CENTRAL EUROPEAN UNIVERSITY

MASTER'S THESIS

# Maximal Subgroups and Character Theory

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I, Trevor Chimpinde, declare that this thesis titled, "Maximal Subgroups and Character Theory" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
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- I have acknowledged all main sources of help.
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Signed : T. C. Chimpinde

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# Contents

# Introduction

In this thesis, we consider two group theoretical problems involving maximal subgroups. The first problem is about the upper bound of the maximal subgroups of a finite solvable group. Newton[10] proved that if a solvable group has order with prime decomposition $p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$ and $p_1^{r_1}$ is the smallest of the ${p_i}^{r_i}$'s, then the number of maximal subgroups of that group is at most $\frac{p_1^{r_1}-1}{p_1-1} + \sum_{i=2}^{n} \frac{p_i^{r_i+1}-1}{p_i-1}$. We follow his ideas but give a slightly different proof of this result. The other problem has to do with the relationship between maximal subgroups and primitive permutation character of solvable groups. These characters are multiplicity-free and any two distinct primitive permutation characters only have the trivial character in common. We will show that if all the irreducible complex characters of a solvable group occur as constituents of primitive permutation characters of the group then the group is either elementary Abelian or is a Frobenius group, whose kernel is elementary Abelian and the complement is a cyclic group of prime order.

A group is a set of elements together with a binary operation that together satisfy the properties of associativity, the identity property, and the inverse property. Groups which arose from the study of polynomial equations are fundamental to the study of the symmetry of mathematical objects. Group theory, the study of the structure of groups, has wide ranging applications in other mathematical disciplines such as algebraic topology and Galois theory, as well as in chemistry and physics. Various techniques have been developed to study the structure of a group. One such technique is the study of their maximal subgroups.

A maximal subgroup of a group is a proper subgroup such that no other proper subgroup contains it. For finite groups, maximal subgroups always exist. There are however groups that contain no maximal subgroups. An example of such a group is the Prüfer group. Studying maximal subgroups can help to understand the structure of a group. Finite groups, all of whose maximal subgroups are normal, have limited structure and are called nilpotent groups. The Frattini subgroup of a group is closely related to its maximal subgroups. It is, by definition, the intersection of all the maximal subgroups of a group and contains exactly the non-generating elements of the group. Maximal subgroups can also help in deciding which groups can be embedded in another group. This problem arises frequently in group theory. If one can find a way of computing maximal subgroups of a group, he can then recursively compute all of its subgroups. O'Nan and Scott[1] have classified the maximal subgroups of the alternating and symmetric groups. The maximal subgroups of the small finite simple groups are listed in the Atlas of Finite Simple Groups.

In studying maximal subgroups, one can also restrict the study to the number of maximal subgroups. This also can tell something about the group. We know for example, that a finite group has only one maximal subgroup if and only if its order is a power of a prime number. The number of maximal subgroups of a cyclic group is equal to the number of prime divisors of its order while the number of maximal subgroups of an elementary Abelian $p$-group of rank $r$ is $\frac{p^r-1}{p-1}$. In this thesis, we will prove an upper bound on the number of maximal subgroups of a solvable group. A solvable group is a group that is made up of cyclic groups of prime order. We will use character theory to give a different proof of Newton's theorem.

A representation of a group assigns to each element of a group a matrix so that the group operation is compatible with matrix multiplication. The character of a group representation is a function on the group that associates to each group element the trace of the corresponding matrix. The character of a group carries essential information of a group representation in a condensed form. Over the field of complex numbers, two representations are isomorphic if and only if they have the same character. We can study the structure of a group by studying irreducible characters, which are the building blocks of all the characters. These characters encode important properties of a group. We can recover all the normal subgroups of a group from its irreducible characters. We can therefore determine if a group is simple from its irreducible characters. Moreover, we can determine from its irreducible characters if a group is simple. Character theory was essential in the classification of finite simple groups, the proof of Frobenius theorem on the structure of Frobenius groups and Feit-Thompson theorem which states that every finite group of odd order is solvable.

The action of a group on a finite set induces a representation of the group. The corresponding character is called the permutation character. If the action is transitive and faithful, and does not preserve a non-trivial partition on the set on which it is acting, we say that the action is primitive and call the corresponding character the primitive permutation character. It can be shown that a primitive permutation character of a solvable group can be written as linear combination of irreducible complex characters with coefficients 0 and 1. Moreover, any two distinct primitive permutation characters, upon decomposing, have only the trivial character in common. In general, it is not true that an irreducible character is a constituent of a primitive character. We will prove that if all the irreducible characters of a solvable group occur as constituents of its primitive permutation characters, then such a group is elementary Abelian or a Frobenius group with elementary Abelian kernel and with complement a cyclic group of prime order.

In the first chapter, we define concepts and theorems we will need for the remaining chapters. This includes solvable groups, theory of characters of a group and primitive permutation characters of a group. The second chapter gives an outline of results concerning the upper bound of the number of maximal subgroups of a solvable group. We also compare different bounds and also give a proof of the theorem by Newton. The last chapter consists of a result of all solvable groups whose irreducible characters are constituents of primitive permutation characters. Before the proof, we give an exposition of Frobenius groups and their character theory.

# Chapter 1

# Preliminaries

This chapter contains some basic definitions, examples and results required for the remaining chapters. We give a brief description of solvable groups, the theory of characters of finite groups and primitive permutation characters.

## 1 Groups

A group is an algebraic structure consisting of a set of elements and a a binary operation that together satisfy certain axioms.

**Definition 1.1.** *A set $G$ together with a binary operation $*$, is called a group if it satisfies the following axioms:*

1. *The binary operation is associative on $G$.*

2. *$G$ has an element $e$ with the property $g * e = e * g = g$ for all elements $g \in G$.*

3. *For each element $g \in G$, there exists an element $h \in G$ called the inverse of $g$ with the property that $g * h = h * g = e$*

For $g \in G$, its inverse will be denoted by $g^{-1}$. If $g, h \in G$, then will simply write $gh$ instead of $g * h$.

Groups are important because they can be used to study symmetry of mathematical objects. This is done by defining an automorphism of the object which is a way of mapping the object to itself while preserving all of its structure. The set of all automorphisms of an object forms a group, called the automorphism group, and is the symmetry group of the object. Examples of groups are:

- The symmetric group, $S_n$. This is the group all of whose elements are all the permutations on the $n$ distinct symbols and binary operation is the composition of these permutations.

- The dihedal group , $D_n$. This is the group of symmetries, rotation and reflections, of an $n$-sided polygon.

In this thesis, we are concerned with finite groups. These are groups with finite number of elements. We will restrict ourselves to finite groups for the rest of this thesis.

**Definition 1.2** (Subgroup). *A non-empty subset $H$ of a group $G$ with binary operation $*$ is called a subgroup of $G$ if $H$ also forms a group under the operation $*$. We write $H \leq G$ to mean $H$ is a subgroup of a group $G$.*

A subgroup $H$ of $G$ is proper if $H \neq G$. The notation for this is $H < G$.

**Definition 1.3** (Maximal Subgroups). *A maximal subgroup $H$ of a group $G$ is a proper subgroup, such that no proper subgroup $K$ of $G$ strictly contains $H$.*

Maximal Subgroups are a central theme of this thesis. Maximal Subgroups are important because they help provide information about the group.

**Definition 1.4** (Normal Subgroup). *A subgroup $N$ of $G$ is called normal, written $N \triangleleft G$, if for all $g \in G$ and for all $n \in N$, $gng^{-1} \in N$.*

The notion of normal subgroup is important because it is closely related to the components or simple groups that make up the group.

**Definition 1.5** (Simple Group). *A simple group is a nontrivial group whose only normal subgroups are the trivial group and the group itself.*

**Theorem 1.1** (Classification of Finite Simple Groups). *Let $G$ be a finite simple group. Then $G$ is isomorphic to one of the following groups:*

1. *A cyclic group of prime order.*
2. *An alternating group of degree at least 5.*
3. *A simple group of Lie type.*
4. *One of the 26 sporadic simple groups.*

*Proof.* See [5] □

**Definition 1.6** (Quotient Group). *Given a group $G$ and a subgroup $H$, and an element $g$ in $G$, the set $gH = \{gh : h \in H\}$ is called the left coset of $H$ in $G$. Let $G/H$ be the set of all left cosets of $H$ in $G$, that is, $G/H = \{gN : g \in G\}$. Define an operation on $G/H$ as follows: If $g_1H$ and $g_2H$ are in $G/H$, then their product is $g_1g_2H$.*

*With this operation $G/H$ becomes a group if and only if $H$ is a normal subgroup of $G$. It is called the quotient group of $G$ by $H$.*

If $N$ is a non-trivial proper normal subgroup of a finite group $G$, then $N$ and $G/N$ are groups whose orders are smaller than $|G|$. We cannot recover $G$ from $N$ and $G/N$. However, using the method of mathematical induction, we can use them to prove something about $G$, see for example 3.11. The notion of quotient groups is also key in showing that every finite group is comprised of simple components called composition factors.

A series, $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$, of subgroups of $G$ is called a normal series if $G_i \triangleleft G_{i+1}$, for $0 \leq i \leq n-1$. For $0 \leq i \leq n-1$, the quotient groups $G_{i+1}/G_i$ are called the factors of the normal series.

A normal series $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$, where $G_i$ is maximal in $G_{i+1}$ is called a composition series. In this case, the factors are simple groups and they are called composition factors.

Every group that is not simple, can be broken into two smaller groups, a normal subgroup and the quotient group. The process can be repeated so that we eventually arrive at simple groups. The Jordan–Hölder theorem tells us that no matter how we break up our group, these simple groups are uniquely determined.

**Theorem 1.2** (Jordan–Hölder)**.** *Let $G$ be a non-trivial finite group. The set of composition factors in a composition series of a group are unique, up to isomorphism.*

*Proof.* See [9, p. 30] □

# 2 Solvable Groups

In this thesis, we are interested in the number of maximal subgroups of solvable groups. Solvable groups are those groups whose composition factors are groups of prime order.

**Definition 1.7** (Solvable Group)**.** *A group is solvable if all its composition factors are cyclic groups of prime order.*

The above definition is equivalent to saying that the group has a normal series all whose factors are Abelian. We examine these groups in a little more detail.

Solvability of a group is closely related to a subgroup called the derived subgroup.

**Definition 1.8.** *Define the commutator of two elements $g$ and $h$ of a group G, denoted by $[g, h]$, to be the element $g^{-1}h^{-1}gh$ . The derived subgroup of a group G which will be denoted by $G'$ is the subgroup*

$$G' = \langle [g, h] \mid g, h \in G \rangle .$$

From the above we define the derived series of a group $G$ to be a series of subgroups

$$G^{(0)} = G, \quad G^{(1)} = G', \quad G^{(2)} = {G^{(1)}}', \quad \cdots$$

**Theorem 1.3.** *The derived subgroup is the smallest normal subgroup such that the quotient group of the original group by this subgroup is Abelian. That is, $G/N$ is Abelian if and only if $N$ contains the derived subgroup.*

*Proof.* See **??**. □

Putting everything together, we get that the derived series terminates. By this we mean there exists $n \in \mathbb{N}$ such that $G^{(n)} = 1$, if and only if $G$ is a solvable group. Furthermore, the derived series is minimal of all the normal series of $G$ with abelian factors. For the rest of the thesis, we will restrict ourselves to finite solvable groups.

**Theorem 1.4** (Feit-Thompson)**.** *Let $G$ be a group. If $G$ has odd order, then $G$ is solvable.*

*Proof.* See [4]. □

Feit-Thompson theorem also tells us that the order of a finite non-Abelian simple group is either a prime number or an even number.

**Theorem 1.5** (Burnside)**.** *For primes $p$ and $q$, every group of order $p^a q^b$ is solvable.*

*Proof.* See [9, theorem 7.8]. □

**Theorem 1.6.** *The minimal normal subgroup of a solvable group is elementary Abelian.*

*Proof.* The proof proceeds by showing that the minimal normal subgroup is Abelian and then that its order is divisible by only one prime number.

Let $N$ be a minimal normal subgroup of a group $G$. Since $N$ is solvable, we must have that the derived subgroup $N'$ is a proper subgroup. Also $N'$ is a characteristic subgroup of $N$, hence normal in G. But $N$ is minimal, thus we have $N' = 1$.

Let $p$ be a prime divisor of $|N|$ and $P$ be a Sylow $p$-subgroup of $N$. $P$ is characteristic subgroup of $N$ since it is the unique Sylow $p$-subgroup. Thus, $P$ is a normal subgroup in $G$ implying that $P = N$.

So far we have shown that $N$ is an Abelian $p$-group. To show it is elementary Abelian, consider the $\{x \in N \mid x^p = 1\}$ of $G$. This subgroup is characteristic in $N$ and hence normal in G. Thus, $\{x \in N \mid x^p = 1\}$ is $N$. It cannot be trivial since Cauchy theorem guarantees the existence of some element of $N$ with prime order. □

**Theorem 1.7.** *Every maximal subgroup of a finite solvable group has prime power index.*

*Proof.* Let $M$ be a maximal subgroup of $G$ and $N$ be a minimal normal subgroup of $G$. If $N \not\leq M$, then $NM = G$ and $N \cap M = 1$. The first equality is because $M$ is maximal and $N \not\leq M$. The second equality follows because $N \cap M$ is normal in $M$ and also normal in $N$ as it is Abelian. Thus $[G : M] = |N|$, and from the previous theorem the minimal normal subgroup is elementary Abelian.

If $N \leq M$, then we can work in the quotient group $G/N$, and use induction. □

## 3 The Character Theory of Finite Groups

We would now like to review some results in character theory of finite groups. These results will be crucial when we come to prove our main results.

A character of a group is closely associated to its representation. A representation describes a group in terms of linear transformations of vector spaces while the character of that group representation associates to each group element the trace of the corresponding matrix. The character is significant because it carries the essential information about the representation in a more condensed form.

**Definition 1.9.** *Let $G$ be a group and $V$ be a vector space over the field of complex numbers. A linear representation of $G$ over $V$ is a group homomorphism from $G$ to $GL(V)$, the general linear group on V. In other words, a representation is a map $\rho\colon G \to \mathrm{GL}(V)$ such that*

$$\rho(g_1g_2) = \rho(g_1)\rho(g_2), \qquad \textit{for all } g_1, g_2 \in G.$$

- The dimension of the vector space $V$ is called the degree of the representation. If $\dim(V) < \infty$, then $\rho$ is called a finite dimensional representation. In this case, we can choose a basis for $V$ and identify $GL(V)$ with $GL(n, \mathbb{C})$, the group of $n \times n$ invertible matrices over $\mathbb{C}$.
- The kernel of a representation $\rho$ of a group $G$ is defined as

  $$\ker \rho = \{g \in G \mid \rho(g) \text{ is the the identity transformation}\}.$$

  It is a normal subgroup of $G$. A representation $\rho$ is faithful if $\ker \rho = \{e\}$.

Another way of looking at a representation $\rho\colon G \to \mathrm{GL}(V)$ is that it is a linear action of $G$ on $V$. That is to say that $\rho\colon G \times V \to V$ is an action such that

1. $\forall v_1, v_2 \in V$ and $\forall g \in G, \qquad \rho(g, v_1 + v_2) = \rho(g, v_1) + \rho(g, v_2).$
2. $\forall \lambda \in \mathbb{C}, g \in G, v \in V, \qquad \rho(g, \lambda \cdot v) = \lambda \cdot \rho(g, v).$

It is for this reason that we will sometimes write $g \cdot v$ or simply $gv$ to mean $\rho(g)(v)$ when it is clear that $\rho$ is the representation we are considering.

Given two complex vector spaces $V$ and $W$, two representations $\rho : G \to GL(V)$ and $\pi : G \to GL(W)$ are said to be isomorphic if there exists a vector space isomorphism $\alpha\colon V \to W$ so that for all $g$ in $G$,

$$\alpha \circ \rho(g) \circ \alpha^{-1} = \pi(g).$$

Isomorphic representations provide the same information about the group and are considered to be the same.

**Example 1.** For any group G, the map

$$\rho : G \to \mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times, \quad g \mapsto 1, \quad \forall g \in G$$

is a representation. This is called the trivial representation of G.

**Example 2.**

Let $G = C_2 = 1, g$ be the cyclic group of order 2.

$$\rho : G \to \mathbb{C}^\times, \quad g \mapsto -1$$

is a representation.

**Definition 1.10.** *Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of $G$ and $W$ be a linear subspace of $V$ that is preserved by the action of $G$, that is, $g{\cdot}w \in W$ for all $w \in W$. Then denote by $\rho_W : G \to GL(W)$, the map that sends each $g$ to the restriction $\rho(g)|_W$. Then $\rho_W : G \to GL(W)$, becomes a representation of $G$ and is called a subrepresentation of the representation $\rho$.*

If $\rho\colon G \to \mathrm{GL}(V)$ has exactly two subrepresentations, namely the trivial subspace $\{0\}$ and $V$ itself, then the representation is said to be irreducible.

**Definition 1.11.** *Let $V$ be a finite-dimensional complex vector space and let $\rho : G \to GL(V)$ be a representation of a group $G$ on $V$. The character of $\rho$ is the function $\chi_\rho : G \to \mathbb{C}$ given by*

$$\chi_\rho(g) = \mathrm{Tr}(\rho(g)),$$

*where* $\mathrm{Tr}$ *is the trace.*

- A character $\chi_\rho$ is called irreducible if $\rho$ is an irreducible representation.
- The degree of the character $\chi$ is the dimension of $\rho$: this is equal to the value $\chi(1)$.
- The kernel of the character $\chi_\rho$ is the normal subgroup

  $$\ker \chi_\rho := \{g \in G \mid \chi_\rho(g) = \chi_\rho(1)\},$$

  which is precisely the kernel of the representation $\rho$.
- Two representations are isomorphic if and only if they have the same character.
- Let $\chi_\rho$ be the character that affords a representation $\rho$ of $G$. If $g, h \in G$, then

  $$\chi_\rho\left(hgh^{-1}\right) = \mathrm{Tr}\left(\rho(h)\,\rho(g)\,\rho\left(h^{-1}\right)\right) = \mathrm{Tr}\left(\rho(g)\right) = \chi_\rho(g).$$

  We see that a character takes a constant value on a given conjugacy class. Functions on a given group $G$ into $\mathbb{C}$ with this property are called class functions. The set of irreducible characters of a group $G$ forms a basis of the $\mathbb{C}$-vector space of all class functions $G \to \mathbb{C}$.

**Theorem 1.8.** *Let $G$ be a finite group. Then the number of non-isomorphic finite-dimensional irreducible representations is equal to the number of conjugacy classes of $G$.*

**Definition 1.12.** *Let $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$. The direct sum of these two representations is the map $\rho_1 \oplus \rho_2 : G \to GL(V_1 \oplus V_2)$ given by $\rho_1 \oplus \rho_2(g) = (\rho_1(g), \rho_2(g))$ for all $g \in G$.*

**Theorem 1.9** (Maschke)**.** *Any representation of a finite group can be written as a direct sum of irreducible representations.*

*Proof.* See [8, theorem 1.9] □

If a representation is the direct sum of subrepresentations, then the corresponding character is the sum of the characters of those subrepresentations.

**Theorem 1.10.** *Let $G$ be a finite group and $\chi_1, \chi_2, \cdots, \chi_n$ be the complete set of the characters of the non-isomorphic finite-dimensional representations of $G$ over $\mathbb{C}$. If $d_i$ is the degree of $\chi_i$, then*

$$|G| = {d_1}^2 + {d_2}^2 \cdots + {d_r}^2.$$

*Proof.* See [8] □

We now look at results that study the relationship between characters of a given group $G$ and the characters of its subgroups.

If $H \leq G$ and $\chi$ is a character of $G$. Then the restriction of $\chi$ to $H$, $\chi_{_H}$, is a character of $H$. This is the character of the restricted representation. If $\chi_{_H} \in \mathrm{Irr}(H)$, then $\chi \in \mathrm{Irr}(G)$. The converse is not true. Not much can said be about $\chi_{_H}$ if $H$ is not a normal subgroup. Alfred H. Clifford proved the following result on the restriction of finite-dimensional irreducible representations from a group to a normal subgroup of finite index.

Let $N \triangleleft G$ and $\theta \in \mathrm{Irr}\,(G)$. For $g \in G$, we define $\theta^g : N \to \mathbb{C}$ by

$$\theta^g\,(n) = \theta\left(gng^{-1}\right)$$

$\theta^g$ is an irreducible character and it follows that G acts by permutations on the set $\mathrm{Irr}(N)$. In fact, since $N$ acts trivially on $\mathrm{Irr}\,(N)$ it is more precise to say that $G/N$ acts on $\mathrm{Irr}(N)$.

By the Orbit-Stabiliser theorem,

$$|\mathrm{Orb}_G(\theta)||\mathrm{Stab}_G(\theta)| = |G|.$$

As $N$ acts trivially on $\mathrm{Irr}\,(N)$, we have that $N \leq \mathrm{Stab}_G(\theta) \leq G$. This subgroup $\mathrm{Stab}_G(\theta)$, is called the inertia subgroup of $\theta$ by $G$ and is denoted by $I_G\,(\theta)$.

**Theorem 1.11** (Clifford). *Let $N \triangleleft G$ and let $\chi \in Irr(G)$. Let $\theta$ be an irreducible constituent of $\chi_N$. Further, let $\theta = \theta_1, \theta_2, \ldots, \theta_{[G:I_G(\theta)]}$ be the distinct conjugates of $\theta$ under the action of $G$. Then*

$$\chi_{_N} = \langle \chi_{_N}, \theta \rangle \sum_{i=1}^{[G:I_G(\theta)]} \theta_i$$

*Proof.* See [8, theorem 6.2] □

We now look at a way of obtaining a character of a group from a character of its subgroup.

**Definition 1.13.** *Let $H \leq G$ and $\psi$ be a character of $H$. Then we define the induced character of $\psi$ on $H$ , denoted by $\psi^G$, as*

$$\psi^G\,(g) = \frac{1}{|H|} \sum_{t \in G,\; t^{-1}gt \in H} \psi\left(t^{-1}gt\right) = \frac{1}{|H|} \sum_{t \in G} \psi^0\left(t^{-1}gt\right).$$

*where*

$$\psi^0(x) = \begin{cases} \psi(x) & : x \in H0 \\ 0 & : x \neq H \end{cases}$$

Below we justify why $\psi^G$ a character of $G$. Observe that $\psi^G$ is constant on the conjugacy classes of $G$.

Let $G$ be a finite group, $H$ a subgroup and $(\psi, V)$ a representation of $H$. We define $V^G$ to be the vector space of all functions $f : G \to V$ such that $f(hx) = \psi(h)f(x)$ when $h \in H$ and $x \in G$. Define, for $g \in G$

$$(\psi^G(g)f)(x) = f(xg).$$

That is $g$ acts on $V^G$ by right translation.

We can easily show that if $f \in V^G$ and $g \in G$ then $\psi^G(g)f \in V^G$ and $\psi^G(g_1g_2) = \psi^G(g_1)\psi^G(g_2)$ so that the pair $(\psi^G, V^G)$ is a representation of $G$. It is called the induced representation and it affords the character defined in definition 1.13.

**Proposition 1.1.** *Let $H$ be a subgroup of a group $G$ and $\psi$ be a character of $H$. Then*

$$\ker\left(\psi^G\right) = \cap_{g\in G}(\ker\psi)^g := core_G(ker\psi).$$

*Proof.* $g \in \ker\left(\psi^G\right)$ if and only if

$$\sum_{t\in G}\psi^0\left(t^{-1}gt\right) = \sum_{t\in G}\psi(1).$$

Now $|\psi^0(t^{-1}gt)| \leq \psi(1)$. Therefore, $g \in \ker\left(\psi^G\right)$ if and only if $\psi^0(t^{-1}gt) = \psi(1)$ for all $x \in G$. This is equivalent to requiring that $g \in (\ker\psi)^t$ for all $t \in G$ and the proof is complete. □

We now state the Mackey Decomposition theorem which concerns the way a character induced from a subgroup $H$ of a finite group $G$ behaves when restricted to a subgroup $K$ of $G$.

**Theorem 1.12** (Mackey Decomposition). *Let $H$ and $K$ be subgroups of a finite group $G$. Let $\theta$ be a character of $H$ and let*

$$G = \bigcup_{t\in T} HtK$$

*be partition of $G$ into double cosets. Then*

$$\left(\theta^G\right)_K = \sum_{t\in T}\left(\left[\theta^t\right]_{t^{-1}Ht\cap K}\right)^K.$$

*Proof.* See [8, Problem 5.6]. □

**Theorem 1.13** (Frobenius Reciprocity). *Let $H \leq G$ and suppose that $\varphi$ is a character on $H$ and that $\theta$ is a character on $G$. Then*

$$\langle\varphi, \theta_H\rangle = \left\langle\varphi^G, \theta\right\rangle.$$

*Proof.* See [8, lemma 5.2] □

In the language of representation theory, Frobenius reciprocity states that given representations $\psi$ of $H$ and $\rho$ of G, the space of $H$-equivariant linear maps from $\psi$ to $\rho_{_H}$ has the same dimension over $\mathbb{C}$ as that of $G$-equivariant linear maps from $\psi^G$ to $\rho$.

The inertia subgroup is important because of the following theorem.

**Theorem 1.14.** *Let $N \lhd G$ and let $\theta \in Irr(N)$ and $T = I_G(\theta)$. Let*

$$\mathcal{A} = \{\psi \in Irr\,(T) \mid \langle \psi_{_N}, \theta\rangle \neq 0\} \quad \text{and} \quad \mathcal{B} = \{\chi \in Irr\,(T) \mid \langle \chi_{_N}, \theta\rangle \neq 0\}\,.$$

*Then $\psi^G$ is irreducible and the map $\psi \to \psi^G$ is a bijection of $\mathcal{A}$ onto $\mathcal{B}$.*

*Proof.* See [8, theorem 6.11] □

## 4 Permution Characters

Recall that if a group $G$ acts as a group of permutations on a set $X$ if

i. $e \cdot x = x, \quad \forall x \in X$, where $e$ is the identity element of $G$.

ii. $g \cdot (h \cdot x) = gh \cdot x, \quad \forall g, h \in G$ and $\forall x \in X$.

If $|X| < \infty$, we can define a finite dimensional representation as follows:

Let $V = \mathbb{C}X$ be the vector space defined to have $X$ as its basis. $V$ consists of elements of the form

$$\sum_{x \in X} c_x x$$

Then $\rho\colon G \to \mathrm{GL}\,(V)$ given by

$$\rho\,(g)\left(\sum_{x\in X} c_x x\right) = \sum_{x\in X} c_x g \cdot x$$

is a called a permutation representation of $G$. Moreover, $\rho\,(G)$ consists of permutation matrices. Let $\mathrm{fix}\,(g) = \{x \in X \mid g \cdot x = x\}$. Let $\chi_\rho$ be the character of the above defined representation. Since $X$ forms a basis of $V$ by definition, we get that

$$\chi_\rho\,(g) = |\mathrm{fix}\,(g)|\,.$$

**Lemma 1.1** (Orbit-Counting lemma)**.** *Let $G$ be a permutation group on the finite set $X$. Then the number of orbits of $G$ on $X$ is equal to the average number of fixed points of an element of $G$, that is,*

$$\frac{1}{|G|}\sum_{g\in G} \mathit{fix}\,(g)\,.$$

An action is said to be transitive if it posseses only a single orbit. This means that for any pair of elements $x, y \in X$, there exists a $g \in G$ such that $g \cdot x = y$.

Let $H$ be a subgroup of $G$. The coset space $H \backslash G$ is the set of left cosets $\{xH\colon x \in G\}$. The following theorem is a classification of transitive actions of a group.

**Theorem 1.15.** *Let $G$ act transitively on $X$. Let $x \in X$ and $H = Stab_G(x)$. Then*

*i Then this action on $X$ isomorphic to the natural action of the coset space $H \backslash G$,*

*ii The group action on two coset spaces $H \backslash G$ and $K \backslash G$ are isomorphic if and only if $H$ and $K$ are conjugate subgroups of $G$.*

Let $G$ be a group acting of a finite set $X$. Further, let $x \in X$ and $H = \mathrm{Stab}_G(x)$. Denote by $\mathbf{1}_H$ the trivial character of $H$ and by $\mathbf{1}_H^G$ the induced character of $\mathbf{1}_H$ to $G$. Let $T$ be the set of representatives for the cosets of $H$ in $G$. Then

$$\begin{aligned}\mathbf{1}_H^G(g) &= \frac{1}{|H|} \sum_{t \in G,\, t^{-1}gt \in H} \mathbf{1}_H\left(t^{-1}gt\right) \\ &= \sum_{t \in T} \mathbf{1}_H\left(t^{-1}gt\right) \\ &= |\mathrm{fix}(g)|\,.\end{aligned}$$

We get that $\mathbf{1}_H^G$ is the permutation character associated to the action of a group $G$ on the coset space $H \backslash G$.

## 5 Primitive Permutation Characters

The last thing we would like to discuss in this chapter is primitive permutation characters. These are central in this thesis and we will discuss them at some length.

Let $G$ act transitively on $X$. If $Y \subset X$ then define $gY = \{g \cdot y \mid y \in Y\}$. A block is a non-empty subset $\triangle$ of $X$ such that $g\triangle = \triangle$ or $\triangle \cap g\triangle = \emptyset$. for all $g \in G$ . Observe that singletons subset and the whole set $X$ are blocks. These are called trivial blocks.

A group action is called primitive if it is transitive and it has no non-trivial blocks. A transitive group action that is not primitive is called imprimitive.

Above, we classified all transitive actions of a given group. Below we classify primitive actions of a group.

**Theorem 1.16.** *A transitive group action of a group $G$ on a set $X$ is primitive if and only if $Stab_G(x)$ is a maximal subgroup of $G$ for some $x \in X$.*

*Proof.* Suppose that $Stab_G(x) < H < G$ and consider the the set

$$\triangle = \{h\, Stab_G(x) : h \in H\}\,.$$

Then $g\triangle = \triangle$ for $g \in H$ and $g\triangle \cap \triangle = \emptyset$ for $g \notin H$.

Conversely, if $\triangle$ is a non-trivial block containing $x$, then the setwise stabiliser

$$Stab_G(\triangle) = \{g \in G \mid g \cdot x \in \Delta\}$$

is a subgroup and $Stab_G(x) < Stab_G(\triangle) < G$. □

We will call the permutation character associated to a primitive group action a primitive permutation character.

**Theorem 1.17.** *If $G$ is solvable and $M, N < G$ are maximal subgroups then either $MN = G$ or $M$ and $N$ are conjugate.*

*Proof.* See [3, theorem A.16.2] □

We define one more concept. A group $G$ acts regularly of a set $X$ if the action is transitive and if only the identity element has a fixed point. By the structure theorem for transitive groups1.15, if $G$ acts regularly on $X$, then $X$ is isomorphic to the space of left cosets of the trivial subgroup. This set can be identified with $G$ with the action being left multiplication.

**Theorem 1.18.** *Let $G$ be a solvable group acting primitively and faithfully on a set $X$. Further, let $x \in X$ and $N$ be a minimal subgroup of $G$. Then $N$ is elementary Abelian (by 1.6) and it acts regularly on $X$. Hence, $N \cap Stab_G(x) = 1$. Since $Stab_G(x)$ is a maximal subgroup of $G$, $NStab_G(x) = G$.*

*Proof.* See [12, 7.2.6]. □

**Theorem 1.19.** *Primitive permutation characters of solvable groups are multiplicity-free.*

*Proof.* Let $G$ be a solvable group with maximal subgroup $M$. Then $G$ acts primitively on the space of left cosets of $M$ with kernel $\text{core}_G(M)$. Thus, $G/\text{core}_G(M)$ acts faithfully and primitively on $M\backslash G$. By 1.18, a minimal normal subgroup $N/\text{core}_G(M)$ of $G/\text{core}_G(M)$ has order $[G : M]$. Consider the primitive permutation character $\mathbf{1}_M^G$. We have

$$\mathbf{1}_M^G(n) = \begin{cases} [G:M] & : n = 1 \\ 0 & : n \in N, n \neq 1 \end{cases}$$

Thus $\mathbf{1}_M^G|_N$ is the regular character of $N$ which is multiplicity-free. Since restriction is additive, we infer that $\mathbf{1}_M^G$ is also multiplicity-free. □

**Lemma 1.2.** *Let $M, N$ be non-conjugate maximal subgroups of a group $G$. Then*

$$\left\langle \mathbf{1}_M^G, \mathbf{1}_N^G \right\rangle = 1.$$

*Proof.* By Frobenius reciprocity1.13, we have that $\left\langle \mathbf{1}_M^G, \mathbf{1}_N^G \right\rangle = \left\langle \mathbf{1}_M, \mathbf{1}_N^G|_M \right\rangle$. By theorem 1.17, $MN = G$. Thus, apply Mackey decomposition theorem 1.12, we get

$$\left\langle \mathbf{1}_M, \mathbf{1}_N^G|_M \right\rangle = \left\langle \mathbf{1}_M, \mathbf{1}_{M\cap N}^M \right\rangle = \left\langle \mathbf{1}_{M\cap N}, \mathbf{1}_{M\cap N} \right\rangle = 1$$

and the proof is complete. □

We can infer from proposition 1.1, that

$$\ker\left(\mathbf{1}_M^G\right) = \text{core}_G\left(M\right).$$

**Lemma 1.3.** *Let $M$ be a maximal subgroup of a group $G$. If $\chi \in Irr\left(G\right)$ is a non-trivial constituent of $\mathbf{1}_M^G$, then*

$$\ker\left(\mathbf{1}_M^G\right) = \ker\chi.$$

*Proof.* Since $\ker\left(\mathbf{1}_M^G\right) = \cap\left\{\chi \in \text{Irr}\left(G\right) \mid \left\langle \mathbf{1}_M^G, \chi\right\rangle = 1\right\}$, we have that $\ker\left(\mathbf{1}_M^G\right) \leq \ker\chi$. Since $\ker\left(\mathbf{1}_M^G\right) = \text{core}_G\left(M\right)$ and $\text{core}_G\left(M\right)$ which is the largest normal subgroup of $G$ contained in the maximal subgroup $M$, either $\ker\chi$ is $\text{core}_G\left(M\right)$ or $G$. But $\chi$ is non-trivial, thus $\ker\left(\mathbf{1}_M^G\right) = \ker\chi$. □

In the next chapter, we will be applying some of these ideas to examine the number of maximal subgroups in Solvable groups.

# Chapter 2

# Upper Bound for the Number of Maximal Subgroups of Finite Solvable Groups

In this chapter, we will use some of the results we introduced in the previous chapter to give an upper bound for maximal subgroups in solvable groups.

We first present the results already existing in literature.

**Theorem 2.1** (Wall)**.** *For a finite solvable group $G$, $|m(G)| \leq |G|$*

*Proof.* See [14] $\square$

Below is a result which is an improvement on the above theorem.

**Theorem 2.2** (Cook, Wiegold and Williamson)**.** *Let $p$ be the smallest prime divisor of the order a finite solvable group $G$. Then,*

$$|m(G)| \leq \frac{|G| - 1}{p - 1}. \tag{2.1}$$

*The bound is achieved if and only if $G$ is elementary Abelian.*

Pál Hegedus[7] proved the above using character theory. We present his proof below because it is closely related to the our main result in chapter 3.

*Proof.* The idea of the proof is counting the number of maximal subgroups using the irreducible characters that occur as constituents of primitive permutation characters.

Let

1. $\mathcal{N}$, the set of maximal subgroups of $G$ that are normal.
2. $\mathcal{M}$, the full set of representatives from the conjugacy classes of non-normal maximal subgroups of $G$.

By theorem 1.10, we have

$$|G| - 1 = \sum_{\mathbf{1}_G \neq \chi \in \mathrm{Irr}(G)} \chi(1)^2 .$$

Primitive permutation characters are multiplicity-free (theorem 1.19). Moreover, if the maximal subgroup is normal, then its corresponding primitive permutation character consists of only linear characters. If the maximal subgroup is non-normal, then all the non-trivial constituents of its primitive permutation character have degree greater than 1.

We also know the the degree of an irreducible character divides $|G|$. Therefore, if $p$ is the smallest prime divisor of $|G|$, then

$$\begin{aligned}
|G|-1 = \sum_{\mathbf{1}_G \neq \chi \in \mathrm{Irr}(G)} \chi(1)^2 &\geq \sum_{N\in\mathcal{N}} \sum_{\langle \chi, \mathbf{1}_N^G\rangle = 1} \chi(1)^2 + \sum_{M\in\mathcal{M}} \sum_{\langle \chi, \mathbf{1}_M^G\rangle = 1} \chi(1)^2 \\
&= \sum_{N\in\mathcal{N}} \sum_{\langle \chi, \mathbf{1}_N^G\rangle = 1} \chi(1) + \sum_{M\in\mathcal{M}} \sum_{\langle \chi, \mathbf{1}_M^G\rangle = 1} \chi(1)^2 \\
&= \sum_{N\in\mathcal{N}} \sum_{\langle \chi, \mathbf{1}_N^G\rangle = 1} \chi(1) + \sum_{M\in\mathcal{M}} \sum_{\langle \chi, \mathbf{1}_M^G\rangle = 1} p\chi(1) \\
&= \sum_{N\in\mathcal{N}} (|G:N|-1) + \sum_{M\in\mathcal{M}} p(|G:M|-1) \\
&\geq \sum_{N\in\mathcal{N}} (p-1) + \sum_{M\in\mathcal{M}} (p-1)|G:M| \\
&= (p-1)\left(|\mathcal{N}| + \sum_{M\in\mathcal{M}} |G:M|\right)
\end{aligned}$$

and the proof is complete. □

In chapter 3 we will be characterising those solvable groups such that all their irreducible characters occur as constituents of primitive permutation characters. In the notation of the theorem, it means those groups $G$ such that

$$|G|-1 = \sum_{\mathbf{1}_G \neq \chi \in \mathrm{Irr}(G)} \chi(1)^2 = \sum_{N\in\mathcal{N}} \sum_{\langle \chi, \mathbf{1}_N^G\rangle = 1} \chi(1)^2 + \sum_{M\in\mathcal{M}} \sum_{\langle \chi, \mathbf{1}_M^G\rangle = 1} \chi(1)^2 .$$

We now look another related theorem.

**Theorem 2.3** (Herzog-Manz). *Let $G$ be a finite solvable group. If $p$ is the smallest prime divisor of $|G|$ and $q$ is the largest prime divisor of $|G|$, then*

$$|m(G)| \leq \frac{q|G|-p}{p(q-1)}. \tag{2.2}$$

Theorem 2.2 and 2.3 are equivalent for elementary Abelian groups. Further, the bound in 2.3 is attained for certain types of Frobenius groups(See 1 for a section on Frobenius groups). Let $p$ and $q$ be prime numbers such that $p \mid q-1$. Further, let $Q_i \cong \mathbb{Z}_q$, $1 \leq i \leq n$ for $n \in \mathbb{Z}$ and $P \cong \mathbb{Z}_p$. Consider the group $Q_1 \times Q_2 \times \cdots \times Q_n \rtimes P$. This group has order $q^n p$ and $P$ acts fixed-point-freely in the same way on each $Q_i$. This group has a unique maximal subgroup isomorphic to $Q_1 \times Q_2 \times \cdots \times Q_n$. The number of non-conjugate non-normal maximal subgroups of $G$ is $\frac{q^n-1}{q-1}$. Thus,

$$m(G) = q\left(\frac{q^n-1}{q-1}\right) + 1 = \frac{q^{n+1}-1}{q-1} = \frac{q|G|-p}{p(q-1)}.$$

It is also worth mentioning that the bound in theorem 2.3 is better than that of theorem 2.2 for groups which are not elementary Abelian. If $G$ is a $p$-group which is not elementary Abelian, then $\Phi(G) \neq 1$. Moreover, $\pi(G) = \pi(G/\Phi(G))$ and $m(G) = m(G/\Phi(G))$. Thus,

$$m(G) = m(G/\Phi(G)) \leq \frac{|G/\Phi(G)|-1}{p-1} < \frac{|G|-1}{p-1}.$$

If $G$ is not a $p$-group, then $p^2 \leq |G|$, which is equivalent to

$$\frac{(p+1)|G|}{p^2} \leq \frac{|G|-1}{p-1}.$$

Let $q$ be the largest prime divisor of $G$. Then $q \geq p+1$. This is equivalent to the inequality $\frac{q}{q-1} \leq \frac{p+1}{p}$. Thus,

$$\frac{q|G|-p}{p(q-1)} = \frac{q|G|}{p(q-1)} - \frac{1}{q-1} < \frac{q|G|}{p(q-1)} \leq \frac{(p+1)|G|}{p^2} \leq \frac{|G|-1}{p-1}.$$

**Theorem 2.4** (Newton). *Let $G$ be a finite solvable group with $|G| = {p_1}^{r_1}{p_2}^{r_2}\cdots{p_m}^{r_m}$ for distinct primes $p_1, p_2, \ldots, p_m$. If ${p_i}^{r_i} = \min\{{p_1}^{r_1} \mid 1 \leq i \leq m\}$ then*

$$|m(G)| \leq \frac{{p_1}^{r_1}-1}{p_1-1} + \sum_{i=2}^{m} \frac{{p_i}^{r_i+1}-p_i}{p_i-1}. \tag{2.3}$$

Below we present the proof of theorem 2.4. Our only contribution is to give a different proof of lemma 2.1 to that given by Newton[10] by using using techniques in character theory. Our attempts to give a completely different proof was unsuccessful.

**Lemma 2.1.** *Let $p$ be a prime number and $P$ be a finite $p$-group. Suppose that $\mathcal{P}$ is a set of proper subgroups of $P$ such that $AB = P$ for $A, B \in P$. Then*

$$\sum_{A \in \mathcal{P}} |P : A| \leq \frac{p(|P|-1)}{p-1}$$

*Proof.*

$$\begin{aligned}
\frac{p\left(|P|-1\right)}{p-1} &= \sum_{\chi\neq\mathbf{1}_G} \frac{p\chi(1)^2}{p-1} \\
&= \sum_{\chi\neq\mathbf{1}_G} \left(\frac{p\chi(1)}{p-1} - \sum_{A\in\mathcal{P}} \left\langle \mathbf{1}_A, \chi_A \right\rangle + \sum_{A\in\mathcal{P}} \left\langle \mathbf{1}_A^G, \chi \right\rangle\right)\chi(1) \\
&= \sum_{\chi\neq\mathbf{1}_G} \left(\frac{p\chi(1)}{p-1} - \sum_{A\in\mathcal{P}} \left\langle \mathbf{1}_A, \chi_A \right\rangle\right)\chi(1) + \sum_{\chi\neq\mathbf{1}_G}\sum_{A\in\mathcal{P}} \left\langle \mathbf{1}_A^G, \chi \right\rangle \chi(1) \\
&= \sum_{\chi\neq\mathbf{1}_G} \left(\frac{p\chi(1)}{p-1} - \sum_{A\in\mathcal{P}} \left\langle \mathbf{1}_A, \chi_A \right\rangle\right)\chi(1) + \sum_{A\in\mathcal{P}} \left(\mathbf{1}_A^G(1) - 1\right) \\
&\geq \sum_{\chi\neq\mathbf{1}_G} \left(\frac{p\chi(1)}{p-1} - 1\right)\chi(1) + \sum_{A\in\mathcal{P}} \left(\mathbf{1}_A^G(1) - 1\right) \\
&\geq \sum_{\chi\neq\mathbf{1}_G} \frac{\chi(1)}{p-1} + \sum_{A\in\mathcal{P}} \left(\mathbf{1}_A^G(1) - 1\right) \\
&\geq \sum_{A\in\mathcal{P}} \mathbf{1}_A^G(1) = \sum_{A\in\mathcal{P}} |P:A|
\end{aligned}$$

$\sum_{A\in\mathcal{P}} \left\langle \mathbf{1}_A, \chi_A \right\rangle \leq 1$ since $AB = P$ for any two distinct subgroups $A, B \in \mathcal{P}$. Also, $\sum_{\chi\neq\mathbf{1}_G} \frac{\chi(1)}{p-1} \geq \sum_{A\in\mathcal{P}} 1$. ☐

Let $G$ be a group and $p$ be a prime number. Define $\mathbf{O}^p(G)$ to be the unique smallest normal subgroup of $G$ of $p$-power index in $G$. That is,

$$\mathbf{O}^p(G) = \bigcap \left\{N \lhd G \mid N \text{ is a normal subgroup of } p\text{-power index in } G\right\}.$$

We present a lemma by Newton [10].

**Lemma 2.2.** *Let $M_1$ and $M_2$ be maximal subgroups of a finite group $G$, both of which have $p$-power index in $G$ and neither of which is normal in $G$. If $P_0 \in Syl_p\left(\mathbf{O}^p(G)\right)$, then*

$$\left(M_1 \cap P_0\right)\left(M_2 \cap P_0\right) = P_0$$

*Proof.* We begin by showing that $\left(M_1 \cap M_2\right)\mathbf{O}^p(G) = G$. From the definition of $\mathbf{O}^p(G)$ and since every subgroup of a finite group is contained in a maximal subgroup, it suffices to show that $M_1 \cap M_2$ cannot be in a normal maximal subgroup of index $p$ in $G$. Let $N$ be a normal maximal subgroup of index $p$ in $G$. By [3, Corollary A.16.7], $M_1 \cap M_2$ is a maximal subgroup of $M_2$, without loss of generality. Thus, if $M_1 \cap M_2 \leq N$, then $M_1 \cap M_2 \leq N \cap M_2 \leq M_2$. This implies that $M_1 \cap M_2 = N \cap M_2$ and thus $M_1 \cap M_2 \lhd M_2$. Therefore, $M_2 \leq \mathbf{N}_G\left(M_1 \cap M_2\right)$, the normaliser of $M_1 \cap M_2$ in $G$. Theorem 1.17 gives us that $M_1M_2 = G$. Thus,

$$\left(M_1 \cap M_2\right)^G = \left\{g^{-1}mg \mid g \in G \text{ and } m \in M_1 \cap M_2\right\} \leq M_1$$

The intersection of two maximal subgroups of $p$-power index is also of $p$-power index. This implies that $[G : (M_1 \cap M_2)^G]$ is also a power of $p$ and hence $\mathbf{O}^p(G) \leq (M_1 \cap M_2)^G \leq M_1$. This contradicts the fact that $M_1$ is maximal but not normal in $G$. Thus, $(M_1 \cap M_2)\,\mathbf{O}^p(G) = G$.

Let $P_0 \in \mathrm{Syl}_p(\mathbf{O}^p(G))$. We now wish to show that $(M_1 \cap P_0)(M_2 \cap P_0) = P_0$. Observe that $[\mathbf{O}^p(G) : (M_1 \cap M_2) \cap \mathbf{O}^p(G)] = [G : M_1 \cap M_2]$. Also, $((M_1 \cap M_2) \cap \mathbf{O}^p(G))\, P_0 = \mathbf{O}^p(G)$ so that

$$[P_0 : (M_1 \cap M_2) \cap P_0] = [G : M_1 \cap M_2].$$

We can use the same argument to show that $[P_0 : M_1 \cap P_0] = [G : M_1]$ and $[P_0 : M_2 \cap P_0] = [G : M_2]$. Putting these equations together we get

$$[P_0 : (M_1 \cap M_2) \cap P_0] = [G : M_1 \cap M_2] = [G : M_1]\,[G : M_2] = [P_0 : M_1 \cap P_0]\,[P_0 : M_2 \cap P_0].$$

Therefore, $(M_1 \cap P_0)(M_2 \cap P_0) = P_0$. □

**Theorem 2.5.** *Let $p$ be a prime number and let $G$ be a finite solvable group of order $p^k m$, where $p$ does not divide the natural number $m$. If $|G : \mathbf{O}^p(G)| = p^r$, Then*

$$\mathbf{m}_p(G) \leq \frac{p^r - 1}{p - 1} + \frac{p^{k-r+1-p}}{p-1}$$

*Proof.* The idea of the proof is to partition the set $\mathbf{m}_p(G)$ into two: those elements that are normal and those that are not. We will then realise those maximal subgroups that are normal in the $p$-group $G/\mathbf{O}^p(G)$ and those that are not in $P_0 \in \mathrm{Syl}_p(\mathbf{O}^p(G))$.

Let $\mathcal{M}$ be the set of maximal subgroups of $G$ of $p$-power index and let $\mathcal{A} = \{M \in \mathcal{M} \mid M \triangleleft G\}$ and $\mathcal{B} = \{M \in \mathcal{M} \mid M \ntriangleleft G\}$. There is a one-one correspondence between normal maximal subgroups of index $p$ and maximal subgroups of $G/\mathbf{O}^p(G)$. Thus,

$$|\mathcal{A}| \leq \frac{p^r - 1}{p - 1},$$

by theorem 2.2.

Let $M_1, M_2, \ldots, M_t$ be the complete set of conjugacy representatives of non-normal maximal subgroups of $G$. By lemma 2.2. $(M_i \cap P_0)(M_j \cap P_0) = P_0$ for $i \neq j$. Applying lemma 2.1

$$|\mathcal{B}| = \sum_{i=1}^{t} [G : M_i] \leq \frac{p^{k-r+1} - p}{p - 1}.$$

Thus, $\mathbf{m}_p(G) \leq \frac{p^r-1}{p-1} + \frac{p^{k-r+1}-p}{p-1}$. Moreover, observe that for a fixed value of $k$, $\mathbf{m}_p(G)$ is greatest when $r = 0$, in which case $\mathbf{m}_p(G) \leq \frac{p^{k+1}-p}{p-1}$. Also, if $\mathbf{O}^p(G) \leq G$, then $r \geq 0$ and $\mathbf{O}^p(G)$ is greatest when $r = 1$ or $r = k$. In this case, $\mathbf{m}_p(G) \leq \frac{p^k-1}{p-1}$. □

We are now ready to present the proof of Newton's theorem.

*Proof of theorem 2.4.* By theorem 1.7, we have that

$$\mathbf{m}\left(G\right)=\mathbf{m}_{p_1}\left(G\right)+\cdots+\mathbf{m}_{p_m}\left(G\right).$$

A solvable group has proper normal subgroup of prime power index. Therefore, $\mathbf{O}^{p_j}\left(G\right)\leq G$ and so $\mathbf{m}_{p_j}\left(G\right)\leq\frac{{p_j}^{r_j}-1}{p_j-1}$. Therefore,

$$\mathbf{m}\left(G\right)=\mathbf{m}_{p_1}\left(G\right)+\cdots+\mathbf{m}_{p_m}\left(G\right)\leq\frac{{p_j}^{r_j}-1}{p_j-1}+\sum_{i\neq j}\frac{{p_i}^{r_i+1}-p_i}{p_i-1}=\sum_{i=1}^{m}\frac{{p_i}^{r_i+1}-p_i}{p_i-1}-{p_j}^{r_j}+1.$$

Since ${p_1}^{r_1}\leq{p_i}^{r_j}$ we have

$$\mathbf{m}\left(G\right)\leq\sum_{i=1}^{m}\frac{{p_i}^{r_i+1}-p_i}{p_i-1}-{p_1}^{r_1}+1=\frac{{p_1}^{r_1}-1}{p_1-1}+\sum_{i=2}^{m}\frac{{p_i}^{r_i+1}-p_i}{p_i-1}.$$

□

Having review some results on upper bounds of the number of maximal subgroups in solvable group, we will now look at a result which relates primitive permutation characters and irreducible characters of a solvable group.

# Chapter 3

# Constituents of Primitive Permutation Characters for Solvable Groups

We now present a result concerning solvable groups all of whose complex irreducible characters are constituents of primitive permutation characters. This idea grew out of Hegedus' proof in which he gave a bound on the number of maximal subgroups of a group by looking at primitive permutation characters of the group. His main observation in that proof are, the trivial character is the only irreducible character which is a common constituent of distinct primitive permutation characters. Since the order of the group equals the sum of squares of the degrees of the irreducible characters of the group, we can relate the number of maximal subgroup with the order of the group. However, an irreducible character need not be a constituent of a primitive permutation character. In this chapter, we will characterise those groups such that all irreducible characters are constituents of a primitive permutation characters of the group. We will show that such a group is either elementary Abelian or a Frobenius group with elementary Abelian kernel and complement a cyclic group of prime order.

We begin with an exposition of the structure and character theory of Frobenius groups.

## 1 The Structure of a Frobenius Group

**Definition 3.1.** *A Frobenius group $G$ is a permutation group acting transitively of a set $\Omega$ such that for all $\omega \in \Omega$, $Stab_G(\omega) \neq 1$, and if $\omega_1 \neq \omega_2$ for $\omega_1, \omega_2 \in \Omega$ then*

$$Stab_G(\omega_1) \cap Stab_G(\omega_2) = 1.$$

**Example 3.** The symmetric group on three elements, $S_3$, is a Frobenius group.

$S_3 = \{(1), (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$ acts transitively of the set $\{1, 2, 3\}$ in a natural way. The stabilizer of the points $1, 2$ and $3$ are the subgroups $\{(1), (2,3)\}$, $\{(1), (1,3)\}$ and $\{(1), (2,3)\}$, respectively. These subgroups intersect trivially pairwise.

If a group $G$ acts transitively on $\Omega$, this action is isomorphic to the the action of $G$ on the coset space, $H\backslash G$ for some subgroup $H$ of $G$. For this reason, we can also say that a group $G$ is a Frobenius group if and only if it has a nontrivial subgroup $H$ such that $H \cap x^{-1}Hx = 1$ for all $x$ in $G \backslash H$. The subgroup $H$ is a called the Frobenius complement and observe that $N_G(H) = H$.

Let $N = G \backslash \{H^x \mid x \in G\} \cup 1$. The non-trivial elements of this set are those which do not fix any element of $\Omega$.

In 1906, Frobenius proved that $N$ is a normal subgroup. It is called the Frobenius kernel.

**Theorem 3.1** (Frobenius)**.** *The Frobenius kernel of a Frobenius group is a normal subgroup*

*Proof.* See [8, Theorem 7.2] □

As $N_G(H) = H$ there are $[G : H]$ distinct conjugates of $H$ in $G$. Therefore,

$$|\cup \{H^x \mid x \in G\}| = [G : H]\,(|H| - 1) + 1 = |G| - [G : H] + 1.$$

Hence

$$|N| = |G| - (|G| - [G : H] + 1) + 1 = [G : H].$$

Thus, $G$ is the semidirect product of $N$ and $H$. In symbols, $G = N \rtimes H$. Since $H \cap H^x = 1$, for $x \in G \backslash H$, we get that $C_G(n) \leq N$, for $1 \neq n \in N$. For if $x \in C_G(n)$ but $x \notin N$, then there is a $g \in G$ such that $x \in H^g$. This implies that ${}^g x \in H \cap C_G({}^g n) = 1$. That is $x = 1$, a contradiction.

As a Frobenius group is a semidirect product of its kernel $N$ and complement $H$, $H$ acts by conjugation on $N$ so that for $1 \neq n \in N$, $Stab_H(n) = C_H(n) = 1$. By the orbit-stabiliser theorem, this action has one orbit of size one and the others have size $|H|$. It follows therefore that $|H|$ divides $|N| - 1$.

The converse is also true, that is, if a group $G$ has a non-trivial normal subgroup $N$ such that $C_G(n) \leq N$, for $1 \neq n \in N$, then $G$ is a Frobenius group.

**Theorem 3.2.** *A finite group $G$ is Frobenius if and only if it has a non-trivial proper normal subgroup $N$ such that if $1 \neq n \in N$ then $C_G(n) \leq N$.*

*Proof.* See [6, Theorem 9.2.1] □

The Frobenius kernel and the Frobenius complement have very restricted structures.

**Theorem 3.3** (Thompson)**.** *The Frobenius kernel is a nilpotent group.*

*Proof.* See [13]. □

This theorem implies that the Frobenius kernel of a Frobenius group is solvable.

**Theorem 3.4.** *The Frobenius kernel of a Frobenius group is unique.*

*Proof.* See [6, Theorem 9.2.8] □

**Theorem 3.5.** *If $G$ is Frobenius with complement $H$, then no subgroup of $H$ is Frobenius.*

*Proof.* See [6, Theorem 9.2.7]. □

**Theorem 3.6.** *Suppose that $G$ is Frobenius with complement $H$ and $p, q$ are prime numbers. If $K \leq H$ and $|K| = pq$, then $K$ is cyclic.*

*Proof.* See [6, Propostition 9.2.9]. □

**Theorem 3.7.** *Suppose that $G$ is Frobenius with complement $H$ and $p$ be a prime divisor of $|H|$.*

- *If $p$ is odd, then a Sylow $p$-subgroup of $H$ is cyclic.*
- *If $p$ is even, then a Sylow $p$-subgroup of $H$ is a cyclic group or a generalised quaternion group of order greater or equal to 8.*

*Proof.* See [6, Theorem 9.2.10]. □

**Theorem 3.8** (Zassenhaus). *Let $G$ be a Frobenius group with kernel $N$ and complement $H$. If $H$ is not a solvable group then it has a normal subgroup $H_0$ of index at most two such that*

$$H_0 \cong SL(2,5) \times M, \quad \textit{where } M \textit{ is a metacyclic group.}$$

*Proof.* See [11, Theorem 18.6] □

# 2 Character Theory of Frobenius Groups

Frobenius groups have interesting character theory. Stating that, we can obtain all its irreducible characters from its kernel and complement by induction and inflation, respectively.

**Theorem 3.9** (Brauer). *Let $G$ and $H$ be groups. Suppose that $G$ acts on Irr$(H)$ and on the set of conjugacy classes of $H$, Cl$(H)$. If $h$ a is conjugacy class representative and $g \in G$, the action of $g$ on $h$ is denoted by $h^g$. If*

$$\chi(h) = \chi^g(h^g), \quad \forall \chi \in \textit{Irr}(H), g \in G, h \in H,$$

*then*

$$\left|\textit{fix}_{\textit{Irr}(H)}(g)\right| = \left|\textit{fix}_{\textit{Cl}(H)}(g)\right|, \quad \forall g \in G.$$

*Proof.* See [8, Theorem 6.32] □

**Proposition 3.1.** *Let $G$ be a Frobenius group with kernel $N$. If $1_N \neq \varphi \in \textit{Irr}(N)$, then $I_G(\varphi) = N$.*

*Proof.* We will show that for $g \in G \backslash N$, $|\text{fix}_{\text{Irr}(N)}(g)| = 1$, that is, the only irreducible character which is stabilized by elements of $G \backslash N$ is the trivial character. This will then imply our result.

Let $g \in G$, $n \in N$ and $\varphi \in \text{Irr}(N)$ then

$$\varphi(n) = \varphi^g\left(g^{-1}ng\right) = \varphi^g(n^g).$$

Thus, theorem 3.9 applies and showing that $|\mathrm{fix}_{\mathrm{Cl}(N)}(g)| = 1$ implies that $|\mathrm{fix}_{\mathrm{Irr}(N)}(g)| = 1$ for $g \in G \backslash N$. Let $n$ be a conjugacy representative of congugacy class of $N$ and suppose that there is a $g \in G \backslash N$ such that $n^g = n$. This means that there is an $n' \in N$ such that $n^g = n^{n'}$. This gives that $\left(gn'^{-1}\right)^{-1} n \left(gn'^{-1}\right) = n$, that is $\left(gn'^{-1}\right) \in \mathrm{C}_G(n)$ or $g \in \mathrm{C}_G(n)\, n'$. Unless $n = 1$, we get a contradiction as $C_G(x) \leq N$ for $1 \neq n \in N$ by theorem 3.2. □

For a Frobenius group $G$ with kernel $N$, let $\varphi \in \mathrm{Irr}(N) \setminus \{1_N\}$. We would like to show that $\varphi^G$ is irreducible. Let $\chi \in \mathrm{Irr}(G)$ be a constituent of $\varphi^G$. Then $\varphi$ is a constituent of $\chi_N$ and thus by Clifford's theorem

$$\chi_N = \left\langle \chi, \varphi^G \right\rangle \sum_{i=1}^{[G:N]} \varphi_i$$

where $\varphi_i \in \mathrm{Orb}_G(\varphi)$. Thus,

$$\left\langle \chi, \varphi^G \right\rangle [G:N] \varphi(1) = \chi_N(1) = \chi(1) \leq \varphi^G(1) = [G:N]\varphi(1).$$

From this we get that $\left\langle \chi, \varphi^G \right\rangle = 1$ and even more, that $\chi = \varphi^G$.

**Theorem 3.10.** *Let $G$ be a Frobenius group with kernel $N$ and complement $H$. Then*

1. *$\mathrm{Irr}(G) = \mathrm{Irr}(H) \cup \left\{\varphi^G \mid 1_N \neq \varphi \in \mathrm{Irr}(N)\right\}$.*
2. *Let $\varphi_1, \varphi_2 \in \mathrm{Irr}(N) \setminus \{1_N\}$. Then $\varphi_1^G = \varphi_2^G$ if and only if $\varphi_2 \in \mathrm{Orb}_G(\varphi_1)$.*

*Proof.* 1. Let $\chi \in \mathrm{Irr}(G)$. Then either $N \not\subseteq \ker \chi$ or $N \subseteq \ker \chi$. If $N \not\subseteq \ker \chi$ then there exist $\varphi \in \mathrm{Irr}(N) \setminus \{1_N\}$ which is a constituent of $\chi_N$. By Frobenius reciprocity,

$$\langle \chi_N, \varphi \rangle = \left\langle \chi, \varphi^G \right\rangle.$$

Thus, $\chi$ is a constituent of $\varphi^G$. Since $\varphi^G$ is irreducible, we get $\varphi^G = \chi$.

If $N \subseteq \ker \chi$ then $\chi$ may be viewed as a character of $G/N \cong H$ as there is a one to one correspondence between irreducible characters of $G$ which contain $N$ in their kernel and irreducible characters of $G/N$.

2. $\varphi_2 \in \mathrm{Orb}_G(\varphi_1)$ if and only if there exists $h \in H$ such that $\varphi_2 = \varphi_1^h$. Therefore,

$$\begin{aligned}
\varphi_2^G(g) &= \frac{1}{|N|} \sum_{x \in G} \varphi_2^x(g) \\
&= \frac{1}{|N|} \sum_{hx \in G} \varphi_2^{hx}(g) \\
&= \frac{1}{|N|} \sum_{hx \in G} \varphi_1^x(g) \\
&= \frac{1}{|N|} \sum_{x \in G} \varphi_1^x(g) \\
&= \varphi_2^G(g).
\end{aligned}$$

□

As the stabiliser, $I_H(\varphi) = 1$, by the orbit-stabiliser theorem, $|\mathrm{Orb}_H(\varphi)| = |H|$. Thus,

$$|\mathrm{Irr}\,(G)| = |\mathrm{Irr}\,(H)| + \frac{(|\mathrm{Irr}\,(N)| - 1)}{|H|}.$$

# 3 Frobenius Groups of a Special Type

We would like to examine the structure of special kind of Frobenius groups which play a central role in the main theorem of this chapter.

Let $p$ and $q$ be prime numbers and $n$ be a natural number. Also, let $N \cong \mathbb{Z}_q^n$ and $H \cong \mathbb{Z}_p$ be an elementary Abelian group and a cyclic group of prime order, respectively. Suppose that $H$ acts fixed-point-freely on $N$. Then $N \rtimes H$ is a Frobenius group and $p \mid q^n - 1$.

This group has a unique normal maximal subgroup $N$. Let $e$, $1 \leq e \leq n$ be the smallest such that $p \mid q^e - 1$ and $M$ be a non-normal maximal subgroup. Then $|M| = q^{n-e}p$ and the number of non-conjugate non-normal maximal subgroups of $G$ is $\frac{q^n-1}{q^e-1}$. For the rest of this chapter, we refer to such groups as groups of type X.

# 4 Constituents of Primitive Permutation Characters

We are now ready to present the main result of this chapter.

**Lemma 3.1.** *Let $N$ be a normal subgroup of $G$. If every irreducible character of $G$ is a constituent of a primitive permutation character of $G$. Every irreducible character of $G/N$ is a constituent of a primitive permutation character of $G/N$.*

*Proof.* Let $\chi$ be an irreducible character of $G/N$ and $\chi'$ be the inflation of $\chi$ to G. There exists a maximal subgroup $M < G$, such that

$$\left\langle \chi', \mathbf{1}_M^G \right\rangle = \left\langle {\chi'}_M, \mathbf{1}_M \right\rangle = 1.$$

Since $N \leq \ker \chi'$ and $\ker \chi \leq M$, we get that $\left\langle \chi_{M/N}, \mathbf{1}_{M/N} \right\rangle = 1$ and hence $\left\langle \chi, \mathbf{1}_{M/N}^{G/N} \right\rangle = 1$. □

**Theorem 3.11.** *Let $G$ be a finite solvable group. Every irreducible complex character of $G$ is a constituent of a primitive permutation character of $G$ if and only if $G$ is elementary Abelian or $G$ is a group of type X.*

*Proof.* To see that this property holds for elementary Abelian groups and Frobenius groups of our type, we will use that fact primitive permutation characters are multiplicity-free and count the number of primitive permutation characters and irreducible characters and check that the degrees add up.

- Let $G$ be an elementary Abelian group and let $|G| = p^n$. Then $G$ has $\frac{p^n-1}{p-1}$ of maximal subgroups all of have order $p^{n-1}$. If $M$ be a maximal subgroup of $G$, then $\dim 1_M^G = p$ and since we know that the primitive permutation characters are multiplicity free and that an irreducible character of a group $G$ occurs in at most one primitive permutation character we get

$$(p-1)\frac{p^n-1}{p-1} = p^n - 1.$$

  This equals the number of irreducible characters of $G$

- If $G = N \rtimes H$ is a group of type X, then the $|H|$ linear characters all of which are constituents of the permutation character $1_N^G$. It has $\frac{|N|-1}{|H|}$ non-linear characters each of degree $|H|$.

  Let $|N| = p^n$ and $|H| = q$. If $e$ is the smallest number such that $q \mid p^e - 1$. If $M$ is a non-normal maximal subgroup of $G$, then $|M| = p^{n-e}q$. Adding the degrees of those non-linear irreducible which are constituents of a primitive permutation character, we get

$$\left(\frac{p^n q}{p^{n-e}q} - 1\right) \times \frac{p^n-1}{p^e-1} = p^n - 1 = |N| - 1,$$

  which equals the sum of degrees of the non-principal constituents of $G$.

Conversely, suppose $G$ is a finite solvable group such that each irreducible character of $G$ is a constituent of a primitive permutation character of $G$.

Suppose $G$ is Abelian, then all its irreducible characters are linear. Let $\chi \in \mathrm{Irr}(G)$. By assumption, there exist $M < G$ maximal such that $\langle 1_M^G, \chi \rangle = 1$. By lemma 1.3, $\ker \chi = M$. Thus, requiring that all the irreducuble characters of $G$ are constituents of primitive permutation characters of $G$ implies that all the non-trivial irreducible characters have kernels which are maximal in $G$. It follows that $G$ is an elementary Abelian group.

Now suppose that $G$ is not Abelian and that $|G| \geq 6$. We will proceed by induction on the order of $G$ to prove that $G$ is of type X.

Let $N$ be a minimal normal subgroup of $G$. We know that it is elementary Abelian since $G$ is solvable. By 3.1, the factor group $G/N$ satisfies our assumption. By the induction hypothesis, either $G/N$ is elementary Abelian or $G/N$ is Frobenius with elementary Abelian kernel and complement of prime order.

Since the intersection of the kernels of all the irreducible characters of a group is the trivial group, requiring that all the irreducible characters occur as constituents of primitive permutation characters implies that the intersection of all non-conjugate maximal subgroups of the group is trivial. Therefore, $\Phi(G) = 1$. Thus, there exists a maximal $H$ such that and $N \nleq H$. And thus, $NH = G$ and $H \cap N = 1$ since $N \cap H$ is a normal subgroup in $H$ and also in $N$. Therefore, $G$ is a direct product of $N$ and $H$ that is, $G = N \rtimes H$ and $G/N \cong H$.

**Case** 1. Suppose that $G/N$ is elementary Abelian. Let $N \cong \mathbb{Z}_p^n$ and $H \cong \mathbb{Z}_q^m$. Now, $p \neq q$ since $G$ cannot be a $p$-group as all they have non-trivial

Frattini subgroups. Now, $\mathbb{Z}_q^m$ acts on $\mathbb{Z}_p^n$ irreducibly and thus has kernel isomorphic to $\mathbb{Z}_q^{m-1}$. So that is $m > 1$, then this kernel will be a non-trivial normal subgroup of G. Thus, $G = \left(\mathbb{Z}_p^n \ltimes \mathbb{Z}_q\right) \times \mathbb{Z}_q^{m-1}$. This group has two types of maximal subgroups:

(a) A Sylow $q$-subgroup isomorphic to $\mathbb{Z}_q^m$.

(b) Maximal subgroups containing the minimal normal subgroup $N$.

We have that

$$\mathbf{1}_N^G = \mathbf{1}_{\mathbb{Z}_q}^{N \ltimes \mathbb{Z}_q}.$$

Therefore, this group have $q^m - 1$ non-trivial linear irreducible characters and $\frac{p^n-1}{q}$ non-linear irreducible character each of dimension $q$. We want

$$\frac{p^n-1}{q}q^2 + q^m - 1 = p^n q^m - 1.$$

This simplifies to

$$\left(p^n - 1\right)\left(q^m - q\right) = 0.$$

Thus, $m = 1$ and $G = \mathbb{Z}_p^n \ltimes \mathbb{Z}_q$, which is what we want.

**Case** 2. Suppose that $G/N$ is a group of type X. We have two possibilities. Either $C_H\left(N\right) = 1$ or $C_H\left(N\right) \neq 1$.

**Case** 2.1. If $C_H\left(N\right) = 1$ then $G$ is Frobenius with kernel $N$ and complement $H$. Now, a Frobenius complement cannot be a Frobenius group or or elementary Abelian of order $r^m$ for some prime $r$ and $m > 1$. Thus, $H$ has prime order and our result follows.

**Case** 2.2. Suppose that $C_H\left(N\right) \neq 1$. Let $M = C_G\left(N\right) = C_H\left(N\right) \times N$. We know that $C_H\left(N\right)$ is normal in $G$ and is the kernel of the action by conjugation of $H$ on $N$. Let $G/C_H\left(N\right) = N \rtimes K$. Then $K$ acts faithfully on $N$, and thus $N \rtimes K$ is Frobenius and $K$ has prime order by case A. Furthermore, by the induction hypothesis, $H$ is Frobenius with Frobenius kernel $C_H\left(N\right)$ since

$$G/M \cong K$$

Let $N \cong \mathbb{Z}_p^n$, $C_H\left(N\right) \cong \mathbb{Z}_q^m$ and $K \cong \mathbb{Z}_r$ for prime numbers $p, q, r$ and integers $n, m$. Suppose $p \neq q$. Let $m_1$ be the smallest integer such that $r \mid q^{m_1} - 1$ and $m_2 = m - m_1$. Then $G \cong \mathbb{Z}_p^n \times \mathbb{Z}_q^m \rtimes Z_r$ has unique normal maximal subgroup $\mathbb{Z}_p^n \times \mathbb{Z}_q^m$ and non-normal maximal subgroups isomorphic to either $\mathbb{Z}_p^n \times \mathbb{Z}_q^{m_2} \rtimes \mathbb{Z}_r$ or $\mathbb{Z}_q^m \rtimes \mathbb{Z}_r$. The subgroups $\mathbb{Z}_p^n \times \mathbb{Z}_q^{m_2} \rtimes \mathbb{Z}_r$ and $\mathbb{Z}_p^m \rtimes \mathbb{Z}_r$ have core $\mathbb{Z}_p^n \times \mathbb{Z}_q^{m_2}$ and $\mathbb{Z}_q^m$, respectively.

We will find an irreducible character of $G = \mathbb{Z}_p^n \times \mathbb{Z}_q^{m_2} \rtimes \mathbb{Z}_r$ whose kernel is neither isomorphic to $\mathbb{Z}_p^n \times \mathbb{Z}_q^{m_2}$ nor $\mathbb{Z}_q^m$. Let $\chi$

be a non-trivial irreducible character of $\mathbb{Z}_p^n \times \mathbb{Z}_q^m$ with kernel isomorphic to $\mathbb{Z}_p^{n-1} \times \mathbb{Z}_q^{m-1}$. Since $\mathbb{Z}_r$ acts faithfully on $\mathbb{Z}_p^n \times \mathbb{Z}_q^m$, we have that $I_G(\chi) = \mathbb{Z}_p^n \times \mathbb{Z}_q^m$. Therefore, $\chi^G$ is irreducible and has kernel isomorphic to the $\text{core}_G\left(\mathbb{Z}_p^{n-1} \times \mathbb{Z}_q^{m-1}\right)$. It is not possible to have $\text{core}_G\left(\mathbb{Z}_p^{n-1} \times \mathbb{Z}_q^{m-1}\right) \cong \mathbb{Z}_p^n \times \mathbb{Z}_q^{m_2}$ or $\text{core}_G\left(\mathbb{Z}_p^{n-1} \times \mathbb{Z}_q^{m-1}\right) \cong \mathbb{Z}_q^m$
We therefore must have that $p = q$ and the proof is complete.

□

This whole thesis grew out of idea of using character theory to prove results in finite group theory. Initially, we started out to prove a known result, Newton's result, using ideas in characters of groups. Although this was not as successful as we envisioned, we stumbled on the idea of characterising all those solvable groups all whose irreducible characters occurred at constituents of primitive permutation characters. We found that these groups are elementary Abelian groups and Frobenius groups of a special type.

It would be worthwhile to investigate more properties relating maximal subgroups of a group and its characters in finite groups.

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