = \lambda_0 w_1(x) + (1-\lambda_0) w_2(x)$. Lemma A.5 in [25] shows that

$$\bar{w} \in L^{\Psi}(\Omega) \Rightarrow \psi(|\bar{w}|) \in L^{\Psi^*}(\Omega),$$

which combined with the Hölder-type inequality for Orlicz spaces leads to

$$\begin{aligned} |H(w_1) - H(w_2)| &\leq \int_\Omega |h(x, w_1(x)) - h(x, w_2(x))| \, dx \\ &= \int_\Omega |\bar{\xi}(x)||w_1(x) - w_2(x)| \, dx \\ &\leq \int_\Omega [b(x) + \psi(|\bar{w}(x)|)] \, |w_1(x) - w_2(x)| \, dx \\ &\leq [|b|_{\Psi^*} + c \, |\psi(|\bar{w}|)|_{\Psi^*}] \, |w_1 - w_2|_\Psi . \end{aligned}$$

In order to prove that H is Lipschitz continuous on bounded domains we only need to show that $|\psi(|\bar{w}|)|_{\Psi^*}$ is bounded above by a constant independent of w_1 and w_2. Clearly we may assume $|\psi(|\bar{w}|)|_{\Psi^*} > 1$. Since w_1 and w_2 belong to a bounded subset of $L^\Psi(\Omega)$ and $\bar{w}$ is a convex combination of them, then there exists a constant $m > 1$, independent of w_1 and w_2, such that $|\bar{w}|_\Psi \leq m$. On the other hand, (2.8) and the fact that (see [25] Corollary C.7)

$$\frac{1}{\psi^+} + \frac{1}{(\psi^{-1})^-} = 1,$$

assure that

$$1 < |\psi(|\bar{w}|)|_{\Psi^*} \leq |\psi(|\bar{w}|)|_{\Psi^*}^{\frac{\psi^+}{\psi^+ - 1}} = |\psi(|\bar{w}|)|_{\Psi^*}^{(\psi^{-1})^-} \leq \int_\Omega \Psi^*(\psi(|\bar{w}|)) \, dx.$$

Using Young's inequality, see (1.7), we have

$$\Psi^*(\psi(t)) \leq \Psi(t) + \Psi^*(\psi(t)) = t\psi(t) \leq \int_t^{2t} \psi(s) \, ds \leq \Psi(2t),$$

and from the Δ_2-condition we get

$$\int_\Omega \Psi^*(\psi(|\bar{w}|)) \, dx \leq c_1 + c_2 \int_\Omega \Psi(|\bar{w}|) \, dx.$$

Combining relations (2.7) and (2.10) with the fact that $|\bar{w}|_\Psi \leq m$ we get

$$\int_\Omega \Psi(|\bar{w}|) \, dx \leq m^{\psi^+},$$

hence

$$|\psi(\bar{w})|_{\Psi^*} \leq c_1 + c_2 m^{\psi^+},$$

with c_1, c_2, m suitable constants independent of w_1 and w_2.

The definition of the generalized directional derivative shows that the map $x \mapsto h^0(x, w(x); z(x))$ is measurable on Ω. Moreover, each of the conditions $(h1)$, $(h2)$ implies the integrability of $h^0(x, w(x); z(x))$. Let us check now that

$$H^0(w; z) \leq \int_\Omega h^0(x, w(x); z(x))\, dx, \quad \text{for all } w, z \in L^\Psi(\Omega). \tag{2.14}$$

If $(h1)$ is assumed, then (2.14) follows directly from Fatou's lemma. On the other hand, if we assume $(h2)$ to hold, then by Lebourg's mean value theorem, for each $\lambda > 0$ we have

$$\frac{h(x, w(x) + \lambda z(x)) - h(x, w(x))}{\lambda} = \langle \xi_x, z \rangle,$$

for some $\xi_x \in \partial_C h(x, \bar{w}(x))$, with $\bar{w}(x) = \mu_0 w(x) + (1 - \mu_0)\,[w(x) + \lambda z(x)]$, $0 < \mu_0 < 1$. Again, (2.14) follows by applying Fatou's lemma.

In order to prove (2.13) let us fix $\xi \in \partial_C H(w)$. Then (see e.g. Remark 2.170 in Carl, Le & Motreanu [19])

$$\xi \in \partial H^0(w; \cdot)(0),$$

where ∂ stands for the subdifferential in the sense of convex analysis. The latter and relation (2.14) show that ξ also belongs to the subdifferential at 0 of the convex map

$$L^\Psi(\Omega) \ni z \mapsto \int_\Omega h^0(x, w(x); z(x))\, dx,$$

and (2.13) follows from the subdifferentiation under the the integral for convex integrands (see e.g. Denkowski, Migorski & Papageorgiou [40]).

For the final part of the Theorem, let us assume that $h(x, \cdot)$ is regular at $w(x)$ for a.e. $x \in \Omega$.

Then, we can apply Fatou's lemma to get

$$\begin{aligned} H^0(w;z) &= \limsup_{\substack{\bar{z}\to z\\ \lambda\downarrow 0}} \frac{H(w+\lambda\bar{z})-H(w)}{\lambda} \\ &\geq \liminf_{\lambda\downarrow} \frac{H(w+\lambda z)-H(w)}{\lambda} \\ &\geq \int_\Omega \liminf_{\lambda\downarrow 0} \frac{h(x,w(x)+\lambda z(x))-h(x,w(x))}{\lambda}\, dx \\ &= \int_\Omega h'(x,w(x);z(x))\, dx \\ &= \int_\Omega h^0(x,w(x);z(x))\, dx \\ &\geq H^0(w;z), \end{aligned}$$

which shows that the directional derivative $H'(w;z)$ exists and

$$H'(w;z) = H^0(w;z) = \int_\Omega h^0(x,w(x);z(x))\, dx, \quad \text{for every } z \in L^\Psi(\Omega).$$

□

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Chapter 3

Elliptic differential inclusions depending on a parameter

Throughout this chapter we study some elliptic differential inclusions of the following type

$$-Au + f \in \lambda \partial_C \Phi(u) + \partial_C \Psi(u), \tag{3.1}$$

in a real Banach space X. Here, $\lambda > 0$ is a real parameter, $f \in X^*$ is given, $A : X \to X^*$ is a nonlinear (single-valued) operator and $\Phi, \Psi : X \to \mathbb{R}$ are locally Lipschitz functionals, while ∂_C stands for Clarke's generalized gradient.

We study boundary value problems with various boundary conditions whose variational formulation (in the sense of distributions) lead to a differential inclusion of the type (3.1), in the case when X is a space of functions defined on an open, bounded and connected subset Ω of $\mathbb{R}^N$ and A is a differential operator which may be viewed as a generalization of the Laplace operator.

We say that $u \in X$ is a solution for problem (3.1) if there exist $\xi \in \partial_C \Phi(u)$ and $\zeta \in \partial_C \Psi(u)$ such that

$$\langle f, w \rangle = \langle Au, w \rangle + \lambda \langle \xi, w \rangle + \langle \zeta, w \rangle, \quad \text{for all } w \in X. \tag{3.2}$$

In order prove that problem (3.2) possesses at least one solution we can adopt two strategies:

- transforming (3.2) into a *hemivariational inequality*, by taking into account the definition of Clarke's generalized gradient (see Chapter 1,) and replacing $w = v - u$ to get
  Find $u \in X$ such that

$$\langle f, v-u\rangle \leq \langle Au, v-u\rangle + \lambda\Phi^0(u; v-u) + \Psi^0(u; v-u), \quad \text{for all } v \in X, \tag{3.3}$$

- using the *nonsmooth critical point theory*, developed by Chang [21], by defining the *energy functional* $\mathcal{E} : X \to \mathbb{R}$ as follows

$$\mathcal{E}_\lambda(u) = F(u) + \lambda\Phi(u) + \Psi(u), \tag{3.4}$$

  with $F : X \to \mathbb{R}$ a $C^1(X, \mathbb{R})$ function which satisfies $F'(u) = Au - f$ and seek for critical points of this functional.

3.1 The $p(\cdot)$-Laplace operator with Steklov-type boundary condition

In this section we are concerned with the study of a differential inclusion of the type

$$(\mathbf{P_1}) : \begin{cases} -\text{div}\,(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u \in \partial_C\phi(x,u), & \text{in } \Omega, \\ \frac{\partial u}{\partial n_{p(x)}} \in \lambda\partial_C\psi(x,u), & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $\lambda > 0$ is a real parameter, $p : \overline{\Omega} \to \mathbb{R}$ is a continuous function such that $\inf_{x\in\overline{\Omega}} p(x) > N$, $\phi : \Omega\times\mathbb{R} \to \mathbb{R}$ and $\psi : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ are locally Lipschitz functionals with respect to the second variable and

$$\frac{\partial u}{\partial n_{p(x)}} = |\nabla u|^{p(x)-2}\nabla u \cdot n,$$

n being the unit outward normal on $\partial\Omega$.

In the case when $p(x) \equiv p$, $\phi(x,t) \equiv 0$ and $\psi(x,t) = \frac{1}{q}|t|^q$ the problem $(\mathbf{P_1})$ becomes

$$(\mathcal{P}) : \begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = \lambda|u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

and it was studied by J. Fernández Bonder and J.D. Rossi [49] in the case $1 < q < p^* = \frac{p(N-1)}{N-p}$ by using variational arguments combined with the Sobolev trace inequality. In [49] it is also proved that if $p = q$ then problem $(\mathcal{P})$ possesses a sequence of eigenvalues $\{\lambda_n\}$, such that $\lambda_n \to \infty$ as $n \to \infty$. Furthermore, S. Martinez and J.D. Rossi [76] proved that the first eigenvalue λ_1 of problem $(\mathcal{P})$ (that is, $\lambda_1 \leq \lambda$ for any other eigenvalue) when $p = q$ is isolated and simple. In the linear case, that is $p = q = 2$, problem $(\mathcal{P})$ is known in the literature as the *Steklov* problem (see e.g. I. Babuška and J. Osborn [8]).

Remark 3.1. *If $N < p^- \leq p(x)$ for any $x \in \overline{\Omega}$, then Theorem 2.2 from [46] ensures that the space $W^{1,p(\cdot)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$, and, since $N < p^-$ it follows that $W^{1,p(\cdot)}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Therefore, there exists a positive constant $c_\infty > 0$ such that*

$$\|u\|_\infty \leq c_\infty \|u\|, \quad \textit{for all } u \in W^{1,p(\cdot)}(\Omega), \tag{3.5}$$

where by $\|\cdot\|_\infty$ we have denoted the usual norm on $C(\overline{\Omega})$, that is $\|u\|_\infty = \sup_{x\in\overline{\Omega}} |u(x)|$.

Definition 3.1. *We say that $u \in W^{1,p(\cdot)}(\Omega)$ is a solution of problem $(\mathbf{P_1})$ if there exist $\xi(x) \in \partial_C \phi(x, u(x))$ and $\zeta(x) \in \partial_C \psi(x, u(x))$ for a.e. $x \in \overline{\Omega}$ such that for all $v \in W^{1,p(\cdot)}(\Omega)$ we have*

$$\int_\Omega \Big(-\mathrm{div}\, (|\nabla u(x)|^{p(x)-2} \nabla u(x)) + |u(x)|^{p(x)-2} u(x) \Big) v(x)\, dx = \int_\Omega \xi(x) v(x)\, dx$$

and

$$\int_{\partial\Omega} \frac{\partial u}{\partial n_{p(\cdot)}} v(x)\, d\sigma = \lambda \int_{\partial\Omega} \zeta(x) v(x)\, d\sigma.$$

Here, and hereafter we shall assume the the following hypotheses hold:

$(\mathcal{H}_5)$ $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a functional such that

- (i) $\phi(x, 0) = 0$ for a.e. $x \in \Omega$;
- (ii) the function $x \mapsto \phi(x, t)$ is measurable for every $t \in \mathbb{R}$;
- (iii) the function $t \mapsto \phi(x, t)$ is locally Lipschitz for a.e. $x \in \Omega$;

(iv) there exist $c_\phi > 0$ and $q \in C(\overline{\Omega})$ with $1 < q(x) \le q^+ < p^-$ such that

$$|\xi(x)| \le c_\phi |t|^{q(x)-1},$$

for a.e. $x \in \Omega$, every $t \in \mathbb{R}$ and every $\xi(x) \in \partial_C \phi(x,t)$.

(v) there exists $\delta_1 > 0$ such that $\phi(x,t) \le 0$ when $0 < |t| \le \delta_1$, for a.e. $x \in \Omega$.

$(\mathcal{H}_6)$ $\psi : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a functional such that

(i) $\psi(x,0) = 0$ for a.e. $x \in \partial\Omega$;

(ii) the function $x \mapsto \psi(x,t)$ is measurable for every $t \in \mathbb{R}$;

(iii) the function $t \mapsto \psi(x,t)$ is locally Lipschitz for a.e. $x \in \partial\Omega$;

(iv) there exist $c_\psi > 0$ and $r \in C(\partial\Omega)$ with $1 < r(x) \le r^+ < p^-$ such that

$$|\zeta(x)| \le c_\psi |t|^{r(x)-1}$$

for a.e. $x \in \partial\Omega$, every $t \in \mathbb{R}$ and every $\zeta(x) \in \partial_C \psi(x,t)$;

(v) there exists $\delta_2 > 0$ such that $\psi(x,t) \le 0$ when $0 < |t| \le \delta_2$, for a.e. $x \in \partial\Omega$.

$(\mathcal{H}_7)$ There exists $\eta > \max\{\delta_1, \delta_2\}$ such that $\eta^{p(x)} \le p(x)\phi(x,\eta)$ for a.e. $x \in \Omega$ and $\psi(x,\eta) > 0$ for a.e. $x \in \partial\Omega$.

$(\mathcal{H}_8)$ There exists $m \in L^1(\Omega)$ such that $\phi(x,t) \le m(x)$ for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$.

$(\mathcal{H}_9)$ There exists $\mu > \max\left\{c_\infty(p^+\|m\|_{L^1(\Omega)})^{1/p^-}; c_\infty(p^+\|m\|_{L^1(\Omega)})^{1/p^+}\right\}$ such that

$$\sup_{|t|\le\mu} \psi(x,t) \le \psi(x,\eta) < \sup_{t\in\mathbb{R}} \psi(x,t).$$

The main result of this section is given by the following theorem.

Theorem 3.1. (N.C. & C. Varga [37]) *Assume that $(\mathcal{H}_5)$-$(\mathcal{H}_7)$ hold true. Then for each $\lambda > 0$ problem $(\mathbf{P_1})$ possesses at least two non-zero solutions. If in addition $(\mathcal{H}_8)$ and $(\mathcal{H}_9)$ hold, then there exists $\lambda^* > 0$ such that problem $(\mathbf{P_1})$ possesses at least three non-zero solutions.*

Proof. Let us denote $X = W^{1,p(\cdot)}(\Omega)$, $Y = Z = C(\overline{\Omega})$ and consider $T : X \to Y$, $S : X \to Z$ to be the embedding operators. It is clear that T, S are compact operators and for the sake of simplicity, everywhere below, we will omit to write Tu and Su to denote the above operators, writing u instead of Tu or Su. We introduce next $L : X \to \mathbb{R}$, $J_1 : Y \to \mathbb{R}$ and $J_2 : Z \to \mathbb{R}$ as follows

$$L(u) = \int_\Omega \frac{1}{p(x)} \left[|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)} \right] \, dx, \quad \text{for } u \in X,$$

$$J_1(y) = \int_\Omega \phi(x, y(x)) \, dx, \quad \text{for } y \in Y,$$

and

$$J_2(z) = \int_{\partial\Omega} \psi(x, z(x)) \, d\sigma, \quad \text{for } z \in Z.$$

We point out the fact that L is sequentially weakly lower semicontinuous and $L' : X \to X^*$,

$$\langle L'(u), v \rangle = \int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x) \cdot \nabla v(x) + |u(x)|^{p(x)-2} u(x) v(x) \, dx$$

has the $(\mathcal{S})_+$ property according to X.L. Fan and Q.H. Zhang (see [45], Theorem 3.1).

The idea is to prove that the functional $\mathcal{E}_\lambda : X \to \mathbb{R}$ defined by

$$\mathcal{E}_\lambda(u) = L(u) - J_1(u) - \lambda J_2(u),$$

satisfies the conditions of Theorem 2.1. Standard arguments show that each critical point of this functional is a solution of problem $(\mathbf{P_1})$ in the sense of Definition 3.1. With this end in view we go through the following steps.

STEP 1. The functionals J_1 and J_2 defined above are locally Lipschitz.

This follows directly from Lebourg's mean value theorem.

STEP 2. $u_0 = 0$ satisfies hypothesis $(\mathcal{H}_1)$.

Indeed, $L(0) = J_1(0) = J_2(0) = 0$ and for each $R > 0$ we have

$$L(u) > 0, \quad \text{for all } u \in \bar{B}_X(0; R) \setminus \{0\},$$

which shows that $u_0 = 0$ is a strict minimum point for L.

STEP 3. The functional $\mathcal{E}_\lambda$ is coercive.

Let $u \in X$ be fixed. A simple computation, combined with Lebourg's mean value theorem yields

$$J_1(u) \leq c_\phi \int_\Omega \|u\|_\infty^{q(x)}\, dx,$$

and

$$J_2(u) \leq c_\psi \int_{\partial\Omega} \|u\|_\infty^{r(x)}\, d\sigma.$$

Hence for $u \in X$ with $\|u\| > 1$ and $\|u\|_\infty > 1$ we have

$$\begin{aligned}
\mathcal{E}_\lambda(u) &= L(u) - J_1(u) - \lambda J_2(u) \\
&= \int_\Omega \frac{1}{p(x)} \left[|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)} \right] dx - \int_\Omega \phi(x, u(x))dx - \lambda \int_{\partial\Omega} \psi(x, u(x))d\sigma \\
&\geq \frac{1}{p^+}\|u\|^{p^-} - c_\phi \text{ meas}(\Omega)\|u\|_\infty^{q^+} - \lambda c_\psi \text{ meas}(\Omega)\|u\|_\infty^{r^+} \\
&\geq \frac{1}{p^+}\|u\|^{p^-} - c_\phi \text{ meas}(\Omega)c_\infty^{q^+}\|u\|^{q^+} - \lambda c_\psi \text{ meas}(\Omega)c_\infty^{r^+}\|u\|^{r^+}.
\end{aligned}$$

We conclude that $\mathcal{E}_\lambda(u) \to \infty$ as $\|u\| \to \infty$ since $r^+ < p^-$ and $q^+ < p^-$.

STEP 4. There exists $\bar{u}_0 \in X$ such that $\mathcal{E}_\lambda(\bar{u}_0) < 0$.

Choosing $\bar{u}_0(x) = \eta$ for all $x \in \overline{\Omega}$ and taking into account $(\mathcal{H}_7)$ we conclude that

$$\begin{aligned}
\mathcal{E}_\lambda(\bar{u}_0) &= L(\bar{u}_0) - J_1(\bar{u}_0) - \lambda J_2(\bar{u}_0) \\
&= \int_\Omega \frac{1}{p(x)} \eta^{p(x)}\, dx - \int_\Omega \phi(x, \eta)\, dx - \lambda \int_{\partial\Omega} \psi(x, \eta)\, d\sigma < 0.
\end{aligned}$$

STEP 5. There exists $R_0 > 0$ such that $J_1(u) \leq L(u)$ and $J_2(u) \leq 0$ for all $u \in B(0; R_0) \setminus \{0\}$.

Let us define $R_0 < \min\left\{\frac{\delta_1}{c_\infty}; \frac{\delta_2}{c_\infty}\right\}$ where c_∞ is given in (3.5) and δ_1, δ_2 are given in $(\mathcal{H}_5)$ and $(\mathcal{H}_5)$, respectively. For an arbitrarily fixed $u \in B(0; R_0)$, taking into account the way we defined the operators T and S, we have

$$|u(x)| \leq \|u\|_\infty \leq c_\infty\|u\| \leq c_\infty R_0 < \delta_1, \quad \text{for all } x \in \Omega$$

and

$$|u(x)| \leq \|u\|_\infty \leq c_\infty\|u\| \leq c_\infty R_0 < \delta_2, \quad \text{for all } x \in \partial\Omega.$$

Hypotheses $(\mathcal{H}_5)$ and $(\mathcal{H}_6)$ ensure that $\phi(x, u(x)) \leq 0$ and $\psi(x, u(x)) \leq 0$ for all $u \in B(0; R_0)$, therefore $J_1(u) \leq 0 < L(u)$ and $J_2(u) \leq 0$ for all $u \in B(0; R_0) \setminus \{0\}$.

STEP 6. There exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda>0} \inf_{u\in X} \lambda \left[L(u) - J_1(u) + \rho\right] - J_2(u) < \inf_{u\in X} \sup_{\lambda>0} \lambda \left[L(u) - J_1(u) + \rho\right] - J_2(u).$$

Using the same arguments as B. Ricceri [106] (see the proof of Theorem 2) we conclude that it suffices to find $\rho \in \mathbb{R}$ and $\bar{u}_1, \bar{u}_2 \in X$ such that

$$L(\bar{u}_1) - J_1(\bar{u}_1) < \rho < L(\bar{u}_2) - J_1(\bar{u}_2) \tag{3.6}$$

and

$$\frac{\sup_{u\in A} J_2(u) - J_2(\bar{u}_1)}{\rho - L(\bar{u}_1) + J_1(\bar{u}_1)} < \frac{\sup_{u\in A} J_2(u) - J_2(\bar{u}_2)}{\rho - L(\bar{u}_2) + J_1(\bar{u}_2)}, \tag{3.7}$$

where $A = (L - J_1)^{-1}((-\infty, \rho])$.

Let us define $\bar{u}_1 \equiv \eta$ and choose $\bar{u}_2$ such that

$$\psi(x, \bar{u}_2(x)) > \sup_{|t|\leq\mu} \psi(x, t).$$

We point out the fact that a $\bar{u}_2$ satisfying the above relation exists due to $(\mathcal{H}_9)$. Next we define

$$\rho = \min\left\{\frac{1}{p^+}\left(\frac{\mu}{c_\infty}\right)^{p^+} - \|m\|_{L^1(\Omega)}; \frac{1}{p^+}\left(\frac{\mu}{c_\infty}\right)^{p^-} - \|m\|_{L^1(\Omega)}\right\}$$

and observe that $\rho > 0$.

Taking into account inequality (3.5) and the properties of the modular, we are able to prove that

$$\|u\|_\infty \leq \mu, \quad \text{for all } u \in A.$$

We only have to check that (3.6) and (3.7) hold for $\bar{u}_1$ and $\bar{u}_2$ chosen as above. From above we conclude that $\bar{u}_2 \notin A$ and thus

$$\sup_{u\in A} J_2(u) \leq \sup_{\|u\|_\infty\leq\mu} J_2(u) \leq J_2(\bar{u}_1), \quad \sup_{u\in A} J_2(u) \leq \sup_{\|u\|_\infty\leq\mu} J_2(u) \leq J_2(\bar{u}_2),$$

and

$$L(\bar{u}_1) - J_1(\bar{u}_1) \leq 0 < \rho < L(\bar{u}_2) - J_1(\bar{u}_2).$$

The above steps show that the hypotheses of Theorem 2.1 are fulfilled. □

Remark 3.2. *In the previous Theorem conditions $(\mathcal{H}_5)-(iii)$ and $(\mathcal{H}_5)-(iv)$ can be replaced with the following condition*

- *there exists a constant $k_\phi > 0$ such that*

$$|\phi(x,t_1) - \phi(x,t_2)| \leq k_\phi |t_1 - t_2|, \text{ for all } t_1, t_2 \in \mathbb{R}.$$

We can also replace conditions $(\mathcal{H}_6)-(iii)$ and $(\mathcal{H}_6)-(iv)$ with the following condition

- *there exists a constant $k_\psi > 0$ such that for*

$$|\psi(x,t_1) - \psi(x,t_2)| \leq k_\psi |t_1 - t_2|, \text{ for all } t_1, t_2 \in \mathbb{R}.$$

Example 3.1. *Let us provide next an example of two functions $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\psi : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ which satisfy the conditions required in Theorem 3.1. Let Ω be an open bounded subset of $\mathbb{R}^N$ with smooth boundary and assume* meas $(\Omega) \geq 1$. *Let $p, q \in C_+(\overline{\Omega})$ be such that $p^- > N$ and $q^+ < p^-$ and $r \in C(\partial\Omega)$ such that $1 < r(x) < r^+ < p^-$. We consider $\mu > 1$ sufficiently large, $0 < \delta < \min\left\{\frac{1}{3}, \left(\frac{N}{2^{p^-}}\right)^{1/(p^- - q^+)}\right\}$. We consider now $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\psi : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ to be two nonsmooth locally Lipschitz functionals defined by*

$$\phi(x,t) = \begin{cases} 0, & t \leq \delta \\ (t-\delta)^{q(x)}, & \delta \leq t < \delta + \frac{\pi}{2} \\ \left(\frac{\pi}{2}\right)^{q(x)} \sin(t-\delta), & \delta + \frac{\pi}{2} \leq t, \end{cases}$$

and

$$\psi(x,t) = \begin{cases} |t+\mu|^{r(x)}, & t \leq -\mu \\ 0, & -\mu < t \leq \delta \\ (t-\delta)(3\delta - t), & \delta \leq t < 3\delta \\ 0, & 3\delta \leq t, \end{cases}$$

and prove that hypotheses $(\mathcal{H}_5)$-$(\mathcal{H}_9)$ are satisfied.

Note that

$$\partial_C \phi(x,t) = \begin{cases} 0, & t \le \delta \\ q(x)(t-\delta)^{q(x)-1}, & \delta < t < \delta + \frac{\pi}{2} \\ \left[0, q(x)\left(\frac{\pi}{2}\right)^{q(x)-1}\right], & t = \delta + \frac{\pi}{2} \\ \left(\frac{\pi}{2}\right)^{q(x)} \cos(t-\delta), & t > \delta + \frac{\pi}{2} \end{cases}$$

and

$$\partial_C \psi(x,t) = \begin{cases} -r(x)(-t-\mu)^{r(x)-1}, & t < -\mu \\ 0, & -\mu \le t < \delta \\ [0, 2\delta], & t = \delta \\ -2t + 4\delta, & \delta < t < 3\delta \\ [-2\delta, 0], & t = 3\delta \\ 0, & t > 3\delta. \end{cases}$$

Thus, for any $\xi(x) \in \partial_C \phi(x,t)$ and any $\zeta(x) \in \psi(x,t)$, we have

$$|\xi(x)| \le \begin{cases} 0 < |t|^{q(x)-1}, & t \le \delta \\ q^+|t-\delta|^{q(x)-1} < q^+ < q^+\left(\frac{|t|}{\delta}\right)^{q(x)-1} < \frac{q^+}{\delta^{q^+-1}}|t|^{q(x)-1}, & \delta < t < \delta+1 \\ q^+|t-\delta|^{q(x)-1} < q^+|t|^{q(x)-1}, & \delta + 1 \le t < \delta + \frac{\pi}{2} \\ q^+\left(\frac{\pi}{2}\right)^{q^+-1} < q^+\left(\frac{\pi}{2}\right)^{q^+-1}|t|^{q(x)-1}, & t = \delta + \frac{\pi}{2} \\ \left(\frac{\pi}{2}\right)^{q^+} < \left(\frac{\pi}{2}\right)^{q^+}|t|^{q(x)-1}, & t > \delta + \frac{\pi}{2} \end{cases}$$

and

$$|\zeta(x)| \le \begin{cases} r^+|t+\mu|^{r(x)-1} < r^+|t|^{r(x)-1}, & t < -1-\mu \\ r^+|t+\mu|^{r(x)-1} < r^+ < r^+\left(\frac{|t|}{\mu}\right)^{r(x)-1} < \frac{r^+}{\mu^{r^--1}}|t|^{r(x)-1}, & -1-\mu < t < -\mu \\ 0 < |t|^{r(x)-1}, & -\mu \le t < \delta \\ 2\delta \le 2\delta\left(\frac{|t|}{\delta}\right)^{r(x)-1} < \frac{2\delta}{\delta^{r^--1}}|t|^{r(x)-1}, & \delta \le t \le 3\delta \\ 0 < |t|^{r(x)-1}, & t > 3\delta. \end{cases}$$

It is clear from above that $(\mathcal{H}_5)$ and $(\mathcal{H}_6)$ hold. In order to see that $(\mathcal{H}_7)$-$(\mathcal{H}_9)$ are satisfied we point out that the functional ϕ is bounded and choose $\eta = 2\delta < 1$. We have

$$\eta^{p(x)} = (2\delta)^{p(x)} \le (2\delta)^{p^-} \le N\delta^{q^+} \le N\delta^{q(x)} \le p(x)\delta^{q(x)} = p(x)\phi(x,\eta).$$

On the other hand we observe that $\psi(x,t)$ attains its maximum at $t = 2\delta$ on $[\delta, 3\delta]$, $\psi(x, 2\delta) = \delta^2 > 0$ and $\psi(x,t) = 0$ on $[-\mu, \delta] \cup [3\delta, \mu]$, which shows that $\sup_{|t|\leq\mu} \psi(x,t) = \psi(x,\eta)$, *while* $\sup_{t\in\mathbb{R}} \psi(x,t) = \infty$.

We close this section by pointing out the fact that the nonsmooth Ricceri-type multiplicity results presented in Chapter 2 can be successfully applied to other kind of problems. An example is the following ordinary differential inclusion with periodic boundary conditions

$$(\mathbf{ODI}) : \begin{cases} -u'' + u \in \lambda\alpha(t)\partial_C F(u) + \beta(t)\partial_C G(u) & \text{in } [0,1] \\ u(0) = u(1) \\ u'(0) = u'(1) \end{cases}$$

where $\lambda > 0$ is a real parameter, $F, G : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz and $\alpha, \beta : [0,1] \to [0,\infty)$ are nonconstant functions. It can be proved that, under suitable assumptions, for each $\lambda > 0$ problem (**ODI**) has at least two nonzero solutions and there exists $\lambda^* > 0$ for which problem (**ODI**) has at least three nonzero solutions. Problems of this type have been insvetigated by F. Faraci and A. Iannizzotto [48].

Another example is the following differential inclusion on the whole space $\mathbb{R}^N$

$$(\tilde{\mathcal{P}}_\lambda) : \begin{cases} -\Delta_p u + |u|^{p-2}u \in \lambda\alpha(x)\partial_C F(u(x)) + \beta(x)\partial_C G(u(x)) & \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

where λ is a positive real parameter, $\alpha, \beta : \mathbb{R}^N \to \mathbb{R}$ are given and $F, G : \mathbb{R} \to \mathbb{R}$ are two locally Lipschitz functionals. A similar problem was studied by A. Kristály, W. Marzantowicz and Cs. Varga [66] using the *principle of symmetric criticality*. We point out the fact that slightly modifying the conditions imposed in [66] we can apply Theorem 2.1 to obtain two (or even three) nonzero solutions for problem $(\tilde{\mathcal{P}}_\lambda)$.

3.2 The p-Laplace-like operators with mixed boundary conditions

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 3)$ having smooth boundary and $2 \leq p < +\infty$. We denote $\partial\Omega = \Gamma$ the boundary of Ω and assume that $\Gamma_1,\ \Gamma_2$ are two open measurable parts that form a partition of Γ (i.e. $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$) such that $\operatorname{meas}(\Gamma_2) > 0$.

We are interested in boundary value problems involving a quasilinear elliptic differential operator, a nonsmooth potential and mixed boundary conditions of the following type:

$$(\mathbf{P_2}): \quad \begin{cases} \operatorname{div}(a(x,\nabla u)) \in \lambda\partial_C F(x,u) - h(x), & \text{in } \Omega \\ -a(x,\nabla u)\cdot n \in \mu(x,u)\partial_C G(x,u), & \text{on } \Gamma_1, \\ u = 0, & \text{on } \Gamma_2, \end{cases}$$

where $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is of the form $a(x,\xi) = (a_1(x,\xi), \dots, a_N(x,\xi))$, with $a_i : \Omega \times \mathbb{R}^N \to \mathbb{R}$ for $i \in \{1,\dots,N\}$, $\lambda > 0$ is a real parameter, $F : \Omega \times \mathbb{R} \to \mathbb{R}$ and $G : \Gamma_1 \times \mathbb{R} \to \mathbb{R}$ are locally Lipschitz functionals with respect to the second variable, $\mu : \Gamma_1 \times \mathbb{R} \to \mathbb{R}$ and $h : \Omega \to \mathbb{R}$ can be viewed as perturbation functions and n is the unit outward normal to $\partial\Omega$. Here and hereafter, the symbols $\partial_C F(x,t)$ and $\partial_C G(x,t)$ stand for the Clarke generalized gradients of the mappings $t \mapsto F(x,t)$ and $t \mapsto G(x,t)$, respectively.

Example 3.2. *Set $a(x,\xi) = |\xi|^{p-2}\xi$. Then $a(x,\xi)$ is the continuous derivative with respect to the second variable of the mapping $\mathcal{A}(x,\xi) = \frac{1}{p}|\xi|^p$, i.e. $a(x,\xi) = \nabla_\xi \mathcal{A}(x,\xi)$. Then we get the $p-$Laplace operator*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Example 3.3. *Set $a(x,\xi) = (1+|\xi|^2)^{(p-2)/2}\xi$. Then $a(x,\xi)$ is the continuous derivative with respect to the second variable of the mapping $\mathcal{A}(x,\xi) = \frac{1}{p}\left[(1+|\xi|^2)^{p/2} - 1\right]$, i.e. $a(x,\xi) = \nabla_\xi \mathcal{A}(x,\xi)$. Then we get the the mean curvature operator*

$$\operatorname{div}\left((1+|\nabla u|^2)^{(p-2)/2}\nabla u\right).$$

We point out the fact that our operator is not necessarily a potential operator, but we have chosen these examples due to the fact that boundary value problems involving the above mentioned operators were studied intensively in the last decades since quasilinear operators can model a variety of physical phenomena (e.g. the p-Laplacian is used in non-Newtonian fluids, reaction-diffusion problems as well as in flow through porous media).

Let us turn now our attention towards the terms given by Clarke's generalized gradient. To our best knowledge differential inclusions similar to $(\mathbf{P_2})$ have been studied in the past either with Neumann condition, or with Dirichlet condition on the entire boundary. This cases can be obtained when $\Gamma_1 = \Gamma$, or $\Gamma_2 = \Gamma$. We present next several particular cases of our problem that have been treated in the last years by various authors.

CASE 1. $\Gamma_1 = \Gamma$ (Neumann problem).

- If F and G are primitives of some Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$

$$F(x,t) = \int_0^t f(x,s)\, ds \quad \text{and} \quad G(x,t) = \int_0^t g(x,s)\, ds$$

  then the functions $t \mapsto F(x,t)$ and $t \mapsto G(x,t)$ are differentiable. Thus $\partial_C F(x,t) = \{f(x,t)\}$, $\partial_C G(x,t) = \{g(x,t)\}$ and $(\mathbf{P_2})$ reduces to the following eigenvalue problem

$$\begin{cases} \operatorname{div}(a(x,\nabla u)) = \lambda f(x,u) - h(x) & \text{in } \Omega \\ -a(x,\nabla u)\cdot n = \mu(x,u) g(x,u) & \text{on } \partial\Omega \end{cases} \tag{3.8}$$

  A particular case of problem (3.8) was studied by Y.X. Huang [59] (there the author studies the case when $a(x,\xi) = |\xi|^{p-2}\xi$, $f(x,t) = m(x)|t|^{p-2}t$, $g \equiv 0$ and $h \equiv 0$).

- In the case when the functionals f and g from the previous example are only locally bounded, i.e. $f \in L^\infty_{\text{loc}}(\Omega \times \mathbb{R})$ and $g \in L^\infty_{\text{loc}}(\partial\Omega \times \mathbb{R})$ then $t \mapsto F(x,t)$ and $t \mapsto G(x,t)$ are locally Lipschitz functionals and, according to Proposition 1.7 in [90] we have

$$\partial_C F(x,t) = \left[\, \underline{f}(x,t), \overline{f}(x,t) \,\right] \quad \text{and} \quad \partial_C G(x,t) = \left[\, \underline{g}(x,t), \overline{g}(x,t) \,\right],$$

  where

$$\underline{f}(x,t) = \lim_{\delta\downarrow 0} \operatorname{ess} \inf_{|s-t|<\delta} f(x,s) \qquad \overline{f}(x,t) = \lim_{\delta\downarrow 0} \operatorname{ess} \sup_{|s-t|<\delta} f(x,s)$$

  and

$$\underline{g}(x,t) = \lim_{\delta\downarrow 0} \operatorname{ess} \inf_{|s-t|<\delta} g(x,s) \qquad \overline{g}(x,t) = \lim_{\delta\downarrow 0} \operatorname{ess} \sup_{|s-t|<\delta} g(x,s).$$

  In this case problem $(\mathbf{P_2})$ reduces to

$$\begin{cases} \operatorname{div}(a(x,\nabla u)) \in \lambda \left[\, \underline{f}(x,u), \overline{f}(x,u) \,\right] - h(x) & \text{in } \Omega \\ -a(x,\nabla u)\cdot n \in \mu(x,u) \left[\, \underline{g}(x,u), \overline{g}(x,u) \,\right] & \text{on } \partial\Omega \end{cases} \tag{3.9}$$

A particular case of problem (3.9) was studied by F. Papalini [102] in the case of the p-Laplacian. The approach is variational and is based on the nonsmooth critical point theory for locally Lipschitz functionals developed by K.-C. Chang in [21].

- In the case when $h \equiv 0$ and $\mu(x,t) \equiv \mu > 0$ problem ($\mathbf{P_2}$) becomes

$$\begin{cases} \operatorname{div}(a(x,\nabla u)) \in \lambda \partial_C F(x,u), & \text{in } \Omega \\ -a(x,\nabla u)\cdot n \in \mu \partial_C G(x,u), & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

A problem similar to (3.10) was studied by A. Kristály, W. Marzantowicz and Cs. Varga in [66] where the authors use a nonsmooth three critical points theorem to prove that there exists a compact interval $[a,b]$ with the property that for every $\lambda \in [a,b]$ there exists $\mu_0 \in (0, \lambda+1)$ such the for each $\mu \in [0,\mu_0]$, the studied problem possesses at least three distinct solutions.

CASE 2. $\Gamma_2 = \Gamma$ (Dirichlet problem).

In this case our problem can be rewritten equivalently as follows:

$$u \in W_0^{1,p}(\Omega): \quad \mathcal{A}u + \lambda \partial_C F(\cdot, u) \ni h \quad \text{in } W^{-1,p'}(\Omega), \tag{3.11}$$

where $\mathcal{A}u(x) = -\operatorname{div}\, a(x, \nabla u(x))$.

Problem (3.11) was treated in the case $\lambda = 1$ and $h \equiv 0$ by S. Carl and D. Motreanu [20] who used the method of sub and supersolutions to obtain general comparison results. We also remember the work of Z. Liu and G. Liu [75] and J. Wang [121] who studied eigenvalue problems for elliptic hemivariational inequalities that can be rewritten equivalently as differential inclusions similar to (3.11). In [75] and [121] the authors used the surjectivity of multivalued pseudomonotone operators to prove the existence of solutions.

As we have seen above, in most papers dealing with differential inclusions of the type ($\mathbf{P_2}$) nonsmooth critical point theory, or the pseudomonotonicity of a certain multivalued operator play an essential role in obtaining the existence of solutions. However, in all the works we are aware of, additional assumptions on the structure of the elliptic operator and/or the generalized Clarke's gradient are needed to obtain the existence of the solution (e.g. the elliptic operator is of potential type, or the locally Lipschitz functional is required to be regular, or to satisfy some conditions of Landesman-Lazer type, or the Clarke's generalized gradient is supposed to satisfy more restrictive growth conditions). Here, our approach is topological and the novelty is that we are able to obtain the existence of at least one weak solution for any $\lambda \in (0, +\infty)$ without assuming any of the above restrictions.

We present next the conditions that need to be imposed in order to prove the main result of this section.

$(\mathcal{H}_{10})$ Let $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be an operator of the form $a(x, \xi) = (a_1(x, \xi), \ldots, a_N(x, \xi))$ which satisfies

(i) for each $i \in \{1, \ldots, N\}$ $a_i : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function and there exists $c_0 > 0$ and $\alpha \in L^{p'}(\Omega)$ such that

$$|a_i(x, \xi)| \leq \alpha(x) + c_0|\xi|^{p-1},$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$;

(ii) there exist $c_1 > 0$ and $\beta \in L^1(\Omega)$ such that

$$a(x, \xi) \cdot \xi \geq c_1|\xi|^p - \beta(x),$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$;

(iii) for a.e. $x \in \Omega$ and all $\xi_1, \xi_2 \in \mathbb{R}^N$

$$[a(x, \xi_1) - a(x, \xi_2)] \cdot (\xi_1 - \xi_2) \geq 0.$$

$(\mathcal{H}_{11})$ Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function which satisfies:

(i) for all $t \in \mathbb{R}$ the function $x \mapsto F(x, t)$ is measurable;

(ii) for a.e. $x \in \Omega$ the function $t \mapsto F(x, t)$ is locally Lipschitz;

(iii) there exists $c_2 > 0$ such that for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$|\partial_C F(x,t)| \le c_2(1 + |t|^{p-1});$$

(iv) there exists $\gamma_1 \in L^p(\Omega)$ such that for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$|F^0(x,t;-t)| \le \gamma_1(x)|t|^{p-1}.$$

$(\mathcal{H}_{12})$ Let $G : \Gamma_1 \times \mathbb{R} \to \mathbb{R}$ be measurable with respect to the first variable and assume there exists $\gamma_2 \in L^{p'}(\Gamma_1)$ such that

$$|G(x,t_1) - G(x,t_2)| \le \gamma_2(x)|t_1 - t_2|,$$

for a.e. $x \in \Gamma_1$ and all $t_1, t_2 \in \mathbb{R}$.

$(\mathcal{H}_{13})$ $\mu : \Gamma_1 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exists $\mu^* > 0$ such that

$$0 \le \mu(x,t) \le \mu^*,$$

for a.e. $x \in \Gamma_1$ and all $t \in \mathbb{R}$.

$(\mathcal{H}_{14})$ $h \in L^{p'}(\Omega)$.

Let us introduce the functional space

$$V = \left\{v \in W^{1,p}(\Omega) : \ \gamma v = 0 \text{ on } \Gamma_2\right\}$$

where $\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ is the Sobolev trace operator. For simplicity, everywhere below, we will omit to write γv to indicate the Sobolev trace on the boundary, writing v instead of γv. Since $\text{meas}(\Gamma_2) > 0$, it is well known that V is a closed subspace of $W^{1,p}(\Omega)$ and can be endowed with the norm

$$\|v\|_V = \|\nabla v\|_{L^p(\Omega)},$$

which is equivalent to the usual norm on $W^{1,p}(\Omega)$ due to the Poincaré-Friedrichs inequality (see e.g. Proposition 2.94 in [19]).

Definition 3.2. *We say that $u \in V$ is a weak solution for problem $(\mathbf{P_2})$ if there exist $\zeta_1 \in L^{p'}(\Omega)$ satisfying $\zeta_1(x) \in \partial_C F(x, u(x))$ for a.e. $x \in \Omega$ and $\zeta_2 \in L^{p'}(\Gamma_1)$ satisfying $\zeta_2(x) \in \partial_C G(x, u(x))$ for a.e. $x \in \Gamma_1$ such that*

$$\int_\Omega a(x, \nabla u) \cdot (\nabla v - \nabla u)\, dx + \lambda \int_\Omega \zeta_1 (v - u)\, dx + \int_{\Gamma_1} \mu(x, u) \zeta_2 (v - u)\, d\sigma = \int_\Omega h(x)(v - u)\, dx,$$

for all $v \in V$.

The main result of this section is the following theorem.

Theorem 3.2. (N.C., I. FIROIU & F.D. PREDA [27]) *Suppose that conditions $(\mathcal{H}_{10}) - (\mathcal{H}_{14})$ are fulfilled. Then for each $\lambda \in (0, +\infty)$ problem $(\mathbf{P_2})$ possesses at least one weak solution.*

Before proving Theorem 3.2 we introduce the operator $A : V \to V^*$ defined by

$$\langle Au, v \rangle = \int_\Omega a(x, \nabla u) \cdot \nabla v\, dx, \tag{3.12}$$

and denote by ϕ the element of V^* given by

$$\langle \phi, v \rangle = \int_\Omega h(x) v\, dx.$$

We have the following proposition which characterizes the weak solutions of problem $(\mathbf{P_2})$.

Proposition 3.1. *An element $u \in V$ is a weak solution for problem $(\mathbf{P_2})$ if and only if it solves the following hemivariational inequality*

$(HI)_\lambda$ *Find $u \in V$ such that*

$$\langle Au, v - u \rangle + \lambda \int_\Omega F^0(x, u; v - u)\, dx + \int_{\Gamma_1} \mu(x, u) G^0(x, u; v - u)\, d\sigma \geq \langle \phi, v - u \rangle,$$

for all $v \in V$.

Finally, we point out the fact that we do not deal with a classical hemivariational inequality due to the presence of the term $\int_{\Gamma_1} \mu(x, u) G^0(x, u; v - u)\, d\sigma$ in the left-hand side of the inequality and consequently several difficulties occur in determining the existence of solutions since the classical methods fail to be applied directly.

Proof of Theorem 3.2. First we point out the fact that under $(\mathcal{H}_{10})$ the operator $A : V \to V^*$ defined in (3.12) is well defined and satisfies the following properties:

- there exists $c_3 > 0$ such that $\langle Au, u\rangle \geq c_1\|u\|_V^p - c_3$, for all $u \in V$;
- $\langle Av - Au, v - u\rangle \geq 0$ for all $u, v \in V$;
- $\langle Au_m, v\rangle \to \langle Au, v\rangle$ for all $v \in V$, whenever $u_m \to u$ in V.

Let us fix $\lambda > 0$. We shall prove next that there exists at least one $u \in V$ which solves $(HI)_\lambda$. In order to do this let us fix $R > 0$ and define $K = \bar{B}_V(0, R) = \{u \in V : \|u\|_V \leq R\}$ and

$$\mathcal{P} = \left\{ (v,u) \in K \times K \;\middle|\; \begin{array}{l} \langle Au, v-u\rangle + \lambda \int_\Omega F^0(x,u;v-u)\,dx + \\ \int_{\Gamma_1} \mu(x,u) G^0(x,u;v-u)\,d\sigma \geq \langle \phi, v-u\rangle \end{array} \right\}.$$

After some computations we are able to show that:

- ◦ For each $v \in K$ the set $\Lambda(v) = \{u \in K : (v,u) \in \mathcal{P}\}$ is weakly closed;
- ◦ For each $u \in K$ the set $\Theta(u) = \{v \in K : (v,u) \notin \mathcal{P}\}$ is either empty or convex;
- ◦ The set $B = \{u \in K : (v,u) \in \mathcal{P} \text{ for all } v \in K\}$ is weakly compact.

The above statements show that we can apply Lin's theorem (see [73], Theorem), for the weak topology of the space V, with $K_0 = K = \bar{B}_V(0, m)$ and obtain the existence of an element $u_m \in \bar{B}_V(0, m)$ such that $\bar{B}_V(0, m) \times \{u_m\} \subseteq \mathcal{P}$, which can be rewritten equivalently as

$$\langle Au_m, v - u_m\rangle + \lambda \int_\Omega F^0(x, u_m, v - u_m)\,dx + \int_{\Gamma_1} G^0(x, u_m; v - u_m)\,d\sigma \geq \langle \phi, v - u_m\rangle, \quad (3.13)$$

for all $v \in \bar{B}_V(0, m)$, which means that, for each positive integer m, the restriction of $(HI)_\lambda$ to $\bar{B}_V(0, m)$ possesses at least one solution.

In order to compete the proof we need to prove that there exists $m^* > 0$ such that

$$u_{m^*} \in B_V(0, m^*) \quad (3.14)$$

and this u_{m^*} solves $(HI)_\lambda$. This can be easily done as follows.

Arguing by contradiction let us assume that $\|u_m\|_V = m$ for all $m > 0$. Taking $v = 0$ in (3.13) we obtain

$$\begin{aligned}\langle Au_m, u_m\rangle &\leq \langle \phi, u_m\rangle + \lambda\int_\Omega F^0(x, u_m; -u_m)\, dx + \int_{\Gamma_1} \mu(x, u_m)G^0(x, u_m; -u_m)\, d\sigma \\ &\leq \|\phi\|_{V^*}\|u_m\|_V + \lambda\int_\Omega \gamma_1(x)|u_m|^{p-1}\, dx + \mu^*\int_{\Gamma_1}\gamma_2(x)|u_m|\, d\sigma \\ &\leq \|\phi\|_{V^*}\|u_m\|_V + \lambda\|\gamma_1\|_{L^p(\Omega)}\|u_m\|^{p-1}_{L^p(\Omega)} + \mu^*\|\gamma_2\|_{L^{p'}(\Gamma_1)}\|u_m\|_{L^p(\Gamma_1)} \\ &\leq \tilde{c}_1\|u_m\|_V + \tilde{c}_2\|u_m\|_V^{p-1},\end{aligned}$$

for some suitable constants $\tilde{c}_1, \tilde{c}_2 > 0$. On the other hand, we know that

$$\langle Au_m, u_m\rangle \geq c_1\|u_m\|_V^p - c_3.$$

Combining the above estimates and keeping in mind that $1 < p$ and $\|u_m\|_V = m$ for all $m > 0$ we arrive at

$$c_1 m^p - c_3 \leq \tilde{c}_1 m^{p-1} + \tilde{c}_2 m.$$

Dividing by m^{p-1} and letting $m \to \infty$ we get a contradiction as the left-hand term of the inequality diverges while the right-hand term remains bounded, which is impossible. This contradiction shows that (3.14) holds.

Now let $v \in V$ be fixed. We know that $\|u_{m^*}\|_V < m^*$ which allows us to choose $t \in (0, 1)$ such that $w = u_{m^*} + t(v - u_{m^*}) \in \bar{B}_V(0, m^*)$. Plugging w in (3.13) we have

$$\begin{aligned}t\langle \phi, v - u_{m^*}\rangle &= \langle \phi, w - u_{m^*}\rangle \\ &\leq \langle Au_{m^*}, w - u_{m^*}\rangle + \lambda\int_\Omega F^0(x, u_{m^*}; w - u_{m^*})\, dx + \\ &\quad \int_{\Gamma_1}\mu(x, u_{m^*})G^0(x, u_{m^*}; w - u_{m^*})\, d\sigma \\ &= t\Big[\langle Au_{m^*}, v - u_{m^*}\rangle + \lambda\int_\Omega F^0(x, u_{m^*}; v - u_{m^*})\, dx + \\ &\quad \int_{\Gamma_1}\mu(x, u_{m^*})G^0(x, u_{m^*}; v - u_{m^*})\, d\sigma\Big].\end{aligned}$$

Dividing the above relation by $t > 0$ we conclude that u_{m^*} is indeed a solution for $(HI)_\lambda$.

□

3.3 The $\overrightarrow{p}(\cdot)$-Laplace operator with the Dirichlet boundary condition

In this section we study the weak solvability of a differential inclusion involving a nonhomogeneous anisotropic differential operator of the following type

$$(\mathbf{P_3}) : \begin{cases} -\sum_{i=1}^{n} \partial_i \left(|\partial_i u|^{p_i(x)-2} \partial_i u \right) \in \lambda \partial_C \alpha(x,u) + \partial_C \beta(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a bounded open set with smooth boundary, $\lambda > 0$ is a real parameter, $\alpha, \beta : \Omega \times \mathbb{R} \to \mathbb{R}$ are two locally Lipschitz functions with respect to the second variable and, for each $i \in \{1, \ldots, n\}$, $p_i : \bar{\Omega} \to \mathbb{R}$ is a continuous function such that $2 \leq p_i(x) < n$ for all $x \in \bar{\Omega}$. The notation $\partial_i u$ stands for the partial derivative of u with respect to the x_i component, that is $\partial u / \partial x_i$, while $\partial_C \alpha(x,t)$ denotes the Clarke generalized gradient of the function $t \mapsto \alpha(x,t)$. The definition and main properties of the Clarke generalized gradient will be given in the next section.

We point out the fact that, if $\alpha(x,t) = \frac{1}{q(x)} |t|^{q(x)}$ and $\beta \equiv const.$, then problem $(\mathbf{P_3})$ reduces to the following nonhomogeneous anisotropic eigenvalue problem

$$\begin{cases} -\sum_{i=1}^{n} \partial_i \left(|\partial_i u|^{p_i(x)-2} \partial_i u \right) = \lambda |u|^{q(x)-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.15}$$

which was studied by Mihăilescu, Pucci and Rădulescu [84, 85]. In these papers the authors show that the "competition" between the growth rates of the functions p_i and q deeply influence the existence or nonexistence of the weak solutions. To our best knowledge these are the first papers dealing with the *anisotropic variable exponent $\overrightarrow{p}(\cdot)$-Laplace operator*, i.e.

$$\Delta_{\overrightarrow{p}(\cdot)} u = \sum_{i=1}^{n} \partial_i \left(|\partial_i u|^{p_i(x)-2} \partial_i u \right),$$

where $\overrightarrow{p} : \bar{\Omega} \to \mathbb{R}^n$ is the vectorial function $\overrightarrow{p}(\cdot) = (p_1(\cdot), \ldots, p_n(\cdot))$. Also in these papers it was introduced for the first time the anisotropic exponent Sobolev space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ that allowed an accurate study of problems of the type (3.15). We point out that the aforementioned space can be viewed as a natural generalization of the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$

(when $p_1(\cdot) = \cdots = p_n(\cdot) = p(\cdot)$) as well as a natural generalization of the classical anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$ (when p_i are constant functions, $i \in \{1, \ldots, n\}$).

On the other hand, let us consider the case when $\alpha \equiv const.$ and β is the primitive of some Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$

$$\beta(x,t) = \int_0^t f(x,s)\, ds.$$

Then the function $t \mapsto \beta(x,t)$ is differentiable and thus $\partial_C \beta(x,t) = \{f(x,t)\}$ and problem $(\mathbf{P_3})$ reduces to the following nonhomogeneous anisotropic problem

$$\begin{cases} -\sum\limits_{i=1}^{n} \partial_i \left(|\partial_i u|^{p_i(x)-2} \partial_i u \right) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.16}$$

which was studied recently by Boureanu, Pucci and Rădulescu [12], by using the symmetric mountain-pass theorem of Ambrosetti and Rabinowitz.

The abstract framework required to study differential inclusions of the type $(\mathbf{P_3})$ lies at the interface of three important directions in analysis:

- *the nonsmooth analysis*: the need for such theory comes naturally whenever we deal with functions which are not differentiable everywhere, but are convex or locally Lipschitz (see, e.g., Andrei, Costea and Matei [4], Chang [21], Clake [24], Costea and Varga [37, 38], Kristály, Rădulescu and Varga [67], Motreanu and Panagiotopoulos [90], Motreanu and Rădulescu [91], Naniewicz and Panagiotopoulos [94], Panagiotopoulos [98, 99, 100]).

- *the variable exponent Lebesgue-Sobolev spaces theory*: problems involving the isotropic $p(x)$-Laplace operator have captured special attention in the last decades since they can model various phenomena which arise in elastic mechanics (see, e.g., Zhikov [123]), image restoration (see, e.g., Chen, Levine and Rao [23]) or electrorheological fluids (see, e.g., Acerbi and Mingione [1], Diening [42], Diening, Harjulehto, Hästö and Ružička [43], Halsey [55], Ružička [109], Costea and Mihăilescu [33], Mihăilescu and Rădulescu [86, 87]).

- *the anisotropic Sobolev spaces theory*: the need for such theory comes naturally whenever we deal with materials possessing inhomogeneities that have different behavior on different space directions (see, e.g., Edmunds and Edmunds [44], Nikol'skii [97], Rákosník [104, 105], Troisi [119]).

Although the $\overrightarrow{p}(\cdot)$-Laplace operator was introduced recently, in 2007 by Mihăilescu, Radulescu and Pucci, problems involving this operator, or similar operators, have captured special attention in the last years (see Boureanu, Pucci and Rădulescu [12], Fan [45], Mihăilescu and Moroşanu [81], Mihăilescu, Moroşanu and Rădulescu [82, 83], Stancu-Dumitru [117]). However, in all the works we are aware of, the *energy functional* attached to the problem is smooth, while differential inclusions like problem $(\mathbf{P_3})$, for which the attached energy functional is only locally Lipschitz and not differentiable, have not yet been studied.

In this section we prove a multiplicity result concerning the weak solutions of problem $(\mathbf{P_3})$. Before defining the concept of weak solution we denote by X the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and by $\|\cdot\|$ the norm defined on this space, that is $\|\cdot\|_{\overrightarrow{p}(\cdot)}$.

Definition 3.3. *A function $u \in X$ is called a weak solution for problem $(\mathbf{P_3})$ if, for a.e. $x \in \Omega$, there exist $\xi(x) \in \partial_C \alpha(x, u(x))$ and $\zeta(x) \in \partial_C \beta(x, u(x))$ such that*

$$\int_\Omega \sum_{i=1}^n |\partial_i u|^{p_i(x)-2} \partial_i u \partial_i v \, dx = \lambda \int_\Omega \xi v \, dx + \int_\Omega \zeta v \, dx, \quad \text{for all } v \in X.$$

In order to obtain our main result we shall assume fulfilled the following hypotheses.

$(\mathcal{H}_{15})$ For each $i \in \{1, \ldots, n\}$ the function $p_i \in C_+(\bar{\Omega})$ is log-Hölder continuous, $2 \le p_i(x) < n$ for all $x \in \bar{\Omega}$ and $p_M^+ < P^*$;

$(\mathcal{H}_{16})$ The function $\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

(i) $\alpha(x, 0) = 0$ for a.e. $x \in \Omega$;

(ii) $x \mapsto \alpha(x, t)$ is measurable for all $t \in \mathbb{R}$;

(iii) $t \mapsto \alpha(x, t)$ is locally Lipschitz for a.e. $x \in \Omega$;

(iv) there exist $c_\alpha > 0$ and $q \in C_+(\bar{\Omega})$ such that $1 < q^- \leq q^+ < p_m^-$ and

$$|\xi(x)| \leq c_\alpha |t|^{q(x)-1}$$

for a.e. $x \in \Omega$, all $t \in \mathbb{R}$ and all $\xi(x) \in \partial_C \alpha(x,t)$;

(v) there exist $\mu \in (0,1)$, $\alpha_0 > 0$ and $t_0 > 0$ such that

$$\alpha(x,t) \leq 0, \qquad \text{for all } |t| < \mu \text{ and a.e. } x \in \Omega,$$

and

$$\alpha(x,t_0) \geq \alpha_0 > 0, \qquad \text{for a.e. } x \in \Omega.$$

$(\mathcal{H}_{17})$: The function $\beta : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

(i) $\beta(x,0) = 0$ for a.e. $x \in \Omega$;

(ii) $x \mapsto \beta(x,t)$ is measurable for all $t \in \mathbb{R}$;

(iii) there exists $r \in C_+(\bar{\Omega})$ with the property that $1 < r(x) < P^*$ and $K \in L^{r'(\cdot)}(\Omega)$ such that

$$|\beta(x,t_1) - \beta(x,t_2)| \leq K(x)|t_1 - t_2|$$

for a.e. $x \in \Omega$ and all $t_1, t_2 \in \mathbb{R}$;

(iv) $\beta(x,t) \leq 0$, for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$.

The main result of this section is given by the following theorem.

Theorem 3.3. (N.C. & G. MOROŞANU [31]) *Assume that $(\mathcal{H}_{15})$, $(\mathcal{H}_{16})$ and $(\mathcal{H}_{17})$ hold. Then there exists $\lambda^* > 0$ such that for any $\lambda \in (\lambda^*, +\infty)$ problem $(\mathbf{P_3})$ possesses at least two non-zero weak solutions.*

Proof. Let us introduce the functionals $J : X \to \mathbb{R}$, $\Lambda : L^{q(\cdot)}(\Omega) \to \mathbb{R}$ and $\Theta : L^{r(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$J(u) := \int_\Omega \sum_{i=1}^n \frac{|\partial_i u|^{p_i(x)}}{p_i(x)}\, dx, \qquad \Lambda(w) := \int_\Omega \alpha(x, w(x))\, dx, \qquad \Theta(z) := \int_\Omega \beta(x, z(x))\, dx.$$

Standard arguments can be employed in order to conclude that the functional $\mathcal{E}_\lambda : X \to \mathbb{R}$ defined by

$$\mathcal{E}_\lambda(u) = J(u) - \Theta(u) - \lambda\Lambda(u),$$

is locally Lipschitz and each critical point of this functional is a solution of problem ($\mathbf{P_3}$). Moreover, it can be shown that $\mathcal{E}_\lambda$ is sequentially weakly lower semicontinuous, coercive and satisfies the (PS)-condition. The following claims complete the proof.

CLAIM 1. There exists $\lambda^* > 0$ such that for any $\lambda > \lambda^*$ we can determine $u_0 \in X$ for which $\mathcal{E}_\lambda(u_0) < 0$.

Let $x_0 \in \operatorname{int}\Omega$ be such that the distance from x_0 to the boundary of Ω is maximal and let R_0 be this distance ($R_0 = \max_{x\in\Omega} \min_{y\in\partial\Omega} |x-y|$). Clearly, for $0 < R < R_0/2$, we have $\bar{B}(x_0; 2R) \subseteq \Omega$. It can be easily seen that there exists $u_0 \in C_0^\infty(B(x_0; 2R))$ such that

$$\begin{cases} u_0(x) = t_0 & \text{for } x \in \bar{B}(x_0; R) \\ 0 \le u_0(x) \le t_0 & \text{for } x \in B(x_0; 2R) \setminus B(x_0; R). \end{cases}$$

Since $u_0 \in C_0^\infty(B(x_0; 2R))$, for $i \in \{1, \dots, n\}$, there exists $m_i > 0$ such that $|\partial_i u_0(x)| \le m_i$ in $B(x_0; 2R)$. Then for $m := \max\{1, m_1 \dots, m_n\}$ we have

$$\begin{aligned} \mathcal{E}_\lambda(u_0) &= \int_\Omega \sum_{i=1}^n \frac{|\partial_i u_0(x)|^{p_i(x)}}{p_i(x)}\, dx - \int_\Omega \beta(x, u_0(x))\, dx - \lambda \int_\Omega \alpha(x, u_0(x))\, dx \\ &\le \int_{B(x_0;2R)} \sum_{i=1}^n \frac{m^{p_M^+}}{p_m^-}\, dx - \int_{B(x_0;2R)} \beta(x, u_0(x))\, dx - \lambda \int_{B(x_0;2R)} \alpha(x, u_0(x))\, dx. \end{aligned}$$

Obviously,

$$\begin{aligned} -\int_{B(x_0;2R)} \beta(x, u_0(x))\, dx &= \int_{B(x_0;2R)} \beta(x,0) - \beta(x, u_0(x))\, dx \\ &\le \int_{B(x_0;2R)} K(x) u_0(x)\, dx \le \beta_1, \end{aligned}$$

for a suitable constant $\beta_1 > 0$.

On the other hand, splitting $B(x_0; 2R)$ into the sets

$$D_1 = \{x \in B(x_0; 2R) : \ \alpha(x, u_0(x)) \le 0\}$$

and

$$D_2 = \{x \in B(x_0; 2R) : \ \alpha(x, u_0(x)) > 0\},$$

we observe that $B(x_0; R) \subset D_2$. Applying Lebourg's mean value theorem and taking into account hypothesis $(\mathcal{H}_{16})$ one can easily prove that

$$\int_{B(x_0;2R)} \alpha(x, u_0(x))\, dx \geq -\alpha_1 + \alpha_0 \frac{\omega_n R^n}{n},$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$ and $\alpha_1 > 0$ is a suitable constant.

Thus

$$\mathcal{E}_\lambda(u_0) \leq \frac{m^{p_M^+} 2^n \omega_n R^n}{p_m^-} + \beta_1 + \alpha_1 - \lambda \alpha_0 \frac{\omega_n R^n}{n} < 0,$$

for any $\lambda > \frac{n 2^n m^{p_M^+} \omega_n R^n + n(\beta_1 + \alpha_1) p_m^-}{\alpha_0 \omega_n R^n}$.

CLAIM 2. The functional $\mathcal{E}_\lambda$ possesses two critical points $u_1, u_2 \in X \setminus \{0\}$, provided that $\lambda \in (\lambda^*, +\infty)$.

The facts that $\mathcal{E}_\lambda$ is sequentially lower semicontinuous and coercive allow us to apply Theorem 1.2 to obtain the existence of an element $u_1 \in X$ such that $\mathcal{E}_\lambda(u_1) = \min_{u \in X} \mathcal{E}_\lambda(u)$. Obviously u_1 is a critical point of $\mathcal{E}_\lambda$ as it is a global minimizer, while CLAIM 1 ensures that $\mathcal{E}_\lambda(u_1) < 0$, which means that $u_1 \neq 0$. Furthermore, if there exists $\rho \in (0, \|u_1\|)$ such that

$$\inf_{\partial B(0;\rho)} \mathcal{E}_\lambda \geq 0 = \max\{\mathcal{E}_\lambda(0), \mathcal{E}_\lambda(u_1)\},$$

then we can apply Theorem 1.3 to obtain another critical point $u_2 \in X \setminus \{0, u_1\}$.

Let us consider $s \in C_+(\bar{\Omega})$ such that $p_M^+ < s^- \leq s^+ < P^*$ and choose $\rho > 0$ such that $\rho < \min\{1, 1/c_s, \|u_1\|\}$, where $c_s > 0$ is the constant given by the compact inclusion $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$, i.e.,

$$\|u\|_{s(\cdot)} \leq c_s \|u\|, \quad \text{for all } u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

Then for each $u \in \partial B(0;\rho)$ we have $\|\partial_i u\|_{p_i(\cdot)} < 1$ $(1 \le i \le n)$ and $\|u\|_{s(\cdot)} < 1$. Thus for $u \in X$ such that $\|u\| = \rho$ we have

$$\begin{aligned}\mathcal{E}_\lambda(u) &= J(u) - \Theta(u) - \lambda\Lambda(u)\\ &\geq \frac{1}{p_M^+}\sum_{i=1}^{n}\int_\Omega |\partial_i u|^{p_i(x)}\,dx - \int_\Omega \beta(x,u(x))\,dx - \lambda\int_\Omega \alpha(x,u(x))\,dx\\ &\geq \frac{1}{p_M^+}\sum_{i=1}^{n}\|\partial_i u\|_{p_i(\cdot)}^{p_M^+} - \lambda\int_\Omega \alpha(x,u(x))\,dx.\end{aligned}$$

Using the convexity of the function $h:\mathbb{R}_+ \to \mathbb{R}_+$ defined by $h(t) = t^{p_M^+}$ we deduce that

$$\sum_{i=1}^{n}\|\partial_i u\|_{p_i(\cdot)}^{p_M^+} \geq \frac{1}{n^{p_M^+-1}}\|u\|^{p_M^+}.$$

Defining $\Omega_1 = \{x \in \Omega :\ |u(x)| < \mu\}$ and $\Omega_2 = \{x \in \Omega :\ |u(x)| \geq \mu\}$ and by the use of hypothesis $(\mathcal{H}_{16}) - (v)$ and Lebourg's mean value theorem we can prove that

$$\int_\Omega \alpha(x,u(x))\,dx \leq c\mu^{q^+-s^-}\|u\|_{s(\cdot)}^{s^-} \leq c\mu^{q^+-s^-}c_s^{s^-}\|u\|^{s^-},$$

for a suitable constant $c > 0$.

Thus, for $u \in \partial B(0;\rho)$

$$\mathcal{E}_\lambda(u) \geq \frac{1}{p_M^+ n^{p_M^+-1}}\rho^{p_M^+} - \lambda c\mu^{q^+-s^-}c_s^{s^-}\rho^{s^-} = \rho^{p_M^+}\left(\frac{1}{p_M^+ n^{p_M^+-1}} - \lambda c\mu^{q^+-s^-}c_s^{s^-}\rho^{s^--p_M^+}\right).$$

Finally, we observe that the function $h:[0,1]\to\mathbb{R}$ defined by

$$h(t) = \frac{1}{p_M^+ n^{p_M^+-1}} - \lambda c\mu^{q^+-s^-}c_s^{s^-}t^{s^--p_M^+}$$

is continuous on $[0,1]$ and $h(0) = \frac{1}{p_M^+ n^{p_M^+-1}} > 0$, hence $h > 0$ in a small neighborhood at the right of the origin. Choosing $\rho > 0$ such that ρ belongs to this neighborhood and $\rho < \min\{1, 1/c_s, \|u_1\|\}$ we deduce that $\mathcal{E}_\lambda(u) > 0$ for all $u \in \partial B(0;\rho)$ and this completes the proof.

□

We close this section with several examples of functions $\alpha, \beta : \Omega \times \mathbb{R} \to \mathbb{R}$ for which conditions $(\mathcal{H}_{16})$ and $(\mathcal{H}_{17})$ are fulfilled.

Example 3.4. *Let $\{\varepsilon_k\}$ be a sequence of positive real numbers such that $\varepsilon_k \downarrow 0$ as $k \to +\infty$. Let $q \in C_+(\bar{\Omega})$ be such that $1 < q^- \le q^+ < p_m^-$ and let $\alpha_k, \beta_k : \Omega \times \mathbb{R} \to \mathbb{R}$ to be defined by*

$$\alpha_k(x,t) = \begin{cases} \frac{1}{q(x)}|t+\varepsilon_k|^{q(x)}, & \text{for } t \in (-\infty, -\varepsilon_k], \\ 0, & \text{for } t \in (-\varepsilon_k, \varepsilon_k), \\ \frac{1}{q(x)}|t-\varepsilon_k|^{q(x)}, & \text{for } t \in [\varepsilon_k, +\infty), \end{cases}$$

and

$$\beta_k(x,t) = 0, \text{ for all } (x,t) \in \Omega \times \mathbb{R}.$$

Then, $\partial_C \beta(x,t) = 0$ for all $(x,t) \in \Omega \times \mathbb{R}$, while

$$\partial_C \alpha_k(x,t) = \begin{cases} |t+\varepsilon_k|^{q(x)-2}(t+\varepsilon_k), & \text{for } t \in (-\infty, -\varepsilon_k), \\ 0, & \text{for } t \in [-\varepsilon_k, \varepsilon_k], \\ |t-\varepsilon_k|^{q(x)-2}(t-\varepsilon_k), & \text{for } t \in (\varepsilon_k, +\infty). \end{cases}$$

Thus, for any $\xi(x) \in \partial_C \alpha(x,t)$, we have

$$|\xi(x)| \le \begin{cases} |t|^{q(x)-1} \le \frac{1}{\varepsilon_k^{q^+-1}}|t|^{q(x)-1}, & \text{for } |t| \in [1+\varepsilon_k, +\infty), \\ 1 \le \left(\frac{|t|}{\varepsilon_k}\right)^{q(x)-1} \le \frac{1}{\varepsilon_k^{q^+-1}}|t|^{q(x)-1}, & \text{for } |t| \in (\varepsilon_k, 1+\varepsilon_k), \\ 0 \le \frac{1}{\varepsilon_k^{q^+-1}}|t|^{q(x)-1}, & \text{for } |t| \in [0, \varepsilon_k). \end{cases}$$

We point out the fact that when $k \to +\infty$ then problem $(\mathbf{P_3})$ with α_k, β_k defined above reduces to problem (3.15), hence this example shows that slightly perturbing problem (3.15) around the origin we can obtain two nontrivial weak solutions instead of only one weak solution as Theorem 3 [85] states.

Example 3.5. *Let $\mu \in (0,1)$, $q_1, q_2 \in C_+(\bar{\Omega})$ be such that $1 < q_1^- \le q_1^+ < q_2^- \le q_2^+ < p_m^-$ and let $a \in L^\infty(\Omega)$ be such that $a(x) < 0$ for a.e. $x \in \Omega$. We consider the functions $\alpha, \beta : \Omega \times \mathbb{R} \to \mathbb{R}$ to be defined by*

$$\alpha(x,t) = \begin{cases} 0, & \text{for } t \in (-\infty, \mu), \\ \max\left\{(t-\mu)^{q_1(x)}, (t-\mu)^{q_2(x)}\right\}, & \text{for } t \in [\mu, +\infty), \end{cases}$$

and

$$\beta(x,t) = a(x)|t|.$$

Then

$$\partial_C\alpha(x,t) = \begin{cases} 0, & \text{for } t \in (\infty, \mu], \\ q_1(x)(t-\mu)^{q_1(x)-1}, & \text{for } t \in (\mu, 1+\mu), \\ [q_1(x), q_2(x)], & \text{for } t = 1+\mu, \\ q_2(x)(t-\mu)^{q_2(x)-1}, & \text{for } t \in (1+\mu, +\infty), \end{cases}$$

and

$$\partial_C\beta(x,t) = \begin{cases} -a(x), & \text{for } t \in (-\infty, 0), \\ [a(x), -a(x)], & \text{for } t = 0, \\ a(x), & \text{for } t \in (0, +\infty). \end{cases}$$

Thus for every $\xi(x) \in \partial_C\alpha(x,t)$ we have

$$|\xi(x)| \leq \begin{cases} 0 \leq \frac{q_2^+}{\mu^{q_2^+-1}}|t|^{q_2(x)-1}, & \text{for } t \in (-\infty, \mu], \\ q_1^+ \leq q_2^+\left(\frac{|t|}{\mu}\right)^{q_2(x)-1} \leq \frac{q_2^+}{\mu^{q_2^+-1}}|t|^{q_2(x)-1}, & \text{for } t \in (\mu, 1+\mu), \\ q_2^+ \leq \frac{q_2^+}{\mu^{q_2^+-1}}|t|^{q_2(x)-1}, & \text{for } t = 1+\mu, \\ q_2^+|t-\mu|^{q_2(x)-1} \leq \frac{q_2^+}{\mu^{q_2^+-1}}|t|^{q_2(x)-1}, & \text{for } t \in [1+\mu, +\infty). \end{cases}$$

Example 3.6. *Let $f, g \in L^\infty_{\text{loc}}(\Omega \times \mathbb{R})$ and consider $\alpha, \beta : \Omega \times \mathbb{R}$ be defined by*

$$\alpha(x,t) = \int_0^t f(x,s)\, ds \quad \text{and} \quad \beta(x,t) = \int_0^t g(x,s)\, ds.$$

Obviously, $t \mapsto \alpha(x,t)$ and $t \mapsto \beta(x,t)$ are locally Lipschitz and according to Proposition 1.7 in Motreanu and Panagiotopoulos [90], we have

$$\partial_C\alpha(x,t) = [\, \underline{f}(x,t), \overline{f}(x,t)\,] \quad \text{and} \quad \partial_C\beta(x,t) = [\, \underline{g}(x,t), \overline{g}(x,t)\,],$$

where for a function $h \in L^\infty_{\text{loc}}(\Omega \times \mathbb{R})$ we denote by

$$\underline{h}(x,t) = \lim_{\delta \downarrow 0} \inf_{|s-t|<\delta} h(x,t) \quad \text{and} \quad \overline{h}(x,t) = \lim_{\delta \downarrow 0} \sup_{|s-t|<\delta} h(x,t).$$

Clearly, there are many ways in which we can choose f and g such that conditions $(\mathcal{H}_{16})$ and $(\mathcal{H}_{17})$ are fulfilled.

Chapter 4

Differential inclusions in Orlicz-Sobolev spaces

4.1 Formulation of the problem

In this chapter we establish an existence result for differential inclusions involving quasilinear elliptic operators in divergence form of the following type

$$Au := \mathrm{div}\,(a(|\nabla u|)\nabla u)\,, \tag{4.1}$$

subjected to Dirichlet boundary conditions, in a bounded domain with smooth boundary. More exactly, we are interested in the existence of weak solutions for the problem

$$(\mathcal{P}): \quad \begin{cases} -\mathrm{div}\,(a(|\nabla u|)\nabla u) \in \partial_C F(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded, open subset of $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary $\partial\Omega$ and $a : [0,\infty) \to [0,\infty)$ is a mapping such that the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) = a(|t|)t,$$

is continuous, strictly increasing and onto. Here, $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function with respect to the second variable and, as usual, $\partial_C F(x,t)$ denotes the Clarke generalized gradient of the mapping $t \mapsto F(x,t)$.

As Ruf [108] pointed out, in dealing with variational problems of the type $(\mathcal{P})$ two questions arise naturally:

(Q1): *Which is the appropriate function space for the problem to be well-posed, hence solvable?*

(Q2): *What kind of growth conditions on F would ensure the existence of weak solutions?*

The answer to (Q1) is determined by two competing factors: on the one hand the space should be "large enough" such that the functional attached to the problem satisfies an appropriate compactness condition (e.g. Palais-Smale or Cerami condition) which ensures that a sequence for which the functional converges to a critical value has a convergent subsequence; and, on the other hand, the space should be "small enough" such that the functional has the desired regularity (locally Lipschitz in our case). In the classical case of the p-Laplacian, i.e. $a(s) = s^{p-2}$ and

$$Au = \Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

the suitable space in which the problem is studied is the Sobolev space $W^{1,p}(\Omega)$. However, in our framework, the space $L^p(\Omega)$ needs to be replaced by the Orlicz space $L^\Phi(\Omega)$ in which the role of the convex function t^p is played by the N-function Φ. It is also worth mentioning that these spaces fill a gap in the Sobolev embedding theorem as, for $mp = N$,

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \text{ for all } q \in [p, \infty),$$

but

$$W^{m,p}(\Omega) \not\hookrightarrow L^\infty(\Omega),$$

hence there is no smallest L^q space in which $W^{m,p}(\Omega)$ can be embedded. Trudinger [120] showed that the best target to embed the space $W^{m,p}(\Omega)$ is the Orlicz space $L^\Phi(\Omega)$ with

$$\Phi(t) = \exp\left(|t|^{p/(p-1)}\right) - 1.$$

Regarding (Q2), it is well known that in the case of the p-Laplacian an important role is played by the critical exponent in the Sobolev embedding theorem $p^* = Np/(N-p)$ for which the space setup works and one is able to prove existence results if subcritical growth assumptions are imposed, while in the critical case we may have nonexistence results. For example let us consider the particular case

$$a(s) = 1,\ \forall s \geq 0 \text{ and } F(t) = \int_0^t f(s)ds$$

with $f : \mathbb{R} \to \mathbb{R}$ a continuous function. Then, F is differentiable and the $\partial_C F(t) = \{f(t)\}$ and problem $(\mathcal{P})$ becomes

$$\begin{cases} -\Delta u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Clearly, the suitable choice for the function space is $H_0^1(\Omega)$ and if f has "subcritical growth", that is

$$|f(t)| \leq c_1 + c_2|t|^q, \quad 1 < q < 2^* - 1,$$

where $2^* = \frac{2N}{N-2}$, then one can prove the existence of solutions, while in the "critical growth" case

$$f(t) = t^{2^*-1},$$

one can use Pohozaev's identity to prove that problem (4.2) has only the trivial solution if Ω is bounded and starshaped.

In this chapter, we consider the case when $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = a(|t|)t$, defines an admissible function and the nonlinearity F has subcritical growth and we will employ variational methods to prove the existence of at least one weak solution for our problem. Let us start by specifying what we understand by weak solution for problem $(\mathcal{P})$.

Definition 4.1. *A function $u \in W_0^1 L^\Phi(\Omega)$ is called weak solution for problem $(\mathcal{P})$ if, for a.e. $x \in \Omega$, there exists $\xi(x) \in \partial_C F(x, u(x))$ such that*

$$\int_\Omega a(|\nabla u|)\nabla u \cdot \nabla v \, dx = \int_\Omega \xi v \, dx, \text{ for all } v \in W_0^1 L^\Phi(\Omega).$$

4.2 An existence result

In order to apply the Direct Method in the Calculus of Variations and show that the energy functional attached to problem $(\mathcal{P})$ has a global minimizer, hence a critical point, we will impose the following hypotheses.

(H_0) The functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ are admissible and satisfy

(i) $\varphi(t) = a(|t|)t$;

(ii) $1 < \varphi^- \le \varphi^+ < \infty$ and $1 < \psi^- \le \psi^+ < \infty$;

(iii) The corresponding N-function of ψ, i.e. $\Psi(t) = \int_0^t \psi(s)\ ds$, increases essentially slower than Φ_* whenever

$$\int_1^\infty \frac{\phi^{-1}(t)}{t^{(N+1)/N}}\ dt = \infty.$$

(H_F) $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(i) $F(x, 0) = 0$ for a.e. $x \in \Omega$;

(ii) $t \mapsto F(x, t)$ is locally Lipschitz for a.e. $x \in \Omega$;

(iii) there exists $b \in L^{\Psi^*}(\Omega)$ such that

$$|\xi| \le b(x) + \psi(|t|),$$

for a.e. $x \in \Omega$, all $t \in \mathbb{R}$ and all $\xi \in \partial_C F(x, t)$.

Let us consider the functionals $J_\Phi : W_0^1 L^\Phi(\Omega) \to \mathbb{R}$ and $J_F : L^\Psi(\Omega) \to \mathbb{R}$ defined by

$$J_\Phi(u) = \int_\Omega \Phi(|\nabla u|)\ dx,$$

and

$$J_F(w) = \int_\Omega F(x, w(x))\ dx.$$

The energy functional corresponding to problem $(\mathcal{P})$, $E : W_0^1 L^\Phi(\Omega) \to \mathbb{R}$, is given by

$$E(u) = J_\Phi(u) - J_F(u). \tag{4.3}$$

Condition (H_0) ensures that $W_0^1L^\Phi(\Omega)$ is compactly embedded into $L^\Psi(\Omega)$, hence E is well defined. The following lemma guarantees the fact that, in order to solve problem $(\mathcal{P})$, it suffices to seek for critical points, in the sense of Definition 1.10, of the energy functional attached to our problem.

Lemma 4.1. *Assume (H_0) and (H_F) hold. Then the functional $E : W_0^1L^\Phi(\Omega) \to \mathbb{R}$ defined in (4.3) has the following properties:*

(i) E is locally Lipschitz;

(ii) E is weakly lower semicontinuous;

(iii) each critical point of E is a weak solution for problem $(\mathcal{P}_1)$.

Proof. (i) According to Lemma A.6 in [25] $J_\Phi \in C^1\left(W_0^1L^\Phi(\Omega), \mathbb{R}\right)$ and

$$\langle J'_\Phi(u), v\rangle = \int_\Omega a(|\nabla u|)\nabla u \cdot \nabla v \, dx \tag{4.4}$$

Since $W_0^1L^\Phi(\Omega)$ is compactly embedded into $L^\Psi(\Omega)$ there exists $c_1 > 0$ such that

$$|u|_\Psi \leq c_1\|u\|, \quad \text{for all } u \in W_0^1L^\Phi(\Omega), \tag{4.5}$$

with $\|\cdot\|$ defined as in Section 1.3, that is $\|u\| = |\,|\nabla u|\,|_\Phi$. Let us fix now $u_0 \in W_0^1L^\Phi(\Omega)$ and prove that there exists $r > 0$ sufficiently small such that E is Lipschitz continuous on $\bar{B}_{W_0^1L^\Phi(\Omega)}(u_0, r) = \left\{v \in W_0^1L^\Phi(\Omega) : \|v - u_0\| \leq r\right\}$. Theorem 2.3 ensures the existence of an $r_0 > 0$ such that J_F is Lipschitz continuous on $\bar{B}_{L^\Psi(\Omega)}(u_0, r_0)$, hence there exists a positive constant L such that

$$|J_F(w_1) - J_F(w_2)| \leq L|w_1 - w_2|_\Psi, \text{ for all } w_1, w_2 \in \bar{B}_{L^\Psi(\Omega)}(u_0, r_0). \tag{4.6}$$

From (4.5) and (4.6) we get

$$|J_F(u_1) - J_F(u_2)| \leq Lc_1\|u_1 - u_2\|, \text{ for all } u_1, u_2 \in \bar{B}_{W_0^1L^\Phi(\Omega)}(u_0, r_0/c_1). \tag{4.7}$$

On the other hand, for $r = r_0/c_1$ and $u_1, u_2 \in \bar{B}_{W_0^1 L^\Phi(\Omega)}(u_0, r)$ we have

$$\begin{aligned} |J_\Phi(u_1) - J_\Phi(u_2)| &\leq |\langle J'_\Phi(\bar{u}), u_1 - u_2\rangle| \\ &\leq \|J'_\Phi(\bar{u})\|_{[W_0^1 L^\Phi(\Omega)]^*} \|u_1 - u_2\|, \end{aligned}$$

for some $\bar{u} \in \{\lambda u_1 + (1-\lambda)u_2 : \ \lambda(0,1)\}$. The space $W_0^1 L^\Phi(\Omega)$ is reflexive, hence the ball $\bar{B}_{W_0^1 L^\Phi(\Omega)}(u_0, r)$ is weakly compact, therefore there exists $m > 0$ such that

$$\|J'_\Phi(\bar{u})\|_{[W_0^1 L^\Phi(\Omega)]^*} \leq m. \tag{4.8}$$

Taking into account (4.7) and (4.8) we conclude that E is a Lipschitz continuous functional on $\bar{B}_{W_0^1 L^\Phi(\Omega)}(u_0, r)$.

(ii) Let $\{u_n\} \subset W_0^1 L^\Phi(\Omega)$ be such that $u_n \rightharpoonup u$ in $W_0^1 L^\Phi(\Omega)$. Reasoning as in Lemma 3.2 in [52] we infer that J_Φ is weakly lower semicontinuous, hence

$$J_\Phi(u) \leq \liminf_{n\to\infty} J_\Phi(u_n).$$

On the other hand, $u_n \to u$ in $L^\Psi(\Omega)$ and by Fatou's lemma

$$\begin{aligned} \limsup_{n\to\infty} J_F(u_n) &= \limsup_{n\to\infty} \int_\Omega F(x, u_n(x))\, dx \\ &\leq \int_\Omega \limsup_{n\to\infty} F(x, u_n(x))\, dx \\ &= \int_\Omega F(x, u(x))\, dx \\ &= J_F(u), \end{aligned}$$

which shows that J_F is weakly upper semicontinuous on $W_0^1 L^\Phi(\Omega)$.

(iii) Let $u \in W_0^1 L^\Phi(\Omega)$ be a critical point of E. According to Propositions 2.3.1 and 2.3.2 in Clarke [24]

$$0 \in \partial_C E(u) \subseteq J'_\Phi(u) - \partial_C J_F(u),$$

hence there exists $\eta \in \partial_C J_F(u)$ such that

$$\langle J'_\Phi(u), v\rangle = \langle \eta, v\rangle, \quad \text{for all } v \in W_0^1 L^\Phi(\Omega). \tag{4.9}$$

Theorem 2.3 ensures the existence of a $\xi \in L^{\Psi^*}(\Omega)$ which satisfies

$$\begin{cases} \xi(x) \in \partial_C F(x, u(x)), & \text{for a.e. } x \in \Omega, \\ \langle \eta, v \rangle = \int_\Omega \xi(x) v(x)\, dx, & \text{for all } v \in W_0^1 L^\Phi(\Omega). \end{cases} \tag{4.10}$$

The conclusion follows at once from (4.4), (4.9) and (4.10).

□

Theorem 4.1. *Suppose* (H_0) *and* (H_F) *are fulfilled and assume in addition that* $\psi^+ < \varphi^-$. *Then problem* $(\mathcal{P})$ *has a nontrivial weak solution.*

Proof. The idea is to prove that E is coercive and apply Theorem 1.2 to conclude that E possesses a global minimum point. To this end, let $u \in W_0^1 L^\Phi(\Omega)$ be such that $\|u\| > 1$. Then, from (2.8), we have

$$J_\Phi(u) \geq |\,|\nabla u|\,|_\Phi^{\varphi^-} = \|u\|^{\varphi^-}. \tag{4.11}$$

On the other hand, we can apply Lebourg's mean value theorem to deduce that there exist $\lambda_0 \in (0,1)$ and $\bar{\zeta} \in \partial_C F(x, \lambda_0 u)$ such that

$$F(x, u(x)) = F(x, u(x)) - F(x, 0) = \bar{\zeta}(x) u(x), \quad \text{for a.e. } x \in \Omega.$$

Thus,

$$\begin{aligned} J_F(u) &= \int_\Omega F(x, u(x))\, dx \\ &\leq \int_\Omega |\bar{\zeta}||u|\, dx \\ &\leq \int_\Omega [b(x) + \psi(\lambda_0 |u|)]\, |u|\, dx \\ &\leq 2|b|_{\Psi^*}|u|_\Psi + \frac{1}{\lambda_0} \int_\Omega \lambda_0 |u| \psi(\lambda_0 |u|)\, dx. \end{aligned}$$

Using Young's inequality we get

$$\lambda_0 |u| \psi(\lambda_0 |u|) \leq \int_{\lambda_0 |u|}^{2\lambda_0 |u|} \psi(s)\, ds \leq \Psi(2\lambda_0 |u|),$$

and the Δ_2-condition yields

$$\int_\Omega \lambda_0|u|\psi(\lambda_0|u|)\,dx \le \int_\Omega \Psi(2\lambda_0|u|)\,dx \le C_1 + C_2\int_\Omega \Psi(\lambda_0|u|)\,dx,$$

for some suitable positive constants C_1 and C_2. Clearly, relations (2.8) and (2.9) imply

$$\int_\Omega \Psi(\lambda_0|u|)\,dx \le \lambda_0\left(|u|_\Psi^{\psi^+} + |u|_\Psi^{\psi^-}\right).$$

The above relations together with (4.5) lead to the following estimate

$$J_F(u) \le C_3 + C_4\|u\| + C_5\|u\|^{\psi^-} + C_6\|u\|^{\psi^+}, \tag{4.12}$$

for some suitable positive constants C_3, C_4, C_5, C_6. Taking into account (4.11) and (4.12) we infer that $E(u) \to \infty$ as $\|u\| \to \infty$. □

Chapter 5

Variational-like inequality problems governed by set-valued operators

In 1989, J. Parida, M. Sahoo and S. Kumar [103] introduced a new type of inequality problem of variational type which had the form

Find $u \in K$ such that

$$\langle A(u), \eta(v, u)\rangle \geq 0, \quad \text{for all } v \in K, \tag{5.1}$$

where $K \subseteq \mathbb{R}^n$ is a nonempty closed and convex set and $A : K \to \mathbb{R}^n$, $\eta : K \times K \to \mathbb{R}^n$ are two continuous maps. The authors called (5.1) *variational-like inequality problem* and showed that this kind of inequalities can be related to some mathematical programming problems. In the recent years, various extensions of (5.1) have been proposed and analyzed by many authors (see e.g. R. Ahmad and S.S. Irfan [7], M.-R. Bai et al. [9], N. Costea and V. Rădulescu [34], N.H. Dien [41], Y.P. Fang and N.J. Huang [47], A.H. Siddiqi, A. Khaiq and Q.H. Ansari [116]) who showed that variational-like inequality problems can be successfully applied in operations research, optimization, mathematical programming and contact mechanics. For various iterative schemes and algorithms for finding approximate solutions for variational-like inequalities as well as convergence results we refer to Q.H. Ansari and J.-C Yao [6] and C.-H. Lee, Q.H. Ansari and J.-C. Yao [72].

The goal of this section is to extend the results obtained in [103] to the following setting: X is a Banach space (not necessarily reflexive) with X^* and $X^{**} = (X^*)^*$ its dual and bidual, respectively, K is a nonempty closed and convex subset of X^{**} and $A : K \to 2^{X^*}$ is a set-valued map. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between a Banach space and its dual; if $u \in X$, $f \in X^*$ and $\xi \in X^{**}$ by $\langle f, u \rangle$ we understand $f(u)$, while by $\langle f, \xi \rangle$ we understand $\xi(f)$. We are interested in finding solutions for the following inequality problems

Find $u \in K \cap \mathcal{D}(\phi)$ such that

$$\exists u^* \in A(u) : \quad \langle u^*, \eta(v, u) \rangle + \phi(v) - \phi(u) \geq 0, \quad \text{for all } v \in K \tag{5.2}$$

and

Find $u \in K$ such that

$$\exists u^* \in A(u) : \quad \langle u^*, \eta(v, u) \rangle \geq 0, \quad \text{for all } v \in K \tag{5.3}$$

where $\eta : K \times K \to X^{**}$, $A : K \to 2^{X^*}$ is a set-valued map, $\phi : X^{**} \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semicontinuous functional such that $K_\phi := K \cap \mathcal{D}(\phi) \neq \emptyset$. Here $\mathcal{D}(\phi)$ stands for the effective domain of the functional ϕ, i.e. $\mathcal{D}(\phi) = \{u \in X^{**} : \phi(u) < +\infty\}$. If ϕ is the indicator function of the set K, i.e.

$$\phi(u) = \begin{cases} 0, & u \in K \\ +\infty, & u \notin K, \end{cases}$$

then problem (5.2) reduces to inequality problem (5.3). Moreover, if A is a single-valued operator, then (5.2) becomes a *generalized variational-like inequality* which was introduced by N.H. Dien [41]. If A is a single-valued operator and $\eta(v, u) = v - u$ then inequality problem (5.3) becomes a *variational inequality* whose study began in the 1960's (see e.g. G. Fichera [50], F.E. Browder [14], P. Hartman and G. Stampacchia [57], J.L. Lions and G. Stampacchia [74]). For more information and connections regarding such types of inequality problems we refer to F. Giannessi, A. Maugeri and P.M. Pardalos [54].

Definition 5.1. *A solution $u_0 \in K \cap \mathcal{D}(\phi)$ of inequality problem (5.2) is called strong if $\langle u^*, \eta(v, u_0) \rangle + \phi(v) - \phi(u_0) \geq 0$ holds for all $v \in K$ and all $u^* \in A(u_0)$.*

It is clear from the above definition that if A is a single-valued operator, then the concepts of solution and strong solution are one and the same.

5.1 The case of nonreflexive Banach spaces

Throughout this subsection X will denote a nonreflexive Banach space and K will stand for a nonempty closed and convex subset of X^{**}. Before stating the results concerning the existence of solutions for problem (5.2) we indicate below some hypotheses that will be needed in the sequel.

($\mathcal{H}_{18}$) $A : K \to 2^{X^*}$ is a set-valued map which is l.s.c. from K endowed with the strong topology into X^* endowed with the w^*-topology and has nonempty values;

($\mathcal{H}_{19}$) $A : K \to 2^{X^*}$ is a set valued map which is u.s.c. from K endowed with the strong topology into X^* endowed with the w^*-topology and has nonempty w^*-compact values;

($\mathcal{H}_{20}$) $\phi : X^{**} \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous functional such that $K_\phi := K \cap \mathcal{D}(\phi)$ is nonempty;

($\mathcal{H}_{21}$) $\eta : K \times K \to X^{**}$ is such that

- for all $v \in K$ the map $u \mapsto \eta(v,u)$ is continuous;
- for all $u, v, w \in K$ and all $w^* \in A(w)$, the map $v \mapsto \langle w^*, \eta(v,u)\rangle$ is convex and $\langle w^*, \eta(u,u)\rangle \geq 0$;

Theorem 5.1. (N.C., D.A. ION & C. LUPU [28]) *Let X be a nonreflexive Banach space and $K \subseteq X^{**}$ nonempty closed and convex. Assume that ($\mathcal{H}_{20}$), ($\mathcal{H}_{21}$) and either ($\mathcal{H}_{18}$) or ($\mathcal{H}_{19}$) hold. If the set K_ϕ is not compact, assume in addition that there exists a nonempty compact convex subset C of K_ϕ such that for each $u \in K_\phi \setminus C$ there exists $u_0^* \in A(u)$ and $\bar{v} \in C$ with the property that*

$$\langle u_0^*, \eta(\bar{v}, u)\rangle + \phi(\bar{v}) - \phi(u) < 0.$$

Then inequality problem (5.2) has at least one strong solution.

Proof. Arguing by contradiction let us assume that (5.2) has no strong solution. Then, for each $u \in K_\phi$ there exist $\bar{u}^* \in A(u)$ and $v = v(u, \bar{u}^*) \in K$ such that

$$\langle \bar{u}^*, \eta(v,u) \rangle + \phi(v) - \phi(u) < 0. \tag{5.4}$$

It is clear that the element v for which (5.4) takes place satisfies $v \in \mathcal{D}(\phi)$, therefore $v \in K_\phi$. We consider next the set-valued map $F : K_\phi \to 2^{K_\phi}$ defined by

$$F(u) = \{ v \in K_\phi : \ \langle \bar{u}^*, \eta(v,u) \rangle + \phi(v) - \phi(u) < 0 \},$$

where $\bar{u}^* \in A(u)$ is given in (5.4).
Using the hypotheses we are able to prove that the following statements hold:

CLAIM 1. For each $u \in K_\phi$ the set $F(u)$ is nonempty and convex;

CLAIM 2. For each $v \in K_\phi$ the set $F^{-1}(v) = \{u \in K_\phi : \ v \in F(u)\}$ is open;

CLAIM 3. $K_\phi = \bigcup\limits_{v \in K_\phi} \mathrm{int}_{K_\phi} F^{-1}(v)$.

If the K_ϕ is not compact then the last condition of our theorem implies that for each $u \in K_\phi \setminus C$ there exists $\bar{v} \in C$ such that $u \in F^{-1}(\bar{v}) = \mathrm{int}_{K_\phi} F^{-1}(\bar{v})$. This observation and the above Claims ensure that all the conditions of Theorem 1.5 are satisfied for $S = T = F$ and we deduce that the set-valued map $F : K_\phi \to 2^{K_\phi}$ has a fixed point $u_0 \in K_\phi$, i.e. $u_0 \in F(u_0)$. This can be rewritten equivalently as

$$0 \leq \langle \bar{u}_0^*, \eta(u_0, u_0) \rangle + \phi(u_0) - \phi(u_0) < 0.$$

We have obtained thus a contradiction which completes the proof. □

5.2 The case of reflexive Banach spaces

Throughout this subsection X will denote a real reflexive Banach space and $K \subseteq X$ will be a nonempty closed and convex set. In order to prove our existence results, throughout this subsection, we shall use some of the following hypotheses:

($\mathcal{H}_{22}$) $A : K \to 2^{X^*}$ is a set-valued map which l.s.c. from K endowed with the strong topology into X^* endowed with the w-topology and has nonempty values;

($\mathcal{H}_{23}$) $A : K \to 2^{X^*}$ is a set valued map which is u.s.c. from K endowed with the strong topology into X^* endowed with the w-topology and has nonempty w-compact values;

($\mathcal{H}_{24}$) $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous functional such that $K_\phi := K \cap \mathcal{D}(\phi)$ is nonempty;

($\mathcal{H}_{25}$) $\eta : K \times K \to X$ is such that

- for all $v \in K$ the map $u \mapsto \eta(v, u)$ is continuous;
- for all $u, v, w \in K$ and all $w^* \in A(w)$ the map $v \mapsto \langle w^*, \eta(v, u)\rangle$ is convex and $\langle w^*, \eta(u, u)\rangle \geq 0$;

($\mathcal{H}_{26}$) $\eta : K \times K \to X$ is such that

- $\eta(u, v) + \eta(v, u) = 0$ for all $u, v \in K$;
- for all $u, v, w \in K$ and all $w^* \in A(w)$ the map $v \mapsto \langle w^*, \eta(v, u)\rangle$ is convex and lower semicontinuous;

($\mathcal{H}_{27}$) $\alpha : X \to \mathbb{R}$ is weakly lower semicontinuous functional such that $\limsup\limits_{\lambda \downarrow 0} \frac{\alpha(\lambda v)}{\lambda} \geq 0$ for all $v \in X$;

($\mathcal{H}_{28}$) $\alpha : X \to \mathbb{R}$ is a such that

- $\alpha(0) = 0$;
- $\limsup\limits_{\lambda \downarrow 0} \frac{\alpha(\lambda v)}{\lambda} \geq 0$, for all $v \in X$;
- $\alpha(u) \leq \limsup \alpha(u_\lambda)$, whenever $u_\lambda \rightharpoonup u$ in X;

The following theorem is a variant of Theorem 5.1 in the framework of reflexive Banach spaces.

Theorem 5.2. (N.C., D.A. ION & C. LUPU [28]) *Let X be a real reflexive Banach space and $K \subseteq X$ nonempty compact and convex. Assume that $(\mathcal{H}_{24})$, $(\mathcal{H}_{25})$ and either $(\mathcal{H}_{22})$ or $(\mathcal{H}_{23})$ hold. Then inequality problem (5.2) has at least one strong solution.*

The proof of Theorem 5.2 follows basically the same steps as the proof of Theorem 5.1, therefore we shall omit it.

We point out the fact that in the above case when K is a compact convex subset of X we do not impose any monotonicity conditions on the set-valued operator A. However, in applications, most problems lead to an inequality whose solution is sought in a closed and convex subset of the space X. Weakening the hypotheses on K by assuming that K is only bounded, closed and convex, we need to impose certain monotonicity properties on A. In the last decades more and more efforts were made to introduce various generalizations of the monotonicity concept from which we remember pseudomonotonicity, quasimonotonicity, semimonotonicity, relaxed α monotonicity, relaxed $\eta - \alpha$ monotonicity, and these concepts were used to prove existence results for various inequality problems, see e.g. M.-R. Bai et al. [9], M. Bianchi, N. Hadjisavvas and S. Schaible [10], Y.Q. Chen [22], N. Costea and A. Matei [29, 30], N. Costea and V. Rădulescu [35, 36], Y.P. Fang and N.J. Huang [47], S. Karamardian and S. Schaible [60], S. Karamardian, S. Schaible and J.P. Crouzeix [61], I.V. Konov and S. Schaible [68] and the references therein.

Definition 5.2. *Let $\eta : K \times K \to X$ and $\alpha : X \to \mathbb{R}$ be two single-valued maps. A set-valued map $T : K \to 2^{X^*}$ is said to be*

- *relaxed $\eta - \alpha$ monotone, if for all $u, v \in K$, all $v^* \in T(v)$ and all $u^* \in T(u)$ we have*

$$\langle v^* - u^*, \eta(v,u) \rangle \geq \alpha(v-u); \tag{5.5}$$

- *relaxed $\eta - \alpha$ pseudomonotone, if for all $u, v \in K$, all $v^* \in T(v)$ and all $u^* \in T(u)$ we have*

$$\langle u^*, \eta(v,u) \rangle \geq 0 \text{ implies } \langle v^*, \eta(v,u) \rangle \geq \alpha(v-u); \tag{5.6}$$

- *relaxed $\eta - \alpha$ quasimonotone, if for all $u, v \in K$, $u \neq v$, all $v^* \in T(v)$ and all $u^* \in T(u)$ we have*

$$\langle u^*, \eta(v,u) \rangle > 0 \text{ implies } \langle v^*, \eta(v,u) \rangle \geq \alpha(v-u). \tag{5.7}$$

SPECIAL CASES.

1. $\eta(v-u) = v-u$. Then

    - (5.5) reduces to: for all $u, v \in K$, all $v^* \in T(v)$ and all $u^* \in T(u)$ we have

    $$\langle v^* - u^*, v-u \rangle \geq \alpha(v-u),$$

    and T is said to be *relaxed* α *monotone*;

    - (5.6) reduces to: for all $u, v \in K$, all $v^* \in T(v)$ and all $u^* \in T(u)$ we have

    $$\langle u^*, v-u \rangle \geq 0 \text{ implies } \langle v^*, v-u \rangle \geq \alpha(v-u),$$

    and T is said to be *relaxed* α *pseudomonotone*;

    - (5.7) reduces to: for all $u, v \in K$, $u \neq v$, all $v^* \in T(v)$ and all $u^* \in T(u)$ we have

    $$\langle u^*, v-u \rangle > 0 \text{ implies } \langle v^*, v-u \rangle \geq \alpha(v-u),$$

    and T is said to be *relaxed* α *quasimonotone*.

2. $\eta(v-u) = v-u$ and $\alpha \equiv 0$. Then

    - (5.5) reduces to: for all $u, v \in K$, all $v^* \in T(v)$ and all $u^* \in T(u)$ we have

    $$\langle v^* - u^*, v-u \rangle \geq 0,$$

    and T is said to be *monotone*;

    - (5.6) reduces to: for all $u, v \in K$, all $v^* \in T(v)$ and all $u^* \in T(u)$ we have

    $$\langle u^*, v-u \rangle \geq 0 \text{ implies } \langle v^*, v-u \rangle \geq 0,$$

    and T is said to be *pseudomonotone*;

    - (5.7) reduces to: for all $u, v \in K$, $u \neq v$, all $v^* \in T(v)$ and all $u^* \in T(u)$ we have

    $$\langle u^*, v-u \rangle > 0 \text{ implies } \langle v^*, v-u \rangle \geq 0,$$

    and T is said to be *quasimonotone*.

Example 5.1. *Let $a > 0$ be a real number, $K = [0, a] \times [0, a] \times \{0\} \subset \mathbb{R}^3$ and define the set-valued operators $T_1, T_2, T_3 : K \to 2^{\mathbb{R}^3}$ as follows*

$$T_1(u) = \{(0, u_2^*, u_3^*) : \ u_2^* = u_2 \text{ and } -u_1 \le u_3^* \le u_1\},$$

$$T_2(u) = \{(0, u_2^*, u_3^*) : \ a/2 \le u_2^* \le a \text{ and } -u_1 \le u_3^* \le u_1\},$$

$$T_3(u) = \{(0, u_2^*, u_3^*) : \ -a \le u_2^* \le 0 \text{ and } -u_1 \le u_3^* \le u_1\}.$$

Then that the following assertions hold true

- T_1 *is monotone;*
- T_2 *is pseudomonotone, but not monotone;*
- T_3 *is quasimonotone, but not pseudomonotone.*

Defining $\alpha : \mathbb{R}^3 \to \mathbb{R}$ by $\alpha(u) = 0$ for all $u \in \mathbb{R}^3$ it is easy to see that

- T_1 *is relaxed α monotone;*
- T_2 *is relaxed α pseudomonotone, but not relaxed α monotone;*
- T_3 *is relaxed α quasimonotone, but not relaxed α pseudomonotone.*

Considering $\eta : K \times K \to \mathbb{R}^3$ defined by $\eta(v, u) = v - u$ and $\alpha : \mathbb{R}^3 \to \mathbb{R}$ as above we conclude that

- T_1 *is relaxed $\eta - \alpha$ monotone;*
- T_2 *is relaxed $\eta - \alpha$ pseudomonotone, but not relaxed $\eta - \alpha$ monotone;*
- T_3 *is relaxed $\eta - \alpha$ quasimonotone, but not relaxed $\eta - \alpha$ pseudomonotone.*

Example 5.2. *Let $a > 0$ be a real number, $K = [0, a] \times [0, a] \times \{0\} \subset \mathbb{R}^3$ and define $T_4 : K \to 2^{\mathbb{R}^3}$ as follows*

$$T_4(u) = \left\{(0, u_2^*, u_3^*) : \ u_2^* = -u_2^2 \text{ and } -u_1 \le u_3^* \le u_1\right\}.$$

Let $\alpha : \mathbb{R}^3 \to \mathbb{R}$ be defined by $\alpha(u) = -2a\|u\|^2$. Then, T_4 is relaxed α monotone and not monotone.

Example 5.3. *Let $a, p > 1$ be real numbers, $K = [0, a] \times [0, a] \times \{0\} \subset \mathbb{R}^3$ and define $T_5 : K \to 2^{\mathbb{R}^3}$ as follows*

$$T_5(u) = \{(0, u_2^*, u_3^*) : \ u_2^* = -u_2^p \text{ and } -u_1 \le u_3^* \le u_1\}.$$

Let $\eta : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\eta(v, u) = (u_1^p - v_1^p, u_2^p - v_2^p, u_3^p - v_3^p)$, $\alpha : \mathbb{R}^3 \to \mathbb{R}$ be defined by $\alpha(u) = -\frac{1}{2}\|u\|$. Then T_5 is relaxed $\eta - \alpha$ monotone, but not relaxed α monotone.

On the other hand, if we define $\eta : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ by $\eta(v, u) = (u_1^p - v_1^p, u_2^p - v_2^p, u_3^p - v_3^p)$ and $\alpha : \mathbb{R}^3 \to \mathbb{R}$ by $\alpha(u) = 0$ is easy to check that T_5 is relaxed $\eta - \alpha$ pseudomonotone and not relaxed α monotone.

Example 5.4. *Let $a > 0$ be a real number, $K = [-a, a] \times [-a, a] \times \{0\} \subset \mathbb{R}^3$ and define $T_6 : K \to 2^{\mathbb{R}^3}$ as follows*

$$T_6(u) = \{(0, u_2^*, u_3^*) : \ u_2^* = -u_2 \text{ and } -a \le u_3^* \le a\}.$$

Then, T_6 is relaxed α pseudomonotone with $\alpha : \mathbb{R}^3 \to \mathbb{R}$, $\alpha(u) = -\|u\|^2$ and T_6 is not pseudomonotone.

Example 5.5. *Let $K = [0, 2\pi] \times [0, 2\pi] \times \{0\} \subset \mathbb{R}^3$ and define $T_7 : K \to 2^{\mathbb{R}^3}$ as follows*

$$T_7(u) = \{(0, u_2^*, u_3^*) : \ u_2^* = \cos u_2 \text{ and } -u_1 \le u_3^* \le u_1\}.$$

We define $\alpha : \mathbb{R}^3 \to \mathbb{R}$ by $\alpha(u) = -m\|u\|^2$ where $m > 1$ is a constant and we claim that T_7 is relaxed α quasimonotone, but not quasimonotone.

On the other hand, if we consider $\eta : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\eta(v, u) = (u_1 \cos u_1, u_2 \cos u_2, u_3 \cos u_3)$$

and $\alpha : \mathbb{R}^3 \to \mathbb{R}$ defined by $\alpha(u) = 0$ we observe that T_7 is relaxed $\eta - \alpha$ quasimonotone, but not relaxed α monotone.

From the above definitions and examples we have the following implications (while the inverse of each implication is not true) which highlight the relations between different kinds of generalized monotonicity for set-valued maps.

monotone	$\rightarrow$	pseudomonotone	$\rightarrow$	quasimonotone
$\downarrow$		$\downarrow$		$\downarrow$
relaxed α monotone	$\rightarrow$	relaxed α pseudomonotone	$\rightarrow$	relaxed α quasimonotone
$\downarrow$		$\downarrow$		$\downarrow$
relaxed $\eta - \alpha$ monotone	$\rightarrow$	relaxed $\eta - \alpha$ pseudomonotone	$\rightarrow$	relaxed $\eta - \alpha$ quasimonotone

We are now able to formulate another main result concerning the existence of solutions of inequality problem (5.2) on bounded, closed and convex subsets.

Theorem 5.3. (N.C., D.A. ION & C. LUPU [28]) *Let K be a nonempty bounded closed and convex subset of the real reflexive Banach space X. Let $A : K \to 2^{X^*}$ be a relaxed $\eta - \alpha$ monotone map and assume that $(\mathcal{H}_{24})$, $(\mathcal{H}_{26})$ and $(\mathcal{H}_{27})$ hold. If in addition*

- $(\mathcal{H}_{22})$ *holds, then inequality problem (5.2) has at least one strong solution;*
- $(\mathcal{H}_{23})$ *holds, then inequality problem (5.2) has at least one solution.*

Proof. We shall apply Mosco's Theorem for the weak topology of X. First we note that K is weakly compact as it is a bounded closed and convex subset of the real reflexive space X and $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is weakly lower semicontinuous as it is convex and lower semicontinuous. We define $\xi, \zeta : X \times X \to \mathbb{R}$ as follows

$$\xi(v,u) = -\inf_{v^* \in A(v)} \langle v^*, \eta(v,u)\rangle + \alpha(v-u)$$

and

$$\zeta(v,u) = \sup_{u^* \in A(u)} \langle u^*, \eta(u,v)\rangle.$$

Let us fix $u, v \in X$ and choose $\bar{v}^* \in A(v)$ such that $\langle \bar{v}^*, \eta(v,u)\rangle = \inf_{v^* \in A(v)} \langle v^*, \eta(v,u)\rangle$. For an arbitrary fixed $u^* \in A(u)$ we have

$$\begin{aligned}
\zeta(v,u) - \xi(v,u) &= \sup_{u^* \in A(u)} \langle u^*, \eta(u,v)\rangle + \inf_{v^* \in A(v)} \langle v^*, \eta(v,u)\rangle - \alpha(v-u) \\
&\geq \langle u^*, \eta(u,v)\rangle + \langle \bar{v}^*, \eta(v,u)\rangle - \alpha(v-u) \\
&= \langle \bar{v}^*, \eta(v,u)\rangle - \langle u^*, \eta(v,u)\rangle - \alpha(v-u) \\
&\geq 0.
\end{aligned}$$

It is easy to check that the conditions imposed on η and α ensure that the map $u \mapsto \xi(v,u)$ is weakly lower semicontinuous, while the map $v \mapsto \zeta(v,u)$ is concave. Applying Theorem 1.8 for $\mu = 0$ we conclude that there exists $u_0 \in K \cap \mathcal{D}(\phi)$ such that

$$\xi(v,u_0) + \phi(u_0) - \phi(v) \leq 0, \quad \text{for all } v \in X,$$

since $\zeta(v,v) = 0$ for all $v \in X$. A simple computation shows that for each $w \in K$ we have

$$\langle w^*, \eta(w,u_0)\rangle + \phi(w) - \phi(u_0) \geq \alpha(w-u_0), \quad \text{for all } w^* \in A(w). \tag{5.8}$$

Let us fix $v \in K$ and define $w_\lambda = u_0 + \lambda(v-u_0)$, with $\lambda \in (0,1)$. Then for a fixed $w_\lambda^* \in A(w_\lambda)$ from (5.8) we have

$$\begin{aligned}\alpha(\lambda(v-u_0)) &\leq \langle w_\lambda^*, \eta(w_\lambda,u_0)\rangle + \phi(w_\lambda) - \phi(u_0)\\ &\leq \lambda\langle w_\lambda^*, \eta(v,u_0)\rangle + (1-\lambda)\langle w_\lambda^*, \eta(u_0,u_0)\rangle + \lambda\phi(v) + (1-\lambda)\phi(u_0) - \phi(u_0)\\ &= \lambda\left[\langle w_\lambda^*, \eta(v,u_0)\rangle + \phi(v) - \phi(u_0)\right],\end{aligned}$$

which leads to

$$\frac{\alpha(\lambda(v-u_0))}{\lambda} \leq \langle w_\lambda^*, \eta(v,u_0)\rangle + \phi(v) - \phi(u_0). \tag{5.9}$$

CASE 1. $(\mathcal{H}_{22})$ holds.

We shall prove next that u_0 is a strong solution of inequality problem (5.2). Let $u_0^* \in A(u_0)$ be arbitrarily fixed. Combining the fact that $w_\lambda \to u_0$ as $\lambda \downarrow 0$ with the fact that A is l.s.c. from K endowed with the strong topology into X^* endowed with the $w-$topology we deduce that for each $\lambda \in (0,1)$ we can find $w_\lambda^* \in A(w_\lambda)$ such that $w_\lambda^* \rightharpoonup u_0^*$ as $\lambda \downarrow 0$. Taking the superior limit in (5.9) as $\lambda \downarrow 0$ and keeping in mind (H_α^1) we get

$$\begin{aligned}0 &\leq \limsup_{\lambda\downarrow 0} \frac{\alpha(\lambda(v-u_0))}{\lambda}\\ &\leq \limsup_{\lambda\downarrow 0}\left[\langle w_\lambda^*, \eta(v,u_0)\rangle + \phi(v) - \phi(u_0)\right]\\ &= \langle u_0^*, \eta(v,u_0)\rangle + \phi(v) - \phi(u_0),\end{aligned}$$

which shows that u_0 is a strong solution of (5.2), since $v \in K$ and $u_0^* \in A(u_0)$ were arbitrarily fixed.

CASE 2. $(\mathcal{H}_{23})$ holds.

We shall prove that u_0 is a solution of (5.2). Reasoning as in the proof of Theorem 5.1-CASE 2 we infer that there exists $\bar{u}_0^* \in A(u_0)$ and a subnet $\{w_\lambda^*\}_{\lambda \in J}$ of $\{w_\lambda^*\}_{\lambda \in (0,1)}$ such that $w_\lambda^* \rightharpoonup \bar{u}_0^*$ as $\lambda \downarrow 0$. Combining this with relation (5.9) and hypothesis (H_α^1) we conclude that

$$\begin{aligned} 0 &\leq \limsup_{\lambda \downarrow 0} \frac{\alpha(\lambda(v-u_0))}{\lambda} \\ &\leq \limsup_{\lambda \downarrow 0} \left[\langle w_\lambda^*, \eta(v,u_0)\rangle + \phi(v) - \phi(u_0)\right] \\ &= \langle \bar{u}_0^*, \eta(v,u_0)\rangle + \phi(v) - \phi(u_0), \end{aligned}$$

which shows that u_0 is a solution of (5.2), since $v \in K$ was arbitrarily fixed.

□

Weakening even more the hypotheses by assuming that the set-valued map $A : K \to 2^{X^*}$ is *relaxed* $\eta - \alpha$ *quasimonotone* instead of being *relaxed* $\eta - \alpha$ *monotone* the existence of solutions for inequality problem (5.2) is an open problem in the case when K is nonempty bounded closed and convex. However, in this case we can prove the following existence result concerning inequality problem (5.3).

Theorem 5.4. (N.C., D.A. ION & C. LUPU [28]) *Let K be a nonempty bounded closed and convex subset of the real reflexive Banach space X. Let $A : K \to 2^{X^*}$ be a relaxed $\eta - \alpha$ quasimonotone map and assume that $(\mathcal{H}_{26})$ and $(\mathcal{H}_{28})$ hold. If in addition*

- $(\mathcal{H}_{22})$ *holds, then inequality problem (5.3) has at least one strong solution;*
- $(\mathcal{H}_{23})$ *holds, then inequality problem (5.3) has at least one solution.*

Proof. Define $G : K \to 2^X$ in the following way:

$$G(v) = \{u \in K : \ \langle v^*, \eta(v,u)\rangle \geq \alpha(v-u) \text{ for all } v^* \in A(v)\}.$$

First of all, let us observe that $v \in G(v)$ for all $v \in K$ and thus $G(v)$ is nonempty for all $v \in K$. Now, we prove that $G(v)$ is weakly closed for all $v \in K$. Let $\{u_\lambda\}_{\lambda \in I} \subset G(v)$ be a net such that u_λ converges weakly to some $u \in K$. Then, we have

$$\begin{aligned}
\alpha(v-u) &\leq \limsup \alpha(v-u_\lambda) \\
&\leq \limsup \langle v^*, \eta(v,u_\lambda)\rangle \\
&= \limsup\, [-\langle v^*, \eta(u_\lambda,v)\rangle] \\
&= -\liminf \langle v^*, \eta(u_\lambda,v)\rangle \\
&\leq -\langle v^*, \eta(u,v)\rangle \\
&= \langle v^*, \eta(v,u)\rangle,
\end{aligned}$$

for all $v^* \in A(v)$. It follows that $u \in G(v)$, so $G(v)$ is weakly closed.

CASE 1. G is a KKM map.

Since K is bounded closed and convex in X which is reflexive, it follows that K is weakly compact and thus $G(v)$ is weakly compact for all $v \in K$ as it is a weakly closed subset of K. Applying the KKM Theorem (see Chapter 1, Theorem 1.6), we have $\bigcap_{v\in K} G(v) \neq \emptyset$ and the set of solutions of problem (5.3) is nonempty. In order to see that let $u_0 \in \bigcap_{v\in K} G(v)$. This implies that for each $w \in K$ we have

$$\langle w^*, \eta(w,u)\rangle \geq \alpha(w-u), \qquad \text{for all } w^* \in A(w).$$

Let v be fixed in K and for $\lambda \in (0,1)$ define $w_\lambda = u_0 + \lambda(v-u_0)$. We infer that

$$\alpha(\lambda(v-u_0)) \leq \lambda\langle w_\lambda^*, \eta(v,u_0)\rangle,$$

for all $w_\lambda^* \in A(w_\lambda)$.

Employing the same arguments as in the previous proof we conclude that u_0 is a strong solution of inequality problem (5.3) if $(\mathcal{H}_{22})$ holds, while if $(\mathcal{H}_{23})$ holds then u_0 is a solution of inequality problem (5.3).

CASE 2. G is not a KKM map.

Consider $\{v_1, v_2, \ldots, v_N\} \subseteq K$ and $u_0 = \sum_{j=1}^N \lambda_j v_j$ with $\lambda_j \in [0,1]$ and $\sum_{j=1}^N \lambda_j = 1$ such that $u_0 \notin \bigcup_{j=1}^N G(v_j)$. The existence of such u_0 is guaranteed by the fact that G is not a KKM map. This implies that for all $j \in \{1, \ldots, N\}$ there exists $\bar{v}_j^* \in A(v_j)$ such that

$$\langle \bar{v}_j^*, \eta(v_j, u_0)\rangle < \alpha(v_j - u_0) \tag{5.10}$$

Now, we claim that there exists a neighborhood U of u_0 such that (5.10) takes place for all $w \in U \cap K$, that is

$$\langle \bar{v}_j^*, \eta(v_j, w)\rangle < \alpha(v_j - w), \quad \text{for all } w \in U \cap K.$$

Arguing by contradiction let us assume that for any neighborhood U of u_0 there exists an index $j_0 \in \{1, \ldots, N\}$ and an element $w_0 \in U \cap K$ such that

$$\langle v_{j_0}^*, \eta(v_{j_0}, w_0)\rangle \geq \alpha(v_{j_0} - w_0), \quad \text{for all } v_{j_0}^* \in A(v_{j_0}). \tag{5.11}$$

Choose $U = \bar{B}_X(u_0; \lambda)$ and for each $\lambda > 0$ one can find a $j_0 \in \{1, \ldots, N\}$ and $w_\lambda \in \bar{B}_X(u_0; \lambda) \cap K$ such that

$$\langle v_{j_0}^*, \eta(v_{j_0}, w_\lambda)\rangle \geq \alpha(v_j - w_\lambda) \quad \text{for all } v_{j_0}^* \in A(v_{j_0}).$$

Let us fix $v_{j_0}^* \in A(v_{j_0})$. Using the fact that $w_\lambda \to u_0$ as $\lambda \downarrow 0$ and taking the superior limit in the above relation, we obtain

$$\begin{aligned}
\alpha(v_{j_0} - u_0) &\leq \limsup_{\lambda \downarrow 0} \alpha(v_{j_0} - w_\lambda) \\
&\leq \limsup_{\lambda \downarrow 0} \langle v_{j_0}^*, \eta(v_{j_0}, w_\lambda)\rangle \\
&= -\liminf_{\lambda \downarrow 0} \langle v_{j_0}^*, \eta(w_\lambda, v_{j_0})\rangle \\
&\leq -\langle v_{j_0}^*, \eta(u_0, v_{j_0})\rangle \\
&= \langle v_{j_0}^*, \eta(v_{j_0}, u_0)\rangle,
\end{aligned}$$

which contradicts relation (5.10) and this contradiction completes the proof of the claim. Now, using the fact that A is relaxed $\eta - \alpha$ quasimonotne map, we prove that

$$\langle w^*, \eta(v_j, w)\rangle \leq 0, \quad \text{for all } w \in K \cap U, \text{ all } w^* \in A(w) \text{ and all } j \in \{1, \ldots, N\}. \tag{5.12}$$

In order to prove (5.12) let us assume by contradiction that there exists $w_0 \in K \cap U$, $w_0^* \in A(w_0)$ and $j_0 \in \{1, \ldots, N\}$ such that $\langle w_0^*, \eta(v_{j_0}, w_0)\rangle > 0$. From the fact that A is relaxed $\eta - \alpha$ quasimonotone it follows that

$$\langle v_{j_0}^*, \eta(v_{j_0}, w_0)\rangle \geq \alpha(v_{j_0} - w_0), \quad \text{for all } v_{j_0}^* \in A(v_{j_0}),$$

which contradicts the fact that (5.10) holds for all $w \in U \cap K$ and all $j \in \{1, \ldots, N\}$. On the other hand, for arbitrary fixed $w \in K \cap U$ and $\bar{w}^* \in A(w)$ we have

$$\begin{aligned} \langle \bar{w}^*, \eta(u_0, w)\rangle &= \left\langle \bar{w}^*, \eta\left(\sum_{j=1}^N \lambda_j v_j, w\right)\right\rangle \\ &\leq \sum_{j=1}^N \lambda_j \langle \bar{w}^*, \eta(v_j, w)\rangle \\ &\leq 0. \end{aligned}$$

Thus, we obtain

$$0 \leq \langle \bar{w}^*, -\eta(u_0, w)\rangle = \langle \bar{w}^*, \eta(w, u_0)\rangle.$$

But $\bar{w}^* \in A(w)$ was choosen arbitrary and thus for each $w \in U \cap K$ we have

$$\langle w^*, \eta(w, u_0)\rangle \geq 0, \quad \text{for all } w^* \in A(w) \tag{5.13}$$

We shall prove next that u_0 solves inequality problem (5.3). Consider $v \in K$ to be arbitrary fixed.

CASE 2.1. $v \in U$.

In this case the entire line segment $(u_0, v) = \{u_0 + \lambda(v - u_0) : \lambda \in (0, 1)\}$ is contained in $U \cap K$ and, according to (5.13), for each $w_\lambda \in (u_0, v)$ and each $w_\lambda^* \in A(w_\lambda)$ we have

$$\begin{aligned} 0 &\leq \langle w_\lambda^*, \eta(w_\lambda, u_0)\rangle \\ &\leq \lambda\langle w_\lambda^*, \eta(v, u_0)\rangle + (1 - \lambda)\langle w_\lambda^*, \eta(u_0, u_0)\rangle \\ &= \lambda\langle w_\lambda^*, \eta(v, u_0)\rangle \end{aligned}$$

Let us assume that $(\mathcal{H}_{22})$ holds and fix $u^* \in A(u)$. Then for each $\lambda \in (0,1)$ we can determine $\bar{w}^*_\lambda \in A(w_\lambda)$ such that $\bar{w}^*_\lambda \rightharpoonup u^*$ as $\lambda \downarrow 0$.

If $(\mathcal{H}_{23})$ holds, then there exists $\bar{u}^*_0 \in A(u_0)$ for which we can determine a subnet $\{w^*_\lambda\}_{\lambda \in J}$ of $\{w^*_\lambda\}_{\lambda \in (0,1)}$ such that $w^*_\lambda \rightharpoonup \bar{u}^*_0$ in X^* as $\lambda \downarrow 0$.

Dividing by $\lambda > 0$ the above relation and taking into account the previous observation we conclude (after passing to the limit as $\lambda \downarrow 0$) that u_0 is a strong solution of problem (5.3) if $(\mathcal{H}_{22})$ holds (u_0 is a solution of problem (5.3) if $(\mathcal{H}_{23})$ holds).

CASE 2.2. $v \in K \setminus U$.

Since K is convex and $u_0, v \in K$, then we have that $(u_0, v) \subseteq K$. From $v \notin U$ there exists $\lambda_0 \in (0,1)$ such that $v_0 = u_0 + \lambda_0(v - u_0) \in (u_0, v)$ and has the property that the entire line segment (u_0, v_0) is contained in $U \cap K$. Thus, for each $\lambda \in (0,1)$ the element $w_\lambda = u_0 + \lambda(v_0 - u_0) \in K \cap V$, but $v_0 = u_0 + \lambda_0(v - u_0)$, hence $w_\lambda = u_0 + \lambda_0\lambda(v-u) \in K \cap V$ and $w_\lambda \to u_0$ as $\lambda \downarrow 0$. Applying the same arguments as in CASE 2.1 we infer that u_0 is a strong solution of problem (5.3) if $(\mathcal{H}_{22})$ holds (u_0 is a solution of problem (5.3) if $(\mathcal{H}_{23})$ holds) and this completes our proof.

□

Let us turn our attention towards the case when K is a unbounded closed and convex subset of X. We shall establish next some sufficient conditions for the existence of solutions of problems (5.2) and (5.3). For every $r > 0$ we define

$$K_r = \{u \in K : \ \|u\| \leq r\} \quad \text{and} \quad K^-_r = \{u \in K : \ \|u\| < r\},$$

and consider the problems

Find $u_r \in K_r \cap \mathcal{D}(\phi)$ such that

$$\exists u^*_r \in A(u_r): \quad \langle u^*_r, \eta(v, u_r)\rangle + \phi(v) - \phi(u_r) \geq 0, \quad \text{for all } v \in K_r, \tag{5.14}$$

and

Find $u_r \in K_r$ such that

$$\exists u_r^* \in A(u_r): \quad \langle u_r^*, \eta(v, u_r)\rangle \geq 0 \quad \text{for all } v \in K_r. \tag{5.15}$$

It is clear from above that the solution sets of problems (5.14) and (5.15) are nonempty. We have the following characterization for the existence of solutions in the case of unbounded closed and convex subsets.

Theorem 5.5. (N.C., D.A. ION & C. LUPU [28]) *Assume that the same hypotheses as in Theorem 5.3 hold without the assumption of boundedness of K. Then each of the following conditions is sufficient for inequality problem (5.2) to admit at least one strong solution (solution):*

($\mathcal{H}_{29}$) *there exists $r_0 > 0$ and $u_0 \in K_{r_0}^-$ such that u_{r_0} solves (5.14).*

($\mathcal{H}_{30}$) *there exists $r_0 > 0$ such that for each $u \in K \setminus K_{r_0}$ there exists $\bar{v} \in K_{r_0}$ such that*

$$\langle u^*, \eta(\bar{v}, u)\rangle + \phi(\bar{v}) - \phi(u) \leq 0, \quad \text{for all } u^* \in A(u).$$

($\mathcal{H}_{31}$) *there exists $\bar{u} \in K$ and a function $c : \mathbb{R}_+ \to \mathbb{R}_+$ with the property that $\lim_{r\to+\infty} c(r) = +\infty$ such that*

$$\inf_{u^* \in A(u)} \langle u^*, \eta(u, \bar{u})\rangle \geq c(\|u\|)\|u\|, \quad \text{for all } u \in K.$$

Chapter 6

A system of nonlinear hemivariational inequalities

6.1 Formulation of the problem and existence results

Let n be a positive integer, let $X_1, \ldots, X_n$ be real reflexive Banach spaces and let $Y_1, \ldots, Y_n$ be real Banach spaces such that there exist linear and compact operators $T_k : X_k \to Y_k$, for $k \in \{1, \ldots, n\}$.

Our aim is to study the following *system of nonlinear hemivariational inequalities* :

(SNHI) Find $(u_1, \ldots, u_n) \in K_1 \times \ldots \times K_n$ such that for all $(v_1, \ldots, v_n) \in K_1 \times \ldots \times K_n$

$$\begin{cases} \psi_1(u_1, \ldots, u_n, v_1) + J^0_{,1}(\hat{u}_1, \ldots, \hat{u}_n; \hat{v}_1 - \hat{u}_1) \geq \langle F_1(u_1, \ldots, u_n), v_1 - u_1 \rangle_{X_1} \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \psi_n(u_1, \ldots, u_n, v_n) + J^0_{,n}(\hat{u}_1, \ldots, \hat{u}_n; \hat{v}_n - \hat{u}_n) \geq \langle F_n(u_1, \ldots, u_n), v_n - u_n \rangle_{X_n}, \end{cases}$$

where for each $k \in \{1, \ldots, n\}$

- $K_k \subseteq X_k$ is a nonempty closed and convex subset;
- $\psi_k : X_1 \times \ldots \times X_k \times \ldots \times X_n \times X_k \to \mathbb{R}$ is a nonlinear functional;
- $J : Y_1 \times \ldots \times Y_n \to \mathbb{R}$ is a regular locally Lipschitz functional;

- $F_k : X_1 \times \ldots \times X_k \times \ldots \times X_n \to X_k^*$ is a nonlinear operator;
- $\hat{u}_k = T_k(u_k)$.

In order to establish the existence of at least one solution for problem **(SNHI)** we shall assume fulfilled the following hypotheses:

$(\mathcal{H}_{32})$ For each $k \in \{1, \ldots, n\}$, the functional $\psi_k : X_1 \times \ldots \times X_k \times \ldots \times X_n \times X_k \to \mathbb{R}$ satisfies

(i) $\psi_k(u_1, \ldots, u_k, \ldots, u_n, u_k) = 0$ for all $u_k \in X_k$;

(ii) For each $v_k \in X_k$ the mapping $(u_1, \ldots, u_n) \mapsto \psi_k(u_1, \ldots, u_n, v_k)$ is weakly upper semicontinuous;

(iii) For each $(u_1, \ldots, u_n) \in X_1 \times \ldots \times X_n$ the mapping $v_k \mapsto \psi_k(u_1, \ldots, u_n, v_k)$ is convex.

$(\mathcal{H}_{33})$ For each $k \in \{1, \ldots, n\}$, $F_k : X_1 \times \ldots \times X_k \times \ldots \times X_n \to X_k^*$ is a nonlinear operator such that

$$\liminf_{m \to \infty} \left\langle F_k\left(u_1^m, \ldots, u_n^m\right), v_k - u_k^m \right\rangle_{X_k} \geq \left\langle F_k\left(u_1, \ldots, u_n\right), v_k - u_k \right\rangle_{X_k}$$

whenever $(u_1^m, \ldots, u_n^m) \rightharpoonup (u_1, \ldots, u_n)$ as $m \to \infty$ and $v_k \in X_k$ is fixed.

The first main result of this chapter refers to the case when the sets K_k are bounded, closed and convex and it is given by the following theorem.

Theorem 6.1. (N.C. & C. VARGA [38]) *For each $k \in \{1, \ldots, n\}$ let $K_k \subset X_k$ be a nonempty, bounded, closed and convex set and let us assume that conditions $(\mathcal{H}_{32})$-$(\mathcal{H}_{33})$ hold. Then, the system of nonlinear hemivariational inequalities* **(SNHI)** *admits at least one solution.*

The existence of solutions for our system will be a direct consequence of the fact that the inequality formulated below admits solutions. Let us introduce the following notations:

- $X = X_1 \times \ldots \times X_n$, $K = K_1 \times \ldots \times K_n$ and $Y = Y_1 \times \ldots \times Y_n$;
- $u = (u_1, \ldots, u_n)$ and $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_n)$;
- $\Psi : X \times X \to \mathbb{R}, \quad \Psi(u, v) = \sum\limits_{k=1}^{n} \psi_k(u_1, \ldots, u_k, \ldots, u_n, v_k)$;

- $F : X \to X^*, \quad \langle Fu, v\rangle_X = \sum_{k=1}^{n} \langle F_k(u_1, \ldots, u_n), v_k\rangle_{X_k}$.

and formulate the following vector hemivariational inequality

(VHI) Find $u \in K$ such that

$$\Psi(u, v) + J^0(\hat{u}; \hat{v} - \hat{u}) \geq \langle Fu, v - u\rangle_X, \quad \text{for all } v \in K.$$

Remark 6.1. *If* $(\mathcal{H}_{32})$-(i) *holds, then any solution* $u^0 = (u_1^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_n$ *of the vector hemivariational inequality* **(VHI)** *is also a solution of the system* **(SNHI)**.

Indeed, if for a $k \in \{1, \ldots, n\}$ we fix $v_k \in K_k$ and for $j \neq k$ we consider $v_j = u_j^0$, using Proposition 1.5 and the fact that u^0 solves **(VHI)** we obtain

$$\begin{aligned}
\left\langle F_k\left(u_1^0, \ldots, u_n^0\right), v_k - u_k^0\right\rangle_{X_k} &= \sum_{j=1}^{n} \left\langle F_j\left(u_1^0, \ldots, u_n^0\right), v_j - u_j^0\right\rangle_{X_j} \\
&= \left\langle Fu^0, v - u^0\right\rangle_X \\
&\leq \Psi\left(u^0, v\right) + J^0\left(\hat{u}^0; \hat{v} - \hat{u}^0\right) \\
&\leq \sum_{j=1}^{n} \psi_j\left(u_1^0, \ldots, u_j^0, \ldots, u_n^0, v_j\right) + \sum_{j=1}^{n} J_{,j}^0\left(\hat{u}_1^0, \ldots, \hat{u}_n^0; \hat{v}_j - \hat{u}_j^0\right) \\
&= \psi_k\left(u_1^0, \ldots, u_k^0, \ldots, u_n^0, v_k\right) + J_{,k}^0\left(\hat{u}_1^0, \ldots, \hat{u}_n^0; \hat{v}_k - \hat{u}_k^0\right).
\end{aligned}$$

As $k \in \{1, \ldots, n\}$ and $v_k \in K_k$ were arbitrarily fixed, we conclude that $(u_1^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_n$ is a solution of our system **(SNHI)**.

Proof of Theorem 6.1. According to Remark 6.1 it suffices to prove that problem **(VHI)** admits a solution. With this end in view we consider the set $\mathcal{A} \subset K \times K$ as follows

$$\mathcal{A} = \left\{(v, u) \in K \times K : \ \Psi(u, v) + J^0(\hat{u}; \hat{v} - \hat{u}) - \langle Fu, v - u\rangle_X \geq 0\right\}.$$

We prove next that the set $\mathcal{A}$ satisfies the conditions required in Theorem 1.7 for the weak topology of the space X, that is,

- For each $v \in K$ the set $\mathcal{N}(v) = \{u \in K : \ (v, u) \in \mathcal{A}\}$ is weakly closed in K.

- For each $u \in K$ the set $\mathcal{M}(u) = \{v \in K : \ (v, u) \notin \mathcal{A}\}$ is either convex or empty.

- $(u, u) \in \mathcal{A}$ for each $u \in K$.
- The set $B = \{u \in K : (v, u) \in \mathcal{A} \text{ for all } v \in K\}$ is compact.

We are now able to apply Lin's theorem and conclude that there exists $u^0 \in B \subseteq K$ such that $K \times \{u^0\} \subset \mathcal{A}$. This means that

$$\Psi(u^0, v) + J^0(\hat{u}^0; \hat{v} - \hat{u}^0) \geq \langle Fu^0, v - u^0 \rangle_X, \quad \text{for all } v \in K,$$

therefore u^0 solves problem **(VHI)** and, accordingly to Remark 6.1, it is a solution of our system of nonlinear hemivariational inequalities **(SNHI)**, the proof of Theorem 6.1 being now complete. □

We will show next that if we change the hypotheses on the nonlinear functionals ψ_k we obtain another existence result for our inequality system. Let us consider that instead of $(\mathcal{H}_{32})$ we have the following set of hypotheses

$(\mathcal{H}_{34})$ For each $k \in \{1, \ldots, n\}$, the functional $\psi_k : X_1 \times \ldots \times X_k \times \ldots \times X_n \times X_k \to \mathbb{R}$ satisfies

(i) $\psi_k(u_1, \ldots, u_k, \ldots, u_n, u_k) = 0$ for all $u_k \in X_k$;

(ii) For each $k \in \{1, \ldots, n\}$ and any pair $(u_1, \ldots, u_k, \ldots, u_n),\ (v_1, \ldots, v_k, \ldots, v_n) \in X_1 \times \ldots \times X_k \times \ldots \times X_n$ we have

$$\psi_k(u_1, \ldots, u_k, \ldots, u_n, v_k) + \psi_k(v_1, \ldots, v_k, \ldots, v_n, u_k) \geq 0;$$

(iii) For each $(u_1, \ldots, u_n) \in X_1 \times \ldots \times X_n$ the mapping $v_k \mapsto \psi_k(u_1, \ldots, u_n, v_k)$ is weakly lower semicontinuous;

(iv) For each $v_k \in X_k$ the mapping $(u_1, \ldots, u_n) \mapsto \psi_k(u_1, \ldots, u_n, v_k)$ is concave.

We are now in position to state our second main result of this chapter, which concerns the case when the sets K_k are bounded, closed and convex for each $k \in \{1, \ldots, n\}$.

Theorem 6.2. (N.C. & C. VARGA [38]) *For each $k \in \{1, \ldots, n\}$ let $K_k \subset X_k$ be a nonempty, bounded, closed and convex set and let us assume that conditions (H2)-(H3) hold. Then, the system of nonlinear hemivariational inequalities* **(SNHI)** *admits at least one solution.*

In order to prove Theorem 6.2 we will need the following lemma.

Lemma 6.1. *Assume that (H3) holds. Then*

(a) $\Psi(u, v) + \Psi(v, u) \geq 0$ *for all* $u, v \in X$;

(b) For each $v \in X$ *the mapping* $u \mapsto -\Psi(v, u)$ *is weakly upper semicontinuous;*

(c) For each $u \in X$ *the mapping* $v \mapsto -\Psi(v, u)$ *is convex.*

Proof of Theorem 6.2. Let us consider the set $\mathcal{A} \subset K \times K$ defined by

$$\mathcal{A} = \{(v, u) \in K \times K : -\Psi(v, u) + J^0(\hat{u}; \hat{v} - \hat{u}) - \langle Fu, v - u \rangle_X \geq 0\}.$$

Lemma 6.1 ensures that we can follow the same steps as in the proof of Theorem 6.1 to conclude that the conditions required in Lin's theorem are fulfilled. Thus we get the existence of an element $u^0 \in K$ such that $K \times \{u^0\} \subset \mathcal{A}$ which is equivalent to

$$-\Psi(v, u^0) + J^0(\hat{u}^0; \hat{v} - \hat{u}^0) \geq \langle Fu^0, v - u^0 \rangle_X \quad \text{for all } v \in K. \tag{6.1}$$

On the other hand Lemma 6.1 tells us that

$$\Psi(u^0, v) + \Psi(v, u^0) \geq 0, \quad \text{for all } v \in K. \tag{6.2}$$

Combining relations (6.1) and (6.2) we deduce that u^0 solves problem **(VHI)**, therefore it is a solution of problem **(SNHI)**. □

Let us consider now the case when at least one of the subsets K_k is unbounded and either conditions $(\mathcal{H}_{32})$-$(\mathcal{H}_{33})$ or $(\mathcal{H}_{33})$-$(\mathcal{H}_{34})$ hold. We shall denote next by $\bar{B}_E(0; R)$ the closed ball of the space E centered in the origin and of radius R, that is

$$\bar{B}_E(0; R) = \{v \in E : \|v\|_E \leq R\}.$$

Let $R > 0$ be such that the set $K_{k,R} = K_k \cap \bar{B}_{X_k}(0; R)$ is nonempty for every $k \in \{1, \ldots, n\}$. Then, for each $k \in \{1, \ldots, n\}$ the set $K_{k,R}$ is nonempty, bounded, closed and convex and according to Theorem 6.1 or Theorem 6.2 the following problem

(SR) Find $(u_1, \ldots, u_n) \in K_{1,R} \times \ldots \times K_{n,R}$ such that for all $(v_1, \ldots, v_n) \in K_{1,R} \times \ldots \times K_{n,R}$

$$\begin{cases} \psi_1(u_1, \ldots, u_n, v_1) + J^0_{,1}(\hat{u}_1, \ldots, \hat{u}_n; \hat{v}_1 - \hat{u}_1) \geq \langle F_1(u_1, \ldots, u_n), v_1 - u_1 \rangle_{X_1} \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \psi_n(u_1, \ldots, u_n, v_n) + J^0_{,n}(\hat{u}_1, \ldots, \hat{u}_n; \hat{v}_n - \hat{u}_n) \geq \langle F_n(u_1, \ldots, u_n), v_n - u_n \rangle_{X_n}, \end{cases}$$

admits at least one solution.

We have the following existence result concerning the case of at least one unbounded subset.

Theorem 6.3. (N.C. & C. VARGA [38]) *For each $k \in \{1, \ldots, n\}$ let $K_k \subset X_k$ be a nonempty, closed and convex set and assume that there exists at least one index $k_0 \in \{1, \ldots, n\}$ such that K_{k_0} is unbounded. Assume in addition that either $(\mathcal{H}_{32})$-$(\mathcal{H}_{33})$ or $(\mathcal{H}_{33})$-$(\mathcal{H}_{34})$ hold. Then, the system of nonlinear hemivariational inequalities* **(SNHI)** *admits at least one solution if and only if the following condition holds true:*

$(\mathcal{H}_{35})$ *there exists $R > 0$ such that $K_{k,R}$ is nonempty for every $k \in \{1, \ldots, n\}$ and at least one solution $(u^0_1, \ldots, u^0_n)$ of problem* **(SR)** *satisfies*

$$u^0_k \in B_{X_k}(0; R), \quad \text{for all } k \in \{1, \ldots, n\}.$$

Corollary 6.1. *For each $k \in \{1, \ldots, n\}$ let $K_k \subset X_k$ be a nonempty, closed and convex set and assume that there exists at least one index $k_0 \in \{1, \ldots, n\}$ such that K_{k_0} is unbounded. Assume in addition that either $(\mathcal{H}_{32})$-$(\mathcal{H}_{33})$ or $(\mathcal{H}_{33})$-$(\mathcal{H}_{34})$ hold. Then, a sufficient condition for* **(SNHI)** *to admit at least one solution is*

$(\mathcal{H}_{36})$ *there exists $R_0 > 0$ such that K_{k,R_0} is nonempty for every $k \in \{1, \ldots, n\}$ and for each $(u_1, \ldots, u_n) \in K_1 \times \ldots \times K_n \setminus K_{1,R_0} \times \ldots \times K_{n,R_0}$ there exists $(v^0_1, \ldots, v^0_n) \in K_{1,R_0} \times \ldots \times K_{n,R_0}$ such that*

$$\psi_k(u_1, \ldots, u_n, v^0_k) + J^0_{,k}(\hat{u}_1, \ldots, \hat{u}_n; \hat{v}^0_k - \hat{u}_k) < \langle F_k(u_1, \ldots, u_n), v^0_k - u_k \rangle_{X_k}, \tag{6.3}$$

for all $k \in \{1, \ldots, n\}$.

In order to simplify some computations let us assume next that $0 \in K_k$ for each $k \in \{1, \ldots, n\}$. In this case $K_{k,R} \neq \emptyset$ for every $k \in \{1, \ldots, n\}$ and every $R > 0$.

Corollary 6.2. *For each $k \in \{1, \ldots, n\}$ let $K_k \subset X_k$ be a nonempty, closed and convex set and assume that there exists at least one index $k_0 \in \{1, \ldots, n\}$ such that K_{k_0} is unbounded and either $(\mathcal{H}_{32})$-$(\mathcal{H}_{33})$ or $(\mathcal{H}_{33})$-$(\mathcal{H}_{34})$ hold. Assume, in addition, that for each $k \in \{1, \ldots, n\}$ the following conditions hold*

$(\mathcal{H}_{37})$ *There exists a function $c : \mathbb{R}_+ \to \mathbb{R}_+$ with the property that $\lim_{t\to\infty} c(t) = +\infty$ such that*

$$-\sum_{k=1}^{n} \psi_k(u_1, \ldots, u_k, \ldots, u_n, 0) \geq c(\|u\|_X)\|u\|_X,$$

for all $(u_1, \ldots, u_n) \in X_1 \times \ldots \times X_n$, where $u = (u_1, \ldots, u_n)$ and $\|u\|_X = \left(\sum_{k=1}^{n} \|u_k\|_{X_k}^2\right)^{1/2}$;

$(\mathcal{H}_{38})$ *There exists $M_k > 0$ such that*

$$J_{,k}^0(w_1, \ldots, w_k, \ldots, w_n; -w_k) \leq M_k\|w_k\|_{Y_k}, \quad \textit{for all } (w_1, \ldots, w_n) \in Y_1 \times \ldots \times Y_n;$$

$(\mathcal{H}_{39})$ *There exists $m_k > 0$ such that*

$$\|F_k(u_1, \ldots, u_k, \ldots, u_n)\|_{X_k^*} \leq m_k, \quad \textit{for all } (u_1, \ldots, u_n) \in X_1 \times \ldots \times X_n.$$

Then the system **(SNHI)** *admits at least one solution.*

6.2 Existence of Nash generalized derivative points

Let $E_1, \ldots, E_n$ be Banach spaces and for each $k \in \{1, \ldots, n\}$ let K_k be a nonempty subset of E_k. We also assume that $g_k : K_1 \times \ldots \times K_k \times \ldots \times K_n \to \mathbb{R}$ are given functionals. We recall below the notion of Nash equilibrium point (see [95, 96]).

Definition 6.1. *An element $(u_1, \ldots, u_k, \ldots, u_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ is a Nash equilibrium point for the functionals $g_1, \ldots, g_k, \ldots, g_n$, if for every $k \in \{1, \ldots, n\}$ and every $(v_1, \ldots, v_k, \ldots, v_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ we have*

$$g_k(u_1, \ldots, v_k, \ldots, u_n) \geq g_k(u_1, \ldots, u_k, \ldots, u_n).$$

Let $D_k \subset E_k$ be an open set such that $K_k \subset D_k$ for all $k \in \{1, \ldots, n\}$. For each $k \in \{1, \ldots, n\}$ we consider the functional $g_k : K_1 \times \ldots \times D_k \times \ldots \times K_n \to \mathbb{R}$ such that $u_k \mapsto g_k(u_1, \ldots, u_k, \ldots, u_n)$ is locally Lipschitz. The following notion was introduced by Kristály in [65].

Definition 6.2. *An element $(u_1, \ldots, u_k, \ldots, u_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ is a Nash generalized derivative point for the functionals $g_1, \ldots, g_k, \ldots, g_n$ if for every $k \in \{1, \ldots, n\}$ and every $(v_1, \ldots, v_k, \ldots, v_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ we have*

$$g^0_{k,k}(u_1, \ldots, u_k, \ldots, u_n; v_k - u_k) \geq 0.$$

We point out the fact that the above definition coincides with the notion of *Nash stationary point* introduced by Kassay, Kolumbán and Páles in [62] if every functional g_k is differentiable with respect to the k^{th} variable. Moreover, every Nash equilibrium point is a Nash generalized derivative point.

1. For each $k \in \{1, \ldots, n\}$ let $D_k \subseteq X_k$ be an open and consider the functional

$$g_k : K_1 \times \ldots \times D_k \times \ldots \times K_n \to \mathbb{R},$$

such that g_k is locally Lipschitz with respect to the k^{th} variable and for each $v_k \in X_k$ the mapping $(u_1, \ldots, u_k, \ldots, u_n) \mapsto g^0_{k,k}(u_1, \ldots, u_k, \ldots, u_n; v_k)$ is weakly upper semicontinuous. Let us choose next $J \equiv 0$, $F_k \equiv 0$ and

$$\psi_k(u_1, \ldots, u_k, \ldots, u_n, v_k) = g^0_{k,k}(u_1, \ldots, u_k, \ldots, u_n; v_k - u_k).$$

(i) If for each $k \in \{1, \ldots, n\}$ the set $K_k \subset X_k$ is nonempty, bounded, closed and convex, then Theorem 6.1 implies that there exists at least one point

$$(u^0_1, \ldots, u^0_k, \ldots, u^0_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$$

such that for all $(v_1, \ldots, v_k, \ldots, v_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ we have

$$g^0_{k,k}(u^0_1, \ldots, u^0_k, \ldots, u^0_n; v_k - u^0_k) \geq 0, \quad \text{for all } k \in \{1, \ldots, n\},$$

that is, $(u^0_1, \ldots, u^0_k, \ldots, u^0_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ is a Nash generalized derivative point for the functionals $g_1, \ldots, g_k, \ldots g_n$.

(ii) Let us assume now that the sets K_k are nonempty, closed and convex and at least one of them is unbounded. Assume in addition that there exists $R_0 > 0$ such that K_{k,R_0} is nonempty for every $k \in \{1, \ldots, n\}$ and for each

$$(u_1, \ldots, u_k, \ldots, u_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n \setminus K_{1,R_0} \times \ldots \times K_{k,R_0} \times \ldots \times K_{n,R_0}$$

there exists $(v_1^0, \ldots, v_k^0, \ldots, v_n^0) \in K_{1,R_0} \times \ldots \times K_{k,R_0} \times \ldots \times K_{n,R_0}$ such that

$$g_{k,k}^0(u_1, \ldots, u_k, \ldots, u_n; v_k^0 - \hat{u}_k) < 0, \text{ for all } k \in \{1, \ldots, n\}.$$

Then, according to Corollary 6.1, there exists at least one point $(u_1^0, \ldots, u_k^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ such that for all $(v_1, \ldots, v_k, \ldots, v_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ we have

$$g_{k,k}^0(u_1^0, \ldots, u_k^0, \ldots, u_n^0; v_k - u_k^0) \geq 0, \quad \text{ for all } k \in \{1, \ldots, n\},$$

which means that $(u_1^0, \ldots, u_k^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ is a Nash generalized derivative point for the functionals $g_1, \ldots, g_k, \ldots, g_n$.

(iii) Let us assume now that the sets K_k are nonempty, closed and convex and at least one of them is unbounded. Assume in addition that there exists a function $c : \mathbb{R}_+ \to \mathbb{R}_+$ with the property that $\lim_{t \to \infty} c(t) = +\infty$ such that

$$-\sum_{k=1}^{n} g_{k,k}^0(u_1, \ldots, u_k, \ldots, u_n; -u_k) \geq c(\|u\|_X)\|u\|_X,$$

for all $(u_1, \ldots, u_n) \in K_1 \times \ldots \times K_n$.

Then, according to Corollary 6.2, there exists at least one point $(u_1^0, \ldots, u_k^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ such that for all $(v_1, \ldots, v_k, \ldots, v_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ we have

$$g_{k,k}^0(u_1^0, \ldots, u_k^0, \ldots, u_n^0; v_k - u_k^0) \geq 0,$$

for all $k \in \{1, \ldots, n\}$, which means that $(u_1^0, \ldots, u_k^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ is a Nash generalized derivative point for the functionals $g_1, \ldots, g_k, \ldots g_n$.

2. Let us consider that for each $k \in \{1, \ldots, n\}$ we have $\psi_k \equiv 0$, $J \equiv 0$ and $F_k : X_1 \times \ldots \times X_k \times \ldots \times X_n \to X_k^*$ a nonlinear operator such that (H2) holds.

   (i) For each $k \in \{1, \ldots, n\}$ we assume that the set $K_k \subset X_k$ is nonempty, bounded, closed and convex. Then Theorem 6.1 implies that there exists at least one point $(u_1^0, \ldots, u_k^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ such that for all $(v_1, \ldots, v_k, \ldots, v_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ we have

$$-\langle F_k(u_1^0, \ldots, u_k^0, \ldots, u_n^0), v_k - u_k^0 \rangle_{X_k} \geq 0, \quad \text{for all } k \in \{1, \ldots, n\},$$

   which means that $(u_1^0, \ldots, u_k^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ is a Nash stationary point for the functionals $g_1, \ldots, g_k, \ldots, g_n$, where $g_k : K_1 \times \ldots \times X_k \times \ldots \times K_n \to \mathbb{R}$ is differentiable with respect to the k^{th} variable and $g'_{k,k} = -\tilde{F}_k$ (here $\tilde{F}_k$ is the restriction of F_k to $K_1 \times \ldots \times X_k \times \ldots \times K_n$).

   (ii) Let us assume now that the sets K_k are nonempty, closed and convex and at least one of them is unbounded. Assume in addition that there exists $R_0 > 0$ such that K_{k,R_0} is nonempty for every $k \in \{1, \ldots, n\}$ and for each $(u_1, \ldots, u_k, \ldots, u_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n \setminus K_{1,R_0} \times \ldots \times K_{k,R_0} \times \ldots \times K_{n,R_0}$ there exists $(v_1^0, \ldots, v_k^0, \ldots, v_n^0) \in K_{1,R_0} \times \ldots \times K_{k,R_0} \times \ldots \times K_{n,R_0}$ such that

$$\langle F_k(u_1, \ldots, u_k, \ldots, u_n), v_k^0 - u_k \rangle_{X_k} > 0, \quad \text{for all } k \in \{1, \ldots, n\}.$$

   Then, according to Corollary 6.1, there exists at least one point $(u_1^0, \ldots, u_k^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ such that for all $(v_1, \ldots, v_k, \ldots, v_n) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ we have

$$-\langle F_k(u_1^0, \ldots, u_k^0, \ldots, u_n^0), v_k - u_k^0 \rangle_{X_k} \geq 0, \quad \text{for all } k \in \{1, \ldots, n\},$$

   which means that $(u_1^0, \ldots, u_k^0, \ldots, u_n^0) \in K_1 \times \ldots \times K_k \times \ldots \times K_n$ is a Nash stationary point for the functionals $g_1, \ldots, g_n$, where $g_k : K_1 \times \ldots \times X_k \times \ldots \times K_n \to \mathbb{R}$ is differentiable with respect to the k^{th} variable and $g'_{k,k} = -\tilde{F}_k$.

Chapter 7

Weak solvability for some contact problems

7.1 Frictional problems for piezoelectric bodies in contact with a conductive foundation

This subsection focuses on the weak solvability of a mechanical model describing the contact between a piezoelectric body and a conductive foundation. The piezoelectric effect is characterized by the coupling between the mechanical and the electrical properties of the materials. This coupling leads to the appearance of electric potential when mechanical stress is present and, conversely, mechanical stress is generated when electric potential is applied.

Before describing the problem let us first present some notations and preliminary material which will be used throughout this subsection.

Let m be a positive integer and denote by $\mathcal{S}_m$ the linear space of second order symmetric tensors on $\mathbb{R}^m$ ($\mathcal{S}_m = \mathbb{R}_s^{m\times m}$). We recall that the inner product and the corresponding norm on $\mathcal{S}_m$ are given by

$$\tau : \sigma = \tau_{ij}\sigma_{ij}, \quad \|\tau\|_{\mathcal{S}_m} = \sqrt{\tau : \tau}, \quad \text{for all } \tau, \sigma \in \mathcal{S}_m.$$

Here, and hereafter the summation over repeated indices is used, all indices running from 1 to

m.

Let $\Omega \subset \mathbb{R}^m$ be an open bounded subset with a Lipschitz boundary Γ and let ν denote the outward unit normal vector to Γ. We introduce the spaces

$$\begin{aligned}
&H = L^2(\Omega;\mathbb{R}^m),\\
&\mathcal{H} = \big\{\tau = (\tau_{ij}) :\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\big\} = L^2(\Omega;\mathcal{S}_m),\\
&H_1 = \{u \in H :\ \varepsilon(u) \in \mathcal{H}\} = H^1(\Omega;\mathbb{R}^m),\\
&\mathcal{H}_1 = \{\tau \in \mathcal{H} :\ \text{Div}\ \tau \in H\},
\end{aligned}$$

where $\varepsilon : H_1 \to \mathcal{H}$ and Div $: \mathcal{H}_1 \to H$ denote the deformation and the divergence operators, defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)),\ \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad \text{Div}\ \tau = \left(\frac{\partial \tau_{ij}}{\partial x_j}\right),$$

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are Hilbert spaces endowed with the following inner products

$$\begin{aligned}
(u,v)_H &= \int_\Omega u_i v_i\ dx, \qquad (\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma : \tau\ dx,\\
(u,v)_{H_1} &= (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \qquad (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (\text{Div}\ \sigma, \text{Div}\ \tau)_H .
\end{aligned}$$

The associated norms in H, $\mathcal{H}$, H_1, $\mathcal{H}_1$ will be denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively.

Given $v \in H_1$ we denote by v its trace γv on Γ, where $\gamma : H^1(\Omega;\mathbb{R}^m) \to H^{1/2}(\Gamma;\mathbb{R}^m) \subset L^2(\Gamma;\mathbb{R}^m)$ is the Sobolev trace operator. Given $v \in H^{1/2}(\Gamma;\mathbb{R}^m)$ we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ, that is $v_\nu = v\cdot\nu$ and $v_\tau = v - v_\nu \nu$. Similarly, for a regular tensor field $\sigma : \Omega \to \mathcal{S}_m$, we define its normal and tangential components to be the normal and the tangential components of the Cauchy vector $\sigma\nu$, that is $\sigma_\nu = (\sigma\nu)\cdot\nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$. Recall that the following Green formula holds:

$$(\sigma,\varepsilon(v))_{\mathcal{H}} + (\text{Div}\ \sigma, v)_H = \int_\Gamma \sigma\nu \cdot v\ d\Gamma, \quad \text{for all } v \in H_1. \tag{7.1}$$

We shall describe next the model for which we shall derive a variational formulation. Let us consider body $\mathcal{B}$ made of a piezoelectric material which initially occupies an open bounded subset $\Omega \subset \mathbb{R}^m$ ($m = 2,3$) with a smooth boundary $\partial\Omega = \Gamma$. The body is subjected to volume forces of density f_0 and has volume electric charges of density q_0, while on the boundary we impose mechanical and electrical constraints. In order to describe these constraints we consider two partitions of Γ: the first partition is given by three mutually disjoint open parts Γ_1, Γ_2 and Γ_3 such that $\mathrm{meas}(\Gamma_1 > 0)$ and the second partition consists of three disjoint open parts Γ_a, Γ_b and Γ_c such that $\mathrm{meas}(\Gamma_a) > 0$, $\Gamma_c = \Gamma_3$ and $\overline{\Gamma}_a \cup \overline{\Gamma}_b = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$. The body is clamped on Γ_1 and a surface traction of density f_2 acts on Γ_2. Moreover, the electric potential vanishes on Γ_a and a surface electric charge of density q_b is applied on Γ_b. On $\Gamma_3 = \Gamma_c$ the body comes in frictional contact with a conductive obstacle, called foundation which has the electric potential φ_F.

Denoting by $u : \Omega \to \mathbb{R}^m$ the displacement field, by $\varepsilon(u) = (\varepsilon_{ij}(u))$ the strain tensor, by $\sigma : \Omega \to \mathcal{S}_m$ the stress tensor, by $D : \Omega \to \mathbb{R}^m$, $D = (D_i)$ the electric displacement field and by $\varphi : \Omega \to \mathbb{R}$ the electric potential we can now write the strong formulation of the problem which describes the above process:

$(\mathcal{P}_M)$ Find a displacement field $u : \Omega \to \mathbb{R}^m$ and an electric potential $\varphi : \Omega \to \mathbb{R}$ such that

$$\mathrm{Div}\, \sigma + f_0 = 0 \ \text{ in } \Omega, \tag{7.2}$$

$$\mathrm{div}\, D = q_0 \ \text{ in } \Omega, \tag{7.3}$$

$$\sigma = \mathcal{E}\varepsilon(u) + \mathcal{P}^T \nabla\varphi \ \text{ in } \Omega, \tag{7.4}$$

$$D = \mathcal{P}\varepsilon(u) - \mathcal{B}\nabla\varphi \ \text{ in } \Omega, \tag{7.5}$$

$$u = 0 \ \text{ on } \Gamma_1, \tag{7.6}$$

$$\varphi = 0 \ \text{ on } \Gamma_a, \tag{7.7}$$

$$\sigma\nu = f_2 \ \text{ on } \Gamma_2, \tag{7.8}$$

$$D \cdot \nu = q_b \ \text{ on } \Gamma_b, \tag{7.9}$$

$$-\sigma_\nu = S \ \text{ on } \Gamma_3, \tag{7.10}$$

$$-\sigma_\tau \in \partial_2 j(x, u_\tau) \ \text{ on } \Gamma_3, \tag{7.11}$$

$$D \cdot \nu \in \partial_2 \phi(x, \varphi - \varphi_F) \ \text{ on } \Gamma_3. \tag{7.12}$$

We point out the fact that once the displacement field u and the electric potential φ are determined, the stress tensor σ and the electric displacement field D can be obtained via relations

(7.4) and (7.5), respectively.

Let us now explain the meaning of the equations and the conditions (7.2)-(7.12) in which, for simplicity, we have omitted the dependence of the functions on the spatial variable x.

First, equations (7.2)-(7.3) are the governing equations consisting of the equilibrium conditions, while equations (7.4)-(7.5) represent the electro-elastic constitutive law.

We assume that $\mathcal{E} : \Omega \times \mathcal{S}_m \to \mathcal{S}_m$ is a nonlinear elasticity operator, $\mathcal{P} : \Omega \times \mathcal{S}_m \to \mathbb{R}^m$ and $\mathcal{P}^T : \Omega \times \mathbb{R}^m \to \mathcal{S}_m$ are the piezoelectric operator (third order tensor field) and its transpose, respectively and $\mathcal{B} : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ denotes the electric permittivity operator (second order tensor field) which is considered to be linear. The tensors $\mathcal{P}$ and $\mathcal{P}^T$ satisfy the equality

$$\mathcal{P}\tau \cdot \zeta = \tau : \mathcal{P}^T\zeta, \quad \text{for all } \tau \in \mathcal{S}_m \text{ and all } \zeta \in \mathbb{R}^m$$

and the components of the tensor $\mathcal{P}^T$ are given by $p^T_{ijk} = p_{kij}$.

When $\tau \mapsto \mathcal{E}(x, \tau)$ is linear, $\mathcal{E}(x, \tau) = \mathcal{C}(x)\tau$ with the elasticity coefficients $\mathcal{C} = (c_{ijkl})$ which may be functions indicating the position in a nonhomogeneous material. The decoupled state can be obtained by taking $p_{ijk} = 0$, in this case we have purely elastic and purely electric deformations.

Conditions (7.6) and (7.7) model the fact that the displacement field and the electrical potential vanish on Γ_1 and Γ_a, respectively, while conditions (7.8) and (7.9) represent the traction and the electric boundary conditions showing that the forces and the electric charges are prescribed on Γ_2 and Γ_b, respectively.

Conditions (7.10)-(7.12) describe the contact, the frictional and the electrical conductivity conditions on the contact surface Γ_3, respectively. Here, S is the normal load imposed on Γ_3, the functions $j : \Gamma_3 \times \mathbb{R}^m \to \mathbb{R}^m$ and $\phi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ are prescribed and φ_F is the electric potential of the foundation.

The strong formulation of problem $(\mathcal{P}_M)$ consists in finding $u : \Omega \to \mathbb{R}^m$ and $\varphi : \Omega \to \mathbb{R}$ such that (7.2)-(7.12) hold. However, it is well known that, in general, the strong formulation of a contact problem does not admit any solution. Therefore, we reformulate problem $(\mathcal{P}_M)$ in a weaker sense, i.e. we shall derive its variational formulation. With this end in view, we introduce the functional spaces for the displacement field and the electrical potential

$$V = \left\{ v \in H^1(\Omega;\mathbb{R}^m) :\ v = 0 \text{ on } \Gamma_1 \right\}, \qquad W = \left\{ \varphi \in H^1(\Omega) :\ \varphi = 0 \text{ on } \Gamma_a \right\}$$

which are closed subspaces of H_1 and $H^1(\Omega)$. We endow V and W with the following inner products and the corresponding norms

$$(u,v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \qquad \|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}}$$
$$(\varphi,\chi)_W = (\nabla\varphi, \nabla\chi)_H, \qquad \|\chi\|_W = \|\nabla\chi\|_H$$

and conclude that $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ are Hilbert spaces.

Assuming sufficient regularity of the functions involved in the problem, using the Green formula (7.1), the relations (7.2)-(7.12), the definition of the Clarke generalized gradient and the equality

$$\int_{\Gamma_3} (\sigma\nu)\cdot v \, d\Gamma = \int_{\Gamma_3} \sigma_\nu v_\nu \, d\Gamma + \int_{\Gamma_3} \sigma_\tau \cdot v_\tau \, d\Gamma$$

we obtain the following variational formulation of problem $(\mathcal{P}_M)$ in terms of the displacement field and the electric potential:

$(\mathcal{P}_V)$ Find $(u,\varphi) \in V \times W$ such that for all $(v,\chi) \in V \times W$

$$\begin{aligned} (\mathcal{E}\varepsilon(u), \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} + \left(\mathcal{P}^T\nabla\varphi, \varepsilon(v) - \varepsilon(u)\right)_{\mathcal{H}} + \int_{\Gamma_3} j^0_{,2}(x, u_\tau; v_\tau - u_\tau)\, d\Gamma \quad &\geq \quad (f, v-u)_V \\ (\mathcal{B}\nabla\varphi, \nabla\chi - \nabla\varphi)_H - (\mathcal{P}\varepsilon(u), \nabla\chi - \nabla\varphi)_H + \int_{\Gamma_3} \phi^0_{,2}(x, \varphi - \varphi_F; \chi - \varphi)\, d\Gamma \quad &\geq \quad (q, \chi - \varphi)_W, \end{aligned}$$

where $f \in V$ and $q \in W$ are the elements given by the Riesz's representation theorem as follows

$$(f, v-u)_V = \int_\Omega f_0 \cdot v \, dx + \int_{\Gamma_2} f_2 \cdot v \, d\Gamma - \int_{\Gamma_3} S v_\nu \, d\Gamma,$$

$$(q,\chi)_W = \int_\Omega q_0 \chi \, dx - \int_{\Gamma_b} q_2 \chi \, d\Gamma.$$

In the study of problem $(\mathcal{P}_V)$ we shall assume fulfilled the following hypotheses:

$(\mathcal{H}_\mathcal{E})$ The elasticity operator $\mathcal{E} : \Omega \times \mathcal{S}_m \to \mathcal{S}_m$ is such that

(i) $x \mapsto \mathcal{E}(x,\tau)$ is measurable for all $\tau \in \mathcal{S}_m$;

(ii) $\tau \mapsto \mathcal{E}(x,\tau)$ is continuous for a.e. $x \in \Omega$;

(iii) there exist $c_1 > 0$ and $\alpha \in L^2(\Omega)$ such that $\|\mathcal{E}(x,\tau)\|_{\mathcal{S}_m} \leq c(\alpha(x) + \|\tau\|_{\mathcal{S}_m})$ for all $\tau \in \mathcal{S}_m$ and a.e. $x \in \Omega$;

(iv) $\tau \mapsto \mathcal{E}(x,\tau) : (\sigma - \tau)$ is weakly upper semicontinuous for all $\sigma \in \mathcal{S}_m$ and a.e. $x \in \Omega$;

(v) there exists $c_2 > 0$ such that $\mathcal{E}(x,\tau) : \tau \geq c\|\tau\|^2_{\mathcal{S}_m}$ for all $\tau \in \mathcal{S}_m$ and a.e. $x \in \Omega$.

$(\mathcal{H}_\mathcal{P})$ The piezoelectric operator $\mathcal{P} : \Omega \times \mathcal{S}_m \to \mathbb{R}^m$ is such that

(i) $\mathcal{P}(x,\tau) = p(x)\tau$ for all $\tau \in \mathcal{S}_m$ and a.e. $x \in \Omega$;

(ii) $p(x) = (p_{ijk}(x))$ with $p_{ijk} = p_{ikj} \in L^\infty(\Omega)$.

$(\mathcal{H}_\mathcal{B})$ $\mathcal{B} : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is such that

(i) $\mathcal{B}(x,\zeta) = \beta(x)\zeta$ for all $\zeta \in \mathbb{R}^m$ and almost $x \in \Omega$;

(ii) $\beta(x) = (\beta_{ij}(x))$ with $\beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$;

(iii) there exists $m > 0$ such that $(\beta(x)\zeta) \cdot \zeta \geq m|\zeta|^2$ for all $\zeta \in \mathbb{R}^m$ and a.e. $x \in \Omega$.

$(\mathcal{H}_j)$ $j : \Gamma_3 \times \mathbb{R}^m \to \mathbb{R}$ is such that

(i) $x \mapsto j(x,\zeta)$ is measurable for all $\zeta \in \mathbb{R}^m$;

(ii) $\zeta \mapsto j(x,\zeta)$ is locally Lipschitz for a.e. $x \in \Gamma_3$;

(iii) there exist $c_3 > 0$ such that $|\partial_2 j(x,\zeta)| \leq c_3(1 + |\zeta|)$ for all $\zeta \in \mathbb{R}^m$ and a.e. $x \in \Gamma_3$;

(iv) there exists $c_4 > 0$ such that $j^0_{,2}(x,\zeta;-\zeta) \leq c_4|\zeta|$ for all $\zeta \in \mathbb{R}^m$ and a.e. $x \in \Gamma_3$;

(v) $\zeta \mapsto j(x,\zeta)$ is regular for a.e. $x \in \Gamma_3$.

$(\mathcal{H}_\phi)$ $\phi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $x \mapsto \phi(x,t)$ is measurable for all $t \in \mathbb{R}$;

(ii) $\zeta \mapsto \phi(x,\zeta)$ is locally Lipschitz for a.e. $x \in \Gamma_3$;

(iii) there exist $c_5 > 0$ such that $|\partial_2 \phi(x,t)| \leq c_5|t|$ for all $t \in \mathbb{R}$ and a.e. $x \in \Gamma_3$;

(iv) $t \mapsto \phi(x,t)$ is regular for a.e. $x \in \Gamma_3$.

$(\mathcal{H}_{f,q})$ $f_0 \in H$, $f_2 \in L^2(\Gamma_2;\mathbb{R}^m)$, $q_0 \in L^2(\Omega)$, $q_b \in L^2(\Gamma_2)$, $S \in L^\infty(\Gamma_3)$, $S \geq 0$, $\varphi_F \in L^2(\Gamma_3)$.

The main result of this subsection is given by the following theorem.

Theorem 7.1. (N.C. & C. VARGA [38]) *Assume fulfilled conditions $(\mathcal{H}_\mathcal{E})$, $(\mathcal{H}_\mathcal{P})$, $(\mathcal{H}_\mathcal{B})$, $(\mathcal{H}_j)$, $(\mathcal{H}_\phi)$ and $(\mathcal{H}_{f,q})$. Then problem $(\mathcal{P}_V)$ admits at least one solution.*

Proof. We observe that problem $(\mathcal{P}_V)$ is in fact a system of two coupled hemivariational inequalities. The idea is to apply one of the existence results obtained in Section 2. with suitable choice of ψ_k, J, and F_k $(k \in \{1,2\})$.

First, let us take $n = 2$ and define $X_1 = V$, $X_2 = W$, $Y_1 = L^2(\Gamma_3;\mathbb{R}^m)$, $Y_2 = L^2(\Gamma_3)$, $K_1 = X_1$ and $K_2 = X_2$. Next we introduce $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$ defined by

$$T_1 = i_\tau \circ \gamma_m \circ i_m|_{\Gamma_3}, \qquad T_2 = \gamma \circ i|_{\Gamma_3},$$

where $i_m : V \to H_1 = H^1(\Omega;\mathbb{R}^m)$ is the embedding operator, $\gamma_m : H_1 \to H^{1/2}(\Gamma;\mathbb{R}^m)$ is the Sobolev trace operator, $i_\tau : H^{1/2}(\Gamma;\mathbb{R}^m) \to L^2(\Gamma_3;\mathbb{R}^m)$ is the operator defined by $i_\tau(v) = v_\tau$, $i : W \to H^1(\Omega)$ is the embedding operator and $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$ is the Sobolev trace operator. Clearly T_1 and T_2 are linear and compact operators. We consider next $\psi_1 : X_1 \times X_2 \times X_1 \to \mathbb{R}$ and $\psi_2 : X_1 \times X_2 \times X_2 \to \mathbb{R}$ defined by

$$\psi_1(u,\varphi,v) = (\mathcal{E}\varepsilon(u), \varepsilon(v) - \varepsilon(u))_\mathcal{H} + \left(\mathcal{P}^T\nabla\varphi, \varepsilon(v) - \varepsilon(u)\right)_\mathcal{H},$$

$$\psi_2(u,\varphi,\chi) = (\mathcal{B}\nabla\varphi, \nabla\chi - \nabla\varphi)_H - (\mathcal{P}\varepsilon(u), \nabla\chi - \nabla\varphi)_H,$$

$J : Y_1 \times Y_2 \to \mathbb{R}$ defined by

$$J(w,\eta) = \int_{\Gamma_3} j(x, w(x))\, d\Gamma + \int_{\Gamma_3} \phi(x, \eta(x) - \varphi(x))\, d\Gamma,$$

and $F_1 : X_1 \times X_2 \to X_1^*$ and $F_2 : X_1 \times X_2 \to X_2^*$ defined by

$$F_1(u,\varphi) = f, \qquad F_2(u,\varphi) = q.$$

It is easy to infer from the above definitions that if $(\mathcal{H}_{\mathcal{E}})$, $(\mathcal{H}_{\mathcal{P}})$, $(\mathcal{H}_{\mathcal{B}})$ hold, then the functionals ψ_1, ψ_2 satisfy conditions (H1) and (H6). Taking $(\mathcal{H}_j)$ and $(\mathcal{H}_\phi)$ into account we conclude that J is a regular locally Lipschitz functional which satisfies

$$J^0_{,1}(w,\eta;z) = \int_{\Gamma_3} j^0_{,2}(x, w(x); z(x))\, d\Gamma$$

$$J^0_{,2}(w,\eta;\zeta) = \int_{\Gamma_3} \phi^0_{,2}(x, \eta(x) - \varphi(x); \zeta(x))\, d\Gamma.$$

Obviously conditions (H2), (H7), (H8) are fulfilled, therefore we can apply Corollary 6.2 to conclude that problem $(\mathcal{P}_V)$ admits at least one solution. □

7.2 The bipotential method for contact problems with nonmonotone boundary conditions

This section focuses on the weak solvability of a general mathematical model which describes the contact between a body and an obstacle. The process is assumed to be static and we work under the small deformations hypothesis. The behavior of the materials is described by a possibly multivalued constitutive law written as a subdifferential inclusion, while the contact between the body and the foundation is described by two inclusions, corresponding to the normal and the tangential directions, each inclusion involving the sum of a Clarke subdifferential and the normal cone of a nonempty, closed and convex set.

Inspired and motivated by some recent papers in the literature we consider a variational formulation in terms of bipotentials for our model. This leads to a system of two inequalities: a hemivariational inequality related to the equilibrium law and a variational inequality related to the functional extension of the constitutive law. The unknown of the system is a pair (u, σ) consisting of the displacement field and the Cauchy stress field. A key role in our approach is played by the separable bipotential that can be defined as the sum of the constitutive map and its Fenchel conjugate. Bipotentials were introduced in 1991 by de Saxcé & Feng [110] and within a very short period of time this theory has undergone a remarkable development both in pure and applied mathematics as bipotentials were successfully applied in addressing various problems arising in mechanics (non-associated Drücker-Prager models in plasticity [18, 114], cam-clay models in soil mechanics [113, 124], cyclic plasticity [11, 112] and viscoplasticity of metals with kinematical hardening rule [58], Coulomb's friction law [15, 70, 78], displacement-traction models for elastic materials [79], contact models with Signorini's boundary condition [77]). For more details and connections regarding the theory of bipotentials see also [16, 17, 115]. The bipotential approach has the advantage that it allows to approximate simultaneously the displacement field and the Cauchy stress tensor and facilitated the implementation of new and efficient numerical algorithms (see e.g. [51, 111]). However, in all the works we are aware of, the bipotential method has been used only for problems with monotone boundary conditions, mostly expressed as inclusions involving the subdifferential of a proper, convex and lower semicontinuous function. Thus, the variational formulation for these problems leads to a coupled system of variational inequalities. In this paper, due to the nonmonotone boundary conditions two major differences arise:

- the set of admissible stress tensors is defined with respect to a given displacement field and depends explicitly on this displacement field, in contrast to the case of monotone boundary conditions when the set of admissible stress tensors is the same for all displacement fields;
- the variational formulation leads to a system of inequalities consisting of a hemivariational inequality and a variational inequality.

Consequently, several difficulties occur in determining the existence of weak solutions since the classical methods fail to be applied directly.

7.2.1 The mechanical model and its variational formulation

Let us consider a body $\mathcal{B}$ which occupies the domain $\Omega \subset \mathbb{R}^m$ ($m = 2, 3$) with a sufficiently smooth boundary Γ (e.g. Lipschitz continuous) and a unit outward normal ν. The body is acted upon by forces of density f_0 and it is mechanically constrained on the boundary. In order to describe these constraints we assume Γ is partitioned into three Lebesgue measurable parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that Γ_1 has positive Lebesgue measure. The body is clamped on Γ_1, hence the displacement field vanishes here, while surface tractions of density f_2 act on Γ_2. On Γ_3 the body may come in contact with an obstacle which will be referred to as foundation. The process is assumed to be static and the behavior of the material is modeled by a (possibly multivalued) constitutive law expressed as a subdifferential inclusion. The contact between the body and the foundation is modeled with respect to the normal and the tangent direction respectively, to each corresponding an inclusion involving the sum between the Clarke subdifferential of a locally Lipschitz function and the normal cone of a nonempty, closed and convex set.

It is well known that the subdifferential of a convex function is a monotone set-valued operator, while the Clarke subdifferential is a set-valued operator which is not monotone in general. This is why we say that the constitutive law is monotone and the boundary conditions are nonmonotone.

The mathematical model which describes the above process is the following. For simplicity we omit the dependence of some functions of the spatial variable.

(P) Find a displacement $u : \Omega \to \mathbb{R}^m$ and a stress tensor $\sigma : \Omega \to \mathcal{S}^m$ such that

$$-\text{Div}\, \sigma = f_0, \qquad \text{in } \Omega \tag{7.13}$$

$$\sigma \in \partial\phi(\varepsilon(u)), \quad \text{a.e. in } \Omega \tag{7.14}$$

$$u = 0, \qquad \text{on } \Gamma_1 \tag{7.15}$$

$$\sigma\nu = f_2, \qquad \text{on } \Gamma_2 \tag{7.16}$$

$$-\sigma_\nu \in \partial_C^2 j_\nu(x, u_\nu) + N_{C_1}(u_\nu), \qquad \text{on } \Gamma_3 \tag{7.17}$$

$$-\sigma_\tau \in h(x, u_\tau)\partial_C^2 j_\tau(x, u_\tau) + N_{C_2}(u_\tau), \qquad \text{on } \Gamma_3 \tag{7.18}$$

where $\phi : \mathcal{S}^m \to \mathbb{R}$ is convex and lower semicontinous, $j_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ and $j_\tau : \Gamma_3 \times \mathbb{R}^m \to \mathbb{R}$ are locally Lipschitz with respect to the second variable and $h : \Gamma_3 \times \mathbb{R}^m \to \mathbb{R}$ is a prescribed function. Here, $C_1 \subset \mathbb{R}$ and $C_2 \subset \mathbb{R}^m$ are nonempty closed and convex subsets and N_{C_k} denotes the normal cone of C_k ($k = 1, 2$). For a Banach space E and a nonempty, closed and convex subset $K \subset E$, the normal cone of K at x is defined by

$$N_K(x) = \left\{\xi \in E^* : \langle \xi, y - x \rangle_{E^* \times E} \leq 0, \text{ for all } y \in K\right\}.$$

It is well known that

$$N_K(x) = \partial I_K(x),$$

where I_K is the indicator function of K, that is,

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

Relation (7.13) represents the equilibrium equation, (7.14) is the constitutive law, (7.15)-(7.16) are the displacement and traction boundary conditions and (7.17)-(7.18) describe the contact between body and the foundation.

Relations between the stress tensor σ and the strain tensor ε of the type (7.14) describe the constitutive laws of the deformation theory of plasticity, of Hencky plasticity with convex yield function, of locking materials with convex locking functions etc. For concrete examples and their physical interpretation one can consult Sections 3.3.1 and 3.3.2 in Panagiotopoulos [98] (see also Section 3.1 in [99]). A particular case of interest regarding (7.14) is when the constitutive map ϕ is Gâteaux differentiable, thus the subdifferential inclusion reduces to

$$\sigma = \phi'(\varepsilon(u)), \tag{7.19}$$

which corresponds to nonlinear elastic materials.
Some classical constitutive laws which can be written in the form (7.19) are presented below:

(i) Assume that ϕ is defined by

$$\phi(\mu) = \frac{1}{2}\mathcal{E}\mu : \mu,$$

where $\mathcal{E} = (\mathcal{E}_{ijkl})$, $1 \leq i, j, k, l \leq m$ is a fourth order tensor which satisfies the symmetry property

$$\mathcal{E}\mu : \tau = \mu : \mathcal{E}\tau, \text{ for all } \mu, \tau \in \mathcal{S}^m,$$

and the ellipticity property

$$\mathcal{E}\mu : \mu \geq c|\mu|^2, \text{ for all } \mu \in \mathcal{S}^m.$$

In this case (7.19) reduces to *Hooke's law*, that is, $\sigma = \mathcal{E}\varepsilon(u)$, and corresponds to linearly elastic materials.

(ii) Assume that ϕ is defined by

$$\phi(\mu) = \frac{1}{2}\mathcal{E}\mu : \mu + \beta\,|\mu - P_{\mathcal{K}}\mu|^2\,,$$

where $\mathcal{E}$ is the elasticity tensor and satisfies the same properties as in the previous example, $\beta > 0$ is a constant coefficient of the material, $P : \mathcal{S}^m \to \mathcal{K}$ is the projection operator and $\mathcal{K}$ is the nonempty, closed and convex von Mises set

$$\mathcal{K} = \left\{\mu \in \mathcal{S}^m : \; \frac{1}{2}\mu^D : \mu^D \leq a^2, \; a > 0\right\}.$$

Here the notation μ^D stands for the deviator of the tensor μ, that is, $\mu^D = \mu - \frac{1}{m}Tr(\mu)I$, with I being the identity tensor.

In this case (7.19) becomes

$$\sigma = \mathcal{E}\varepsilon(u) + 2\beta(I - P_{\mathcal{K}})\varepsilon(u),$$

which is known in the literature as the *piecewise linear constitutive law* (see e.g. Han & Sofonea [56]).

(iii) Assume ϕ is defined by

$$\phi(\mu) = \frac{k_0}{2}Tr(\mu)I : \mu + \frac{1}{2}\varphi\left(\left|\mu^D\right|^2\right),$$

where $k_0 > 0$ is a constant and $\varphi : [0, \infty) \to [0, \infty)$ is a continuously differentiable constitutive function.

In this case (7.19) becomes

$$\sigma = k_0 Tr(\varepsilon(u))I + \varphi'\left(\left|\varepsilon^D(u)\right|^2\right)\varepsilon^D(u),$$

and this describes the behavior of the Hencky materials (see e.g. Zeidler [122]).

Boundary conditions of the type (7.17) and (7.18) model a large class of contact problems arising in mechanics and engineering. For the case $h \equiv 1$ many examples of nonmonotone laws of the type

$$-\sigma_\nu \in \partial_C j_\nu(u_\nu) \text{ and } -\sigma_\tau \in \partial_C j_\tau(u_\tau),$$

can be found in [99] Section 2.4, [94] Section 1.4 or [53] Section 2.8.

The case when the function h actually depends on the second variable allows the study of contact problems with *slip-dependent friction law* (see e.g. [29, 80] for antiplane models and [30] for general 3D models). This friction law reads as follows

$$-\sigma_\tau \le \mu(x, |u_\tau|),\ -\sigma_\tau = \mu(x, |u_\tau|)\frac{u_\tau}{|u_\tau|} \text{ if } u_\tau \neq 0, \tag{7.20}$$

where $\mu : \Gamma_3 \times [0, +\infty) \to [0, +\infty)$ is the sliding threshold and it is assumed to satisfy

$$0 \le \mu(x, t) \le \mu_0, \text{ for a.e. } x \in \Gamma_3 \text{ and all } t \ge 0,$$

for some positive constant μ_0. It is easy to see that (7.18) can be put in the form (7.20) simply by choosing

$$h(x, u_\tau) = \mu(x, |u_\tau|) \text{ and } j_\tau(x, u_\tau) = |u_\tau|.$$

We point out the fact that the above example cannot be written in the form $-\sigma_\tau \in \partial_C j_\tau(u_\tau)$ as, in general, for two locally Lipschitz functions h, g there does not exists j such that $\partial_C j(u) = h(u)\partial_C g(u)$. We would also like to point out that many boundary conditions of classical elasticity are particular cases of (7.17) and (7.18), in most of these cases the functions j_ν and j_τ being convex, hence leading to monotone boundary conditions. We list below some examples:

(a) *The Winkler boundary condition*

$$-\sigma_\nu = k_0 u_\nu,\ k_0 > 0.$$

This law is used in engineering as it describes the interaction between a deformable body and the soil and can be expressed in the form (7.17) by setting

$$C_1 = \mathbb{R} \text{ and } j_\nu(x, u_\nu) = \frac{k_0}{2}u_\nu^2.$$

More generally, if we want to describe the case when the body may lose contact with the foundation, we can consider the following law

$$\begin{cases} u_\nu < 0 \Rightarrow \sigma_\nu = 0, \\ u_\nu \geq 0 \Rightarrow -\sigma_\nu = k_0 u_\nu. \end{cases}$$

The first relation corresponds to the case when there is no contact, while the second models the contact case. Obviously the above law can be expressed in the form (7.17) by choosing

$$C_1 = \mathbb{R} \text{ and } j_\nu(x, u_\nu) = \begin{cases} 0, & \text{if } u_\nu < 0, \\ \frac{k_0}{2} u_\nu^2, & \text{if } u_\nu \geq 0. \end{cases}$$

In [93] the following nonmonotone boundary conditions were imposed to model the contact between a body and a Winkler-type foundation which may sustain limited values of efforts

$$\begin{cases} u_\nu < 0 \Rightarrow \sigma_\nu = 0, \\ u_\nu \in [0, a) \Rightarrow -\sigma_\nu = k_0 u_\nu, \\ u_\nu = a \Rightarrow -\sigma_\nu \in [0, k_0 a], \\ u_\nu > a \Rightarrow \sigma_\nu = 0. \end{cases}$$

This means that the rupture of the foundation is assumed to occur at those points in which the limit effort is attained. The first condition holds in the noncontact zone, the second describes the zone where the contact occurs and it is idealized by the Winkler law. The maximal value of reactions that can be maintained by the foundation is given by $k_0 a$ and it is accomplished when $u_\nu = a$, with k_0 being the Winkler coefficient. The fourth relation holds in the zone where the foundation has been destroyed. The above Winkler-type law can be written as an inclusion of the type (7.17) by setting

$$C_1 = \mathbb{R} \text{ and } j_\nu(x, u_\nu) = \begin{cases} 0, & \text{if } u_\nu < 0, \\ \frac{k_0}{2} u_\nu^2, & \text{if } 0 \leq u_\nu < a, \\ \frac{k_0}{2} a^2, & \text{if } u_\nu \geq a. \end{cases}$$

Since all of the above examples only describe what happens in the normal direction, in order to complete the model we must combine these with boundary conditions concerning σ_τ, u_τ, or both. The simplest cases are $u_\tau = 0$ (which corresponds to $C_2 = \{0\}$) and $\sigma_\tau = S_\tau$, where $S_\tau = S_\tau(x)$ is given (which corresponds to $j_\tau(x, u_\tau) = -S_\tau \cdot u_\tau$).

(b) *The Signorini boundary conditions*, which hold if the foundation is rigid and are as follows

$$\begin{cases} u_\nu < 0 \Rightarrow \sigma_\nu = 0, \\ u_\nu = 0 \Rightarrow \sigma_\nu \leq 0, \end{cases}$$

or equivalently,

$$u_\nu \leq 0,\ \sigma_\nu \leq 0 \text{ and } \sigma_\nu u_\nu = 0.$$

This can be written equivalently in form (7.17) by setting

$$C_1 = (-\infty, 0] \text{ and } j_\nu \equiv 0.$$

(c) In [78] the following *static version of Coulomb's law of dry friction with prescribed normal stress* was considered

$$\begin{cases} -\sigma_\nu(x) = F(x) \\ |\sigma_\tau| \leq k(x)|\sigma_\nu|, \\ \sigma_\tau = -k(x)|\sigma_\nu|\frac{u_\tau}{|u_\tau|}, \text{ if } u_\tau(x) \neq 0. \end{cases}$$

We can write the above law in the form of (7.17) and (7.18) simply by setting

$$C_1 = \mathbb{R},\ C_2 = \mathbb{R}^m,\ j_\nu(x, u_\nu) = F(x)u_\nu,\ h(x, u_\tau) = k(x)|F(x)| \text{ and } j_\tau(x, u_\tau) = |u_\tau|.$$

The assumptions on the functions $f_0,\ f_2,\ \phi,\ h,\ j_\nu$ and j_τ required to prove our main result are listed below.

$(\mathbf{H_C})$ The constraint sets C_1 and C_2 are convex cones, i.e.

$$0 \in C_k \quad \text{and} \quad \lambda C_k \subset C_k \text{ for all } \lambda > 0, \quad k = 1, 2.$$

($\mathbf{H_f}$) The density of the volume forces and the traction satisfy $f_0 \in H$ and $f_2 \in L^2(\Gamma_2; \mathbb{R}^m)$.

($\mathbf{H}_\phi$) The constitutive function $\phi : \mathcal{S}^m \to \mathbb{R}$ and its Fenchel conjugate $\phi^* : \mathcal{S}^m \to (-\infty, +\infty]$ satisfy

(i) ϕ is convex and lower semicontinuous;

(ii) there exists $\alpha_1 > 0$ such that $\phi(\tau) \geq \alpha_1 |\tau|^2$, for all $\tau \in \mathcal{S}^m$;

(iii) there exists $\alpha_2 > 0$ such that $\phi^*(\mu) \geq \alpha_2 |\mu|^2$, for all $\mu \in \mathcal{S}^m$;

(iv) $\phi(\varepsilon(v)) \in L^1(\Omega)$, for all $v \in V$ and $\phi^*(\tau) \in L^1(\Omega)$, for all $\tau \in \mathcal{H}$.

($\mathbf{H_h}$) The function $h : \Gamma_3 \times \mathbb{R}^m \to \mathbb{R}$ is such that

(i) $\Gamma_3 \ni x \mapsto h(x, \zeta)$ is measurable for each $\zeta \in \mathbb{R}^m$;

(ii) $\mathbb{R}^m \ni \zeta \mapsto h(x, \zeta)$ is continuous for a.e. $x \in \Gamma_3$;

(iii) there exists $h_0 > 0$ such that $0 \leq h(x, \zeta) \leq h_0$ for a.e. $x \in \Gamma_3$ and all $\zeta \in \mathbb{R}^m$.

($\mathbf{H_{j_\nu}}$) The function $j_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $\Gamma_3 \ni x \mapsto j_\nu(x, t)$ is measurable for each $t \in \mathbb{R}$;

(ii) there exists $p \in L^2(\Gamma_3)$ such that for a.e. $x \in \Gamma_3$ and all $t_1, t_2 \in \mathbb{R}$

$$|j_\nu(x, t_1) - j_\nu(x, t_2)| \leq p(x)|t_1 - t_2|;$$

(iii) $j_\nu(x, 0) \in L^1(\Gamma_3)$.

($\mathbf{H_{j_\tau}}$) The function $j_\tau : \Gamma_3 \times \mathbb{R}^m \to \mathbb{R}$ is such that

(i) $\Gamma_3 \ni x \mapsto j_\tau(x, \zeta)$ is measurable for each $\zeta \in \mathbb{R}^m$;

(ii) there exist $q \in L^2(\Gamma_3)$ such that for a.e. $x \in \Gamma_3$ and all $\zeta_2, \zeta_2 \in \mathbb{R}^m$

$$|j_\tau(x, \zeta_1) - j_\tau(x, \zeta_2)| \leq q(x)|\zeta_1 - \zeta_2|;$$

(iii) $j_\tau(x,0) \in L^1(\Gamma_3;\mathbb{R}^m)$.

The strong formulation of problem **(P)** consists in finding $u : \Omega \to \mathbb{R}^m$ and $\sigma : \Omega \to \mathcal{S}^m$, regular enough, such that (7.13)-(7.18) are satisfied. However, it is a fact that for most contact problems the strong formulation has no solution. Therefore, it is useful to reformulate problem **(P)** in a weaker sense, i.e. we shall derive a variational formulation. With this end in mind, we consider the following function space

$$V = \{v \in H_1 : \ v = 0 \text{ a.e. on } \Gamma_1\} \tag{7.21}$$

which is a closed subspace of H_1, hence a Hilbert space. Since the Lebesgue measure of Γ_1 is positive, it follows from Korn's inequality that the following inner product

$$(u,v)_V = (\varepsilon(u),\varepsilon(v))_{\mathcal{H}} \tag{7.22}$$

generates a norm on V which is equivalent with the norm inherited from H_1.

Let us provide a variational formulation for problem $(\mathbf{P})$. To this end, we consider u a strong solution, $v \in V$ a test function and we multiply the first line of **(P)** by $v-u$. Using the Green formula (7.1) we have

$$\begin{aligned}
(f_0, v-u)_H &= -\,(\mathrm{Div}\ \sigma, v-u)_H \\
&= -\int_\Gamma (\sigma\nu)\cdot(v-u)\ d\Gamma + (\sigma, \varepsilon(v)-\varepsilon(u))_{\mathcal{H}} \\
&= -\int_{\Gamma_2} f_2\cdot(v-u)\ d\Gamma - \int_{\Gamma_3} \left[\sigma_\nu(v_\nu-u_\nu)+\sigma_\tau\cdot(v_\tau-u_\tau)\right]\ d\Gamma + (\sigma,\varepsilon(v)-\varepsilon(u))_{\mathcal{H}}
\end{aligned}$$

for all $v \in V$. Since $V \ni v \mapsto (f_0,v)_H + \int_{\Gamma_2} f_2\cdot v\ d\Gamma$ is linear and continuous, we can apply Riesz's representation theorem to conclude that there exists a unique element $f \in V$ such that

$$(f,v)_V = (f_0,v)_H + \int_{\Gamma_2} f_2\cdot v\ d\Gamma. \tag{7.23}$$

Consider now the following nonempty, closed and convex subset of V

$$\Lambda = \{v \in V : \ v_\nu(x) \in C_1 \text{ and } v_\tau(x) \in C_2 \text{ for a.e. } x \in \Gamma_3\},$$

which is called the *set of admissible displacement fields*.

Since C_1, C_2 are convex cones, it follows that Λ is also a convex cone. Moreover, taking into account Definitions 1.7 and 1.8, we deduce that for all $v \in \Lambda$ the following inequalities hold

$$-\int_{\Gamma_3} \sigma_\nu(v_\nu - u_\nu)\, d\Gamma \le \int_{\Gamma_3} j_\nu^0(x, u_\nu; v_\nu - u_\nu)\, d\Gamma \tag{7.24}$$

and

$$-\int_{\Gamma_3} \sigma_\tau \cdot (v_\tau - u_\tau)\, d\Gamma \le \int_{\Gamma_3} h(x, u_\tau) j_\tau^0(x, u_\tau; v_\tau - u_\tau)\, d\Gamma. \tag{7.25}$$

Here, and hereafter, the generalized derivatives of the functions j_ν and j_τ are taken with respect to the second variable, i.e. of the functions $\mathbb{R} \ni t \mapsto j_\nu(x,t)$ and $\mathbb{R}^m \ni \zeta \mapsto j_\tau(x,\zeta)$ respectively, but for simplicity we omit to mention that in fact these are partial generalized derivatives. On the other hand, taking Proposition 1.2 into account we can rewrite (7.14) as

$$\varepsilon(u) \in \partial\phi^*(\sigma), \text{ a.e. in } \Omega,$$

and which after integration over Ω leads to

$$-(\varepsilon(u), \mu - \sigma)_{\mathcal{H}} + \int_\Omega \phi^*(\mu) - \phi^*(\sigma)\, dx \ge 0, \text{ for all } \mu \in \mathcal{H}. \tag{7.26}$$

Let us define the operator $L : V \to \mathcal{H}$ by $Lv = \varepsilon(v)$ and denote by $L^* : \mathcal{H} \to V$ its adjoint, that is,

$$(L^*\mu, v)_V = (\mu, Lv)_{\mathcal{H}}, \text{ for all } v \in V \text{ and all } \mu \in \mathcal{H}.$$

Using (7.23)-(7.26) we arrive at the following system of inequalities

($\tilde{\mathbf{P}}$) Find $u \in \Lambda$ and $\sigma \in \mathcal{H}$ such that

$$(L^*\sigma, v - u)_V + \int_{\Gamma_3} \left[j_\nu^0(x, u_\nu; v_\nu - u_\nu) + h(x, u_\tau) j_\tau^0(x, u_\tau; v_\tau - u_\tau) \right] d\Gamma \ge (f, v - u)_V, \tag{7.27}$$

$$-(Lu, \mu - \sigma)_{\mathcal{H}} + \int_\Omega \phi^*(\mu) - \phi^*(\sigma)\, dx \ge 0, \tag{7.28}$$

for all $(v, \mu) \in \Lambda \times \mathcal{H}$.

The inequality (7.27) is related to the equilibrium relation, while (7.28) represents the functional extension of the constitutive law (7.14). It is well-known (see e.g. [53], Theorem 1.3.21) that relation (7.28) implies $Lu \in \partial\phi^*(\sigma)$ a.e. where in Ω.

Proposition 1.2 allows us to construct the separable bipotential $a : \mathcal{S}^m \times \mathcal{S}^m \to (-\infty, +\infty]$, which connects the constitutive law, the function ϕ and its conjugate ϕ^*, as follows

$$a(\tau, \mu) = \phi(\tau) + \phi^*(\mu), \text{ for all } \tau, \mu \in \mathcal{S}^m.$$

Using the bipotential a let us define $A : V \times \mathcal{H} \to \mathbb{R}$ by

$$A(v, \mu) = \int_\Omega a(Lv, \mu)\, dx, \text{ for all } v \in V, \mu \in \mathcal{H}.$$

and note that, due to $(\mathbf{H}_\phi)$, A is well defined and

$$A(v, \mu) \geq \alpha_1 \|v\|_V^2 + \alpha_2 \|\mu\|_{\mathcal{H}}^2, \text{ for all } v \in V, \mu \in \mathcal{H}.$$

Moreover, Proposition 1.2 ensures that

$$A(u, \sigma) = (L^*\sigma, u)_V \text{ and } A(v, \mu) \geq (L^*\mu, v)_V, \text{ for all } v \in V,\ \mu \in \mathcal{H}. \tag{7.29}$$

Combining (7.27) and (7.29) we get

$$A(v, \sigma) - A(u, \sigma) + \int_{\Gamma_3} \left[j_\nu^0(x, u_\nu; v_\nu - u_\nu) + h(x, u_\tau) j_\tau^0(x, u_\tau; v_\tau - u_\tau) \right] d\Gamma \geq (f, v - u)_V, \tag{7.30}$$

for all $v \in \Lambda$.

Let us define now the *set of admissible stress tensors with respect to the displacement* u, to be the following subset of $\mathcal{H}$

$$\Theta_u = \left\{ \mu \in \mathcal{H} :\ (L^*\mu, v)_V + \int_{\Gamma_3} \left[j_\nu^0(x, u_\nu; v_\nu) + h(x, u_\tau) j_\tau^0(x, u_\tau; v_\tau) \right] d\Gamma \geq (f, v)_V,\ \forall v \in \Lambda \right\}.$$

Let $w \in \Lambda$ be fixed. Choosing $v = u + w \in \Lambda$ in (7.27) shows that $\sigma \in \Theta_u$, hence $\Theta_u \neq \emptyset$. It is easy to check that Θ_u is an unbounded, closed and convex subset of $\mathcal{H}$. Taking into account (7.29) we have

$$A(u, \mu) + \int_{\Gamma_3} \left[j_\nu^0(x, u_\nu; u_\nu) + h(x, u_\tau) j_\tau^0(x, u_\tau; u_\tau) \right] d\Gamma \geq (f, u)_V, \text{ for all } \mu \in \Theta_u,$$

while for $v = 0 \in \Lambda$ in (7.27) we have

$$-A(u, \sigma) + \int_{\Gamma_3} \left[j_\nu^0(x, u_\nu; -u_\nu) + h(x, u_\tau) j_\tau^0(x, u_\tau; -u_\tau) \right] d\Gamma \geq -(f, u)_V.$$

Adding the above relations, for all $\mu \in \Theta_u$ we have

$$\begin{aligned} 0 \;\leq\; & A(u,\mu) - A(u,\sigma) + \int_{\Gamma_3} j_\nu^0(x,u_\nu;u_\nu) + j_\nu^0(x,u_\nu;-u_\nu)d\Gamma \\ & + \int_{\Gamma_3} h(x,u_\tau)\left(j_\tau^0(x,u_\tau;u_\tau) + j_\tau^0(x,u_\tau;-u_\tau)\right)d\Gamma. \end{aligned}$$

On the other hand, Proposition 1.3 and $(\mathbf{H_h})$ ensure that

$$\int_{\Gamma_3} \left[j_\nu^0(x,u_\nu;u_\nu) + j_\nu^0(x,u_\nu;-u_\nu) + h(x,u_\tau)\left(j_\tau^0(x,u_\tau;u_\tau) + j_\tau^0(x,u_\tau;-u_\tau)\right)\right] \, d\Gamma \geq 0. \quad (7.31)$$

$\left(\mathcal{P}_{var}^b\right)$ Find $u \in \Lambda$ and $\sigma \in \Theta_u$ such that

$$\begin{cases} A(v,\sigma) - A(u,\sigma) + \int_{\Gamma_3}\left[j_\nu^0(x,u_\nu;v_\nu - u_\nu) + h(x,u_\tau)j_\tau^0(x,u_\tau;v_\tau - u_\tau)\right]d\Gamma \geq (f,v-u)_V, \\ A(u,\mu) - A(u,\sigma) \geq 0, \end{cases}$$

for all $(v,\mu) \in \Lambda \times \Theta_u$.

Each solution $(u,\sigma) \in \Lambda \times \Theta_u$ of problem $\left(\mathcal{P}_{var}^b\right)$ is called a *weak solution* for problem $(\mathbf{P})$.

7.2.2 The connection with classical variational formulations

In this section we prove an existence result concerning the solutions of problem $\left(\mathcal{P}_{var}^b\right)$ by using a recent result due to Costea & Varga [38]. First we highlight the connection between the variational formulation in terms of bipotentials and other variational formulations such as the primal and dual variational formulations. As we have seen in the previous section, multiplying the first line of problem $(\mathbf{P})$ by $v-u$, integrating over Ω and then taking the functional extension of the constitutive law, we get a coupled system of inequalities, namely problem $(\tilde{\mathbf{P}})$. The primal variational formulation consists in rewriting $(\tilde{\mathbf{P}})$ as an inequality which depends only on the displacement field u, while the dual variational formulation consists in rewriting $(\tilde{\mathbf{P}})$ in terms of the stress tensor σ. The primal variational formulation can be derived by reasoning in the following way.

The second line of $(\tilde{\mathbf{P}})$ implies that $Lu \in \partial\phi^*(\sigma)$ and this can be written equivalently as $\sigma \in \partial\phi(Lu)$, hence

$$\sigma : (\mu - Lu) \leq \phi(\mu) - \phi(Lu), \text{ for all } \mu \in \mathcal{S}^m.$$

For each $v \in \Lambda$, taking $\mu = Lv$ in the previous inequality and integrating over Ω yields

$$(L^*\sigma, v-u)_V \leq \int_\Omega \phi(Lv) - \phi(Lu)\ dx, \text{ for all } v \in \Lambda.$$

Now, combining the above relation and the first line of $(\tilde{\mathbf{P}})$ we get the following problem

$(\mathcal{P}^p_{var})$ Find $u \in \Lambda$ such that

$$F(v) - F(u) + \int_{\Gamma_3} \left[j^0_\nu(x, u_\nu; v_\nu - u_\nu) + h(x, u_\tau) j^0_\tau(x, u_\tau; v_\tau - u_\tau)\right]\ d\Gamma \geq (f, v-u)_V, \forall v \in \Lambda,$$

where $F : V \to \mathbb{R}$ is the convex and lower semicontinous function defined by

$$F(v) = \int_\Omega \phi(Lv)\ dx.$$

Problem $(\mathcal{P}^p_{var})$ is called the *primal variational formulation* of problem $(\mathbf{P})$.

Conversely, in order to transform $(\tilde{\mathbf{P}})$ into a problem formulated in terms of the stress tensor we reason in the following way. First let us define $G : \mathcal{H} \to \mathbb{R}$ by

$$G(\mu) = \int_\Omega \phi^*(\mu)\ dx,$$

and for a fixed $w \in \Lambda$ let Θ_w be the following subset of $\mathcal{H}$

$$\Theta_w = \left\{\mu \in \mathcal{H} :\ (L^*\mu, v)_V + \int_{\Gamma_3} \left[j^0_\nu(x, w_\nu; v_\nu) + h(x, w_\tau) j^0_\tau(x, w_\tau; v_\tau)\right]\ d\Gamma \geq (f, v)_V, \forall v \in \Lambda\right\}.$$

Let us consider the following inclusion

$\left(\mathcal{P}^d_w\right)$ Find $\sigma \in \mathcal{H}$ such that

$$0 \in \partial G(\sigma) + \partial I_{\Theta_w}(\sigma),$$

which we call the *dual variational formulation with respect to* w.

Now, looking at the first line of $(\tilde{\mathbf{P}})$, i.e. relation (7.27), and keeping in mind the above notations, we deduce that $\Theta_u \neq \emptyset$ as $\sigma \in \Theta_u$. Moreover, for each $\mu \in \Theta_u$ we have

$$\begin{aligned} -(L^*(\mu - \sigma), u)_V \quad \leq \quad & \int_{\Gamma_3} j^0_\nu(x, u_\nu; u_\nu) + j^0_\nu(x, u_\nu; -u_\nu) d\Gamma \\ & + \int_{\Gamma_3} h(x, u_\tau) \left(j^0_\tau(x, u_\tau; u_\tau) + j^0_\tau(x, u_\tau; -u_\tau)\right) d\Gamma, \end{aligned}$$

which combined with the second line of $(\tilde{\mathbf{P}})$ leads to

$$\begin{aligned} G(\mu)-G(\sigma) \quad \geq \quad & -\int_{\Gamma_3} j_\nu^0(x,u_\nu;u_\nu)+j_\nu^0(x,u_\nu;-u_\nu)d\Gamma \\ & -\int_{\Gamma_3} h(x,u_\tau)\left(j_\tau^0(x,u_\tau;u_\tau)+j_\tau^0(x,u_\tau;-u_\tau)\right)d\Gamma, \end{aligned}$$

for all $\mu \in \Theta_u$.

A simple computation shows that any solution of $\left(\mathcal{P}_u^d\right)$ will also solve the above problem.

A particular case of interest regarding problem $\left(\mathcal{P}_w^d\right)$ is if the set Θ_w does not actually depend on w. In this case problem $\left(\mathcal{P}_w^d\right)$ will be simply denoted $\left(\mathcal{P}^d\right)$ and will be called *the dual variational formulation of problem* $(\mathbf{P})$. For example, this case is encountered when the functions j_ν and j_τ are convex and positive homogeneous, as it is the case of examples (a)-(c) presented in the previous section.

In the above particular case, problem $\left(\tilde{\mathbf{P}}\right)$ reduces to the following system of variational inequalities

$(\tilde{\mathbf{P}}')$ Find $u \in \Lambda$ and $\sigma \in \mathcal{H}$ such that

$$\begin{cases} (L^*\sigma, v-u)_V + H(v) - H(u) \geq (f, v-u)_V, & \text{for all } v \in \Lambda \\ -(Lu, \mu-\sigma)_{\mathcal{H}} + G(\mu) - G(\sigma) \geq 0, & \text{for all } \mu \in \mathcal{H}, \end{cases}$$

where $H = j \circ T$, $j : L^2(\Gamma_3; \mathbb{R}^m) \to \mathbb{R}$ is defined by

$$j(y) = \int_{\Gamma_3} j_\nu(x,y_\nu) + j_\tau(x,y_\tau)\, d\Gamma,$$

and $T : V \to L^2(\Gamma_3;\mathbb{R}^m)$ is given by $Tv = [(\gamma \circ i)(v)]|_{\Gamma_3}$, with $i : V \to H_1$ being the embedding operator and $\gamma : H_1 \to H^{1/2}(\Gamma;\mathbb{R}^m)$ being the trace operator. On the other hand, for each $w \in \Lambda$,

$$\Theta_w := \Theta = \{\mu \in \mathcal{H} : \ (L^*\mu, v)_V + H(v) \geq (f,v)_V, \ \text{ for all } v \in \Lambda\},$$

and thus by taking $v = 2u$ and $v = 0$ in the first line of $(\tilde{\mathbf{P}}')$ we get

$$(L^*\sigma, u)_V + H(u) = (f,u)_V,$$

hence

$$-(Lu, \mu - \sigma)_{\mathcal{H}} \leq 0, \text{ for all } \mu \in \Theta.$$

Combining this and the second line of $(\tilde{\mathbf{P}}')$ we get

$$G(\mu) - G(\sigma) \geq 0, \text{ for all } \mu \in \Theta,$$

which can be formulated equivalently as

$\left(\mathcal{P}^d\right)$ Find $\sigma \in \mathcal{H}$ such that

$$0 \in \partial G(\sigma) + \partial I_\Theta(\sigma).$$

7.2.3 The existence of weak solutions

The following proposition points out the connection between the variational formulations presented above.

Proposition 7.1. *A pair $(u, \sigma) \in V \times \mathcal{H}$ is a solution for $\left(\mathcal{P}^b_{var}\right)$ if and only if u solves $(\mathcal{P}^p_{var})$ and σ solves $\left(\mathcal{P}^d_u\right)$.*

The main result of this section is given by the following theorem.

Theorem 7.2. (N.C., M. CSIRIK & C. VARGA [26]) *Assume $(\mathbf{H_C})$, $(\mathbf{H_f})$, $(\mathbf{H_h})$, $(\mathbf{H_{j_\nu}})$, $(\mathbf{H_{j_\tau}})$ and $(\mathbf{H}_\phi)$ hold. Then problem $\left(\mathcal{P}^b_{var}\right)$ has at least one solution.*

Before proving the main result we need the following Aubin-Clarke type result concerning the Clarke subdifferential of integral functions. Let us consider the function $j : L^2\left(\Gamma_3; \mathbb{R}^m\right) \times L^2\left(\Gamma_3; \mathbb{R}^m\right) \to \mathbb{R}$ defined by

$$j\left(y, z\right) = \int_{\Gamma_3} j_\nu\left(x, z_\nu\right) + h\left(x, y_\tau\right) j_\tau\left(x, z_\tau\right)\ d\Gamma. \tag{7.32}$$

Lemma 7.1. *Assume $(\mathbf{H_h})$, $(\mathbf{H_{j_\nu}})$ and $(\mathbf{H_{j_\tau}})$ are fulfilled. Then, for each $y \in L^2(\Gamma_3; \mathbb{R}^m)$, the function $z \mapsto j\left(y, z\right)$ is Lipschitz continuous and*

$$j^0_{,2}\left(y, z; \bar{z}\right) \leq \int_{\Gamma_3} j^0_\nu\left(x, z_\nu; \bar{z}_\nu\right) + h\left(x, y_\tau\right) j^0_\tau\left(x, z_\tau; \bar{z}_\tau\right)\ d\Gamma. \tag{7.33}$$

In order to prove Theorem 7.2 we consider the following system of nonlinear hemivariational inequalities.

$(\mathcal{S}_{K_1,K_2})$ Find $(u,\sigma)\in K_1\times K_2$ such that

$$\begin{cases} \psi_1(u,\sigma,v)+J^0_{,1}(Tu,S\sigma;Tv-Tu)\geq (F_1(u,\sigma),v-u)_{X_1}, & \text{for all } v\in K_1,\\ \psi_2(u,\sigma,\mu)+J^0_{,2}(Tu,S\sigma;S\mu-S\sigma)\geq (F_2(u,\sigma),\mu-\sigma)_{X_2}, & \text{for all } \mu\in K_2,\end{cases}$$

where

- $X_1=V$, $X_2=\mathcal{H}$, $K_i\subset X_i$ is closed and convex $(i=1,2)$, $Y_1=L^2(\Gamma_3;\mathbb{R}^m)$, $Y_2=\{0\}$;
- $\psi_1: X_1\times X_2\times X_1\to\mathbb{R}$ is defined by $\psi_1(u,\sigma,v)=A(v,\sigma)-A(u,\sigma)$;
- $\psi_2: X_1\times X_2\times X_2\to\mathbb{R}$ is defined by $\psi_2(u,\sigma,\mu)=A(u,\mu)-A(u,\sigma)$;
- $T: X_1\to Y_1$ is defined by $Tv=[(\gamma\circ i)(v)]|_{\Gamma_3}$, with $i: V\to H_1$ the embedding operator and $\gamma: H_1\to H^{1/2}(\Gamma;\mathbb{R}^m)$ is the trace operator;
- $S: X_2\to Y_2$ is defined by $S\tau=0$, for all $\tau\in X_2$;
- $J: Y_1\times Y_2\to\mathbb{R}$ is defined by $J\left(y^1,y^2\right)=j\left(y^0,y^1\right)$, where $j: L^2(\Gamma_3;\mathbb{R}^m)\times L^2(\Gamma_3;\mathbb{R}^m)\to\mathbb{R}$ is as in (7.32) and y^0 is a fixed element of $L^2(\Gamma_3;\mathbb{R}^m)$;
- $F_1: X_1\times X_2\to X_1$ is defined by $F_1(v,\mu)=f$;
- $F_2: X_1\times X_2\to X_2$ is defined by $F_2(v,\mu)=0$.

Lemma 7.2. *Assume* $(\mathbf{H_h})$, $(\mathbf{H_{j_\nu}})$, $(\mathbf{H_{j_\tau}})$ *and* $(\mathbf{H_\phi})$ *are fulfilled. Then the following statements hold:*

(i) $\psi_1(u,\sigma,u)=0$ *and* $\psi_2(u,\sigma,\sigma)=0$, *for all* $(u,\sigma)\in X_1\times X_2$;

(ii) for each $v\in X_1$ *and each* $\mu\in X_2$ *the maps* $(u,\sigma)\mapsto\psi_1(u,\sigma,v)$ *and* $(u,\sigma)\mapsto\psi_2(u,\sigma,\mu)$ *are weakly upper semicontinuous;*

(iii) for each $(u,\sigma)\in X_1\times X_2$ *the maps* $v\mapsto\psi_1(u,\sigma,v)$ *and* $\mu\mapsto\psi_2(u,\sigma,\mu)$ *are convex;*

(iv) $\liminf_{k\to+\infty}(F_1(u_k,\sigma_k),v-u_k)_{X_1}\geq(F_1(u,\sigma),v-u)_{X_1}$ *and* $\liminf_{k\to+\infty}(F_2(u_k,\sigma_k),\mu-\sigma_k)_{X_2}\geq(F_2(u,\sigma),\mu-\sigma)_{X_2}$ *whenever* $(u_k,\sigma_k)\rightharpoonup(u,\sigma)$ *as* $k\to+\infty$;

(v) there exists $c : \mathbb{R}_+ \to \mathbb{R}_+$ *with the property* $\lim_{t\to+\infty} c(t) = +\infty$ *such that*

$$\psi_1(u,\sigma,0)+\psi_2(u,\sigma,0) \leq -c\left(\sqrt{\|u\|_{X_1}^2 + \|\sigma\|_{X_2}^2}\right)\sqrt{\|u\|_{X_1}^2 + \|\sigma\|_{X_2}^2},\ \textit{for all } (u,\sigma) \in X_1 \times X_2;$$

(vi) The function $J : Y_1 \times Y_2 \to \mathbb{R}$ *is Lipschitz with respect to each variable. Moreover, for all* $\left(y^1, y^2\right), \left(z^1, z^2\right) \in Y_1 \times Y_2$ *we have*

$$J_{,1}^0\left(y^1, y^2; z^1\right) = j_{,2}^0\left(y^0, y^1; z^1\right)$$

and

$$J_{,2}^0\left(y^1, y^2; z^2\right) = 0;$$

(vii) There exists $M > 0$ *such that*

$$J_{,1}^0\left(y^1, y^2; -y^1\right) \leq M\left\|y^1\right\|_{Y_1},\ \textit{for all}\ \left(y^1, y^2\right) \in Y_1 \times Y_2;$$

(viii) there exist $m_i > 0$*,* $i = 1, 2$*, such that* $\|F_i(u,\sigma)\|_{X_i} \leq m_i$*, for all* $(u,\sigma) \in X_1 \times X_2$*.*

Proof of Theorem 7.2 The proof will be carried out in three steps as follows.

STEP 1. Let $K_1 \subset X_1$ and $K_2 \subset X_2$ be closed and convex sets. Then $(\mathcal{S}_{K_1,K_2})$ admits at least one solution.

This will be done by applying a slightly modified version of Corollary 3.7 in [38]. Lemma 7.2 ensures that all the conditions of the aforementioned corollary are satisfied except the regularity of J. We point out the fact that in our case this condition needs not to be imposed because the only reason it is imposed in the paper of Costea & Varga is to ensure the following inequality

$$J^0\left(y^1, y^2; z^1, z^2\right) \leq J_{,1}^0\left(y^1, y^2; z^1\right) + J_{,2}^0\left(y^1, y^2; z^2\right)$$

which in this paper is automatically fulfilled because J does not depend on the second variable and the following equalities take place

$$J^0\left(y^1, y^2; z^1, z^2\right) = J_{,1}^0\left(y^1, y^2; z^1\right)$$

and

$$J^0_{,2}\left(y^1, y^2; z^2\right) = 0,$$

and this completes the first step.

STEP 2. Let $K_1^1, K_1^2 \subset X_1$ and $K_2^1, K_2^2 \subset X_2$ be closed and convex sets and let $\left(u^1, \sigma^1\right)$ and $\left(u^2, \sigma^2\right)$ be solutions for $\left(\mathcal{S}_{K_1^1, K_2^1}\right)$ and $\left(\mathcal{S}_{K_1^2, K_2^2}\right)$, respectively. Then $\left(u^1, \sigma^2\right)$ solves $\left(\mathcal{S}_{K_1^1, K_2^2}\right)$ and $\left(u^2, \sigma^1\right)$ solves $\left(\mathcal{S}_{K_1^2, K_2^1}\right)$.

The fact that $\left(u^1, \sigma^1\right)$ solves $\left(\mathcal{S}_{K_1^1, K_2^1}\right)$ means

$$\begin{cases} \psi_1(u^1, \sigma^1, v) + J^0_{,1}(Tu^1, S\sigma^1; Tv - Tu^1) \geq (F_1(u^1, \sigma^1), v - u^1)_{X_1}, & \forall v \in K_1^1 \\ \psi_2(u^1, \sigma^1, \mu) + J^0_{,2}(Tu^1, S\sigma^1; S\mu - S\sigma^1) \geq (F_2(u^1, \sigma^1), \mu - \sigma^1)_{X_2}, & \forall \mu \in K_2^1 \end{cases} \tag{7.34}$$

while the fact that $\left(u^2, \sigma^2\right)$ solves $\left(\mathcal{S}_{K_1^2, K_2^2}\right)$ shows

$$\begin{cases} \psi_1(u^2, \sigma^2, v) + J^0_{,1}(Tu^2, S\sigma^2; Tv - Tu^2) \geq (F_1(u^2, \sigma^2), v - u^2)_{X_1}, & \forall v \in K_1^2 \\ \psi_2(u^2, \sigma^2, \mu) + J^0_{,2}(Tu^2, S\sigma^2; S\mu - S\sigma^2) \geq (F_2(u^2, \sigma^2), \mu - \sigma^2)_{X_2}, & \forall \mu \in K_2^2 \end{cases} \tag{7.35}$$

Putting together the first line of (7.34) and the second line of (7.35) we get

$$\begin{cases} \psi_1(u^1, \sigma^1, v) + J^0_{,1}(Tu^1, S\sigma^1; Tv - Tu^1) \geq (F_1(u^1, \sigma^1), v - u^1)_{X_1}, & \forall v \in K_1^1 \\ \psi_2(u^2, \sigma^2, \mu) + J^0_{,2}(Tu^2, S\sigma^2; S\mu - S\sigma^2) \geq (F_2(u^2, \sigma^2), \mu - \sigma^2)_{X_2}, & \forall \mu \in K_2^2 \end{cases} \tag{7.36}$$

On the other hand, keeping in mind the way $\psi_1, \psi_2, J, F_1, F_2$ were defined is it easy to check that for any $(v, \mu) \in K_1^1 \times K_2^2$ the following equalities hold

$$\psi_1(u^1, \sigma^1, v) = \psi_1(u^1, \sigma^2, v) \text{ and } \psi_2(u^2, \sigma^2, \mu) = \psi_2(u^1, \sigma^2, \mu),$$

$$J^0_{,1}(Tu^1, S\sigma^1; Tv - Tu^1) = J^0_{,1}(Tu^1, S\sigma^2; Tv - Tu^1)$$

$$J^0_{,2}(Tu^2, S\sigma^2; S\mu - S\sigma^2) = J^0_{,2}(Tu^1, S\sigma^2; S\mu - S\sigma^1)$$

$$F_1(u^1, \sigma^1) = F_1(u^1, \sigma^2) \text{ and } F_2(u^2, \sigma^2) = F_2(u^1, \sigma^2).$$

Using these equalities and (7.36) we obtain

$$\begin{cases} \psi_1(u^1,\sigma^2,v)+J^0_{,1}(Tu^1,S\sigma^2;Tv-Tu^1)\geq (F_1(u^1,\sigma^2),v-u^1)_{X_1}, & \forall v\in K^1_1\\ \psi_2(u^1,\sigma^2,\mu)+J^0_{,2}(Tu^1,S\sigma^2;S\mu-S\sigma^2)\geq (F_2(u^1,\sigma^2),\mu-\sigma^2)_{X_2}, & \forall \mu\in K^2_2\end{cases}$$

hence (u^1,σ^2) solves $\left(\mathcal{S}_{K^1_1,K^2_2}\right)$. In a similar way we can prove that (u^2,σ^1) solves $\left(\mathcal{S}_{K^2_1,K^1_2}\right)$.

STEP 3. There exist $u\in\Lambda$ and $\sigma\in\Theta_u$ such that (u,σ) solves $\left(\mathcal{P}^b_{var}\right)$.

Let us choose $K^1_1=\Lambda$ and $K^1_2=X_2$. According to STEP 1 there exists a pair (u^1,σ^1) which solves $\left(\mathcal{S}_{K^1_1,K^1_2}\right)$. Next, we choose $K^2_1=\Lambda$ and $K^2_2=\Theta_{u^1}$ and use again STEP 1 to deduce that there exists a pair (u^2,σ^2) which solves $\left(\mathcal{S}_{K^2_1,K^2_2}\right)$. Then, according to STEP 2, the pair (u^1,σ^2) will solve $\left(\mathcal{S}_{K^1_1,K^2_2}\right)$. Invoking the way $\psi_1,\psi_2,J,F_1,F_2,K^1_1,K^2_2$ were defined, it is clear that the pair $(u,\sigma)=(u^1,\sigma^2)\in\Lambda\times\Theta_u$ is a solution of the system

$$\begin{cases} A(v,\sigma)-A(u,\sigma)+j^0_{,2}\left(y^0,Tu;Tv-Tu\right)\geq (f,v-u)_V, & \text{for all } v\in\Lambda,\\ A(u,\mu)-A(u,\sigma)\geq 0, & \text{for all } \mu\in\Theta_u,\end{cases}$$

for all $y^0\in L^2(\Gamma_3;\mathbb{R}^m)$, since y^0 was arbitrary fixed. Choosing $y^0=Tu$ an taking into account (7.33) we conclude that $(u,\sigma)\in\Lambda\times\Theta_u$ solves $\left(\mathcal{P}^b_{var}\right)$, hence the proof is complete.

□

We close this section with some comments and remarks concerning the particular case when the boundary conditions (7.17) and (7.18) reduce to the Signorini boundary condition combined with a frictionless condition, that is $\sigma_\tau=0$. In this case

$$C_1=(-\infty,0],\ C_2=\mathbb{R}^m \text{ and } j_\nu,j_\tau,h\equiv 0,$$

while

$$\Lambda=\{v\in V:\ v_\nu\leq 0 \text{ on } \Gamma_3\},$$

and

$$\Theta=\{\mu\in\mathcal{H}:\ (\mu,\varepsilon(v))_{\mathcal{H}}\geq (f,v)_V \text{ for all } v\in\Lambda\}.$$

Problem $\left(\mathcal{P}^b_{var}\right)$ reduces to the following system of variational inequalities

Find $(u,\sigma) \in \Lambda \times \Theta$ such that for all $(v,\mu) \in \Lambda \times \Theta$

$$\begin{cases} A(v,\sigma) - A(u,\sigma) \geq (f, v-u)_V, \\ A(u,\mu) - A(u,\sigma) \geq 0. \end{cases} \tag{7.37}$$

This case was studied recently by Matei [77] who used the Direct Method in the Calculus of Variations to prove that the functional $\mathcal{L} : \Lambda \times \Theta \to \mathbb{R}$

$$\mathcal{L}(v,\mu) = A(v,\mu) - (f,v)_V,$$

admits a global minimizer and each minimizer (u,σ) of $\mathcal{L}$ is in fact a solution for (7.35). Our proof is different, so even in this particular case our approach is new and supplements the result obtained by Matei in [77]. Furthermore, as far as we are aware, there are no papers in the literature in which the existence of the solutions for the variational approach via bipotentials is proved by using systems of hemivariational inequalities. It would be interesting to consider constitutive laws that involve bipotentials which are not separable.

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I hereby declare that the dissertation contains no material accepted for any other degree in any other institution.

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I hereby declare that Nicuşor COSTEA is the primary author of the following joint paper which is based on the material contained in this thesis:

N. COSTEA & G. MOROŞANU, A multiplicity result for an elliptic anisotropic differential inclusion involving variable exponents, *Set-Valued Var. Anal.* **21** (2013), 311-332.

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I hereby declare that Nicuşor COSTEA is the primary author of the following joint papers which is based on the material contained in this thesis:

1. N. COSTEA & C. VARGA, Systems of nonlinear hemivariational inequalities and applications, *Topol. Methods Nonlinear Anal.* **41** (2013), 39-65.

2. N.COSTEA & C.VARGA, Multiple critical points for non-differentiable parametrized functionals and applications to differential inclusions, *J. Global Optim.* **56** (2013), 399-416.

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N. COSTEA, D.A. ION & C. LUPU, Variational-like inequality problems involving set-valued maps and generalised monotonicity, *J. Optim. Theory Appl.* **155** (2012), 79-99.

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We hereby declare that Nicuşor COSTEA is the primary author of the following joint paper which is based on the material contained in this thesis:

N.COSTEA, M. CSIRIK & C. VARGA, Weak solvability via bipotential method for contact models with nonmonotone boundary conditions, *Z. Angew. Math. Phys.* (2015), DOI 10.1007/s00033-015-0513-2.

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Statement of objection

Student's name: Nicuşor COSTEA

Program: Ph.D. in Mathematics

Dissertation title: Existence results for some differential inclusions and related problems

Disseration supervisor(s): Gheorghe MOROŞANU

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